04.25.2023

Finite sound velocity $\Rightarrow$ superfluidity.

Notion of superfluidity

Consider a classical heavy particle with mass M ($\gg m$) moving through the condensate with velocity $\vec{v}$. It may decelerate by emitting excitations in the Bose system:

$$E_{\vec{p}} = \frac{p^2}{2M} \quad ; \qquad \underbrace{E_{\vec{p}} - E_{\vec{p}-\vec{q}} = \varepsilon_{\vec{q}}} \; ; \quad \varepsilon_{\vec{q}} = q\sqrt{c_1^2 + q^2/4m^2}\Big|_{q \ll p^*} \simeq c_1 q$$

$$E_{\vec{p}} - E_{\vec{p}-\vec{q}} = \frac{\partial E}{\partial \vec{p}}\vec{q} = \vec{v}\cdot\vec{q}$$

$$\vec{v}\cdot\vec{q} = c_1 q$$

$\vec{v}\cdot\vec{q} \leq vq \quad \Rightarrow \quad c_1 q \leq vq$ for the emission to be possible.

Emission of excit. is possible only if $v \geq c_1$ (critical velocity for a condensate); no deceleration at $v<c$.

Reading

8.2. Non-uniform interacting bose gas at T=0. Gross-Pitaevskii Equation.

Heisenberg eq. of motion for Bose field operator:

$$i\hbar \frac{\partial \hat{\varphi}(\bar{r},t)}{\partial t} = [\hat{\varphi}(\bar{r},t), \mathcal{H}]; \quad \mathcal{H} = \hat{T} + \hat{\mathcal{H}}_{int} + \hat{V}_{ext};$$

$$\mathcal{H}_{int} = \frac{1}{2}\int d\bar{r}_1 d\bar{r}_2\, g(\bar{r}_1 - \bar{r}_2)\, \varphi^+(r_1)\varphi^+(r_2)\varphi(r_2)\varphi(r_1)$$

$$\hat{V}_{ext} = \int d\bar{r}\, V_{ext}(r,t)\, \hat{\varphi}^+(\bar{r},t)\hat{\varphi}(\bar{r},t)$$

$$i\hbar \frac{\partial \hat{\varphi}(\bar{r},t)}{\partial t} = \left\{ -\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\bar{r},t) + \int d\bar{r}'\, \hat{\varphi}^+(\bar{r}',t) g(\bar{r}'-\bar{r}) \hat{\varphi}(\bar{r}',t) \right\} \hat{\varphi}(\bar{r},t)$$

Suppose $V_{ext}(\bar{r},t)$ is smooth on scale of scatt. length a; also assume weak interaction, use Born approx. for $a \Rightarrow$

$$\Rightarrow g \equiv \int d^3r\, g(r) = \frac{4\pi\hbar^2}{m}\cdot a; \quad g \ll \hbar^2/n^{1/3}m \Leftrightarrow na^3 \ll 1.$$

Replace $\hat{\varphi}(\bar{r},t)$ with a classical field $\psi(\bar{r},t)$ (same approx. as replacing a_0, a_0^+ by $\sqrt{N}$ in the uniform gas theory)

$$\boxed{i\hbar \frac{\partial \psi(\bar{r},t)}{\partial t} = \left\{ -\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\bar{r},t) + g|\psi(\bar{r},t)|^2 \right\}\psi(\bar{r},t)}$$

Gross-Pitaevskii equation (GPE, 1961)

$n(r,t) = |\psi(\bar{r},t)|^2$ - density of a (non-uniform) condensate

GPE $\Leftrightarrow$ Hartree approx. In stationary case

$$\psi(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N) = \prod_{i=1}^{N}\left(\frac{1}{\sqrt{N}}\,\psi_0(\bar{r}_i)\right)$$

Consider a time-indep. potential $V_{ext}(\bar{r})$ and look for stationary solutions of GPE

$$\psi_0(\bar r,t) = \psi_0(\bar r)\cdot e^{-i\frac{\mu}{\hbar}t} \quad ; \qquad \mu = \partial E/\partial N$$

Plug in GPE:

$$\begin{cases} \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\bar r) - \mu + g|\psi_0(\bar r)|^2\right)\psi_0(\bar r) = 0 \\ n(\bar r) = |\psi_0(\bar r)|^2 \; ; \quad N = \int d^3\bar r\, n(\bar r) \end{cases}$$

$\mu \to \psi(r) \to n(r) \to N$

$\mu(N)$

GPE defines $n(\bar r)$ and μ in terms of N

Conservation laws associated with GPE

Particle # conservation $\to$ Continuity equation:

$$\frac{\partial n}{\partial t} + \operatorname{div} j = 0$$

allows to derive current density

$$\vec j(\bar r,t) = -\frac{i\hbar}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \equiv n(\bar r,t)\cdot\frac{\hbar}{m}\bar\nabla S(\bar r,t)$$

re-write

Cast $\psi(\bar r,t)$ in the form:

$$\psi(\bar r,t) = |\psi(\bar r,t)|\, e^{iS(\bar r,t)} = \sqrt{n(\bar r,t)}\, e^{iS(\bar r,t)}$$

$$j(\bar r,t) = n(\bar r,t)\cdot\vec v(\bar r,t) \; ; \qquad \vec v(\bar r,t) \equiv \frac{\hbar}{m}\bar\nabla S(\bar r,t)$$ – velocity of the condensate

$n(\bar r,t)$ – density of the condensate

Relation of linearized GPE to Bogoliubov excitations

$$\psi(\bar r,t) = \psi_0(\bar r,t) + \delta\psi(\bar r,t) \; ; \qquad \psi_0(\bar r,t) = \psi_0(\bar r)\, e^{-i\frac{\mu}{\hbar}t}$$

Seek to solve for $\delta\psi$ in the form:

$$\delta\psi = e^{-i\frac{\mu}{\hbar}t}\left(u(\bar r)e^{-i\omega t} - v^*(\bar r)e^{i\omega t}\right)$$

Substitute $\psi(\bar r,t)$ in GPE, linearize wrt $u(\bar r)$, $v(\bar r)$:

$$\begin{cases} \hbar\omega\, u(\bar r) = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\bar r) - \mu + 2g n_0(\bar r)\right)u(\bar r) + g\left(\psi_0(\bar r)\right)^2 v(\bar r) \\ -\hbar\omega\, v(\bar r) = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\bar r) - \mu + 2g n_0(\bar r)\right)v(\bar r) + g\left(\psi_0(\bar r)\right)^2 u(\bar r) \end{cases}$$

For a spatially-uniform system ($V_{ext}=0$) $\Rightarrow$ $|\psi_0|^2 = n_0$ is independent of $\vec{r}$, and $\mu = g\cdot n_0$; $u_{\vec{k}}(\vec{r}), v_{\vec{k}}(\vec{r}) \sim e^{i\vec{k}\cdot\vec{r}}$

$$\begin{cases} \hbar\omega u_{\vec{k}} = \dfrac{\hbar^2 k^2}{2m} u_{\vec{k}} + gn(u_{\vec{k}} + v_{\vec{k}}) \\ -\hbar\omega v_{\vec{k}} = \dfrac{\hbar^2 k^2}{2m} v_{\vec{k}} + gn(u_{\vec{k}} + v_{\vec{k}}) \end{cases}$$

Matrix form: $$\begin{pmatrix} \dfrac{\hbar^2 k^2}{2m} + gn - \hbar\omega_{\vec{k}} & gn \\ gn & \dfrac{\hbar^2 k^2}{2m} + gn + \hbar\omega_{\vec{k}} \end{pmatrix} \begin{pmatrix} u_{\vec{k}} \\ v_{\vec{k}} \end{pmatrix} = 0$$

$$\Rightarrow \quad \hbar\omega_{\vec{k}} = \sqrt{\left(\frac{\hbar^2 k^2}{2m} + gn\right)^2 - (gn)^2} \equiv \varepsilon_k$$

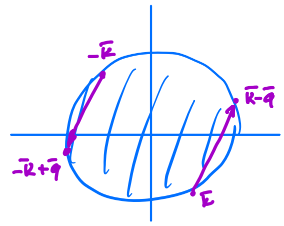
$\mathrm{Det}\{\ldots\} = 0$

9. Superconductivity, BCS model

9.1. Scattering in Cooper channel

Consider 2-body interaction between fermions

$$\hat{V} = \sum_{\bar{k}\bar{p}\bar{q}} V_q c^{+}_{\bar{k}-\bar{q}} c^{+}_{\bar{p}+\bar{q}} c_{\bar{p}} c_{\bar{k}}$$



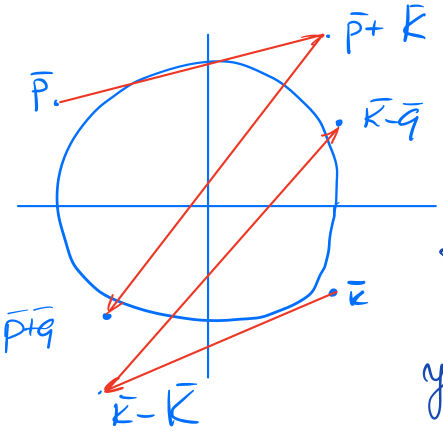
① Scattering between particles with $\bar{p} = -\bar{k}$ is special because the conservation of momentum and energy allows for a large interval of momentum transfers, $|q| \leq 2k$

② The corresp. scattering amplitude (being evaluated to order V^2) is divergent at $E(\vec{k}) \to E_F$

$$\hat{T} = \hat{V} + \hat{V} \frac{1}{E - \hat{\mathcal{H}}_0 - i0} \hat{V} + \ldots$$

Born approx.: $t_1 \sim V$ (dispense with the q-dep. of V_q)

Second order in $\hat{V}$



(Pauli blocking of virtual states)

$$t_2(\bar{k},\bar{p}) \sim \sum_{\vec{K}} \frac{V^2\, \theta(E_{\bar{k}-\bar{K}} - E_F)\,\theta(E_{\bar{p}+\bar{K}} - E_F)}{E_{\bar{k}} + E_{\bar{p}} - \underbrace{(E_{\bar{k}-\bar{K}} + E_{\bar{p}+\bar{K}})}_{\text{energy of virtual state}}}$$

Two conditions on $\vec{K}$: $E_{\bar{k}-\bar{K}} \geq E_F$, $E_{\bar{p}+\bar{K}} \geq E_F$

yield the same constraint on $\vec{K}$ if $\bar{k} = -\bar{p}$

the same condition!

$(E_{\bar{k}-\bar{K}} > E_F,\ E_{-\bar{k}+\bar{K}} \equiv E_{\bar{k}-\bar{K}} > E_F)$

The result of $\sum_{\vec{K}} \ldots$:

$$t_2 \simeq -\frac{V^2}{4\pi^2}\cdot\frac{k_F^2}{\hbar v_F}\ln\frac{\omega}{\max(\epsilon,\eta)}$$

(ω is defined by upper cut-off in $\sum_{\bar{k}}\ldots$)

$\epsilon = \xi_k + \xi_p$, where $\xi_k = E_{\bar{k}} - E_F = \hbar v_F (k - k_F)$

$\eta = \hbar v_F |\bar{k} + \bar{p}|$

9.1a. The Cooper problem

Reading

$$\mathcal{H} = \sum_{\substack{k>k_F\\ \sigma}} \xi_k c^+_{k\sigma} c_{k\sigma} + \frac{1}{L^3}\sum_{k_F \le k,p < k_F + \omega/v_F} V c^+_{k\uparrow} c^+_{-k\downarrow} c_{-p\downarrow} c_{p\uparrow}$$

important constraint!

$$\xi_k = \frac{k^2}{2m} - \frac{k_F^2}{2m}$$

Consider a 2-particle state: $\Psi = \sum_{q>k_F} A_q c^+_{q\uparrow} c^+_{-q\downarrow} |GS\rangle$

$\hat{\mathcal{H}}\Psi = E\Psi$ (eq. for eigenvalues of energy E)

$(E - \hat{\mathcal{H}})\Psi = 0$

$$\sum_q \left(A_q (E - 2\xi_q) - \frac{1}{L^3} V \sum_{k_F \le k \le k_F + \omega/v_F} A_k \right) c^+_{q\uparrow} c^+_{-q\downarrow} |0\rangle = 0$$

$$(E - 2\xi_q) A_q - \frac{V}{L^3} \sum_{k_F \le k \le k_F + \omega/v_F} A_k = 0 \quad \text{(eigenvalue equation)}$$

Denote $\sum_k A_k = C \Rightarrow A_q = \frac{1}{E - 2\xi_q}\cdot\frac{V}{L^3}\cdot C$

$$\cancel{C} = \sum_{k_F \le k < k_F + \frac{\omega}{v_F}} \frac{1}{E - 2\xi_k}\cdot\frac{V}{L^3}\cancel{C} \Rightarrow 1 = V\cdot\frac{1}{L^3}\sum_{k_F \le k < k_F + \omega/v_F}\frac{1}{E - 2\xi_k}$$

(this is eq. for E)

$$1 = V\cdot\nu(E_F)\int_0^{\omega}\frac{d\xi}{E - 2\xi}; \quad \nu(E_F) \equiv \nu_0, \quad \omega \ll E_F$$

For $V < 0$: $1 = V\cdot\nu_0\cdot(-1)\ln\frac{2\omega}{|E|}$, $E < 0$

$$\boxed{E = -2\omega \exp\left\{-\frac{1}{|V|\nu_0}\right\}}$$ — Bound state! (discrete energy level)

note $E(<0)$ is measured from $2E_F$ (min energy of 2 free fermions)

L. N. Cooper PR 104, 1189 (1956)

Sec. 2-2 p.28 of Theory of Supercond. by J. Schrieffer, Westview press 1999

Tinkham, Intro to SC, Sec. 3.1 p.44 (McGraw-Hill, 1996)

9.2. The model Hamiltonian for many-body system, mean-field solution

$$\mathcal{H}_M = \hat{\mathcal{H}} - \mu \hat{N} = \sum_{\vec{k}\sigma} \xi_k c^+_{\vec{k}\sigma} c_{\vec{k}\sigma} - \underbrace{\frac{1}{L^3} V_0 \sum_{|\xi_{k'}|<\hbar\omega_D} c^+_{\vec{k}'\uparrow} c^+_{-\vec{k}'\downarrow} \sum_{|\xi_k|<\hbar\omega_D} c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow}}$$

$\xi_k = \varepsilon_k - \mu$

1. dominant term ($\vec{k}, -\vec{k} \to \vec{k}', -\vec{k}'$)
2. a model for interaction accounting for the dynamic screening (see HW 7) $\to$ attraction

Try self-consistent field with

$$V_0 \sum_{|\xi_k|<\hbar\omega_D} \langle c^+_{\vec{k}'\uparrow} c^+_{-\vec{k}'\downarrow} \rangle_{BCS} \equiv L^3 \cdot \Delta^*, \quad V_0 \sum_{|\xi_{k'}|<\hbar\omega_D} \langle c_{-\vec{k}'\downarrow} c_{\vec{k}\uparrow} \rangle_{BCS} = L^3 \Delta$$

$$\mathcal{H}_{BCS} = \sum_{\vec{k}\sigma} \xi_k c^+_{k\sigma} c_{\vec{k}\sigma} - \Delta^* \sum_{|\xi_k|<\hbar\omega_D} c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} - \Delta \sum_{|\xi_k|<\hbar\omega_D} c^+_{\vec{k}\uparrow} c^+_{-\vec{k}\downarrow} + \frac{|\Delta|^2}{V_0} L^3$$

(note sign, comes from $\langle \mathcal{H}_{BCS} \rangle = \langle \mathcal{H}_M \rangle$)

Bogolibov transformation:

$$c_{\vec{k}\uparrow} = u_k b_{\vec{k}\uparrow} + v_k b^+_{-\vec{k}\downarrow}; \quad b_{\vec{k}\downarrow} = u_k c_{\vec{k}\downarrow} + v_k c^+_{-\vec{k}\uparrow};$$

$$c_{\vec{k}\downarrow} = u_k b_{\vec{k}\downarrow} - v_k b^+_{-\vec{k}\uparrow}; \quad b_{\vec{k}\uparrow} = u_k c_{k\uparrow} - v_k c^+_{-k\downarrow}.$$

See de Gennes, Supercond. of Metals and Alloys, Addison-Wesley 1989 pp. 140, 144 for important generalizations (incl. TRS-violation)

Assume Δ is real $\Rightarrow$ real u, v.

Canonical transformation $\Rightarrow$ $\boxed{u_k^2 + v_k^2 = 1}$

$$\mathcal{H}_{BCS} = \sum_{\vec{k}} \xi_k \Big[(u_k^2 - v_k^2)(b^+_{\vec{k}\uparrow} b_{\vec{k}\uparrow} + b^+_{-\vec{k}\downarrow} b_{-\vec{k}\downarrow}) + 2v_k^2 + 2u_k v_k b_{-k\downarrow} b_{k\uparrow}$$

$$+ 2u_k v_k b^+_{\vec{k}\uparrow} b^+_{-k\downarrow} \Big] + \sum_{\vec{k}} \Big[2\Delta u_k v_k (b^+_{k\uparrow} b_{k\uparrow} + b^+_{-k\downarrow} b_{-k\downarrow} - 1)$$

$$+ \Delta (v_k^2 - u_k^2) b_{-k\downarrow} b_{k\uparrow} + \Delta (v_k^2 - u_k^2) b^+_{k\uparrow} b^+_{-k\downarrow} \Big] + \frac{\Delta^2}{V_0} \cdot L^3$$

Diagonalization of $\mathcal{H}_{BCS}$:

$$\boxed{2\xi_k u_k v_k + \Delta(v_k^2 - u_k^2) = 0}$$

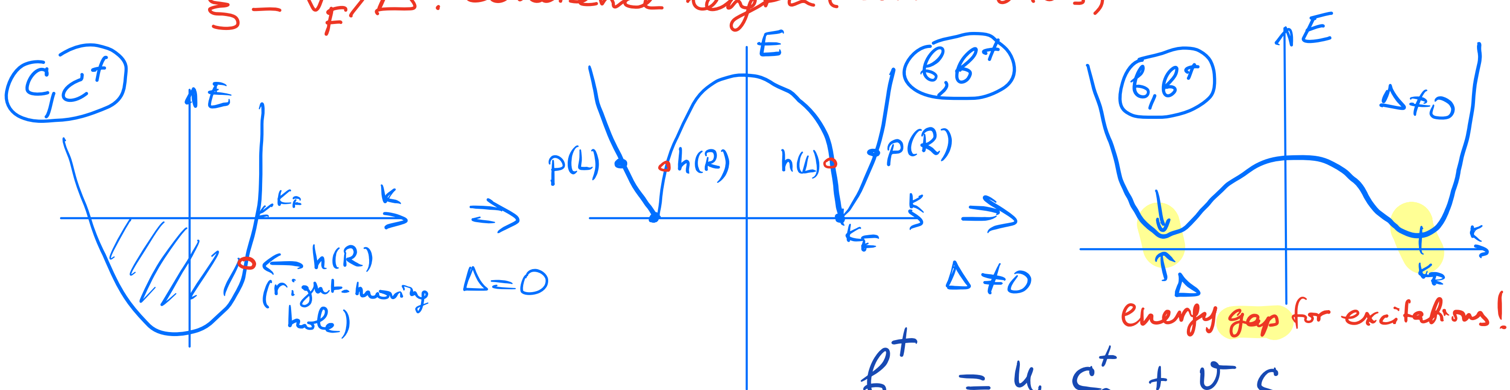
$$\begin{cases} u_k^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{\varepsilon_k}\right) \\ v_k^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{\varepsilon_k}\right) \end{cases} \qquad \varepsilon_k = \sqrt{\xi_k^2 + \Delta^2}, \quad \xi_k = \hbar v_F(|\vec{k}| - k_F) \ (\text{at } |k - k_F| \ll k_F)$$

in case of complex-valued Δ:

$$\varepsilon_k = \sqrt{\xi_k^2 + |\Delta|^2}$$

$$\mathcal{H}_{BCS} = \sum_k \varepsilon_k (b^+_{k\uparrow} b_{k\uparrow} + b^+_{k\downarrow} b_{k\downarrow}) + \sum_k (\xi_k - \varepsilon_k) + \frac{\Delta^2}{V_0} \cdot L^3$$

$\xi^* = v_F/\Delta$: coherence length (corr. radius)



$$b^+_{k\downarrow} = u_k c^+_{k\downarrow} + v_k c_{-k\uparrow}$$

$$\Delta = \frac{1}{L^3} V_0 \sum_k u_k v_k \left(1 - \sum_\sigma \langle b^+_{k\sigma} b_{k\sigma}\rangle\right)$$

At $T = 0$, $\langle b^+_{k\sigma} b_{k\sigma}\rangle = 0$: $\quad \Delta = \frac{1}{L^3} V_0 \sum_k \frac{\Delta}{2\varepsilon_k}$

Transform $\frac{1}{L^3}\sum_k \ldots \longrightarrow \int \frac{d^3k}{(2\pi)^3} \ldots$: $\quad \Delta = \nu_0 V_0 \cdot \frac{1}{2} \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{\Delta \cdot d\xi}{\sqrt{\xi^2 + \Delta^2}}$, $\nu_0 \equiv \nu(E_F)$

per orbital state (2 spin directions)

$$\nu_0 V_0 \int_0^{\hbar\omega_D} \frac{d\xi}{\sqrt{\xi^2 + \Delta^2}} = 1 \Rightarrow \frac{1}{\nu_0 V_0} = \sinh^{-1}\frac{\hbar\omega_D}{\Delta},$$

has solution only at $V_0 > 0$ (attraction)

Weak interaction : $\nu_0 V_0 \ll 1 \Rightarrow \boxed{\Delta \simeq 2\hbar\omega_D \exp\left\{-\frac{1}{\nu_0 V_0}\right\}}$

04. 27. 2023

Ground state energy (measured from normal state)

$$E_S - E_N = \sum_k (|\xi_k| - \varepsilon_k) + \frac{\Delta^2}{V_0} \cdot L^3 = L^3 \left\{ 2\nu_0 \int_0 d\xi \left(|\xi| - \sqrt{\xi^2 + \Delta^2}\right) \right.$$

$$+\nu_0 \int_0^{\hbar\omega_D} d\xi \, \frac{\Delta^2}{\sqrt{\xi^2+\Delta^2}} \Bigg\} = 2L^3\nu_0 \int_0^{\hbar\omega_D} d\xi \left(\xi + \frac{\Delta^2}{2\sqrt{\xi^2+\Delta^2}} - \sqrt{\xi^2+\Delta^2}\right)$$

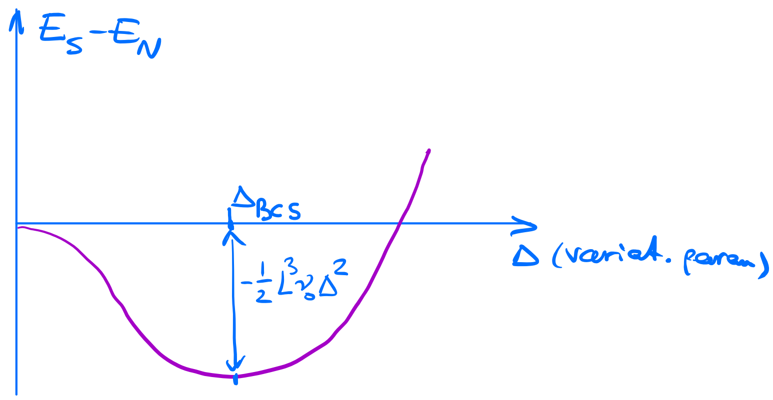
(converges at $\hbar\omega_D \to \infty$)

$$\simeq 2L^3\nu_0 \int_0^{\infty} d\xi \left(\xi + \frac{\Delta^2}{2\sqrt{\xi^2+\Delta^2}} - \sqrt{\xi^2+\Delta^2}\right) = -\frac{1}{2}L^3\nu_0\Delta^2$$

(reducing energy by $\sim \Delta^2/E_F$ per particle in the Fermi sea).

Equivalent consideration: variational principle. The trial ground state wave function: $|\Psi\rangle = \prod_{\vec{k}}(u_{\vec k} + v_{\vec k}\, c^+_{k\uparrow} c^+_{-k\downarrow})|0\rangle$; $u_{\vec k}, v_{\vec k}$ - functions of variational parameter Δ. [Exercise: check that $b_{\vec p}|\Psi\rangle = 0, \ \forall \vec p$]

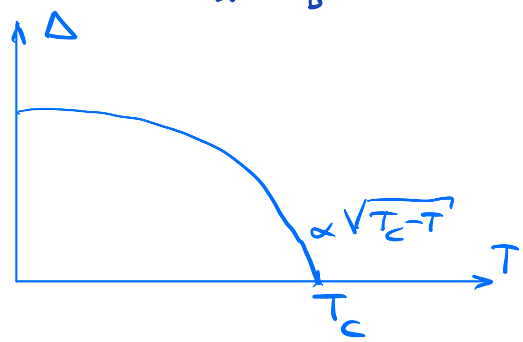
$$\langle\Psi|\mathcal{H}|\Psi\rangle_{\Delta} - \langle\Psi|\mathcal{H}|\Psi\rangle_{\Delta=0} \equiv E_S - E_N$$

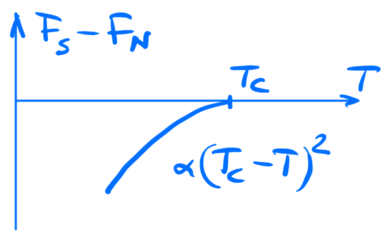


① Finite temperatures:

$$\Delta = V_0 \int_{|\xi|<\hbar\omega_D} \frac{d^3k}{(2\pi)^3} \cdot \frac{\Delta}{2\varepsilon_k} \cdot \left(1 - 2 f_F(\varepsilon_k)\right); \quad f_F(\varepsilon_k) = \frac{1}{e^{\varepsilon_k/T}+1}$$

$$\varepsilon_k = \sqrt{\xi_k^2 + \Delta^2}$$





2nd-order phase transition (mean-field theory)

② Critical velocity

$$\hbar \vec v \cdot \vec k = 2\varepsilon_k \Rightarrow \boxed{v_{cr} = \frac{|\Delta|}{\hbar k_F}} \Rightarrow j_{cr} = en \cdot v_{cr}$$

crit. current density in "clean limit" (mean free path ($l \gg \xi^*$))

(threshold to excite 2 quasiparticles from condensate)

③ Phase of Δ (order parameter) is arbitrary, $\Delta = |\Delta| e^{i\varphi}$

$$v_k = |v_k| e^{i\varphi}$$

(4) $$\langle \psi^+(r_1)\psi^+(r_2)\rangle_{GS} \propto \frac{1}{r^{\frac{d-1}{2}}} e^{-r/\xi^*}; \quad r = |\bar{r}_1 - \bar{r}_2|; \quad \xi^* = \frac{\hbar v_F}{\Delta}$$

coherence length

BEC: similar, $\xi^* \to \xi^* = \frac{\hbar}{p^*} = \frac{\hbar}{mc}$ (healing length)

c ← (speed of sound)

(5) Applicability of the mean-field theory:

$$\frac{\langle (\hat{\mathcal{H}}_M - \langle \mathcal{H}_M\rangle_{BCS})^2\rangle_{BCS}}{(E_S - E_N)^2} \ll 1 \Rightarrow \Delta \ll E_F, \text{ or (same) } n(\xi^*)^3 \gg 1$$

(BEC: $n(\xi^*)^3 \gg 1$)

(9.3) Josephson effect. (9.4) Quantum fluctuations of phase, qubits

$$|\Psi_\varphi\rangle = \prod_k (u_k + |v_k| e^{i\varphi} c^+_{k\uparrow} c^+_{-k\downarrow})|0\rangle \qquad (|\Psi_\varphi\rangle = |\Psi_{\varphi+2\pi}\rangle)$$

Energy E_S independent of φ!

$$|\Psi_{N_p}\rangle = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-iN_p\varphi} |\Psi_\varphi\rangle$$

(de Gennes, Supercond. of Metals and Alloys, Westview Press, 1999, p.p. 107-109)
(Tinkham, Intro to Superconductivity, 2nd ed., MGH 1996)

Operator of # of pairs in two representations:

$$\hat{N}_p |\Psi_{N_p}\rangle \equiv N_p \cdot |\Psi_{N_p}\rangle$$

φ-rep: $(N_p \to N)$

$$\hat{N}|\Psi_N\rangle = N\int_0^{2\pi}\frac{d\varphi}{2\pi} e^{-iN\varphi}|\Psi_\varphi\rangle = \int_0^{2\pi}\frac{d\varphi}{2\pi}\left(i\frac{d}{d\varphi}e^{-iN\varphi}\right)|\Psi_\varphi\rangle$$

$$= \int_0^{2\pi}\frac{d\varphi}{2\pi} e^{-iN\varphi}\left(-i\frac{d}{d\varphi}|\Psi_\varphi\rangle\right) \Rightarrow \boxed{\hat{N} = -i\frac{d}{d\varphi}}$$

Consider operator raising the # of pairs by 1:

$$\hat{T} \equiv \sum_N |N+1\rangle\langle N|$$

In φ-representation:

$$|\psi_{N+1}\rangle = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-i(N+1)\varphi} |\psi_\varphi\rangle = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-i\varphi} \left(e^{-iN\varphi}|\psi_\varphi\rangle\right)$$

$$|\psi_N\rangle = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-iN\varphi} |\psi_\varphi\rangle \qquad \boxed{\hat{T} = e^{-i\varphi}}$$

In any representation (at the level of operators): $\boxed{[\hat{N}, e^{-i\hat{\varphi}}] = e^{-i\hat{\varphi}}}$

Brief summary on $\hat{N}, \hat{\varphi}$ reps:

$$|\psi_\varphi\rangle = \prod_k \left(u_k + |v_k| e^{i\varphi} c^+_{k\uparrow} c^+_{-k\downarrow}\right)|0\rangle, \quad \psi(\varphi+2\pi) = \psi(\varphi)$$

yields exact value of G.S. energy in the limit $n\xi^3 \to \infty$, assuming thermodynamic limit $L^3 \underbrace{\nu\Delta}_{\text{density of Cooper pairs}} \to \infty$

$$|\psi_{N_p}\rangle = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-iN_p\varphi} |\psi_\varphi\rangle \text{ - fixed \# of particles}$$

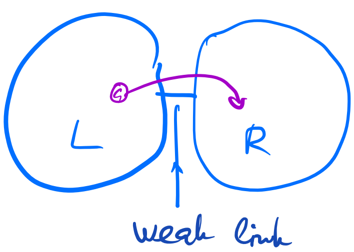
(we neglect energies ~ level spacing for single particle, $\frac{1}{\nu L^3}$, vs. Δ)

$\hat{N}$ and $\hat{\varphi}$ reps. (in the G.S. manifold): $\psi(\varphi+2\pi) = \psi(\varphi)$

$$\boxed{\hat{N} = -i\frac{d}{d\varphi}} \qquad \hat{T} \equiv \sum_N |N+1\rangle\langle N| \qquad \boxed{\hat{T} = e^{-i\varphi}}$$

$$\boxed{[\hat{N}, e^{-i\hat{\varphi}}] = e^{-i\hat{\varphi}}}$$

Connecting superconductors



Phenomenological description of a link between two superconductors

$$\mathcal{H}_J = \sum_{\ell=1}^{\infty} \left(\gamma_\ell (\hat{T}_R^+)^\ell (\hat{T}_L)^\ell + \gamma_\ell^* (\hat{T}_L^+)^\ell \hat{T}_R^\ell\right)$$

Weak tunneling:

$|\underbrace{\gamma_1}_{\text{1 pair}}| >> |\underbrace{\gamma_2}_{\text{2 pairs}}| \dots$ $\Rightarrow$ keep only γ_1 (real-valued, if TRS is observed)

$$\hat{\mathcal{H}}_J = -E_J \cos\hat{\varphi}, \quad \hat{\varphi} = \hat{\varphi}_L - \hat{\varphi}_R \quad (\varphi\text{-rep.}), \quad E_J > 0$$