Simple application of FDT: Johnson–Nyquist noise

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Reminders: Kubo formula:

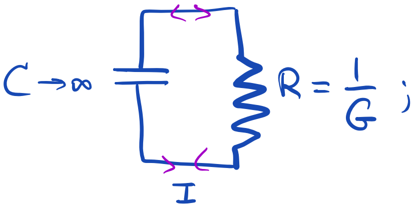
Weak perturbation, $\mathcal{H}_1 = e^{0\cdot t} f(t)\,\hat{A}$; $\quad A(\omega) = \Pi_R(\omega) f_\omega$

$A \equiv \langle \hat{A} \rangle$; $\langle \ldots \rangle$ – over perturbed density matrix.

FDT:

$$\int_{-\infty}^{\infty} dt\, \frac{1}{2} \underbrace{\langle \hat{A}(0)\hat{A}(t) + \hat{A}(t)\hat{A}(0) \rangle_0}_{\text{equilibrium } \langle \ldots \rangle_0} e^{i\omega t - 0\cdot|t|} \equiv \langle A^2 \rangle_\omega$$

$$\boxed{\langle A^2 \rangle_\omega = -\coth\left(\frac{\hbar\omega}{2k_BT}\right) \cdot \hbar \cdot \operatorname{Im} \Pi_R(\omega)}$$



G: conductance $I = G\cdot V$; $\hat{I} = \frac{d}{dt}\hat{Q}$

($\hat{Q}$ is charge passed through the resistor, it is the dynamical variable that couples to the external field V)

$\hat{\mathcal{H}}_1 = -\hat{Q}\cdot V$ $\Pi_R(\omega)$: response of $\hat{Q}$ to voltage V, $\langle Q\rangle_\omega \equiv Q_\omega = \Pi_R V_\omega$

Relate $\Pi_R(\omega)$ to G: Use relation $I_\omega = -i\omega\langle\hat{Q}\rangle_\omega$ and $I_\omega = G(\omega)V_\omega$

$$-i\omega Q_\omega = G\cdot V_\omega \Rightarrow Q_\omega = \frac{G}{-i\omega}\cdot V_\omega \Rightarrow \Pi_R(\omega) = \frac{G(\omega)}{-i\omega}$$

Here we promoted G to ω-dependent $G(\omega)$

We are interested in the current noise: $S_\omega \equiv \langle I^2\rangle_\omega = -\omega^2\langle Q^2\rangle_\omega$;

Use FDT: $\langle Q^2\rangle_\omega = -\hbar\coth\frac{\hbar\omega}{2k_BT}\cdot \operatorname{Im}\frac{G(\omega)}{i\omega} \Rightarrow \langle I^2\rangle_\omega = \hbar\omega\coth\frac{\hbar\omega}{2k_BT}\operatorname{Re}G(\omega)$

$(\operatorname{Im}\omega = 0)$

Classical limit (thermal noise) $\hbar\omega\to 0$ $\langle I^2\rangle_\omega = 2k_BT\cdot G$ (Nyquist noise)

5.4. Dynamic density structure factor (DSF)

Linear response of $\hat{n}(r,t)$ to a field coupled to $\hat{n}$ is $\underbrace{}_{\text{determined by}}$ $\langle n(\bar{r}_1,t)\,n(\bar{r}_2,0)\rangle_0$

(we assume $\mathcal{H}_0$ is independent of time). Its Fourier transform is:

$$\int_{-\infty}^{\infty}dt\, e^{i\omega t}\int dr_1\int dr_2\, e^{i\bar{q}_1\bar{r}_1}\cdot e^{i\bar{q}_2\bar{r}_2}\langle n(\bar{r}_1,t)\,n(\bar{r}_2,0)\rangle_0$$

$\bar{r} = \bar{r}_1 - \bar{r}_2$

$$= \int_{-\infty}^{\infty}dt\, e^{i\omega t}\int dr\int dr_2\, e^{i\bar{q}_1\bar{r}}\cdot e^{i(\bar{q}_1+\bar{q}_2)\bar{r}_2}\langle n(\bar{r}_1,t)\,n(\bar{r}_2,0)\rangle_0$$

in a translationally-invariant system, we simplify further:

$$= \int_{-\infty}^{\infty}dt\, e^{i\omega t}\int dr\, e^{iqr}\langle n(\bar{r},t)\,n(0,0)\rangle_0\cdot V\cdot\delta_{\bar{q}_1,-\bar{q}_2} \Rightarrow \bar{q}_2 = -\bar{q}_1$$

We define DSF as:

$$S(\bar q,\omega)=\underbrace{\int_{-\infty}^{\infty}dt\, e^{i\omega t}\int d\bar r\, e^{i\bar q\bar r}\langle \hat n(\bar r,t)\hat n(0,0)\rangle_0}_{\text{intensive (i.e., finite in limit } V\to\infty)}=\int_{-\infty}^{\infty}dt\, e^{i\omega t}\langle \hat n_{\bar q}(t)\hat n_{-\bar q}(0)\rangle_0$$

continuous $\bar q$ normalization on $\delta(q_1-q_2)$, not on δ_{q_1,q_2}

($\hat n(\bar r,t)$ is the operator of particles density)

$S(\bar q,\omega)$ can be related $\Pi_R^{n_{\bar q}n_{-\bar q}}(\omega)\equiv\Pi_R(\bar q,\omega)$

Indeed, consider a perturbation

$$\mathcal{H}_1=-\left(\int d\bar r'\, v(\bar r',t)\,\hat n(\bar r',t)\,e^{-i\omega t}+h.c.\right)=-\left(\sum_{\bar q} v_{\bar q}\hat n_{-\bar q}e^{-i\omega t}+h.c.\right)$$

Introduce $\hat V_q=-v_q\hat n_{-q}e^{-i\omega t}$; $\mathcal{H}_1=\sum_{\bar q}(\hat V_{\bar q}+\hat V_{\bar q}^+)$

Using Kubo formula, we may write

$$\langle n_q(t)\rangle=-v_q e^{i\omega t}\int_0^{\infty}dt_1\,\Pi_R^{n_{\bar q}n_{-\bar q}}(t_1)\,e^{-i\omega t_1}$$

$$\equiv -v_q e^{i\omega t}\int_{-\infty}^{\infty}dt_1\,\Pi_R(\bar q,t_1)\,e^{-i\omega t_1}$$

with

$$\Pi_R(\bar q,t)=-\frac{i}{\hbar}\theta(t)\langle[n_{\bar q}(t),n_{-\bar q}(0)]\rangle_0$$

$$\langle n_{\bar q}(t)\rangle=-v_{\bar q}e^{i\omega t}\,\Pi_R(\bar q,\omega)\qquad (\langle n_q\rangle_0=0 \text{ at } q\neq 0)$$

Use FDT:

$$\mathrm{Im}\,\Pi_R(\bar q,\omega)=-\frac{\pi}{\hbar}(1-e^{-\beta\hbar\omega})\sum_m\frac{e^{-\beta E_m}}{Z}\sum_n|(n_{\bar q})_{mn}|^2\,\delta(\omega+\omega_m-\omega_n)$$

Now recall

$$\int_{-\infty}^{\infty}dt\,\langle A(t)A(0)\rangle_0\,e^{i\omega t}=2\pi\sum_m\frac{e^{-\beta E_m}}{Z}\sum_n|A_{mn}|^2\,\delta(\omega+\omega_m-\omega_n)$$

Replace $\langle A(t)A(0)\rangle_0 \to \langle n_q(t)\, n_{-q}(0)\rangle_0$ and use $\hat{n}_{\bar{q}} = \hat{n}^+_{-\bar{q}}$

follows from $\hat{n}(\bar{r}) = \hat{n}^+(\bar{r})$

to obtain

$$\int_{-\infty}^{\infty} dt\, \langle n_{\bar{q}}(t)\, n_{-\bar{q}}(0)\rangle_0\, e^{i\omega t} = \int_{-\infty}^{\infty} dt\, \langle n_{\bar{q}}(t)\, n^+_{\bar{q}}(0)\rangle_0\, e^{i\omega t}$$

$$= 2\pi \sum_m \frac{e^{-\beta E_m}}{Z} \sum_n |(n_q)_{mn}|^2\, \delta(\omega+\omega_m-\omega_n)$$

$$S(\bar{q},\omega) = 2\pi \sum_m \frac{e^{-\beta E_m}}{Z} \sum_n |(n_q)_{mn}|^2\, \delta(\omega+\omega_m-\omega_n)$$

$$\boxed{S(\bar{q},\omega) = -2\hbar\, \frac{\mathrm{Im}\, \Pi_R(\bar{q},\omega)}{1-e^{-\beta\hbar\omega}}}$$

$S(\bar{q},\omega)$ vs. $\mathrm{Im}\,\Pi_R(\bar{q},\omega)$: $S(\bar{q},\omega)$ characterizes the rate of absorption of incoming photons while $\mathrm{Im}\,\Pi_R(\bar{q},\omega)$ characterizes absorbed power, affected also by emission of photons at $T \neq 0$ (difference grows with T).

Further reading:

D. Pines, P. Nozieres, The Theory of Quantum liquids, v. 1, Sec. 2.1–2.3. Adison-Wesley 1988

K. Sturm, Dynamic structure factor: An Introduction Z. Naturforsch. 48a, 233–242 (1993)

Comment: DSF vs. Spectral function

The following is only a schematic exposition.

DSF builds on density-density corr. function,

$$\langle \hat n(\bar r,t)\,\hat n(0,0)\rangle = \langle \hat\psi^+(r,t)\,\hat\psi(r,t)\,\psi^+(0,0)\,\psi(0,0)\rangle$$

in the absence of interaction,

$$\langle \hat\psi^+(r,t)\,\hat\psi(r,t)\,\psi^+(0,0)\,\psi(0,0)\rangle \to \langle\psi^+(r,t)\psi(0,0)\rangle\langle\psi(r,t)\psi^+(0,0)\rangle$$

$$\langle\psi^+(r,t)\psi(0,0)\rangle\langle\psi(r,t)\psi^+(0,0)\rangle \to \underbrace{A_p(k,\omega) * A_h(k-q,\omega-\Omega)}_{\text{convolution of spectral functions}}$$

Recall ARPES: We had for the fermion field operators

$$\hat\Psi^+ = \hat\psi(r,t) + \underbrace{\hat\varphi(r,t)}_{\text{free fermion}}$$

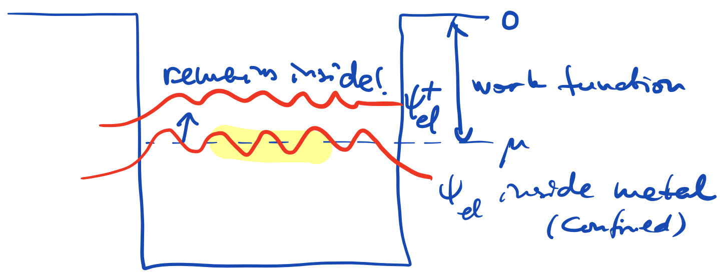
, we assumed no interaction between el-ns inside the solid ($\hat\psi$) and emitted ones ($\hat\varphi$).

Evaluation of $w_{\bar q,\bar k}$ amounted to finding a fourier transform for the off-diag component of $\hat n(r,t)$:

$$\hat n_{\text{off-diag}} \sim \hat\psi^+(r,t)\,\hat\varphi(r,t),$$

$$\langle \hat n_{\text{off-diag}}(r,t)\,\hat n_{\text{off-diag}}(0,0)\rangle \to \langle\hat\psi^+(\bar r,t)\,\hat\varphi(\bar r,t)\,\hat\varphi^+(0,0)\,\hat\psi(0,0)\rangle$$

$$\to \underbrace{\langle\psi^+(r,t)\,\psi(0,0)\rangle}_{\hookrightarrow A_h(\bar p,\omega)}\ \underbrace{\langle\varphi(r,t)\,\varphi^+(0,0)\rangle}_{\delta(\varepsilon-k^2/2m)}$$



$$w_{\bar q,\vec k} =$$

$\varphi^+(r) \Leftarrow \hat\varphi^+ \to \psi^+$ in DSF

$$= \frac{2\pi}{\hbar}|\lambda|^2\,\frac{1}{Z}\sum_n e^{-\beta E_n}\sum_\ell \left|\langle \ell,\vec k|\int d\vec r\, e^{i\vec q\vec r}\,\hat\varphi^+(\vec r)\,\psi(\vec r)\,|n,0\rangle\right|^2 \times \delta\left(E_n - E_\ell - \frac{\hbar^2k^2}{2m} + \hbar\omega_{\bar q}\right)$$

$$\langle \hat n_{\text{off-diag}}(r,t)\,\hat n_{\text{off-diag}}(0,0)\rangle$$

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5.5. Free fermion response function

$$\Pi_R(q,\omega^+) \equiv -\frac{i}{\hbar}\int_0^\infty dt\, e^{i\omega^+ t} \langle [\hat{n}_{\bar q}(t), \hat{n}_{-\bar q}(0)]\rangle_0$$

$$= -\frac{i}{\hbar}\int_0^\infty dt\, e^{i\omega^+ t} \langle \hat{n}_{\bar q}(t)\hat{n}_{-\bar q}(0) - \hat{n}_{-\bar q}(0)\hat{n}_{\bar q}(t)\rangle_0$$

$$= -\frac{i}{\hbar}\int_0^\infty dt\, e^{i\omega^+ t} \langle \frac{1}{L^3}\sum_{k_1} a^+_{\bar k_1}(t) a_{\bar k_1+\bar q}(t) \sum_{\bar k_2} a^+_{\bar k_2}(0) a_{\bar k_2-\bar q}(0)$$

$$- \frac{1}{L^3}\sum_{\bar k_2} a^+_{\bar k_2}(0) a_{\bar k_2-\bar q}(0) \sum_{k_1} a^+_{\bar k_1}(t) a_{\bar k_1+\bar q}(t)\rangle_0$$

$$= -\frac{i}{\hbar L^3}\int_0^\infty dt\, e^{i\omega^+ t} \times \sum_{\bar k_1 \bar k_2} \langle a^+_{\bar k_1} a_{\bar k_1+\bar q} a^+_{\bar k_2} a_{\bar k_2-\bar q} - a^+_{\bar k_2} a_{\bar k_2-\bar q} a^+_{\bar k_1} a_{\bar k_1+\bar q}\rangle_0$$

$$\times \exp\left\{\frac{i}{\hbar}(\varepsilon_{\bar k_1} - \varepsilon_{\bar k_1+\bar q})t\right\}$$

ε_k : single-fermion energy

$$= -\frac{i}{\hbar L^3}\int_0^\infty dt\, e^{i\omega^+ t} \sum_{\bar k_1 \bar k_2} \left(\langle a^+_{\bar k_1} a_{\bar k_2-\bar q}\rangle_0 \langle a_{\bar k_1+\bar q} a^+_{\bar k_2}\rangle_0 - \langle a^+_{\bar k_2} a_{\bar k_1+\bar q}\rangle_0 \langle a_{\bar k_2-\bar q} a^+_{k_1}\rangle_0 \right)$$

$\bar k_2 = \bar k_1 + \bar q$

$$\times \exp\left\{\frac{i}{\hbar}(\varepsilon_{\bar k_1} - \varepsilon_{\bar k_1+\bar q})t\right\};\quad \langle a^+_{\bar k_1} a_{\bar k_2-\bar q}\rangle = \langle a^+_{\bar k_1} a_{\bar k_1}\rangle \delta_{\bar k_1, \bar k_2-\bar q};\quad \langle a^+_{\bar k_1} a_{\bar k_1}\rangle \equiv n_{\bar k_1}$$

$$\Pi_R(q,\omega^+) = -\frac{i}{\hbar L^3}\int_0^\infty dt\, e^{i\omega^+ t} \times \sum_{\bar k_1}\left(n_{\bar k_1}(1-n_{\bar k_1+\bar q}) - n_{\bar k_1+\bar q}(1-n_{\bar k_1})\right)$$

$$\times \exp\left\{\frac{i}{\hbar}(\varepsilon_{\bar k_1} - \varepsilon_{\bar k_1+\bar q})t\right\}$$

$$\Pi_R(\bar q,\omega^+)= -\frac{i}{\hbar}\cdot\frac{1}{L^3}\sum_{\bar k_1}\frac{n_{\bar k_1}-n_{\bar k_1+\bar q}}{-i\left(\omega^+ + \frac{\varepsilon_{\bar k}-\varepsilon_{\bar k_1+\bar q}}{\hbar}\right)}$$

$$\boxed{\begin{aligned}&\Pi_R(\bar q,\omega^+)= -\frac{1}{\hbar}\cdot\frac{1}{L^3}\cdot\sum_{\bar k_1}\frac{n_{\bar k_1}-n_{\bar k_1+\bar q}}{\frac{\varepsilon_{\bar k_1+\bar q}}{\hbar}-\frac{\varepsilon_{\bar k_1}}{\hbar}-(\omega+i\delta)}, \quad \delta\to+0\\ &\operatorname{Im}\Pi_R(\vec q,\omega) = -\pi\int\frac{d^d k}{(2\pi)^d}\,(n_{k}-n_{k+q})\,\delta(\varepsilon_{k+q}-\varepsilon_k-\hbar\omega)\end{aligned}}$$

Here:

We may use

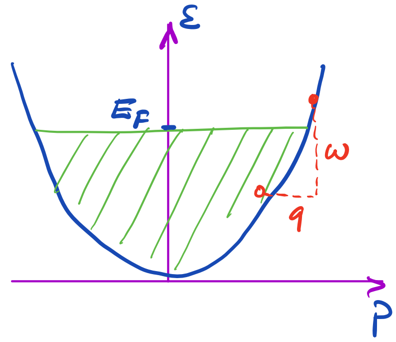
$$n_{\vec k} = n_F(\varepsilon_{\vec k}) \text{ – Fermi distribution;} \qquad n_{\bar k+\bar q} = n_F(\varepsilon_{\bar k+\bar q}) = n_F(\varepsilon_k+\hbar\omega)$$

$$\underbrace{\delta(\varepsilon_{k+q}-\varepsilon_k-\hbar\omega)}$$

$T=0 \Rightarrow n_{\vec k} = \theta(E_F-\varepsilon_{\vec k})$

Clear relation to the absorption of a photon $(\bar q,\omega)$

(note that at $T=0$
$S(\bar q,\omega) = -2\hbar\operatorname{Im}\Pi(q,\omega)$)



Detailed evaluation of $\Pi_R(\bar q,\omega)$ at $T=0$: FW p.158 (Sec. 12)

+ Mahan Sec. 5.5

We will concentrate on several limits.

First note that at $T=0$

$$(n_k-n_{k+q})\,\delta(\varepsilon_{k+q}-\varepsilon_k-\hbar\omega) = \underbrace{\Big(\theta(E_F-\varepsilon_k)-\theta(E_F-\hbar\omega-\varepsilon_k)\Big)}_{\neq 0 \text{ only if } \varepsilon_k<E_F,\ \varepsilon_k+\hbar\omega>E_F\ (\text{as } \omega>0)\text{, therefore}}\delta(\varepsilon_{k+q}-\varepsilon_k-\hbar\omega)$$

$$= \Theta(E_F - \varepsilon_k)\,\Theta(\varepsilon_k + \hbar\omega - E_F)\,\delta(\varepsilon_{\bar{k}+\bar{q}} - \varepsilon_{\bar{k}} - \hbar\omega)$$

$$= \Theta(E_F - \varepsilon_k)\,\Theta(\varepsilon_{\bar{k}+\bar{q}} - E_F)\,\delta(\varepsilon_{\bar{k}+\bar{q}} - \varepsilon_{\bar{k}} - \hbar\omega) = \Theta(k_F - k)\,\Theta(|\bar{k}+\bar{q}| - k_F)$$

(isotropy $\Rightarrow \varepsilon(\bar{k}) = \varepsilon(|\bar{k}|)$)

$$\times\, \delta(\varepsilon_{\bar{k}+\bar{q}} - \varepsilon_{\bar{k}} - \hbar\omega)$$

Using these identities, we may re-write

$$\mathrm{Im}\,\Pi_R(\bar{q},\omega)\Big|_{T=0} = -\pi \int \frac{d^d k}{(2\pi)^d}\, \Theta(k_F - k)\,\Theta(|\bar{k}+\bar{q}| - k_F)\,\delta(\varepsilon_{\bar{k}+\bar{q}} - \varepsilon_{\bar{k}} - \hbar\omega).$$

Sometimes, this form is convenient.

Response function in limiting cases (at $T=0$)

1. Domain of (ω, q) plane where $\mathrm{Im}\,\Pi_R(\bar{q},\omega) \neq 0$

For example, take $\varepsilon(k) = \hbar^2 k^2 / 2m^*$. In that example,

$$\varepsilon(|\bar{k}+\bar{q}|) = \frac{\hbar^2}{2m^*}(k^2 + 2\bar{k}\cdot\bar{q} + q^2) = \frac{\hbar^2 k^2}{2m^*} + \frac{\hbar^2 q^2}{2m^*} + \frac{\hbar^2 kq}{m^*}\cos\theta$$

(θ: angle $\widehat{\bar{k}\,\bar{q}}$, any dimension $d \geq 2$)

$$\varepsilon(|\bar{k}+\bar{q}|) - \varepsilon(k) - \hbar\omega = \frac{\hbar^2 kq}{m^*}\cos\theta + \frac{\hbar^2 q^2}{2m^*} - \hbar\omega$$

$$\delta(\varepsilon(|\bar{k}+\bar{q}|) - \varepsilon(k) - \hbar\omega) = \delta\left(\frac{\hbar^2 kq}{m^*}\cos\theta + \frac{\hbar^2 q^2}{2m^*} - \hbar\omega\right)$$

To satisfy the δ-function: $\dfrac{\hbar^2 kq}{m^*}\cos\theta + \dfrac{\hbar^2 q^2}{2m^*} - \hbar\omega = 0$

$$\Rightarrow \cos\theta = \frac{\hbar\omega - \hbar^2 q^2/2m^*}{\hbar^2 k q/m^*}$$

Constraint: $|\cos\theta| \leq 1$

$$\Downarrow$$

$$-\frac{\hbar^2 q}{m^*}k \leq \hbar\omega - \frac{\hbar^2 q^2}{2m^*} \leq \frac{\hbar^2 q}{m^*}k \Rightarrow k \geq \frac{m^*}{\hbar^2 q}\left|\hbar\omega - \frac{\hbar^2 q^2}{2m^*}\right|$$

Now recall that

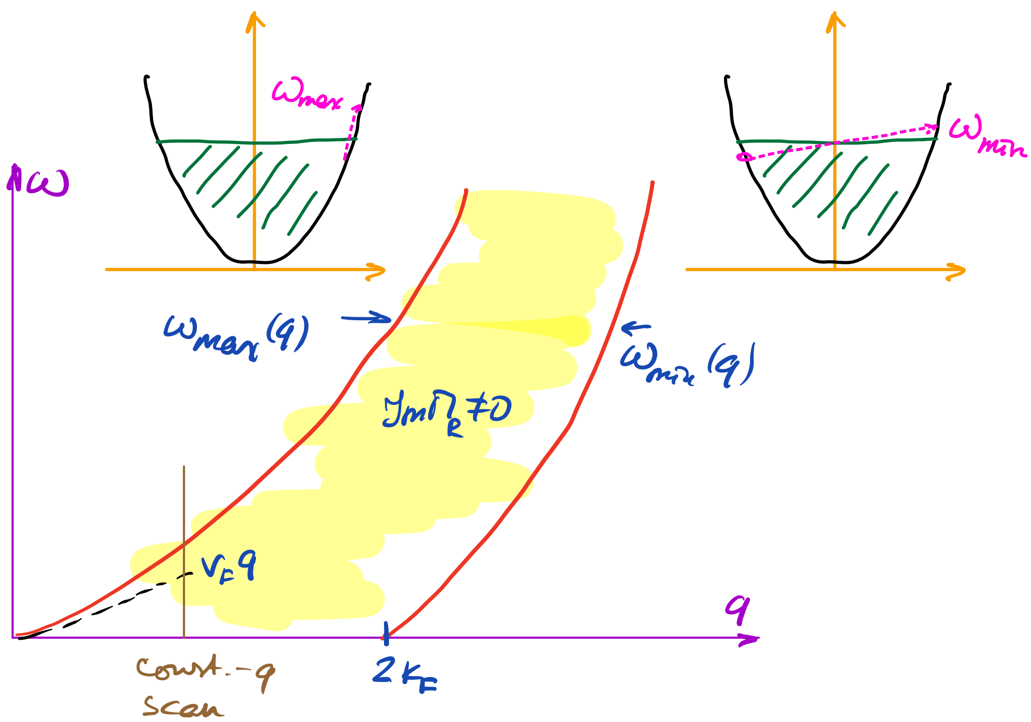
$$\mathrm{Im}\,\Pi_R(\bar{q},\omega)\Big|_{T=0} = -\pi \int \frac{d^d k}{(2\pi)^d}\, \underbrace{\theta(k_F - k)}_{k \le k_F}\, \theta(|\bar{k}+\bar{q}| - k_F)\, \delta(\varepsilon_{\bar{k}+\bar{q}} - \varepsilon_{\bar{k}} - \hbar\omega).$$

$$\underbrace{k_F \geq \frac{m^*}{\hbar^2 q}\left|\hbar\omega - \frac{\hbar^2 q^2}{2m^*}\right|}_{\text{solve the inequality wrt } \omega}$$

condition on ω, q only

Constraints on the (ω, q) domain for $\mathrm{Im}\,\Pi_R(\bar{q},\omega)\Big|_{T=0} \neq 0$:

$$\begin{cases} \omega < \omega_{max}(q) = \dfrac{\hbar k_F q}{m^*} + \dfrac{\hbar q^2}{2m^*} & \left(\cos\theta = 1;\ q + k_F > k_F \Rightarrow \theta(|\bar{k}+\bar{q}| - k_F) \neq 0\right) \\ \omega > \omega_{min}(q) = -\dfrac{\hbar k_F q}{m^*} + \dfrac{\hbar q^2}{2m^*} & \left(\cos\theta = -1;\ q > 2k_F \text{ for } \omega_{min}(q) > 0 \Rightarrow \theta(|\bar{k}+\bar{q}| - k_F) \neq 0\right) \end{cases}$$



2. Asymptote of $\mathrm{Im}\,\Pi_R(\vec{q},\omega)\big|_{T=0}$ at $\omega \to 0$.

$$\mathrm{Im}\,\Pi_R(\vec{q},\omega) = -\pi \int \frac{d^d k}{(2\pi)^d} \left(n_F(\varepsilon_{\vec{k}}) - n_F(\varepsilon_{\vec{k}} + \hbar\omega)\right) \delta(\varepsilon_{\vec{k}+\vec{q}} - \varepsilon_{\vec{k}} - \hbar\omega)$$

Small ω:

$$n_F(\varepsilon_{\vec{k}}) - n_F(\varepsilon_{\vec{k}} + \hbar\omega) \cong -\frac{\partial n_F}{\partial \varepsilon} \cdot \hbar\omega = \delta(\varepsilon_{\vec{k}} - E_F) \cdot \hbar\omega \quad , \quad T = 0$$

$$-\mathrm{Im}\,\Pi_R(\vec{q},\omega) = \pi \cdot \hbar\omega \cdot \int \frac{d^d k}{(2\pi)^d} \delta(\varepsilon_{\vec{k}} - E_F)\, \delta(\varepsilon_{\vec{k}+\vec{q}} - \varepsilon_{\vec{k}} - \hbar\omega)$$

indep. of direction!

3D: $$= \pi\hbar\omega \int \frac{k^2 dk}{(2\pi)^3} \delta(\varepsilon_k - E_F) \int dO\, \delta(\varepsilon_{\vec{k}+\vec{q}} - \varepsilon_{\vec{k}} - \hbar\omega)$$