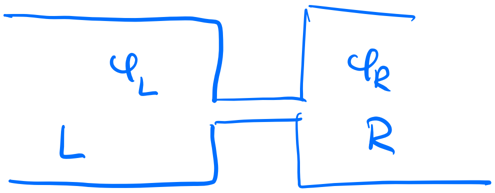
05.01.2023

Josephson current:




$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_L + \hat{\mathcal{H}}_R + \hat{\mathcal{H}}_J$$

$$[\hat{\mathcal{H}}_L + \hat{\mathcal{H}}_R, \hat{N}_L - \hat{N}_R] = 0$$

$$\hat{I} = \frac{2e}{2}\frac{d}{dt}\left(\hat{N}^L - \hat{N}^R\right) = e \cdot \frac{1}{i\hbar}[\hat{N}^L - \hat{N}^R, \hat{\mathcal{H}}]$$

$$= e \cdot \frac{1}{i\hbar}[\hat{N}^L - \hat{N}^R, \hat{\mathcal{H}}_J] = \frac{2e}{\hbar} E_J \sin\hat{\varphi}$$

$$I = \frac{2e}{\hbar} E_J \sin\varphi \quad \left(\text{or more general, } I = \frac{2e}{\hbar}\frac{\partial E_{GS}}{\partial \varphi}\right)$$

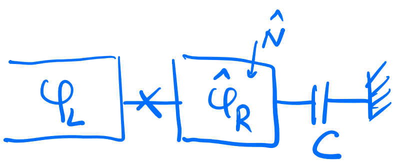
Josephson current (ground-state property, non-dissipative), B.D. Josephson, 1962

Ambegaokar, Baratoff (Phys. Rev. Lett. 10, 486, 1963):

$$I = I_c \sin(\varphi_R - \varphi_L), \qquad I_c = \frac{\pi\Delta}{2e} G_n \Rightarrow I_c R_n = \frac{\pi}{2e}\Delta$$

$G_n$ – junction's conductance ($1/R_n$) in normal state

## Charging energy effect on Josephson tunneling




Josephson energy: $-E_J \cos(\hat{\varphi}_R - \hat{\varphi}_L)$

Charging energy $\frac{1}{2C}(2e)^2\hat{N}^2 = \frac{1}{2}E_c\left(\frac{1}{i}\frac{\partial}{\partial\varphi_R}\right)^2$,

$$E_c \equiv \frac{4e^2}{C}$$

set $\varphi_L = 0$ (no loss of generality); $\hat{\varphi}_R = \hat{\varphi}$

Hamiltonian of the system (GS manifold, $\varepsilon \ll \Delta$ !):

$$\hat{\mathcal{H}} = \underbrace{\frac{1}{2C}(2e)^2\hat{N}^2}_{\text{charging energy}} - \underbrace{E_J\cos(\hat{\varphi}_R - \hat{\varphi}_L)}_{\text{Josephson energy}}$$

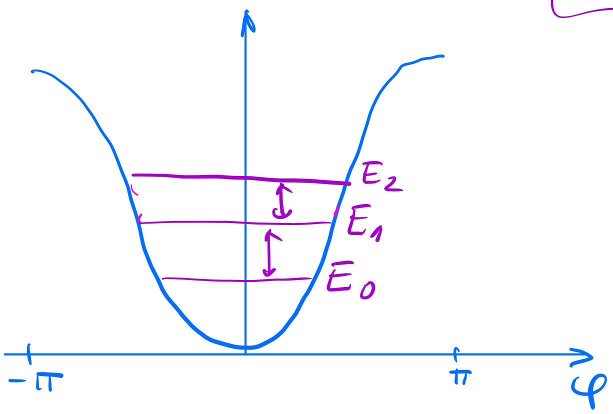
non-commuting terms

# An elementary superconducting qubit

$$\hat{\mathcal{H}} = -\frac{1}{2}E_C\left(\frac{\partial}{\partial\varphi}\right)^2 - \underbrace{E_J\cos\varphi}_{\varphi^2 - \dots \varphi^4},$$

$\psi(\varphi) = \psi(\varphi + 2\pi)$

$E_C \ll E_J$




$E_2 - E_1 \underset{(<)}{\neq} \underbrace{E_1 - E_0}$ can be used as a qubit (two-level, addressable system)

arXiv: 2003.04366, sections 1, 2.

# 10. Green's functions at $T=0$ (LL v.9, Sec. 2.7, 2.8)

$\hat{\mathcal{H}}' = \hat{\mathcal{H}} - \mu \hat{N}$, $\quad -i \frac{\partial \hat{\psi}}{\partial t} = \hat{\mathcal{H}}' \hat{\psi} - \hat{\psi} \hat{\mathcal{H}}'$ $\quad \hbar \to 1$

($\hat{\psi}(\bar{r}, t)$ is the field operator)

$t, \vec{r} \to X$ (4-vector)

1. Definition of (causal) Green function

$$G_{\alpha\beta}(X_1, X_2) = -i \langle T \hat{\psi}_\alpha(X_1) \hat{\psi}^+_\beta(X_2) \rangle = \begin{cases} -i \langle \hat{\psi}_\alpha(X_1) \psi^+_\beta(X_2) \rangle, & t_1 > t_2 \\ \pm i \langle \psi^+_\beta(X_2) \psi_\alpha(X_1) \rangle, & t_1 < t_2 \end{cases}$$

GS averaging

$T$ — operator of chronological ordering

$+$: F, $-$: B

$\alpha, \beta$ — spin indices

(transl. invariance)

$$G_{\alpha\beta}(X_1 - X_2) = G_{\alpha\beta}(X); \quad X = X_1 - X_2; \quad G_{\alpha\beta}(X) = \delta_{\alpha\beta} G(X) \quad \text{(SU(2) symmetry)}$$

$$G(\omega, \bar{p}) = \int d^3\bar{r}\, dt\, e^{i(\omega t - \bar{p}\bar{r})} G(t, \bar{r})$$

$$G(t, \bar{r}) = \int \frac{d\omega}{2\pi} \frac{d^3\bar{p}}{(2\pi)^3} e^{i(\bar{p}\bar{r} - \omega t)} G(\omega, \bar{p})$$

Density matrix in coordinate rep.:

$$\rho(\bar{r}_1, \bar{r}_2) = \frac{1}{N} \langle \psi^+(t, \bar{r}_2) \psi(t, \bar{r}_1) \rangle$$

(depends only on $\bar{r}_1 - \bar{r}_2$ due to transl. inv.)

(Normalization: $\int d^3r\, \rho(r, r) = \frac{1}{N} \int d^3r \langle \psi^+(r) \psi(r) \rangle = 1$ )

Distribution of particles over momenta:

$$\langle \psi^+_p \psi_p \rangle \equiv N(\bar{p}) = N \int d^3(\bar{r}_1 - \bar{r}_2)\, e^{-i p(\bar{r}_1 - \bar{r}_2)} \rho(r_1, r_2)$$

Using the def. of $G(t, \bar{r})$ in $\rho(\bar{r}_1, \bar{r}_2)$ and performing FT, we find:

$$N(\bar{p}) = -i \lim_{t \to -0} \int \frac{d\omega}{2\pi} e^{-i\omega t} G(\omega, \bar{p})$$

Particle density: $\frac{N}{L^3} = \overset{\text{spin factor}}{2}\int \frac{d^3\bar p}{(2\pi)^3} N(\bar p)$

## Green functions and energy spectrum of the system.

$$\langle n|\hat\Psi(t,\bar r)|m\rangle = e^{i\omega_{mn}t}\langle n|\hat\varphi(\bar r)|m\rangle$$

Eigenstates of $\mathcal{H}'$

$\Psi(t,r) = e^{i\mathcal{H}'t}\psi(r)e^{-i\mathcal{H}'t}$

$$\omega_{mn} = E_n' - E_m' = E_n - E_m - \mu(N_n - N_m) \; ; \quad N_m = N_n + 1 \qquad (1)$$

Operator of translations:

$\hat T(\bar r) = \exp\{-i\bar r\hat{\vec P}\}$ ; $\hat{\vec P}$ : is the momentum operator,

$\hat\psi(r) = T^+(\bar r)\,\hat\psi(0)\,T(\bar r)$ $\qquad [\hat P, \mathcal{H}'] = 0$

$\langle n|\hat\psi(r)|m\rangle = e^{-ik_{nm}\bar r}\langle n|\hat\psi(0)|m\rangle$, classified by momenta $P_n, P_m$ (2)

$\vec k_{nm} = P_n - P_m$

GS averaging (called $|G\rangle$ earlier in the course)

$$\underset{t>0}{G(t,\bar r)} = -i\sum_m \langle 0|\hat\psi(X_1)|m\rangle\langle m|\psi^+(X_2)|0\rangle$$

$$G(t,\bar r) = -i\sum_m |\langle 0|\hat\psi(0)|m\rangle|^2 e^{i(\omega_{0m}t + \bar P_m\bar r)} \quad (P_0 = 0) \qquad (3)$$

Similar: $t<0$

the above eqs. (1)–(3) allow to derive Lehman rep. of $G(\omega,\bar p)$:

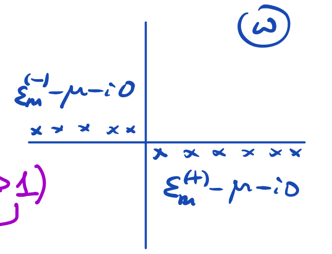
$$G(\omega,\bar p) = 2\pi\sum_m\left\{\frac{A_m\,\delta(\bar p - \bar P_m)}{\omega + \mu - \varepsilon_m^{(+)} + i0} + \frac{B_m\,\delta(\bar p + \bar P_m)}{\omega + \mu - \varepsilon_m^{(-)} - i0}\right\}$$

$A_m = |\langle 0|\hat\psi(\bar r = 0)|m\rangle|^2$, $\quad B_m = |\langle m|\hat\psi(r=0)|0\rangle|^2$

$\varepsilon_m^{(+)} = E_m(N+1) - E_0(N) = E_m(N+1) - E_0(N+1) + \mu$

(energy to add a particle) $\qquad (N \gg 1)$

$\varepsilon_m^{(-)} = E_0(N-1) - E_m(N-1) + \mu$




Poles of $G(\omega,\bar p)$ : excitation spectrum

# Retarded and advanced Green functions:

$G_R(\omega, \bar{p})$: analytic at $\operatorname{Im}\omega > 0$

$G_A(\omega, \bar{p})$: ——— " ——— $\operatorname{Im}\omega < 0$

$$\operatorname{Re} G_R(\omega,\bar{p}) = \operatorname{Re} G_A(\omega,\bar{p}) = \operatorname{Re} G(\omega,\bar{p}) \quad (\text{for } \operatorname{Im}\omega \to 0)$$

$$\operatorname{Im} G_R(\omega,\bar{p}) = \operatorname{Im} G \cdot \operatorname{sign}\omega \quad (\text{for } \operatorname{Im}\omega \to 0)$$

$$\operatorname{Im} G_A(\omega,\bar{p}) = -\operatorname{Im} G \operatorname{sign}\omega \quad (\text{for } \operatorname{Im}\omega \to 0)$$

+ analytic continuation to the proper half-plane

$$G_R(X_1 - X_2) = \begin{cases} -i\langle \psi(X_1)\psi^+(X_2) + \psi^+(X_2)\psi(X_1)\rangle, & t_1 > t_2 \\ 0 & t_1 < t_2 \end{cases}$$

(fermions)

$$G_A(X_1 - X_2) = \begin{cases} 0 & t_1 > t_2 \\ i\langle \psi^+(X_2)\psi(X_1) + \psi(X_1)\psi^+(X_2)\rangle, & t_1 < t_2 \end{cases}$$

Additional sources:

Abrikosov, Gorkov, Dzyaloshinskii Sec. 7.2

Wen, Quant. Field Theory... Sec. 2.2 (finite Temp.)

Doniach, Sondheimer, App. 2 ($G_R$, $G_A$, finite temps.)

$$G\Big|_{t_1 - t_2 \to +0} - G_{t_1 - t_2 \to -0} = (-i)\cdot\delta(\bar{r}_1 - \bar{r}_2)$$ (any of the defs. of G)

$$i\frac{\partial}{\partial t_1} G(X_1, X_2)\Big|_{t_1 \to t_2} = \delta(\bar{r}_1 - \bar{r}_2)\,\delta(t_1 - t_2) \longleftarrow$$ source term in eqs. for GF

Example: free particles $\quad i\frac{\partial}{\partial t} G_0(t,\bar{r}) - \hat{\mathcal{H}}_0' G_0(t,\bar{r}) = \delta(\bar{r})\,\delta(t)$

$$\mathcal{H}_0' = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \bar{r}^2} - \mu \; ; \qquad \boxed{G_0(\omega,\bar{p}) = \frac{1}{\omega - \xi_p + i0\cdot\operatorname{sign}\omega}}$$ (causal GF)

The spectral function (introduced in the context of ARPES):

$$A_h(\bar{p},\omega) = \frac{1}{\pi}\,\mathrm{Im}\,G(\omega,\bar{p}) = -\frac{1}{\pi}\,\mathrm{Im}\,G_R(\omega,\bar{p}) \qquad \text{holes, } \omega<0$$

$$A_p(\bar{p},\omega) = -\frac{1}{\pi}\,\mathrm{Im}\,G(\omega,\bar{p}) = -\frac{1}{\pi}\,\mathrm{Im}\,G_R(\omega,\bar{p}) \qquad \text{particles, } \omega>0$$

$$A(\bar{p},\omega) = -\frac{1}{\pi}\,\mathrm{Im}\,G_R(\omega,\bar{p}), \quad \text{any sign of } \omega$$

05.02.2023

Density response function (and relation to $\Pi_R(\omega,q)$ introduced in the context of Kubo formula)

$$\Pi(t,\bar{r}) \equiv \langle T\, \hat{n}(t,\bar{r})\,\hat{n}(0,0)\rangle$$

$$\Pi(\omega,\bar{q}) = \int dt\, d^3\bar{r}\; \Pi(t,\bar{r})\, e^{-i(\bar{q}\bar{r}-\omega t)}$$

Retarded response function:

$$\left.\begin{aligned} \mathrm{Im}\,\Pi_R(\omega,q) &= \mathrm{Im}\,\Pi(\omega,\bar{q})\,\mathrm{sign}\,\omega \\ \mathrm{Re}\,\Pi_R(\omega,q) &= \mathrm{Re}\,\Pi(\omega,q) \end{aligned}\right\} \Rightarrow \begin{aligned} &\Pi_R(t,r) \propto \Theta(t) \\ &\Pi_R^*(\omega,q) = \Pi_A(\omega,q) \quad (\mathrm{Im}\,\omega\to 0) \end{aligned}$$

Example $\Pi_0(t,r)$ for free fermions


$iG_0(X,0)$ graphic rep.

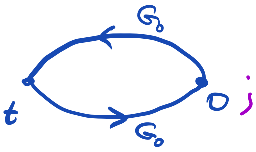
$$\Pi_0(t,r) = \langle T\,\psi^+(r,t)\psi(r,t)\psi^+(0,0)\psi(0,0)\rangle \overset{\text{Wick's th.}}{=} \langle T\,\psi^+(r,t)\psi(0,0)\rangle$$

$$\times\langle T\,\psi(r,t)\psi^+(0,0)\rangle + n^2$$

graphic rep.


$iG_0(0,X)$

$$= -\underbrace{G_0(X,0)G_0(0,X)}_{\text{graphic rep.}} + n^2$$

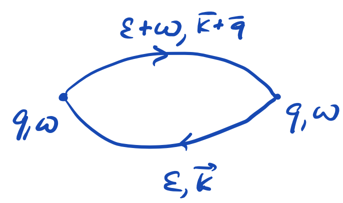



; $\Pi_0(t,\bar{r}) \to \Pi_0(\omega,\bar{q})$

$G_0(X,0) \to G_0(\omega,\bar{k})$

$$\Pi_0(\omega,\bar{q}) \propto \int\frac{d\varepsilon}{2\pi}\int\frac{d^3\bar{k}}{(2\pi)^3}\, G_0(\varepsilon+\omega,\bar{k}+\bar{q})\,G_0(\varepsilon,\bar{k})$$

$(q\neq 0)$



# 11. Perturbation theory for Green functions

$$\mathcal{H}' = \hat{\mathcal{H}}'^{(0)} + \hat{V} \; ; \qquad \hat{\mathcal{H}}'^{(0)} = \hat{\mathcal{H}}_0 - \mu \hat{N}$$

$$G = -i \langle \Phi_0 | T \Psi(X_1) \Psi^+(X_2) | \Phi_0 \rangle$$ (Operators and state $\Phi_0$ are in the Interaction rep.)

$$\Phi_0(t+\delta t) = (1 - i(\delta t)\hat{V}(t))\,\Phi_0(t) \simeq e^{-i\delta t V_0(t)}\,\Phi_0(t) \quad \text{(small } \delta t\text{)}$$

↑ Heisenb. form

$$\Phi_0(t) = \prod_{t_i = t_0}^{t} \exp\{-i\delta t \hat{V}(t_i)\}\,\Phi_0(t_0) \equiv T\exp\left\{i\int_{t_0}^{t} dt_1\, \hat{V}(t_1)\right\}\Phi_0(t_0)$$

Def: $$\hat{S}(t, t_0) = T\exp\left\{-i\int_{t_0}^{t} dt_1\, \hat{V}(t_1)\right\} \; ; \quad \Phi_0(t) = \hat{S}(t,t_0)\,\Phi_0(t_0)$$

$$\hat{S}(t_3, t_2)\,\hat{S}(t_2, t_1) = \hat{S}(t_3, t_1)$$

$$\hat{S}^{-1}(t_2, t_1)\,\hat{S}^{-1}(t_3, t_2) = \hat{S}^{-1}(t_3, t_1)$$

$$\hat{S}^{-1} = \hat{S}^{+} \quad \text{(unitary operator)}$$

Adiabatic switching on of the interaction between $t_0 = -\infty$ and finite $t$: take $\Phi_0(t) = \hat{S}(t,t_0)\,\Phi_0(t_0)$ and consider limit $t_0 \to -\infty$,

$$\Phi_0(t) = \hat{S}(t, -\infty)\,\Phi$$ ← free-particle ground state (no interaction at $t_0 \to -\infty$)

therefore

free-fermion creation operator, evolves with $\mathcal{H}_0'$

$$\hat{\Psi}(t) = \hat{S}^{-1}(t, -\infty)\,\hat{\Psi}_0(t)\,\hat{S}(t, -\infty), \quad \hat{\Psi}^+ = \hat{S}^{-1}(t, -\infty)\,\Psi_0^+(t)\,\hat{S}(t, -\infty)$$

For the Green function $(t_1 > t_2)$ (see Sec. 12 of LL v. 9 for details)

$$G(X_1, X_2) = -i \langle \Phi | \hat{S}^{-1}(t_1, -\infty)\,\Psi_0(t_1)\,\underbrace{\hat{S}(t_1, -\infty)\,\hat{S}^{-1}(t_2, -\infty)}\,\Psi_0^+(t_2)\,\hat{S}(t_2, -\infty) | \Phi \rangle$$

$$\hat{S}(t_2 - \infty)\,\hat{S}^{-1}(t_2 - \infty) = \hat{S}(t_1, t_2)$$

$$\underbrace{\hat{S}^{-1}(t_1, -\infty)\,\hat{S}^{-1}}(\infty, t_1)\,\hat{S}(\infty, t_1) = \hat{S}^{-1}(\infty, -\infty)\,\hat{S}(\infty, t_1)$$

$$G(x_1,x_2) = -i\langle \hat{S}^{-1}(\infty,-\infty)\, \hat{S}(\infty,t_1)\, \hat{\psi}_0(t_1)\, \hat{S}(t_1,t_2)\, \hat{\psi}_0^+(t_2)\, \hat{S}(t_2,-\infty)\rangle$$

$$\boxed{G(x_1,x_2) = -i\langle \hat{S}^{-1} T[\hat{\psi}_0(t_1)\hat{\psi}_0^+(t_2)\hat{S}]\rangle}$$

Here $\hat{S} \equiv \hat{S}(\infty,-\infty) = T\exp\left\{-i\int_{-\infty}^{\infty} dt_1\, \hat{V}_0(t_1)\right\}$

$e^{i\alpha}|\varphi\rangle = \hat{S}(\infty,-\infty)|\varphi\rangle \qquad \langle\varphi|\hat{S}|\varphi\rangle = e^{i\alpha}$ (non-degenerate G.S.)

$$\boxed{G(x_1,x_2) = -i\frac{1}{\langle S\rangle}\langle T[\hat{\psi}_0(x_1)\hat{\psi}_0^+(x_2)\hat{S}]\rangle}$$

## The idea of diagram technique

$$\langle T\psi_0(x)\psi_0^+(x')\hat{S}\rangle = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_n\,\langle T\psi_0(x)\psi^+(x')\hat{V}_0(t_1)\ldots\hat{V}_0(t_n)\rangle$$

from expansion of T-exp.

$$\hat{V}_0(t) = \frac{1}{2}\int d^3\bar{r}_1\, d^3\bar{r}_2\, \psi_0^+(t,r_1)\,\psi_0^+(t,r_2)\, V(\bar{r}_1-\bar{r}_2)\,\psi_0(t,r_2)\,\psi_0(t,r_1)$$

Plug in $\hat{V}_0$ into the $\sum_{n=0}^{\infty}\ldots$, do averaging using Wick's theorem

Allows to express the result in terms of sum of products of G.F.

$$iG(x_1,x_2) = iG^{(0)} + iG^{(1)}_{\alpha\beta} + \ldots$$

$$iG^{(1)} = -\frac{1}{2}i\langle T\hat{\psi}_0(x_1)\psi_0^+(x_2)\int_{-\infty}^{\infty}dt\int d^3r_3\, d^3r_4\, \psi_0^+(t,\bar{r}_3)\psi_0^+(t,\bar{r}_4)\, V(r_3-r_4)$$

$$\times\,\hat{\psi}_0(t,r_4)\,\hat{\psi}_0(t,r_3)\rangle\cdot\frac{1}{\langle S\rangle^{(0)}} + G^{(0)}\cdot\left(\frac{-\langle S\rangle^{(1)}}{(\langle S\rangle^{(0)})^2}\right)$$

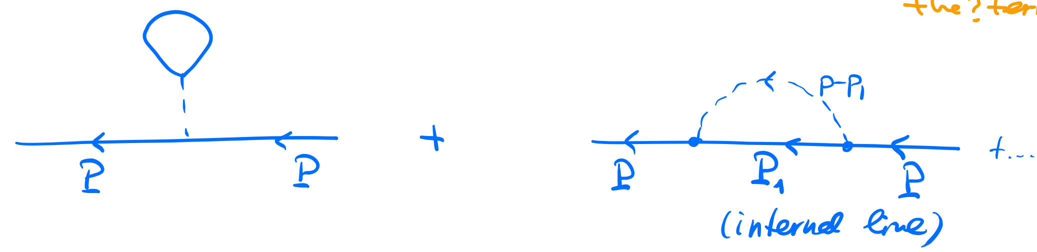
(?term); $\langle S\rangle^{(0)}=1$

(from expansion of $1/\langle S\rangle$)

When applying Wick's theorem, two types of pairings: involving and not involving $\psi^+\psi$ outside the expansion of $\hat{S}$ in orders of $\hat{V}$

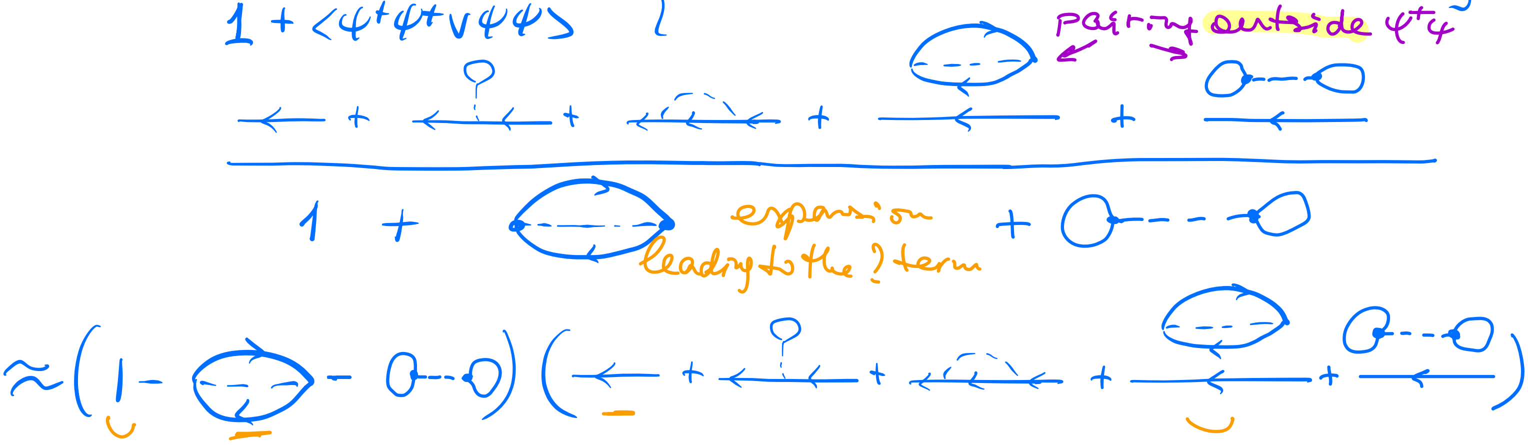
$$iG^{(1)}(P) = i\,n^{(0)}\,V(0)\,G^{(0)}(p)\,G^{(0)}(P) - \int \frac{d^4P_1}{(2\pi)^4}\, G^{(0)}(P)\,G^{(0)}(P_1)\,G^{(0)}(P) \times V(\vec{p}-\vec{p}_1) + \underbrace{\dots}_{\text{the ? term}}$$

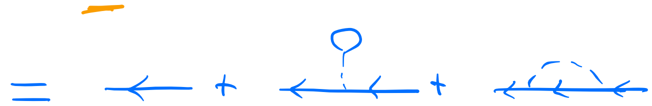
$iG^{(1)}(P) =$




Schematics showing a way to cancel the ? contributions:

$$G = \frac{1}{1+\langle \psi^+\psi^+ V \psi\psi\rangle} \cdot \left\{ \langle T\psi^+\psi\rangle + \langle T\psi^+\psi\ \psi^+\psi^+ V\psi\psi\rangle \right\}$$

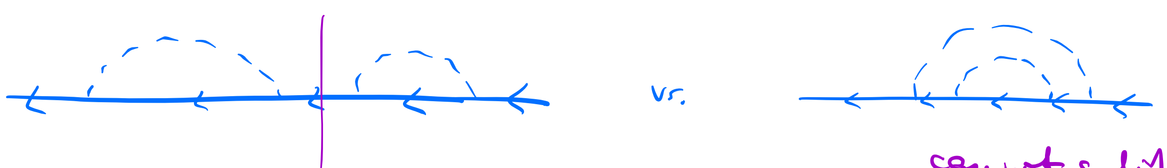



(the ? terms cancel)

Exclude pairings which involve only pairing of terms coming from expansion of $\hat{S}$-operator + do not write $\frac{1}{\langle S\rangle}$ (true to all orders)

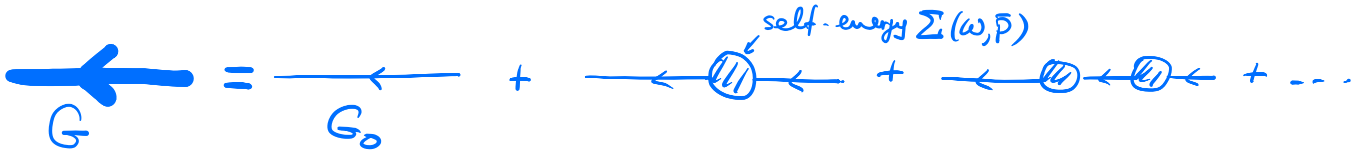
## Reducible vs. irreducible diagrams

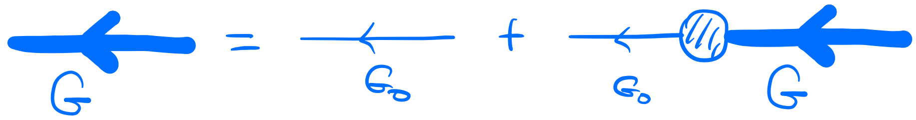



can split in two by cutting a single line

cannot split in two by cutting a single line

**Self-energy:** Sum of all diagrams ending with vertices and which can not be separated in two parts by cutting one line



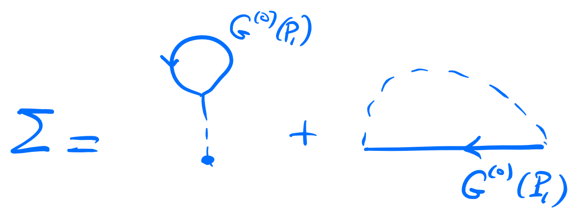




(cf. geom. series)

$$G = G^{(0)} + G^{(0)}\Sigma G \qquad \text{Dyson's equation}$$

$$G = \frac{1}{\omega - \xi_{\bar{p}} - \Sigma(\omega,\bar{p})}$$

1-st order

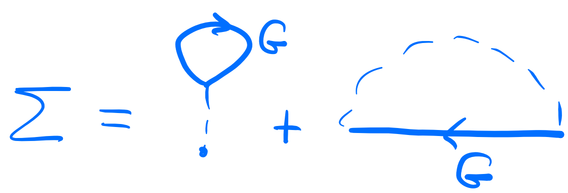



$$\Sigma(\omega=\xi_k, k) = v(0)\sum_p n_F(\xi_p) - \sum_q v(\bar{q})\, n_F(\xi_{\bar{k}+\bar{q}})$$

First order in V for $\Sigma \to$ infinite series in V for G: much better than the first order in V for G. (Similar to perturbation theory for energy $E = E^{(0)} + E^{(1)} + \dots$ being much better than

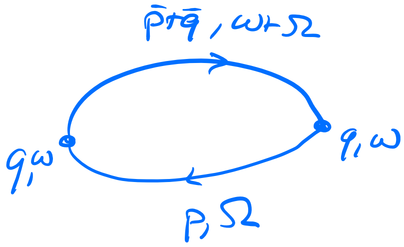
perturbation theory for time-dependent wave function:
$\psi(t) = e^{-i(E^{(0)}+E^{(1)}+\dots)t}\psi(0)$ vs. $\psi(t) \propto e^{-iE^{(0)}t}\psi(0) + \psi^{(1)} + \dots)$

**HF approx recipe:**

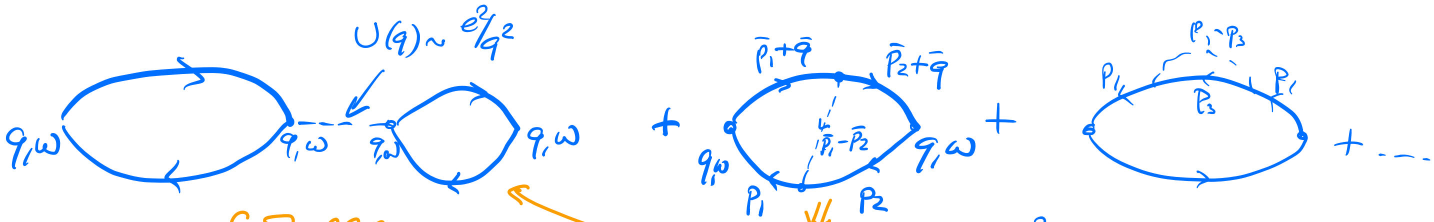



$$G = \frac{1}{\omega - \xi_{\bar{p}} - \Sigma(\omega,\bar{p})}$$

# Justification of RPA:

$\Pi_0(\omega, \bar{q}) \sim$ 

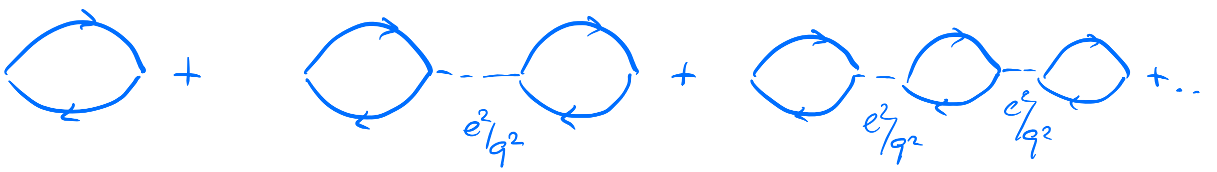

1st order correction:




compare:

integration dulls up the singularity in q

$$\int dp_1 \int dp_2 \, \frac{e^2}{(\bar{p}_1 - \bar{p}_2)^2}$$

$\Pi_{RPA} \sim$ 


Doniach, Sondheimer, Sec. 6.5