Study 3: VAR(1) with Normal Margins — Gaussian vs. Clayton Copula

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0. Summary

0.1 Computational Stability

Both models exhibit excellent computational stability across all simulation conditions. The Normal–Gaussian (NG) model and the Normal–Clayton (NC) model both show no post-warmup divergent transitions and max $\hat{R} \leq 1.01$ in all replications. This stands in contrast to Studies 1–2 where some model-DGP combinations exhibited divergences.

0.2 Model Performance Under Clayton Copula DGP

Normal–Clayton (NC): The correctly specified model is approximately unbiased for all parameters including the copula parameter θ. Coverage is close to nominal (0.95) across conditions, with slight under-coverage for θ at higher dependence levels ($\theta \geq 4$) in small samples.

Normal–Gaussian (NG): The misspecified model recovers the VAR dynamics (Φ) and intercepts (μ) without substantial bias, demonstrating robustness of marginal parameter inference to copula misspecification. The copula parameter ρ has no ground truth under the Clayton DGP, but posterior means increase monotonically with θ, indicating the Gaussian copula "absorbs" the Clayton dependence structure.

0.3 Key Insights

1. **Copula misspecification does not propagate to marginal parameters**: Under normal margins, the NG model recovers μ, Φ, and σ with similar accuracy to the correctly specified NC model.
2. **The Clayton copula captures asymmetric tail dependence**: Unlike the symmetric Gaussian copula, the Clayton copula concentrates dependence in the lower tail. This feature cannot be captured by the NG model's ρ parameter.

3. **Kendall's τ provides a common dependence scale**: The relationship $\tau = \theta/(\theta+2)$ maps Clayton θ to the $[0,1]$ scale, facilitating comparison with Gaussian ρ (where $\tau = (2/\pi)\arcsin(\rho)$).

1. Introduction

This simulation study compares two Bayesian VAR(1) models under a Clayton copula DGP with normal margins: a Normal–Gaussian (NG) model (misspecified copula) and a Normal–Clayton (NC) model (correctly specified). Unlike Studies 1–2, which examined marginal distribution misspecification, Study 3 focuses on **copula misspecification** while keeping margins correctly specified in both models.

1.1. Data Generating Process (DGP)

The DGP follows the same VAR(1) structure as Studies 1–2:

$$Y_t = \mu + \Phi Y_{t-1} + \varepsilon_t, \qquad t = 2, \dots, T,$$

with $\mu = \mathbf{0}$. Innovations $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})$ have standard normal margins (mean 0, variance 1) coupled through a **Clayton copula** with parameter $\theta > 0$.

i Clayton Copula

The Clayton copula is an Archimedean copula with CDF:

$$C(u,v;\theta) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}, \qquad \theta > 0$$

It exhibits **lower-tail dependence**: extreme low values co-occur more frequently than under a Gaussian copula. The lower-tail dependence coefficient is $\lambda_L = 2^{-1/\theta}$.

! Kendall's Tau Relationship

For the Clayton copula, Kendall's τ relates to θ via:

$$\tau = \frac{\theta}{\theta+2}$$

This provides a monotone mapping between $\theta \in (0,\infty)$ and $\tau \in (0,1)$:

θ	0.5	1.0	2.0	4.0	8.0
τ	0.20	0.33	0.50	0.67	0.80

1.2. Simulation Design

Table 2: Summary of the Simulation Design Factors.

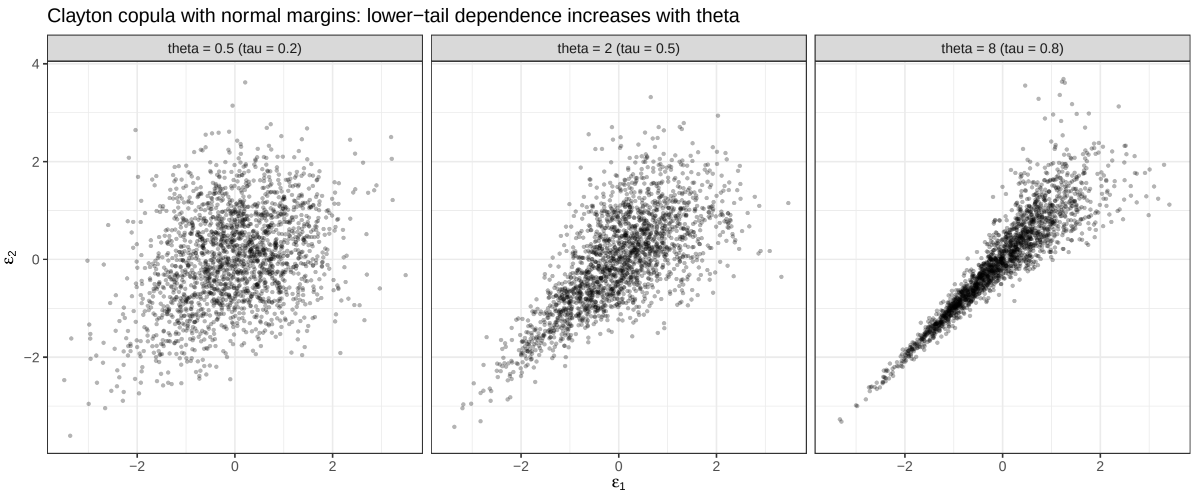
> **Design Simplification**
>
> Study 3 uses only VAR Set A (symmetric) and excludes skewness direction factors since all margins are symmetric normal. This yields $3 \times 5 = 15$ conditions with 200 replications each, for a total of 3,000 simulated datasets.

1.3. True Parameter Values

Table 3: True Parameter Values Used in the Data Generating Process.

> **No Ground Truth for ρ in NG Model**
>
> Under the Clayton copula DGP, the Gaussian copula parameter ρ has no well-defined ground truth. The NG model is misspecified for the copula structure, so bias and coverage metrics are **not computed** for ρ. Instead, we report posterior means descriptively to show how the Gaussian copula absorbs Clayton dependence.

1.4. Visual Check: Clayton Copula Dependence Structure



2. Data Loading and Preparation

2.1. MCMC Classification and Overview

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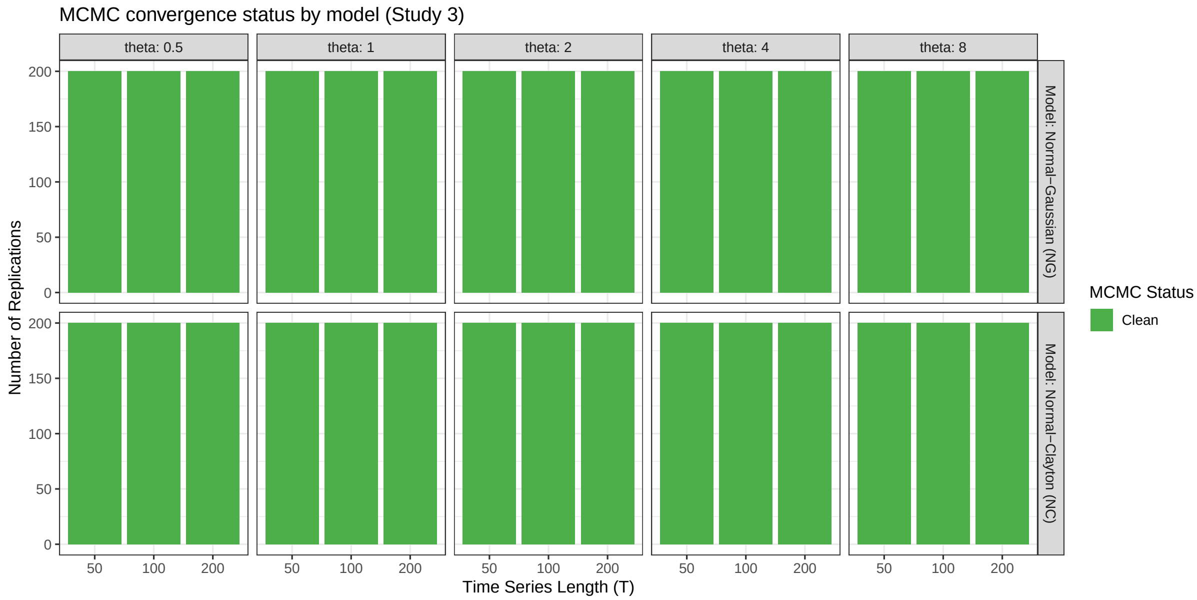
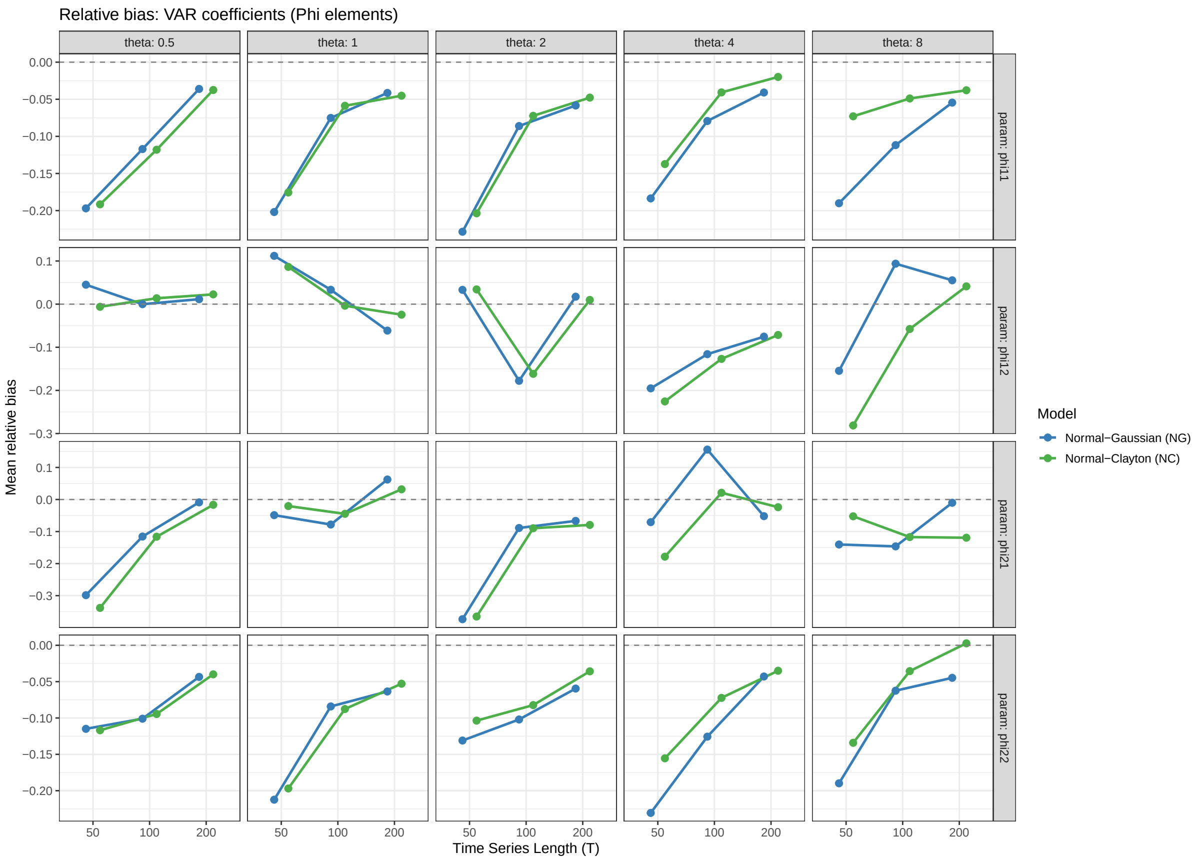


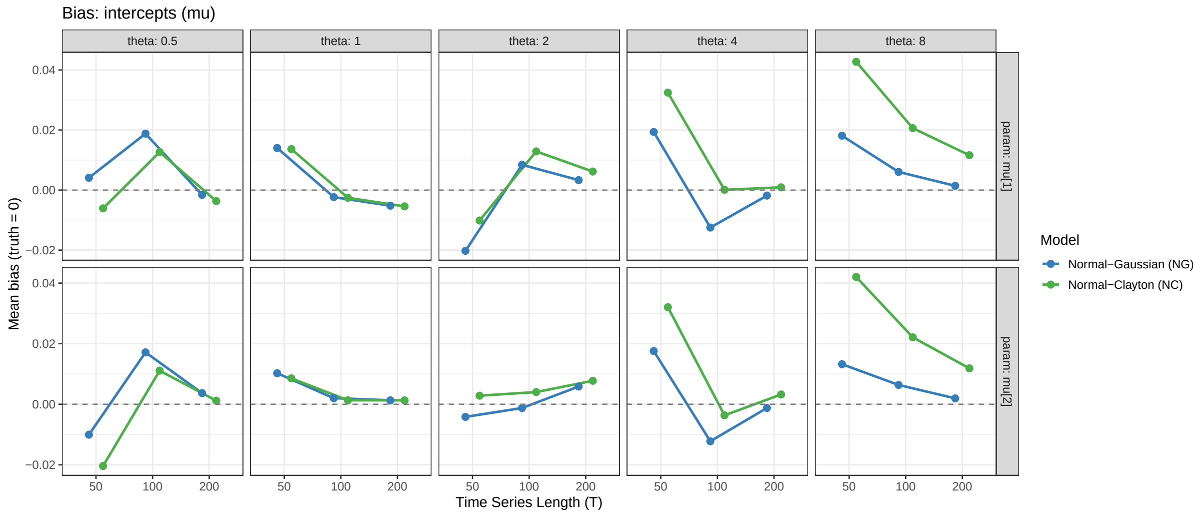
Table 4: Divergence summary: All runs have zero post-warmup divergent transitions.

3. Helper Functions

4. VAR Parameters: NG vs NC Performance

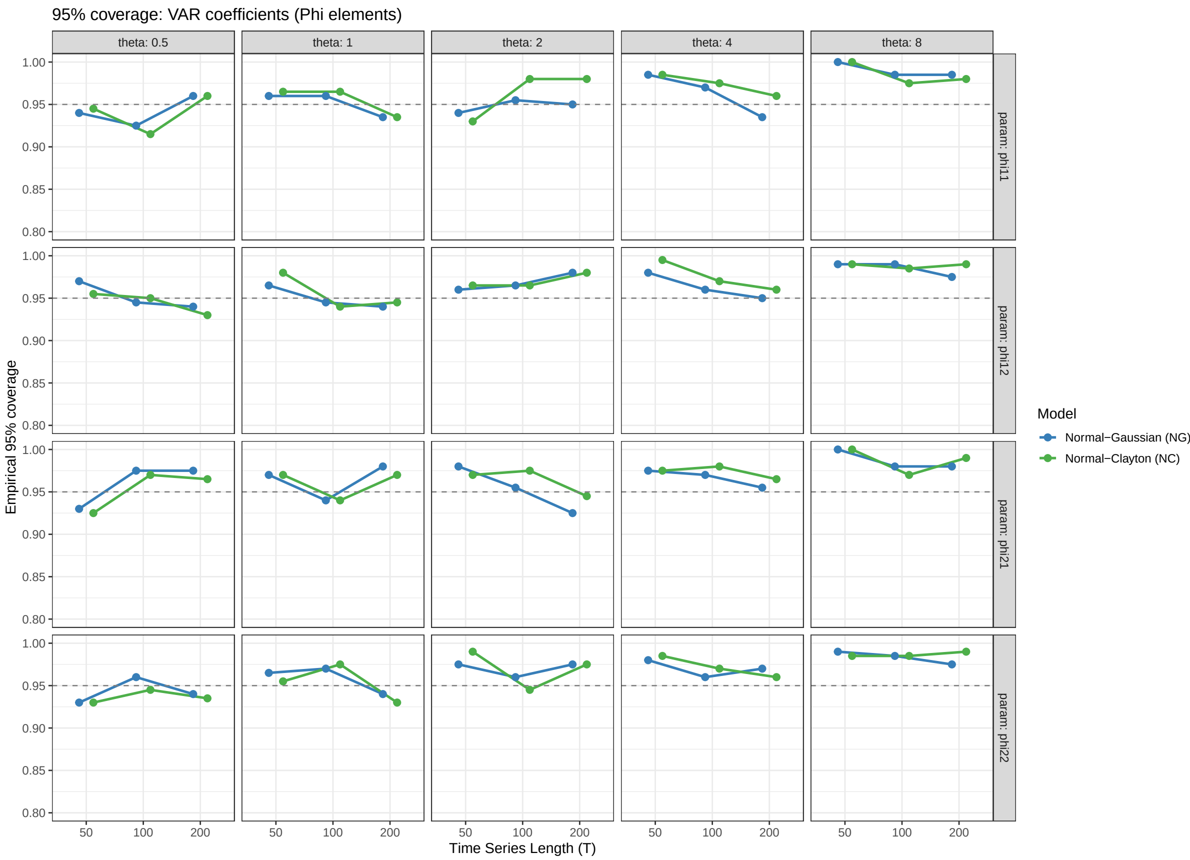
4.1. Relative Bias

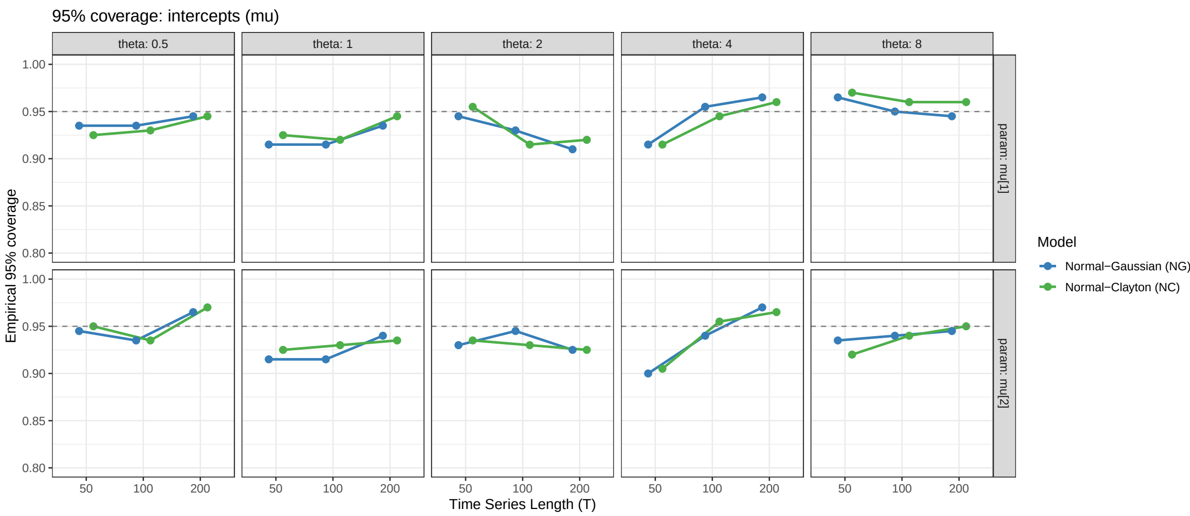




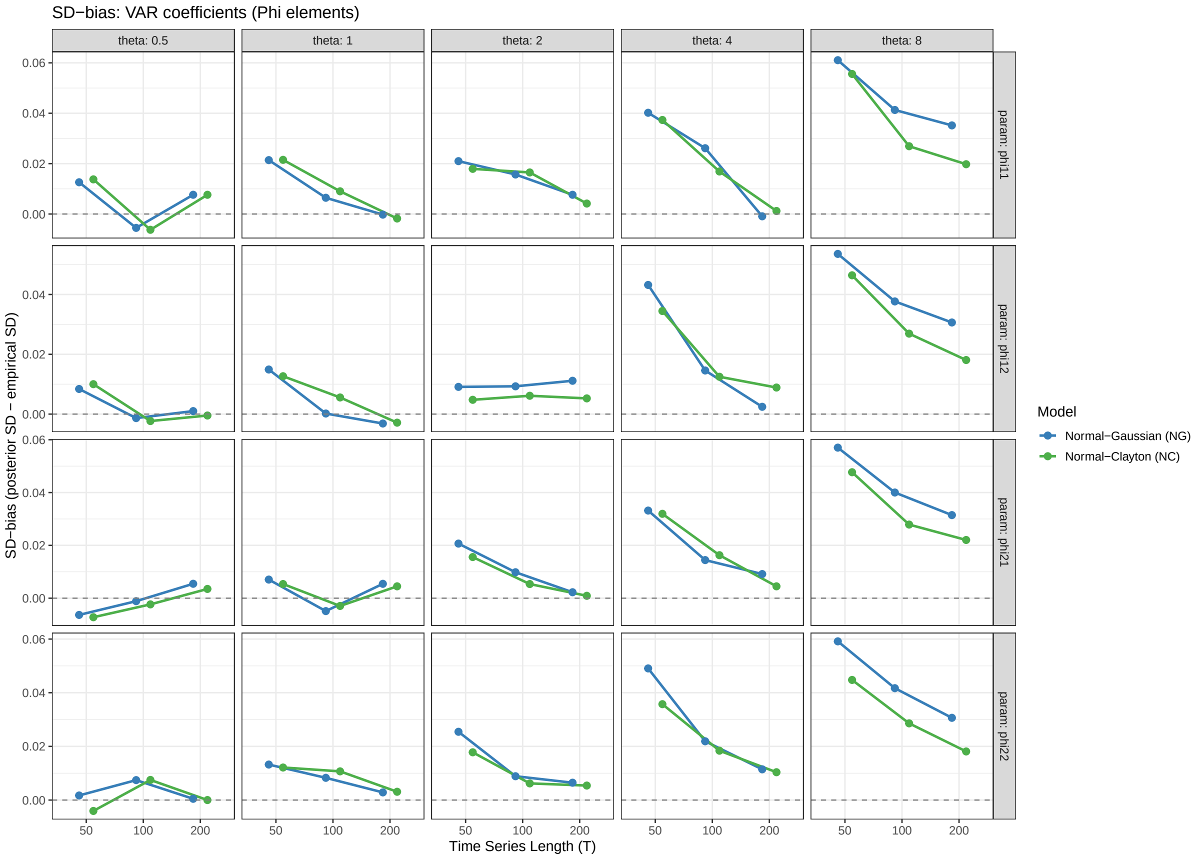
Bias: intercepts (mu)
theta: 0.5
theta: 1
theta: 2
theta: 4
theta: 8
param: mu[1]
param: mu[2]
Mean bias (truth = 0)
0.04
0.02
0.00
−0.02
50
100
200
Time Series Length (T)
Model
Normal−Gaussian (NG)
Normal−Clayton (NC)

4.2. 95% Coverage

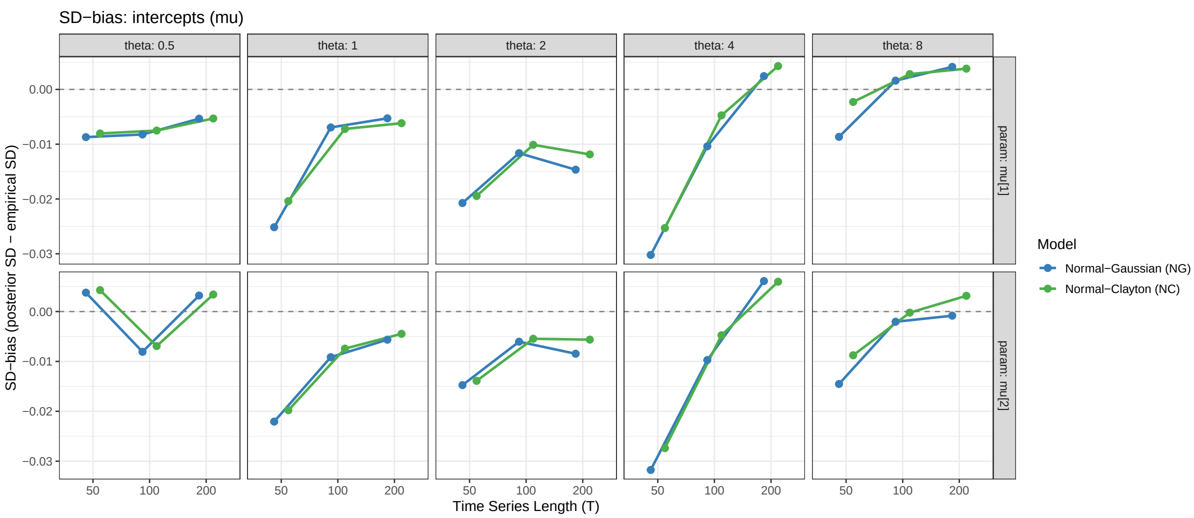




4.3. SD-Bias



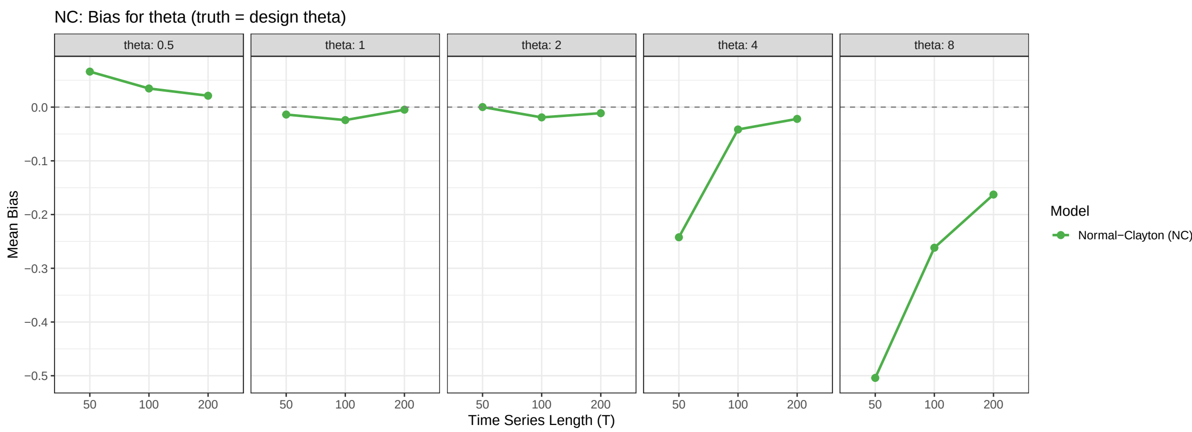
SD−bias: VAR coefficients (Phi elements)
theta: 0.5
theta: 1
theta: 2
theta: 4
theta: 8
param: phi11
param: phi12
param: phi21
param: phi22
SD−bias (posterior SD − empirical SD)
Time Series Length (T)
Model
Normal−Gaussian (NG)
Normal−Clayton (NC)

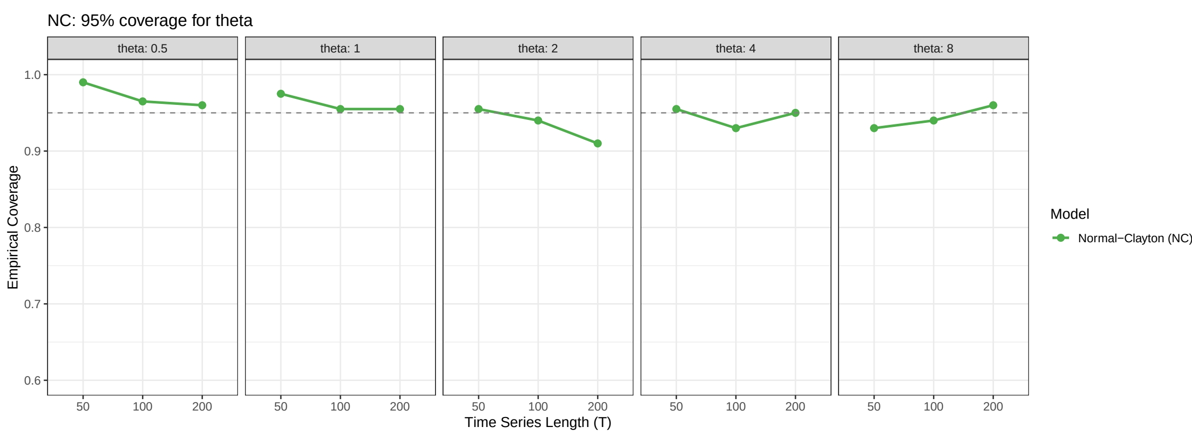


SD−bias: intercepts (mu)
theta: 0.5
theta: 1
theta: 2
theta: 4
theta: 8
param: mu[1]
param: mu[2]
SD−bias (posterior SD − empirical SD)
Time Series Length (T)
Model
Normal−Gaussian (NG)
Normal−Clayton (NC)

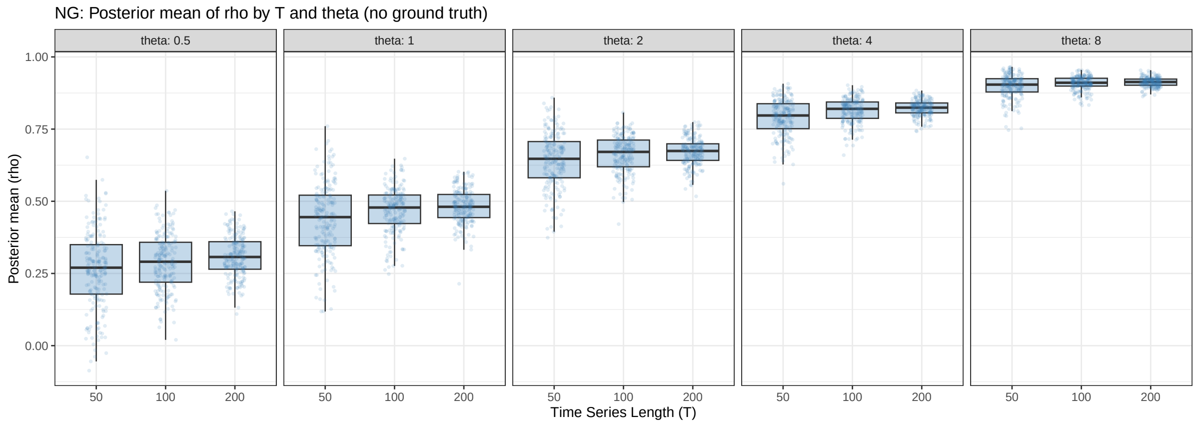
5. Copula Parameters

5.1. NC (Clayton copula): θ Recovery

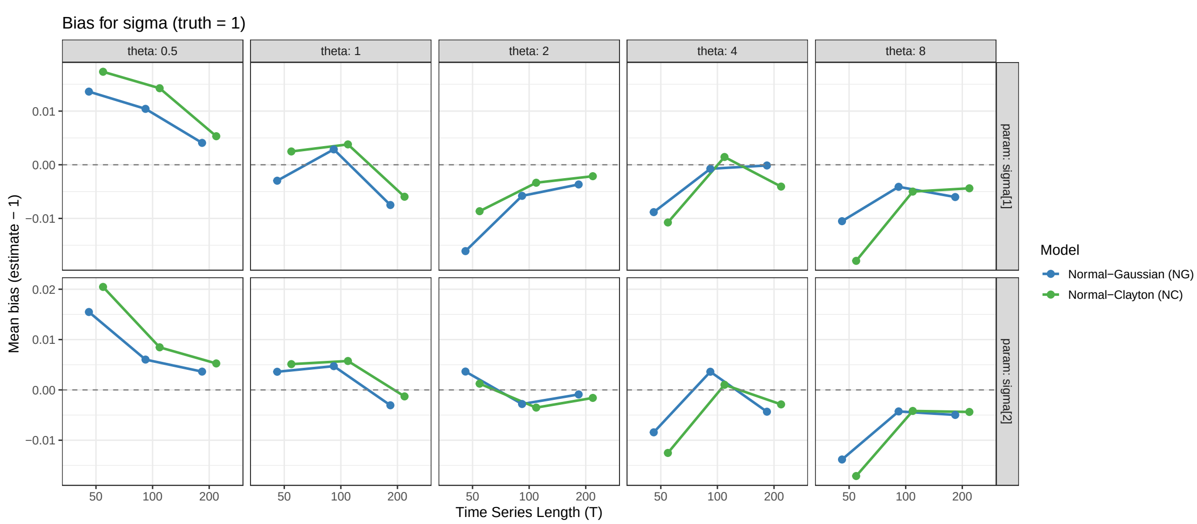


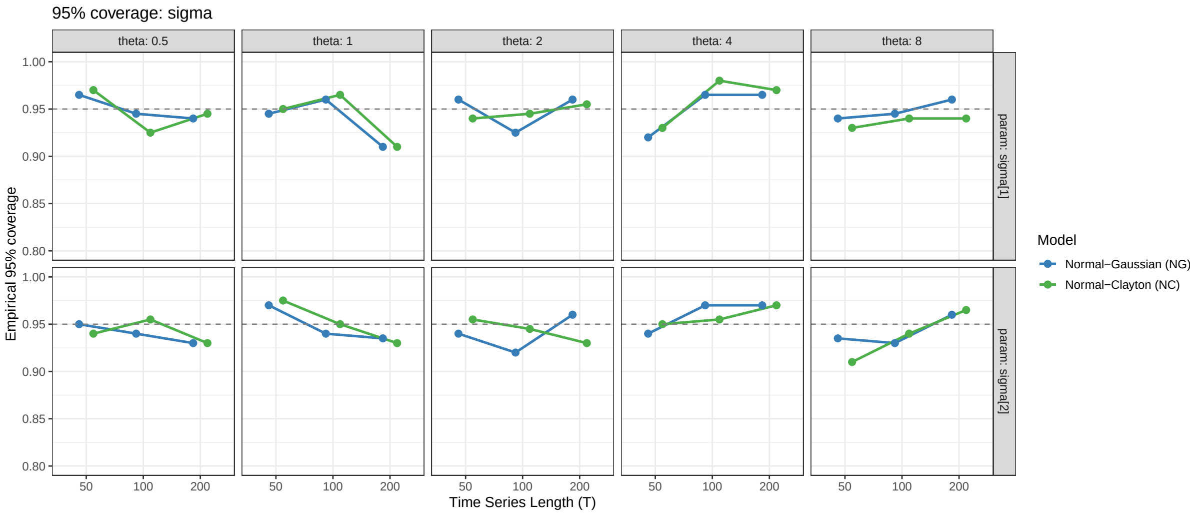


5.2. NG (Gaussian copula): ρ Posterior Means

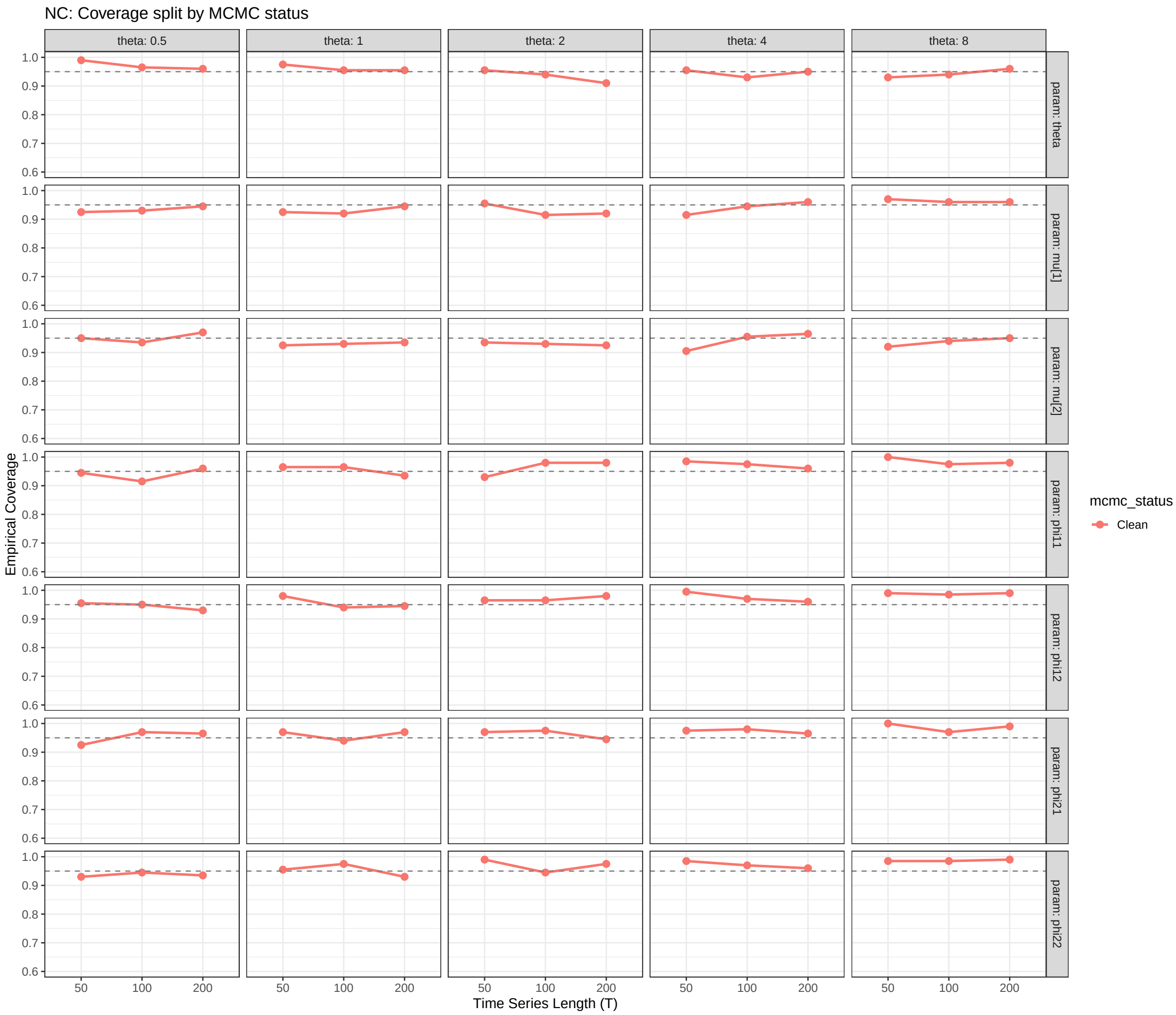


6. Marginal Parameters




95% coverage: sigma
theta: 0.5
theta: 1
theta: 2
theta: 4
theta: 8
param: sigma[1]
param: sigma[2]
Empirical 95% coverage
1.00
0.95
0.90
0.85
0.80
50
100
200
Time Series Length (T)
Model
Normal−Gaussian (NG)
Normal−Clayton (NC)

7. MCMC Diagnostics: Status Split



8. Export Tables

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9. Details

9.1. Prior Specifications

ℹ Log-Normal Prior for Clayton θ

The Log-Normal(0, 1) prior has median 1 and places approximately 95% of its mass in $(0.14, 7.4)$. This covers the design grid $\theta \in \{0.5, 1, 2, 4, 8\}$ well, with only the largest value $(\theta = 8)$ lying slightly in the upper tail.

9.2. MCMC Settings

9.3. Gaussian Copula Log-Density

9.4. Clayton Copula Log-Density

Boundary Clamping

Both copula implementations apply boundary clamping with $\varepsilon = 10^{-9}$:

$$u_{\text{clamped}} = \max(\varepsilon, \min(1 - \varepsilon, u))$$

This prevents numerical issues when $u \to 0$ or $u \to 1$.

9.5. Reproducibility Strategy

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9.6. Ground Truth Under Copula Misspecification