Study 2: Comparative Performance of Normal–Gaussian and Exponential–Gaussian Copula VAR Models Under Exponential Innovations

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0. Summary

0.1 Computational Stability versus Statistical Inference

Both models exhibit excellent computational stability across all simulation conditions. The Normal–Gaussian (NG) model shows no post-warmup divergent transitions and max $\hat{R} \leq 1.01$ in all replications. The Exponential–Gaussian (EG) model is equally well-behaved, with zero divergent transitions and good convergence diagnostics. This is an improvement over the SG model under extremeCHI in Study 1, where divergences were frequent.

0.2 Model Performance Under Exponential Innovations

Exponential–Gaussian (EG): EG is (i) approximately unbiased for the VAR dynamics Φ and intercepts μ, and (ii) well calibrated for dependence when ρ is interpreted on the correct scale in mixed-direction cells (see Section 1.1 for the sign convention under mirroring). Empirical coverage is close to 0.95 in most conditions

Normal–Gaussian (NG): NG is computationally stable but exhibits attenuation in $|\rho|$ and corresponding under-coverage for ρ under Exponential margins. This attenuation arises from PIT distortion (see Study 1, Section 6.1).

0.3 Insights

PIT distortion induced by marginal CDF misspecification is the dominant failure mechanism, consistent with findings from Study 1. When NG assumes Gaussian margins but the true innovations are Exponential, the probability integral transform yields non-uniform PITs, which attenuate the effective dependence seen by the Gaussian copula.

1. Introduction

This simulation study extends the analysis from Study 1 by comparing two Bayesian VAR(1) models under exponential innovations: a Normal–Gaussian (NG) model and an Exponential–Gaussian (EG) model. While Study 1 examined skew-normal and chi-squared innovations, this study focuses on exponential margins to assess whether correctly specifying the marginal distribution recovers the models parameters without bias.

1.1. Data Generating Process (DGP)

The DGP follows the same structure as Study 1:

$$Y_t = \mu + \Phi Y_{t-1} + \varepsilon_t, \qquad t = 2, \dots, T,$$

with $\mu = \mathbf{0}$. Innovations $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})$ have standardized Exponential margins (mean 0, variance 1), with optional mirroring to induce left-skewness, and are coupled through a Gaussian copula with correlation parameter ρ.

> **Standardization of Exponential Innovations**
>
> A standard Exponential(1) random variable X has $\mathbb{E}[X] = 1$ and $\mathrm{Var}(X) = 1$. To standardize:
>
> $$Z = X - 1$$
>
> yields $\mathbb{E}[Z] = 0$ and $\mathrm{Var}(Z) = 1$. Mirroring ($-Z$) produces left-skewed innovations. For the joint density via Gaussian copula, see Study 1, Section 1.1.

1.2. Simulation Design

The study employs a factorial design crossing four factors.

Table 1: Summary of the Simulation Design Factors.

Factor	Levels
DGP Level	Standardized Exponential
Time Series Length (T)	50, 100, 200
Copula Correlation (ρ)	0.30, 0.50 (reported as input; evaluation uses ρ_{eff} in mixed-direction cells)

Factor	Levels
VAR Parameters (Φ)	**Set A**: $\begin{pmatrix} 0.40 & 0.10 \\ 0.10 & 0.40 \end{pmatrix}$
	Set B: $\begin{pmatrix} 0.55 & 0.10 \\ 0.10 & 0.25 \end{pmatrix}$
Skewness Direction	`++` (both right), `--` (both left), `+-` (mixed)

Mixed-Direction Symmetry

Only one mixed-direction case ($+-$) is included. For the asymmetric VAR set B, $+-$ and $-+$ are not equivalent under variable relabeling. See Study 1, Section 1.2 for discussion of this design choice.

1.3. True Parameter Values

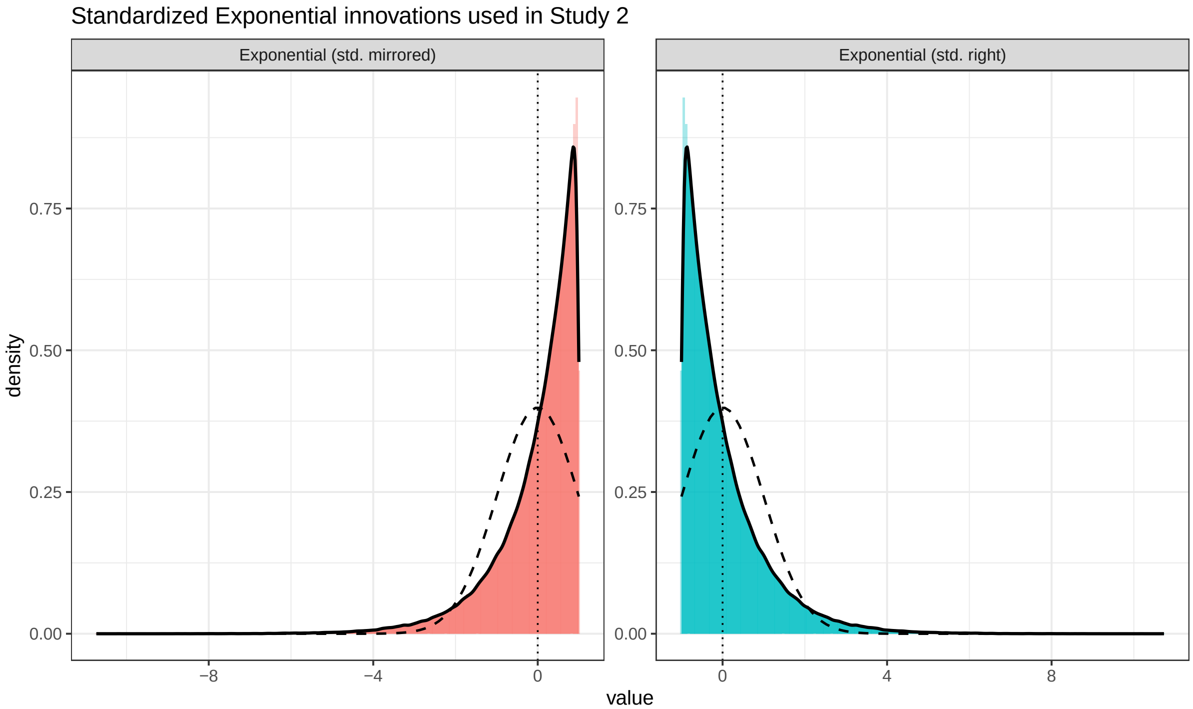
Table 2: True Parameter Values Used in the Data Generating Process.

Parameter	True Value	Notes
μ_1, μ_2	0, 0	Innovations are mean-zero
ϕ_{11} (Set A / Set B)	0.40 / 0.55	Diagonal AR coefficients
$\phi_{12} = \phi_{21}$	0.10	Cross-effects (symmetric)
ϕ_{22} (Set A / Set B)	0.40 / 0.25	Diagonal AR coefficients
ρ	0.30 or 0.50	Copula correlation
σ_1, σ_2 (NG model)	1.0, 1.0	Innovations are unit-variance
$\sigma_{\exp,1}, \sigma_{\exp,2}$ (EG model)	1.0, 1.0	Scale parameter for standardized Exponential

Bias Metric for Intercepts (μ)

Because $\mu = 0$ in the DGP, "relative bias" is undefined. We report absolute bias for μ (i.e., $\hat{\mu} - 0$), using the convention rel_bias = bias when |truth| is near 0.

1.4 Visual Check: Standardized Marginal Innovations (DGP)

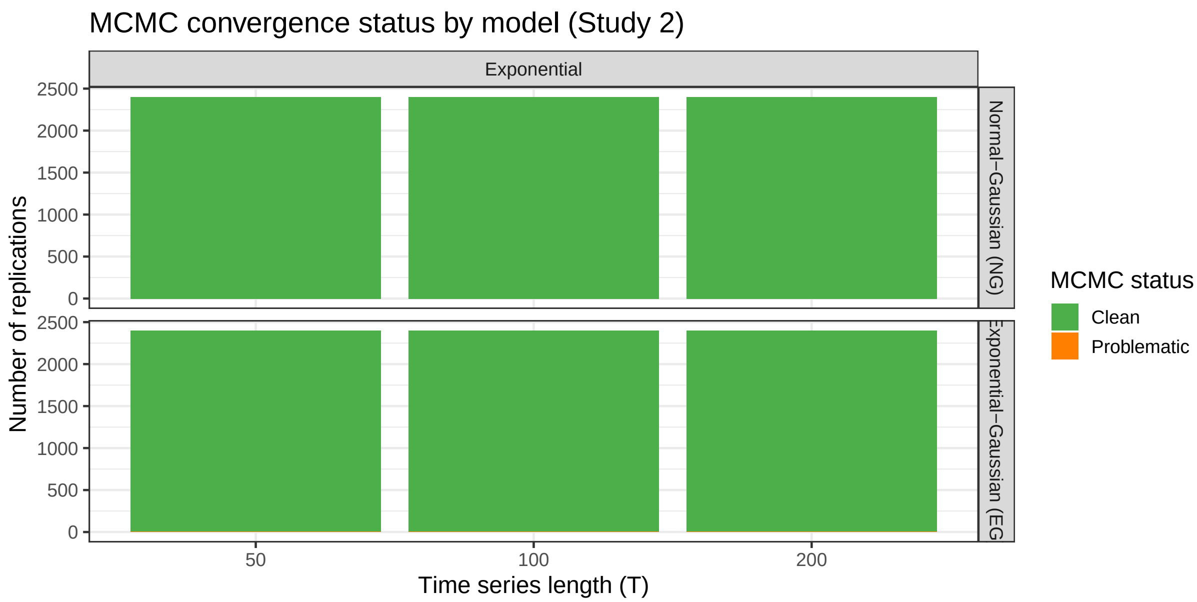


2. Data Loading and Preparation

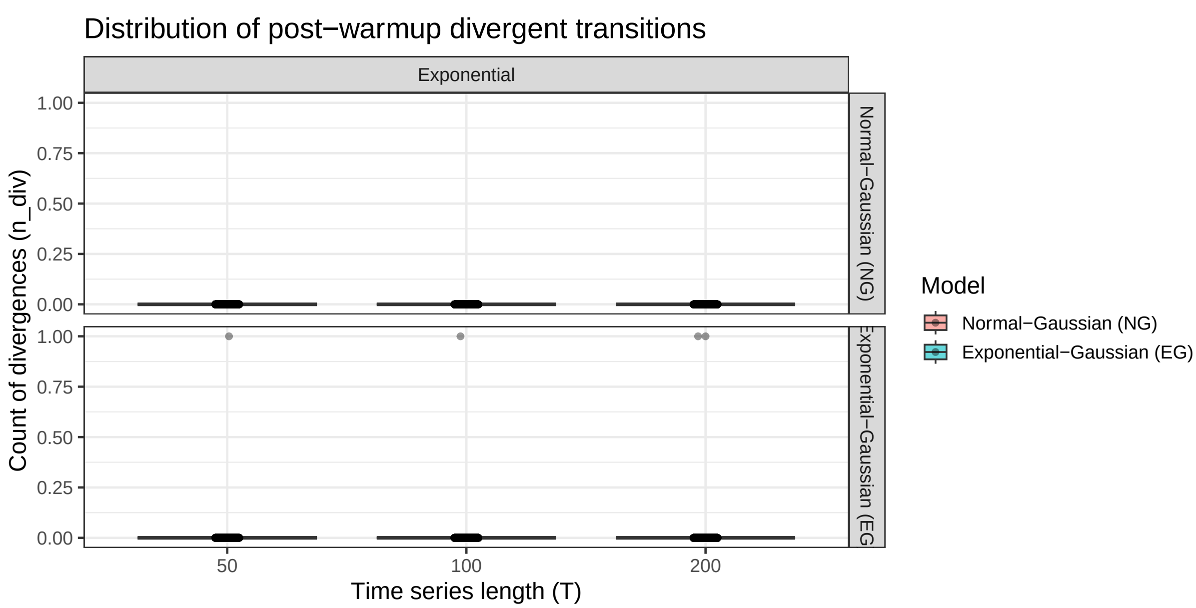
2.1. MCMC Classification and Overview

We classify runs based on MCMC diagnostics ($\hat{R}$ and divergent transitions) using the same criteria as Study 1:

- **Clean:** $\hat{R} \leq 1.01$ and no post-warmup divergences.
- **Problematic:** $\hat{R} > 1.01$ or at least one divergence.
- **Failed/Error:** Non-OK status or missing diagnostics.



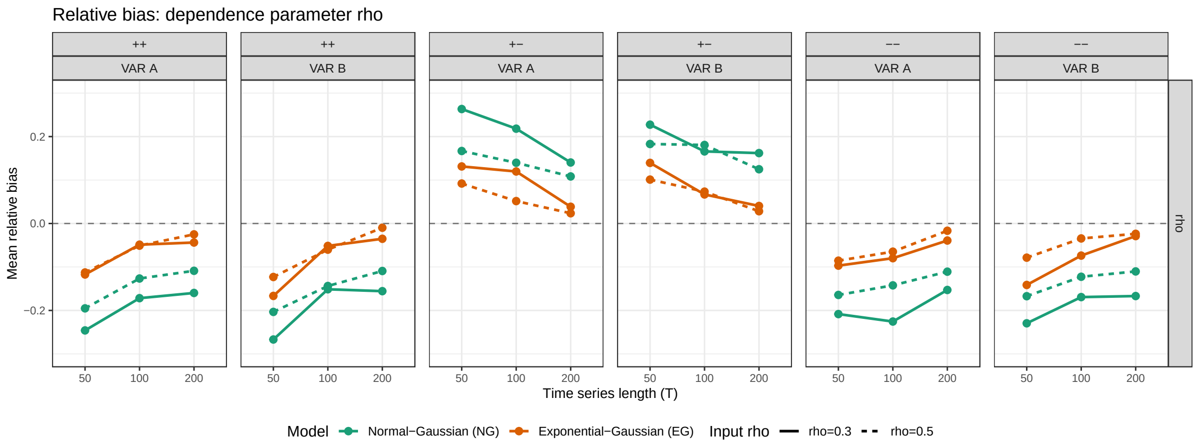
Both NG and EG models exhibit uniformly clean fits across the entire design, with almost no post-warmup divergent transitions. This favorable computational performance contrasts with the SG model under extremeCHI in Study 1, which experienced frequent divergences. The absence of sampling difficulties in the current analysis indicates that correct specification of the marginal distribution, as in EG, mitigates the posterior geometry challenges associated with model misspecification.

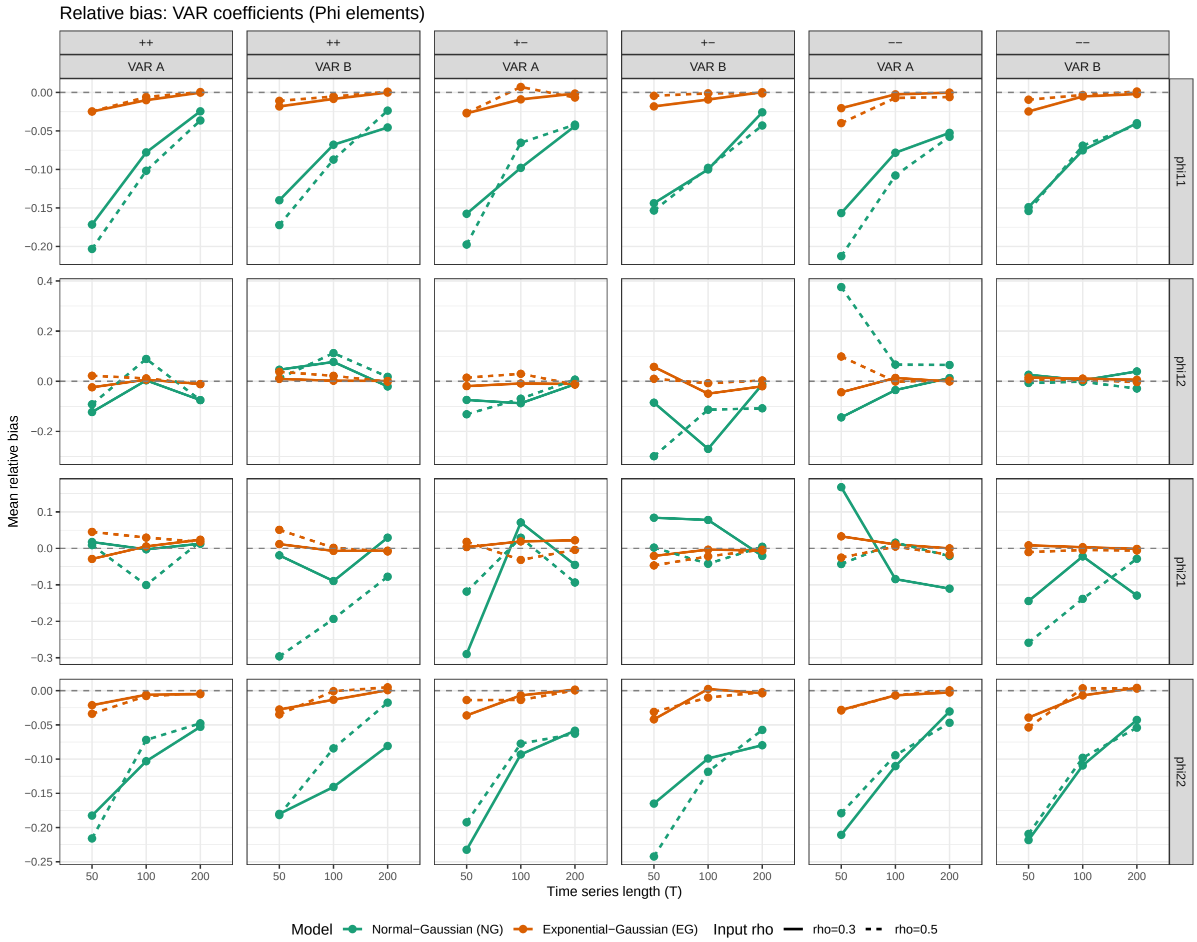


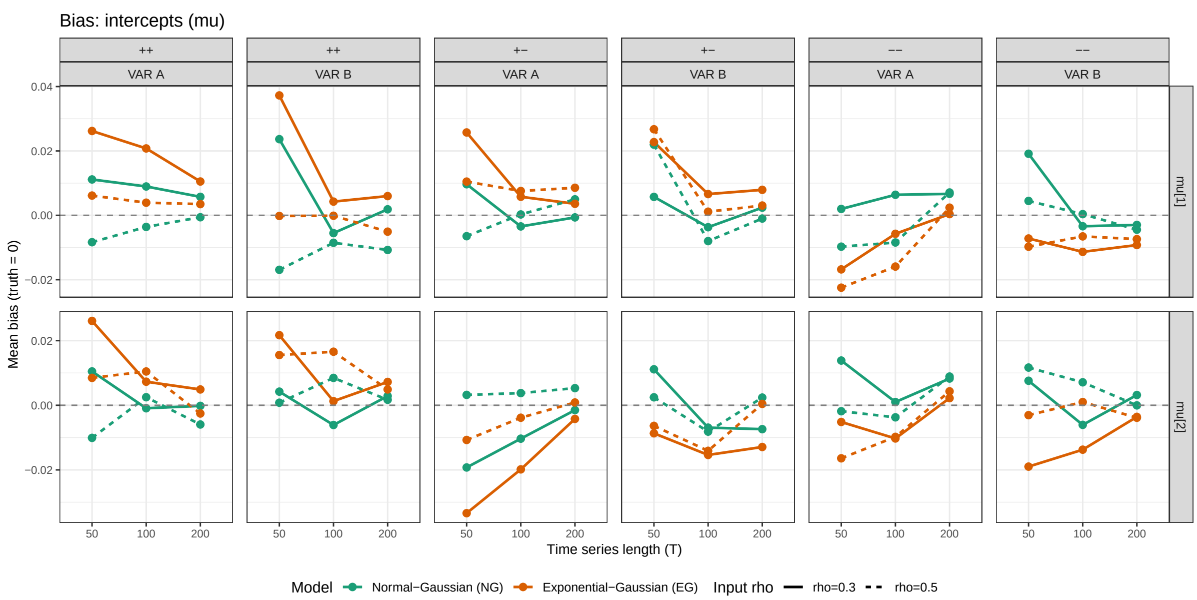
3. Helpers

4. Exponential DGP: NG vs EG Performance

4.1. Relative Bias

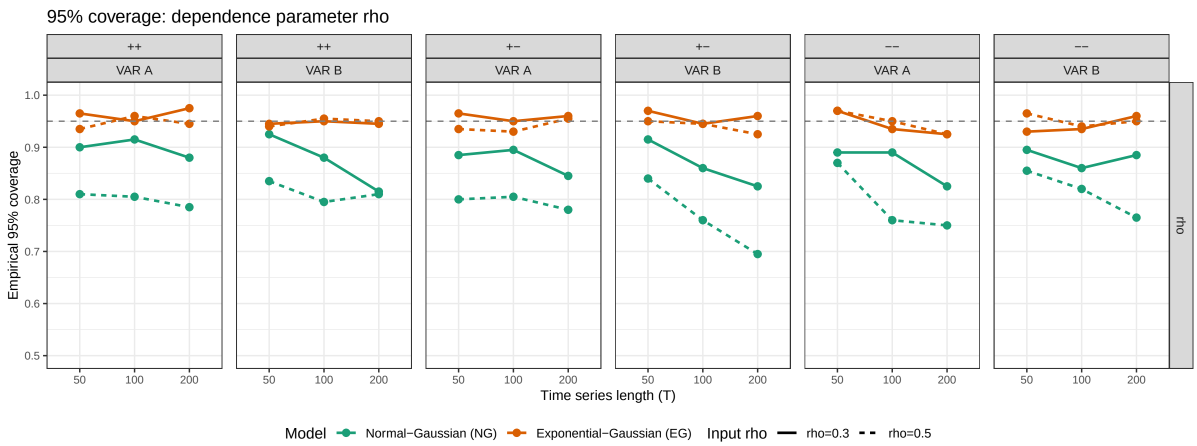


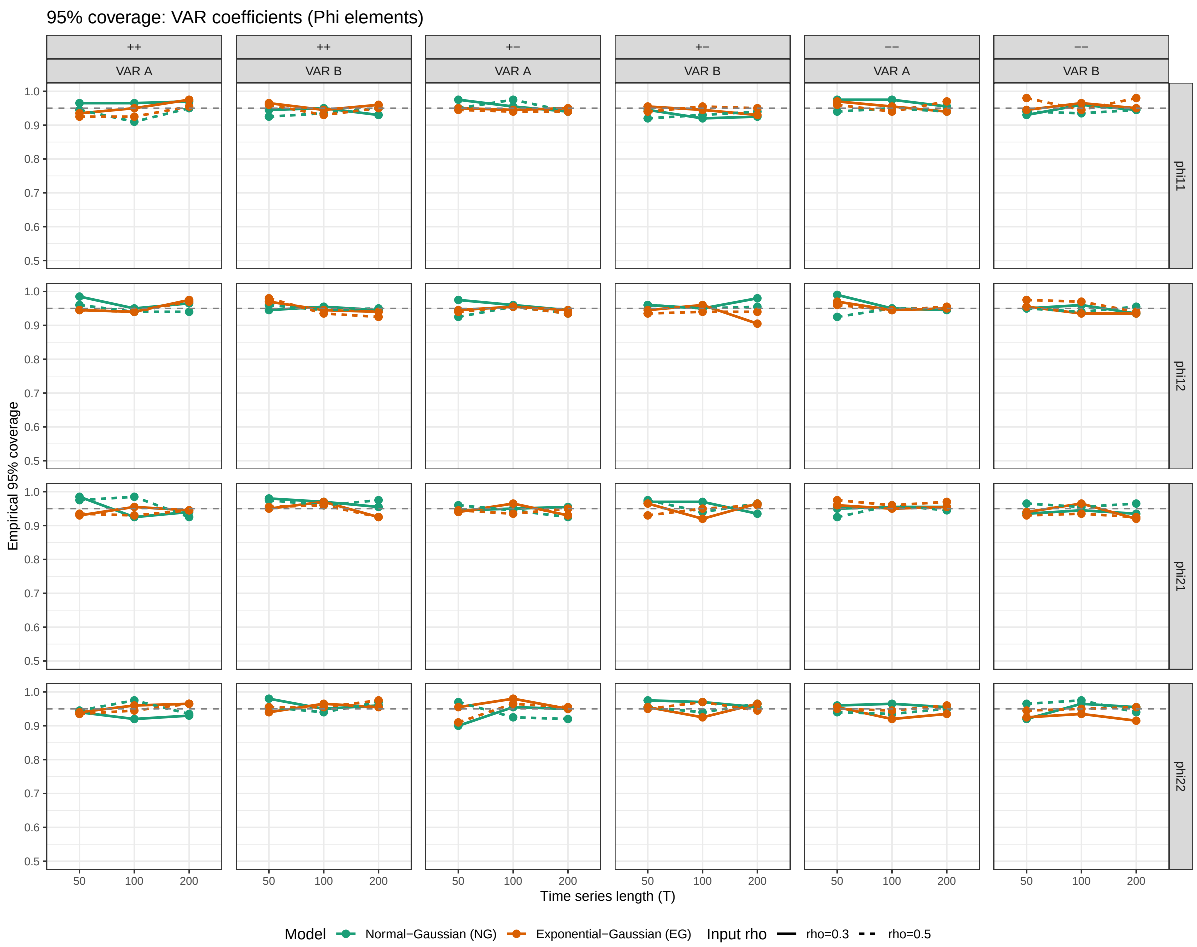

Relative bias: VAR coefficients (Phi elements)
++
VAR A
VAR B
+−
−−
phi11
phi12
phi21
phi22
Mean relative bias
Time series length (T)
50
100
200
Model
Normal−Gaussian (NG)
Exponential−Gaussian (EG)
Input rho
rho=0.3
rho=0.5


Bias: intercepts (mu)
++
VAR A
VAR B
+−
−−
mu[1]
mu[2]
Mean bias (truth = 0)
Time series length (T)
50
100
200
Model
Normal−Gaussian (NG)
Exponential−Gaussian (EG)
Input rho
rho=0.3
rho=0.5

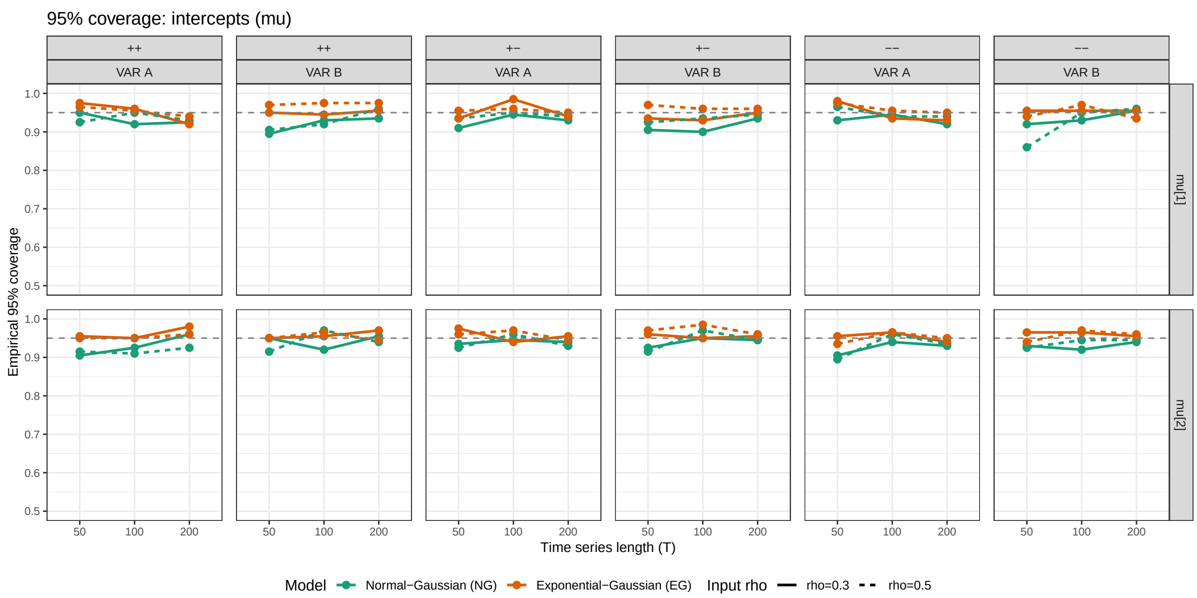
Interpretation. Across T, EG is approximately unbiased for Φ and μ. For ρ, the target in mixed-direction cells is ρ_{eff} (callout above): EG is close to unbiased, while NG exhibits attenuation in $|\rho|$, most pronounced at small T.

4.2. 95% Coverage





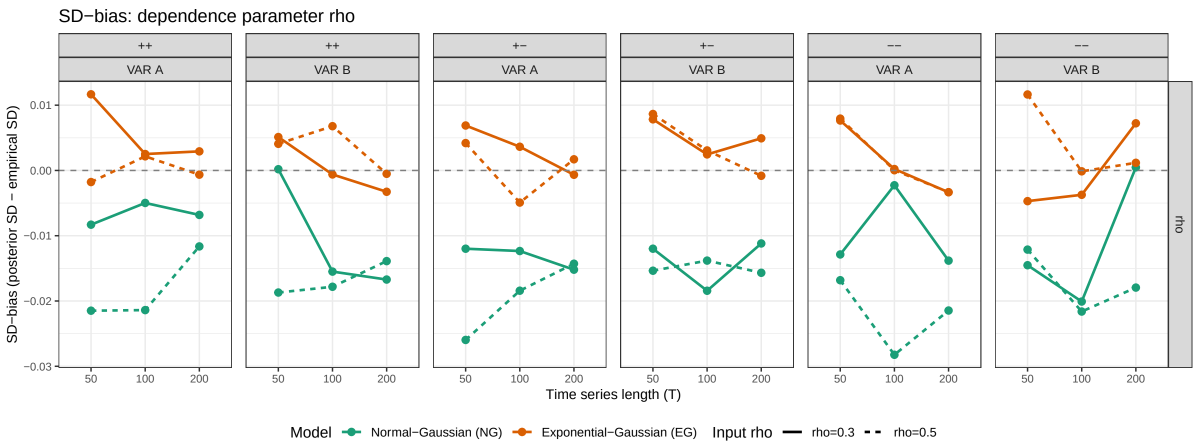
95% coverage: VAR coefficients (Phi elements)
++
+−
−−
VAR A
VAR B
phi11
phi12
phi21
phi22
Empirical 95% coverage
Time series length (T)
Model
Normal−Gaussian (NG)
Exponential−Gaussian (EG)
Input rho
rho=0.3
rho=0.5

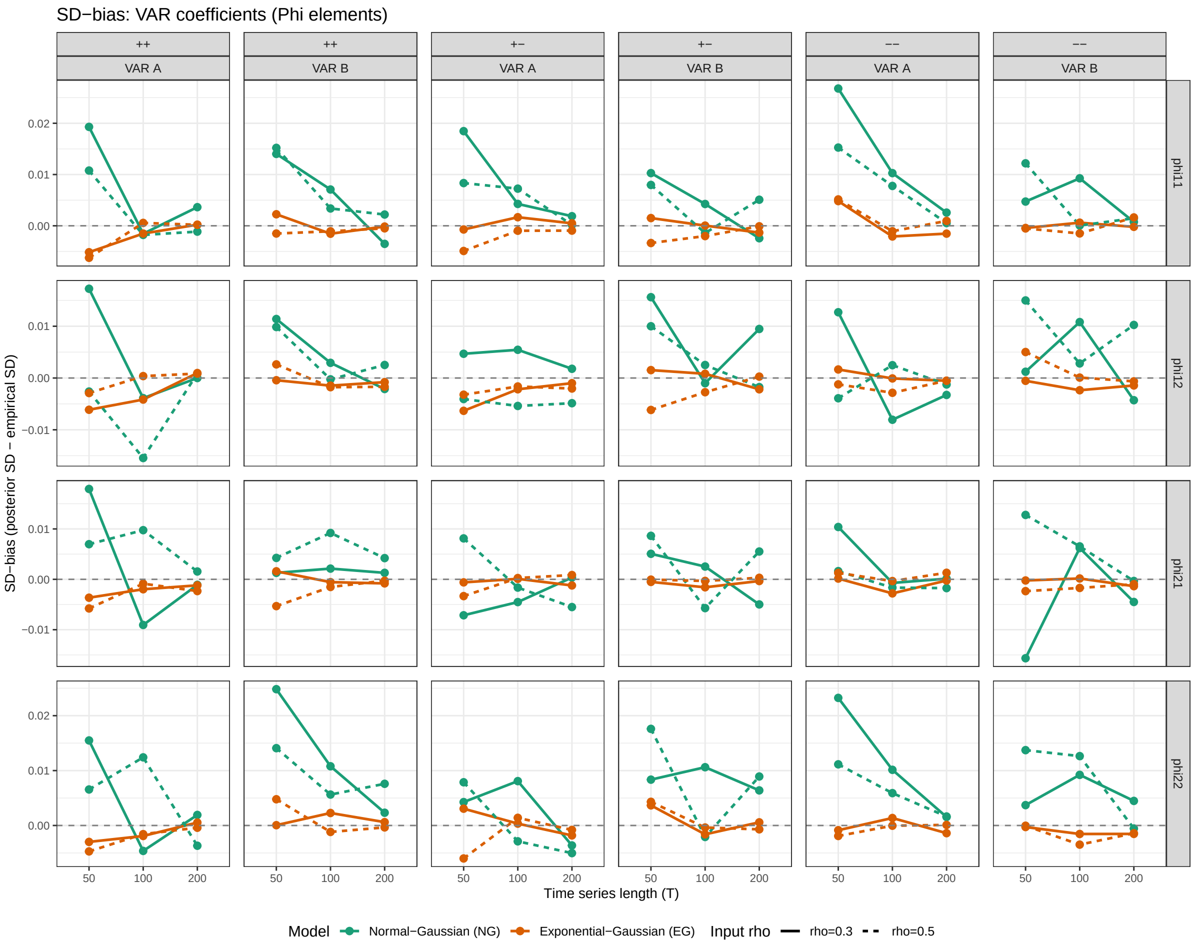


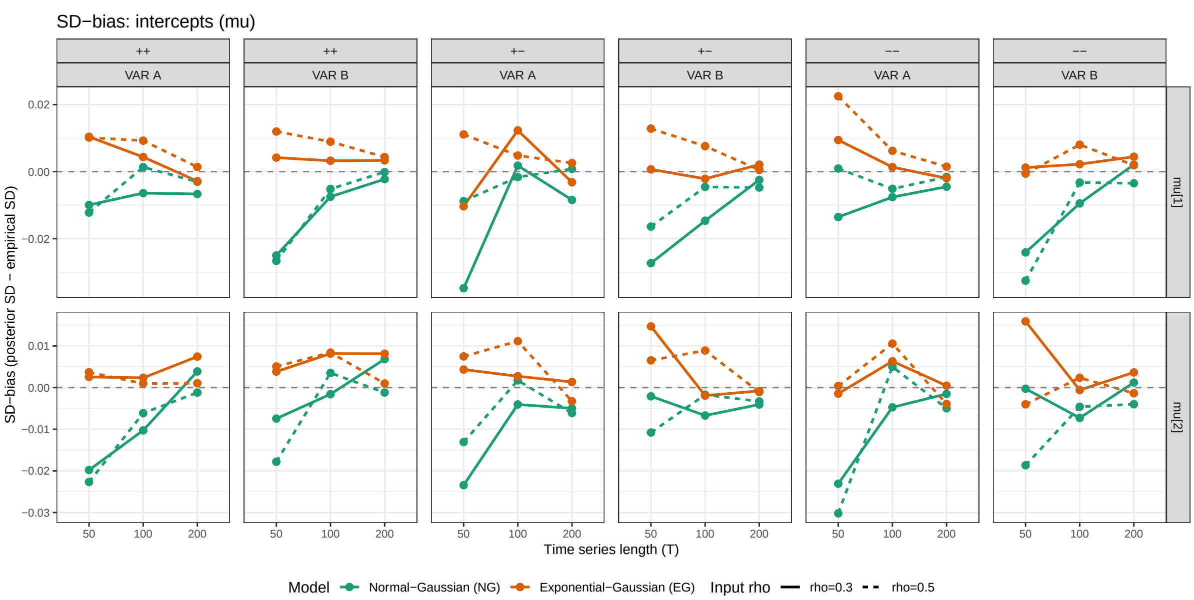
95% coverage: intercepts (mu)
++
+−
−−
VAR A
VAR B
mu[1]
mu[2]
Empirical 95% coverage
Time series length (T)
Model
Normal−Gaussian (NG)
Exponential−Gaussian (EG)
Input rho
rho=0.3
rho=0.5

Interpretation. EG coverage is typically close to nominal for the core parameters. NG coverage shortfalls are concentrated in ρ, consistent with attenuation bias under marginal misspecification.

4.3. SD-Bias



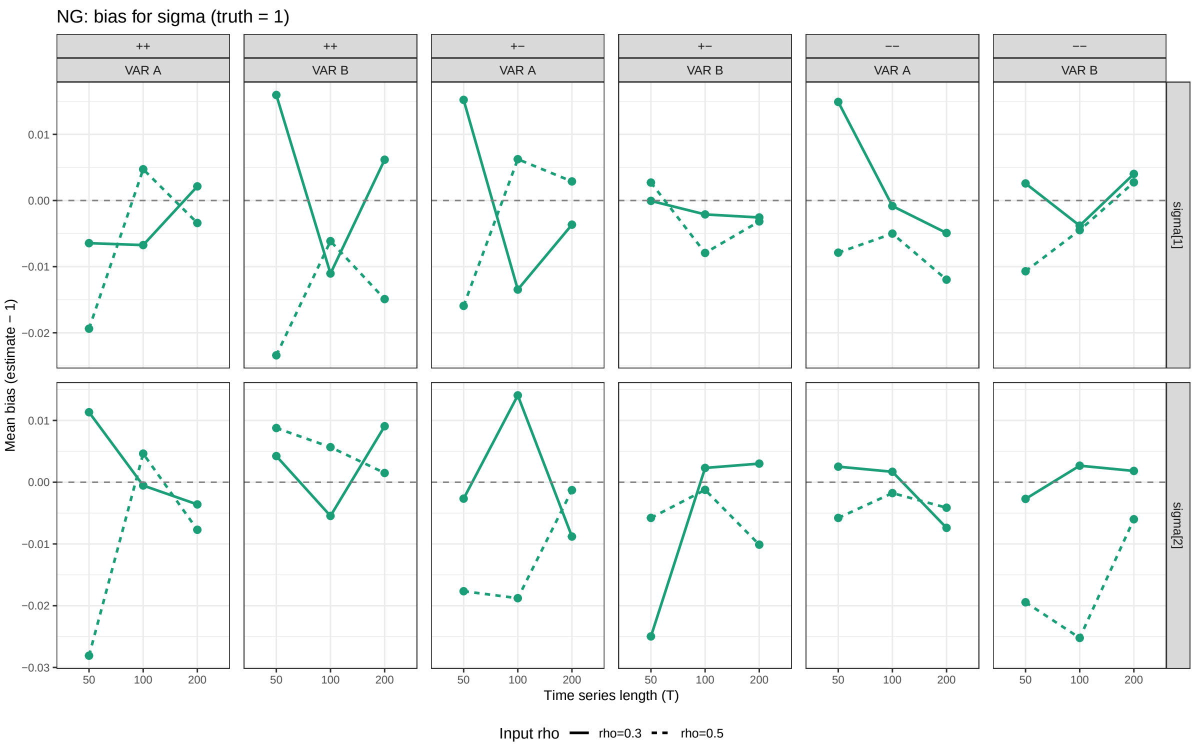

SD−bias: VAR coefficients (Phi elements)
++
VAR A
VAR B
+−
−−
phi11
phi12
phi21
phi22
SD−bias (posterior SD − empirical SD)
Time series length (T)
Model
Normal−Gaussian (NG)
Exponential−Gaussian (EG)
Input rho
rho=0.3
rho=0.5


SD−bias: intercepts (mu)
++
VAR A
VAR B
+−
−−
mu[1]
mu[2]
SD−bias (posterior SD − empirical SD)
Time series length (T)
Model
Normal−Gaussian (NG)
Exponential−Gaussian (EG)
Input rho
rho=0.3
rho=0.5

Interpretation. EG is generally well calibrated (SD-bias near 0). NG frequently shows negative SD-bias for ρ, indicating overconfident posterior uncertainty for dependence under misspecified margins.

5. Marginal Parameters

We report mean bias for the Exponential scale parameters in EG (σ_{exp}) and the innovation standard deviations in NG (σ). Under the standardized DGP, the reference value is 1.



Interpretation. The σ estimates under NG show small but systematic deviations from the true value of 1. As discussed in Study 1 (Section 7.1), these deviations arise from marginal misspecification but are not the mechanism driving ρ attenuation—both phenomena stem from the same root cause (marginal misspecification) but through different pathways.

6. Details

This section provides technical details on the implementation of the simulation study, including prior specifications, MCMC settings, and mathematical derivations specific to the Exponential–Gaussian (EG) model.

6.1. Prior Specifications

Both models use weakly informative priors. The EG model employs a novel reparameterization for the scale parameter to handle the support constraint of exponential marginals.

Table: Prior Specifications for the Normal–Gaussian (NG) Model

Parameter	Prior	Support	Rationale
μ_1, μ_2	Normal$(0, 1)$	$\mathbb{R}$	Weakly informative; centered at true value (0)
$\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}$	Normal$(0, 0.5)$	$(-1, 1)$	Regularizes toward stationarity; truncated by bounds
σ_1, σ_2	Half-Normal$(0, 1)$	$(0, \infty)$	Weakly informative scale prior
ρ	Normal$(0, 0.5)$	$(-1, 1)$	Regularizes toward independence; truncated by bounds

Table: Prior Specifications for the Exponential–Gaussian (EG) Model

Parameter	Prior	Support	Rationale
μ_1, μ_2	Normal$(0, 1)$	$\mathbb{R}$	Weakly informative; centered at true value (0)
$\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}$	Normal$(0, 0.5)$	$(-1, 1)$	Regularizes toward stationarity; truncated by bounds
$\sigma_{\text{exp},1}, \sigma_{\text{exp},2}$	Log-Normal$(0, 0.5)$	(b_i, ∞)	Induced prior via reparameterization (see Section 6.4)
ρ	Normal$(0, 0.5)$	$(-1, 1)$	Regularizes toward independence; truncated by bounds

EG Scale Parameter Prior

The EG model does **not** place a direct prior on σ_{exp}. Instead, it estimates an unconstrained parameter $\eta \in \mathbb{R}$ and applies a change-of-variables transformation:

$$\sigma_{\text{exp}} = b + \exp(\eta)$$

where b is a data-dependent feasibility bound (Section 6.4). The induced prior on σ_{exp} is Log-Normal$(0, 0.5)$ with support shifted to (b, ∞).

6.2. MCMC Settings

All models were fitted using the No-U-Turn Sampler (NUTS) implemented in Stan via the `rstan` package.

Table: MCMC Sampling Configuration

Setting	Value	Description
Chains	4	Number of independent Markov chains
Total iterations	4,000	Iterations per chain (including warmup)
Warmup iterations	2,000	Discarded adaptation period
Post-warmup draws	2,000	Retained samples per chain
`adapt_delta`	0.95	Target acceptance probability
`max_treedepth`	15	Maximum tree depth for NUTS
Parallelization	Outer loop	Replications parallelized; chains run sequentially

6.3. Gaussian Copula Log-Density

The Gaussian copula density implementation is identical to Study 1. For uniform marginals $(u, v) \in (0, 1)^2$ with correlation parameter $\rho \in (-1, 1)$:

Let $z_1 = \Phi^{-1}(u)$ and $z_2 = \Phi^{-1}(v)$ denote the standard normal quantile transforms. The copula log-density is:

$$\log c(u, v; \rho) = -\frac{1}{2} \log(1 - \rho^2) - \frac{1}{2(1 - \rho^2)} \left(z_1^2 - 2\rho z_1 z_2 + z_2^2\right) + \frac{1}{2} \left(z_1^2 + z_2^2\right)$$

i Boundary Clamping

As in Study 1, the implementation applies boundary clamping with $\varepsilon = 10^{-9}$:

$$u_{\text{clamped}} = \max(\varepsilon, \min(1 - \varepsilon, u))$$

This prevents numerical overflow when $\Phi^{-1}(u)$ is evaluated near 0 or 1.

6.4. EG Model: Feasibility-Bound Reparameterization

The Exponential–Gaussian model requires that the shifted residuals have positive support. This section documents the reparameterization strategy that ensures feasibility while maintaining good posterior geometry.

For an Exponential marginal with scale σ_{exp}, the likelihood requires:

$$x_{\text{shifted},t} = \sigma_{\text{exp}} + s \cdot \text{res}_t > 0, \quad \forall\, t$$

where $s \in \{+1, -1\}$ is the skew direction ($+1$ for right-skewed, -1 for left-skewed/mirrored). Naively parameterizing $\sigma_{\text{exp}} > 0$ can lead to divergent transitions when the sampler proposes values that violate this constraint for some t.

Define the feasibility bound:

$$b_i = \max_t \left(-s_i \cdot \text{res}_{t,i}\right)$$

This is the minimum value of $\sigma_{\text{exp},i}$ such that $x_{\text{shifted},t,i} > 0$ for all observations. Any $\sigma_{\text{exp},i} > b_i$ is guaranteed to satisfy the support constraint.

η Reparameterization

Rather than estimating σ_{exp} directly, the EG model estimates an unconstrained parameter $\eta \in \mathbb{R}$ and transforms:

$$\sigma_{\text{exp}} = b + \exp(\eta)$$

where $\exp(\eta) > 0$ represents the "slack" above the feasibility bound.

Induced Prior via Change-of-Variables

To place a Log-Normal$(0, 0.5)$ prior on σ_{exp}, we apply the change-of-variables formula. The Jacobian of the transformation $\sigma_{\text{exp}} = b + \exp(\eta)$ is:

$$\frac{d\sigma_{\text{exp}}}{d\eta} = \exp(\eta)$$

The log-posterior contribution is:

$$\log p(\sigma_{\text{exp}}) + \log \left| \frac{d\sigma_{\text{exp}}}{d\eta} \right| = \text{lognormal\_lpdf}(\sigma_{\text{exp}} \mid 0, 0.5) + \eta$$

Stan Implementation

```
model {
  // compute feasibility bounds: b_i = max_t(-s_i * res_{t,i})
  for (i in 1:2) {
    real m = -skew_direction[i] * residuals[1, i];
    for (t in 2:(T-1)) {
      m = fmax(m, -skew_direction[i] * residuals[t, i]);
    }
    b[i] = m;
  }

  // reparameterized sigma: always feasible
  for (i in 1:2) {
    sigma_exp[i] = b[i] + exp(eta[i]);
  }
  rate_exp = 1.0 ./ sigma_exp;

  // induced prior on sigma_exp via change-of-variables
  for (i in 1:2) {
    target += lognormal_lpdf(sigma_exp[i] | 0, 0.5) + eta[i];
  }

  // likelihood contributions ...
}
```

6.5. CDF Mirroring for Left-Skewed Exponential Margins

For left-skewed (mirrored) Exponential margins, the model must correctly transform the CDF to preserve the copula correlation structure.

The Mirroring Problem

Let $X \sim \text{Exponential}(\lambda)$ with CDF $F_X(x) = 1 - e^{-\lambda x}$. For a mirrored variable $Y = -X$:

$$F_Y(y) = P(Y \leq y) = P(-X \leq y) = P(X \geq -y) = 1 - F_X(-y)$$

On the uniform (PIT) scale, if $U = F_X(X)$, then the PIT of $Y = -X$ is $V = 1 - U$.

Copula Correlation Sign Flip

For a Gaussian copula, the latent normal scores are $Z = \Phi^{-1}(U)$. Under the $U \mapsto 1 - U$ transformation:

$$\Phi^{-1}(1 - U) = -\Phi^{-1}(U) = -Z$$

This negates the latent score, flipping the sign of the copula correlation. If the DGP has $\rho > 0$ and exactly one margin is mirrored, the effective correlation on the observed scale is $\rho_{\text{eff}} = -\rho$.

Stan Implementation

The EG model applies the CDF flip after computing the exponential CDF:

```
for (i in 1:2) {
  // shift residual to positive support
  x_shifted[i] = sigma_exp[i] + skew_direction[i] * res[i];
  target += exponential_lpdf(x_shifted[i] | rate_exp[i]);

  // compute PIT
  u_vec[i] = exponential_cdf(x_shifted[i], rate_exp[i]);

  // flip CDF for left-skewed (mirrored) margins
  if (skew_direction[i] < 0) u_vec[i] = 1.0 - u_vec[i];
}
```

```
target += gaussian_copula_ld(u_vec[1], u_vec[2], rho);
```