9. State-Space Techniques

Canonical Forms
Full-State Feedback
Controllability
Linear Quadratic Regulator (LQR)
Estimator Design

Full-State Feedback (1/4)

1) The first step in the state-space controller design is to find the control law as feedback of a linear combination of the state variables, that is,

$$u = -Kx = -\begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

2) It is assumed that for feedback purposes all the elements of the state vector are available. In most cases, this is not the case and a special calculator called observer is needed.

3) Substituting the feedback law $u = -Kx$ into the system yields

$$\dot{x} = Ax + Bu = Ax - BKx = (A - BK)x$$

4) The characteristic equation of this closed-loop system is

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5) The control-law design consists of picking the gain K so that the roots of the characteristic equation (or the closed-loop poles) are in desirable locations.

Full-State Feedback (2/4)

6) If the desired locations are known as

$$s = s_1, s_2, \cdots, s_n$$

7) Then, the corresponding desired characteristic equation is

$$\alpha_c(s) = (s - s_1)(s - s_2)\cdots(s - s_n) = 0$$

- *Example*

Q: An undamped oscillator with frequency ω_0 and a state-space description given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Find the control law that places the closed-loop poles of the system so that they are both at $-2\omega_0$. In other words, you wish to double the natural frequency and increase the damping ratio from 0 to 1.

Sol: The desired characteristic equation is

$$\alpha_c(s) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Full-State Feedback (3/4)

The characteristic equation of the closed-loop system is

$$\det[sI - (A - BK)] = \det\left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix} \right) \right\}$$

or

$$s^2 + K_2 s + \omega_0^2 + K_1 = 0.$$

Equating the coefficients with like power of s in the two equations yields the system of equations

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

and therefore,

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Finally, the control law becomes

$$K = [K_1 \quad K_2] = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Full-State Feedback (4/4)

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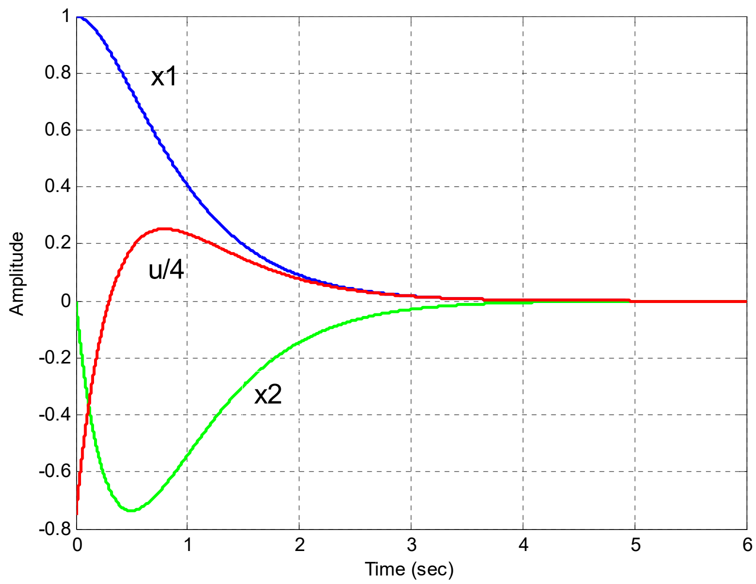
When the initial conditions are x1 = 1.0 and x2 = 0.0, and ω_0= 1,



Two system variables are controlled at the same time !

Pole Placement (1/3)

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1. *Problem with previous example*
   1) Calculating the control gains using the previous technique becomes rather tedious when the order of the system is higher than 3.
   2) It is not also clear whether a solution always exists to the equations that result from matching coefficients of the system's characteristic equation to a desired equation.

2. *Control canonical form*
   1. When the system has been transformed into control canonical form, the equations have a solution that is trivial to find.

$$F_c = \begin{bmatrix} -a_1 & -a_2 & & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \\ & & & \cdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \quad G_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$H_c = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}, \qquad J_c = 0$$

The special structure of this system matrix is referred to as the

________________________ .

Pole Placement (2/3)

2. The closed-loop system matrix $F_c - G_c K_c$ is

$$F_c - G_c K_c = \begin{bmatrix} -a_1 - K_1 & -a_2 - K_2 & \cdots & -a_n - K_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \end{bmatrix}$$

3. The closed-loop characteristic equation is

4. If the desired pole locations result in the characteristic equation given by

$$\alpha_c(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n = 0$$

then, the necessary feedback gains can be found by equating the coefficients in the two polynomials:

Pole Placement (3/3)

3. *Systematic approach*
   1. Transform (F, G) to control canonical form (F_c, G_c) by changing the state using $x = Tz$.
   2. Solve for the control gains by inspection using

      $$K_1 = -a_1 + \alpha_1, \quad K_2 = -a_2 + \alpha_2, \quad \cdots \quad , \quad K_n = -a_n + \alpha_n$$

      to give the control law $u = -K_c z$.
   3. Since the gain is for the state in the control canonical form, the gain must be transformed to the original state to get

      $$K = K_c T^{-1}$$

Ackermann's Formula (1/2)

1. Ackermann (1972) organized the previous three-step process of converting to (F_c, G_c), solving for the gains, and converting back again into a very compact form.

2. Ackermann's formula

$$K = [0 \quad \cdots \quad 0 \quad 1]\, C^{-1} \alpha_c(F)$$

such that $$C = [G \quad FG \quad \cdots \quad F^{n-1}G]$$

where C is the controllability matrix, n gives the order of the system and the number of state variables, and $\alpha_c(F)$ is a matrix defined as

$$\alpha_c(F) = F^n + \alpha_1 F^{n-1} + \alpha_2 F^{n-2} + \cdots + \alpha_n I$$

where the α_i are the coefficients of the desired characteristic polynomial.

Ackermann's Formula (2/2)

- *Example*

Q: Use Ackermann's formula to solve for the gains for the previous undamped oscillator.

Sol: Using the following desired characteristic polynomial coefficients

$$\alpha_1 = 4\omega_0 \,, \quad \alpha_2 = 4\omega_0^2$$

the matrix polynomial equation becomes

$$\alpha_c(F) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} + 4\omega_0 \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} + 4\omega_0^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The controllability matrix and its inverse become

____________________ and ________________

Finally, the control gain matrix is

$$K = [K_1 \quad K_2] = [0 \quad 1] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = [\qquad\qquad]$$

Remarks on Controllability (1/4)

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1. The fact that we can shift the poles of a system by state feedback to any desired location is a rather remarkable result.
2. This shift is possible if we can transform (F, G) to the control canonical form (F_c, G_c), which in turn is possible if the system is ____________ .
3. If the system is uncontrollable, no possible control will yield arbitrary pole locations.
4. Uncontrollable systems have certain modes, or subsystems, that are _____________________ . This usually means that parts of the system are physically disconnected from the input.
5. For example, in modal canonical form for a system with distinct poles, one of the modal state variables is not connected to the input if there is a zero entry in the ____________ .
6. A good physical understanding of the system being controlled would prevent any attempt to design a controller for an uncontrollable system. Although there are algebraic tests for controllability, no mathematical test can replace the control engineer's understanding of the physical system.

Remarks on Controllability (2/4)

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7. Often the physical situation is such that every mode is controllable to some degree, and while the mathematical tests indicate the system is controllable, certain modes are so weakly controllable that designs to control them are virtually useless.
8. For example, pitch motion x_p of an airplane is primarily affected by the elevator δ_e and weakly affected by rolling motion x_r. Rolling motion is essentially only affected by the ailerons δ_a. The state-space description of these relationships is

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} F_p & \varepsilon \\ 0 & F_r \end{bmatrix} \begin{bmatrix} x_p \\ x_r \end{bmatrix} + \begin{bmatrix} G_p & 0 \\ 0 & G_r \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_a \end{bmatrix}$$

where the matrix of small numbers ε represents the weak coupling from rolling motion to pitching motion. A mathematical test of controllability for this system would conclude that pitch plane motion (and therefore altitude) is controllable by the ailerons as well as by the elevator! However, it is impractical to attempt to control an airplane's altitude by rolling the aircraft with the ailerons.

Remarks on Controllability (3/4)

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- *Example*

Q: Find the state feedback gains necessary for placing the poles of the system shown below at the roots of $s^2 + 2\zeta\omega_n s + \omega_n^2$.

Sol: The state description matrices are

$$A_o = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix}, \quad B_o = \begin{bmatrix} 1 \\ -z_o \end{bmatrix}$$

$$C_o = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D_o = 0$$

The closed-loop characteristic equation is:

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By equating this equation to the desired characteristic equation, we get

$$K_1 - z_o K_2 = 2\zeta\omega_n - 7$$

$$-z_o K_1 - (7z_o + 12)K_2 = \omega_n^2 - 12$$

Remarks on Controllability (4/4)

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The solutions to these equations are

$$K_1 = \frac{z_o(14\zeta\omega_n - 37 - \omega_n^2) + 12(2\zeta\omega_n - 7)}{(z_o + 3)(z_o + 4)}$$

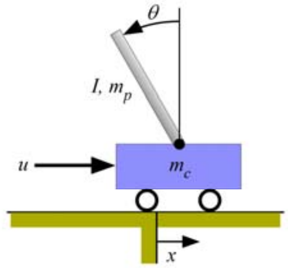
$$K_2 = \frac{z_o(7 - 2\zeta\omega_n) + 12 - \omega_n^2}{(z_o + 3)(z_o + 4)}$$

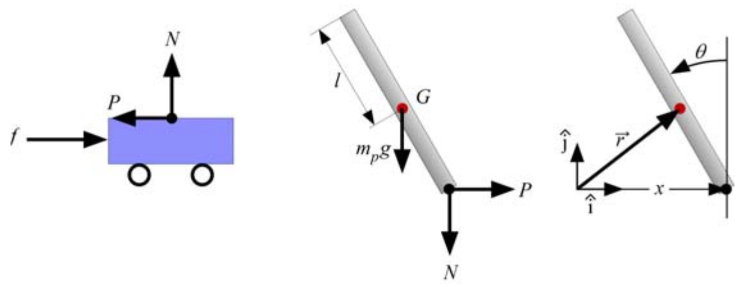
- *Notes*

1. The gains grow as the zero z_o approaches either –3 or –4, the values this system loses controllability. In other words, as controllability is almost lost, the control gains become very large.
2. Therefore, the system has to work harder and harder to achieve control as controllability slips away.
3. Both K_1 and K_2 grow as the desired closed-loop bandwidth given by ω_n is increased.
4. Therefore, to move the poles a long way requires large gains.

Example: Inverted Pendulum (1/6)

- Design a controller that places the closed-loop poles of the inverted pendulum system such that the step response has a damping ratio of 0.707 and a settling time of 1 sec.





Example: Inverted Pendulum (2/6)

- The model of the system is

$$(m_c + m_p)\ddot{x} + m_p\, l\dot{\theta}^2 \sin\theta - m_p\, l\ddot{\theta}\cos\theta = f$$

$$(I + m_p l^2)\ddot{\theta} - m_p g\, l \sin\theta = m_p l\, \ddot{x}\cos\theta$$

- By linearizing the nonlinear differential equations, we get

- Combining the two equations by eliminating x leads to

$$[(m_c + m_p)I + m_c m_p l^2]\ddot{\theta} - (m_c + m_p)m_p g\, l\theta = m_p l\, f$$

$$\Rightarrow \left[\frac{(m_c + m_p)I + m_c m_p l^2}{m_p l}\right]\ddot{\theta} - (m_c + m_p)g\,\theta = f$$

Example: Inverted Pendulum (3/6)

- By introducing new variables, *a* and *b* such that

$$a \equiv \frac{(m_c + m_p)I + m_c m_p l^2}{m_p l}, \quad b \equiv (m_c + m_p)g$$

the differential equation can be put in the state-space form.

$$a\ddot{\theta} - b\theta = f$$

- Use the following system parameters.

m_c = 1kg, m_p = 0.1kg, l = 0.2m, $I = 1.33\times10^{-3}$ kg·m², and g = 9.81m/s²

$$a = 0.27, \quad b = 10.8$$

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 39.5 & 0\end{bmatrix}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} + \begin{bmatrix}0 \\ 3.66\end{bmatrix} f$$

Example: Inverted Pendulum (4/6)

- Controller for the full-state feedback

$$f = -Kx = -[K_1 \quad K_2]\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$$

- The characteristic equation of the closed-loop system

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 39.5 & 0\end{bmatrix}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} - \begin{bmatrix}0 \\ 3.66\end{bmatrix}[K_1 \quad K_2]\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$$

$$= \begin{bmatrix}0 & 1 \\ 39.5 - 3.66K_1 & -3.66K_2\end{bmatrix}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$$

$$\det[sI - (A - BK)] = \det\left(\begin{bmatrix}s & 0 \\ 0 & s\end{bmatrix} - \begin{bmatrix} & \\ & \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} & \\ & \end{bmatrix}\right) =$$

Example: Inverted Pendulum (5/6)

- The design point should be calculated.

$$\zeta = 0.707 \quad \Rightarrow \quad \theta = \sin^{-1}\zeta = \sin^{-1}(0.707) = 45°$$

$$t_s = 1 \quad \Rightarrow \quad \sigma = \frac{4}{1} = 4$$

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- Characteristic equation for the design point

$$(s+4+4j)(s+4-4j) = s^2 + 8s + 32$$

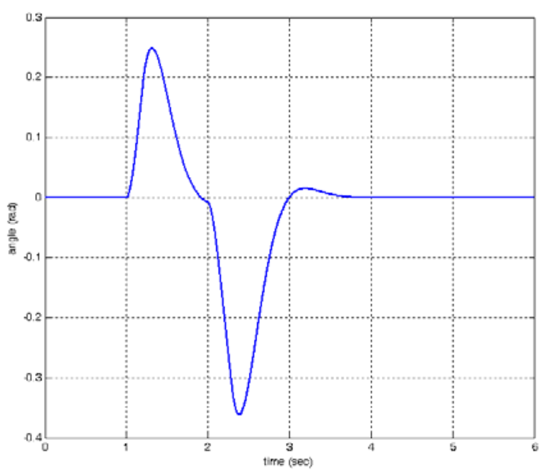
- By comparing term by term:

$$\begin{aligned} 3.66K_1 - 39.5 &= 32 \\ 3.66K_2 &= 8 \end{aligned} \quad \Rightarrow \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

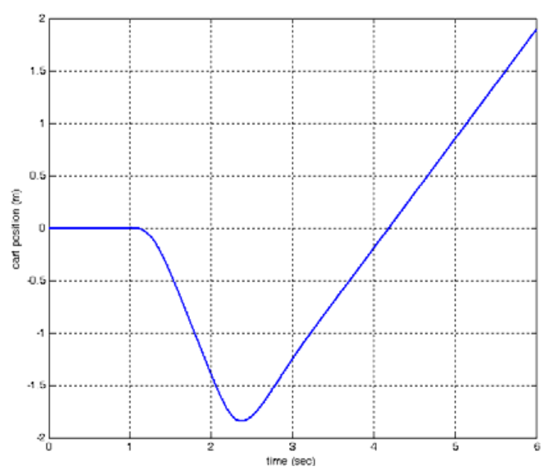
- Finally, the control law becomes

$$f = -[K_1 \quad K_2]\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

Example: Inverted Pendulum (6/6)



Angle of pendulum



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Full-Order Model (1/3)

- The linearized model of the inverted pendulum is

$$(m_c + m_p)\ddot{x} - m_p l\ddot{\theta} = f$$

$$(I + m_p l^2)\ddot{\theta} - m_p g l\theta = m_p l \ddot{x}$$

- By removing $\ddot{x}$ from the two equations, we can get

$$\left[\frac{(m_c + m_p)I + m_c m_p l^2}{m_p l}\right]\ddot{\theta} = [(m_c + m_p)g]\theta + f$$

- By removing $\ddot{\theta}$ from the two equations, we also get

$$\left[\frac{(m_c + m_p)I + m_c m_p l^2}{(I + m_p l^2)}\right]\ddot{x} = \left[\frac{m_p^2 l^2 g}{(I + m_p l^2)}\right]\theta + f$$

- Let's introduce four new parameters

$$a = \frac{(m_c + m_p)I + m_c m_p l^2}{m_p l}, \quad b = (m_c + m_p)g$$

Full-Order Model (2/3)

$$c = \frac{(m_c + m_p)I + m_c m_p l^2}{(I + m_p l^2)}, \quad d = \frac{m_p^2 l^2 g}{(I + m_p l^2)}$$

- Using the four parameters, the two equations can be rewritten as

$$\begin{aligned} a\ddot{\theta} &= b\theta + f \\ c\ddot{x} &= d\theta + f \end{aligned} \quad \Rightarrow \quad \ddot{\theta} = \frac{b}{a}\theta + \frac{1}{a}f \quad \text{and} \quad \ddot{x} = \frac{d}{c}\theta + \frac{1}{c}f$$

- The two linear equations can be put in the state space form

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} f$$

Full-Order Model (3/3)

- Use the following system parameters.
  m_c = 1kg, m_p = 0.1kg, l = 0.2m, $I = 1.33\times10^{-3}$ kg·m², and g = 9.81m/s²

  $$a = 0.27, \quad b = 10.8, \quad c = 1.03, \quad d = 0.74$$

  $$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0.72 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 39.5 & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.98 \\ 0 \\ 3.66 \end{bmatrix} f$$

- Controller for the full-state feedback

  $$f = -Kx =$$

Optimal Control - LQR

- The most effective and widely used technique of linear control systems design is the optimal **Linear Quadratic Regulator (LQR)**.
- In the optimal control, the control purpose is to find the control such that a performance index or cost function such as

  ____

  , where Q and R are penalty matrices, is minimized.
- For a continuous time linear system described by
  $$\dot{x} = Ax + Bu$$
  with a cost function defined as
  $$J = \int_0^\infty [x^T Q x + u^T R u]\, dt$$
  the full-state control law that minimizes the value of the cost is
  $$u = -Kx$$
  where K is given by
  $$K = \_\_\_\_\_\_\_\_\_\_$$
  and P is found by solving the Algebraic Riccati Equation (ARE)
  $$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

Optimal Control

- For the control of inverted pendulum, Matlab script to find the gain *K* is

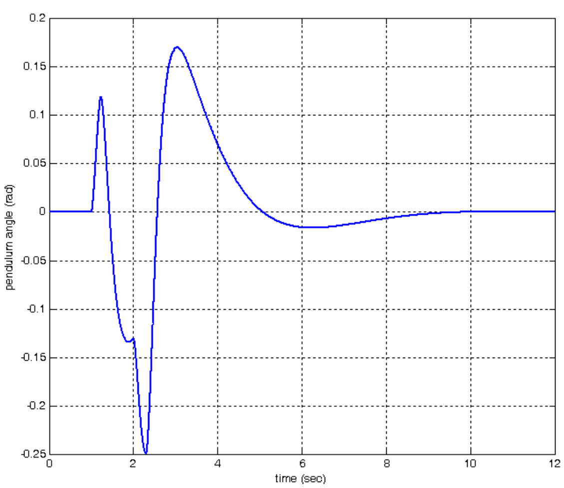
```
A=[0 1 0 0; 0 0 0.72 0; 0 0 0 1; 0 0 39.5 0];
B = [ 0 0.98 0 3.66]';
Q = [1 0 0 0; 0 0 0 0; 0 0 1 0; 0 0 0 0];
R = 1;
[K,S,E]=lqr(A,B,Q,R);

K = -1.0000   -1.8016   26.7877    4.2892
```

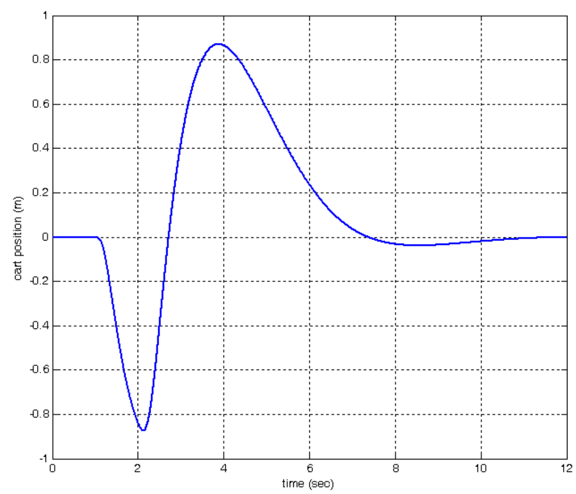
- Controller for the full-state feedback

$$f = -Kx = [1 \quad 1.80 \quad -26.79 \quad -4.29] \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

Inverted Pendulum with LQR



Angle of pendulum



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Full State Feedback with Reference Input (1/5)

1. *Full state feedback with reference input*
   1) Thus far, the control has been given by $u=-Kx$.
   2) An obvious way to include a reference input, $u=-Kx+r$ will not work.
   3) Thus, it is required to compute the steady-state values of the state and the control input that will result in zero output error and then force them to take these values.
2. *Design procedure*
   1. If the desired final values of the state and the control input are x_{ss} and u_{ss}, then the new control formula should be

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      so that when $x = x_{ss}$ (no error), $u = u_{ss}$. To pick the correct final values, we must solve the equations so that the system will have zero steady-state error to any constant input.
   2. The system differential equations are the standard ones:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Full State Feedback with Reference Input (2/5)

3. In the steady state, they reduce to the pair

   ____

4. We want to solve for the values for which $y_{ss} = r_{ss}$ for any value of r_{ss}. To do this we make $x_{ss} = N_x r_{ss}$ and $u_{ss} = N_u r_{ss}$. With these substitutions we can write the above equations as a matrix equation; the common factor r_{ss} cancels out to give the equation for the gains:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} \mathrm{N}_x \\ N_u \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

5. This equation can be solved for $\mathbf{N}_x$ and N_u to get

$$\begin{bmatrix} \mathrm{N}_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

6. With these values we finally have the basis for introducing the reference input so as to get zero steady-state error to a step input:

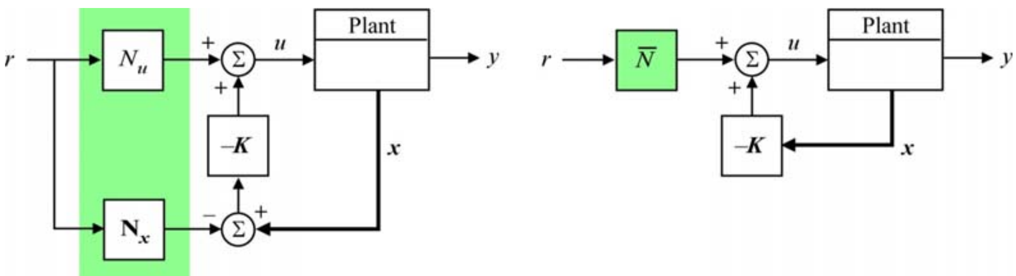
$$u = N_u r - K(x - \mathrm{N}_x r) = -Kx + (N_u + K\mathrm{N}_x)r$$

Full State Feedback with Reference Input (3/5)

7. The coefficient of r in parenthesis is a constant that can be computed beforehand. We give it the symbol $\bar{N}$, so

$$u = -Kx + \bar{N}r$$

8. Block diagrams for introducing the reference input with full-state feedback



(a) with state and control gains (b) with a single composite gain

Full State Feedback with Reference Input (4/5)

- *Example*

Q: Compute the necessary gains for zero steady-state error to a step command at x_1, and plot the resulting unit step response for the oscillator with ω_0=1.

Sol: The state equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u\,, \quad y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So, the equation for the gains becomes:

The solution is $\mathrm{N}_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $N_u = 1$, and for the given control law,

K = [3 4], $\bar{N} = 4$.

Full State Feedback with Reference Input (5/5)

When the initial conditions are x1 = 0.0 and x2 = 0.0, and a unit step input is applied for ω_0= 1,