Study 2: Comparative Performance of Normal–Gaussian and Exponential–Gaussian Copula VAR Models Under Exponential Innovations

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0. Summary

0.1 Computational Stability versus Statistical Inference

Both models exhibit excellent computational stability across all simulation conditions. The Normal–Gaussian (NG) model shows no post-warmup divergent transitions and max $\hat{R} \leq 1.01$ in all replications. The Exponential–Gaussian (EG) model is equally well-behaved, with zero divergent transitions and good convergence diagnostics. This is an improvement over the SG model under extremeCHI in Study 1, where divergences were frequent.

0.2 Model Performance Under Exponential Innovations

Exponential–Gaussian (EG): EG is (i) approximately unbiased for the VAR dynamics Φ and intercepts μ, and (ii) well calibrated for dependence when ρ is interpreted on the correct scale in mixed-direction cells (see Section 1.1 for the sign convention under mirroring). Empirical coverage is close to 0.95 in most conditions

Normal–Gaussian (NG): NG is computationally stable but exhibits attenuation in $|\rho|$ and corresponding under-coverage for ρ under Exponential margins. This attenuation arises from PIT distortion (see Study 1, Section 6.1).

0.3 Insights

PIT distortion induced by marginal CDF misspecification is the dominant failure mechanism, consistent with findings from Study 1. When NG assumes Gaussian margins but the true innovations are Exponential, the probability integral transform yields non-uniform PITs, which attenuate the effective dependence seen by the Gaussian copula.

1. Introduction

This simulation study extends the analysis from Study 1 by comparing two Bayesian VAR(1) models under exponential innovations: a Normal–Gaussian (NG) model and an Exponential–Gaussian (EG) model. While Study 1 examined skew-normal and chi-squared innovations, this study focuses on exponential margins to assess whether correctly specifying the marginal distribution recovers the models parameters without bias.

1.1. Data Generating Process (DGP)

The DGP follows the same structure as Study 1:

$$Y_t = \mu + \Phi Y_{t-1} + \varepsilon_t, \qquad t = 2, \dots, T,$$

with $\mu = \mathbf{0}$. Innovations $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})$ have standardized Exponential margins (mean 0, variance 1), with optional mirroring to induce left-skewness, and are coupled through a Gaussian copula with correlation parameter ρ.

Standardization of Exponential Innovations

A standard Exponential(1) random variable X has $\mathbb{E}[X] = 1$ and $\mathrm{Var}(X) = 1$. To standardize:

$$Z = X - 1$$

yields $\mathbb{E}[Z] = 0$ and $\mathrm{Var}(Z) = 1$. Mirroring $(-Z)$ produces left-skewed innovations. For the joint density via Gaussian copula, see Study 1, Section 1.1.

Copula Sign Under Mirroring

Left-skew Exponential margins are produced by mirroring standardized Exponential innovations (multiplying by -1). For a Gaussian copula, mirroring exactly one margin corresponds to the transformation $u \mapsto 1-u$ on the PIT scale. Since $\Phi^{-1}(1-u) = -\Phi^{-1}(u)$, this flips the sign of the latent Gaussian score and therefore maps

$$\rho \;\mapsto\; -\rho.$$

Accordingly, in mixed-direction settings (e.g., $+-$), the effective copula correlation associated with the observed marginals is

$$\rho_{\mathrm{eff}} \;=\; s_1 s_2\, \rho, \qquad s_j \in \{+1, -1\},$$

where $s_j = -1$ indicates a mirrored (left-skew) margin.

1.2. Simulation Design

The study employs a factorial design crossing four factors.

Table 1: Summary of the Simulation Design Factors.

Factor	Levels
DGP Level	Standardized Exponential
Time Series Length (T)	50, 100, 200
Copula Correlation (ρ)	0.30, 0.50 (reported as input; evaluation uses ρ_{eff} in mixed-direction cells)
VAR Parameters (Φ)	**Set A**: $\begin{pmatrix} 0.40 & 0.10 \\ 0.10 & 0.40 \end{pmatrix}$
	Set B: $\begin{pmatrix} 0.55 & 0.10 \\ 0.10 & 0.25 \end{pmatrix}$
Skewness Direction	`++` (both right), `--` (both left), `+-` (mixed)

Mixed-Direction Symmetry

Only one mixed-direction case ($+-$) is included. For the asymmetric VAR set B, $+-$ and $-+$ are not equivalent under variable relabeling. See Study 1, Section 1.2 for discussion of this design choice.

1.3. True Parameter Values

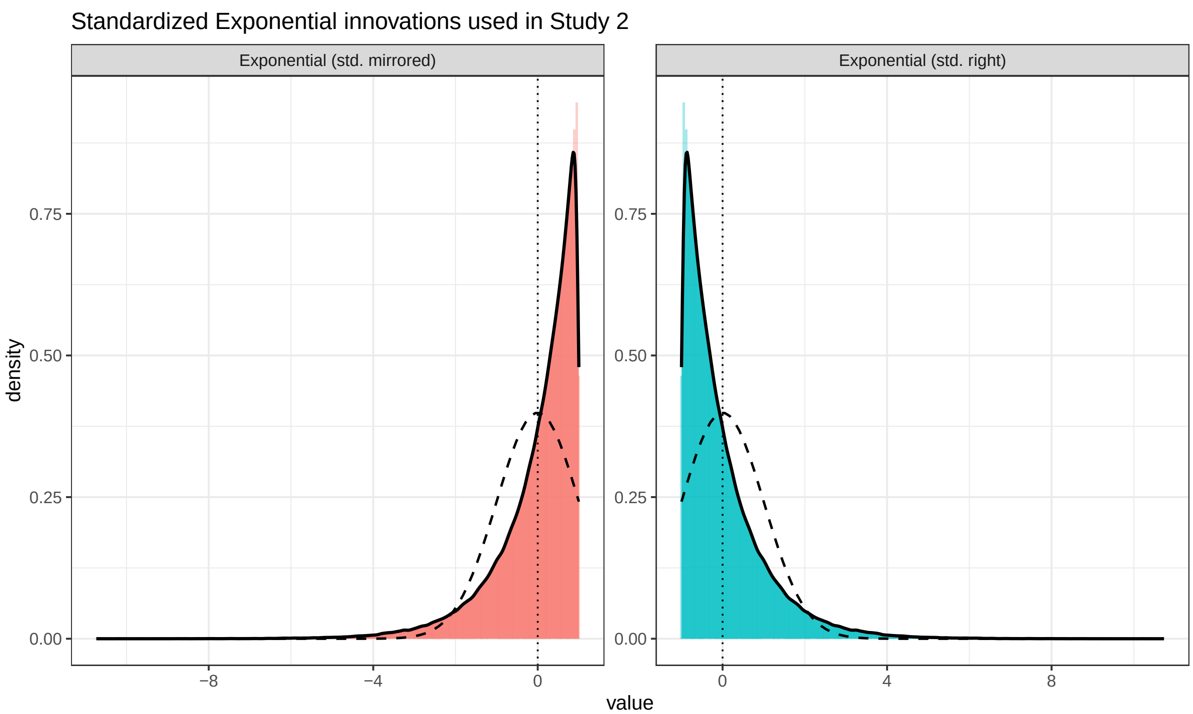
Table 2: True Parameter Values Used in the Data Generating Process.

Parameter	True Value	Notes
μ_1, μ_2	0, 0	Innovations are mean-zero
ϕ_{11} (Set A / Set B)	0.40 / 0.55	Diagonal AR coefficients
$\phi_{12} = \phi_{21}$	0.10	Cross-effects (symmetric)
ϕ_{22} (Set A / Set B)	0.40 / 0.25	Diagonal AR coefficients
ρ	0.30 or 0.50	Copula correlation
σ_1, σ_2 (NG model)	1.0, 1.0	Innovations are unit-variance
$\sigma_{\text{exp},1}, \sigma_{\text{exp},2}$ (EG model)	1.0, 1.0	Scale parameter for standardized Exponential

Bias Metric for Intercepts (μ)

Because $\mu = 0$ in the DGP, "relative bias" is undefined. We report absolute bias for μ (i.e., $\hat{\mu} - 0$), using the convention rel_bias = bias when |truth| is near 0.

1.4 Visual Check: Standardized Marginal Innovations (DGP)

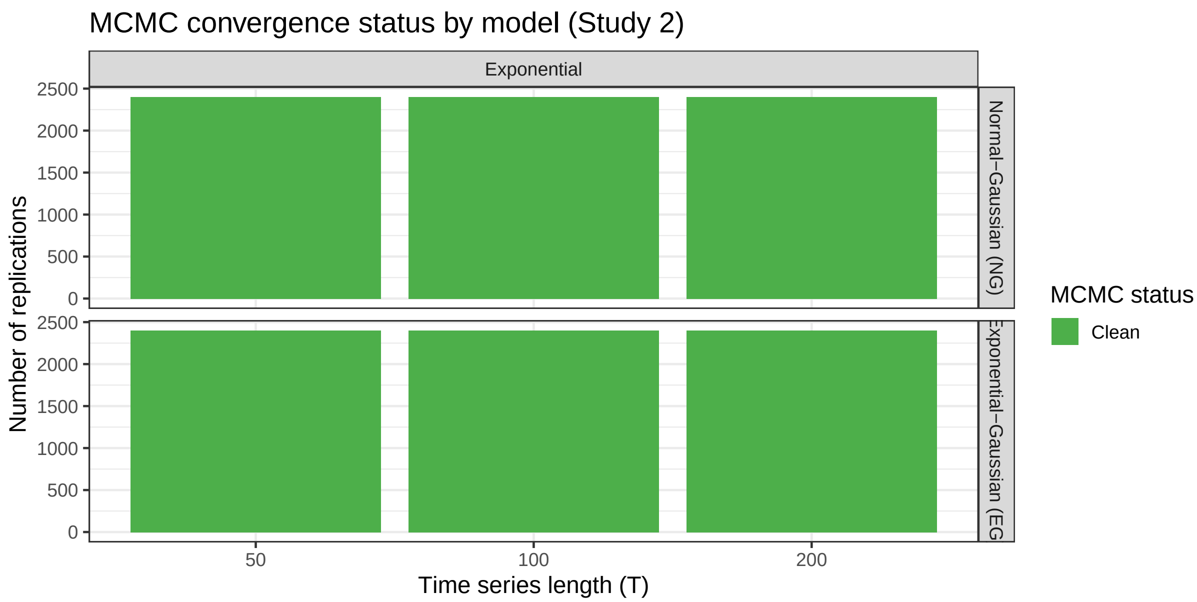


2. Data Loading and Preparation

2.1. MCMC Classification and Overview

We classify runs based on MCMC diagnostics ($\hat{R}$ and divergent transitions) using the same criteria as Study 1:

- **Clean:** $\hat{R} \leq 1.01$ and no post-warmup divergences.
- **Problematic:** $\hat{R} > 1.01$ or at least one divergence.
- **Failed/Error:** Non-OK status or missing diagnostics.



Interpretation: Both NG and EG fits are uniformly Clean across the entire design, with zero post-warmup divergent transitions. This excellent computational behavior contrasts with the SG model under extremeCHI in Study 1, which exhibited frequent divergences. The absence of sampling difficulties here demonstrates that correctly specifying the marginal distribution (EG with Exponential margins) avoids the posterior geometry challenges that arose from model misspecification.

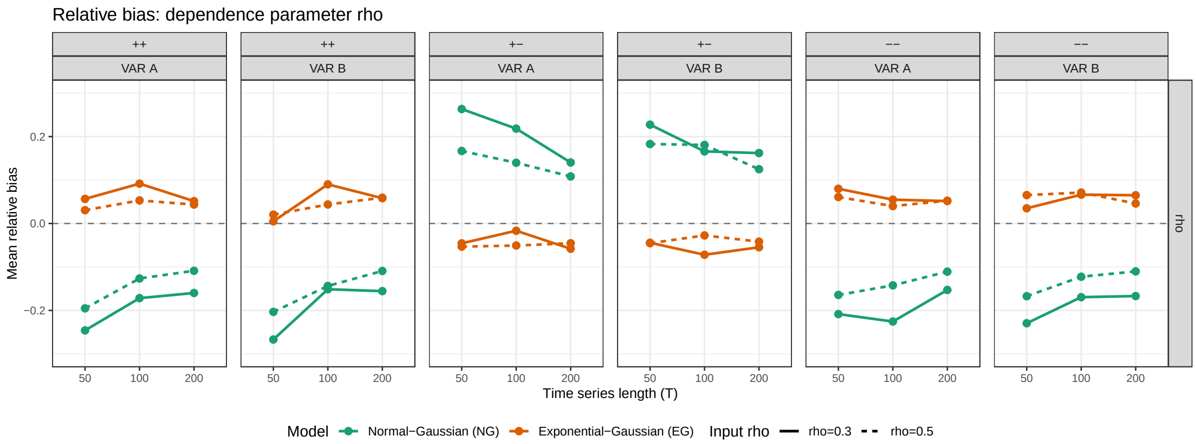
Table 3: Divergence summary: All runs have zero post-warmup divergent transitions.

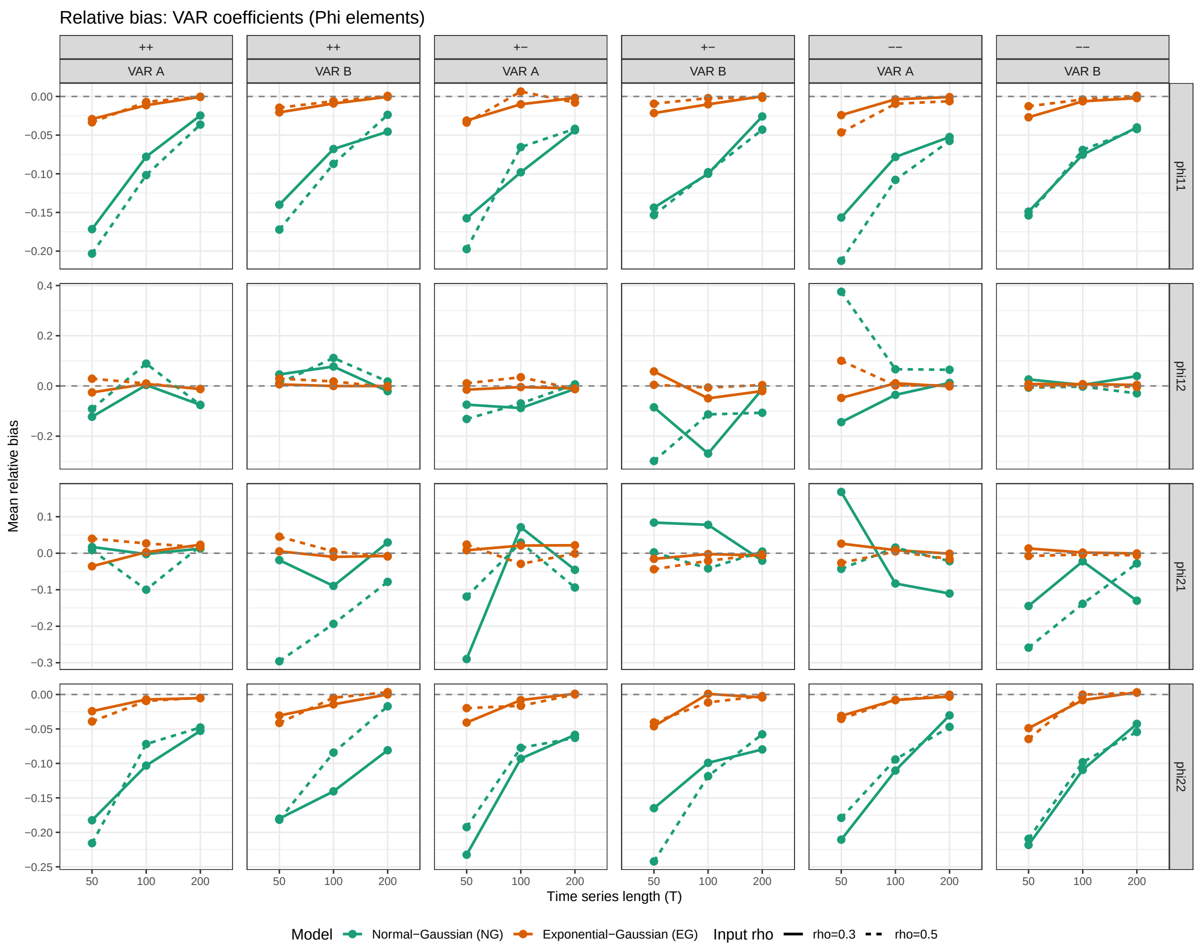
Model	T	N Runs	Total Divergences
Normal–Gaussian (NG)	50	2400	0
Normal–Gaussian (NG)	100	2400	0
Normal–Gaussian (NG)	200	2400	0
Exponential–Gaussian (EG)	50	2400	0
Exponential–Gaussian (EG)	100	2400	0
Exponential–Gaussian (EG)	200	2400	0

3. Helpers

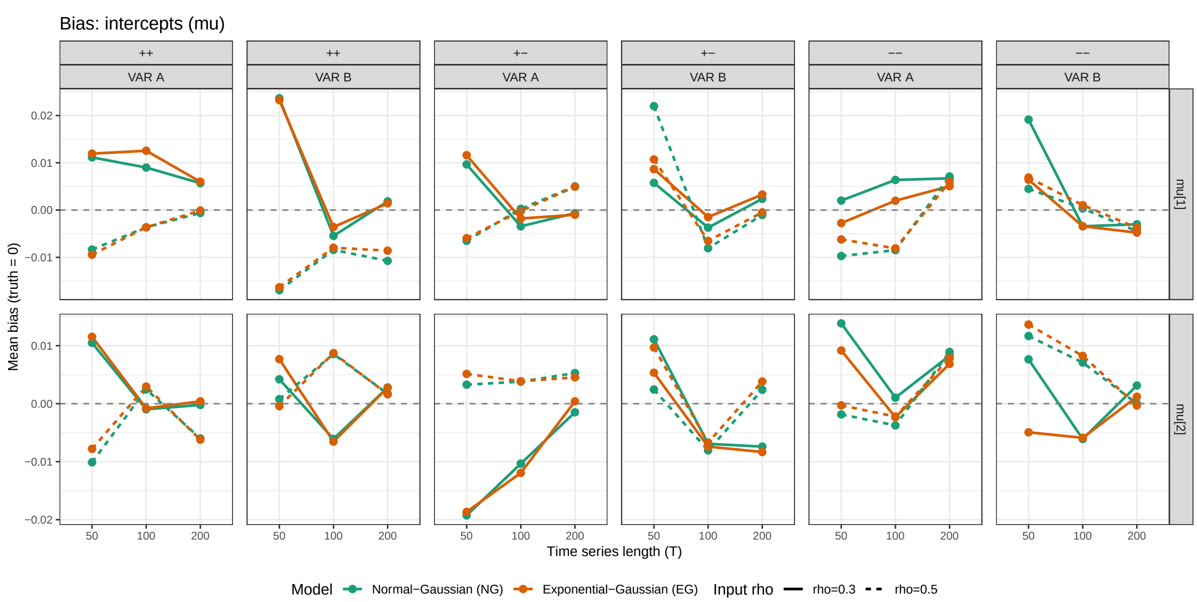
4. Exponential DGP: NG vs EG Performance

4.1. Relative Bias





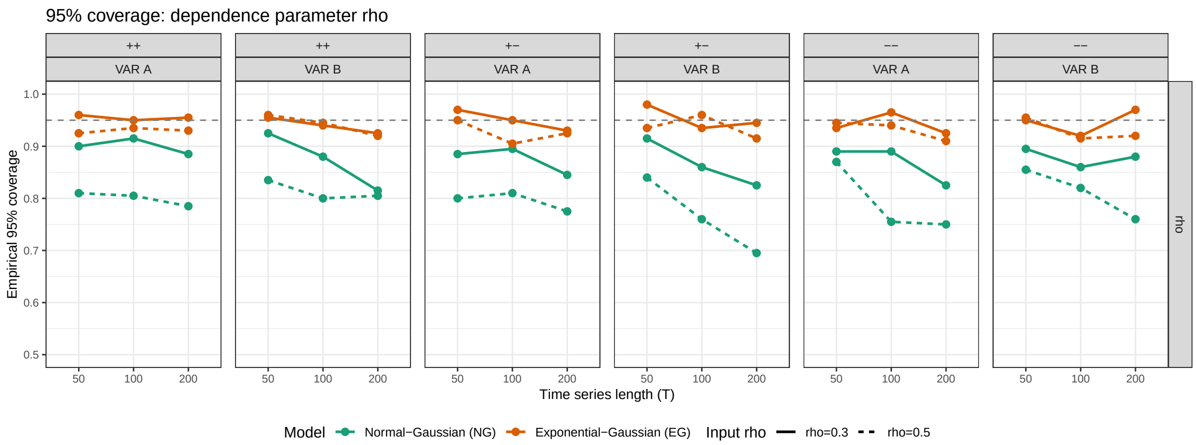
Relative bias: VAR coefficients (Phi elements)
++
+−
−−
VAR A
VAR B
phi11
phi12
phi21
phi22
Mean relative bias
Time series length (T)
Model
Normal−Gaussian (NG)
Exponential−Gaussian (EG)
Input rho
rho=0.3
rho=0.5

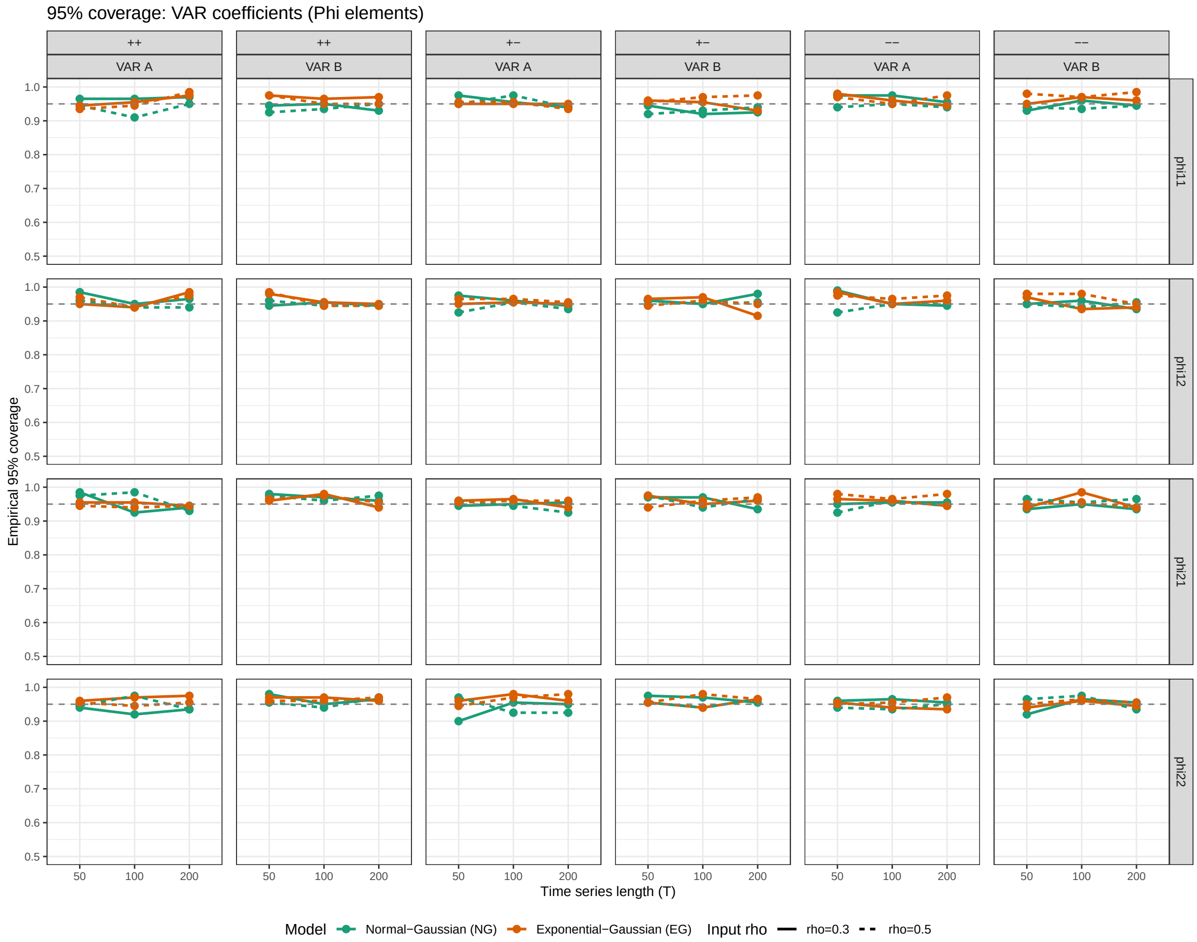


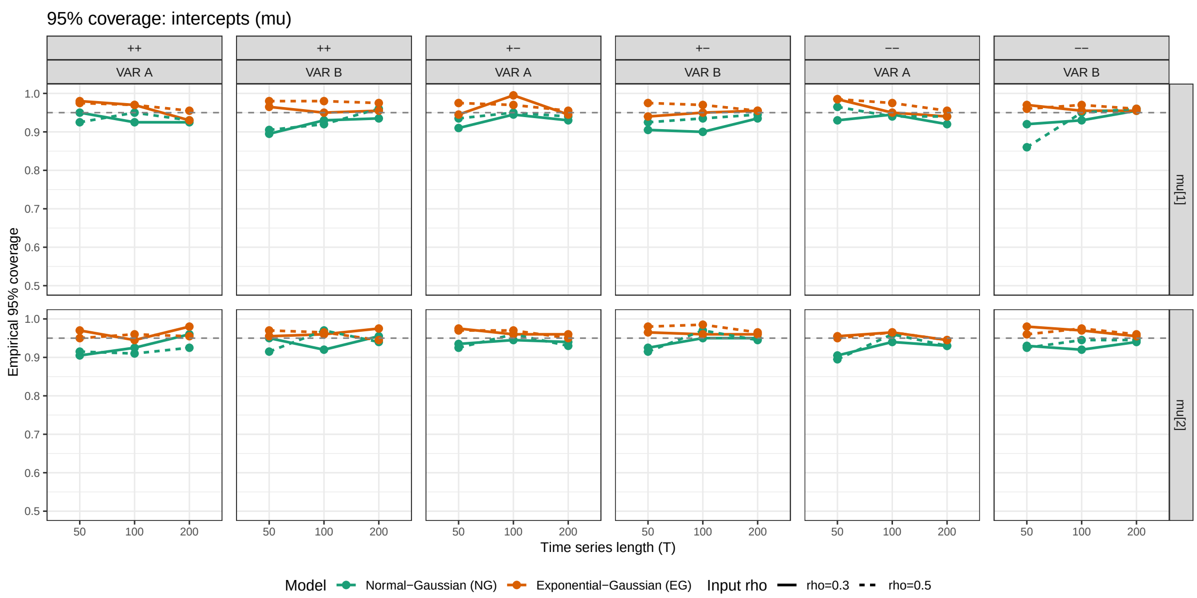
Bias: intercepts (mu)
++
+−
−−
VAR A
VAR B
mu[1]
mu[2]
Mean bias (truth = 0)
Time series length (T)
Model
Normal−Gaussian (NG)
Exponential−Gaussian (EG)
Input rho
rho=0.3
rho=0.5

Interpretation. Across T, EG is approximately unbiased for Φ and μ. For ρ, the target in mixed-direction cells is ρ_{eff} (callout above): EG is close to unbiased, while NG exhibits attenuation in $|\rho|$, most pronounced at small T.

4.2. 95% Coverage

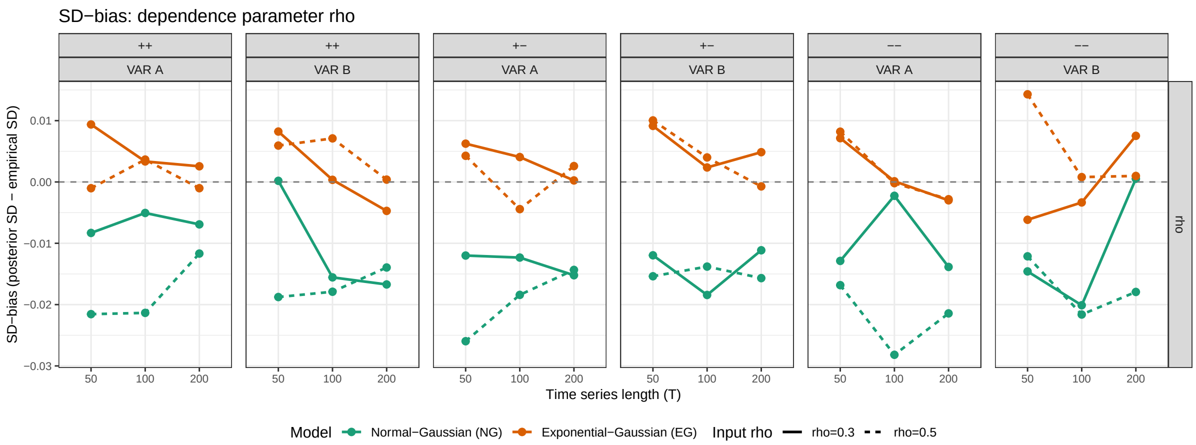


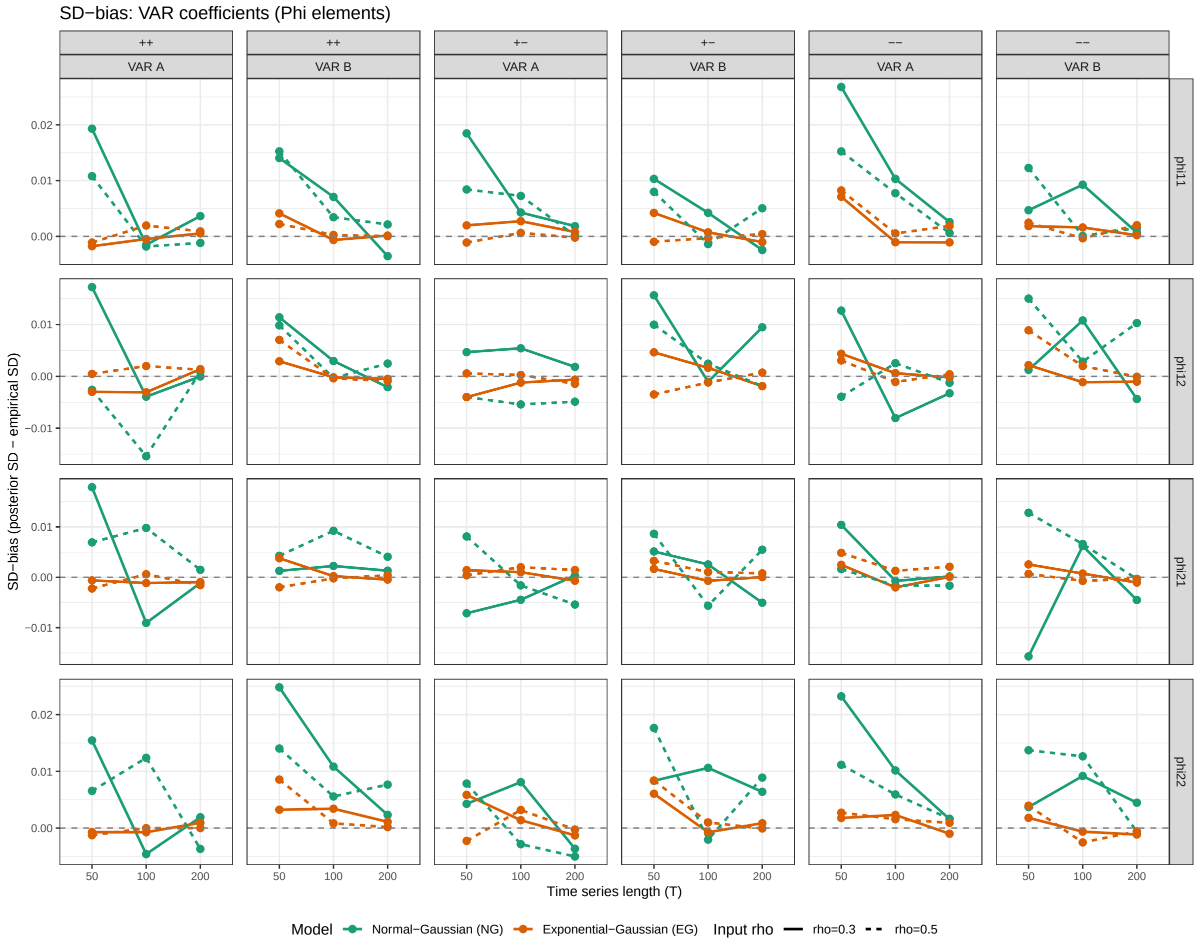

95% coverage: VAR coefficients (Phi elements)
++
+−
−−
VAR A
VAR B
phi11
phi12
phi21
phi22
Empirical 95% coverage
Time series length (T)
Model
Normal−Gaussian (NG)
Exponential−Gaussian (EG)
Input rho
rho=0.3
rho=0.5

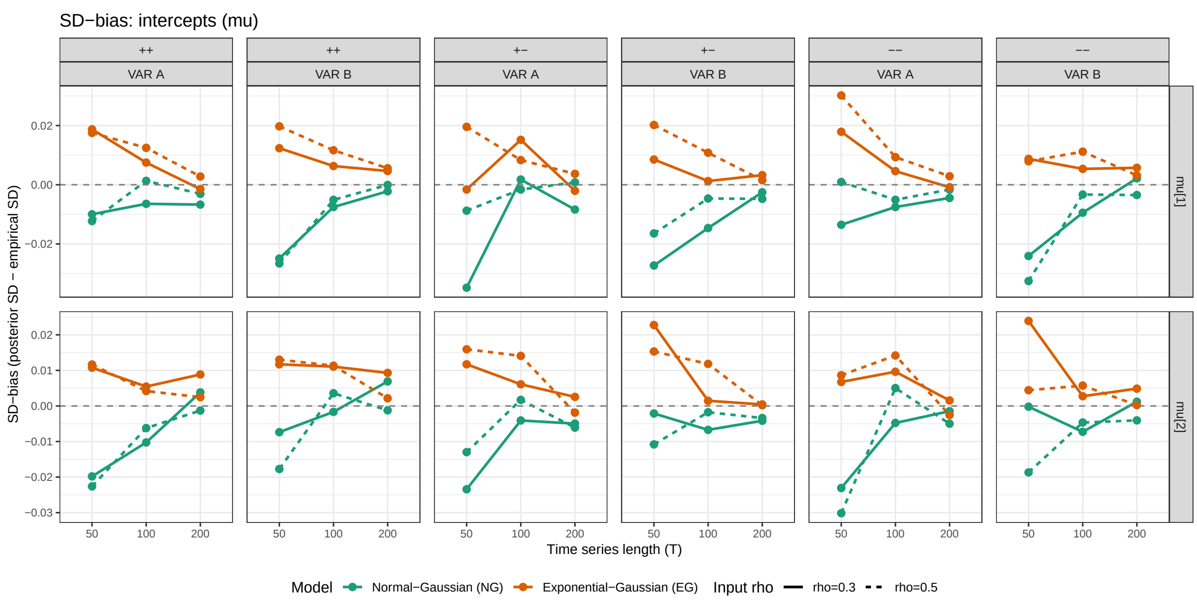

95% coverage: intercepts (mu)
++
+−
−−
VAR A
VAR B
mu[1]
mu[2]
Empirical 95% coverage
Time series length (T)
Model
Normal−Gaussian (NG)
Exponential−Gaussian (EG)
Input rho
rho=0.3
rho=0.5

Interpretation. EG coverage is typically close to nominal for the core parameters. NG coverage shortfalls are concentrated in ρ, consistent with attenuation bias under marginal misspecification.

4.3. SD-Bias



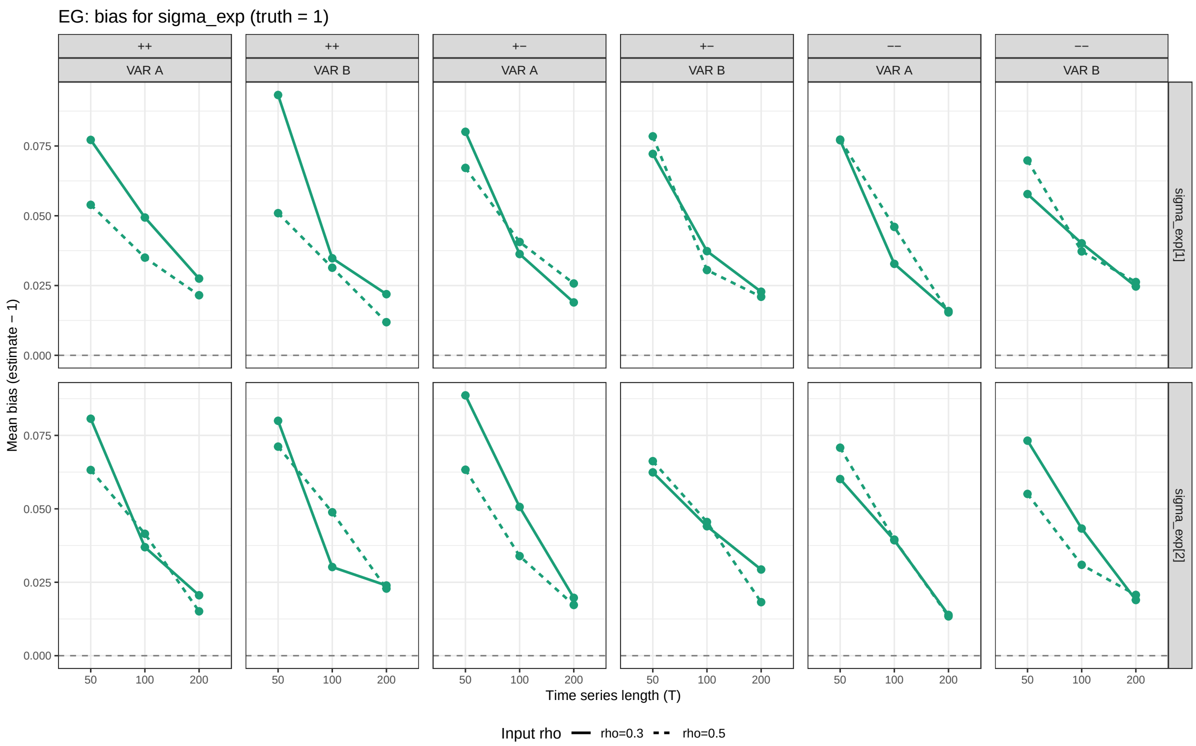

SD−bias: VAR coefficients (Phi elements)
++
VAR A
VAR B
+−
−−
phi11
phi12
phi21
phi22
SD−bias (posterior SD − empirical SD)
Time series length (T)
Model
Normal−Gaussian (NG)
Exponential−Gaussian (EG)
Input rho
rho=0.3
rho=0.5

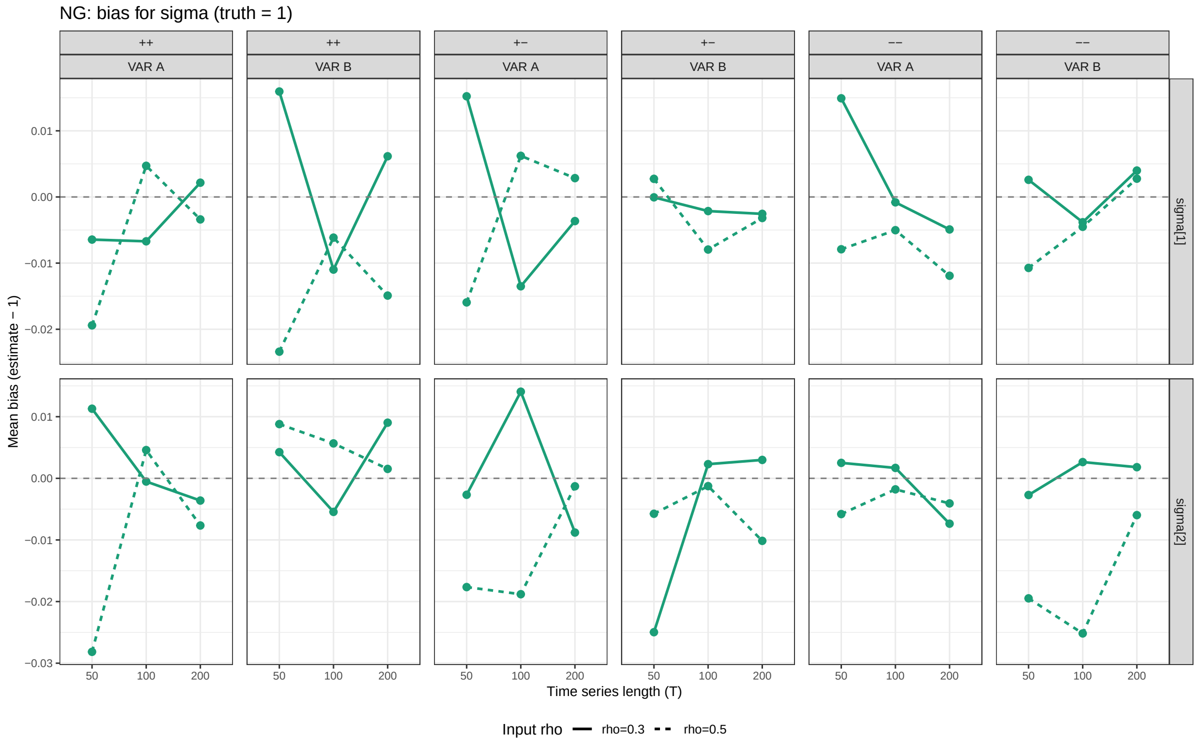

SD−bias: intercepts (mu)
++
VAR A
VAR B
+−
−−
mu[1]
mu[2]
SD−bias (posterior SD − empirical SD)
Time series length (T)
Model
Normal−Gaussian (NG)
Exponential−Gaussian (EG)
Input rho
rho=0.3
rho=0.5

Interpretation. EG is generally well calibrated (SD-bias near 0). NG frequently shows negative SD-bias for ρ, indicating overconfident posterior uncertainty for dependence under misspecified margins.

5. Marginal Parameters

We report mean bias for the Exponential scale parameters in EG (σ_{exp}) and the innovation standard deviations in NG (σ). Under the standardized DGP, the reference value is 1.





Interpretation. The σ estimates under NG show small but systematic deviations from the true value of 1. As discussed in Study 1 (Section 7.1), these deviations arise from marginal misspecification but are not the mechanism driving ρ attenuation—both phenomena stem from the same root cause (marginal misspecification) but through different pathways.