A Derivation of Black-Scholes

Ben Marrow & Filippo Cavaleri

October 2023

1 Introduction to Black-Scholes

Black-Scholes is a famous formula that gives the fair price of an European call option given dynamics of the underlying stock. The formula is typically derived by constructing a partial differential equation (PDE) that must hold under no-arbitrage, and then solving that PDE using boundary conditions pinned down by the terminal payoff of the option. Partial differential equations are in general quite difficult to solve. In the case of Black Scholes, the PDE that describes the evolution of the no-arbitrage portfolio happens to be the same PDE that describes the diffusion of heat from physics ("heat equation"). This made it somewhat straightforward to move from differential equation of the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

$$V_t|_{S=0} = 0,$$

$$\lim_{S\to\infty} V_t = S - K$$

$$V_T = (S - K)^+$$

to the famous Black-Scholes formula:

$$V_t = \Phi(d_1) S_t + \Phi(d_2) Ke^{-r(T-t)}$$

where

$$d_1 = \frac{\ln\frac{S_t}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\frac{S_t}{K} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

I have always found this step a little intractable, both because of the method (I don't know how to solve second order partial differential equations) and the ultimate solution (what are d_1 and d_2 supposed to represent? Why do we have normal CDF functions in the solution?) The goal of this note is to provide a derivation of Black-Scholes that does not rely on any PDEs or replicating portfolios, and in a way that shows intuitively what the objects of the Black-Scholes solution represent.

2 Notation and Environment

Let us begin with notation. We seek to price an option whose value at time t is V_t. The option is defined by it's expiration date, T, a strike price K, and the underlying stock whose price at time t is S_t. The option is then defined by its (terminal) payoff:

$$V_T = \max\{S_T - K, 0\} = (S_T - K)^+$$

We assume the stock evolves according to a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W_t is a one dimensional Brownian motion, and μ and σ are the time-invariant drift and diffusion. There exists a risk-free discount factor with evolution

$$dD_t = -D_t r dt$$

such that the discount rate at time t for a risk-free payoff at time T is

$$D_t = \exp\left(-\int_t^T r_s ds\right) = \exp\left(-r\left(T - t\right)\right)$$

3 Derivation

3.1 Options as a Risk-Neutral Expectation

We first will represent the option as a risk-neutral expectation of a terminal payoff. Prices of any asset are discounted expectations of future payoffs; the challenge is in finding the correct discount rate to use. To address this challenge, we will invoke a change of measure from physical probabilities to risk-neutral probabilities. Under the risk-neutral measure, all payoffs are discounted at the risk free rate induced by D_t.

Girsanov's theoremtells us that there exists a risk-neutral probability measure $\mathbb{Q}$, such that the dynamics of the stock price under $\mathbb{Q}$ satsify

$$\frac{dS_t}{S_t} = rD_t + \sigma dW_t^{\mathbb{Q}}$$

Consider now a self-financing portfolio of stocks and bonds consisting of θ_t shares of the stock, with all residual wealth invested in the bond. The investors wealth X_t in this portfolio evolves as:

$$dX_t = \theta_t dS_t + r\left(X_t - \theta_t S_t\right) dt$$

Substituting in dS_t under measure $\mathbb{Q}$, we have:

$$dX_t = \theta_t \left(rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}\right) + r\left(X_t - \theta_t S_t\right) dt$$
$$= rX_t dt + \theta_t \sigma S_t dW_t^{\mathbb{Q}}$$

Now consider the *discounted* wealth process $D_t X_t$. The dynamics (again, under $\mathbb{Q}$) evolve as:

$$
\begin{aligned}
d\left(D_t X_t\right) &= X_t dD_t + D_t dX_t + dD_t dX_t \\
&= X_t\left(-D_t r dt\right) + D_t dX_t \\
&= -X_t D_t r dt + D_t\left(r X_t dt + \theta_t \sigma S_t dW_t^{\mathbb{Q}}\right) \\
&= \sigma X_t dW_t^{\mathbb{Q}}
\end{aligned}
$$

This tells us that the discounted value of any portfolio of stocks and bonds is a Martingale under $\mathbb{Q}$, since the expectation of the drift is 0:

$$
\mathbb{E}^{\mathbb{Q}}\left[d\left(D_t X_t\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[\sigma X_t dW_t^{\mathbb{Q}}\right] = 0
$$

Intuitively, under the risk neutral measure, the portfolio drifts up by r, but is discounted by an identical negtaive drift in the discounting process. Since the discounted portfolio value is a Martingale, the expected future discounted value (i.e. the present value) is simply the current value:

$$
\mathbb{E}_t^{\mathbb{Q}}\left[D_T X_T\right] = D_t X_t
$$

We can use this formula to price an option V_t, by recognizing that a call option has a replicating portfolio of stocks and bonds. For our purposes, it is not important what the replicating portfolio consists of, merely that such a replicating portfolio exists so that we can replace X_t with V_t. We know from Black-Scholes' original paper that such a replicating portfolio exists, so by the Martingale property above,

$$
\mathbb{E}_t^{\mathbb{Q}}\left[V_T D_T\right] = V_t D_t
$$

Rearranging, and substituting in our expression for the discount rate D_t:

$$
\begin{aligned}
V_t &= \frac{\mathbb{E}_t^{\mathbb{Q}}\left[V_T D_T\right]}{D_t} = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[V_T\right] \\
&= e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\left(S_T - K\right)^+\right]
\end{aligned}
\tag{1}
$$

which follows since $D_T/D_t = e^{-r(T-t)}$.

Equation (1) simply expresses that the price of an option at time t is the risk-neutral expectation of the terminal payoff, $\left(S_T - K\right)^+$, discounted at the risk-free rate, r. The remainder of the problem then becomes one of computing $\mathbb{E}_t^{\mathbb{Q}}\left[\left(S_T - K\right)^+\right]$.

3.2 Probability of an In-the-Money Option

To begin our analysis, we can consider splitting $\mathbb{E}_t^{\mathbb{Q}}\left[\left(S_T - K\right)^+\right]$ from (1) into two terms using the properties of expectations and maximums:

$$
\mathbb{E}_t^{\mathbb{Q}}\left[\left(S_T - K\right)^+\right] = \mathbb{E}_t^{\mathbb{Q}}\left[S_T \mathbf{1}\left\{S_T \geq K\right\}\right] - K\mathbb{E}_t^{\mathbb{Q}}\left[\mathbf{1}\left\{S_T \geq K\right\}\right]
$$

Loosely speaking, the first term is proportional to a conditional expectation of a stock price (conditional

on being in the money). The second term is proportional to the probability the option ends up in the money. Let's begin with the second term.

The expectation of an indicator function is simply the probability the event occurs:

$$\mathbb{E}_t^{\mathbb{Q}}\left[1\left\{S_T \geq K\right\}\right] = \Pr\left(S_T \geq K\right)$$

Observe S_T is a random value whose distribution (and hence expectation) we can compute by solving the differential equation for S under $\mathbb{Q}$:

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t^{\mathbb{Q}}$$

$$d\ln S_t = \left(r - \frac{1}{2}\sigma^2\right)dt + dW_t^{\mathbb{Q}}$$

$$\ln\frac{S_T}{S_t} = \left(\left(r - \frac{1}{2}\sigma^2\right)(T - t) + \sigma\left(W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}}\right)\right)$$

$$\sim N\left(\left(r - \frac{1}{2}\sigma^2\right)(T - t), \sigma^2(T - t)\right)$$

Hence we can write the probability of in-the-money as,

$$\mathbb{E}_t^{\mathbb{Q}}\left[1\left\{S_T \geq K\right\}\right] = \Pr\left(S_T \geq K\right) = \Pr\left(\ln\frac{S_T}{S_t} \geq \ln\frac{K}{S_t}\right)$$

We just showed that $\ln\frac{S_T}{S_t}$ is a normal random variable with mean $\left(r - \frac{1}{2}\sigma\right)(T - t)$ and standard deviation $\sigma\sqrt{T - t}$, so we have:

$$\mathbb{E}_t^{\mathbb{Q}}\left[1\left\{S_T \geq K\right\}\right] = \Phi\left(d_2\right),$$

$$d_2 = \frac{-\ln\frac{K}{S_t} + \left(r - \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}}$$

3.3 Conditional Expectation of the In-the-Money Stock Price

We've now computed the "second" term in our split expectation, related to the probability the option ends up in the money:

$$V_t = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[(S_T - K)\right] = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[S_T 1\left\{S_T \geq K\right\}\right] - \underbrace{e^{-r(T-t)}K\mathbb{E}_t^{\mathbb{Q}}\left[1\left\{S_T \geq K\right\}\right]}_{=e^{-r(T-t)}K\Phi(d_2)}$$

The challenging part is now the first component:

$$e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[S_T 1\left\{S_T \geq K\right\}\right]$$

The future value expectation, $\mathbb{E}_t^{\mathbb{Q}}\left[S_T 1\left\{S_T \geq K\right\}\right]$, is itself a conditional probability multiplied by the probability the option ends in the money:

$$\mathbb{E}_t^{\mathbb{Q}}\left[S_T 1\left\{S_T \geq K\right\}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[S_T \mid S_T \geq K\right]\Pr\left(S_T \geq K\right)$$
$$= \mathbb{E}_t^{\mathbb{Q}}\left[S_T \mid S_T \geq K\right]\Phi\left(d_2\right)$$

It remains to compute the conditional probability $\mathbb{E}_t^{\mathbb{Q}}\left[S_T \mid S_T \geq K\right]$, We can multiply and divide both sides by S_t, since it is known at time t:

$$\mathbb{E}_t^{\mathbb{Q}}\left[S_T \mid S_T \geq K\right] = S_t\mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_T}{S_t} \mid S_T \geq K\right]$$

Next, we're going rewrite S_T/S_t as $\exp\ln\left(S_T/S_t\right)$, and then we are going to apply some symmetric operations to the inequality conditioning the expectation:

$$= S_t\mathbb{E}_t^{\mathbb{Q}}\left[\exp\ln\left(\frac{S_T}{S_t}\right) \mid \frac{S_T}{S_t} \geq \frac{K}{S_t}\right]$$

Now, we know that for a log-normally distributed r.v. X (i.e. $Y = \ln X, X \sim N\left(\mu_X, \sigma_X\right)$), the truncated expectation $\mathbb{E}\left[X \mid X > c\right]$ has the form:

$$\mathbb{E}\left[X \mid X > c\right] = e^{\mu_X + \frac{\sigma_X^2}{2}} \times \frac{\Phi\left(\frac{\mu_X + \sigma_X^2 - \ln c}{\sigma_X}\right)}{1 - \Phi\left(\frac{\ln c - \mu_X}{\sigma_X}\right)}$$

Translating this into our current problem, we have:

$$\mathbb{E}\left[\frac{S_T}{S_t} \mid \frac{S_T}{S_t} > \frac{K}{S_t}\right] = e^{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \frac{1}{2}\sigma^2(T-t)} \times \frac{\Phi\left(\frac{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma^2(T-t) - \ln\left(\frac{K}{S_t}\right)}{\sigma\sqrt{T-t}}\right)}{1 - \Phi\left(\frac{\ln\left(\frac{K}{S_t}\right) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)}$$

$$= e^{r(T-t)}\frac{\Phi\left(d_1\right)}{\Phi\left(d_2\right)}$$

where

$$d_2 = \frac{\ln\frac{S_t}{K} + \left(r - \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}}$$

Taken together, the first term in our option pricing equation is the discount factor $\times$ the conditional probability $\times$ the marginal probability of being in-the-money:

$$e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[S_T 1\left\{S_T \geq K\right\}\right] = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[S_T \mid S_T \geq K\right]\Phi\left(d_2\right)$$
$$= e^{-r(T-t)}S_t\mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_T}{S_t} \mid \frac{S_T}{S_t} \geq \frac{K}{S_t}\right]\Phi\left(d_2\right)$$
$$= e^{-r(T-t)}S_t e^{r(T-t)}\frac{\Phi\left(d_1\right)}{\Phi\left(d_2\right)}\Phi\left(d_2\right)$$
$$= S_t\Phi\left(d_1\right)$$

3.4 Bringing Everything Together

So now have all the ingredients. Options are risk-neutral expectations of the terminal payoff discounted at the risk free rate:

$$V_t = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[(S_T - K)^+\right]$$

The undiscounted expectation of a maximum is the expectation of the union of two events: (1) the option ending up in the money and (2) the spread of the stock price above the strike price. In a slight of abuse of notation, but one that perhaps clarifies the intutition, we have:

Expected Payoff $=$ Expected $((\text{Stock Price - Strike Price}) \text{ and } (\text{In the Money}))$

$= $ Expected $(\text{Stock Price -Strike Price} \mid \text{In the Money}) \times \Pr(\text{In the Money})$

$= $ Expected $(\text{Stock Price} \mid \text{In the Money}) \times \Pr(\text{In the Money}) - \text{Strike Price} \times \Pr(\text{In the Money})$

Mathematically, this is simply:

$$V_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+\right] = \mathbb{E}_t^{\mathbb{Q}}[S_T \mathbf{1}\{S_T \geq K\}] - K\mathbb{E}_t^{\mathbb{Q}}[\mathbf{1}\{S_T \geq K\}]$$

$$= e^{-r(T-t)}\left[\underbrace{\left(S_t e^{r(T-t)}\frac{\Phi(d_1)}{\Phi(d_2)}\right)\underbrace{\Phi(d_2)}_{\Pr(S_T \geq K)}}_{\mathbb{E}_t^{\mathbb{Q}}[S_T \mid S_T \geq K]} - K\underbrace{\Phi(d_2)}_{\Pr(S_T \geq K)}\right]$$

or, using the notation above, where ITM means "In-the-Money",

$$V_t = \underbrace{e^{-r(T-t)}}_{\text{Discounting}}\left[\underbrace{\left(S_t e^{r(T-t)}\frac{\Phi(d_1)}{\Phi(d_2)}\right)}_{\text{Expected stock price given ITM}}\underbrace{\Phi(d_2)}_{\text{Pr ITM}} - \underbrace{K}_{\text{Strike}}\underbrace{\Phi(d_2)}_{\text{Pr ITM}}\right]$$

It is now easy to see that $\Phi(d_2)$ is precisely the risk-neutral probability that the option ends in the money. $\Phi(d_1)$ is slightly less intuitive; it simply arises as the numerator in the conditonal expectation of a log normal distribution.