1. Solve the following differential equations using the Laplace Transform method.

(a) $\ddot{x}+5\dot{x}+4x=0 \qquad x(0)=4 \quad \dot{x}(0)=-10$

- Taking the Laplace transform:

$$[s^2X - sx(0)-\dot{x}(0)]+5[sX-x(0)]+4X=0$$

$$[s^2X-4s+10]+5[sX-4]+4X=0$$

$$(s^2+5s+4)X=4s+10 \;\Rightarrow\; X=\frac{(4s+10)}{(s^2+5s+4)}$$

- Partial fraction expansion:

$$X=\frac{(4s+10)}{(s^2+5s+4)}=\frac{(4s+10)}{(s+1)(s+4)}=\frac{A}{(s+1)}+\frac{B}{(s+4)}$$

Using the cover-up method,

$$A=\left.\frac{(4s+10)}{(s^2+5s+4)}(s+1)\right|_{s=-1}=\left.\frac{(4s+10)}{(s+4)}\right|_{s=-1}=\frac{6}{3}=2$$

$$B=\left.\frac{(4s+10)}{(s^2+5s+4)}(s+4)\right|_{s=-4}=\left.\frac{(4s+10)}{(s+1)}\right|_{s=-4}=\frac{-6}{-3}=2$$

$$X=\frac{2}{(s+1)}+\frac{2}{(s+4)}$$

- Taking the Inverse Laplace Transform:

$$L^{-1}\left(\frac{2}{(s+1)}+\frac{2}{(s+4)}\right)=2e^{-t}+2e^{-4t}$$

$$x(t)=2e^{-t}+2e^{-4t}$$

(b) $\ddot{x}+6\dot{x}+34x=68 \qquad x(0)=1 \quad \dot{x}(0)=0$

- Taking the Laplace transform:

$$[s^2X-sx(0)-\dot{x}(0)]+6[sX-x(0)]+34X=\frac{68}{s}$$

$$[s^2X-s]+6[sX-1]+34X=\frac{68}{s}$$

$$(s^2+6s+34)X=s+6+\frac{68}{s}=\frac{s^2+6s+68}{s} \;\Rightarrow\; X=\frac{(s^2+6s+68)}{s(s^2+6s+34)}$$

- Partial fraction expansion:

$$X=\frac{(s^2+6s+68)}{s(s^2+6s+34)}=\frac{(s^2+6s+68)}{s[(s^2+3)^2+5^2]}=\frac{A}{s}+\frac{B(s+3)}{(s^2+3)^2+5^2}+\frac{C\cdot 5}{(s^2+3)^2+5^2}$$

Using the cover-up method,

$$A=\left.\frac{(s^2+6s+68)}{s(s^2+6s+34)}s\right|_{s=0}=\left.\frac{(s^2+6s+68)}{(s^2+6s+34)}\right|_{s=0}=\frac{68}{34}=2$$

By multiplying the both sides by $s(s^2+6s+34)$,

$$s^2+6s+68=2(s^2+6s+34)+B\,s(s+3)+C\cdot 5s=(2+B)s^2+(12+3B+5C)s+68$$

Equating like powers of *s*,

$$\left.\begin{aligned}1&=2+B\\6&=12+3B+5C\end{aligned}\right\}\Rightarrow\begin{cases}B=-1\\C=-\dfrac{3}{5}\end{cases}$$

$$X=\frac{(s^2+6s+68)}{s[(s^2+3)^2+5^2]}=\frac{2}{s}-\frac{(s+3)}{(s^2+3)^2+5^2}-\frac{3}{5}\frac{5}{(s^2+3)^2+5^2}$$

- Taking the Inverse Laplace Transform:

$$L^{-1}\left(\frac{1}{s}-\frac{(s+3)}{(s^2+3)^2+5^2}-\frac{3}{5}\frac{5}{(s^2+3)^2+5^2}\right)=2-e^{-3t}\cos 5t-\frac{3}{5}e^{-3t}\sin 5t$$

$$x(t)=2-e^{-3t}\cos 5t-\frac{3}{5}e^{-3t}\sin 5t$$

2. Derive the transfer function of each system given below and plot the system poles in the complex plane. What is the damping condition for each system (when applicable)?

(a) $2\dot{x}(t)+8x(t)=f(t),\ x(0)=0$

- Laplace transform: $(2s+8)X(s)=F(s)$
- Transfer function: $T(s)=\dfrac{X(s)}{F(s)}=\dfrac{1}{2s+8}$
- System poles: $2s+8=0$ ➔ $s=-4$
- No damping can be defined for a first-order system.

(b) $\ddot{x}+4\dot{x}(t)+13x(t)=2f(t),\ x(0)=\dot{x}(0)=0$

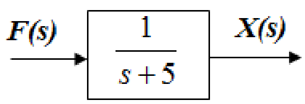
- Laplace transform: $(s^2+4s+13)X(s)=2F(s)$
- Transfer function: $T(s)=\dfrac{X(s)}{F(s)}=\dfrac{2}{s^2+4s+13}$
- System poles: $s^2+4s+13=0$ ➔ $s=-2\pm 3j$
- The system is under damped.

(c) $2\ddot{x}(t)+7\dot{x}(t)+2x(t)=\dot{f}(t)+3f(t),\ x(0)=\dot{x}(0)=0$

- Laplace transform: $(2s^2+7s+2)X(s)=(s+3)F(s)$
- Transfer function: $T(s)=\dfrac{X(s)}{F(s)}=\dfrac{s+3}{2s^2+7s+2}$
- System poles: $2s^2+7s+2=0$ ➔ $s=-0.32,-3.19$
- The system is over damped.

3. Find the unit step response, *x(t)* of the following systems.

(a)



$$X(s)=T(s)F(s)=\frac{1}{s+5}\cdot\frac{1}{s}=\frac{1}{s(s+5)}$$

- Partial fraction expansion:

$$X=\frac{1}{s(s+5)}=\frac{A}{s}+\frac{B}{s+5}$$

Using the cover-up method,

$$A=\frac{1}{s\,(s+5)}s\Bigg|_{s=0}=\frac{1}{(s+5)}\Bigg|_{s=0}=\frac{1}{5}$$

$$B=\frac{1}{s\,(s+5)}(s+5)\Bigg|_{s=-5}=\frac{1}{s}\Bigg|_{s=-5}=-\frac{1}{5}$$

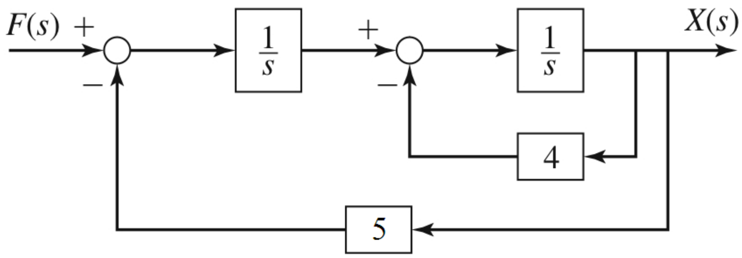
$$X=\frac{1}{5s}-\frac{1}{5(s+5)}$$

- Taking the Inverse Laplace Transform:

$$L^{-1}\left(\frac{1}{5s}-\frac{1}{5(s+5)}\right)=\frac{1}{5}-\frac{1}{5}e^{-5t}$$

$$x(t)=\frac{1}{5}-\frac{1}{5}e^{-5t}$$

(b)



$$\frac{X(s)}{F(s)}=\frac{\frac{1}{s}\left(\frac{1/s}{1+4/s}\right)}{1+5\frac{1}{s}\left(\frac{1/s}{1+4/s}\right)}=\frac{\frac{1}{s}\left(\frac{1}{s+4}\right)}{1+5\frac{1}{s}\left(\frac{1}{s+4}\right)}=\frac{1}{s(s+4)+5}=\frac{1}{s^2+4s+5}$$

$$X(s)=T(s)X(s)=\frac{1}{s^2+4s+5}\cdot\frac{1}{s}=\frac{1}{s(s^2+4s+5)}=\frac{A}{s}+\frac{B(s+2)}{(s+2)^2+(1)^2}+\frac{C}{(s+2)^2+(1)^2}$$

By multiplying the both sides by $s(s^2+4s+5)$,

$$1=A(s^2+4s+5)+Bs(s+2)+C\cdot s=(A+B)s^2+(4A+2B+C)s+5A$$

Equating like powers of s,

$$\left.\begin{aligned}1&=5A\\0&=A+B\\0&=4A+2B+C\end{aligned}\right\}\Rightarrow\left\{\begin{aligned}A&=1/5\\B&=-1/5\\C&=-2/5\end{aligned}\right.$$

$$X(s)=\frac{1}{5}\frac{1}{s}-\frac{1}{5}\frac{(s+2)}{(s+2)^2+(1)^2}-\frac{2}{5}\frac{1}{(s+2)^2+(1)^2}$$

⇨ $x(t)=\frac{1}{5}-\frac{1}{5}e^{-2t}\cos t-\frac{2}{5}e^{-2t}\sin t$