04.04.2023

Introduce density of states (# of states per unit energy, per unit volume)

$$\nu(E)\,dE = \int \frac{d^d k}{(2\pi)^d}\,\delta(\varepsilon_{\vec k} - E)\,dE \underset{\substack{\uparrow \\ \text{3D, isotropic disp. relation}}}{=} \int \frac{k^2 dk}{(2\pi)^3}\,\delta(\varepsilon_k - E_F)\underbrace{\int dO_{\vec k}}_{4\pi}\,dE$$

Then

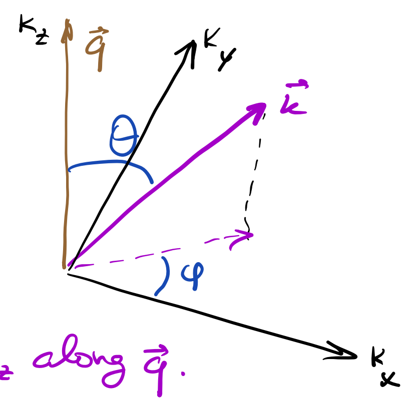
$$-\operatorname{Im}\Pi_R(\bar q,\omega) = \pi\cdot\hbar\omega\cdot\nu(E_F)\int\frac{dO_{\vec k}}{4\pi}\,\delta(\varepsilon_{\vec k+\vec q} - \varepsilon_{\vec k} - \hbar\omega),$$

$\delta(\varepsilon_k - E_F)$ $\Downarrow$ $|\vec k| = k_F$

($dO_{\vec k}$ is the solid angle element; we assumed isotropic dispersion relation, $\varepsilon_{\vec k} = \varepsilon(k)$).

For example, take $\varepsilon(k) = \hbar^2 k^2/2m^*$.

$$\varepsilon(|\vec k+\bar q|) - \varepsilon(k) - \hbar\omega = \frac{\hbar^2 k q}{m^*}\cos\theta + \frac{\hbar^2 q^2}{2m^*} - \hbar\omega$$



3D: We use spherical coordinates for $\vec k$ with k_z along $\vec q$.

$$-\operatorname{Im}\Pi_R(\bar q,\omega) = \pi\hbar\omega\,\nu(E_F)\cdot\frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} \underbrace{\sin\theta\, d\theta}_{-d(\cos\theta)}\cdot\,\delta\!\left(\frac{\hbar^2 k_F q}{m^*}\cos\theta + \frac{\hbar^2 q^2}{2m^*} - \hbar\omega\right)$$

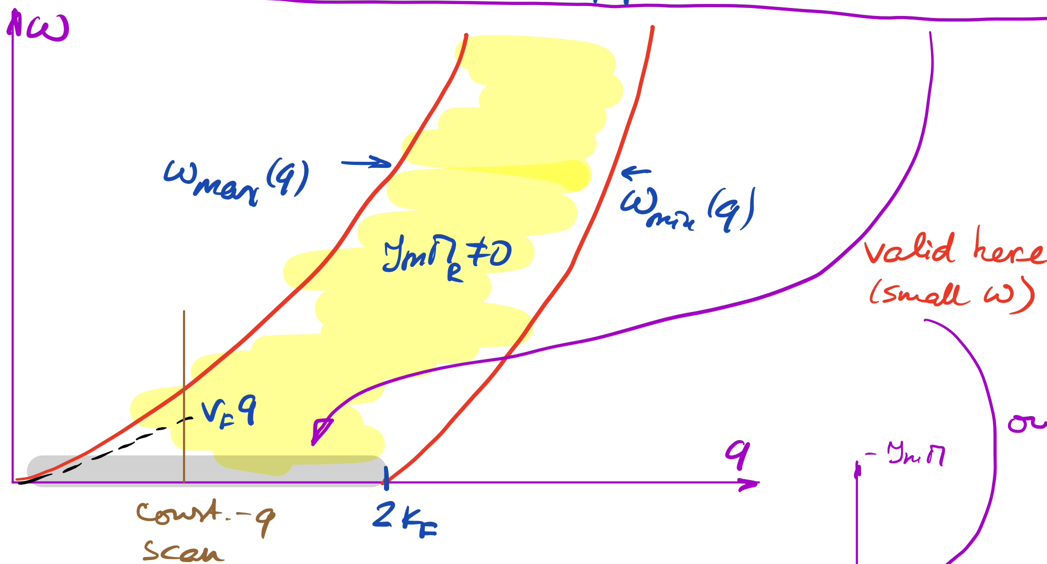
$$-\operatorname{Im}\Pi_R(\bar q,\omega) = \frac{\pi}{2}\,\nu(E_F)\,\frac{m\omega}{\hbar k_F q}\,\theta(\omega - \omega_{min}(q))\,\theta(\omega_{max}(q) - \omega)$$

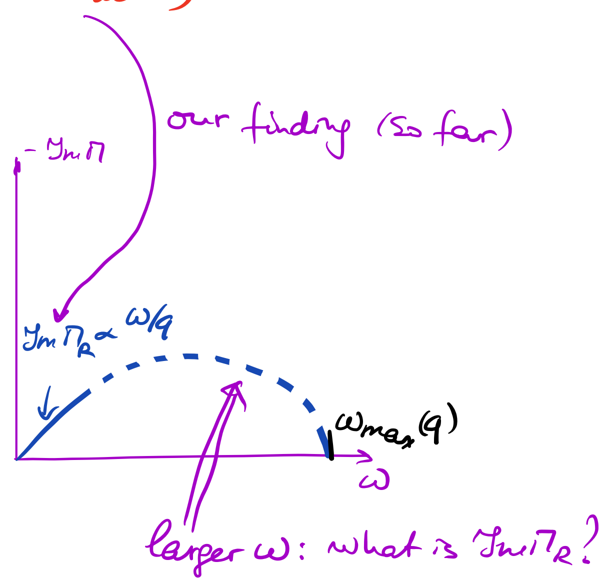
valid at $\omega \ll \omega_{max}(q)$ (the small-ω condition). Note $\mathrm{Im}\,\Pi_R(q,\omega)$ is not singular at $\omega \to 0$, allowing the absorption power $W(\omega)$ tend to 0 at $\omega \to 0$: static potential does not lead to dissipation.

(at $q < 2k_F$, $\omega_{min}(q) = 0$)

$$\boxed{-\mathrm{Im}\,\Pi_R(\bar{q},\omega) = \frac{\pi}{2}\nu(\varepsilon_F)\frac{m\omega}{\hbar k_F q}\,\theta(\omega - \omega_{min}(q))\,\theta(\omega_{max}(q) - \omega)}$$

This form valid for $d \geq 2$.





3. Threshold behavior of $\mathrm{Im}\,\Pi_R(q,\omega)\big|_{T=0}$

$$\mathrm{Im}\,\Pi_R(\bar{q},\omega)\Big|_{T=0} = -\pi\int\frac{d^dk}{(2\pi)^d}\,\theta(k_F - k)\,\theta(|\bar{k}+\bar{q}| - k_F)\,\delta(\varepsilon_{\bar{k}+\bar{q}} - \varepsilon_{\bar{k}} - \hbar\omega)$$

3 D:

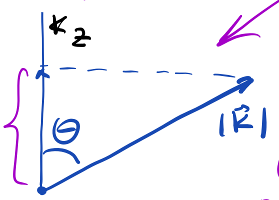
$$-\mathrm{Im}\,\Pi_R(\bar{q},\omega) = \pi\int_0^\infty\frac{k^2dk}{(2\pi)^3}\int_0^{2\pi}d\varphi\int_0^\pi\sin\theta\,d\theta\;\theta(k_F - |\bar{k}|)\,\theta(|\bar{k}+\bar{q}| - k_F)$$

$$\times\,\delta\!\left(\frac{\hbar^2 kq}{m^*}\cos\theta + \frac{\hbar^2q^2}{2m^*} - \hbar\omega\right)$$

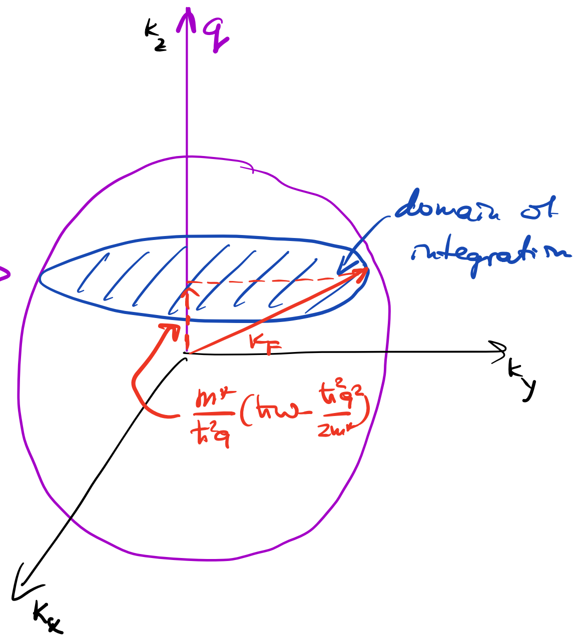
$$\frac{\hbar^2}{m^*}\underbrace{qk\cos\theta}_{\vec{q}\cdot\vec{k}} = \hbar\omega - \frac{\hbar^2 q^2}{2m^*} \Rightarrow \frac{\vec{q}\cdot\vec{k}}{|\vec{q}|} = \frac{m^*}{\hbar^2 q}\left(\hbar\omega - \frac{\hbar^2 q^2}{2m^*}\right)$$

This equation defines a <u>plane</u> in $\vec{k}$-space:

fixed by the rhs of the equation



+ Constraint $|\vec{k}| < k_F$ (and we checked in part 1 that $|\vec{k}+\vec{q}| > k_F$ poses no further limitations at $\omega \to \omega_{\substack{max\\min}}(q)$) $\Rightarrow$



$$-\mathrm{Im}\,\Pi_R(\vec{q},\omega) = \pi \int_0^\infty \frac{k^2 dk}{(2\pi)^3}\int_0^{2\pi} d\varphi \int_0^\pi \sin\theta\, d\theta\; \theta(k_F - |\vec{k}|)\,\underbrace{\theta(|\vec{k}+\vec{q}| - k_F)}_{\text{satisfied at } \omega \to \left\{\substack{\omega_{max}(q)\\ \omega_{min}(q)\\ q>2k_F}\right.}$$

$$\times\, \delta\left(\frac{\hbar^2 k q}{m^*}\cos\theta + \frac{\hbar^2 q^2}{2m^*} - \hbar\omega\right)$$

switch to cylindr. coord:

$k\cos\theta = k_z$

$$= \pi \frac{m^*}{\hbar^2 q}\cdot\frac{1}{8\pi^3}\int_0^{2\pi} d\varphi \int dk_z\, \delta\left(k_z - \frac{\hbar\omega - \hbar^2 q^2/2m^*}{\hbar^2 q/m^*}\right)\int_0^{\sqrt{k_F^2 - k_z^2}} k_\perp dk_\perp$$

$$= \frac{1}{8\pi^2}\frac{m^*}{\hbar^2 q}\cdot \pi\left[k_F^2 - k_z^2\right]\Bigg|_{k_z = \frac{\hbar\omega - \hbar^2 q^2/2m^*}{\hbar q/m^*}}$$

$$= \frac{1}{8\pi}\frac{m^*}{\hbar^2 q}(k_F - k_z)(k_F + k_z)\Bigg|_{k_z = \frac{\hbar\omega - \hbar^2 q^2/2m^*}{\hbar q/m^*}} = \frac{m^*}{8\pi\hbar^3 q}\left(\omega_{max}(q) - \omega\right)\left(\omega - \omega_{min}(q)\right)$$

(near thresholds at $\omega_{max}(q) \geq \omega \geq \omega_{min}(q)$, specific for 3D)

Near thresholds:

$$-\mathrm{Im}\,\Pi_R(\vec{q},\omega) \propto \left(\omega_{max}(q) - \omega\right)\theta\left(\omega_{max}(q) - \omega\right)$$

$$-\operatorname{Im}\Pi_R(\bar q,\omega)\propto(\omega-\omega_{min}(q))\,\theta(\omega-\omega_{min}(q))$$

From analyticity of $\Pi_R(\bar q,\omega)$:

$$\ln z=\ln|z|+i\arg z$$

$$-\operatorname{Re}\Pi_R(\bar q,\omega)\propto(\omega-\omega_{min}(q))\ln(\omega-\omega_{min}(q))\quad\text{(Kohn anomaly)}$$

4. $\Pi_R(\bar q,\omega)$ at $\omega=0$

We already found $\operatorname{Im}\Pi_R(\bar q,\omega=0)=0$

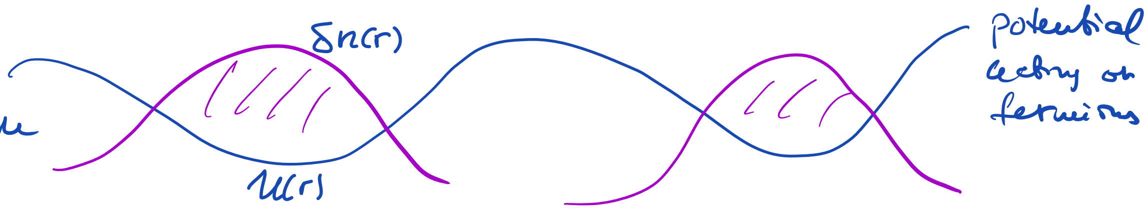
$$\operatorname{Re}\Pi_R(\bar q,0)=\int\frac{d^dk}{(2\pi)^d}\,\frac{n_{\bar k+\bar q}-n_{\bar k}}{\varepsilon_{\bar k+\bar q}-\varepsilon_{\bar k}}=\int\frac{d^dk}{(2\pi)^d}\,\frac{n_F(\varepsilon_{\bar k+\bar q})-n_F(\varepsilon_{\bar k})}{\varepsilon_{\bar k+\bar q}-\varepsilon_{\bar k}}$$

Limit $q\to0$:

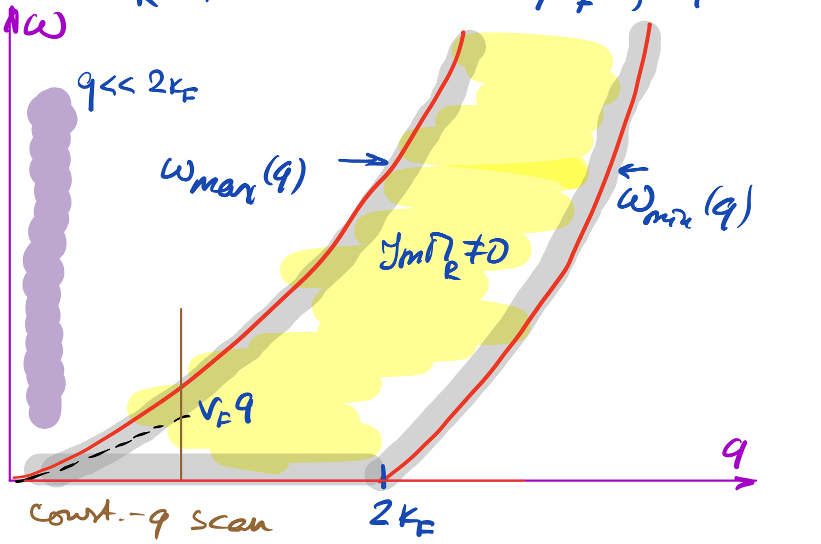
$$\operatorname{Re}\Pi_R(q\to0,0)=\int\frac{d^dk}{(2\pi)^d}\cdot\frac{\partial n_F}{\partial\varepsilon}\cdot\left(\frac{\partial\varepsilon}{\partial\vec k}\cdot\vec q\right)\frac{1}{\left(\frac{\partial\varepsilon}{\partial\vec k}\cdot\vec q\right)}$$

$$=-\int\frac{d^dk}{(2\pi)^d}\,\delta(\varepsilon_{\vec k}-E_F)=-\nu(E_F)\ \leftarrow\ \text{any dimension, } T=0$$

$\delta n=\nu\,\delta\mu$



5. $\operatorname{Re}\Pi_R(\bar q,\omega)$ at $\omega\gg qv_F$, $q\ll 2k_F$



$$\operatorname{Re}\Pi_R(q,\omega)=\int\frac{d^3k}{(2\pi)^3}\,\frac{n_{\bar k+\bar q}-n_{\bar k}}{\underbrace{\varepsilon_{\bar k+\bar q}-\varepsilon_{\bar k}}-\hbar\omega}$$

have to expand in $\bar q$, because

$$\int\frac{d^3k}{(2\pi)^3}\,n_{\bar k+\bar q}=\int\frac{d^3k}{(2\pi)^3}\,n_k$$

at any q

At $q \ll 2k_F$:

$$-\mathrm{Re}\,\Pi_R(q,\omega) = \int \frac{d^3k}{(2\pi)^3}\cdot\frac{\partial n_F}{\partial\varepsilon}\left(\frac{\partial\varepsilon}{\partial\vec k}\cdot\vec q\right)\cdot\underbrace{\frac{1}{\hbar\omega - \frac{\partial\varepsilon}{\partial\vec k}\vec q}}$$

expand to 1st order in $\vec q$; 0-th order term ($\propto 1/\hbar\omega$) does not contribute because $\int\frac{d^3k}{(2\pi)^3}\frac{\partial n_F}{\partial\varepsilon}\left(\frac{\partial\varepsilon}{\partial\vec k}\cdot\vec q\right)\Big|_{q\to 0} = \int\frac{d^3k}{(2\pi)^3}n_{k+q} - \int\frac{d^3k}{(2\pi)^3}n_k = 0$

$$\approx \int\frac{d^3p}{(2\pi\hbar)^3}\frac{\partial n_F}{\partial\varepsilon}\frac{\partial\varepsilon}{\partial p_i}\frac{\partial\varepsilon}{\partial p_j}\cdot\frac{q_iq_j}{\omega^2}$$

04.06.2023

Introduce

$$\left(\frac{1}{m}\right)_{ij} \equiv -\frac{1}{n}\int\frac{d^3\vec p}{(2\pi\hbar)^3}\frac{\partial n_F}{\partial\varepsilon}\frac{\partial\varepsilon}{\partial p_i}\frac{\partial\varepsilon}{\partial p_j}$$

$$\equiv -\frac{1}{n}\int d\varepsilon\int\frac{d^3\bar p}{(2\pi\hbar)^3}\delta(\varepsilon-\varepsilon(\bar p))\frac{\partial n_F}{\partial\varepsilon}\frac{\partial\varepsilon}{\partial p_i}\frac{\partial\varepsilon}{\partial p_j}$$

$$= \int d\varepsilon\left(-\frac{\partial n_F}{\partial\varepsilon}\right)\cdot\underbrace{\frac{1}{n}\int\frac{d^3\bar p}{(2\pi\hbar)^3}\delta(\varepsilon-\varepsilon(\bar p))\,v_i(\bar p)\,v_j(\bar p)} \qquad \vec v \equiv \frac{\partial\varepsilon}{\partial\vec p}$$

$$\left(\frac{1}{m}\right)_{ij}(\varepsilon) \equiv \frac{1}{n}\,\nu(\varepsilon)\,\underbrace{\overline{v_i(\bar p)v_j(\bar p)}}$$

DOS at energy ε

bar stands for averaging over the surface $\varepsilon(\bar p)=\varepsilon$ in $\bar p$-space

$$\mathrm{Re}\,\Pi_R(q,\omega) = n\left(\frac{1}{m}\right)_{ij}\frac{q_iq_j}{\omega^2}$$

$$\left(\frac{1}{m}\right)_{ij} = \int d\varepsilon\left(-\frac{\partial n_F}{\partial\varepsilon}\right)\frac{1}{n}\nu(\varepsilon)\,\overline{v_i(\bar p)v_j(\bar p)}\,; \qquad \left(\frac{1}{m}\right)_{ij}\Bigg|_{T=0} = \left(\frac{1}{m}\right)_{ij}(E_F)$$

Example:

$$\varepsilon(p) = p^2/2m^* \Rightarrow \quad \overline{v_x(\bar p)v_x(\bar p)} = \overline{v_y(\bar p)\cdot v_y(\bar p)} = \overline{v_z(\bar p)v_z(\bar p)} = \frac{1}{3}\frac{2\varepsilon}{m^*}\,;$$

off-diag. components $=0$, therefore:

$$\overline{v_i(\bar p)v_j(\bar p)} = \frac{1}{3}\frac{2\varepsilon}{m^*}\delta_{ij}$$

$$\left(\frac{1}{m}\right)_{ij} = -\frac{1}{n}\int d\varepsilon\, \frac{\partial n_F}{\partial \varepsilon}\, \overbrace{\overbrace{\nu(\varepsilon)}^{\propto\varepsilon^{1/2}}\cdot \overbrace{\frac{1}{3}\cdot\frac{2\varepsilon}{m^*}}^{\propto\varepsilon}}^{\text{product }\propto\,\varepsilon^{3/2}} \delta_{ij} \underset{\uparrow}{=} \frac{1}{n}\int d\varepsilon\, n_F(\varepsilon)\,\nu(\varepsilon)\,\frac{\delta_{ij}}{m^*} = \frac{1}{\cancel{n}}\cdot\cancel{n}\cdot\underbrace{\frac{\delta_{ij}}{m^*}}_{\text{independent of } E_F}$$

by parts $\varepsilon^{3/2}\to\varepsilon^{1/2}\to\nu(\varepsilon)$, $\frac{\partial n_F}{\partial\varepsilon}\to n_F(\varepsilon)$

Isotropic $\varepsilon(\vec{k})$: $1/m$ is diagonal,

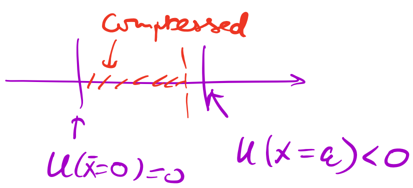
$$\left(\frac{1}{m}\right)_{ij} = \frac{1}{m^*}\delta_{ij}\,. \text{ Therefore: } \qquad \boxed{\operatorname{Re}\Pi_R(q,\omega) = \frac{n}{m^*}\cdot\frac{q^2}{\omega^2}}$$

Corrections in orders of qv_F/ω:

$$\operatorname{Re}\Pi_R(q,\omega) = \frac{n}{m^*}\frac{q^2}{\omega^2}\left(1 + O\left(\frac{qv_F}{\omega}\right)^2\right) \longleftarrow \text{expansion parameter}$$

$\uparrow$ does not contain $\hbar$, a classical quantity

The result for $\Pi_R(q,\omega)$ can be obtained from classical continuous medium mechanics (as one expects in the long-wavelength, $q\to 0$, limit). Introduce a unit-volume displacement $\vec{u}(\vec{r},t)$ of continuous medium (displacement is caused by ext. potential which we define as $-v(\vec{r},t)$, in agreement with the def. of perturbation Hamiltonian $\mathcal{H}_1$, see Lecture Notes of 03.28.2023)



(1) $\delta n = -n\,\mathrm{div}\,\vec{u}$

$\uparrow$ var. of density as a result of non-uniform $u(\vec{r})$

(2) $$n\cdot m^*\frac{d^2u}{dt^2} = n\nabla v(\vec{r},t) \qquad (\text{potential } -v(\vec{r}))$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underbrace{\vec{v}\cdot\vec{\nabla}}_{}$$

neglect (linear response).

(1) $$\delta n(\vec{q},\omega) = i n\,\vec{q}\cdot\vec{u}(\vec{q},\omega)$$

(2) $$-n m^*\omega^2\vec{u} = -in\vec{q}\,v_{\vec{q}}$$

$$\boxed{\delta n(\vec{q},\omega) = -\frac{n}{m^*}\frac{q^2}{\omega^2}v_{\vec{q}}} \quad\Rightarrow\quad \boxed{\operatorname{Re}\Pi_R(q,\omega) = \frac{n}{m^*}\cdot\frac{q^2}{\omega^2}}$$

Continuity equation

$$\frac{\partial n}{\partial t} + \operatorname{div}\vec{j} = 0 \Rightarrow \quad i\omega\,\delta n(\bar{q},\omega) - i\bar{q}\cdot\vec{j}(\bar{q},\omega) = 0$$ (j: particle density current)

$$\vec{j}(\bar{q},\omega)\cdot\frac{\vec{q}}{q} = \frac{\omega}{q}\,\delta n(\bar{q},\omega);$$ use here $\delta n(\bar{q},\omega) = -\frac{n}{m^*}\frac{q^2}{\omega^2}U_{\bar{q}}$

For charge current density response to electric field: $\vec{j}_c = e\vec{j}$; $U_{\bar{q}} = -e\frac{\bar{q}\vec{E}_q}{q^2}$

allows to find longitudinal component of the conductivity tensor

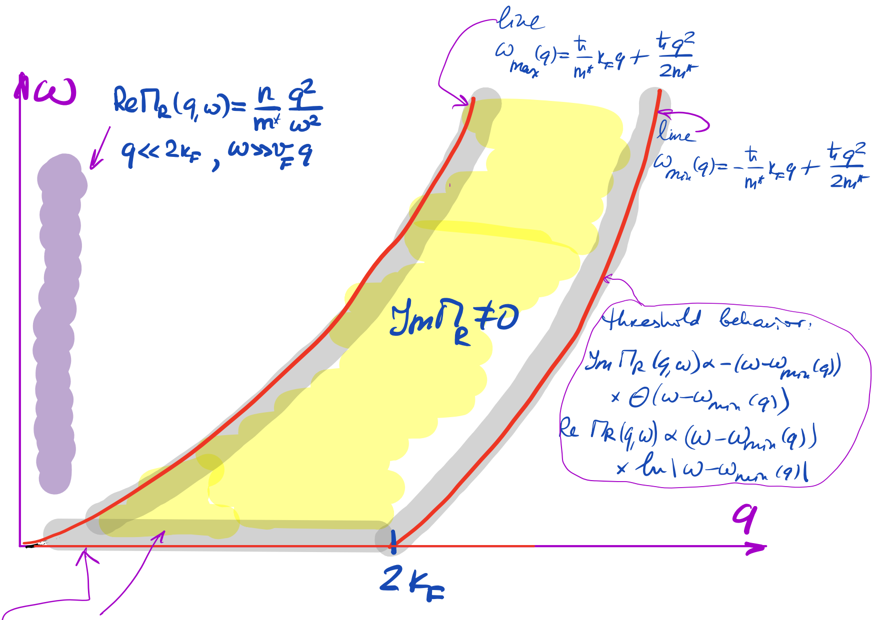
$$\left(j_c(\bar{q},\omega)\right)_i = \sigma_{ij}(\bar{q},\omega)\,E_j(\bar{q},\omega)$$

long-wavelength limit $(q\to 0)$ of longitudinal component $(\vec{j}\parallel\vec{E}\parallel\bar{q})$

$$\sigma_{\parallel}(q=0,\omega) = \frac{ne^2/m^*}{i(\omega - i0)}$$ (Drude peak)

$\hbar\omega \ll E_F$

Summary for the free-fermion $\Pi_R(q,\omega)$



$\omega \ll v_F q$

$$\mathrm{Im}\,\Pi_R(q,\omega) = \frac{\pi}{2}\nu(E_F)\frac{m^*\omega}{\hbar k_F q}$$ ← the ω factor reflects small (at $\omega\to 0$) phase space for excitations

$\omega = 0$, $q \ll 2k_F$ $\quad \mathrm{Re}\,\Pi_R(q\to 0,\omega=0) = -\nu(E_F)$

6. Random Phase Approximation (RPA)

Consider interacting fermions, but assume the interaction is weak and long-ranged. For example, consider Coulomb interaction in a "dense" Fermi gas.

"Typical" interaction of two fermions: $V \sim \frac{e^2}{r}\Big|_{r\sim\lambda_F} \sim e^2 \cdot k_F$ (up to numerical factors)

Fermi energy $E_F = \hbar^2 k_F^2 / 2m^*$ $\Rightarrow \frac{V}{E_F} \sim \frac{e^2 m^*}{\hbar^2 k_F} = \frac{e^2}{\hbar v_F}$

Weak interaction $\frac{V}{E_F} \ll 1 \Rightarrow$ gas parameter $\frac{e^2}{\hbar v_F} \ll 1$ (assume)

But the Coulomb potential is long-ranged, $V_{\vec q} = \frac{4\pi e^2}{q^2}$

$\left[\nabla^2 V(\vec r) = -4\pi e^2 \delta(r) \Rightarrow V_{\vec q} = \frac{4\pi e^2}{q^2}\right]$

Now we want to find the response of interacting fermions to the external potential $v_{\vec q}$ introduced by an external charge en^{ext}, so in our case $\boxed{v_{\vec q} = V_q n_q^{ext}}$

In the following, the response function Π_R of non-interacting fermions is denoted as $\Pi_0(q,\omega)$. The response function of the real system of interacting fermions is $\Pi(q,\omega) \leftarrow$ full response function

Def. of $\Pi(q,\omega)$:

$$\delta n_{\vec q} = \Pi(q,\omega)\, V_q\, n_{\vec q}^{ext}$$

We are after $\Pi(q,\omega)$. Perturbatively in the interaction potential V_q:

$$\begin{cases} \Pi = \Pi^{(0)} + \Pi^{(1)} + \dots & \Pi^{(n)} \propto V^{(n)} \\ \delta n = \delta n^{(1)} + \delta n^{(2)} + \dots & \delta n^{(n)} \propto V^{(n)} \end{cases}$$

In the 1-st order, $\boxed{\delta n^{(1)}_{\bar{q}} = \Pi_0 V_q n^{ext}_{\bar{q}}}$ (as $\Pi^{(0)} = \Pi_0$)

To evaluate the next order, we may try treating $\delta n^{(2)}$ as a bare response (Π_0) to the induced density $\delta n^{(1)}_{\bar{q}}$:

$$\delta n^{(2)}_{\bar{q}} \simeq \Pi_0 V_q \delta n^{(1)} \Rightarrow \Pi^{(1)} = \Pi_0 V_q \Pi_0$$

Justification of keeping only this term in $\Pi^{(1)}$: the strongest divergence at $q \to 0$ in $\Pi^{(1)}(q,\omega)$ (i.e. other terms which are $\propto V_{\bar{k}}$ in $\Pi^{(1)}(q,\omega)$ are less divergent at $q \to 0$, as they contain $\int d\bar{k}\, V_{\bar{k}}\, K(\bar{k}-\bar{q}) \propto V(k \sim k_F)$ at $q \to 0$).

Further iterations yield $\delta n^{(i)}_{\bar{q}} = \Pi_0 V_q \delta n^{(i-1)}_{\bar{q}}$

$$\delta n_{\bar{q}} = \sum_{i=1}^{\infty} \delta n^{(i)}_{\bar{q}} = \sum_{i=1}^{\infty} (\Pi_0 V_q)^n n^{ext}_q = \frac{\Pi_0 V_q}{1 - \Pi_0 V_q} n^{ext}_q$$ (Geometric series)

$$\boxed{\Pi^{RPA}(q,\omega) = \frac{\Pi_0(q,\omega)}{1 - \Pi_0(q,\omega) V_q}}$$

This is a sum of terms $\propto \left(\frac{e^2}{\hbar v_F}\right)^n \cdot \left(\frac{k_F}{q}\right)^{2n}$, i.e. most divergent at $q \to 0$ in each order of the small parameter $e^2/\hbar v_F \ll 1$ (at $\omega = 0$).

To derive $\Pi^{RPA}(q,\omega)$, we considered a response to a special form of perturbation, $U(\bar{q},\omega) = V_{\bar{q}} n^{ext}(\bar{q},\omega)$. Now, once we derived $\Pi^{RPA}(q,\omega)$, we may use it to evaluate linear response to any form of external potential $V_{ext}(\bar{q},\omega)$, as long as we are interested in response at small q (e.g. at $q r_{TF} \ll 1$, see below for the derivation of r_{TF}).