2. Modeling in the Frequency Domain

Laplace Transform
Transfer Function
Block Diagram

Laplace Transform

1. *Definition*
   1) The Laplace transform of a function, $f(t)$ is defined as
   $$L[f(t)] = F(s) = \int_0^{\infty} f(t) e^{-st} dt$$
   2) The transform is a function of the parameter s, which is a complex number, *i.e.*, ____________.
   3) It's a mapping from a function of time t to a function of s, or $L: t \rightarrow s$.
   4) The time function $f(t)$ whose transform is $F(s)$ is defined by ____________________, where the symbol L^{-1} denotes the inverse transform.
2. *Characteristics of Laplace Transform*
   1) The Laplace transform provides a systematic and general method for solving linear ODEs.
   2) The method converts **linear differential equations** into ____________________ that can be handled easily.
   3) Initial conditions of differential equations are implicitly taken care of.
   4) The general solution is found directly so it is not needed to find the complementary solution and the particular solution separately.

Laplace Transforms of Simple Functions

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1. *Step Function*

The unit-step function is defined as $u_s(t)=\begin{cases}0 & t<0\\ 1 & t>0\end{cases}$.

Then, $L[u_s(t)]=\int_0^\infty u_s(t)e^{-st}dt=$

2. *Exponential Function*

$$L[e^{-at}]=\int_0^\infty e^{-at}e^{-st}dt=$$

3. *Sine and Cosine Functions*

$L[\cos\omega t]=$ and $L[\sin\omega t]=$

For derivation, you will need to use the transform of exponential functions.

Derivative Theorem

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1. *Laplace Transform of First Derivative*

By applying integration by parts, we obtain

$$L\left[\frac{dx}{dt}\right]=\int_0^\infty \tfrac{dx}{dt}e^{-st}dt=x(t)e^{-st}\Big|_0^\infty+s\int_0^\infty x(t)e^{-st}dt$$

$$=$$

2. *Laplace Transform of Second Derivative*

$$L\left[\frac{d^2x}{dt^2}\right]=\int_0^\infty \tfrac{d^2x}{dt^2}e^{-st}dt=\tfrac{dx}{dt}e^{-st}\Big|_0^\infty+s\int_0^\infty \tfrac{dx}{dt}e^{-st}dt$$

$$=sL\left[\frac{dx}{dt}\right]-\dot{x}(0)=s[sX(s)-x(0)]-\dot{x}(0)$$

$$=s^2X(s)-sx(0)-\dot{x}(0)$$

Thus, **differentiation** in time domain is converted into _______ _____ in s-domain

3. *Laplace Transform of Higher Derivatives*

$$L\left[\frac{d^nx}{dt^n}\right]=s^nX(s)-s^{n-1}x(0)-s^{n-2}\dot{x}(0)-\cdots x^{(n-1)}(0)$$

Integral Theorem

1. *Laplace Transform of Indefinite Integral*

$$L\left[\int x(t)dt\right] = \int_0^\infty \left[\int x(t)dt\right] e^{-st}dt = \left[\int x(t)dt\right]\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty x(t)\frac{e^{-st}}{-s}dt$$

$$= \frac{1}{s}\int x(t)dt\bigg|_{t=0} + \frac{1}{s}\int_0^\infty x(t)e^{-st}dt$$

$$= \frac{X(s)}{s} + \frac{1}{s}\int x(t)dt\big|_{t=0}$$

2. *Laplace Transform of Definite Integral*

If $\int_0^t x(t)dt$ is used instead of $\int x(t)dt$, the above result becomes

$$L\left[\int_0^t x(t)dt\right] = \int_0^\infty \left[\int_0^t x(t)dt\right] e^{-st}dt = \underline{\qquad\qquad}$$

Thus, **integration** in time domain is converted into ___________ in s-domain

Initial and Final Value Theorems

1. *Initial Value Theorem*
   1) The initial value of the function, $x(t)$ can be found from its Laplace transform such that

$$x(0_+) = \lim_{t \to 0_+} x(t) = \underline{\qquad\qquad\qquad}$$

   2) This theorem is valid when the latter limit exists and the transforms of $x(t)$ and dx/dt exist.

2. *Final Value Theorem*
   1) The value of the function, $x(t)$ when t approaches infinity can be found from its Laplace transform such that

$$x(\infty) = \lim_{t \to \infty} x(t) = \underline{\qquad\qquad\qquad}$$

   2) This theorem is true if the functions $x(t)$ and dx/dt have Laplace transforms and $x(t)$ approaches a constant value as $t \to \infty$. The latter condition will be satisfied if all the roots of the denominator of $sX(s)$ have **negative real parts**.

Partial-Fraction Expansion (1/3)

- In order to invert the Laplace transform of a function back to its original function of time, we need to use the partial-fraction expansion. In general, the Laplace transform can be written as

$$X(s)=\frac{N(s)}{D(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}\quad ,\ n\geq m$$

1. *Distinct Roots Case*

1) If all roots are distinct, $X(s)$ can be expressed in factored form:

$$X(s)=\frac{N(s)}{(s+r_1)(s+r_2)\cdots(s+r_n)}$$

2) This form can be expanded as

$$X(s)=\frac{C_1}{s+r_1}+\frac{C_2}{s+r_2}+\cdots+\frac{C_n}{s+r_n}\ ,\ \text{where}\ \underline{\qquad\qquad\qquad}$$

3) The inverse transform becomes

$$x(t)=C_1 e^{-r_1 t}+C_2 e^{-r_2 t}+\cdots+C_n e^{-r_n t}$$

Example (Distinct Real Roots)

$$X(s)=\frac{(s+2)(s+4)}{s(s+1)(s+3)}$$

Partial-Fraction Expansion (2/3)

2. *Repeated-Roots Case*

   1) Suppose that p of the roots have the same value s=-r_1, and the remaining (n-p) roots are distinct and real. Then $X(s)$ is of the form:

   $$X(s) = \frac{N(s)}{(s+r_1)^p (s+r_{p+1})(s+r_{p+2})\cdots(s+r_n)}$$

   2) The expansion is

   $$X(s) = \frac{C_1}{(s+r_1)^p} + \frac{C_2}{(s+r_1)^{p-1}} + \cdots + \frac{C_p}{s+r_1} + \frac{C_{p+1}}{s+r_{p+1}} + \cdots + \frac{C_n}{s+r_n}$$

   3) The coefficients for the repeated roots are

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   4) The coefficients for the distinct roots are the same as the previous case: $C_i = X(s)(s+r_i)\big|_{s=-r_i}$

   5) The time function is

$$x(t) = C_1 \frac{t^{p-1}}{(p-1)!} e^{-r_1 t} + C_2 \frac{t^{p-2}}{(p-2)!} e^{-r_1 t} + \cdots + C_p e^{-r_1 t} + C_{p+1} e^{-r_{p+1} t} + \cdots + C_n e^{-r_n t}$$

Example (Repeated Roots)

$$X(s) = \frac{s+3}{(s+1)(s+2)^2}$$

Partial-Fraction Expansion (3/3)

3. *Complex Roots Case*

   1) When some of the roots of the transform denominator are complex, it is easier to put the second order term in the square form.
   2) For example,

$$X(s) = \frac{3s+7}{4s^2+24s+136} = \frac{3s+7}{4(s^2+6s+34)} \Rightarrow$$

$$X(s) = \frac{1}{4}\left[\frac{3s+7}{(s+3)^2+5^2}\right] = \frac{1}{4}\left[A\frac{(s+3)}{(s+3)^2+5^2} + B\frac{5}{(s+3)^2+5^2}\right]$$

   3) The coefficients, A and B can be determined by multiplying both sides by the denominator:

$$3s+7 = A(s+3) + B\cdot 5 = As + (3A+5B) \quad \Rightarrow \quad A = 3, B = -\frac{2}{5}$$

$$X(s) = \frac{1}{4}\left[3\frac{(s+3)}{(s+3)^2+5^2} - \frac{2}{5}\frac{5}{(s+3)^2+5^2}\right] = \frac{3}{4}\frac{(s+3)}{(s+3)^2+5^2} - \frac{1}{10}\frac{5}{(s+3)^2+5^2}$$

$$\Rightarrow \quad x(t) = \frac{3}{4}e^{-3t}\cos 5t - \frac{1}{10}e^{-3t}\sin 5t$$

Example (Complex Roots)

$$X(s) = \frac{2s+12}{s^2+2s+5}$$

Transfer Function (1/2)

1. *Linear Time-Invariant system with zero initial conditions*
   1) The Laplace transform of the following second order differential equation $\ddot{x}+a\dot{x}+bx=f(t)$ with zero initial conditions, $x(0)=\dot{x}(0)=0$ is $s^2X(s)+asX(s)+bX(s)=F(s)$.
   2) The ratio $X(s)/F(s)$ is defined as the **Transfer Function** of the system and denoted by $T(s)$. For the above example, the TF is

$$T(s)=\frac{X(s)}{F(s)}= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

2. *Properties of transfer function*
   1) The TF is the Laplace transform of the forced response divided by the Laplace transform of the input.
   2) It can be used as a multiplier to obtain the forced response transform from the input transform: $X(s) = T(s)F(s)$.
   3) The denominator of the transfer function is the characteristic polynomial whose roots are called the system ____________.
   4) The TF possesses the intrinsic characteristics of the system, apart from the effects of the input and initial conditions.

Transfer Function (2/2)

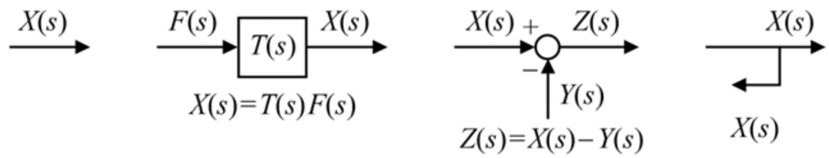
2. *Properties of transfer function (Cont'd)*
   5) The TF is equivalent to the original equation of motion in the form of the ODE. Therefore, the TF can be obtained from the ODE or conversely, the ODE can be obtained from the TF.
3. *Example*
   - Obtain the transfer function for the following equations.

$$5\ddot{x}+30\dot{x}+40x=6f(t)$$ ← **No initial conditions, no specific input.**

$$5\ddot{x}+30\dot{x}+40x=3\dot{f}(t)+2f(t)$$

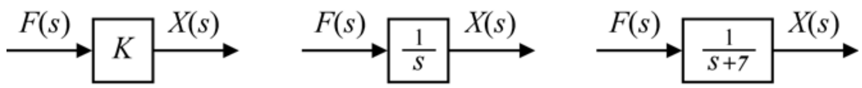
Block Diagrams

1. *Visual representation of a system model*
   1) Block diagram is a visual representation of a dynamic system and can be constructed from the transfer function of the system.
   2) Conversely, the transfer function can be obtained from a block diagram for a given system.
2. *Simple block diagrams*
   1) Four basic symbols used in block diagrams



(a) Arrow (b) Block (c) Summer (d) Branch

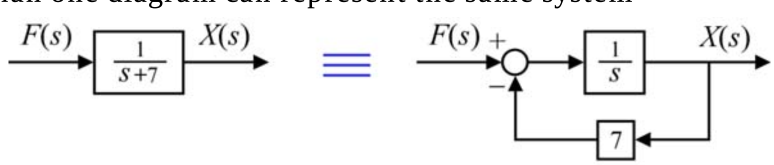
   2) Block multiplications



(a) Gain (b) Integrator (c) Transfer function

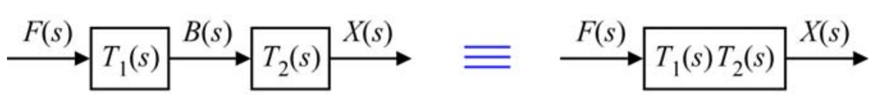
Block Diagram Algebra (1/2)

1. *Equivalent block diagrams*
   1) More than one diagram can represent the same system



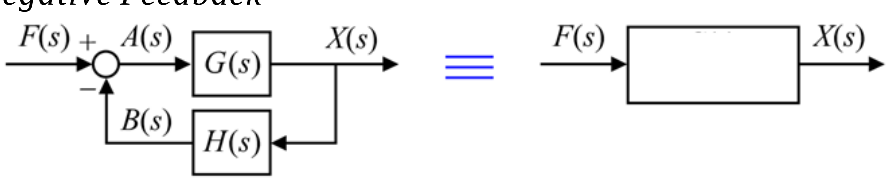
(a) Transfer function (b) Negative feedback

2. *Series connection*



$B(s)=T_1(s)F(s)$, $X(s)=T_2(s)B(s)$ $X(s)=T_1(s)T_2(s)F(s)$

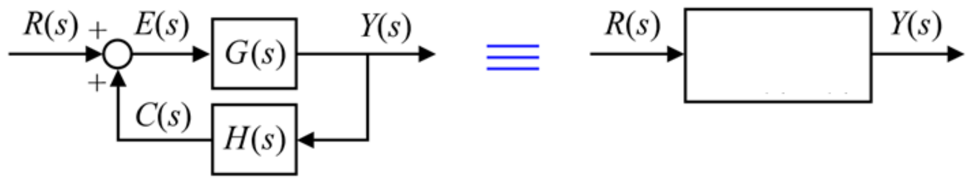
3. *Negative Feedback*



$A(s)=F(s)-B(s)$, $B(s)=H(s)X(s)$, $X(s)=G(s)A(s)$ $X(s)=$ ____________

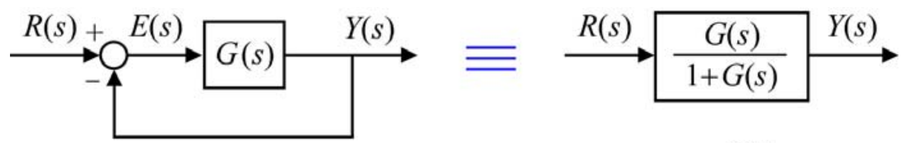
Block Diagram Algebra (2/2)

4. *Positive Feedback*



$E(s) = R(s) + C(s)$, $C(s) = H(s)Y(s)$, $Y(s) = G(s)E(s)$ $\quad Y(s) =$ ________________

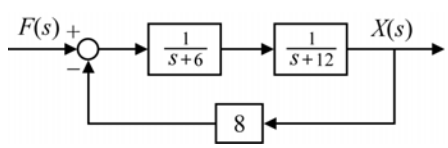
5. *Unity Feedback*



$E(s) = R(s) - Y(s)$, $Y(s) = G(s)E(s)$ $\quad Y(s) = \dfrac{G(s)}{1+G(s)} R(s)$

Example Problems

Determine the transfer function X(s)/F(s) for the systems whose diagrams are shown below.



Example Problems

Determine the transfer function X(s)/F(s) for the systems whose diagrams are shown below.