A Derivation of Black-Scholes

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1 Introduction to Black-Scholes

Black-Scholes is a famous formula that gives the fair price of an European call option given dynamics of the underlying stock. The formula is typically derived by constructing a partial differential equation (PDE) that must hold under no-arbitrage, and then solving that PDE using boundary conditions pinned down by the terminal payoff of the option. Partial differential equations are in general quite difficult to solve. In the case of Black-Scholes, the PDE that describes the evolution of the no-arbitrage portfolio happens to be the same PDE that describes the diffusion of heat from physics (the "heat equation"). This made it somewhat straightforward to move from a differential equation of the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

$$V_t|_{S=0} = 0,$$

$$\lim_{S\to\infty} V_t = S - K$$

$$V_T = (S_T - K)^+$$

to the famous Black-Scholes formula:

$$V_t = \Phi\left(d_1\right) S_t - \Phi\left(d_2\right) K e^{-r(T-t)}$$

where

$$d_1 = \frac{\ln \frac{S_t}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln \frac{S_t}{K} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

I have always found this step a little intractable, both because of the method (I don't know how to solve second order partial differential equations) and the ultimate solution (What are d_1 and d_2 supposed to represent? Why do we have normal CDF functions in the solution? How come the second term has a discount rate ($e^{-r(T-t)}$) but the first term doesn't?) The goal of this note is to provide a derivation of Black-Scholes that does not rely on any PDEs or replicating portfolios, and in a way that shows intuitively what the objects of the Black-Scholes solution represent.

2 Notation and Environment

Let us begin with notation. We seek to price an option whose value at time t is V_t. The option is defined by its expiration date, T, a strike price K, and the underlying stock whose price at time t is S_t. The option is then defined by its (terminal) payoff:

$$V_T = \max\{S_T - K, 0\} = (S_T - K)^+$$

We assume the stock evolves according to a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W_t is a one dimensional Brownian motion, and μ and σ are the time-invariant drift and diffusion. There exists a risk-free discount factor with evolution

$$dD_t = -D_t r dt$$

such that the discount rate at time t for a risk-free payoff at time T is

$$D_t = \exp\left(-\int_t^T r_s ds\right) = \exp\left(-r\left(T-t\right)\right)$$

3 Derivation

3.1 Options as a Risk-Neutral Expectation

We first will represent the option as a risk-neutral expectation of a terminal payoff. Prices of any asset are discounted expectations of future payoffs; the challenge is in finding the correct discount rate to use. To address this challenge, we will invoke a change of measure from physical probabilities to risk-neutral probabilities: this lets us use the risk-free rate (which is known) to discount the terminal payoff. In other words, the goal of this subsection is to show that we can express the price of an option as the discounted expected future payoff,

$$V_t = \underbrace{e^{-r(T-t)}}_{\text{Discounted}} \underbrace{\mathbb{E}_t^{\mathbb{Q}}\left[(S_T - K)^+\right]}_{\text{Expected Payoff}}$$

where $\mathbb{Q}$ is the risk-neutral measure. Again, by writing the equation this way, we avoid the challenge of figuring out what discount rate to the use, since under the *risk-neutral* measure, all payoffs are discounted at the risk free rate induced by D_t.

To move from the physical probability measure to a risk neutral probability measure, we will make use of Girsanov's Theorem. Specifically, Girsanov's theorem tells us that there exists a risk-neutral probability measure $\mathbb{Q}$, such that the dynamics of the stock price under $\mathbb{Q}$ satsify

$$\frac{dS_t}{S_t} = rD_t + \sigma dW_t^{\mathbb{Q}} \tag{1}$$

i.e., under this measure, the stock follows a geometric Brownian motion as before, but with a drift of r, the risk-free drift. Intuitively, moving to a risk-neutral measure strips aways the component of the the drift associated with risk (i.e. with excess returns), leaving just the risk-free component the drift.[1]

We now will explore the dynamics of a discounted, self-financing portfolio. The idea is to show that the value of a discounted portfolio is a martingale under $\mathbb{Q}$, and so the discounted value of an option (which is itself a portfolio of stocks and bonds) will also behave as a Martingale under $\mathbb{Q}$.

Consider a general self-financing portfolio of stocks and bonds consisting of θ_t shares of the stock, with all residual wealth invested in the bond. The investors wealth X_t in this portfolio evolves as:

$$dX_t = \theta_t dS_t + r\left(X_t - \theta_t S_t\right) dt$$

i.e., the θ_t shares in the stock gains the stock wealth, dS_t, and the residual wealth, $X_t - \theta S_t$ gains at the bond drift r. Substituting in dS_t under measure $\mathbb{Q}$, we have:

$$\begin{aligned} dX_t &= \theta_t \left(rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}\right) + r\left(X_t - \theta_t S_t\right) dt \\ &= rX_t dt + \theta_t \sigma S_t dW_t^{\mathbb{Q}} \end{aligned}$$

Now consider the *discounted* wealth process $D_t X_t$. The dynamics (again, under $\mathbb{Q}$) evolve as:

$$\begin{aligned} d\left(D_t X_t\right) &= X_t dD_t + D_t dX_t + dD_t dX_t \\ &= X_t\left(-D_t r dt\right) + D_t dX_t \\ &= -X_t D_t r dt + D_t \left(rX_t dt + \theta_t \sigma S_t dW_t^{\mathbb{Q}}\right) \\ &= \sigma X_t dW_t^{\mathbb{Q}} \end{aligned}$$

This tells us that the discounted value of any portfolio of stocks and bonds is a Martingale under $\mathbb{Q}$, since the expectation of the drift is 0:

$$\mathbb{E}^{\mathbb{Q}}\left[d\left(D_t X_t\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[\sigma X_t dW_t^{\mathbb{Q}}\right] = 0$$

Intuitively, under the risk neutral measure, the portfolio drifts up by r, but is discounted by an identical negative drift in the discounting process. Since the discounted portfolio value is a Martingale, the expected future discounted value (i.e. the present value) is simply the current value:

$$\mathbb{E}_t^{\mathbb{Q}}\left[D_T X_T\right] = D_t X_t$$

We can use this formula to price an option V_t, by recognizing that a call option has a replicating portfolio of stocks and bonds. For our purposes, it is not important what the replicating portfolio consists of, merely that such a replicating portfolio exists so that we can replace X_t with V_t. We know from Black-Scholes' original paper that such a replicating portfolio exists, so by the Martingale property above,

$$\mathbb{E}_t^{\mathbb{Q}}\left[V_T D_T\right] = V_t D_t$$

[1]The risk neutral measures alters the Brownian motion as well, hence $dW_t^{\mathbb{Q}}$ is indexed by $\mathbb{Q}$. In the conclusion, we will revisit the relationship between the Brownian motion under $\mathbb{P}$ and $\mathbb{Q}$.

Rearranging, and substituting in our expression for the discount rate D_t:

$$\begin{aligned} V_t &= \frac{\mathbb{E}_t^{\mathbb{Q}}\left[V_T D_T\right]}{D_t} = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[V_T\right] \\ &= e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[(S_T - K)^+\right] \end{aligned} \tag{2}$$

which follows since $D_T/D_t = e^{-r(T-t)}$.

As desiered, qquation (2) simply expresses that the price of an option at time t is the risk-neutral expectation of the terminal payoff, $(S_T - K)^+$, discounted at the risk-free rate, r. The remainder of the problem then becomes one of computing $\mathbb{E}_t^{\mathbb{Q}}\left[(S_T - K)^+\right]$.

3.2 Probability of an In-the-Money Option

To begin our analysis, we can consider splitting $\mathbb{E}_t^{\mathbb{Q}}\left[(S_T - K)^+\right]$ from (2) into two terms using the properties of expectations and maximums:

$$\mathbb{E}_t^{\mathbb{Q}}\left[(S_T - K)^+\right] = \mathbb{E}_t^{\mathbb{Q}}\left[(S_T - K)\,1\left\{S_T \geq K\right\}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[S_T 1\left\{S_T \geq K\right\}\right] - K\mathbb{E}_t^{\mathbb{Q}}\left[1\left\{S_T \geq K\right\}\right]$$

Loosely speaking, the first term is proportional to a conditional expectation of a stock price (conditional on being in-the-money). The second term is proportional to the probability the option ends up in the money. Let's begin with the second term.

The expectation of an indicator function is simply the probability[2] the event occurs:

$$\mathbb{E}_t^{\mathbb{Q}}\left[1\left\{S_T \geq K\right\}\right] = \Pr\left(S_T \geq K\right)$$

Observe S_T is a random value whose distribution (and hence expectation) we can compute by solving the differential equation for S under $\mathbb{Q}$:

$$\begin{aligned} \frac{dS_t}{S_t} &= r dt + \sigma dW_t^{\mathbb{Q}} \\ d\ln S_t &= \left(r - \frac{1}{2}\sigma^2\right) dt + dW_t^{\mathbb{Q}} \\ \ln\left(\frac{S_T}{S_t}\right) &= \left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}}\right) \\ &\sim N\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2 (T-t)\right) \end{aligned}$$

Hence we can write the probability of in-the-money as,

$$\mathbb{E}_t^{\mathbb{Q}}\left[1\left\{S_T \geq K\right\}\right] = \Pr\left(S_T \geq K\right) = \Pr\left(\ln\frac{S_T}{S_t} \geq \ln\frac{K}{S_t}\right)$$

We just showed that $\ln\frac{S_T}{S_t}$ is a normal random variable with mean $\left(r - \frac{1}{2}\sigma\right)(T-t)$ and standard deviation $\sigma\sqrt{T-t}$, so we have:

[2]All probabilities hereafter are taken under the $\mathbb{Q}$ measure.

$$\mathbb{E}_t^{\mathbb{Q}}\left[1\left\{S_T \geq K\right\}\right] = \Phi\left(d_2\right),$$
$$d_2 = \frac{-\ln\frac{K}{S_t} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

3.3 Conditional Expectation of the In-the-Money Stock Price

We've now computed the "second" term in our split expectation, related to the probability the option ends up in the money:

$$V_t = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\left(S_T - K\right)\right] = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[S_T 1\left\{S_T \geq K\right\}\right] - \underbrace{e^{-r(T-t)}K\mathbb{E}_t^{\mathbb{Q}}\left[1\left\{S_T \geq K\right\}\right]}_{=e^{-r(T-t)}K\Phi(d_2)}$$

The challenging part is now the first component,

$$e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[S_T 1\left\{S_T \geq K\right\}\right]$$

The future value expectation, $\mathbb{E}_t^{\mathbb{Q}}\left[S_T 1\left\{S_T \geq K\right\}\right]$, is itself a conditional probability multiplied by the probability the option ends in the money:

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}\left[S_T 1\left\{S_T \geq K\right\}\right] &= \mathbb{E}_t^{\mathbb{Q}}\left[S_T \mid S_T \geq K\right]\Pr\left(S_T \geq K\right) \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[S_T \mid S_T \geq K\right]\Phi\left(d_2\right)\end{aligned}$$

It remains to compute the conditional probability $\mathbb{E}_t^{\mathbb{Q}}\left[S_T \mid S_T \geq K\right]$. We can multiply and divide both sides by S_t, since it is known at time t:

$$\mathbb{E}_t^{\mathbb{Q}}\left[S_T \mid S_T \geq K\right] = S_t\mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_T}{S_t} \mid S_T \geq K\right]$$

Next, we're going rewrite S_T/S_t as $\exp\left(\ln\left(S_T/S_t\right)\right)$, and then we are going to apply some symmetric operations to the inequality conditioning the expectation:

$$= S_t\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(\ln\left(\frac{S_T}{S_t}\right)\right) \mid \frac{S_T}{S_t} \geq \frac{K}{S_t}\right]$$

Now, we know that for a log-normally distributed r.v. X (i.e. $Y = \ln X, X \sim N\left(\mu_X, \sigma_X\right)$), the expectation of the truncated distribution $\mathbb{E}\left[X \mid X > c\right]$ has the form:

$$\mathbb{E}\left[X \mid X > c\right] = e^{\mu_X + \frac{\sigma_X^2}{2}} \times \frac{\Phi\left(\frac{\mu_X + \sigma_X^2 - \ln c}{\sigma_X}\right)}{1 - \Phi\left(\frac{\ln c - \mu_X}{\sigma_X}\right)}$$

Translating this into our current problem, we have:

$$\mathbb{E}\left[\frac{S_T}{S_t} \mid \frac{S_T}{S_t} > \frac{K}{S_t}\right] = e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\frac{1}{2}\sigma^2(T-t)} \times \frac{\Phi\left(\frac{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma^2(T-t)-\ln\left(\frac{K}{S_t}\right)}{\sigma\sqrt{T-t}}\right)}{1-\Phi\left(\frac{\ln\left(\frac{K}{S_t}\right)-\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)}$$

$$= e^{r(T-t)}\frac{\Phi(d_1)}{\Phi(d_2)}$$

where

$$d_2 = \frac{\ln\frac{S_t}{K} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

Taken together, the first term in our option pricing equation is the discount factor $\times$ the conditional probability $\times$ the marginal probability of being in-the-money:

$$\begin{aligned}
e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[S_T 1\left\{S_T \geq K\right\}\right] &= e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[S_T \mid S_T \geq K\right]\Phi(d_2) \\
&= e^{-r(T-t)}S_t\mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_T}{S_t} \mid \frac{S_T}{S_t} \geq \frac{K}{S_t}\right]\Phi(d_2) \\
&= e^{-r(T-t)}S_t e^{r(T-t)}\frac{\Phi(d_1)}{\Phi(d_2)}\Phi(d_2) \\
&= S_t\Phi(d_1)
\end{aligned}$$

3.4 Bringing Everything Together

So now have all the ingredients. Options are risk-neutral expectations of the terminal payoff, discounted at the risk free rate:

$$V_t = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[(S_T - K)^+\right]$$

The undiscounted expectation of a maximum is the expectation of the union of two events: (1) the option ending up in the money and (2) the spread of the stock price above the strike price. In a slight of abuse of notation, but one that perhaps clarifies the intutition, we have that the time T expected payoff is:

$$\begin{aligned}
\text{Expected Payoff} &= \text{Expected}\left((\text{Stock Price} - \text{Strike Price})\,\text{and}\,(\text{In the Money})\right) \\
&= \text{Expected}\left((\text{Stock Price} - \text{Strike Price}) \mid \text{In the Money}\right) \times \Pr(\text{In the Money}) \\
&= \text{Expected}\left(\text{Stock Price} \mid \text{In the Money}\right) \times \Pr(\text{In the Money}) - \text{Strike Price} \times \Pr(\text{In the Money})
\end{aligned}$$

Mathematically, we express this as:

$$V_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+\right] = \mathbb{E}_t^{\mathbb{Q}}\left[S_T 1\{S_T \geq K\}\right] - K\mathbb{E}_t^{\mathbb{Q}}\left[1\{S_T \geq K\}\right]$$

$$= e^{-r(T-t)}\left[\underbrace{\left(S_t e^{r(T-t)}\frac{\Phi(d_1)}{\Phi(d_2)}\right)}_{\mathbb{E}_t^{\mathbb{Q}}[S_T|S_T \geq K]}\underbrace{\Phi(d_2)}_{\Pr(S_T \geq K)} - K\underbrace{\Phi(d_2)}_{\Pr(S_T \geq K)}\right]$$

or, using the notiation above, where ITM means "In-the-Money",

$$V_t = \underbrace{e^{-r(T-t)}}_{\text{Discounting}}\left[\underbrace{\left(S_t e^{r(T-t)}\frac{\Phi(d_1)}{\Phi(d_2)}\right)}_{\text{Expected stock price given ITM}}\underbrace{\Phi(d_2)}_{\text{Pr ITM}} - \underbrace{K}_{\text{Strike}}\underbrace{\Phi(d_2)}_{\text{Pr ITM}}\right]$$

It is now easy to see that $\Phi(d_2)$ is precisely the risk-neutral probability that the option ends in the money. $\Phi(d_1)$ is slightly less intuitive; it simply arises as the numerator in the conditonal expectation of a log normal distribution. The normal CDFs are present because they convert the random distribution of the terminal stock price into conditional probabilities and conditional expectations under the risk-neutral measure. The first term is not discounted, because the discount rate cancels out with terms from the conditional expectation of a log-normal distirbution. The $\pm\frac{1}{2}\sigma^2$ in the expressions for $d_1$and d_2 arise from the convexity term when taking expectations of a log-normally distributed random variable.

4 Remarks

- Girsanov's theorem shows the relationship between the stock dynamics under the risk-neutral measure and under the physical measure. Under $\mathbb{P}$and $\mathbb{Q}$respectively,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^{\mathbb{P}}$$
$$\frac{dS_t}{S_t} = r dt + \sigma dW_t^{\mathbb{Q}}$$

  Matching like terms and rearranging, we have:

$$\mu dt + \sigma dW_t^{\mathbb{P}} = r dt + \sigma dW_t^{\mathbb{Q}}$$
$$dW_t^{\mathbb{P}} = dW_t^{\mathbb{Q}} - \frac{\mu - r}{\sigma}dt$$

  This illustrates that the Brownian motion under the different measures are not mean 0. Under $\mathbb{P}$,for example, $\mathbb{E}^{\mathbb{P}}\left[dW_t^{\mathbb{Q}}\right] = \frac{\mu-r}{\sigma}dt \geq 0$.

- The relationship between risk-neutral and physical measures explains the Monte Carlo approach to pricing an option. Since the *variance* of the Brownian motion is the same under both measures, one can estimate σ from the data (i.e., under $\mathbb{P}$) and then use that σ to simulate the stock price dynamics under $\mathbb{Q}$:

$$\frac{dS_t}{S_t} = r dt + \sigma dW_t$$

from some starting price S_t. The average terminal payoff across simulated paths – $\frac{1}{N}\sum_{i=1}^{N}(S_T - K)^+$– can then be discounted at the risk free rate.

- We can prove that the (instantaneous) volatility of an option is greater than the (instantaneous) volatility of the underlying. By Ito's Lemma,

$$\begin{aligned}
dV_t(t, S_t) &= \frac{\partial V_t}{\partial t}dt + \frac{\partial V_t}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}(dS_t)^2 \\
&= -\Theta dt + \Delta_t dS_t + \frac{1}{2}\Gamma_t S_t^2 \sigma^2 dt \\
&= -\Theta dt + \Delta_t(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}\Gamma_t S_t^2\sigma^2 dt \\
&= -\Theta dt + \Delta_t \mu S_t dt + \frac{1}{2}\Gamma_t S_t^2\sigma^2 dt + \Delta_t \sigma S_t dW_t \\
\frac{dV_t}{V_t} &= \frac{\left(-\Theta + \Delta_t\mu S_t + \frac{1}{2}\Gamma_t S_t^2\sigma^2\right)dt}{V_t} + \underbrace{\frac{\Delta_t S_t}{V_t}}_{:=\lambda}\sigma dW_t \\
&= \frac{\left(-\Theta + \Delta_t\mu S_t + \frac{1}{2}\Gamma_t S_t^2\sigma^2\right)dt}{V_t} + \lambda\sigma dW_t \\
Var\left(\frac{dV_t}{V_t}\right) &= \lambda^2\sigma^2 dt \\
Var\left(\frac{dS_t}{S_t}\right) &= \sigma^2 dt
\end{aligned}$$

the embedded leverage of an option, λ, is given by:

$$\begin{aligned}
\lambda = \Delta\frac{S}{V} &= \frac{\Phi(d_1)}{\Phi(d_1)S_t - \Phi(d_2)Ke^{-r(T-t)}}S_t \\
&= \frac{\Phi(d_1)}{\Phi(d_1) - \Phi(d_2)\frac{K}{S_t}e^{-r(T-t)}}
\end{aligned}$$

where in the first line, $\Delta = \Phi(d_1)$, and in the second line we divided through by S_t. Note that when $K = 0$, the embedded leverage is 1. When $K > 0$, it is clear than since

$$\Phi(d_2)\frac{K}{S_t}e^{-r(T-t)} > 0$$

(since $S_t > 0$, $K > 0$, $\Phi(\cdot) > 0$), hence $K > 0 \iff \lambda > 1$. So embedded leverage is strictly greater than or equal to 1, $\lambda \geq 1$ and so:

$$Var\left(\frac{dV_t}{V_t}\right) = \lambda^2\sigma^2 dt \geq \sigma^2 dt = Var\left(\frac{dS_t}{S_t}\right)$$