5. Root Locus Techniques

Sketching the Root Locus

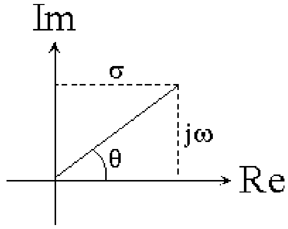
Transient Response Design via Gain Adjustment

Introduction

1. *Control system problem*
   1) Whereas the poles of the open-loop transfer function are easily found and do not change with the system gain, the poles of the ___________ transfer function are more difficult to find, and they change with the system gain.
2. *What is Root Locus*?
   1) A graphical presentation of the closed-loop poles as a system parameter is varied.
   2) Provides a qualitative description of a control system's performance.
   3) Allows control design for systems of order higher than 2.
   4) Gives a graphical representation of a system's ____________.

Complex Numbers

1) Any complex number, s described in Cartesian coordinates can be graphically represented by a vector. It can also be represented in polar or exponential form.



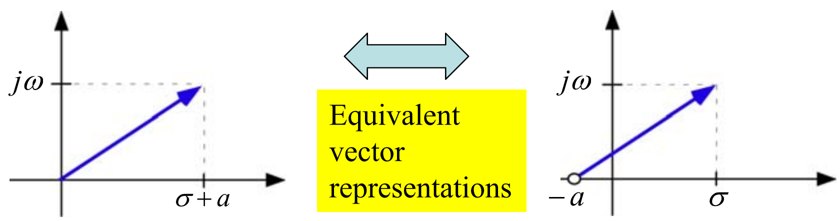
$$s = \sigma + j\omega \quad \Rightarrow \quad M\angle\theta \quad \text{or} \quad M\,e^{j\theta}$$

where $M =$ __________

$\theta =$ __________

2) If the complex number is substituted in a complex function $F(s)$, another complex number will result.

If $F(s) \;=\; s + a$ with $s \;=\; \sigma + j\omega \quad \Rightarrow \quad F(s) = \sigma + a + j\omega$



Complex Functions

- General complex functions can be represented as

$$F(s) = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = \frac{\Pi \text{ numerator's complex factors}}{\Pi \text{ denominator's complex factors}} = \frac{\Pi\,(\qquad\qquad)}{\Pi\,(\qquad\qquad)}$$

- Since each complex factor can be represented by a vector, the magnitude, M of $F(s)$ at any point, s, is

$$M = \frac{\Pi \text{ zero lengths}}{\Pi \text{ pole lengths}} = \frac{\prod_{i=1}^{m}\left|(s+z_i)\right|}{\prod_{j=1}^{n}\left|(s+p_j)\right|}$$

and the angle, θ of $F(s)$ at any point, s is

$$\theta = \sum \text{angles of zeros} - \sum \text{angles of poles}$$

$$= \sum_{i=1}^{m}\angle(s+z_i) - \sum_{j=1}^{n}\angle(s+p_j)\,.$$

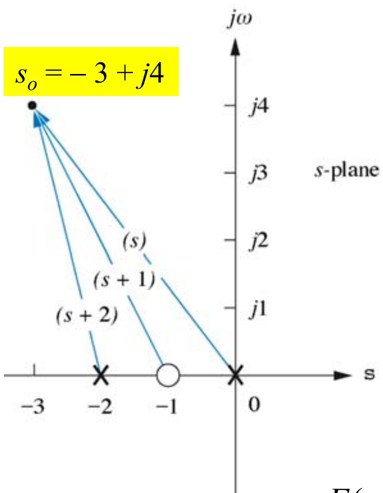
- Thus, __________

Example 1: Complex Functions

- Given

$$F(s) = \frac{s+1}{s(s+2)}$$

find $F(s)$ at the point $s_o = -3 + j4$.



$$M_{z1} = |s_o + 1| = |-3 + j4 + 1| = \sqrt{(-2)^2 + 4^2} = \sqrt{20}$$

$$M_{p1} = |s_o + 0| = |-3 + j4 + 0| = \sqrt{9+16} = 5$$

$$M_{p2} = |s_o + 2| = |-3 + j4 + (2)| = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$$

$$\theta_{z_1} = \tan^{-1}\left(4/_{-2}\right) = 116.57°$$

$$\theta_{p_1} = \tan^{-1}\left(4/_{-3}\right) = 126.87°$$

$$\theta_{p_2} = \tan^{-1}\left(4/_{-1}\right) = 104.04°$$

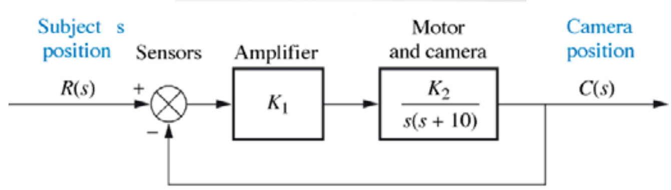
$$F(s_o) = \frac{\sqrt{20}}{5\sqrt{17}} \angle \underline{\qquad\qquad}$$

Root Locus Motivating Example (1/3)

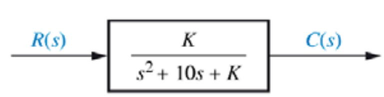
CameraMan® Camera System
Automatically follows a subject who wears infrared sensors on front and back (the front sensor is also a microphone); tracking commands and audio are relayed to CameraMan via a radio frequency link from a unit worn by the subject.

Table 8.1 Pole location as a function of gain for the system

K	Pole 1	Pole 2
0	−10	0
5	−9.47	−0.53
10	−8.87	−1.13
15	−8.16	−1.84
20	−7.24	−2.76
25	−5	−5
30	−5 + j2.24	−5 − j2.24
35	−5 + j3.16	−5 − j3.16
40	−5 + j3.87	−5 − j3.87
45	−5 + j4.47	−5 − j4.47
50	−5 + j5	−5 − j5



(b)

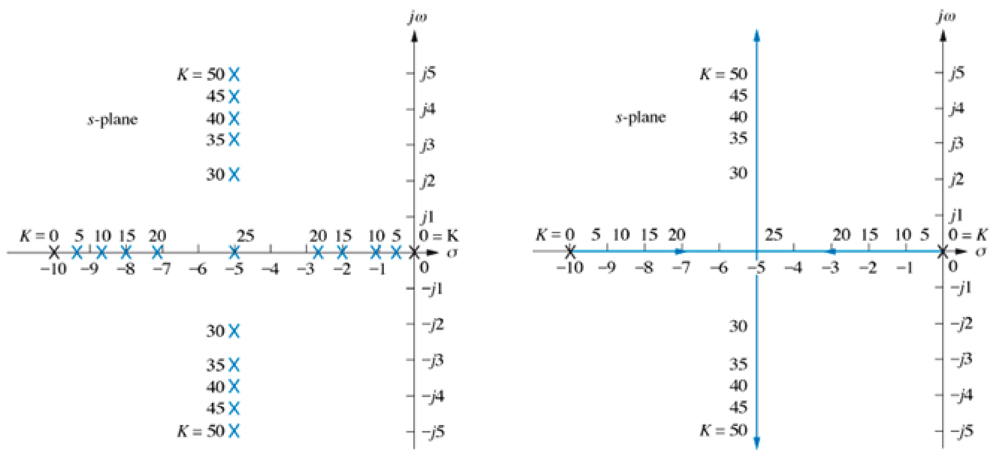


where $K = K_1K_2$

(c)

Root Locus Motivating Example (2/3)

- As the gain, K increases, the closed-loop pole, which is at -10 for $K = 0$, moves toward the _________ and the closed-loop pole, which is at 0 for $K = 0$ moves toward the _______.
- They meet at _____, break away from the real axis, and move into the complex plane. One closed-loop pole moves upward while the other moves downward.

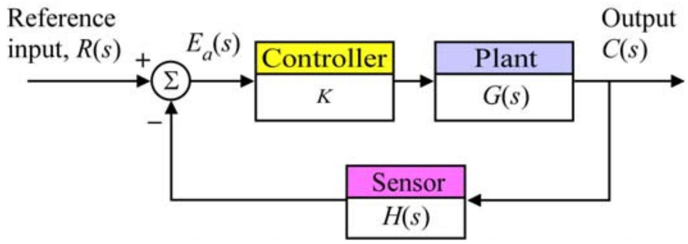


Root Locus Motivating Example (3/3)

1. It is this representation of the ____________________________ **as the gain is varied** that we call a root locus.
2. For this course, the discussion is limited to positive gain, or $K \geq 0$.
3. The root locus shows the changes in the **transient response** as the gain, K varies.
4. The poles are real for gains less than 25. Thus, the system is ________________.
5. At a gain of 25, the poles are real and multiple and hence __________ damped.
6. For gains above 25, the system is ______________.
7. Regardless of the value of gain, the settling time for the system remains the same under all conditions of underdamped responses.
8. As the gain is increased, the damping ratio diminishes and the percent overshoot increases.
9. The damped frequency of oscillation, which is equal to the imaginary part of the pole, also increases with an increase in gain, resulting in a reduction of the peak time.
10. Since the root locus never crosses over into the right half-plane, the system is ___________________.

Root Locus Formulation (1/3)

1. Feedback control system with a proportional controller.



2. The closed-loop transfer function for the control system is

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

3. The closed-loop poles are found by solving the characteristic equation, i.e.,

____________________ or equivalently,

$$KG(s)H(s) = -1 = 1\angle(2k+1)\,180^\circ, \quad k = 0, \pm 1, \pm 2, \cdots$$

Root Locus Formulation (2/3)

4. Thus, the two criteria that the closed-loop poles must meet are
   - **Magnitude criterion**

     $|KG(s)H(s)| =$ ______

   - **Angle criterion**

     $\angle KG(s)H(s) =$ ____________________

5. If the system transfer function can be factored as

$$GH = \frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

then,

$$1 + KG(s)H(s) = 0 \quad \Rightarrow \quad 1 + \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)} = 0$$

$$\boxed{(s+p_1)(s+p_2)\ldots(s+p_n) + K(s+z_1)(s+z_2)\ldots(s+z_m) = 0}$$

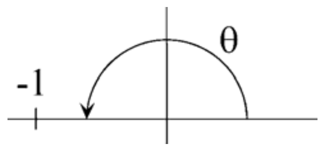
- Thus, the closed-loop poles will change as K varies from 0 to ∞.

Root Locus Formulation (3/3)

6. By the Root Locus method, one can find what the locations (locus) of the closed loop poles when K increases from 0 to ∞. This method also provides information on the transient behavior and system stability.
7. When drawing Root Locus, the _________ **Criterion** is first used to find the values of s that satisfies the characteristic equation.

$$\text{For} \quad G(s)H(s) = \frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

$$\angle KG(s)H(s) = \sum \theta_i = (\theta_{z_1} + \theta_{z_2} + \cdots) - (\theta_{p_1} + \theta_{p_2} + \cdots) = (2k+1)180^\circ$$



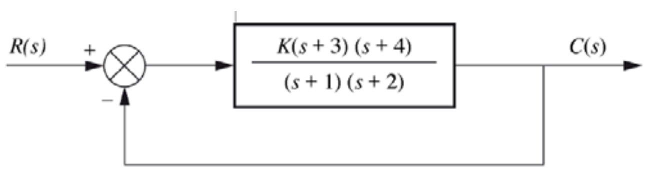
θ must be an ______ multiple of $180°$

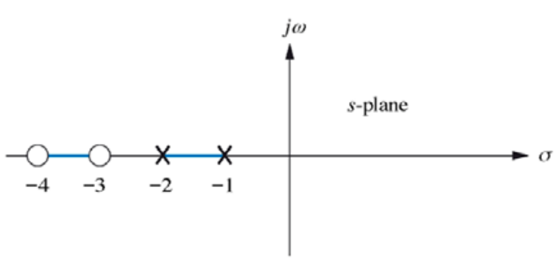
8. Then, the particular value of K for each of s can be found by using the _________ **Criterion** as follows.

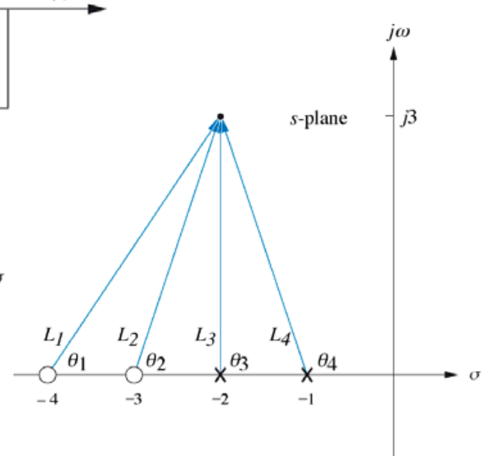
$$K = \frac{\qquad}{\qquad}$$

Example 2: Root Locus Formulation (1/2)

- For the following system, check whether points $s_1 = -2 + j3$ and $s_2 = -2 + j(\sqrt{2}/2)$ are on the root locus of the closed-loop system.







- Solution

$$KGH = \frac{K(s+3)(s+4)}{(s+1)(s+2)} \quad \Rightarrow \quad T(s) = \frac{K(s+3)(s+4)}{(1+K)s^2 + (3+7K)s + (2+12K)}$$

Example 2: Root Locus Formulation (2/2)

- If the point $s_1 = -2 + j3$ is a closed-loop pole, it must satisfy the angle criterion.

$$\angle KG(s_1)H(s_1) = \theta_{z_1} + \theta_{z_2} - (\theta_{p_1} + \theta_{p_1})$$
$$= 56.31^\circ + 71.57^\circ - 90^\circ - 180.43^\circ = -70.55^\circ$$

Therefore, s_1 is not on the root locus.

- The sum of the angles for $s_2 = -2 + j(\sqrt{2}/2)$ is

$$\angle KG(s_2)H(s_2) = \theta_{z_1} + \theta_{z_2} - (\theta_{p_1} + \theta_{p_1})$$
$$= 19.47^\circ + 35.26^\circ - 90^\circ - 144.74^\circ = -180.0^\circ$$

Therefore, s_2 satisfies the Angle Criterion and is a point on the root locus for some value of gain, K.

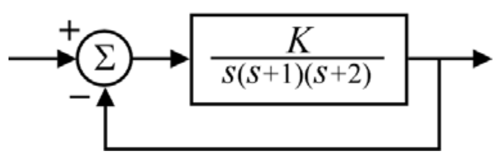
- The value of gain, K for s_2 can be found by using the Magnitude criterion.

$$K = \frac{1}{|G(s_2)H(s_2)|} = \underline{\hspace{6cm}}$$

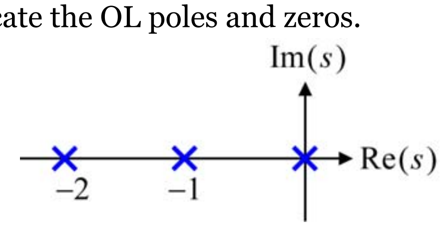
Thus, $s_2 = -2 + j(\sqrt{2}/2)$ is a point on the root locus for a gain of ______.

Root Locus Sketching (1/9)

- Consider drawing Root Locus for the following system.



- **Step 1**: Locate the OL poles and zeros.

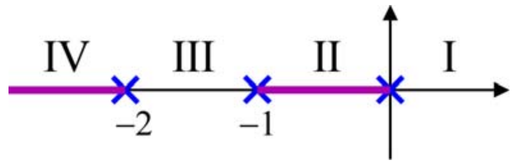


- **Rule** : If there are n poles and m zeros in the OL transfer function, the ____ branches of the locus start at the __________ of the **OL transfer function**. As the gain K increases to infinity, m of these branches approach the zeros of the OL transfer function.

$$\text{OL transfer function} = \frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

Root Locus Sketching (2/9)

- **Step 2**: Determine branches of the Root Locus on the real axis.
- **Rule** : The loci are on the real axis to the _______ of an _____ number of poles and zeros.



Branch	I	II	III	IV
Sum of angles	0	180°	360°	540°

- Sum of all the angles to test points in branches II and IV are odd multiples of 180°, and by the angle criterion, they are on the RL.

Root Locus Sketching (3/9)

- **Step 3**: Determine asymptotes.
- **Rule** : The root locus approaches _____________ as asymptotes as the locus approaches infinity.

a. Number of asymptotes ➔ __________

where n = # of OL poles

m = # of OL zeros

For the current example: ______________

b. Angles of asymptotes

$$\theta_k = \frac{(2k+1)180°}{n-m}, \quad \text{where} \quad k = 0, 1, 2, \cdots$$

For the current example, k = 0, 1, 2 because of 3 asymptotes.

$$\theta_1 = \frac{(1)180°}{n-m} = 60°, \ \theta_2 = \frac{(2+1)180°}{n-m} = 180°, \ \theta_3 = \frac{(5)180°}{n-m} = 300°$$

Root Locus Sketching (4/9)

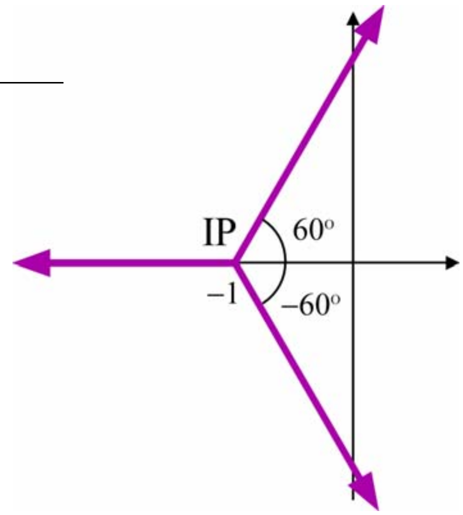
c. Intersection Point (IP) of asymptotes

$$\alpha = \frac{\sum poles - \sum zeros}{n - m}$$

It doesn't matter whether the poles and zeros are real or complex

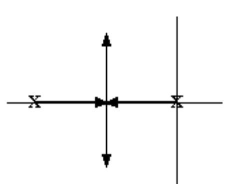
For the current example

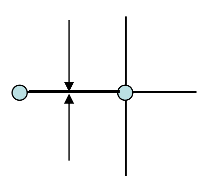
$\alpha =$ ____________________

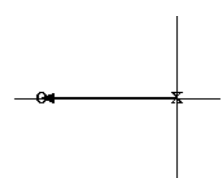


Root Locus Sketching (5/9)

- **Step 4**: Determine break away/ break in points.


breakaway point


break-in point


not a break point

- **Rule:**
  - **a. Breakaway point** on the real axis occurs when the value of K is a ______________ with respect to the values of K on either side.
  - **b. Break-in point** on the real axis occurs when the value of K is a ______________ with respect to the values of K on either side.
- Thus, one first needs to find the maximum and minimum by using the derivative

  ____

  and the solutions of the equation are the breakaway and break-in points.

Root Locus Sketching (6/9)

- For the current example, the denominator is given by

$$1+GH=0=1+\frac{K}{s(s+1)(s+2)}=0$$

and the characteristic equation is

From $K=-s(s+1)(s+2)=-(s^3+3s^2+2s)$,

$$\frac{dK}{ds}=0=\underline{\hspace{4cm}}\quad\Rightarrow\quad s=-0.45\,,-1.7$$

Since -1.7 is not on the RL, – 0.45 is the breakaway point in this example.

- Although -1.7 is an extremum point, since it is not on the valid branch, the point is not on the RL. As can be seen from this example, the derivative condition is a **necessary, but not sufficient** condition to indicate a break point.

Root Locus Sketching (7/9)

- **Step 5**: Determine departure and arrival angles.

The departure and arrival angles can be found by applying the Angle Criterion which states that

$$\underbrace{(\psi_1+\psi_2+\cdots+\psi_m)}_{\text{Angles from the zeros to a test point}}-\underbrace{(\phi_1+\phi_2+\cdots+\phi_n)}_{\text{Angles from the poles to a test point}}=(2k+1)180°$$

- **Rule:**

a. **Departure angle** of a Root Locus from a pole can be found from

$$\phi_d=\underbrace{(\psi_1+\psi_2+\cdots+\psi_m)}_{\text{Angles from the zeros to the \textbf{pole} of interest}}-\underbrace{(\phi_1+\phi_2+\cdots+\phi_n)}_{\text{Angles from the poles to the \textbf{pole} of interest}}-(2k+1)180°$$

b. **Arrival angle:** of a Root Locus to a zero can be found

$$\psi_d=-\underbrace{(\psi_1+\psi_2+\cdots+\psi_m)}_{\text{Angles from the zeros to the \textbf{zero} of interest}}+\underbrace{(\phi_1+\phi_2+\cdots+\phi_n)}_{\text{Angles from the poles to the \textbf{zero} of interest}}+(2k+1)180°$$

Root Locus Sketching (8/9)

- **Step 6**: Determine the frequency and gain at imaginary-axis crossing.

  Method 1: The Routh-Hurwitz criterion can be used to find both the frequency and gain for which the Root Locus crosses the imaginary axis.

  Method 2: Simply substitute ________ in the characteristic equation and set both the real and imaginary parts equal to zero.

- **Example:**

  The characteristic equation: $s^3 + 3s^2 + 2s + K = 0$

  Then, by substituting ________ into the char. eq., one gets

  ____

  $\Rightarrow (K_{cr} - 3\omega^2) + (2\omega - \omega^3)j = 0$

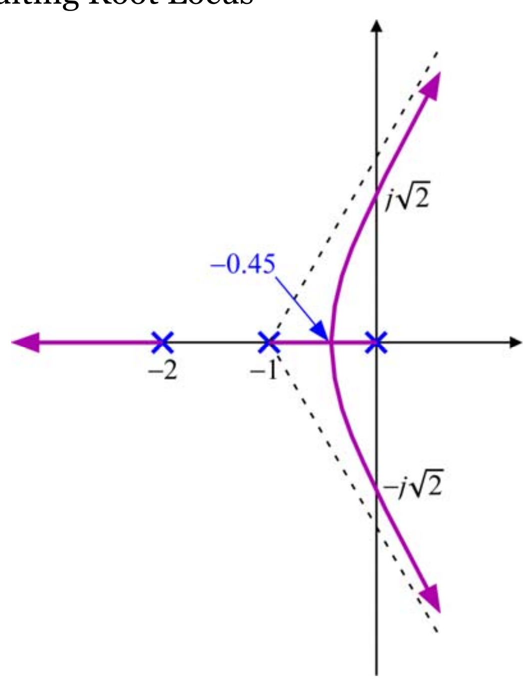
  $\Rightarrow K_{cr} = 3\omega^2$ and $2\omega = \omega^3$

  $\Rightarrow \omega = \sqrt{2}$ and $K_{cr} = 3(\sqrt{2})^2 = 6$

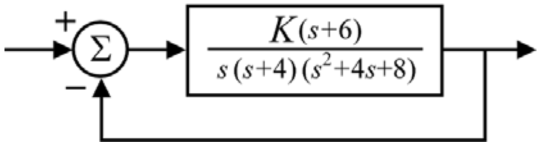
  Thus, the Root Locus crosses at $s = \pm j\sqrt{2}$ when $K = 6$.

Root Locus Sketching (9/9)

- The resulting Root Locus

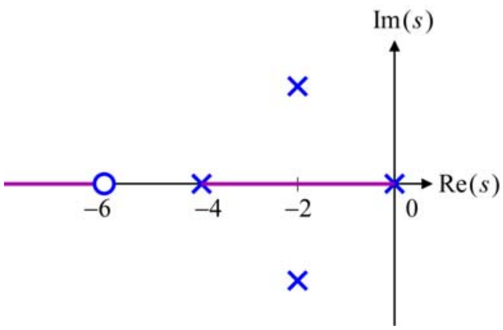


Root Locus Sketching Example #1 (1/4)



Zeros: -6
Poles: $0, -4, -2 \pm j2$

Steps 1,2



Step 3
$n = 4,\ m = 1$
Number of asymptotes = ____________
For k = 0, 1, 2:

$$\theta_1 = 60, \quad \theta_2 = 180, \quad \theta_3 = 300$$

$$\alpha = \frac{-4 + (-2+2j) + (-2-2j) + 6}{3}$$
$$= -\frac{2}{3}$$

Root Locus Sketching Example #1 (2/4)

Steps 4:

$$\frac{K(s+6)}{s(s+4)(s^2+4s+8)} = -1 \quad \Rightarrow \quad K = \underline{\hspace{6cm}}$$

$$\frac{dK}{ds} = 0 \quad \Rightarrow \quad s = -3.08, -7.30 \ \text{(real roots)}$$

(they are both on the root locus!)

Notice that we are only interested in the numerator because the denominator can't make $dK/ds = 0$

$$K = \frac{N(s)}{D(s)} \quad \Rightarrow \quad \frac{dK}{ds} = \underline{\hspace{6cm}}$$

$$0 = D(s)N'(s) - N(s)D'(s)$$

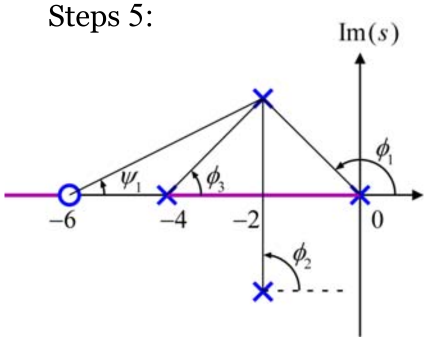
Root Locus Sketching Example #1 (3/4)

$$N(s) = -(s^4 + 8s^3 + 24s^2 + 32s) \qquad D(s) = s + 6$$
$$N'(s) = -(4s^3 + 24s^2 + 48s + 32) \qquad D'(s) = 1$$

The roots of $D(s)N'(s) - N(s)D'(s) = 0$ can be found using Matlab:

```
N = -[1 8 24 32 0];
D =[1 6];
Np = -[4 24 48 32];
Dp = [1];
roots(conv(D,Np)-conv(N,Dp))
```

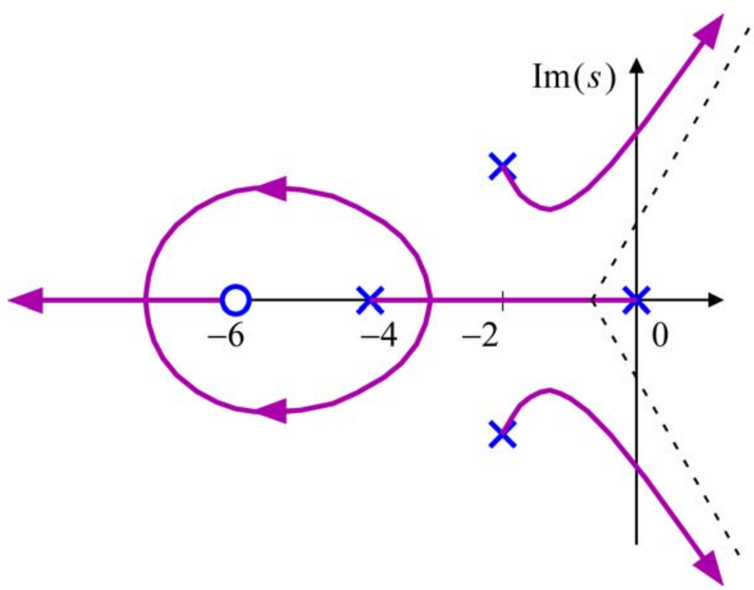
Steps 5:



$$\phi_1 = 135^\circ, \quad \phi_2 = 90^\circ, \quad \phi_3 = 45^\circ$$
$$\psi_1 = \tan^{-1}(2/4) = 26.6^\circ$$
$$\phi_d = \sum\psi - \sum\phi - (2k+1)180^\circ$$
$$= \underline{\qquad\qquad\qquad\qquad}$$
$$= \underline{\qquad\qquad}$$

Root Locus Sketching Example #1 (4/4)

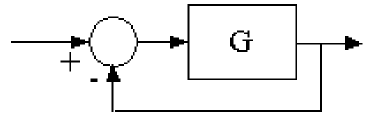
Step 6: sketch

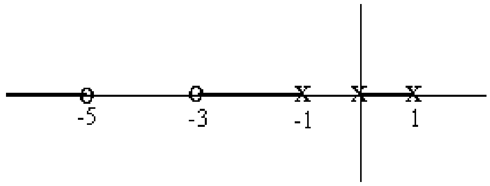


Root Locus Sketching Examples (1/5)

- Show the minimum calculations needed for R.L. of the unity feedback system.

1. $$G(s) = \frac{K(s+3)(s+5)}{s(s^2-1)}$$





3 poles, 2 zeros - so there is only one asymptote

# of asymptotes = 3 - 2 = 1

For 1 asymptote $k = 0$.

$$\theta_k = \frac{(2k+1)180^\circ}{1} = \frac{(2(0)+1)180^\circ}{1} = 180^\circ$$

You don't need to do a calc of the break-in/breakaway points.

No poles off of the real axis, so no departure angle calcs.

Root Locus Sketching Examples (2/5)

When the gain K is increased, the poles enter the stable region.

Find for which value of K, the system becomes stable.

Substitute $s=j\omega$ in the characteristic equation and set both the real and imaginary parts equal to zero.

Char. Eq.: $s(s^2-1)+K(s+3)(s+5) = s^3 - s + K(s^2+8s+15)$

$$(j\omega)^3 - j\omega + K((j\omega)^2 + 8j\omega + 15) = -j\omega^3 - j\omega + K(15-\omega^2+8j\omega) = 0$$

$$\Rightarrow (15K - K\omega^2) + (8K\omega - \omega - \omega^3)j = 0$$

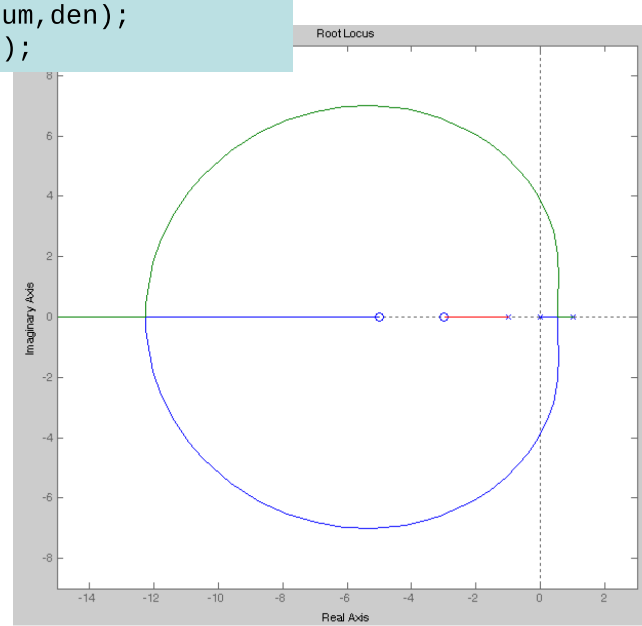
$\Rightarrow$ ________________ and ________________

$\Rightarrow$ $\omega =$ ______ and $K =$ ________________

Thus, the Root Locus crosses at $s = \pm j$____ when $K =$ ____.

Root Locus Sketching Examples (3/5)

```
num = conv([1 3],[1 5]);
den = conv([1 0],[1 0 -1]);
sys = tf(num,den);
rlocus(sys);
```



Root Locus Sketching Examples (4/5)

2. $G(s) = \dfrac{K}{s^2(s+p)}$

3. Poles at 0, – 2, $-3 \pm 2j$

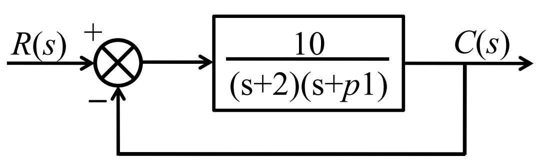
Root Locus Sketching Examples (5/5)

4. $G(s)=\dfrac{K(s+2)^2}{s(s-2)(s+4)}$

5. $G(s)=\dfrac{K}{s(s+4)^3}$

Generalized Root Locus (1/2)

- Using the same Root Locus technique, we can obtain a root locus for variations of system parameters other than the forward-path gain, *K*.
- For example: We can obtain a root locus for variations of the value of ____ in the following system.



$$KG(s)H(s)=\frac{10}{(s+2)(s+p1)}$$

The problem is that p1 is not a ___________________ of the function, as the gain *K*.

- The solution to this problem is to create an equivalent system where $p1$ appears as the _________________, *i.e.*, we effectively want to create an equivalent system whose denominator is $1+p1G(s)H(s)$.
- For the example system, the CL transfer function is

$$T(s)=\frac{KG(s)}{1+KG(s)H(s)}=\frac{10}{s^2+(p1+2)s+2p1+10}$$

Generalized Root Locus (2/2)

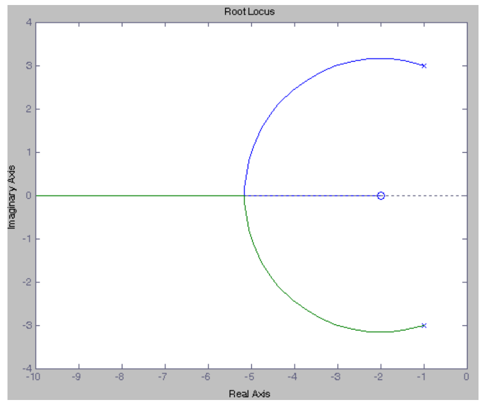
- By isolating $p1$, we have $T(s) = \dfrac{10}{s^2 + 2s + 10 + p1(s+2)}$.
- Now, by dividing numerator and denominator by the term not included with $p1$, $s^2 + 2s + 10$, we obtain,

$$T(s) = \underline{\qquad\qquad\qquad\qquad}$$

- We have reached a system for which

$$KG(s)H(s) = \underline{\qquad\qquad\qquad}$$

and the root locus can now be sketched as a function of $p1$.

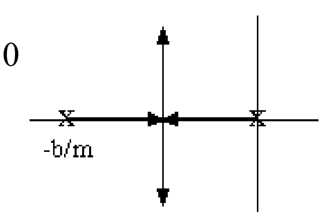


Generalized Root Locus: Example 1

- Given the <u>closed</u> loop expression: $\dfrac{Y}{R}(s) = \dfrac{1}{ms^2 + bs + k_1}$
- We must express the <u>denominator</u> as ________________, where K is the variable being varied.
- If we vary k_1, we must isolate it first.

$$(ms^2 + bs) + k_1 = 0 \quad \Rightarrow \quad 1 + \frac{k_1}{ms^2 + bs} = 0$$

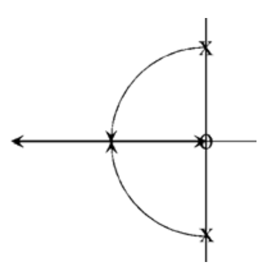
$$\Rightarrow \quad 1 + \underbrace{\frac{k_1}{ms^2 + bs}}_{KG(s)H(s)} = 0$$



- If we vary b,

$$ms^2 + k_1 + bs = 0 \quad \Rightarrow \quad 1 + \frac{bs}{ms^2 + k_1} = 0$$

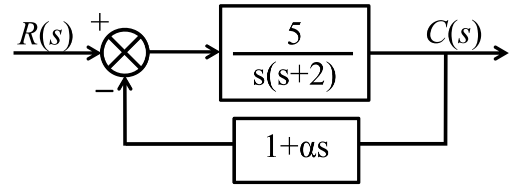
$$\Rightarrow \quad 1 + \underbrace{\frac{bs}{ms^2 + k_1}}_{KG(s)H(s)} = 0$$



Generalized Root Locus: Example 2 (1/2)

- Given the system to the right, plot the root locus for a system as α goes from 0 to ∞.

  ** Notice that α cannot be simply multiplied by a transfer function.*



1. Get the closed loop transfer function C/R in the form

$$\frac{C}{R}(s) = \frac{G(s)}{1+G(s)H(s)}$$ or in our example, $$\frac{C}{R}(s) = \frac{\frac{5}{s(s+2)}}{1+\frac{5(1+\alpha s)}{s(s+2)}}$$

** Note that you don't need to simplify the TF.*

2. Set the denominator equal to zero and isolate the variable in question.

$$1+\frac{5(1+\alpha s)}{s(s+2)} = 0 \Rightarrow \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

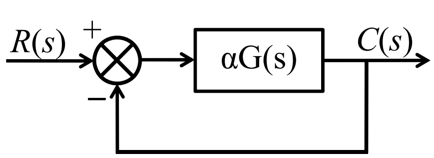
Generalized Root Locus: Example 2 (2/2)

3. Write in the form $1 + KG(s) = 0$ where K is the variable that goes from zero to infinity.

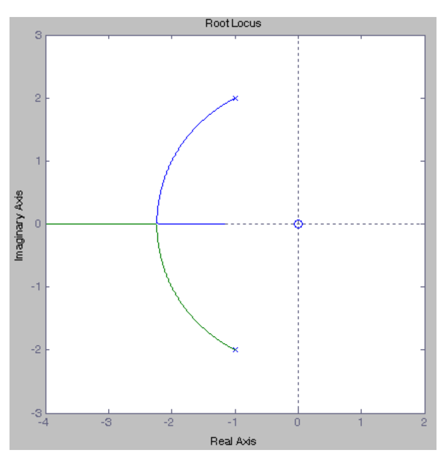
$$1+\frac{\alpha 5s}{s^2+2s+5} = 0$$

** Notice that this is in the correct form and now $K = \alpha$ in this example.*

4. Plot the root locus. Notice that there is a new open loop transfer function $G(s)$. Our system now looks like:



$G(s) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$



Selecting the Gain from the Root Locus

- Root locus is a plot of all possible root locations to the Closed-Loop characteristic equation or equivalently to the following equation:

$$1 + KG(s) = 0$$

- The two criteria that all the points on a RL must meet are
  - **Magnitude criterion**

    $$|KG(s)| = 1$$

  - **Angle criterion**

    $$\angle KG(s) = (2k+1)\,180^\circ$$

- Once the RL is obtained using the **angle criterion**, the gain value, K for each point on the RL can be found by using the **magnitude criterion**.

$$|KG(s)| = 1 \quad \Rightarrow \quad K = \frac{1}{|G(s)|} \quad (K > 0)$$

- K can be found numerically or graphically.

Gain Selection- Method #1 (1/2)

- Q: When the OL transfer function is given as

$$G(s) = \frac{1}{s(s^2+8s+32)} = \frac{1}{s(s^2+8s+16+32-16)}$$
$$= \frac{1}{s((s+4)^2+16)} = \frac{1}{s(s+4+4j)(s+4-4j)},$$

determine the CL gain, K needed to give $\zeta = 0.5$ to the system.

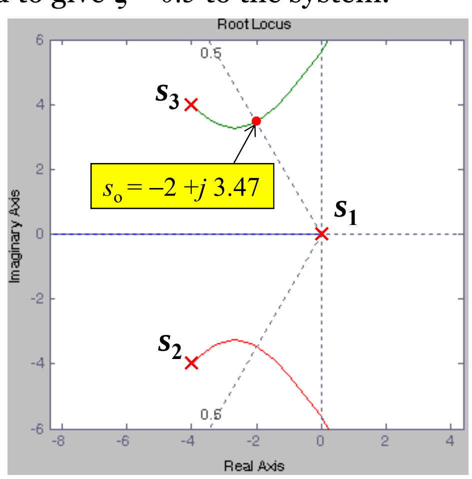
- From the RL graph, one can find that the line of $\zeta = 0.5$ crosses the root locus at

  ________________.

- The distance from each of the poles to s_o is:

  $|s_o - s_1|,\ |s_o - s_2|,\ |s_o - s_3|$

  , where $s_1 = 0$
  $s_2 = -4-4j$
  $s_3 = -4+4j$



Gain Selection- Method #1 (2/2)

- The OL transfer function evaluated at s_o is:

$$G(s_o) = \frac{1}{s_o(s_o - s_2)(s_o - s_3)}$$

- Now, using the magnitude criterion, the gain K can be calculated as

$$K = \frac{1}{|G(s_o)|} = |s_o||s_o - s_2||s_o - s_3| = |s_o||s_o + (4+4j)||s_o + (4-4j)|$$

$$= |-2+3.47j||-2+3.47j+4+4j||-2+3.47j+4-4j|$$

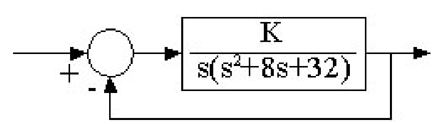
$$= |-2+3.47j||2+7.47j||2-0.53j|$$

$= \underline{\hspace{10cm}}$

$= \underline{\hspace{4cm}}$

Gain Selection- Method #2

- This method works only for a third-order system.
- For a third-order system shown,



$$\frac{Y}{R}(s) = \frac{K}{s(s^2+8s+32)+K} = \frac{K}{s^3+8s^2+32s+K}$$

- Characteristic polynomial is:

$$D_c(s) = s^3 + 8s^2 + 32s + K$$

- This polynomial is compared with the standard cubic equation term by term. $D_c(s) = \underline{\hspace{6cm}}$

$$= s^3 + (2\zeta\omega_n + p)s^2 + (2\zeta\omega_n p + \omega_n^2)s + \omega_n^2 p$$

- We get the following three equations

$$8 = 2\zeta\omega_n + p, \ \ 32 = 2\zeta\omega_n p + \omega_n^2, \ \ K = \omega_n^2 p$$

and the fourth equation is the design specification, $\zeta = 0.5$.

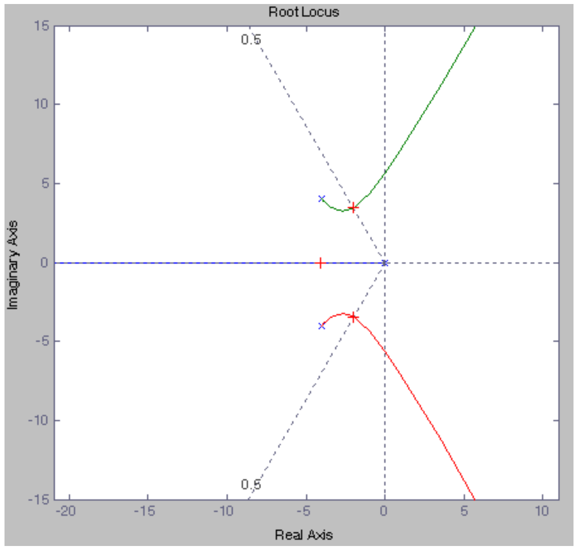
- By solving the four equations simultaneously, one can get

$$p = 4, \ \ \omega_n = 4, \ \ K = 64$$

Gain Selection- Method #3 (1/2)

- Using Matlab, we can find the value of gain K that gives

$$G(s) = \frac{K}{s\,(s^2 + 8s + 32)}, \qquad \zeta = 0.50$$



Gain Selection- Method #3 (2/2)

- Matlab code:

```
clear; clf

num=1;                          % numerator
den=[1 8 32 0];                 % denominator
sys=tf(num,den);
rlocus(sys);                    % draw root locus
zeta=.5;
wn=0;
sgrid(zeta,wn);                 % place grid line at zeta = 0.5
[K,p]=rlocfind(sys)             % find value of K on the root locus
                                % point p at gain K is also found
pause
[sys2]=feedback(K*sys,1,-1)     % closed loop system using gain K
step(sys2)                      % system
```

Gain Selection- Step Response

$$G(s) = \frac{K}{s(s^2 + 8s + 32)} \quad \text{with} \quad \zeta = 0.50$$