Testing

Floating Point

Still looking at fields in Saskatchewan...

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Fields				
Corner 1		Corner 2		Crop
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮

Still looking at fields in Saskatchewan...

Latitude/longitude

Fields				
Corner 1		Corner 2		Crop
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
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Latitude/longitude

Floating point numbers

Fields				
Corner 1		Corner 2		Crop
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮

Still looking at fields in Saskatchewan...

Latitude/longitude

Floating point numbers

That's when trouble starts

Fields				
Corner 1		Corner 2		Crop
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮

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What follows is (over-)simplified

Goldberg (1991): "What Every Computer Scientist Should Know About Floating-Point Arithmetic"

Use sign, magnitude, and exponent

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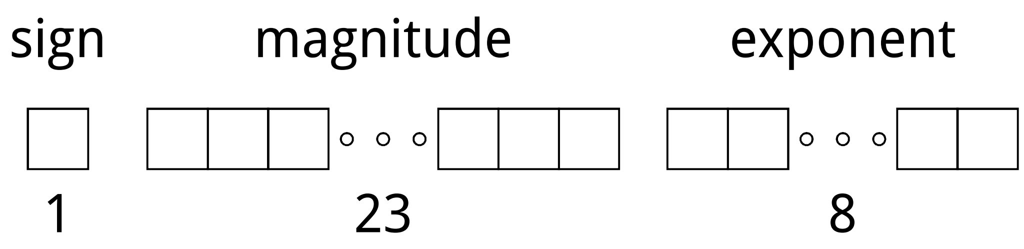
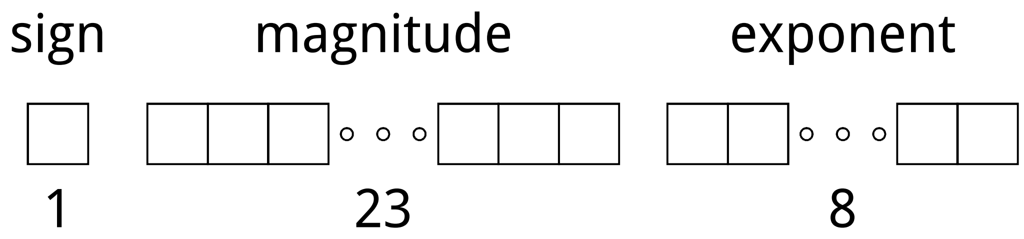
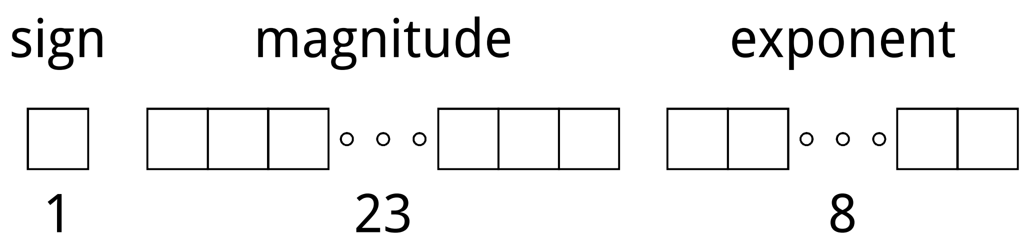


Use sign, magnitude, and exponent



To illustrate problems, we'll use a simpler format

Use sign, magnitude, and exponent



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And only positive values without fractions

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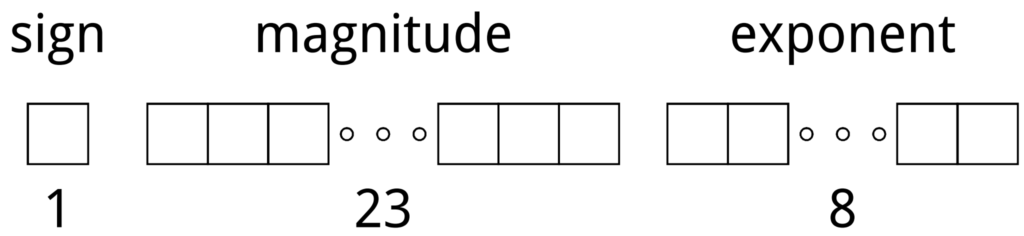

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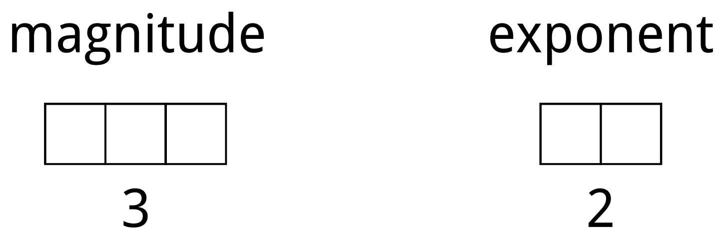

Possible values

		Exponent		
Mantissa	00	01	10	11
000	0	0	0	0
001	1	2	4	8
010	2	4	8	16
011	3	6	12	24
100	4	8	16	32
101	5	10	20	40
110	6	12	24	48
101	7	14	28	56

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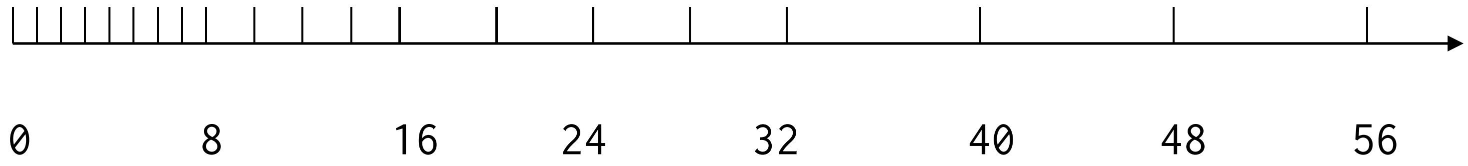
Actual representation doesn't have redundancy

A clearer view of those values



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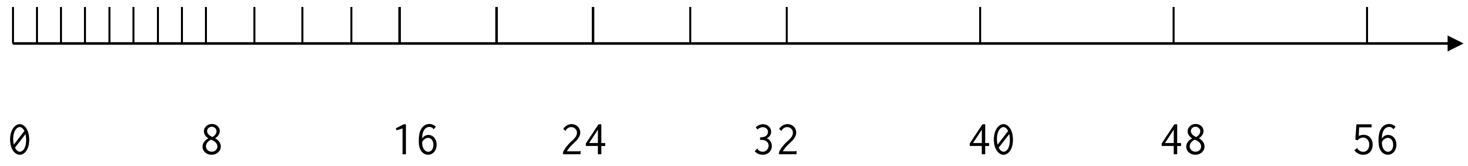

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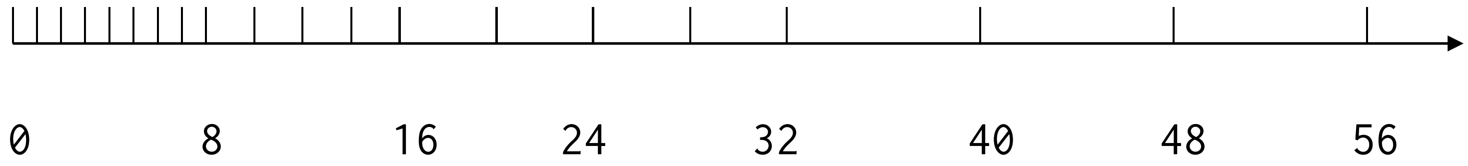

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Just as 1/3 must be 0.3333 or 0.3334 in decimal

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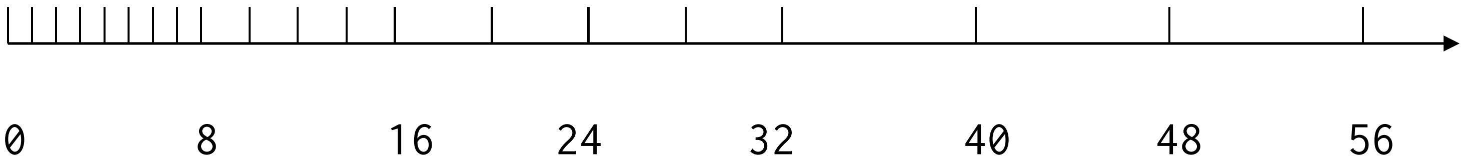

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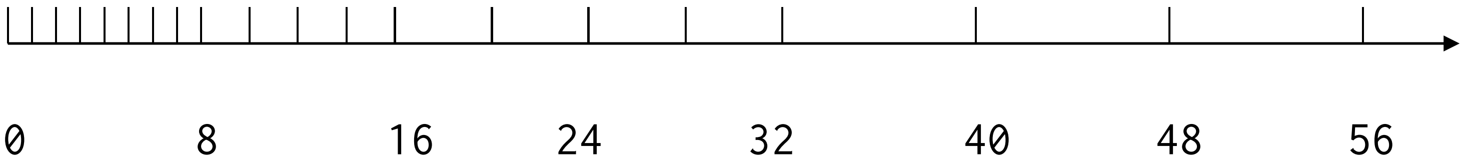

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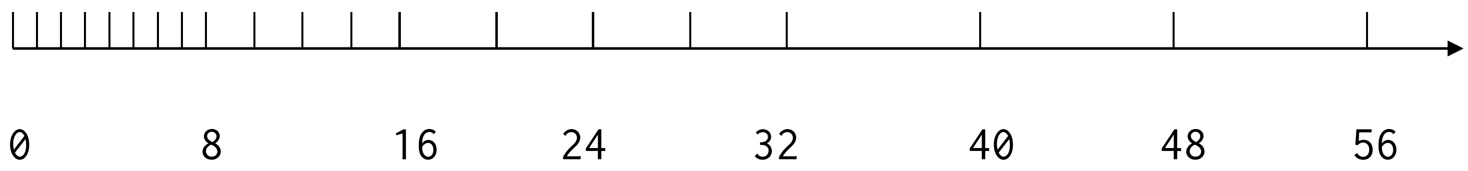

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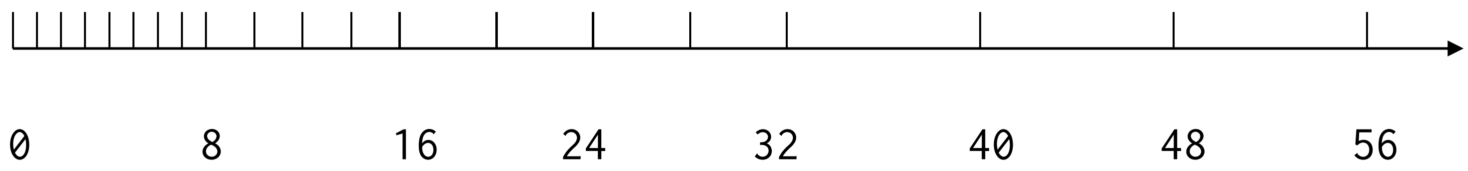

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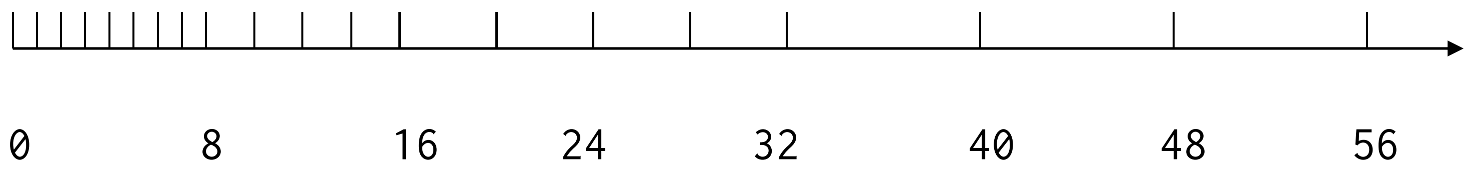

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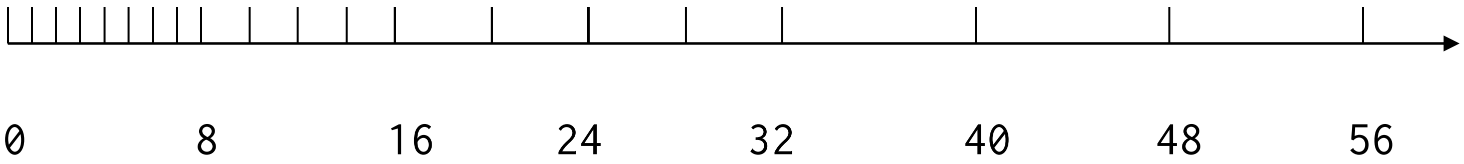

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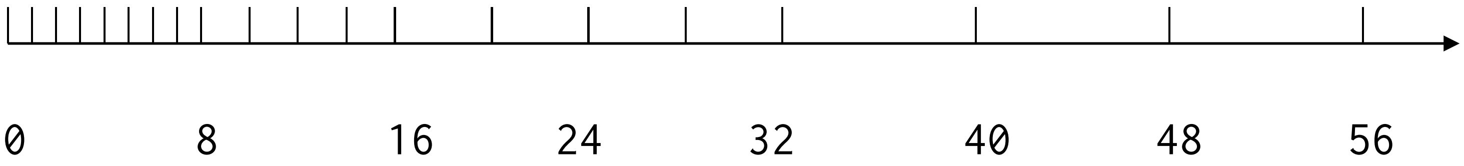
"Sort then sum" would give the same answer...

Another observation



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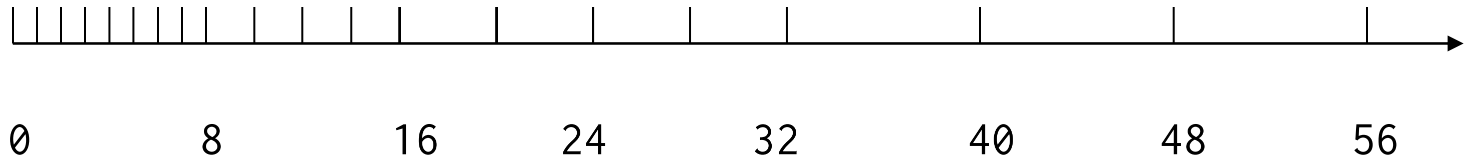

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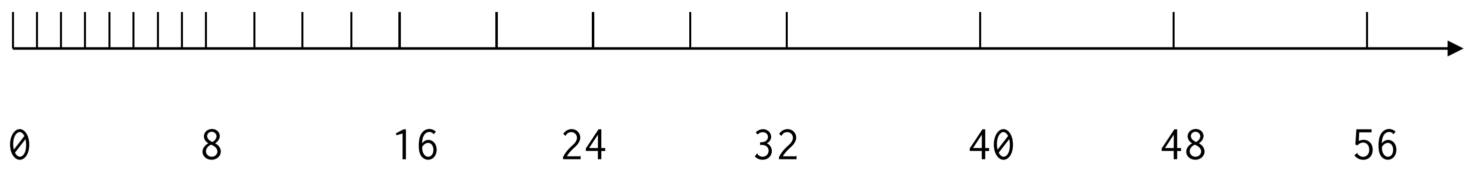

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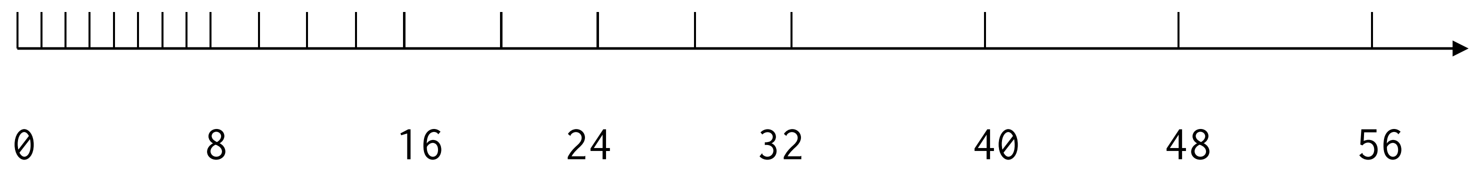

Spacing is uneven

But *relative* spacing stays the same

Because we're multiplying the same few mantissas

by ever-larger exponents

Absolute error = |val – approx|

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Relative error = |val – approx| / val

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Operation	Actual Result	Intended Result	Absolute Error	Relative Error
8+1	8	9	1	1/9 (11.11%)
56+1	56	57	1	1/57 (1.75%)

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Relative error is more useful

Makes little sense to say "off by 0.01" if the value you're approximating is 0.000000001

What does this have to do with testing?

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```
vals = []
for i in range(1, 10):
  number = 9.0 * 10.0 ** -i
  vals.append(number)
  total = sum(vals)
  expected = 1.0 - (1.0 * 10.0 ** i)
  diff = total - expected
  print '%2d  %22.21f  %22.21f' % \
        (i, total, total-expected)
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i = 1, 2, 3, ..., 9

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0.9, 0.09, 0.009, ...

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But is it?

0.9, 0.99, 0.999, ...

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1-0.1, 1-0.01, ...

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Should also make

0.9, 0.99, ...

1-0.1, 1-0.01, ...

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        (i, total, total-expected)
```

Check and print

And the answer is:

```
1  0.90000000000000022204  0.00000000000000000000
2  0.98999999999999991118  0.00000000000000000000
3  0.99899999999999999112  0.00000000000000000000
4  0.99990000000000001101  0.00000000000000000000
5  0.99999000000000004551  0.00000000000000000000
6  0.99999900000000008227  0.00000000000000111022
7  0.99999990000000005264  0.00000000000000000000
8  0.99999999000000006077  0.00000000000000111022
9  0.99999999900000002828  0.00000000000000000000
```

And the answer is:

1	0.9000000000000000022204	0.0000000000000000000000
2	0.9899999999999999991118	0.0000000000000000000000
3	0.9989999999999999999112	0.0000000000000000000000
4	0.99990000000 [illegible]	[illegible] 00000000000000
5	0.999990000000 [illegible]	[illegible] 00000000000000
6	0.9999990000000000082267	0.0000000000000000111022
7	0.9999999000000000052636	0.0000000000000000000000
8	0.9999999900000000060775	0.0000000000000000111022
9	0.9999999990000000028282	0.0000000000000000000000

Already slightly off

And the answer is:

1	0.90000000000000022204	0.00000000000000000000
2	0.98999999999999991118	0.00000000000000000000
3	0.99899999999999999112	0.00000000000000000000
4	0.999	00000
5	0.999	00000
6	0.99999900000000082267	0.00000000000000111022
7	0.99999990000000052636	0.00000000000000000000
8	0.99999999000000060775	0.00000000000000111022
9	0.99999999900000028282	0.00000000000000000000

But at least they're consistent

And the answer is:

1	0.900000000000000022204	0.000000000000000000000
2	0.989999999999999991118	0.000000000000000000000
3	0.998999999999999999112	0.000000000000000000000
4	0.999900000000000011013	0.000000000000000000000
5	0.999990000000000045510	0.000000000000000000000
6	0.999999000000000082267	0.000000000000000111022
7	0.999999900000000052636	0.000000000000000000000
8	0.999999990000000060775	0.000000000000000111022
9	0.999999999000000028282	0.000000000000000000000

Sometimes, they're not

And the answer is:

1	0.9000000000000000022204	0.0000000000000000000000
2	0.9899999999999999991118	0.0000000000000000000000
3	0.9989999999999999999112	0.0000000000000000000000
4	0.9999000000000000011013	0.0000000000000000000000
5	0.9999900000000000045510	0.0000000000000000000000
6	0.9999990000000000082267	0.0000000000000000111022
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Sometimes errors cancel out later

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No one has a good answer

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Fail if the two objects are unequal as determined by
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Is that absolute or relative?

created by

Greg Wilson

August 2010