9. State-Space Techniques

Canonical Forms
Full-State Feedback
Controllability
Linear Quadratic Regulator (LQR)
Estimator Design

Estimator (Observer) Design

1. *Full state feedback*
   1) Thus far, the control law assumed that all the state variables are available for feedback.
   2) However, in most cases, not all the state variables are measured. The cost of the required sensors may be prohibitive, or it may be physically impossible to measure all the state variables, as in, for example, a nuclear power plant.
2. *State estimator or observer*
   1. It is possible to reconstruct _________________ of a system from ___________________.
   2. If the estimate of the state is denoted by $\hat{x}$, it would be convenient if we could replace the true state in the control law with the estimates so that the control becomes

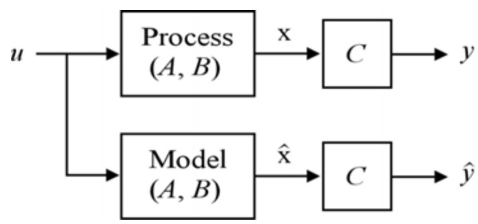
$$u = -K\hat{x} + \overline{N}r$$

Open-Loop Estimator (1/2)

1. One method of estimating the state is to construct a full-order model of the plant dynamics,

$$\dot{\hat{\mathrm{x}}} = A\hat{\mathrm{x}} + Bu$$

where $\hat{\mathrm{x}}$ is the estimate of the actual state x. A, B, and $u(t)$ are all known. Hence this estimator will be satisfactory if we can obtain the correct initial condition x(0) and set $\hat{\mathrm{x}}(0)$ equal to it. This open-loop estimator is depicted as



2. However, it is precisely the lack of information about x(0) that requires the construction of an estimator. Otherwise, the estimated state would track the true state exactly. Thus, if we made a poor estimate for the initial condition, the estimated state would have a continually growing error or an error that goes to zero too slowly.

Open-Loop Estimator (2/2)

3. To study the dynamics of the estimator, we define the error in the estimate to be

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Then the dynamics of this error system are given by

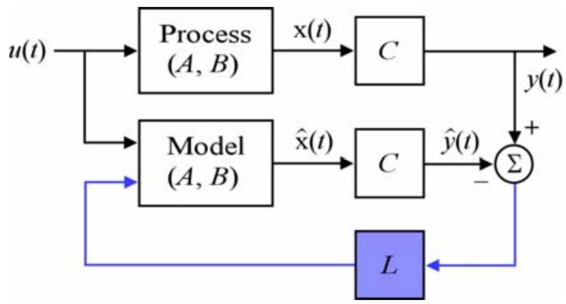
$\dot{\tilde{\mathrm{x}}} =$ ____

or simply, _________ with the initial conditions as $\tilde{\mathrm{x}}(0) = \mathrm{x}(0) - \hat{\mathrm{x}}(0)$.

4. The error converges to zero for a stable system (A stable), but we have no ability to influence the rate at which the state estimate converges to the true state. Furthermore, the error is converging to zero at ____________________________________ . If this convergence rate were satisfactory, no control or estimation would be required.

Closed-Loop Estimator (1/6)

1. Consider feeding back the difference between the measured and estimated outputs and correcting the model continuously with this error signal. This scheme can be shown as



and the corresponding equation is

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Here L is a proportional gain defined as

$$L = [l_1, \quad l_2, \quad \cdots, \quad l_n]^T$$

and is chosen to achieve satisfactory error characteristics.

Closed-Loop Estimator (2/6)

2. As for the open-loop estimator, the dynamics of the error can be obtained by subtracting the estimate from the state to get the error equation,

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = (Ax + Bu) - (A\hat{x} + Bu + L(y - C\hat{x}))$$
$$=$$

or simply _______________ .

3. The characteristic equation of the error is now given by

$$\det[sI - (A - LC)] = 0.$$

If we can choose L so that $A–LC$ has stable and reasonably fast eigenvalues, then $\tilde{x}$ will decay to zero and remain there – independent of the known forcing function $u(t)$ and its effect on the state $x(t)$ and irrespective of the initial condition $\tilde{x}(0)$. This means that $\hat{x}(t)$ will converge to $x(t)$, regardless of the value of $\hat{x}(0)$; furthermore, we can choose the dynamics of the error to be stable as well as much faster than the open-loop dynamics determined by A.

Closed-Loop Estimator (3/6)

4. Note that in deriving the error dynamics equation, we have assumed that *A*, *B*, and *C* are __________ in the physical plant and in the computer implementation of the estimator. If we do not have an accurate model of the plant (*A*, *B*, *C*), the dynamics of the error are no longer governed by the error equation. However, we can typically choose *L* so that the error system is still at least stable and the error remains acceptably small, even with (small) modeling errors and disturbing inputs.
5. The selection of *L* can be approached in exactly the same fashion as *K* is selected in the control-law design. If we specify the desired location of the estimator error poles as

$$s_i = \beta_1,\ \beta_2,\ \cdots,\ \beta_n,$$

then the desired estimator characteristic equation is

$$\alpha_e(s) \equiv (s-\beta_1)(s-\beta_2)\cdots(s-\beta_n)$$

We can then solve for *L* by comparing coefficients in the desired characteristic equation and the estimator characteristic equation.

Closed-Loop Estimator (4/6)

- *Example*

Q: Design an estimator for the simple pendulum. Compute the estimator gain matrix that will place both the estimator error poles at $-10\omega_0$ (five times as fast as the controller poles selected previously).

Sol: The equations of motion are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We are asked to place the two estimator error poles at $-10\omega_0$. The corresponding characteristic equation is

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The characteristic equation of the estimator is

$$\det[sI-(A-LC)] = \underline{\hspace{5cm}}$$

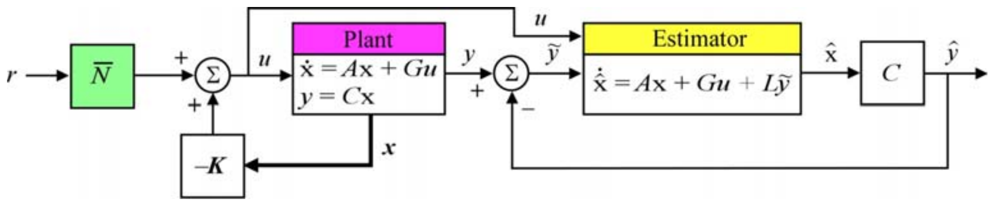
Closed-Loop Estimator (5/6)

Comparing the coefficients in the two characteristic equations, we find that

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

This completes the design of the estimator for this example.

- In order to test the performance of the estimator, the **plant with state feedback** is combined with **the estimator with output feedback** resulting in the following overall system:



Closed-Loop Estimator (6/6)

The response of this closed-loop system with $\omega_0 = 1$, to an initial condition $x_0 = [1.0, 0.0]^T$ and $\hat{x}_0 = [0, 0]^T$ is