# 4. Feedback Control and PID Control

**Properties of feedback control systems**

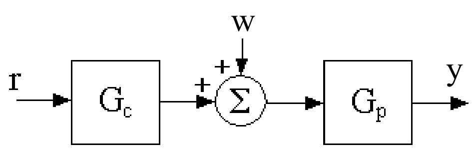
**PID control**

**System types**



## Advantages of Feedback Control

1. *Two advantages of feedback (closed-loop) control over open-loop control*
   1) **Disturbance rejection**: The errors of the system output are less susceptible to disturbance inputs when they are closed-loop controlled than they are open-loop controlled.
   2) **Sensitivity**: In feedback control, the error in the controlled quantity is less sensitive to variations in the system parameters.
2. *Disturbance rejection of open-loop controller*
   1) The control goal is to make the system output, $y$ as close as possible to the reference input, $r$, i.e. $y = r$ (example: cruise control).

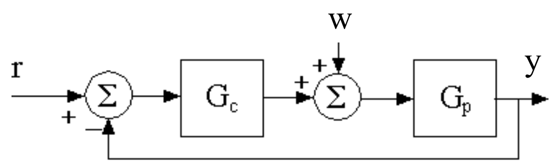



$$Y(s) = G_p(G_c R(s) + W(s))$$
$$= G_p G_c R(s) + G_p W(s)$$

   2) If $G_c = \dfrac{1}{G_p}$ and $G_p$ is ________________, then _________ .
   3) If $G_p$ is large, then $y_{ss} \neq r$. ← restriction of open-loop control.

# Disturbance Rejection

*3. Disturbance rejection of closed-loop controller*




$$Y(s)=\frac{G_pG_c}{1+G_pG_c}R(s)+\frac{\qquad\qquad}{\qquad\qquad}W(s)$$

1) If $G_c >> 1$, $\dfrac{G_pG_c}{1+G_pG_c}\approx 1$ and $\_\_\_\_\_\_\_\_$. Therefore, $Y(s)\approx R(s)$.

$$y_{ss}=\lim_{s\to 0} sY(s)= \_\_\_\_\_\_\_\_\_\_\_\_$$

2) Therefore, with feedback, the system output follows the reference input even with external disturbances. ➔ **Disturbance rejection**



# Sensitivity (1/2)

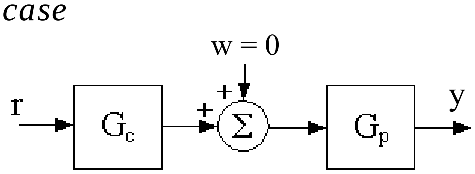
*1. Definition of sensitivity*

1) The sensitivity of a system variable, *A* to the variation of a system parameter, *B* is defined as

$$S_B^A \equiv \frac{\Delta A/A}{\Delta B/B}=\frac{B}{A}\frac{dA}{dB}=\frac{\text{fractional change in variable}}{\text{fractional change in parameter}}$$

2) For example, if $S_B^A=0.5$, then 10% change in parameter *B* gives 5% change in system variable *A*.

*2. Open-Loop case*



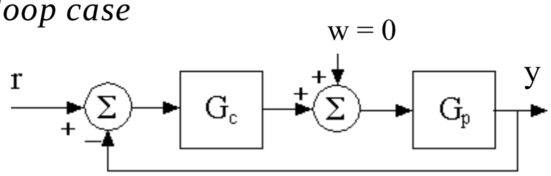

$$Y(s)=G_pG_cR(s)=G_{pc}R(s) \quad \text{where} \quad G_{pc}=G_pG_c$$

$$S_{G_p}^{G_{pc}}=\frac{G_p}{G_{pc}}\frac{dG_{pc}}{dG_p}= \_\_\_\_\_\_\_\_\_\_\_\_$$

# Sensitivity (2/2)

$S_{G_p}^{G_{pc}} = 1$ means that a 10% change in the system parameter will produce a 10% change in the total transfer function.

3. *Closed-loop case*




$$Y(s) = \frac{G_p G_c}{1 + G_p G_c} R(s) = G'_{pc} R(s) \quad \text{where } G'_{pc} \equiv \frac{G_p G_c}{1 + G_p G_c}$$

$$S_{G_p}^{G'_{pc}} = \frac{G_p}{G'_{pc}} \frac{dG'_{pc}}{dG_p} = G_p \left( \frac{1 + G_p G_c}{G_p G_c} \right) \frac{(1 + G_p G_c) G_c - G_p G_c G_c}{(1 + G_p G_c)^2}$$

$$= \underline{\qquad\qquad\qquad\qquad}$$

- If $G_c$ is very large ($G_c >> 1$), $S_{G_p}^{G'_{pc}} \approx 0$ ➔ Closed-loop transfer function is almost insensitive to the variation of the plant parameter.



# Example Problem (Sensitivity)

1. *When* $G_p = \dfrac{1}{s(s+a)}$, *and* $G_c = K$, *find the sensitivity of the system output to the variation of parameter a.*

   1) Open-loop:

$$G_{pc} = G_p G_c = \frac{K}{s(s+a)} \Rightarrow \frac{dG_{pc}}{da} = \frac{-K}{s(s+a)^2} = -\frac{G_{pc}}{(s+a)}$$

$$S_a^{G_{pc}} = \frac{a}{G_{pc}} \frac{dG_{pc}}{da} = \frac{a}{G_{pc}} \left( -\frac{G_{pc}}{s+a} \right) = -\frac{a}{s+a}$$

The sensitivity cannot be adjusted.

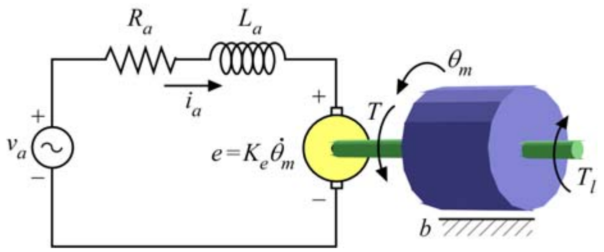
   2) Closed-loop:

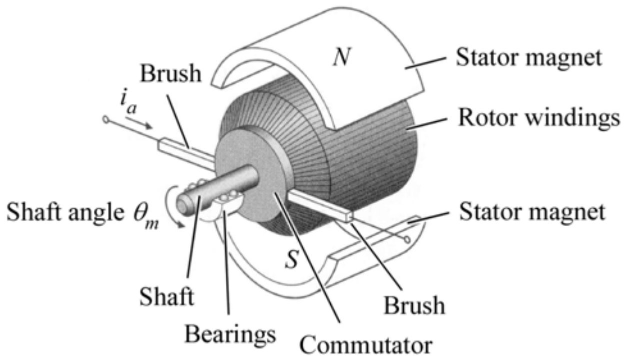
$$G'_{pc} = \frac{G_c G_p}{1 + G_c G_p} = \frac{K}{s^2 + as + K} \Rightarrow \frac{dG'_{pc}}{da} = -\frac{Ks}{(s^2 + as + K)^2}$$

$$S_a^{G'_{pc}} = \underline{\qquad\qquad\qquad\qquad}$$

As $K >> 1$, the sensitivity becomes very small

# Case Study: DC Motor Speed Control




Typical DC motor

$K_t$ = motor torque constant
$K_e$ = electromotive force constant
$R_a$ = armature resistance
$L_a$ = armature inductance
$i_a$ = armature current
$T$ = motor torque
$T_l$ = load torque (disturbance)
$J_m$ = rotor inertia
$v_a$ = applied voltage
$e$ = back emf



# DC Motor Modeling (1/3)

1. *Mechanical part: rotor dynamics*
   1) Torque developed in the rotor in response to the armature current, $i_a$:
   $$T = K_t i_a$$
   2) Dynamics of the rotor:
   $$J_m \ddot{\theta}_m = -b\dot{\theta}_m + T$$
   3) Equation of motion for the rotor with load torque, $T_l$:
   $$J_m \ddot{\theta}_m + b\dot{\theta}_m = K_t i_a + T_l$$
2. *Electrical part: armature circuit*
   1) Back emf, $e$ is induced voltage due to the interaction of the magnetic field and the current in the armature and is proportional to $\dot{\theta}_m$.
   $$e = K_e \dot{\theta}_m$$
   2) KVL loop analysis:
   $$v_a - L_a \frac{di_a}{dt} - R_a i_a - e = 0$$
   3) Differential equation for the current:
   $$L_a \frac{di_a}{dt} + R_a i_a = v_a - K_e \dot{\theta}_m$$

## DC Motor Modeling (2/3)

3. *Laplace transform*

1) Change of variables: $y \equiv \dot{\theta}_m,\ w \equiv T_l$

$$J_m \dot{y} + by = K_t i_a + w$$

$$K_e y + L_a \frac{di_a}{dt} + R_a i_a = v_a$$

2) Laplace transform:

$$sJ_m Y(s) + bY(s) = K_t I_a(s) + W(s)$$

$$sL_a I_a(s) + R_a I_a(s) + K_e Y(s) = V_a(s)$$

3) Solving the 1st equation for $I_a(s)$ and substitute into the 2nd equation leads to:

$$(J_m L_a s^2 + bL_a s + J_m R_a s + bR_a + K_t K_e)Y(s) = K_t V_a(s) + (sL_a + R_a)W(s)$$

or
$$\left( \frac{J_m L_a}{bR_a + K_t K_e} s^2 + \frac{J_m R_a + bL_a}{bR_a + K_t K_e} s + 1 \right) Y(s)$$
$$= \frac{K_t}{bR_a + K_t K_e} V_a(s) + \frac{R_a + L_a s}{bR_a + K_t K_e} W(s)$$



## DC Motor Modeling (3/3)

4. *Transfer function*

1) Roots of the characteristic equation:

$$\frac{J_m L_a}{bR_a + K_t K_e} s^2 + \frac{J_m R_a + bL_a}{bR_a + K_t K_e} s + 1 = 0$$

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4a}}{2a} \quad \text{where} \quad a = \frac{J_m L_a}{bR_a + K_t K_e},\ b = \frac{J_m R_a + bL_a}{bR_a + K_t K_e}$$

2) Using the roots of the characteristic equation, we can rewrite the Laplace transform of the system equation as

$$(\tau_1 s + 1)(\tau_2 s + 1) Y(s) = AV_a(s) + BW(s)$$

where $\tau_1 = -\frac{1}{s_1},\ \tau_2 = -\frac{1}{s_2}$ and $A = \frac{K_t}{bR_a + K_t K_e},\ B = \frac{R_a + L_a s}{bR_a + K_t K_e}$

3) Finally the transfer function is

**2nd - order system**

# Motor Modeling (No Inductance)

1. *No inductance approximation*
   1) The **electrical response** of the circuit is much **faster than** the **mechanical motion** of the rotor ➔ An applied voltage results in essentially an instantaneous change in the current flow ➔ It is possible to neglect the inductance, $L_a$.
   2) Then, the Laplace transform of the system equation becomes

$$\left(\frac{J_m R_a}{bR_a + K_t K_e} s + 1\right) Y(s) = \frac{K_t}{bR_a + K_t K_e} V_a(s) + \frac{R_a}{bR_a + K_t K_e} W(s)$$

2. *Transfer function*

$$Y(s) = \frac{A'}{(\tau' s + 1)} V_a(s) + \frac{B'}{(\tau' s + 1)} W(s)$$

**1st - order system**

where

$$\tau' = \frac{J_m R_a}{bR_a + K_t K_e}, \quad A' = \frac{K_t}{bR_a + K_t K_e}, \quad B' = \frac{R_a}{bR_a + K_t K_e}$$

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# Motor Modeling (Conclusion)

1. *System transfer function*
   1) 1st-order system

$$Y(s) = \frac{A'}{(\tau' s + 1)} V_a(s) + \frac{B'}{(\tau' s + 1)} W(s)$$

   2) 2nd-order system

$$Y(s) = \frac{A}{(\tau_1 s + 1)(\tau_2 s + 1)} V_a(s) + \frac{B}{(\tau_1 s + 1)(\tau_2 s + 1)} W(s)$$

   **3) Order of System**: Depending on the focus of the application, a given system can be modeled in various orders using different degrees of approximation.

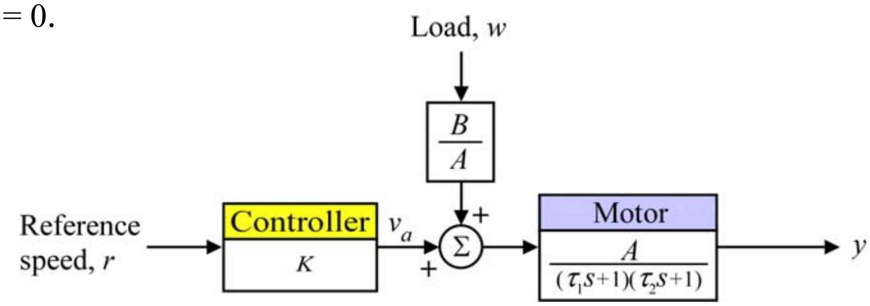
2. *Steady-state response*
   1) If $w$ and $v_a$ are constant, the steady-state response is

$$y_{ss} = \lim_{s \to 0} sY(s) = \lim_{s \to 0} s\left(\frac{A}{(\tau_1 s + 1)(\tau_2 s + 1)} \frac{v_a}{s} + \frac{B}{(\tau_1 s + 1)(\tau_2 s + 1)} \frac{w}{s}\right)$$

$$=$$

## Open-Loop Control for 2nd-Order Model

- *Determine the control gain, $K$ to make the steady-state speed, $y_{ss}$ the same as the reference input, $r$ when $A = 10$, $B = 50$, $r = 100$, and $w = 0$.*




Since $R(s) = \dfrac{100}{s}$ , $Y(s) = \dfrac{KA}{(\tau_1 s+1)(\tau_2 s+1)} R(s) = \dfrac{100KA}{s(\tau_1 s+1)(\tau_2 s+1)}$

Therefore, $y_{ss} = \lim_{s \to 0} sY(s) = 100KA$ and in order for $y_{ss}$ to be equal to $r$, $K = \dfrac{1}{A} = \dfrac{1}{10} = 0.1$.



## Open-Loop Control (Cont'd)

- *When disturbance of $w = -0.1$ added:*

$$Y(s) = \frac{KA}{(\tau_1 s+1)(\tau_2 s+1)} R(s) + \frac{B}{(\tau_1 s+1)(\tau_2 s+1)} W(s)$$

The two inputs are $R(s) = \dfrac{100}{s}$ and $W(s) = \dfrac{-0.1}{s}$ .

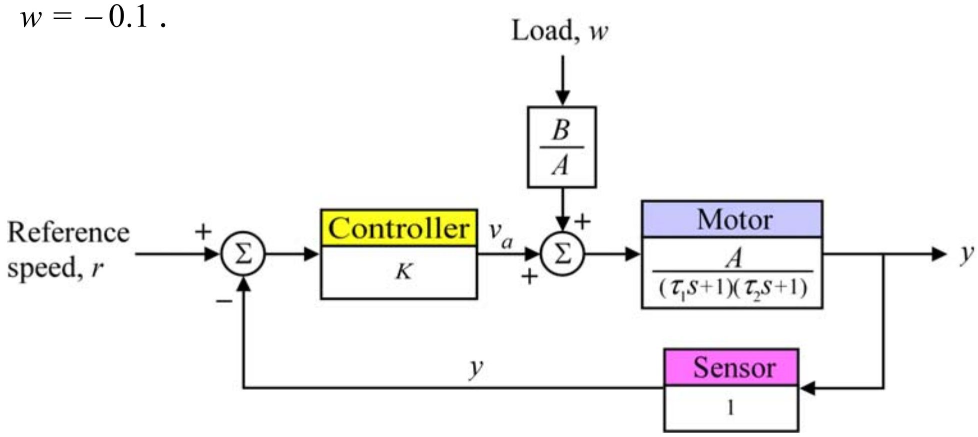
$$Y(s) = \frac{100}{s(\tau_1 s+1)(\tau_2 s+1)} + \frac{50(-.1)}{s(\tau_1 s+1)(\tau_2 s+1)} = \frac{95}{s(\tau_1 s+1)(\tau_2 s+1)}$$

$y_{ss} =$ ______________________

When the disturbance is added, the steady-state speed of 100 cannot be maintained with open loop control. In order to account for disturbances, we need to close the loop. ➔ Closed-loop control

## Closed-Loop Control for 2nd-Order Model

- *By including an output sensor and feeding its signal back to the controller, the system becomes closed-looped. In this case, the sensor is a _____________, which produces a voltage proportional to the shaft speed.*
- *Determine the control gain, K to make the steady-state speed,* $y_{ss}$ *the same as the reference input, r when* $A = 10$, $B = 50$, $r = 100$, and $w = -0.1$ .






## Closed-Loop Control (Cont'd)

- *The closed-loop transfer function of the system is*

$$Y(s) = \frac{\dfrac{KA}{(\tau_1 s+1)(\tau_2 s+1)}}{1+\dfrac{KA}{(\tau_1 s+1)(\tau_2 s+1)}} R(s) \; + \; \frac{\dfrac{B}{(\tau_1 s+1)(\tau_2 s+1)}}{1+\dfrac{KA}{(\tau_1 s+1)(\tau_2 s+1)}} W(s)$$

$$= \frac{KA}{(\tau_1 s+1)(\tau_2 s+1)+KA} R(s) \; + \; \frac{B}{(\tau_1 s+1)(\tau_2 s+1)+KA} W(s)$$

- *Theoretically, if* $K \to \infty$, then $Y(s) \to R(s)$ regardless of $A$, $B$, $R$, $W$, $\tau_1$ and $\tau_2$. Therefore, when $r = 100$, and $w = -0.1$

$$y_{ss} = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sR(s) = \underline{\hspace{6cm}}$$

- *However, for most systems, there is an upper limit on the gain K in order to achieve a well-damped stable response. Therefore, we cannot let K increase to any large number.*

# Closed-Loop Control (Cont'd)

- *If we choose K to be* 10 *:*

  The two inputs are $R(s) = \dfrac{100}{s}$ and $W(s) = \dfrac{-0.1}{s}$ .

$$Y(s) = \frac{KA}{(\tau_1 s+1)(\tau_2 s+1)+KA} R(s) + \frac{B}{(\tau_1 s+1)(\tau_2 s+1)+KA} W(s)$$

$$= \frac{100}{(\tau_1 s+1)(\tau_2 s+1)+100} \frac{100}{s} + \frac{50}{(\tau_1 s+1)(\tau_2 s+1)+100} \frac{-0.1}{s}$$

$$y_{ss} = \lim_{s\to 0} sY(s) = \underline{\hspace{8cm}}$$

Thus, by using the feedback, the effect of the disturbance has been greatly reduced and the steady-state value follows the reference input very closely.