04.11.2023

Introduce the screened potential $v_{sc}(q,\omega)$ modified (compared to $v_{ext}(q,\omega)$) by the medium polarization, and dielectric function $\varepsilon(q,\omega)$:

$$\left.\begin{aligned}\delta n(\bar{q},\omega) &= \Pi(q,\omega)\, v_{ext}(\bar{q},\omega)\\ \delta n(\bar{q},\omega) &= \Pi_0(q,\omega)\, v_{sc}(q,\omega)\end{aligned}\right\} \rightarrow \boxed{v_{sc}(q,\omega) = \frac{v_{ext}(q,\omega)}{\varepsilon(q,\omega)}}$$

definition of $\varepsilon(q,\omega)$ in terms of $v_{sc}(q,\omega)$

these two eqs. define $v_{sc}$ in terms of $\Pi$ and $\Pi_0$

These defs. do not rely on approximations.

$$\boxed{\frac{1}{\varepsilon(q,\omega)} \underset{\substack{\text{def}\\ \text{in terms}\\ \Pi \text{ and } \Pi_0}}{=} \frac{\Pi(q,\omega)}{\Pi_0(q,\omega)} \underset{\substack{\text{in RPA}\\ \text{approx.}}}{=} \frac{1}{1-\Pi_0(q,\omega)V_q} = 1 + \Pi^{RPA}(q,\omega)V_q}$$

## 6.1. Static limit, screening in RPA approximation

$$\omega = 0 \Rightarrow \operatorname{Re}\Pi_0(q,0)\Big|_{q\to 0} = \operatorname{Re}\underset{\substack{\uparrow\\ \text{(non-interact.)}}}{\Pi_R}(q,0)\Big|_{q\to 0} = -\nu(E_F)$$

(as derived earlier)

$(q \ll 2k_F)$

$$\Pi^{RPA}(q,0) = \frac{-\nu(E_F)}{1+\nu(E_F)\cdot 4\pi e^2/q^2}, \qquad q \ll 2k_F$$

$$V_{sc}(q) = \frac{V_q}{1-\Pi_0(q,0)V_q} = \frac{4\pi e^2}{q^2+4\pi e^2\nu(E_F)}; \qquad r_{TF}^{-2} \equiv 4\pi e^2 \nu(E_F)$$

Thomas–Fermi screening radius (Debye screening radius)

$$V_{sc}(q) = \frac{4\pi e^2}{q^2 + r_{TF}^{-2}}$$

Note that $V_{sc}(q\to 0) = 1/\nu(E_F)$ independent of $e$ ("unitary limit")

Upon Fourier transform: $\boxed{V_{sc}(r) \propto \frac{1}{r} e^{-r/r_{TF}}}$

More detail: $V_{sc}(q) = \dfrac{V_q}{1-\Pi_0(q,0)V_q}$ is applicable at $q\to 0$.

At $q \to 2k_F$, the product $\Pi_0(q,0)V(q) \ll 1 \Rightarrow$ ordinary 1st order perturbation theory for $\Pi(q,0)$ is OK in order to find $V_{sc}(q)$ up to 2-nd order in $V_q$. One of the 1st order terms is $V_{sc}(q) = V_q - \Pi_0(q,0)V_q^2 \approx V_q/(1-\Pi_0(q,0)V_q)$.

Consider now

$$V_{sc}(r) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\vec{r}} V_{sc}(q) = \int_0^\infty \frac{q^2 dq}{(2\pi)^2} \cdot \frac{2\sin qr}{qr} \cdot V_{sc}(q)$$

$(r \geq 0)$

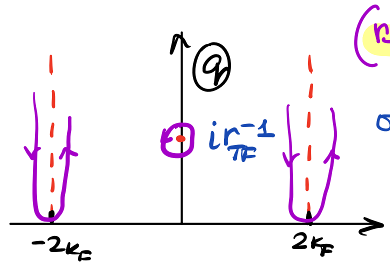
$$= \frac{1}{2} \Im m \int_{-\infty}^{\infty} \frac{q\,dq}{(2\pi)^2} \cdot \frac{e^{iqr}}{r} \cdot \frac{V_q}{1-\Pi_0(q,0)V_q}$$

(used that $\Pi_0(\vec{q},0)$ even, defined $\Pi(-q)=\Pi(q)$)

and use complex variable calculus to evaluate $\int dq\, e^{iqr} F(q)$ with $F(q) = \dfrac{V_q}{1-\Pi_0(q,0)V_q}$ being an analytical function in the $\Im m\, q > 0$ half-plane with a pole at $q_{pole} = i r_{TF}^{-1}$ and a cut at $q = \pm 2k_F$.

(recall the $\Theta(\omega - \omega_{min}(q))$ in $\Pi_0(q,\omega)$)

Upon the deformation of the integration contour:




one obtains the two leading terms at $k_F r \gg 1$: $V_{sc}(r) = V_{TF}(r) + V_{Friedel}(r)$

$V_{TF}(r) = \dfrac{1}{r} e^{-r/r_{TF}}$ (the pole contribution);

$V_{Friedel}(r) \propto \underbrace{\dfrac{e^2}{\hbar v_F} \cdot \dfrac{1}{(k_F r)^3}}_{\text{small at } r\sim r_D} \cos(2k_F r)$ (the Friedel oscillation part comes from the contribution of cuts)

Again, (the origin of the non-analyticity leading to the cut: recall the threshold behavior of $\Pi_R(q,\omega)$:

$$-\mathrm{Re}\ \Pi_R(\vec{q},\omega) \propto (\omega_{max}(q)-\omega)\ln(\omega_{max}(q)-\omega)$$

and set $\omega = 0$. Then $\ln(\omega_{max}(q)-\omega)\big|_{\omega=0} = \underbrace{\ln(q-2k_F)}_{\text{analytical on a plane with a cut.}} + \text{const}$

## 6.2 Plasma oscillations

$$\Pi^{RPA}(q,\omega) = \frac{\Pi_0(q,\omega)}{1-\Pi_0(q,\omega)V_q}$$

$\omega \gg q v_F$, $q \ll 2k_F$

$$\Pi^{RPA}(q,\omega) = \frac{(n/m)(q/\omega)^2\left[1+O\left(\frac{qv_F}{\omega}\right)^2\right]}{1-\underbrace{\frac{4\pi e^2}{\omega^2}\cdot\frac{n}{m}}\cdot\left[1+O\left(\frac{qv_F}{\omega}\right)^2\right]+\underset{\smile}{i0}}$$

(used : $V_q = \frac{4\pi e^2}{q^2}$)

retarded response function

$$\boxed{\omega_p^2 = \frac{4\pi e^2 n}{m}}$$

$\omega_p$ : Plasma oscillations frequency

With an additional account for $O\left(\frac{qv_F}{\omega}\right)^2$:

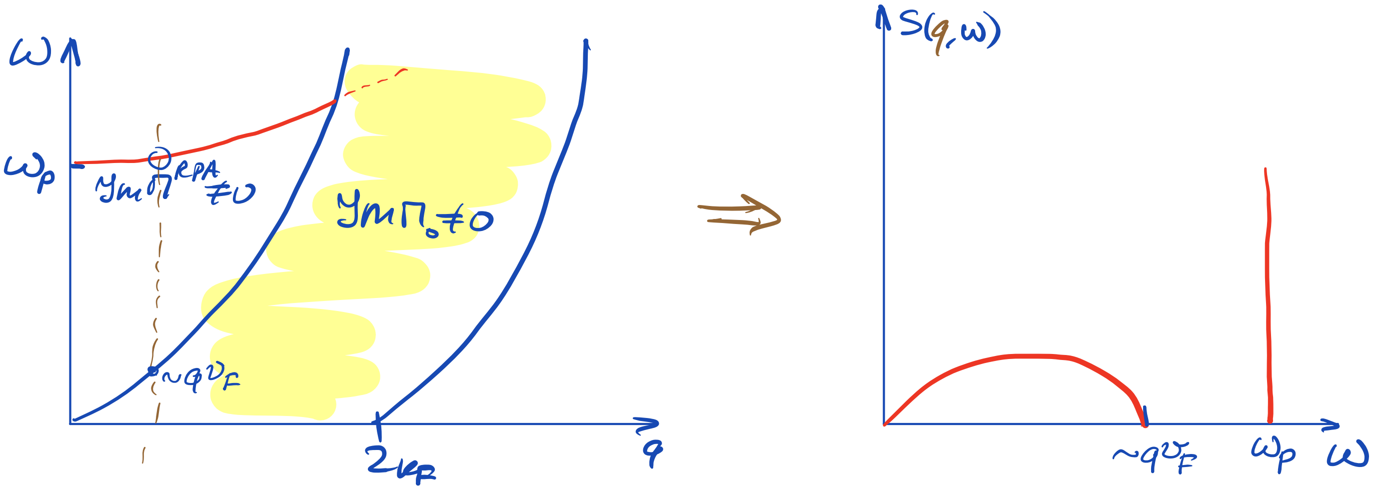
$$\omega_p(q) = \omega_p\left(1+\frac{3}{5}\left(\frac{qv_F}{\omega_p}\right)^2\right)$$

restored coefficient in place of $O(\dots)$.

Keeping the term $\sim (qv_F/\omega_p)^2$ is not exceeding the accuracy of the RPA approximation at $q \ll r_{TF}^{-1}$

Indeed, $(qv_F/\omega_p)^2 \sim (\hbar v_F/e^2)\cdot(q/k_F)^2$. At $q \ll r_{TF}^{-1}$ we find $\left(\frac{qv_F}{\omega_p}\right)^2 \ll 1$, small compared to the first term in the RHS of eq. for $\omega_p(q)$.

Structure factor with an account for plasma oscillations mode




Further reading (Lindhard function, Friedel oscillations, RPA, plasmons): Sec. 4.4 and Sec. 5.3 of G. Giuliani and G. Vignale, Quantum Theory of the Electron Liquid, Cambridge U. Press, 2005.

Complementary treatment: TF screening and plasma osc at $q \to 0$ from continuous medium approx.

Displacement of a unit volume: $\vec{u}(\vec{r}, t)$

Small density perturbation (away from a uniform $n$): $\delta n = -n\,\mathrm{div}\,\vec{u}$

Kinetic energy density of the medium: $K = n\frac{m}{2}(\dot{\vec{u}})^2$

Potential energy density $=$ $\mathcal{U} = \frac{1}{2}\frac{\partial \mu}{\partial n}\underbrace{(\delta n)^2}_{\text{deviation from eq. density}} - \delta n \cdot e\varphi_{ext}$

Lagrangian density:

$$\mathcal{L} = K - \mathcal{U} = n\frac{m}{2}(\dot{\vec{u}})^2 - \frac{1}{2}\frac{\partial \mu}{\partial n}(\delta n)^2 + e\varphi_{ext}(\vec{r})\,\delta n$$

$$= n\frac{m}{2}\left(\frac{d\vec{u}}{dt}\right)^2 - \frac{1}{2}\frac{\partial \mu}{\partial n}n^2(\mathrm{div}\,\vec{u})^2 - en\,\varphi_{ext}(\vec{r})\,\mathrm{div}\,\vec{u}$$

RPA approx. (at $q \to 0$): equivalent to replacing $\frac{\partial \mu}{\partial n} \to \left.\frac{\partial \mu}{\partial n}\right|_{\substack{\text{free particles}\\(V=0)}} \overset{T \ll E_F}{=} 1/\nu(E_F)$

(see also the eqs. below)

and include $e\,\delta n$ in the Poisson equation that defines $\varphi$: $\varphi_{ext}(r) \to \varphi(\vec{r})$

$$\left\{\begin{array}{l} \nabla^2\varphi = 4\pi e(\delta n + n^{ext}) \\ \frac{d}{dt}\left(\frac{\delta \mathcal{L}}{\delta \dot{\vec{u}}}\right) - \frac{\delta \mathcal{L}}{\delta \vec{u}} = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} \nabla^2\varphi = -4\pi e n\,\mathrm{div}\,\vec{u} + 4\pi e n^{ext} \\ nm\frac{d^2\vec{u}}{dt^2} - \frac{n^2}{\nu(E_F)}\,\mathrm{grad}(\mathrm{div}\,\vec{u}) - en\vec{\nabla}\varphi = 0 \end{array}\right.$$

eq. of motion

take div...

$$-m\frac{d^2}{dt^2}(n\,\mathrm{div}\,\vec{u}) + \frac{n}{\nu(E_F)}\nabla^2(n\,\mathrm{div}\,\vec{u}) + en\nabla^2\varphi = 0$$

use Poisson eq. here

$$\nabla^2\varphi = -4\pi e n\,\mathrm{div}\,\vec{u} + 4\pi e n^{ext}$$

$$\boxed{m\frac{\partial^2}{\partial t^2}(\delta n) - \frac{n}{\nu(E_F)}\nabla^2(\delta n) + 4\pi e^2 n\,\delta n + 4\pi e^2 n \cdot n^{ext} = 0}$$

defines the response of $\delta n(r,t)$ to $n^{ext}(r,t)$

Static limit $(d(\delta n)/dt = 0)$

$$\nabla^2 \delta n - \underbrace{4\pi e^2 \nu(E_F)}\,\delta n = 4\pi e^2 \nu(E_F)\, n^{ext}$$

$$\nabla^2 \delta n - \frac{1}{r_{TF}^2}\,\delta n = 4\pi e^2 \nu(E_F)\, n^{ext}$$

$$\delta n_q = \frac{-\nu(E_F)}{1+\nu(E_F)\cdot\frac{4\pi e^2}{q^2}} \cdot \frac{4\pi e^2}{q^2}\, n_q^{ext} \Rightarrow \boxed{V_{sc}(q) = \frac{4\pi e^2}{q^2 + r_{TF}^{-2}}}$$

Spatially-uniform perturbation $\left(\frac{\partial}{\partial x}\delta n = 0\right)$

$$\frac{\partial^2}{\partial t^2}(\delta n) + \frac{4\pi e^2 n}{m}(\delta n) = -\frac{4\pi e^2 n}{m}\, n^{ext}$$

$$\boxed{\omega_p^2 = \frac{4\pi n e^2}{m}}$$

The continuous-medium approx. is convenient for a treatment of a multi-component plasma (+ jellium model of a metal, see e.g. Pines, Elementary excitations in solids, Benjamin inc. 1964).

# 7. Mean-field theory

## 7.1. Variational Method

We will show that

$$\Omega \le \Omega'_{trial}$$

where $\Omega = -T \ln Z$, $Z = \mathrm{Tr}\, e^{-\beta \hat{\mathcal{H}}}$, $\Omega'_{trial} = -T \ln Z'_{trial}$

with

$$Z'_{trial} \equiv \sum_n e^{-\beta \mathcal{H}_{nn}}, \quad \mathcal{H}_{nn} = \langle n|\mathcal{H}|n\rangle \quad \text{and}$$

$|n\rangle$ being eigenstates of some "suitable" Hamiltonian $\mathcal{H}_0$:

$\mathcal{H}_0 |n\rangle = E_n |n\rangle$ (but $|n\rangle$ is <u>not</u> necessarily an eigenstate of $\mathcal{H}$)

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Proof:

Let us write $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{V}$. Then

$$Z = \mathrm{Tr}\, e^{-\beta(\hat{\mathcal{H}}_0 + \hat{V})} = \sum_n \langle n| e^{-\beta(\mathcal{H}_0 + V)} |n\rangle$$ (complete basis of $|n\rangle$)

$$Z'_{trial} \equiv \sum_n e^{-\beta\langle n|\mathcal{H}_0 + V|n\rangle} = \sum_n e^{-\beta E_n} \cdot e^{-\beta V_{nn}}$$

Let us prove that $\Omega \le \Omega'_{trial}$. For that, we need to prove: $Z \ge Z'_{trial}$. It is sufficient to prove that

$$\langle n| e^{-\beta(\mathcal{H}_0 + V)} |n\rangle \ge e^{-\beta E_n} \cdot e^{-\beta V_{nn}}$$

for each term $n$ in the respective sums.

Indeed, define $g(\beta) = \langle n| e^{-\beta\hat{O}} |n\rangle$

with $\hat{O} = \hat{\mathcal{H}}_0 + \hat{V} - \langle n| \hat{\mathcal{H}}_0 + \hat{V} |n\rangle$

Then: $g(0) = 1$ and

insert $\sum_M |M\rangle\langle M|$ $\qquad$ $\sum_{M_1} |M_1\rangle\langle M_1|$

$$\frac{dg}{d\beta} = -\langle n| \hat{O} e^{-\beta\hat{O}} |n\rangle = -\langle n| \hat{O} [e^{-\beta\hat{O}} - 1] |n\rangle, \text{ as } \langle n|\hat{O}|n\rangle = 0$$

Now <u>insert</u> a complete set $|M\rangle$ of $\hat{\mathcal{H}}$ eigenstates ($|M\rangle$ is also an eigenstate of $\hat{O}$: $\hat{O}|M\rangle = \mathcal{E}_M |M\rangle$):

$$\frac{dg}{d\beta} = -\sum_M \underbrace{|\langle n|M\rangle|^2}_{\ge 0} \underbrace{\mathcal{E}_M [e^{-\beta\mathcal{E}_M} - 1]}_{\text{always } \le 0} \ge 0$$

$$g(0) = 1,\ \frac{dg}{d\beta} \ge 0 \Rightarrow g(\beta) \ge 1 \Rightarrow \langle n| e^{-\beta(\hat{\mathcal{H}}_0 + \hat{V})} |n\rangle \cdot e^{\beta(\langle n|\hat{\mathcal{H}}_0 + \hat{V}|n\rangle)} \ge 1$$

$$\Rightarrow \langle n| e^{-\beta(\hat{\mathcal{H}}_0 + \hat{V})} |n\rangle \ge e^{-\beta(\langle n|\hat{\mathcal{H}}_0 + \hat{V}|n\rangle)}$$

So we got $Z \geq Z'_{trial}$, or:

$$-T \ln Z \leq -T \ln \sum_n e^{-\beta \langle n | \hat{\mathcal{H}} | n \rangle} \iff \Omega \leq \Omega'_{trial}$$

If $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{V}$ with "small" $\hat{V}$, then expand the RHS in $\hat{V}$ to obtain

$$\Omega \leq \Omega_0 + \langle \hat{V} \rangle_0 , \quad \text{where} \quad \langle \hat{V} \rangle_0 = \frac{\mathrm{Tr}\, \hat{V} e^{-\beta \mathcal{H}_0}}{\mathrm{Tr}\, e^{-\beta \mathcal{H}_0}} ; \quad \Omega_0 = -T \ln e^{-\beta \mathcal{H}_0}$$

Turns out that

$$\boxed{\Omega \leq \Omega_0 + \langle \mathcal{H} - \mathcal{H}_0 \rangle_0 \equiv \Omega_{trial}}$$

holds even without the smallness of $\langle \hat{\mathcal{H}} - \hat{\mathcal{H}}_0 \rangle$ requirement. (Feynman, Stat. Mech., Benjamin Press 1972; original proof: Bogoliubov 1956; good text: Tiablikov, Methods in Quantum Theory of Magnetism, Ch.4 - Springer 1967)

## 7.1 a. Self energy in Hartree-Fock approximation

Consider spinless fermions with interaction

$$\hat{\mathcal{H}} = \hat{T} + \hat{V} ; \qquad \hat{T} = \sum_{\vec{k}} (\varepsilon(\vec{k}) - \mu) c^+_{\vec{k}} c_{\vec{k}} ; \quad \hat{V} = \frac{1}{2} \sum_{\vec{k}\vec{p}\vec{q}} v(q) c^+_{\vec{k}+\vec{q}} c^+_{\vec{p}-\vec{q}} c_{\vec{p}} c_{\vec{k}}$$

A general soluble $\hat{\mathcal{H}}_0$:

$$\hat{\mathcal{H}}_0 = \sum_{\vec{k}\vec{k}'} \Lambda_{\vec{k}\vec{k}'} c^+_{\vec{k}} c_{\vec{k}'} + \Gamma_{\vec{k}\vec{k}'} c^+_{\vec{k}} c^+_{\vec{k}'} + \Gamma^*_{\vec{k}\vec{k}'} c_{\vec{k}'} c_{\vec{k}}$$

(matrices $\Lambda$ and $\Gamma$ are hermitian)

If we want $\hat{N} = \sum_{\vec{k}} c^+_{\vec{k}} c_{\vec{k}}$ and $\hat{P} = \sum_{\vec{k}} \hbar \vec{k}\, c^+_{\vec{k}} c_{\vec{k}}$ to be conserved

then we need to demand:

$$[\hat{N}, \hat{\mathcal{H}}_0] = 0, \quad [\hat{P}, \hat{\mathcal{H}}_0] = 0 \Rightarrow \Gamma = 0, \quad \Lambda \propto \delta_{\vec{k}\vec{k}'}$$

So, here is the form of $\mathcal{H}_0$ satisfying the cons. laws:

$$\mathcal{H}_0 = \sum_{k} \left( \varepsilon(\vec{k}) + \Sigma(\vec{k}) - \mu \right) c^+_{\vec{k}} c_{\vec{k}}$$

$\Sigma(\bar{k})$ is called self energy; we will find it in Hartree-Fock approximation.

The grand canonical potential for $\hat{\mathcal{H}}_0$:

$$\beta\Omega_0 = -\sum_{\bar{k}} \ln\left[1 + e^{-\beta(\varepsilon(\bar{k})+\Sigma(\bar{k})-\mu)}\right]$$

(recall free fermions stat. mech.)

$$\langle \mathcal{H}_0 \rangle_0 = \frac{\partial(\beta\Omega_0)}{\partial\beta} = \sum_{\bar{k}} (\varepsilon(\bar{k})+\Sigma(\bar{k})-\mu)\, n_F(\varepsilon(\bar{k})+\Sigma(\bar{k})-\mu)$$

$$\langle T \rangle_0 = \sum_{k} (\varepsilon(\bar{k})-\mu)\, n_F(\varepsilon(\bar{k})+\Sigma(\bar{k})-\mu)$$

$$\langle V \rangle_0 = \frac{1}{2}\sum_{\bar{k}\bar{p}\bar{q}} v(\bar{q}) \langle c^+_{\bar{k}+\bar{q}}\, c^+_{\bar{p}-\bar{q}}\, c_{\bar{p}}\, c_{\bar{k}} \rangle_0$$

$\bar{q}=0$

$\bar{k}+\bar{q}=\bar{p}$

use Wick's theorem as $\mathcal{H}_0$ is bilinear

$$\langle V \rangle_0 = \frac{1}{2} v(0) \sum_{\bar{k}\bar{p}} n_F(\varepsilon(\bar{k})+\Sigma(\bar{k})-\mu)\, n_F(\varepsilon(\bar{p})+\Sigma(\bar{p})-\mu)$$

$$- \frac{1}{2}\sum_{\bar{k},\bar{q}} v(\bar{q})\, n_F(\varepsilon(\bar{k}+\bar{q})+\Sigma(\bar{k}+\bar{q})-\mu)\, n_F(\varepsilon(\bar{k})+\Sigma(\bar{k})-\mu)$$

<u>04.14.22</u>

$$\Omega_{trial} = \Omega_0 + \langle \mathcal{H} - \mathcal{H}_0 \rangle_0$$

$$= \sum_{\bar{k}} \Big\{ -\frac{1}{\beta} \ln\left[1 + e^{-\beta(\varepsilon(\bar{k})+\Sigma(\bar{k})-\mu)}\right] \underbrace{- \Sigma(\bar{k})\, n_F(\varepsilon(\bar{k})+\Sigma(\bar{k})-\mu)}_{\langle T - \mathcal{H}_0\rangle_0} \Big\}$$

$$+ \frac{1}{2} v(0) \sum_{\bar{k}\bar{p}} n_F(\varepsilon(\bar{k})+\Sigma(\bar{k})-\mu)\, n_F(\varepsilon(\bar{p})+\Sigma(\bar{p})-\mu) \leftarrow \langle V \rangle_0$$

$$- \frac{1}{2}\sum_{\bar{k},\bar{q}} v(\bar{q})\, n_F(\varepsilon(\bar{k}+\bar{q})+\Sigma(\bar{k}+\bar{q})-\mu)\, n_F(\varepsilon(\bar{k})+\Sigma(\bar{k})-\mu) \leftarrow \langle V \rangle_0$$

Goal: to tighten up constraint on $\Omega$ $\rightarrow$ minimize $\Omega_{trial}$ wrt $\Sigma(\bar{k})$

Extremal point:
$$\frac{\delta \Omega_{trial}}{\delta \Sigma(\bar{k})} = 0$$

yields an equation for self-energy $\Sigma(\vec{k})$

$$\frac{\delta\Omega_{trial}}{\delta\Sigma(\vec{k})} = \cancel{n_F(\varepsilon(\vec{k})+\Sigma(\vec{k})-\mu) - n_F(\varepsilon(\vec{k})+\Sigma(\vec{k})-\mu)} - \Sigma(\vec{k})\frac{\delta n_F}{\delta\Sigma(\vec{k})}$$

$$+ v(0)\sum_{\bar{p}} n_F(\varepsilon(\bar{p})+\Sigma(\bar{p})-\mu)\frac{\delta n_F}{\delta\Sigma(\vec{k})} - \sum_{\bar{q}} v(\bar{q})\, n_F(\varepsilon(\vec{k}+\bar{q})+\Sigma(\vec{k}+\bar{q})-\mu)\frac{\delta n_F}{\delta\Sigma(\vec{k})} = 0$$

The self-consistency eq. for $\Sigma(\vec{k})$:

$$\Sigma(\vec{k}) = v(0)\sum_{\bar{p}} n_F(\varepsilon(\bar{p})+\Sigma(\bar{p})-\mu) - \sum_{\bar{q}} v(\bar{q})\, n_F(\varepsilon(\vec{k}+\bar{q})+\Sigma(\vec{k}+\bar{q})-\mu)$$

(Fermions) Hartree Fock (exchange)

$$\Sigma(\vec{k}) = v(0)\sum_{\bar{p}} n_B(\varepsilon_B(\bar{p})+\Sigma(\bar{p})-\mu) + \sum_{\bar{q}} v(\bar{q})\, n_B(\varepsilon(\vec{k}+\bar{q})+\Sigma(\vec{k}+\bar{q})-\mu)$$

(Bosons)

$$\mathcal{H}_0 = \sum_{\vec{k}}(\varepsilon(\vec{k})+\Sigma(\vec{k})-\mu)c^+_{\vec{k}}c_{\vec{k}}$$

$$\Omega_{trial} = -\sum_{k}\left\{k_BT\ln\left[1+e^{-\beta(\varepsilon(k)+\Sigma(k)-\mu)}\right] + \frac{1}{2}\Sigma(k)\, n_F(\varepsilon(k)+\Sigma(k)-\mu)\right\}$$

Meaning of HF approx: one-particle spectrum renormalization

## 7.3. Self-consistent field theory (another formulation of the mean-field theory)

Recipe:

1. Replace the higher-order terms in $\hat{\mathcal{H}}$ by a bilinear form in $c, c^+$.
2. Find the coeffs. in the replacement self-consistently.

Consider $V \sim c^+_{\vec{k}+\vec{q}}\, c^+_{\vec{p}-\vec{q}}\, c_{\vec{p}}\, c_{\vec{k}}$

① $V \to V_{HF} \equiv \frac{1}{2}\sum_{kpq} v(q)\left\{ c^+_{\vec{k}+\vec{q}}\langle c^+_{p-q}c_p\rangle c_k + \langle c^+_{k+q}c_k\rangle c^+_{\vec{p}-\vec{q}}c_{\vec{p}}\right.$

$$- \langle c^{+}_{k+q} c_p \rangle c^{+}_{p-q} c_k - \langle c^{+}_{p-q} c_k \rangle c^{+}_{k+q} c_p \}$$

Assuming no broken symmetries:

(a) we already used part. cons. $\Rightarrow$ $\langle c_k c_p \rangle = \langle c^{+}_k c^{+}_p \rangle = 0$

(b) we will also assume no broken transl. invariance:

$$\langle c^{+}_{p-q} c_p \rangle = 0 \quad \text{if } q \neq 0$$

we obtain:

$$V_{HF} = v(0) \sum_p \langle c^{+}_p c_p \rangle \sum_k c^{+}_k c_k - \sum_{kq} v(q) \langle c^{+}_{k+q} c_{k+q} \rangle c^{+}_k c_k$$

Cast $V_{HF}$ into form $V_{HF} = \sum_k \left( \Sigma(k) c^{+}_k c_k \right)$

Use $\mathcal{H}_0 \equiv T + V_{HF}$ to evaluate $\langle c^{+}_k c_k \rangle = n_F\left(\varepsilon(k) + \Sigma(k) - \mu\right)$

Substitute $\langle c^{+}_k c_k \rangle = n_F\left(\varepsilon(k) + \Sigma(k) - \mu\right)$ into $\Sigma(\bar{k})$ $\Rightarrow$ results in the self-consist. equation (same as already derived):

$$\Sigma(\bar{k}) = v(0) \sum_{\bar{p}} n_F\left(\varepsilon(\bar{p}) + \Sigma(\bar{p}) - \mu\right) + \sum_{\bar{q}} v(\bar{q})\, n_F\left(\varepsilon(\bar{k}+\bar{q}) + \Sigma(\bar{k}+\bar{q}) - \mu\right)$$

Other formulations of HF approx:

1. HF means that MB wave function is a Slater det. (or permanent for bosons) with a modified single-particle states – see Fetter, Walecka
2. Diagrammatic formulation – see Fetter, Walecka

3. Path integral formulation (Altland & Simons text, Ch. 6) – especially good method, as it allows one to derive conditions of applicability (large # of particles within a correlation radius).