04. 18. 2023

7.2 Fermi Liquid theory in Hartree-Fock approximation

Single-particle excitations near the Fermi level are well-defined (long lifetime) even in the presence of interaction (see the end of Sec. 4). However, the parameters of these excitations (quasiparticles) are renormalized by interaction. We illustrate this using HF approximation.

(i) Effective mass m^*:

Define the Fermi velocity corrected by interaction (isotropic spectrum) and effective mass m^*,

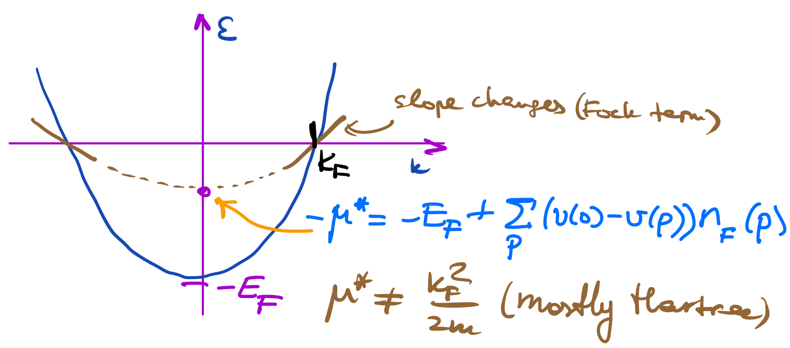
$$v_F \equiv \frac{1}{\hbar}\frac{d}{dk}\left(\varepsilon(k)+\Sigma(k)\right)\Big|_{|\vec{k}|=k_F} \equiv \frac{\hbar k_F}{m^*} \leftarrow \text{definition of effective mass } m^*$$

k_F is independent of $v(q)$ and fixed by the density of fermions

The "bare" value of mass satisfies: $\left.\frac{d\varepsilon}{dk}\right|_{k=k_F} = \frac{\hbar k_F}{m}$. Comparing with v_F, we relate the renormalized mass m^* to the bare value m:

$$\boxed{\frac{m}{m^*} = 1 + \left.\frac{\partial \Sigma(k)}{\partial \varepsilon(k)}\right|_{k=k_F}}$$

fixed by density



Heavy fermions: $m^* \gg m_0$ (example: UPt_3)

The effective mass m^* can be inferred from a measurement of low-temp. ($T/E_F \ll 1$) specific heat $c(T)$.

Recall that for non-interacting fermions $c_0(T) = \frac{\pi^2}{6} k_B^2 \nu_0(E_F) T$.

The density of states here

$$\nu_0(E_F) = \int \frac{d^3k}{(2\pi)^3}\,\delta(\varepsilon(\vec{k})-E_F) = \int d\varepsilon \int \frac{dS_\varepsilon}{(2\pi)^3}\,\delta(\varepsilon(k)-E_F)$$

$$d^3k = d\varepsilon\, \underbrace{dS_\varepsilon} \qquad ; \quad \delta(\alpha x) = \frac{1}{|\alpha|}\delta(x)$$

element of an equienergy surface in k-space

$$\nu_0(E_F) = \int d\varepsilon\, \delta(\varepsilon-E_F) \int \frac{dS_\varepsilon}{(2\pi)^3}\cdot \frac{1}{\left|\frac{\partial\varepsilon}{\partial\vec{k}}\right|} = \int \frac{dS_{E_F}}{(2\pi)^3}\,\frac{1}{\left|\frac{\partial\varepsilon}{\partial\vec{k}}\right|}$$

Interaction renormalizes

$$\varepsilon(k) \to \varepsilon(k) + \Sigma(k) \ ; \qquad \left|\frac{\partial\varepsilon}{\partial\vec{k}}\right| \to \left|\frac{\partial}{\partial\vec{k}}(\varepsilon(k)+\Sigma(k))\Big|_{|\vec{k}|=k_F}\right|$$

Therefore due to interactions $\nu_0(E_F) \to \nu^*(\mu)$,

$$\nu^*(\mu) = \nu_0(E_F)\cdot\frac{1}{1+\frac{\partial\Sigma(k)}{\partial\varepsilon(k)}\Big|_{k=k_F}} \to \nu^*(\mu) = \nu_0(E_F)\cdot\frac{m^*}{m}$$

at the Fermi surface, k_F is fixed by the particle density, $n \propto k_F^3$

Renormalisation $\nu_0 \to \nu^*$ results in the renormalization of $C(T)$:

$$\boxed{C = \frac{\pi^2}{6}\cdot k_B^2\,\nu^*(\mu)\cdot T} \to \frac{m^*}{m}$$

$$\boxed{C = \frac{\pi^2}{6} k_B^2\, T\, \nu_0(E_F)\,\frac{m^*}{m}} = \gamma\cdot T$$

$\leftarrow$ measurable coeff.

(ii). Pressure and compressibility

Reading

Ω - grand canonical therm. potential

$p = -\frac{\Omega}{V}$ use HF approx. to establish the lower bound on p

$$\Omega_{trial} = -\sum_k \left\{ k_BT \ln\left[1+e^{-\beta(\varepsilon(k)+\Sigma(k)-\mu)}\right] + \frac{1}{2}\Sigma(k)\, n_F(\varepsilon(k)+\Sigma(k)-\mu)\right\}$$

$T=0$

$$\Omega_{trial} = \sum_k \left(\varepsilon(k) + \frac{1}{2}\Sigma(k) - \mu\right) \underbrace{n_F(\varepsilon(k)+\Sigma(k)-\mu)}_{\text{taken at } T=0}$$

$$p = -\int \frac{d^3k}{(2\pi)^3} \left(\varepsilon(k) + \frac{1}{2}\Sigma(k) - \mu\right) \Theta(k_F - k)$$

k_F is fixed by density n, independent of interaction → equation for μ:

$$\varepsilon(k_F) - \mu + \Sigma(k_F, \mu) = 0$$

$$\boxed{p = -\int \frac{d^3k}{(2\pi)^3} \left(\varepsilon(k) - \varepsilon(k_F) + \overbrace{\frac{1}{2}\Sigma(k) - \Sigma(k_F)}^{\text{effect of interaction}}\right) \Theta(k_F - k)}$$

(If $\Sigma(k)$ would be k-indep. (e.g. only Hartree term) then $\Delta p = \frac{1}{2}\Sigma \cdot n$)

Compressibility $\varkappa$

$$\frac{\partial p}{\partial n} \equiv \frac{1}{n \cdot \varkappa} \quad (\text{def. of } \varkappa) \Rightarrow \boxed{\varkappa = \frac{1}{n^2}\frac{\partial n}{\partial \mu}}$$

$$\frac{\partial p}{\partial n} = n \frac{\partial \mu}{\partial n}$$

at $T=0$

μ is proxy for density n here

$$\varkappa = \frac{1}{n^2}\frac{\partial}{\partial \mu}\int \frac{d^3k}{(2\pi)^3} \Theta\left(\mu - \varepsilon(k) - \Sigma(k,\mu)\right) = \frac{1}{n^2}\int \frac{d^3k}{(2\pi)^3} \delta\left(\mu - \varepsilon(k) - \Sigma(k,\mu)\right)$$

$$\times \left[1 - \frac{\partial \Sigma(k,\mu)}{\partial \mu}\right] = \frac{1}{n^2} \underbrace{\int_0^\infty d\varepsilon\, \nu_0(\varepsilon)\, \delta\left(\mu - \varepsilon - \Sigma(k(\varepsilon),\mu)\right)}_{\longrightarrow \frac{1}{1 + \left.\frac{\partial \Sigma(k)}{\partial \varepsilon(k)}\right|_{k=k_F}} \cdot \nu_0(E_F)} \left[1 - \frac{\partial \Sigma}{\partial \mu}\right]_{k=k_F}$$

$$\boxed{\frac{\varkappa}{\underset{\text{free-fermion compressibility}}{\underbrace{\varkappa_0}}} = \frac{1 - \left.\dfrac{\partial \Sigma(k,\mu)}{\partial \mu}\right|_{k=k_F}}{1 + \left.\dfrac{\partial \Sigma(k,\mu)}{\partial \varepsilon(k)}\right|_{k=k_F}}}$$

variation of Σ with μ (and, respectively with density) at fixed k

variation of Σ with $|\vec{k}|$ at fixed μ (and therefore fixed density)

Compressibility $\varkappa$ can be inferred from measurement of sound velocity.

Convert $\nabla n \to \nabla P$: $\nabla P = \frac{\partial p}{\partial n} \nabla n$

Use continuous-medium mechanics:

$$m \frac{d}{dt} \vec{J} = -\nabla P \qquad (\text{Newton}) \qquad \operatorname{div} \vec{J} + \frac{\partial n}{\partial t} = 0$$

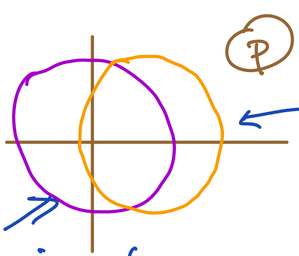
$$-\frac{1}{m} \frac{\partial p}{\partial n} \nabla^2 n + \frac{\partial^2 n}{\partial t^2} = 0 \Rightarrow (c_1)^2 = \frac{1}{m} \cdot \frac{\partial p}{\partial n} = \frac{1}{\varkappa m n}$$

$$\boxed{c_1^2 = \frac{1}{\varkappa m n}}$$ ← 1-st sound velocity (c_1)

For weakly-interacting fermions: $\varkappa \to \varkappa_0$, $c_1 = \frac{1}{\sqrt{3}} v_F < v_F$

(derive by yourself)

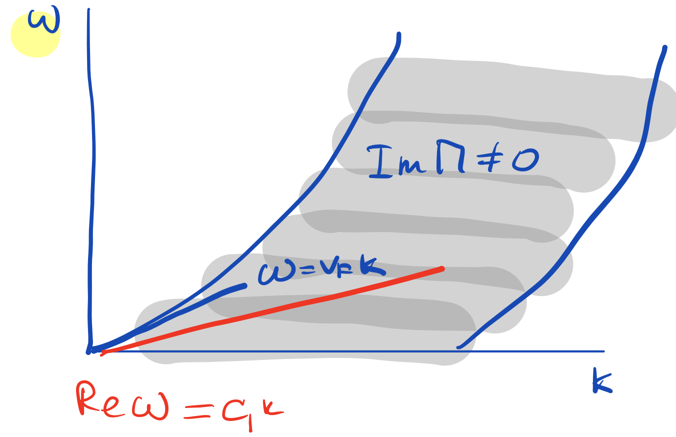
Continuous medium description: equilibrium distribution function in co-moving frame:



← Fermi surface at $\vec{J} \neq 0$

Fermi surface at $\vec{j} = 0$

For that particles need to relax over the cycle of the wave ($\omega \tau_{\text{fermion}} \ll 1$)



First Sound can propagate if its frequency $\omega \ll 1/\tau$, where $1/\tau(T)$ is the relaxation rate for fermions; note that $\frac{1}{\tau(T)} \sim T^2/E_F$

The 1st sound branch is within the continuum $\Rightarrow \operatorname{Im}\omega \neq 0$, $\operatorname{Im}\omega \propto \left(\frac{c_1 k}{T}\right)^2$; equivalently, at given ω the wavevector of the 1st sound wave is complex, $q = \omega/c_1 + i\alpha$, $\alpha \propto \omega^2/T^2$.

(Baym, Pethick, p. 45 in Landau Fermi Liquid theory, Wiley-VCH, 1991)

Further Reading:

Phenomenological FL theory (does not use HF explicitly): LL v. 9 Ch. 1.

7.3. Time-dependent Hartree approximation (another formulation of RPA).

External perturbation $V^{ext} = \sum_q f_q(t)\, v_q^{ext}\, \hat{n}_{-q}$

Linear response of non-interacting system:

$$\langle \delta n_{\bar q}(t) \rangle_0 = v_q^{ext} \int_{-\infty}^{\infty} dt'\, \Pi_R^{(0)}(\bar q, t-t')\, f_q(t')$$

Interaction

$$\hat V = \frac{1}{2} \sum_q v(q) :\hat n_{\bar q} \hat n_{-\bar q}: = \frac{1}{2} \sum_q v(q) \sum_{pk} c^+_{k+q} c^+_{p-q} c_p c_k$$

1. Time-dependent Hartree approx:

$$V \to \bar{V} = \frac{1}{2}\sum_{\bar{q}} v(\bar{q})\left\{ \hat{n}_{\bar{q}} \langle n_{-\bar{q}}(t)\rangle^H + \langle n_{\bar{q}}(t)\rangle^H \hat{n}_{-\bar{q}} \right\} = \sum_{\bar{q}} v(\bar{q})\langle n_{\bar{q}}\rangle^H \hat{n}_{-\bar{q}}$$

(used $v(q) = v(-q)$)

2. Self-consistency:

$$\mathcal{H} = \hat{T} + \hat{V} + \hat{V}^{ext} \longrightarrow \bar{\mathcal{H}} = \hat{T} + \underbrace{\sum_{\bar{q}}\left(v^{ext}_{\bar{q}} f_{\bar{q}}(t) + v(\bar{q})\langle n_{\bar{q}}\rangle^H\right)\hat{n}_{-\bar{q}}}_{\text{response to}}$$

We may use Π_0 to evaluate response to

Self.-const. cond.:

$$\langle n_q(t)\rangle^H = \int_{-\infty}^{\infty} dt' \, \Pi^{(0)}_R(q, t-t')\left\{ v^{ext}_{\bar{q}} f(t') + v(\bar{q})\langle n_{\bar{q}}(t')\rangle^H \right\}$$

Define

$$n(\bar{q},\omega) = \int_{-\infty}^{\infty} dt \, e^{i\omega t}\langle n_q(t)\rangle^H$$

$$n(\bar{q},\omega) = \Pi^{(0)}_R(q, \omega + i\delta)\left\{ v^{ext}_q f(\omega) + v(q)\, n(\bar{q},\omega)\right\}$$

$$\left(1 - v(q)\,\Pi^{(0)}_R(q,\omega+i\delta)\right) n(\bar{q},\omega) = \Pi^{(0)}_R(q,\omega+i\delta)\, v^{ext}_q f(\omega)$$

Define $\Pi_R(\bar{q},\omega)$ by $\quad n(\bar{q},\omega) = \Pi_R(q,\omega+i\delta)\, v^{ext}_q f(\omega)$

$$\boxed{\Pi_R(q,\omega+i\delta) = \frac{\Pi^{(0)}_R(q,\omega+i\delta)}{1 - v(q)\,\Pi^{(0)}_R(q,\omega+i\delta)}}$$

RPA (or t-dep. H approx) accounts for the medium polarization but not for the spectrum renorm. by interactions.

if the product is large, then RPA (or t-dep. H) works.

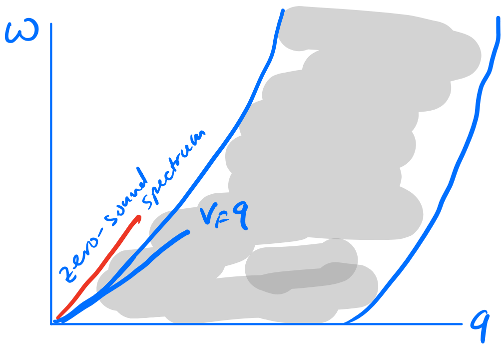
method is good if $v(q) \to \infty$ at $q \to 0$ or near the singularity of $\Pi^{(0)}_R(q,\omega)$

(Reading: Pines & Nozieres, Theory of Quant. Liquids, V. 1, Sec 5.2)

Zero sound in liquid of neutral fermions

(determined by poles of $\Pi_R(q,\omega)$ in the case of short-ranged interaction)

How $v(q)\Pi_R^{(0)}(q,\omega)$ may be large (~ 1) if $v(\vec{q})$ is small at all q?



Reminder

two limits:

(1) $\mathrm{Re}\Pi_R(q,\omega) = \frac{n}{m}\frac{q^2}{\omega^2}$ at $q \ll 2k_F$, $\omega \gg v_F q$

(2) $\mathrm{Re}\,\Pi_R(q\to 0, \omega=0) = -\nu(E_F)$ at $\omega=0$, $q \ll 2k_F$

Yet another limit: $q \to 0$ at fixed $\omega/v_F q > 1$:

$$\Pi_R^{(0)} = -\nu_0\left\{1 + \frac{\omega}{2v_F q}\ln\frac{\omega - v_F q}{\omega + v_F q}\right\}$$

Pole of $\Pi_R(q,\omega)$: $v(q)\,\Pi_R^{(0)}(q,\omega) = 1$

$$-\nu_0 v(q)\left\{1 + \frac{\omega}{2v_F q}\ln\left(\frac{\omega - v_F q}{\omega + v_F q}\right)\right\} = 1 \Leftarrow \text{eq. for } \omega(q)\text{, dispersion of a wave (0-sound wave)}$$

Solve it using smallness of $v(q)$:

$$\frac{\omega}{v_F q} = x \qquad \frac{1}{2}\ln\left(\frac{x-1}{2}\right) = -\frac{1}{\nu_0 v(q)} - 1$$

$$x = 1 + 2\exp\left\{-2\left(\frac{1}{\nu_0 v(0)} + 1\right)\right\}$$

$\omega = c_0 \cdot q$, $\quad c_0$ - zero-sound velocity

$$\boxed{c_0 = v_F \left\{ 1 + 2e^{-2} e^{-2/\nu_0 V(0)} \right\}}$$

limit of weak interaction

(Landau, 1957)

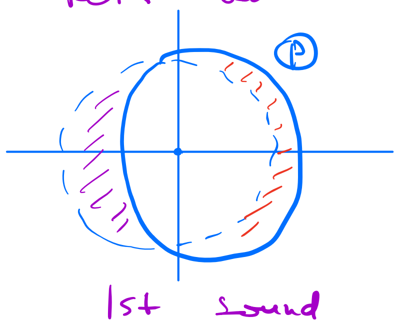
Sov. Phys. JETP $\underline{5}$, 101(57)

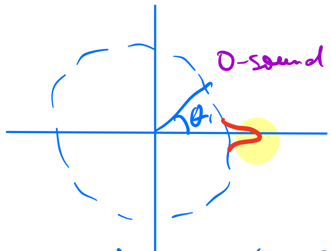
Experiments:

B.E. Keen et al Proc. Roy. Soc. $\underline{A284}$, p. 125 (1965)

W.R. Abel et al PRL $\underline{17}$, 74 (1966)

Perturbation of the Fermi sea in 1st sound and 0-sound waves:





$$\delta n_{(\vec p)} \propto e^{i\vec q \vec r - \omega t} \frac{\delta(|p| - \hbar k_F)\cos\theta}{\frac{c_0}{v_F} - \cos\theta}$$

Reading: Negele, Orland, Section 5.4.

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8. Weakly-interacting Bose gas

1. Uniform gas, Bogoliubov theory

Consider Bosons with a "point-like" interaction

$$\mathcal{H} = \sum_{\bar p} \frac{p^2}{2m} a^+_{\bar p} a_{\bar p} + \frac{g}{2L^3} \sum_{\bar p, \bar p', \bar q} a^+_{\bar p} a^+_{\bar p'} a_{\bar p' + \bar q}\, a_{\bar p - \bar q}$$

g originates from

$$\mathcal{H}_{int} = \frac{1}{2} \int d\bar r_1\, d\bar r_2\; g(\bar r_1 - \bar r_2)\, \psi^+(r_1)\, \psi^+(r_2)\, \psi(r_2)\, \psi(r_1)$$

Interaction is assumed short-ranged, $g(\bar q) \to g = \int d^3\bar r\, g(\bar r)$

$N \sim 10^5$ particles (in a cold-atom trap). Weak interaction $\equiv$ most of the particles are in the $p=0$ state. Call # of particles in state $p=0$ as N_0. At weak interaction $N - N_0 \ll N$. We may approximate $\langle a_0^+ a_0 \rangle \simeq N$. Boson comm. rel.: $a_0 a_0^+ - a_0^+ a_0 = 1$.

We may try a mean-field approx. with $\langle a_0 \rangle \neq 0$

$\langle a_0 \rangle \neq 0$, $N_0 = \langle a_0^+ a_0 \rangle \approx \langle a_0^+ \rangle \langle a_0 \rangle \Rightarrow \langle a_0 \rangle = \langle a_0^+ \rangle = \sqrt{N}$ (dispense with phases of $\langle a_0 \rangle, \langle a_0^+ \rangle$; $N - N_0 \ll N_0$).

Replace a_0, a_0^+ by $\sqrt{N}$ in $\mathcal{H}$:

$$\mathcal{H}_B = \sum_{\bar p} \frac{p^2}{2m} a^+_{\bar p} a_{\bar p} + \frac{g N_0}{L^3} \sum_{\bar p \neq 0} a^+_{\bar p} a_{\bar p} + \frac{g N_0}{2L^3} \sum_{\bar p \neq 0} \left(a^+_{\bar p} a^+_{-\bar p} + a_{-\bar p} a_{\bar p} \right) + \frac{g N_0^2}{2L^3}$$

Bogoliubov transformation:

$$a_{\bar p} = u_{\bar p} \alpha_{\bar p} - v_p \alpha^+_{-\bar p} \qquad a^+_{-\bar p} = -v_{\bar p} \alpha_{\bar p} + u_{\bar p} \alpha^+_{-\bar p}$$

$$a^+_{\bar p} = u_p \alpha^+_{\bar p} - v_p \alpha_{-p} \qquad a_{-\bar p} = -v_p \alpha^+_{\bar p} + u_p \alpha_{-\bar p}$$

Canonical transf.: $[\alpha_{\bar p_1}, \alpha^+_{\bar p_2}] = \delta_{\bar p_1 \bar p_2}$, as well $[a_{\bar p_1}, a^+_{\bar p_2}] = \delta_{\bar p_1 \bar p_2}$

Constraint on $u_{\bar p}, v_{\bar p}$: $\boxed{u_{\bar p}^2 - v_{\bar p}^2 = 1}$ (1)

$$\mathcal{H}_B = \frac{1}{2}\sum_{\bar p} \mathcal{H}_B^{(\bar p)} + \frac{1}{2} g n N;$$

$n = N/L^3$ – density of bosons

$n_0 = N_0/L^3 \simeq n$

Hamiltonian of a single pair of states here:

$$\mathcal{H}_B^{(\bar p)} = \xi_{\bar p}\left(a^+_{\bar p} a_p + a^+_{-\bar p} a_{-\bar p}\right) + gn\left(a^+_{\bar p} a^+_{-\bar p} + a_{-\bar p} a_{\bar p}\right)$$

with
$$\xi_{\bar p} = \frac{p^2}{2m} + gn$$

Substitute $a_{\bar p}, a^+_{\bar p}$ in terms of $\alpha_{\bar p}, \alpha^+_{\bar p}$:

$$\begin{aligned}\mathcal{H}_B^{(\bar p)} = &\left(\alpha^+_{\bar p}\alpha_{\bar p} + \alpha^+_{-\bar p}\alpha_{-\bar p}\right)\left[\xi_{\bar p}\left(u^2_{\bar p} + v^2_{\bar p}\right) - gn\cdot 2u_{\bar p} v_{\bar p}\right] \\ + &\left(\alpha^+_{\bar p}\alpha^+_{-\bar p} + \alpha_{-p}\alpha_p\right)\left[-\xi_{\bar p}\cdot 2u_{\bar p}v_{\bar p} + gn\left(u^2_{\bar p} + v^2_{\bar p}\right)\right] \\ + &\; 2\xi_{\bar p} v^2_{\bar p} - gn\cdot 2u_{\bar p}v_{\bar p}\end{aligned}$$

$\mathcal{H}_B^{(\bar p)}$ becomes diagonal if we require

$$\boxed{-\xi_{\bar p}\cdot 2u_{\bar p}v_{\bar p} + gn\left(u^2_{\bar p} + v^2_{\bar p}\right) = 0} \qquad (2)$$

$$\boxed{u^2_{\bar p} - v^2_{\bar p} = 1} \qquad (1)$$

$$z_{\bar p} = v_{\bar p}/u_{\bar p} \Rightarrow \quad gn\left(z^2_{\bar p} + 1\right) - 2\xi_{\bar p} z_{\bar p} = 0$$

solve it:
$$z_{\bar p} = \frac{1}{gn}\left(\xi_{\bar p} - \sqrt{\xi^2_{\bar p} - (gn)^2}\right)$$

write $v_{\bar p}$ as $v_{\bar p} = u_{\bar p}\cdot z_{\bar p}$, subst. in Eq. (1):

$$\boxed{u^2_{\bar p} = \frac{1}{2}\left(\frac{\xi_{\bar p}}{\varepsilon_{\bar p}} + 1\right); \quad v^2_{\bar p} = \frac{1}{2}\left(\frac{\xi_{\bar p}}{\varepsilon_{\bar p}} - 1\right)} \qquad (3)$$

with
$$\boxed{\varepsilon_{\bar p} = \sqrt{\xi^2_{\bar p} - (gn)^2} = \sqrt{\left(\frac{p^2}{2m} + gn\right)^2 - (gn)^2}} \qquad (4)$$

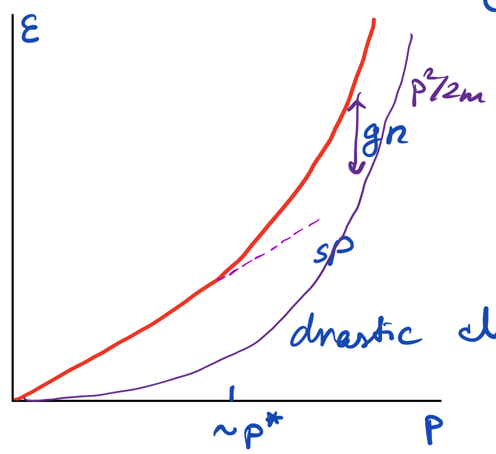
$\mathcal{H}_B^{(\bar p)}$ is diagonal in operators $\alpha_{\bar p}, \alpha^+_{\bar p}$:

$$\mathcal{H}_B^{(\bar p)} = \varepsilon_{\bar p}\left(\alpha^+_{\bar p}\alpha_{\bar p} + \alpha^+_{-\bar p}\alpha_{-\bar p}\right) + 2\xi_{\bar p} v^2_{\bar p} - gn\cdot 2u_{\bar p}v_{\bar p}$$

Excitations spectrum (isotropic, $\varepsilon_{\bar p} = \varepsilon_p$)

$$\varepsilon_p = p\sqrt{\frac{gn}{m} + \frac{p^2}{4m^2}} = \begin{cases} \varepsilon_p = s\cdot p\,; \quad s = \sqrt{\frac{gn}{m}}\,; & p \ll p^* \equiv \sqrt{2mgn} \\ \varepsilon_p = \frac{p^2}{2m} + \underbrace{gn}_{} = \xi_{\bar p} & p \gg p^* \end{cases}$$

Hartree term $\simeq \Sigma(\bar p)$, average interaction potential per particle



$\xi^* = \hbar/p^* \sim \hbar/\sqrt{mgn}$ – "healing length" (i.e. correlation radius)

drastic change of excit. spectrum at $p \ll p^*$ ($\varepsilon_p \ll gn$).

$$\mathcal{H}_B = \sum_{\bar p} \varepsilon_{\bar p}\, \alpha^+_{\bar p} \alpha_{\bar p} + E_0$$

$$E_0 = \frac{gn}{2}\cdot N + \sum_{p\neq 0} \underbrace{\left(2\xi_{\bar p} v_{\bar p}^2 - gn\cdot 2u_p v_{\bar p}\right)}_{\text{use } u_p,\, v_p} = \frac{1}{2} gn\cdot N + \frac{1}{2}\sum_{\bar p \neq 0} \left(\varepsilon_{\bar p} - \xi_{\bar p}\right) \qquad (5)$$

Rough estimate of $\sum_{\bar p\neq 0} \ldots$ by using $\begin{cases} |\varepsilon_{\bar p} - \xi_{\bar p}| \sim gn, & \text{at } p < p^* \\ \varepsilon_{\bar p} - \xi_{\bar p} = 0, & \text{at } p > p^* \end{cases}$

$$\sum_{\bar p \neq 0} \ldots \sim (gn)\cdot (p^*)^3 \cdot L^3/(2\pi\hbar)^3 \sim \left(g^{5/2} n^{3/2}\cdot m^{3/2} N\right)/(2\pi\hbar)^3$$

$$\frac{g^{5/2}\, n^{3/2}\, m^{3/2}}{(2\pi\hbar)^3} \ll gn \Rightarrow \boxed{g \ll \frac{\hbar^2}{m n^{1/3}}}$$

condition of applicability of the small-g mean field theory ($N - N_0 \ll N_0$)

rewrite as $n(\xi^*)^3 \gg 1$ (large # of particles within corr. radius)

Introduce scattering length a

(scatt. cross-sect. $\sigma\big|_{k \ll r_0^{-1}} = 4\pi a^2 \leftarrow$ def. of scatt. length)

In Born approx.: $a = mg/4\pi\hbar^2$

$$\boxed{g \ll \frac{\hbar^2}{m n^{1/3}}} \Longrightarrow \boxed{n a^3 \ll 1}$$

condition of applicability of the small-g mean field theory ($N - N_0 \ll N_0$)

$E_0 \simeq \frac{1}{2} g \frac{1}{L^3} \cdot N^2 \simeq \frac{2\pi\hbar^2}{m} a \frac{N^2}{L^3}$ – ground state energy to the leading order in g.

$T=0$ chemical potential $\mu = \frac{\partial E_0}{\partial N} = gn$ (positive!)

Pressure $P = -\frac{\partial E_0}{\partial V} = \frac{1}{2} g n^2$ $(V = L^3)$

Sound velocity $\boxed{c_1 = \sqrt{\frac{1}{m}\frac{\partial P}{\partial n}} = \sqrt{\frac{gn}{m}}}$ (see the previous lecture notes)

← coincides with s! $\boxed{c_1 = s}$

There is a reason for the coincidence:

Sound (density) waves ⇔ Bogoliubov qp's at $p \ll p^*$

Operator of particle density:

Largest term

$$\hat{n}_{\vec q} = \sum_p a^+_{\vec p} a_{\vec p + \vec q} \simeq a^+_0 a_{\vec q} + \underbrace{a^+_{-\vec q} a_0}_{\text{a "hole" in condensate and a particle above it}} \to \langle a^+_0 \rangle a_{\vec q} + a^+_{-\vec q} \langle a_0 \rangle = \sqrt{N}\,(a_{\vec q} + a^+_{-\vec q})$$

$$= \sqrt{N}\,\{(u_q - v_q)(\alpha_q + \alpha^+_{-q})\}$$

← Bogoliubov quasiparticle

Sound: wave of density ($\hat{n}_q(t)$) ⇔ Bogoliubov quasiparticle (α^+_{-q})

More about the sum $\sum_{\bar p \neq 0} (\varepsilon_{\bar p} - \xi_{\bar p})$: it is divergent at $p \to \infty$.

Using asymptote of $\varepsilon_{\vec p}$ at $p/p^* \gg 1$, we find: $\sum_{p \gg p^*} (\varepsilon_{\bar p} - \xi_{\bar p}) \sim g^2 \sum_{p \gg p^*} \frac{1}{p^2}$,

i.e. (a) it is $\propto g^2$, and (b) it is divergent at the upper limit.

The divergence is cut off by $p_{max} \sim 1/r_0$, where r_0 is the interaction range. However the cut-off does not enter in the result for E_0, if the latter is expressed in terms of scatt. length a (which, unlike g, is a measurable quantity)

To find E_0 in terms of a:

(1) find a to the second order in g

$$a = \frac{m}{4\pi\hbar^2}\left(g - \frac{g^2}{L^3}\sum_{p\neq 0}\frac{m}{p^2}\right) \Rightarrow \underbrace{g = \frac{4\pi\hbar^2 a}{m}\left(1 + \frac{4\pi\hbar^2}{L^3}a\sum_{p\neq 0}\frac{1}{p^2}\right)} \qquad (6)$$

(2) Substitute g in terms of a into the first term of Eq. (5). The divergence in Eq. (6) cancels with the divergent part of the second term in Eq. (5).

The resulting form of E_0 to the lowest two orders in na^3:

small correction

$$\boxed{E_0 = N\cdot\frac{2\pi\hbar a}{m}\left[1 + \frac{128}{15}\sqrt{\frac{na^3}{\pi}}\right]}$$

Lee, Yang 1957

LL v. 9 §25