9. State-Space Techniques

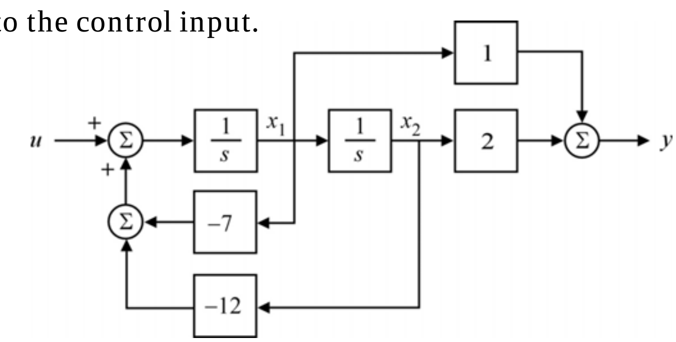
Canonical Forms
Full-State Feedback
Controllability
Linear Quadratic Regulator (LQR)
Estimator Design

Block Diagrams and Canonical Forms

- *Consider a system that has a simple transfer function*

$$G = \frac{b(s)}{a(s)} = \frac{s+2}{s^2+7s+12} = \frac{2}{s+4} + \frac{-1}{s+3}$$

1) The transfer function has been represented in two forms: a _______ __________ and the result of a ______________________.
2) In order to develop a state description of the system, a block diagram is constructed using only isolated integrators as the dynamic elements.
3) **___________________** has each state variable connected by the feedback to the control input.



Control Canonical Form (1/2)

1. *System equations*
   1. Differential equations:

$$\dot{x}_1 = -7x_1 - 12x_2 + u, \quad \dot{x}_2 = x_1 \quad \text{and} \quad y = x_1 + 2x_2$$

   2. Matrix equations:

$$\dot{x} = A_c x + B_c u, \quad y = C_c x + D_c u \quad \text{where}$$

- Notes: the coefficients 1 and 2 of the numerator polynomial $b(s)$ appear in the C_c matrix, and the coefficients 7 and 12 of the denominator polynomial $a(s)$ appear (with opposite signs) as the first row of the A_c matrix.

Control Canonical Form (2/2)

- *General form of the state matrices in control canonical form*
  1. If $b(s) = b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n$ and $a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n$
  2. then the MATLAB steps are:

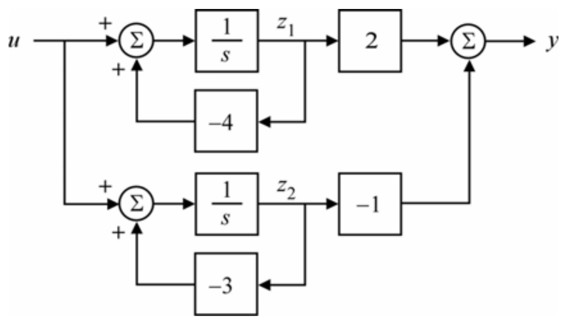
$$b = [b_1 \quad b_2 \quad \cdots \quad b_n]$$
$$a = [1 \quad a_1 \quad a_2 \quad \cdots \quad a_n]$$
$$[A_c, B_c, C_c, D_c] = \mathrm{tf2ss}(b, a)$$

  3. The result will be

$$A_c = \begin{bmatrix} -a_1 & -a_2 & & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \\ & & & \cdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
$$C_c = [b_1 \quad b_2 \quad \cdots \quad b_n], \qquad D_c = 0$$

Modal Canonical Form (1/2)

1. *Modal canonical form*
   1. Block diagram corresponding to the ______________________ :



   2. Matrix equations:

$$\dot{z} = A_m z + B_m u$$
$$y = C_m z + D_m u$$

where

____

Modal Canonical Form (2/2)

2. *Properties of modal canonical form*
   1) The poles of the system transfer function are sometimes called the _______________ of the system ➔ **modal canonical form.**
   2) Systems poles (here -4 and -3) appear as the elements along _____ _____________________ , and the residues, the numerator terms in the partial-fraction expansion, (here 2 and -1) appear in the C_m matrix.
3. *Complicated cases*
   1. When the poles of the system are complex, the elements of the matrices will be complex ➔ The complex poles of the partial-fraction expansion need to be expressed as conjugate pairs in second-order terms. The corresponding A_m matrix will have 2×2 blocks along the main diagonal representing the local coupling between the variables of the complex-pole set.
   2. When the partial-fraction expansion has repeated poles ➔ The corresponding state variables need to be coupled so that the poles appear along the diagonal with off-diagonal terms indicating the coupling.

State Transformation (1/5)

1. *State transformation*
   1) It is possible to find the relationship between matrices in two different forms (and their corresponding state variables).
   2) Thus, it is possible to calculate the desired canonical form without obtaining the transfer function first.
2. *Transformation equations*
   1. Consider a system described by the state equations:

$$\dot{x} = Fx + Gu$$
$$y = Hx + Ju$$

   2. Consider a change of state from x to a new state z that is a linear transformation of x. For a nonsingular matrix T we let

$$x = Tz$$

   3. By substituting the above equation into the matrix equation, we have the equations of motion in terms of the new state z:

$$\dot{x} = T\dot{z} = \_\_\_\_\_\_\_\_\_ \qquad \text{where} \quad A = \_\_\_\_\_\_\_\_\_$$
$$\Rightarrow \dot{z} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \qquad B = \_\_\_\_\_\_\_\_\_$$
$$\Rightarrow \dot{z} = \_\_\_\_\_\_\_\_\_\_\_\_$$

State Transformation (2/5)

   4. The output equation becomes

$$y = HTz + Ju \qquad \text{where} \quad C = \_\_\_\_\_\_$$
$$= Cz + Du \qquad D = \_\_\_\_$$

3. *Finding transformation matrix, T*
   1. Given the general matrices F, G, and H and scalar J, the transformation matrix T can be found such that A, B, C, and D are in a particular form, for example, control canonical form.
   2. To find such a T, it is assumed that A, B, C, and D are already in the required form, further that the transformation T has a general form, and match terms.
   3. Example: for a third-order system,

      $A = T^{-1}FT$ can be rewritten as _______________

   4. If A is in control canonical form, T^{-1} can be described as a matrix with rows t_1, t_2, and t_3:

$$\begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} t_1F \\ t_2F \\ t_3F \end{bmatrix}$$

State Transformation (3/5)

5. Working out the third and second rows gives the matrix equations

$$t_2 = t_3 F,$$
$$t_1 = t_2 F = t_3 F^2.$$

6. From $T^{-1}G = B$, assuming that B is also in control canonical form, we have the relation

$$\begin{bmatrix} t_1 G \\ t_2 G \\ t_3 G \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

7. Combining these results, we get

$$t_3 G = \underline{\quad\quad}$$
$$t_2 G = \underline{\quad\quad\quad\quad\quad}$$
$$t_1 G = \underline{\quad\quad\quad\quad\quad}$$

8. These equations can in turn be written in matrix form as

$$\underline{\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad}$$

State Transformation (4/5)

9. The previous equation can be rewritten as

$$t_3 = [0 \quad 0 \quad 1] C^{-1}$$

where the controllability matrix $C = [G \quad FG \quad F^2 G]$. With t_3, all the other rows of T^{-1} can be constructed.

4. *In summary, the conversion process of a general state description to control canonical form is as follows:*

1. From F and G, form the controllability matrix C as

$$C = [G \quad FG \quad \cdots \quad F^{n-1}G]$$

2. Compute the last row of the inverse of the transformation matrix as

$$t_n = [0 \quad 0 \quad \cdots \quad 1] C^{-1}$$

3. Construct the entire transformation matrix as

$$T^{-1} = \begin{bmatrix} t_n F^{n-1} \\ t_n F^{n-2} \\ \vdots \\ t_n \end{bmatrix}$$

State Transformation (5/5)

4. Compute the new matrices from T^{-1} using

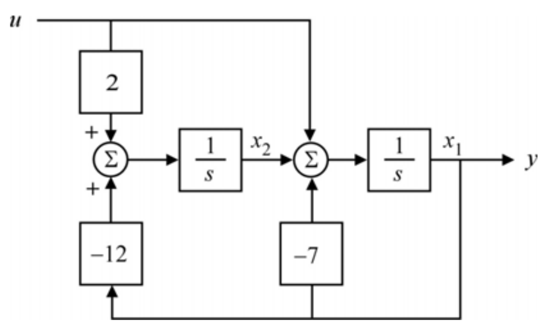
$$A = T^{-1}FT \qquad C = HT$$
$$\text{and}$$
$$B = T^{-1}G \qquad D = J$$

5. *Conclusion*
   1. One can always transform a given state description to control canonical form if and only if the **controllability matrix $\boldsymbol{C}$ is ____________** .
   2. If the controllability matrix C is nonsingular, the corresponding F and G matrices are said to be controllable.
   3. If a state representation is **controllable**, an external input can move the internal state of the system from any initial state to any other final state in a finite time interval.
   4. A change of state by a nonsingular linear transformation does not change controllability.

$$C_z = [B \quad AB \quad \cdots \quad A^{n-1}B]$$
$$= [T^{-1}G \quad T^{-1}FTT^{-1}G \quad \cdots \quad T^{-1}F^{n-1}TT^{-1}G] = T^{-1}C$$

Observer Canonical Form (1/3)

1. *Observer canonical form*
   1. Block diagram



   2. Matrices:

$$A_o = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix}, \quad B_o = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$C_o = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D_o = 0$$

- Notes: All the feedback is from the output to the state variables.

Observer Canonical Form (2/3)

2. *Consideration of the controllability of this system*

   1. To see what happens if the zero at -2 is varied, the second element 2 of B_o is replaced by the variable zero location $-z_o$ and then, the controllability matrix becomes:

$$C = [B_o \quad A_o B_o] = \begin{bmatrix} 1 & -7 - z_o \\ -z_o & -12 \end{bmatrix}$$

   2. The determinant of this matrix is a function of z_o:

$$\det(C) = -12 + (z_o)(-7 - z_o) = -(z_o^2 + 7z_o + 12)$$

   3. This polynomial is zero for $z_o = -3$ or -4, implying that controllability is lost for these values. Let's consider this phenomenon.
   4. In terms of the parameter z_o, the transfer function is

$$G(s) = \frac{s - z_o}{(s+3)(s+4)}$$

Observer Canonical Form (3/3)

If $z_o = -3$ or -4, there is a pole-zero cancellation and the transfer function reduces from a second-order system to a first-order one. When $z_o = -3$, for example, the mode at -3 is decoupled from the input and control of this mode is lost.

- *Notes:*
  1. We have taken a transfer function and given it two realizations, one in control canonical form and one in observer canonical form.
  2. The control form is always controllable for any value of the zero, while the observer form loses controllability if the zero cancels either of the poles.
  3. Thus, these two forms may represent the same transfer function, but it may not be possible to transform the state of one to the state of the other. ➔ No one-to-one mapping transformation
  4. While a transformation of state cannot affect controllability, the particular state selected from a transfer function can.

  ❖ *Controllability is a function of the state representation of the system and cannot be decided from a transfer function.*