1. A system model was derived as $6\dot{x} + 2x = 3f(t)$, where $f(t)$ is a unit step and $x(0) = 0$.

   (a) What is the time constant of the system?
   Characteristic equation: $6s + 2 = 0$ ➔ System pole: $s = -1/3$
   Therefore, $\tau = 3$ s.

   (b) How long does it take for the system to reach 98% of its final value?
   Since the time constant $\tau = 3$ s, it takes $4\tau = 4\times 3 =$ 12 s for the system to reach 98% of the difference.

   (c) How long does it take for the system to reach 80% of its steady-state value?
   $x(t) = 1.5 - 1.5e^{-1/3t} = 1.5 \times 0.8 = 1.2$ ➔ $1.5e^{-1/3t} = 0.3$ ➔ $e^{-1/3t} = 0.2$ ➔ $-\frac{1}{3}t = \ln 0.2$
   ➔ $t = 4.83$ sec

2. A SDOF mass-spring-damper system has $m = 1$ kg, $c = 6$ Ns/m, and $k = 13$ N/m with the initial conditions $x(0) = \dot{x}(0) = 0$. A step force of 20 N is applied at $t = 0$.

   (a) Find the response of the system.
   $\ddot{x} + 6\dot{x} + 13x = 20$, $x(0) = 0$ and $\dot{x}(0) = 0$.

   - Take Laplace Transform of the equation:
     - $s^2X + 6sX + 13X = \frac{20}{s}$
     - $X(s) = \frac{20}{s(s^2+6s+13)}$
   - Find the poles:
     - $s(s^2 + 6s + 13) = 0$ ➔ $s = 0, -3 \pm 2j$
   - Partial fraction expansion:
     - $X(s) = \frac{A}{s} + \frac{B(s+3)}{(s+3)^2+2^2} + \frac{C(2)}{(s+3)^2+2^2}$
     - To find A, B, and C, combine the three terms on the right side:
       $$X(s) = \frac{A(s^2+6s+13)+Bs(s+3)+2Cs}{s[s^2+6s+13]} = \frac{(A+B)s^2+(6A+3B+2C)s+13A}{s[s^2+6s+13]}$$
     - Obtain the following three simultaneous equations:
       $$A + B = 0$$
       $$6A + 3B + 2C = 0$$
       $$13A = 20$$
     - Solving these equations:
       $$A = 1.54,\ B = -1.54,\ C = -2.31$$
   - Therefore, $x(t) = 1.54 - 1.54e^{-3t}\cos(2t) - 2.31e^{-3t}\sin(2t)$.

(b) Indicate the transient and the steady-state responses.

Transient: $-1.54e^{-3t}\cos(2t)-2.31e^{-3t}\sin(2t)$,

Steady-State: 1.54

(c) Indicate the natural and forced responses.

Natural: $-1.54e^{-3t}\cos(2t)-2.31e^{-3t}\sin(2t)$,

Forced: 1.54

3. Show the allowable region in the s-plane for the poles of a transfer function of a system if the system response requirements are $t_s \le 2$ sec and $M_p \le 3\%$. In your equation derivation, clearly show the directions of the inequalities.

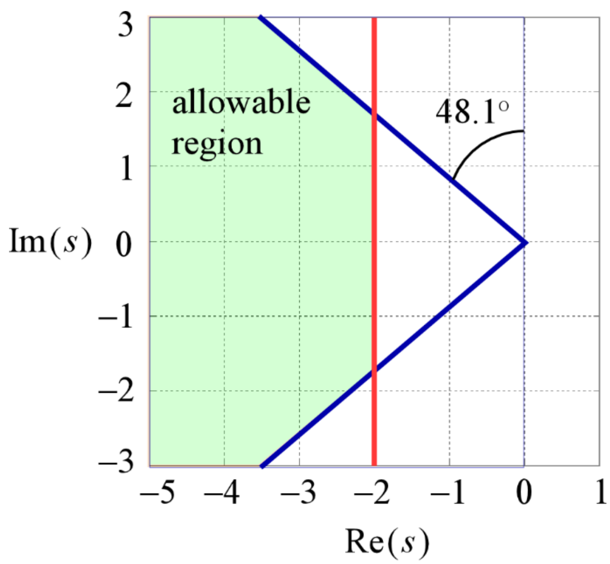
$$t_s = \frac{4}{\sigma} \le 2 \rightarrow \sigma \ge \frac{4}{2} = 2$$

$$M_p = 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}} \rightarrow 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}} \le 3 \rightarrow \ln e^{-\pi\zeta/\sqrt{1-\zeta^2}} \le \ln 0.03 \rightarrow$$

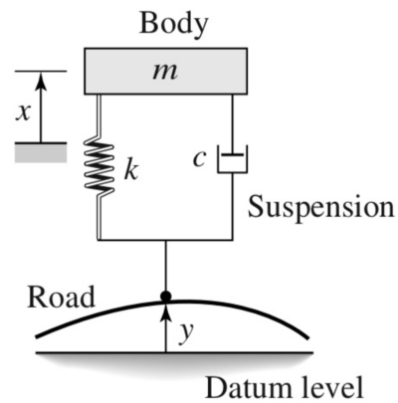
$$-\pi\zeta/\sqrt{1-\zeta^2} \le \ln 0.03 \rightarrow -\pi\zeta \le (\ln 0.03)\sqrt{1-\zeta^2} \rightarrow \pi^2\zeta^2 \ge (\ln 0.03)^2(1-\zeta^2) \rightarrow$$

$$\zeta^2 \ge \frac{(\ln 0.03)^2}{\pi^2 + (\ln 0.03)^2} \rightarrow \zeta \ge 0.7448 \rightarrow \theta \ge \sin^{-1}\zeta = \sin^{-1} 0.7448 = 0.8402 = 48.1^\circ \rightarrow$$

$$\theta \ge 48.1^\circ$$



4. Consider the single-mass full car model shown to the right. The mass is 1200 kg and the spring constant is 6000 N/m.
   (a) If we need to design the damper such that the car shows less than 5% overshoot, what would be the smallest value for *c*?
   (b) Using the damping coefficient found in (a), show the location of the system poles in the **S-plane** with the corresponding semi-circle (natural frequency) and lines (damping ratio).



(a)

$$M_p = 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}} \Rightarrow 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}} \le 5 \Rightarrow \ln e^{-\pi\zeta/\sqrt{1-\zeta^2}} \le \ln 0.05 \Rightarrow$$

$$-\pi\zeta\Big/\sqrt{1-\zeta^2} \le \ln 0.05 \Rightarrow -\pi\zeta \le (\ln 0.05)\sqrt{1-\zeta^2} \Rightarrow \pi^2\zeta^2 \ge (\ln 0.05)^2(1-\zeta^2)$$

$$\Rightarrow \zeta^2 \ge \frac{(\ln 0.05)^2}{\pi^2 + (\ln 0.05)^2} \Rightarrow \zeta \ge 0.69 \Rightarrow \theta \ge \sin^{-1}\zeta = \sin^{-1} 0.69 = 0.76 \Rightarrow$$

$$\theta \ge 43.6°$$

For the smallest *c* value, $\zeta = \dfrac{c}{2\sqrt{mk}} \ge 0.69 \Rightarrow$

$$c \ge 0.69 \times 2\sqrt{mk} = 0.69 \times 2\sqrt{6000 \times 1200} = 3703$$

$\therefore\ c = 3703\ \text{N·s/m}$

(b)

$$1200\ddot{x} + 3703\dot{x} + 6000x = f \Rightarrow 1200s^2 + 3703s + 6000 = 0$$

$$s = -1.54 \pm j1.62$$

with

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{6000}{1200}} = 2.24\ \text{rad/sec}$$

$$\theta \ge 43.6°$$