3. System Response

Poles, Zeros, and System Response
First-Order Systems
Second-Order Systems

Poles, Zeros, and System Response

1. *Poles of a transfer function are:*
   1) The values of the Laplace transform variable, *s*, that cause the transfer function to become infinite.
   2) ___________________________ of the transfer function.
   3) Poles are shown as ×'s in the complex *s*-plane.
   4) A pole of the input function generates the form of the __________ ______________ .
   5) A pole of the transfer function generates the form of the ______ or ______________________ .
2. *Zeros of a transfer function are:*
   1) The values of the Laplace transform variable, *s*, that cause the transfer function to become zero.
   2) Roots of the numerator of the transfer function.
   3) Zeros are shown as o's in the complex *s*-plane.
   4) The zeros and poles generate the *amplitudes* for both the forced and natural responses.

Example 1: Poles and Zeros

1) Unit step response of the system $G(s)$ in (a) is:

$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}$$

$$= \frac{\qquad}{s} + \frac{\qquad}{s+5}$$

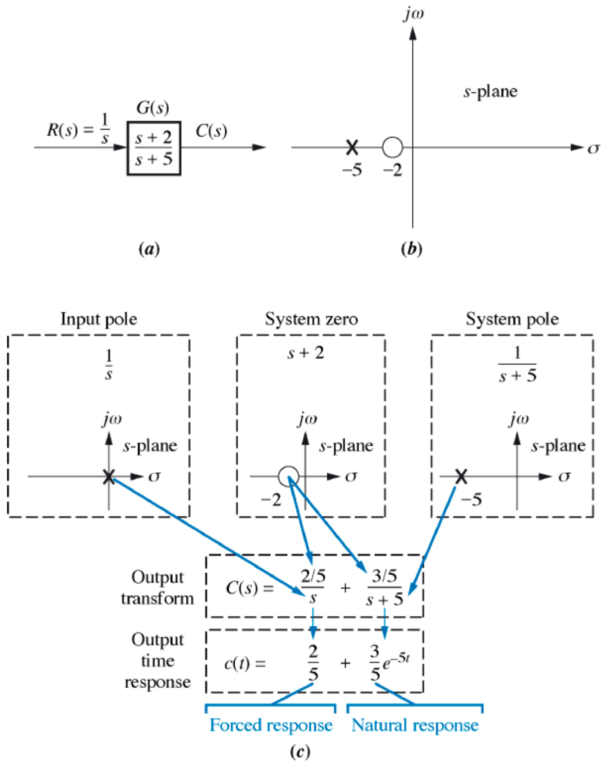
where

$$A = \left.\frac{(s+2)}{(s+5)}\right|_{s\to 0} =$$

$$B = \left.\frac{(s+2)}{s}\right|_{s\to -5} =$$

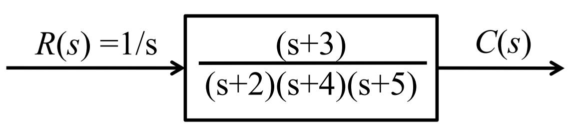
Thus,

$$c(t) =$$



Response vs. Poles

- *Given the system, write the output and specify which terms are related to the input and which terms are from the system's own characteristics.*



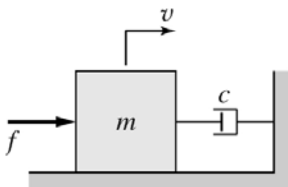
1. *By inspection, each system pole generates an exponential as part of the natural response.*

$$C(s) \equiv \underbrace{\frac{K_1}{s}} + \underbrace{\frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+5)}}$$

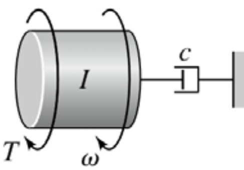
2. *Taking the inverse Laplace transform, we get*

$$c(t) \equiv \underbrace{K_1} + \underbrace{K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}}$$

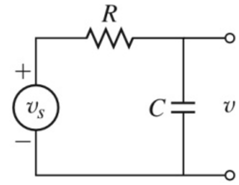
First-Order Systems



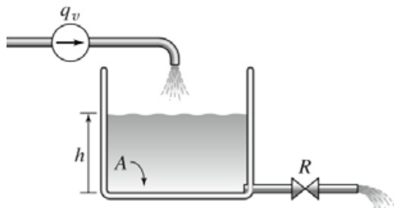
$$m\frac{dv}{dt}+cv=f$$
$$\tau=\frac{m}{c}$$



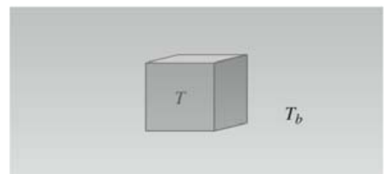
$$I\frac{d\omega}{dt}+c\omega=T$$
$$\tau=\frac{I}{c}$$



$$RC\frac{dv}{dt}+v=v_s$$
$$\tau=RC$$



$$AR\frac{dh}{dt}+gh=Rq_v$$
$$\tau=\frac{AR}{g}$$



$$mc_pR\frac{dT}{dt}+T=T_b$$
$$\tau=mc_pR$$

First-Order System (1/2)

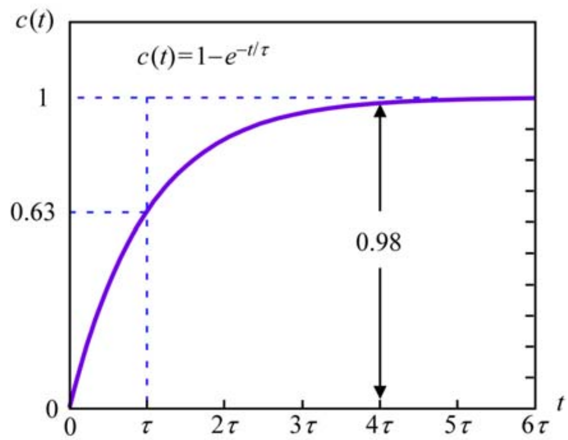
1. *Step response of first-order system*
   1) Consider the following first-order system.
   $$\dot{x}+ax=af(t)$$
   2) The characteristic equation is ____________ and its root, ________ is called the system _ _ _ _ _ _ _ _ _ _.
   3) For a **step response**, a unit step function, $u(t)$ is applied to the first-order system. The resulting system equation becomes
   $$\dot{x}+ax=au_s(t)$$
   4) The solution of the equation with zero initial conditions is

   ____

2. *Time constant*
   1) The time constant, τ is defined as _______ (positive reciprocal of the system pole) and the step response can be rewritten as $x(t)=1-e^{-t/\tau}$.
   2) Systems with **smaller** time constants respond _ _ _ _ _ _ _ _ _ _ than those with larger time constants.

First-Order System (2/2)

2. *Time constant (cont'd)*
   3) At $t = \tau$, $e^{-t/\tau}$ decays to 37% of its initial value and $x(t)$ rises to ______ of its final value.
   4) At $t = 4\tau$, the response is ________ of the steady-state value, and at $t = 5\tau$, it is 99% of steady state.
   5) For most engineering purposes, we regard that $x(t)$ reaches steady state at $t = 4\tau$.



Time (t)	$1-e^{-t/\tau}$
$t = 0$	0
$t = \tau$	0.63
$t = 2\tau$	0.86
$t = 3\tau$	0.95
$t = 4\tau$	0.98
$t = 5\tau$	0.99

Example: First-Order System

❑ *A system model is found to be* $\dot{x} + 50x = 50f$ *. Find the time constant and the time that the system takes to reach* 50% *and* 98% *of the steady-state value when a unit step input is applied.*

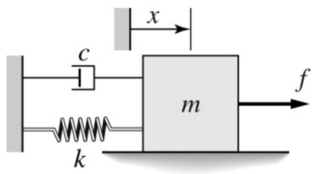
1. *Time constant,* $\tau =$ ____________________________.
2. *When* t=4τ, *the system reaches* 98% *of the final value. Thus, after* ______________, *the response rises to* 98% *of the final value.*
3. *In order to find the time that it reaches* 50% *of the final value, the following equation needs to be solved.*

$$1 - e^{-50t} = 0.5 \quad \Rightarrow \quad e^{-50t} = 0.5 \quad \Rightarrow \quad \ln(e^{-50t}) = \ln(0.5)$$

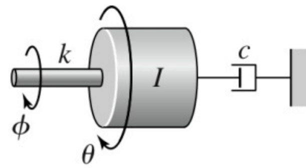
$$\Rightarrow \quad t = \underline{\hspace{6cm}}$$

First order systems do not exhibit any oscillatory behavior!

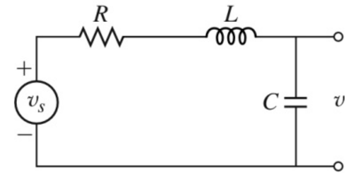
Second-Order Systems



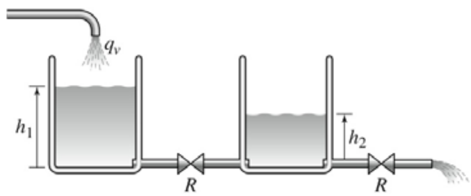
$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=f$$



$$I\frac{d^2\theta}{dt^2}+c\frac{d\theta}{dt}+k\theta=k\phi$$

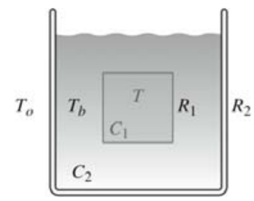


$$LC\frac{d^2v}{dt^2}+RC\frac{dv}{dt}+v=v_s$$



$$RA\frac{dh_1}{dt}+g(h_1-h_2)=Rq_v$$

$$RA\frac{dh_2}{dt}+g(h_2-h_1)+gh_2=0$$

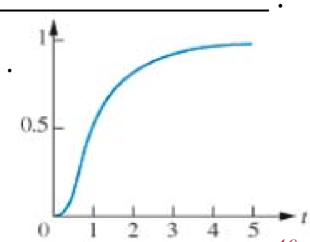


$$R_1C_1\frac{dT}{dt}+T=T_b$$

$$R_1R_2C_2\frac{dT_b}{dt}+(R_1+R_2)T_b=R_2T+R_1T_o$$

Second-Order System (1/3)

1. *Step response of second-order system*
   1) Consider the following second-order system.
   $$\ddot{x}+a\dot{x}+bx=bf(t)$$
   2) The characteristic equation is $s^2+as+b=0$ and the system **poles** are __________________________.
   2) For a step response, a unit step function, $u(t)$ is applied to the second-order system. The resulting system equation becomes
   $$\ddot{x}+a\dot{x}+bx=bu_s(t)$$
2. *Overdamped case*
   1) If $a^2-4b>0$, the char. eq. has two distinct real roots, $-\sigma_1$ and $-\sigma_2$.
   2) The general step response is ______________________________________ .
   3) Two time constants exist: $\tau_1=1/\sigma_1$ and $\tau_2=1/\sigma_2$.
   4) Example: If $a=b=9$,
   $$x(t)=1+0.171e^{-7.854t}-1.171e^{-1.146t}$$



Second-Order System (2/3)

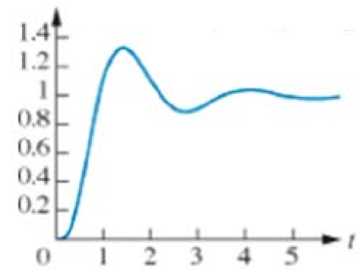
3. *Underdamped case*

   1) If $a^2 - 4b < 0$, the char. eq. has two complex roots, $-\sigma_d \pm j\omega_d$.
   2) The general step response is

      ____

   3) Time constant: _______________.
   4) Example: If $a = 2$ and $b = 9$,

      $$x(t) = 1 - e^{-t}\cos(\sqrt{8}t) - \frac{\sqrt{8}}{8}e^{-t}\sin(\sqrt{8}t)$$



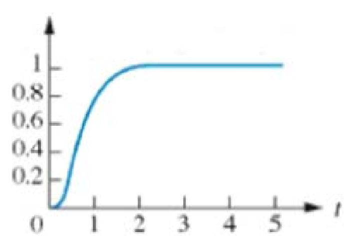
   5) Response characteristics: Damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's real part. The radian frequency of the sinusoid, the __________ ____________ of oscillation, is equal to the imaginary part of the poles.

Second-Order System (3/3)

4. *Critically-Damped case*

   1) If $a^2 - 4b = 0$, the char. eq. has _______ roots, $-\sigma_1$.
   2) The general step response is ________________________.
   3) Time constant: $\tau = 1/\sigma_1$.
   4) Example: If $a = 6$ and $b = 9$,

      $$x(t) = 1 - 3te^{-3t} - e^{-3t}$$



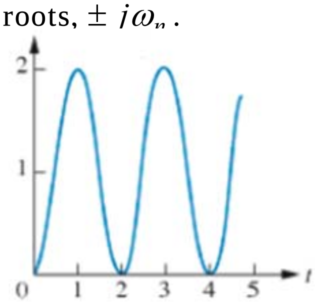
5. *Undamped case*

   1) If $a = 0$, the char. eq. has two imaginary roots, $\pm j\omega_n$.
   2) The general step response is

      $$x(t) = C_0 + C_1\sin(\omega_n t) + C_2\cos(\omega_n t) = C_0 + A\sin(\omega_n t + \phi)$$

   3) Example: If $a = 0$ and $b = 9$,

      $x(t) = 1 - \cos 3t$



Example (Overdamped case)

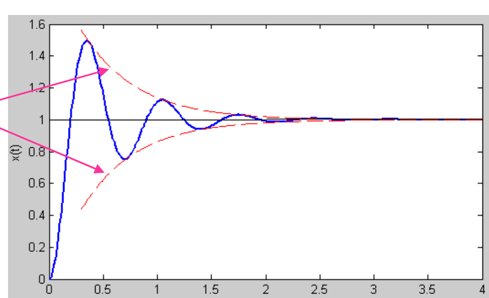
Identify the transient, steady-state, free, and forced responses of the following equation with $\dot{x}(0)$ and $x(0)$ *as I.C.'s.*

$$\ddot{x} + 7\dot{x} + 10x = c$$

Example (Underdamped Case)

Determine the response of the SDOF mass-spring-damper system when m=1 *kg*, c=4 *Ns/m*, *and* k=85 *N/m with the initial conditions* $x(0) = 0$ *and* $\dot{x}(0) = 0$, *and the external input* f= 85.

Vibration decays exponentially !



Example (Zero-Damping Case)

Determine the response of the SDOF mass-spring-damper system when m=1 kg, c= 0, and k=85 N/m with the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$, and the external input f= 85.

Solution) $\ddot{x} + 85x = 85 \quad x(0) = 0 \quad \dot{x}(0) = 0$

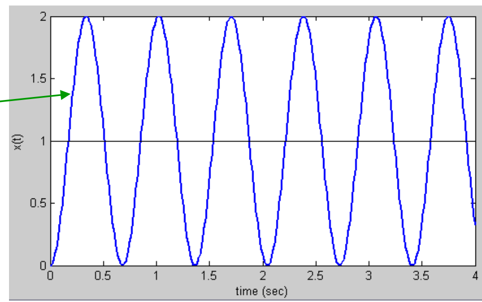
$x_c = A\cos\sqrt{85}t + B\sin\sqrt{85}t$ and $x_p = 1$.

The general solution becomes $x(t) = A\cos\sqrt{85}t + B\sin\sqrt{85}t + 1$.

By applying the I.C.'s, the final solution is found to be

$$x(t) = \underbrace{\qquad\qquad\qquad}_{\text{steady-state response}}$$

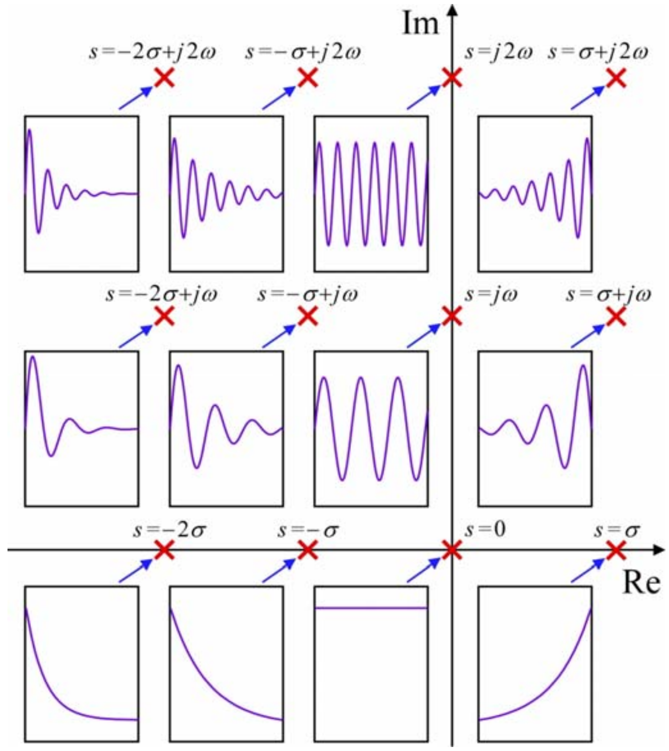
Without damping, no energy is dissipated and vibration continues forever !



Effect of Pole Location

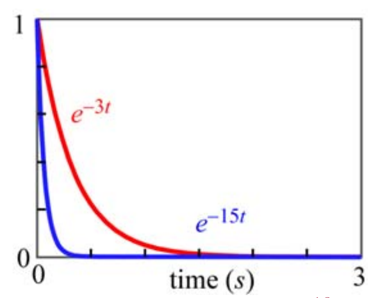
1. *Location of system poles (characteristic roots)*
   - The system behavior is dependent on the location of the roots of the characteristic equation. They are also called the **system poles**.
2. *General tendency*
   1) Unstable behavior occurs when any pole lies in the ______________.
   2) Neutrally stable behavior occurs when a pole lies on the imaginary axis.
   3) Since the system parameters $\zeta, \omega_n, \omega_d, \sigma,$ and τ are normally for stable systems only, it is assumed here that all poles are located in the ______________ (stable region) when those system parameters are defined.
   4) The response oscillates only when the pole has a nonzero ______________.
   5) The greater the imaginary part, the _________ the frequency of the oscillation. (Imaginary part of system pole = Frequency of oscillation)
   6) The farther to the left the root lies in the S-plane, the ________ the response decays. (Real part of system pole = Decay rate of response)

Effect of Pole Location (Free Response)



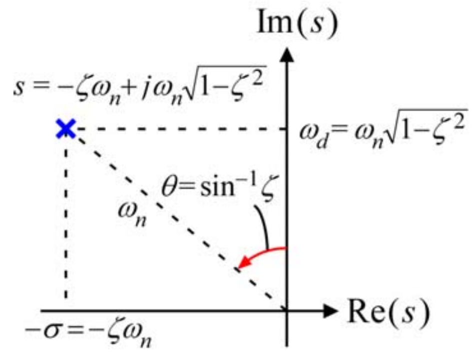
Dominant-Root Analysis

1. *Dominant root*
   1) If the real parts of two different characteristic roots are $-\sigma_1$ and $-\sigma_2$, the response will have the exponentials $e^{-\sigma_1 t}$ and $e^{-\sigma_2 t}$ associated with each of these roots.
   2) If both roots are negative, the system is stable and time constants can be defined such that $\tau_1 = 1/\sigma_1$ and $\tau_2 = 1/\sigma_2$.
   3) Since a time constant τ is a measure of the decay rate of an exponential $e^{-t/\tau}$, the two exponentials will decay at different rates.
   4) The exponential having the largest time constant decays the _______ and thus _________________. Its corresponding root is thus called the dominant root, and its time constant is the dominant time constant.
2. *Example*
   1) If the system poles of a second-order system are $s=-3, -15$, the associated time constants are __________ and _____________. Thus, the second exponential will disappear five times faster than the first exponential.



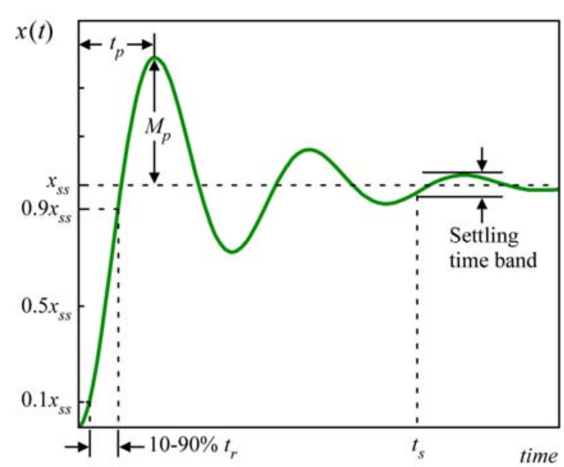
Pole Location vs. System Parameters

- *System response vs. pole location*
  1) To visualize the connection between the pole location and the system behavior, the poles are plotted in the complex plane or S-plane.
  2) The real part of the system pole is _______________ and determines how fast the response decays.
  3) The imaginary part of the system pole is _______________ and is the damped natural frequency.
  3) The distance between the origin and the pole is _______, the natural frequency.
  4) The damping ratio, ζ is related to θ , the angle measured from the imaginary axis such that ___________.



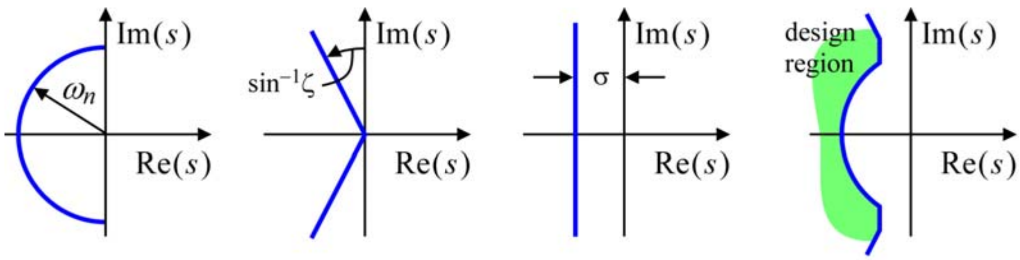
Time Domain Specifications

1. *Definitions*
   1) Maximum or peak __________, M_p: the maximum deviation of the output above its steady-state value, x_{ss}. It is sometimes expressed as a percentage of the final value.
   2) **Peak time, t_p**: the time at which the maximum overshoot occurs.
   3) _______________, t_s: the time required for the oscillations to stay within some specified small percentage (2% or 5%) of the final value.
   4) ___________, t_r: the time required for the output to rise from a certain low value (0% or 10%) to a certain high value (100% or 90%).



Pole Location vs. Time Domain Specifications

2. *General design guideline*
   1) **Peak time, t_p :** $t_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$
   2) **Overshoot, M_p (%) :** $M_p = 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}}$
   3) **Settling time (2%), t_s :** $t_s = \dfrac{4}{\sigma} = \dfrac{4}{\zeta\omega_n}$
   4) **Rise time (10% – 90%), t_r :** $t_r \cong \dfrac{1.8}{\omega_n}$
3. *Graphs of regions in s-plane for certain transient requirements*



(a) rise time (b) overshoot (c) settling time (d) composite

Example: Specifications to the S-Plane

- *Find the allowable regions in the s-plane for the poles of a system if the system response requirements are $t_r \le 0.6$ sec, $M_p \le 10\%$, and $t_s \le 3$ sec.*

Solution)

$M_p = 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}}$ ⇨ $100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 10$ ⇨ ____________

⇨ $\theta \ge \sin^{-1}\zeta =$ ____________

$t_r \cong \dfrac{1.8}{\omega_n}$ ⇨ $\omega_n \ge$ ____________

$t_s = \dfrac{4}{\sigma}$ ⇨ $\sigma \ge$ ____________