8. Modeling in Time Domain

State-Space Representation
Conversion between Transfer Function and State-Space Representation

Classical & Modern Controls

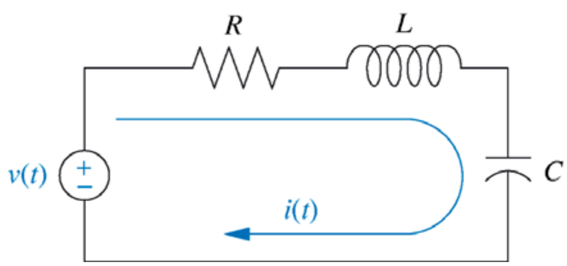
1. *Classical, frequency-domain technique:* ____________________
   1) A system's differential equation is converted to a transfer function. This way, a mathematical model is generated to algebraically relate a representation of the output to a representation of the input.
   2) Advantages: They rapidly provide stability and transient response information. It's easy to see the effects of varying system parameters using a graphic presentation or by a few quick calculations.
   3) Disadvantages: It can be applied only to Single-Input Single-Output (SISO), Linear, Time-Invariant (LTI) systems.

2. *Modern, time-domain technique:* _______________
   1) A system's differential equation can be simply rearranged into the state-space formulation.
   2) Advantages: A unified method for modeling, analyzing, and designing a wide range of systems including many nonlinear systems (backlash, saturation, and dead zone), time-varying systems (rockets), and Multi-Input Multi-Output (MIMO) systems.
   3) Disadvantages: It's not intuitive as the classical approach.

Procedure for State-Space Representation

1. *Obtain a differential equation for a given system using appropriate physical laws (e.g.: Newton's law, Kirchhoff's law).*
2. *Select a particular subset of all possible system variables. The variables in this subset are called _____________. For an n^{th}-order system, n linearly independent system variables are required to describe the system.*
3. *For an n^{th}-order system, write n simultaneous, first-order differential equations in terms of the state variables. They are called ________________.*
4. *The ______________ can be formed by algebraically combining the state variables with the system's input. Using the output equations, all of the other system variables can be found for $t>t_0$.*
5. *If the system is linear, the state and output equations can be written in ________________.*

Example 1: Electrical System



1. *Differential equation:* Kirchhoff's Voltage Law

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int i\,dt=v(t)\overset{i=dq/dt}{\Rightarrow}\quad \underline{\hspace{8cm}}$$

2. *State variables:* For the second-order system, two linearly independent state variables need to be selected. $\boxed{q \text{ and } \dot{q}=i}$
3. *State equations:* For the second-order system, form two simultaneous first-order differential equations.

$$\frac{dq}{dt}=i \quad \text{and} \quad \underline{\hspace{6cm}}$$

Electrical System (Cont'd)

4. *Output equations: one can solve for all other network variables in terms of the state variables and the input. For example, the voltage across the inductor can be written as*

$$v_L(t) = -\frac{1}{C}q(t) - Ri(t) + v(t)$$

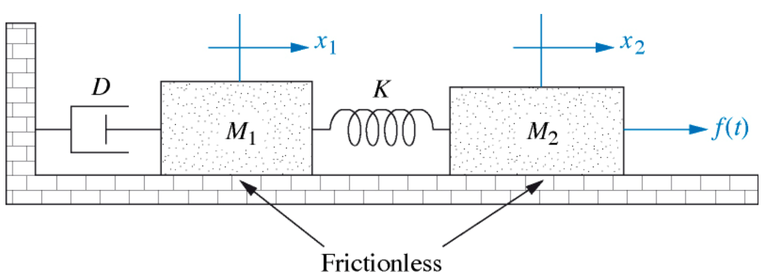
5. *Since the circuit is a linear system, the state and output equations can be written as*

$$\begin{bmatrix} dq/dt \\ d\dot{q}/dt \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v(t)$$

$$v_L(t) = \begin{bmatrix} -1/C & -R \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + v(t)$$

or $\dot{x} = Ax + Bu$, $y = Cx + Du$ *where*

Example 2: Mechanical System



1. *Differential equation:* Two second-order differential equations can be obtained using Newton's Law.

$$M_1 \frac{d^2x_1}{dt^2} + D\frac{dx_1}{dt} + Kx_1 - Kx_2 = 0$$

$$M_2 \frac{d^2x_2}{dt^2} - Kx_1 + Kx_2 = f(t)$$

2. *State variables:* From each of the second-order differential equations, two linearly independent state variables need to be selected. In this problem, they are ______________.

Mechanical System (Cont'd)

3. *State equations:* Form two simultaneous first-order differential equations from each of the two second-order differential eq's.

$$\frac{dx_1}{dt} = \dot{x}_1$$

$$\frac{d\dot{x}_1}{dt} = -\frac{K}{M_1}x_1 - \frac{D}{M_1}\dot{x}_1 + \frac{K}{M_1}x_2$$

$$\frac{dx_2}{dt} = \dot{x}_2$$

$$\frac{d\dot{x}_2}{dt} = \frac{K}{M_2}x_1 - \frac{K}{M_2}x_2 + \frac{1}{M_2}f(t)$$

5. *In matrix form:*

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}\begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \quad \end{bmatrix} f(t)$$

Comments on State-Space Representation

1. *A minimum number of state variables must be selected as components of the state vector. This minimum number of state variables is ___________ to describe completely the state of the system.*

2. *The components of the state vector (that is, the minimum number of state variables) must be ____________________. Variables and their successive derivatives are linearly independent. (e.g.: velocity and position are linearly independent)*

3. *If an **n**th-order differential equation describes the system, then **n** simultaneous, first-order differential equations are required along with **n** state variables.*

4. *For mechanical systems, position and velocity of each mass are typically selected for state variables.*

Conversion to State-Space (1/3)

1. *When a system represented by a transfer function is simulated on the digital computer, it is converted to the state-space form first.*
2. *The transfer function needs to be converted to a differential equation first by cross-multiplying and taking the inverse Laplace transform, assuming zero initial conditions.*
3. *An n^{th}-order linear differential equation can be conveniently converted to state-space form using **phase variables**, where each subsequent state variable is defined to be the derivative of the previous state variable.*

Conversion to State-Space (2/3)

4. *For the following diff. eq.,*

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_0 u$$

the phase variables are chosen as

$x_1 = y, x_2 =$ ______________________________

which leads to

$\dot{x}_1 = \frac{dy}{dt} = x_2,$ ______________________________

5. *The previous relationships can be written as*

$\dot{x}_1 = x_2$

Conversion to State-Space (3/3)

6. *In matrix form, the state and output equations are*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Example 3: Conversion to State-Space

- *Find the state-space representation in phase-variable form for the following transfer function.*

$$R(s) \longrightarrow \boxed{\frac{24}{s^3+9s^2+26s+24}} \longrightarrow C(s)$$

1. *The corresponding diff. eq. is found by taking inverse Laplace transform assuming zero I.C.*

____

2. *Choose the state variables as successive derivatives.*

$$x_1 = c, \quad x_2 = \dot{c}, \quad x_3 = \ddot{c}$$

3. *Differentiate the above equations and use the differential eq.*

$$\dot{x}_1 = \dot{c} = x_2$$
$$\dot{x}_2 = \ddot{c} = x_3$$
$$\dot{x}_3 = \dddot{c} = \text{____________________}$$

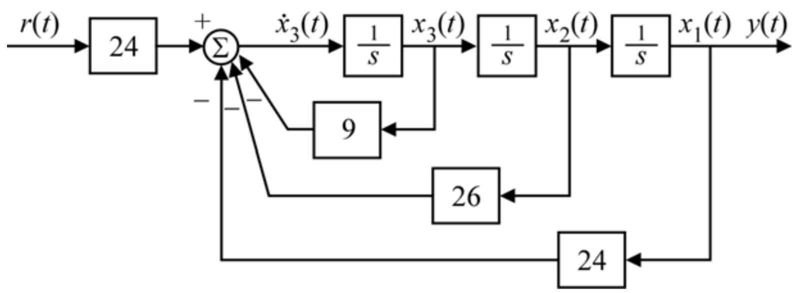
Example 3 (Cont'd)

4. *In matrix form, the state and output equations are*

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 24\end{bmatrix}r$$

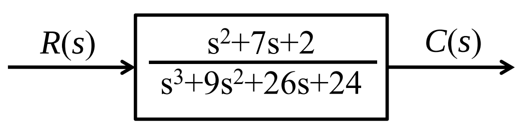
$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

5. *An equivalent block diagram showing phase variables is*

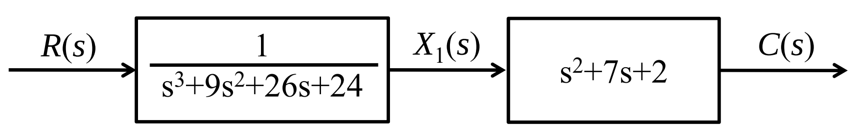


Example 4: TF with Numerator Dynamics

- *Find the state-space representation in phase-variable form for the following transfer function.*



1. *Separate the system into two cascaded blocks such that the first block contains the denominator and the second block contains the numerator.*



2. *Find the state equations for the first block.*

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}r$$

Not 24

Example 4 (Cont'd)

3. *Introduce the effect of the second block. First derive the differential equation.*

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \Rightarrow \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

4. *Using* $x_1 = x_1,\ \dot{x}_1 = x_2,\ \ddot{x}_1 = x_3$,

$$y(t) = c = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

5. *In matrix form, the output equation becomes*

$$y = [\qquad\qquad\qquad] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

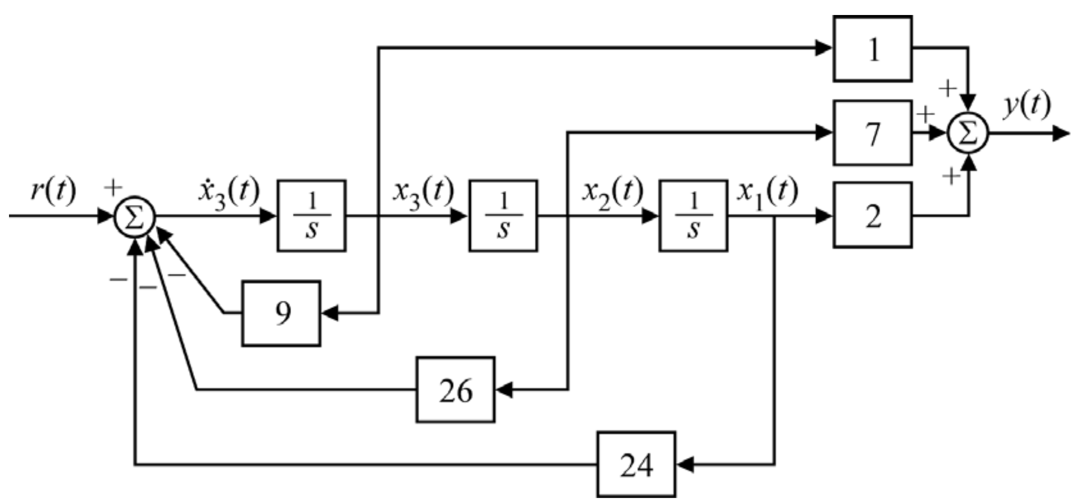
- *Therefore,*

Numerator in Transfer Function ➔ ____________ in State-Space form

Denominator in Transfer Function ➔ ____________ in S-S form

Example 4 (Cont'd)

6. *An equivalent block diagram is*



Conversion to Transfer Function

- *A state-space representation of a dynamic system can be converted into a transfer function.*

1. *Given the state and output equations*

$$\dot{x} = Ax + Bu \qquad (1)$$

$$y = Cx + Du \qquad (2)$$

take the Laplace transform assuming zero initial conditions

$$sX(s) = AX(s) + BU(s) \qquad (3)$$

$$Y(s) = CX(s) + DU(s) \qquad (4)$$

2. *Solving for* $X(s)$ *in Eq.(3) leads to*

$$\underline{\qquad\qquad\qquad\qquad} \Rightarrow \underline{\qquad\qquad\qquad\qquad} \qquad (5)$$

3. *Substituting Eq.(5) into Eq.(4) yields*

Transfer function matrix

$$Y(s) = [\boxed{\qquad\qquad\qquad}]\, U(s)$$

4. *If the input, U(s) and the output, Y(s) are scalars, the transfer function is*

$$T(s) = \frac{Y(s)}{U(s)} = \underline{\qquad\qquad\qquad\qquad} \qquad (6)$$

Example 5: Conversion to TF

- *Find the transfer function* $T(s)=Y(s)/U(s)$ *for the following system.*

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0]\, x$$

Example 5 (Cont'd)

Matlab code to obtain the state-space and transfer function objects (A, B, C, and D defined based on the previous example).

```
A = [0  1  0; 0  0  1; –1 –2 –3];
B = [10  0  0]′;
C = [1  0  0];
D = 0;
SYS = ss(A,B,C,D);
T = tf(SYS);
```