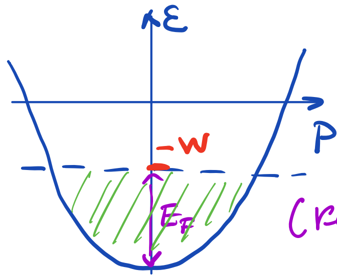
Now let us look at $A_h(\vec{p},\omega)$ for non-interacting fermions



$$\varepsilon(\vec{p}) = \frac{p^2}{2m^*} - (E_F + W) \; ; \quad E_F \equiv \frac{p_F^2}{2m^*} \; ; \; \varepsilon(p_F) = -W$$

(ref. point for energy is set by $k^2/2m$ for free electron)

$\hbar = 1$

$$\langle G | \psi^+(0,0)\, \psi(\vec{r},t) | G \rangle = \frac{1}{L^3} \sum_{\vec{p}_1,\vec{p}_2} \langle G | a^+_{\vec{p}_1} e^{i\mathcal{H}_0 t} a_{\vec{p}_2} e^{-i\mathcal{H}_0 t} | G \rangle e^{i\vec{p}_2\vec{r}}$$

$$= \frac{1}{L^3} \sum_{\vec{p}_1,\vec{p}_2} e^{i\vec{p}_2\vec{r}} \langle G | a^+_{\vec{p}_1} \exp\Big\{ i \sum_{\vec{p}_3} \varepsilon_{\vec{p}_3} a^+_{\vec{p}_3} a_{\vec{p}_3} t \Big\} a_{\vec{p}_2} \exp\Big\{ -i \sum_{\vec{p}_4} \varepsilon_{\vec{p}_4} a^+_{\vec{p}_4} a_{\vec{p}_4} t \Big\} | G \rangle$$

$$| G \rangle = \prod_{|\vec{k}| < p_F} a^+_k | 0 \rangle$$

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$$= \frac{1}{L^3} \sum_{\vec{p}_1 \vec{p}_2} \underbrace{\exp\Big\{ -i \sum_{|\vec{p}_4| < p_F} \varepsilon_{\vec{p}_4} t \Big\} \exp\Big\{ i \sum_{\substack{|\vec{p}_3| < p_F \\ p_3 \neq p_2}} \varepsilon_{\vec{p}_3} t \Big\}}_{e^{-i\varepsilon_{\vec{p}_2} t}} \underbrace{\langle G | a^+_{\vec{p}_1} a_{\vec{p}_2} | G \rangle}_{\delta_{\vec{p}_1\vec{p}_2} \cdot \theta(\varepsilon_F - \varepsilon(\vec{p}_2))} e^{i\vec{p}_2\vec{r}}$$

$$\varepsilon_F \equiv \varepsilon(p_F)$$

$$\langle G | \psi^+(0,0)\, \psi(\vec{r},t) | G \rangle = \frac{1}{L^3} \sum_{p_1} e^{i\vec{p}_1\vec{r}} e^{-i\varepsilon_{\vec{p}_1} t} \, \theta(\varepsilon_F - \varepsilon(\vec{p}_1))$$

Now use the def of $A_h(\vec{p},\omega)$:

$$A_h(\vec{p},\omega) = \frac{1}{\pi}\mathrm{Re}\int_{-\infty}^{0} dt\, e^{i\omega t+0.t} \int_{V=L^3} d\vec{r}\, e^{-i\vec{p}\vec{r}} \frac{1}{L^3}\sum_{\vec{p}_1} e^{i\vec{p}_1\cdot\vec{r}} e^{-i\varepsilon_{\vec{p}_1}t}\,\theta(\varepsilon_F-\varepsilon(\vec{p}_1))$$

$$= \frac{1}{\pi}\mathrm{Re}\int_{-\infty}^{0} dt\, e^{i\omega t+0.t} \sum_{\vec{p}_1} \underbrace{\frac{1}{L^3}\int_{V=L^3} d\vec{r}\, e^{-i(\vec{p}_1-\vec{p})\cdot\vec{r}}}_{\delta_{\vec{p}\vec{p}_1}} e^{-i\varepsilon_{\vec{p}_1}t}\,\theta(\varepsilon_F-\varepsilon(\vec{p}_1))$$

$$= \underbrace{\frac{1}{\pi}\mathrm{Re}\int_{-\infty}^{0} dt\, e^{i\omega t+0.t}\cdot e^{-i\varepsilon(\vec{p})t}}_{\delta\left(\omega-\frac{\varepsilon(\vec{p})}{\hbar}\right)}\,\theta(\varepsilon_F-\varepsilon(\vec{p}))$$

(we restored $\hbar$ here)

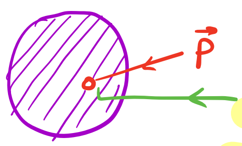
$$\boxed{A_h(\vec{p},\omega) = \theta(\varepsilon_F-\varepsilon(\vec{p}))\,\delta\left(\omega-\frac{\varepsilon(\vec{p})}{\hbar}\right)}$$

$$\boxed{A_h(\vec{p},\omega) = \theta(\varepsilon_F-\varepsilon(\vec{p}))\,\delta\left(\omega-\frac{\varepsilon(\vec{p})}{\hbar}\right)}$$

non-zero only at $\varepsilon(\vec{p})<\varepsilon_F$ (comes from the occup. of states)

reveals the single-particle spectrum $\varepsilon(\vec{p})$ for the holes

pictorial rep. of a hole

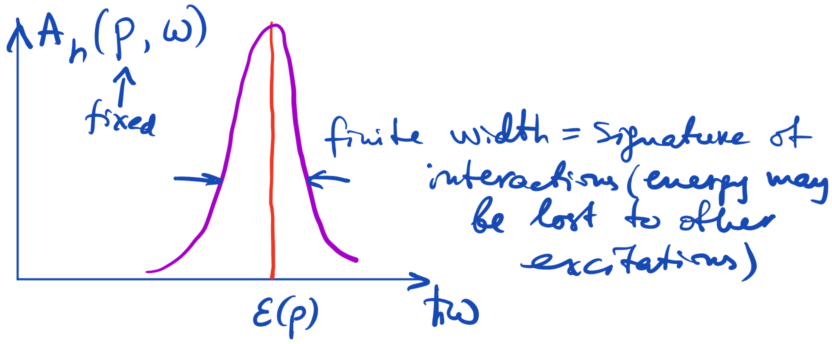


$\vec{p} = \vec{k}-\vec{q}$

$\varepsilon(\vec{p}) = \hbar\omega = \frac{\hbar^2k^2}{2m} - \hbar\omega_q$

For simplicity, focus on a 2D metal ($\vec{p}\to\vec{p}_{\parallel}$, $\vec{k}\to\vec{k}_{\parallel}$) so there is no question re the momentum non-conservation in ARPES.

One may measure $\omega_{\vec{p}\to\vec{k}}$, say, at fixed $\vec{k}-\vec{q}\equiv\vec{p}$ and variable $\hbar\omega = \frac{\hbar^2k^2}{2m}-\hbar cq$



Fixed $\vec p$, variable $\hbar\omega$:
Energy distr. curve (EDC)

Fixed $\hbar\omega$ variable $\vec p$:
Momentum distr. curve (MDC)

Returning to the general expression ($T=0$, but interactions may be present):

$$A_h(\vec p,\omega) = \frac{1}{\pi}\,\mathrm{Re}\int_{-\infty}^{0} dt\, e^{i\omega t + 0\cdot t}\int d\vec r\, e^{-i\vec p\vec r}\, \langle \psi^+(0,0)\,\psi(\vec r,t)\rangle_{T=0}$$

$T=0 \to \langle G|\ldots|G\rangle$

At arbitrary interaction:

$$\psi(\vec r,t) = \frac{1}{L^{3/2}}\sum_{\vec p} e^{i\vec p\vec r}\, e^{i\hat{\mathcal H}_e t/\hbar}\, a_{\vec p}\, e^{-i\hat{\mathcal H}_e t/\hbar}$$

$$\langle G|\psi^+(0,0)\psi(\vec r,t)|G\rangle = \frac{1}{L^3}\sum_{\vec p_1,\vec p_2}\langle G|a^+_{\vec p_1}\, e^{i\hat{\mathcal H}et/\hbar}\, a_{\vec p_2}\, e^{-i\hat{\mathcal H}et/\hbar}\, e^{i\vec p_2\vec r}|G\rangle$$

$$= \frac{1}{L^3}\sum_{\vec p_1,\vec p_2}\langle G|a^+_{\vec p_1}\, e^{i\mathcal H et/\hbar}\underbrace{\sum_n |n\rangle\langle n|}_{=\hat{I}}\, a_{\vec p_2}\,|G\rangle \cdot e^{-iE_g t/\hbar}\, e^{i\vec p_2\vec r}$$

$=\hat I$ (as a sum over all eigenstates of $\mathcal H_e$)

$$= \frac{1}{L^3}\sum_{\vec p_1,\vec p_2,n}\langle G|a^+_{\vec p_1}|n\rangle\langle n|a_{\vec p_2}|G\rangle\, e^{i(E_n-E_g)t/\hbar}\, e^{i\vec p_2\vec r}$$

Momentum conservation for the entire system $\Rightarrow \vec p_1 = \vec p_2$

$$\langle G|\psi^+(0,0)\psi(\vec r,t)|G\rangle = \frac{1}{L^3}\sum_{\vec p_2,n}|\langle n|a_{\vec p_2}|G\rangle|^2\, e^{i\vec p_2\vec r}\, e^{i(E_n-E_g)t/\hbar}$$

$$\boxed{A_h(\vec p,\omega) = \sum_n |\langle n|a_{\vec p}|G\rangle|^2\,\delta\!\left(\omega + \frac{1}{\hbar}\left(E_n(N-1) - E_g(N)\right)\right)}$$

Independent of N at $N\to\infty$

Now we analyze $E_n - E_g$ at $N >> 1$

$$E_n - E_g \equiv E_n(N-1) - E_g(N) = \left(E_n(N-1) - E_g(N-1)\right) + E_g(N-1) - E_g(N)$$

$$\equiv -\xi - \mu$$

$(\xi < 0)$ $(N >> 1)$

Def.: $-\xi \equiv E_n(N-1) - E_g(N-1) = E_n(N) - E_g(N)\Big|_{N>>1}$ (Independent of N in the limit $N>>1$)

$$-\mu = -\frac{\partial E_g}{\partial N} = E_g(N-1) - E_g(N)\Big|_{N>>1}$$

Note that $E_n(N) > E_g(N)$, because n is an excited state for N particles, while g is the ground state for the same # of particles. Therefore $\xi < 0$. This statement is gauge-invariant, i.e. independent of the origin chosen for measuring energy. On the contrary, μ does depend on such origin. Therefore $A_h(\bar{p}, \omega)$ is not gauge-invariant, as

$A_h(\vec{p}, \omega) = \sum_n \dots \delta(\omega - \frac{1}{\hbar}(\mu + \xi))$ depends on μ. We may circumvent this inconvenience by re-defining ω:

$\omega - \frac{1}{\hbar}\mu \to \omega$ (i.e., measure ω from μ):

$$A_h(\vec{p}, \omega) = \sum_n \dots \delta(\omega - \tfrac{1}{\hbar}(\mu + \xi)) \xrightarrow[\text{from } \mu]{\text{measure } \omega} \sum_n \dots \delta(\omega - \tfrac{1}{\hbar}\xi)$$

$A_h \neq 0$ only at $\omega < 0$

Similarly, we define the spectral function for particles

$$\boxed{A_p(\bar{p}, \omega) = \sum_n |\langle n | a^+_{\vec{p}} | G \rangle|^2 \, \delta\left(\omega + \tfrac{1}{\hbar}\left(E_g^{(N+1)} - E_n^{(N)}\right)\right)}$$

Independent of N at $N \to \infty$

$$E_g - E_n = E_g(N) - E_n(N+1) = E_g(N) - E_g(N+1) - (E_n(N+1) - E_g(N+1)) = -\mu - \xi$$

$(N \gg 1)$ $(\xi > 0)$

$$-\mu = E_g(N) - E_g(N+1) = -\frac{\partial E_g}{\partial N}\Big|_{N \gg 1}$$

Def.:

$$\xi = E_n(N+1) - E_g(N+1) = E_n(N) - E_g(N)\Big|_{N \gg 1}$$

(note the sign change, cf. holes)

$$A_p(\vec{p}, \omega) = \sum \dots \delta\left(\omega - \frac{1}{\hbar}(\mu + \xi)\right) \xrightarrow[\text{from } \mu]{\text{measure } \omega} \sum \dots \delta\left(\omega - \frac{\xi}{\hbar}\right) \Rightarrow A_{\vec{p}} \neq 0 \text{ only at } \omega > 0$$

We define the $T=0$ fermion spectral function as:

$$\boxed{A(\vec{p}, \omega) = A_p(\vec{p}, \omega)\,\Theta(\omega) + A_h(\vec{p}, \omega)\,\Theta(-\omega)}$$

works for all ω

For free fermions: $A(\vec{p}, \omega) = \delta\left(\omega - \frac{1}{\hbar}\xi(\vec{p})\right)$; $\xi(\vec{p}) = \varepsilon(p) - \varepsilon(p_F)$

(note we used the convention for ω here, $\mu = \varepsilon(p_F)$)

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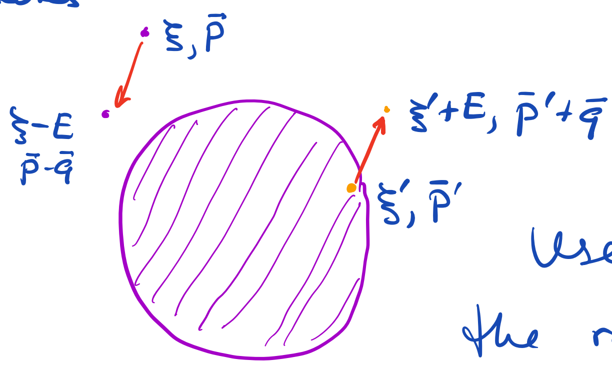
4.4. Notion of the Fermi liquid (Landau, 1957)

Consider the effect of weak interactions on single-particle states



$$\hat{V} = \frac{1}{2} \sum_{\vec{p}\vec{p}'\vec{q}} V_{\vec{q}}\, a^+_{\vec{p}-\vec{q}}\, a^+_{\vec{p}'+\vec{q}}\, a_{\vec{p}'}\, a_{\vec{p}}$$

$\vec{q} \neq 0$

Use Fermi's Golden rule to evaluate the rate $(1/\tau)$ of the transitions away from the initial state

$T=0$

$$\frac{1}{\tau} = \frac{2\pi}{\hbar}\int\frac{d\bar{p}'}{(2\pi\hbar)^3}\int\frac{d\bar{q}}{(2\pi\hbar)^3}\,|V_{\bar q}|^2\,\theta(-\xi(p'))\,\theta(\xi(p'+q))\,\theta(\xi(\bar p-\bar q))\cdot\Theta(\xi_p)$$
$$\times\,\delta(\xi_{\bar p}-\xi_{\bar p-\bar q}+\xi_{\bar p'}-\xi_{\bar p'+\bar q})$$

We are interested in the dependence of $1/\tau$ on ξ

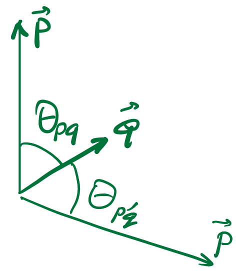
Introduce energy E transferred in a 2-particle collision:

$$\frac{1}{\tau} = \frac{2\pi}{\hbar}\int\frac{d\bar{p}'}{(2\pi\hbar)^3}\int\frac{d\bar{q}}{(2\pi\hbar)^3}\,|V_{\bar q}|^2\,\theta(-\xi(p'))\,\theta(\overbrace{\xi(p'+q)}^{\xi(p')+E})\,\theta(\overbrace{\xi(\bar p-\bar q)}^{\xi(p)-E})\cdot\Theta(\xi_p)$$
$$\times\,\underbrace{\delta(\xi_{\bar p}-\xi_{\bar p-\bar q}+\xi_{\bar p'}-\xi_{\bar p'+\bar q})}$$

$$= \frac{2\pi}{\hbar}\int\frac{d\bar{p}'}{(2\pi\hbar)^3}\int\frac{d\bar{q}}{(2\pi\hbar)^3}\,|V_{\bar q}|^2\int_{-\infty}^{\infty}dE\,\delta(\xi_{\bar p}-\xi_{\bar p-\bar q}-E)\,\delta(\xi_{\bar p'}-\xi_{\bar p'+\bar q}+E)$$
$$\times\,\theta(-\xi(p'))\,\theta(\xi(p')+E)\,\theta(\xi(p)-E)\cdot\Theta(\xi_p)$$

$\xi_{p'}<0$ $\quad E>0$ $\quad E<\xi_p$ $\quad \xi_p>0$

Now we change variables:



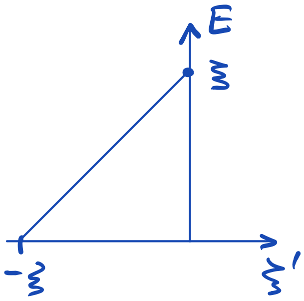
$$\frac{d\vec p'}{(2\pi\hbar)^3}\to\frac{d(|\vec p'|,\theta_{p'q},\varphi_{p'q})}{(2\pi\hbar)^3}\to\frac{1}{(2\pi\hbar)^3}\underbrace{\frac{dp'}{d\xi'}}_{\frac{\nu(\varepsilon')}{4\pi}}d\xi',\ \underbrace{dO'}_{\text{solid angle element}}$$

$$\nu(\varepsilon)\equiv\int\frac{d^3p}{(2\pi\hbar)^3}\delta(\varepsilon-\varepsilon(\vec p))$$ Density of states (DOS)

and use

$$\xi_{\vec p}-\xi_{\vec p-\vec q}\simeq\frac{\partial\xi}{\partial\vec p}\cdot\vec q=\vec v(\vec p)\cdot\vec q \qquad (\vec v(\vec p)=\vec p/m^*)$$

$$\frac{1}{\tau(\xi)}=\frac{2\pi}{\hbar}\int_0^{\xi_{\vec p}}dE\int_{-E}^{0}\nu(\xi')d\xi'\int\frac{dO'}{4\pi}\int\frac{q^2dq}{(2\pi\hbar)^3}\underbrace{dO}_{\text{solid angle element}}\,|V(q)|^2\,\delta(\underbrace{\vec v(\vec p)\cdot\vec q}_{v_Fq\cos\theta})\,\delta(\underbrace{\vec v(\vec p')\cdot\vec q}_{v_Fq\cos\theta'})$$



$$dO' = \sin\theta' d\theta' d\varphi' \qquad \delta(f(\theta)) = \frac{1}{|f'(\theta_0)|}\delta(\theta-\theta_0)$$

$$\frac{1}{\tau(\xi)} \sim \frac{2\pi}{\hbar}\nu(E_F)\int_0^{\xi} dE \int_{-E}^{0} d\xi' \int \frac{q^2 dq}{(2\pi)^3\hbar^3 q^2}\frac{|V(q)|^2}{v_F^2} \propto \nu(E_F)\,\xi^2 \int \frac{q^2 dq}{q^2}\frac{|V(q)|^2}{v_F^2}$$

$$\boxed{\frac{1}{\tau} \propto \xi^2} \qquad \frac{1}{\tau} \propto \xi^2 \ln E_F/\xi \qquad \boxed{\xi\cdot\tau(\xi) \propto \frac{1}{\xi}\Big|_{\xi\to 0} \to \infty}$$

(3D) (2D)

single-particle excitations well-defined near the Fermi level

$$\boxed{\frac{1}{\tau(\xi)} \sim \frac{e^2}{\hbar v_F}\cdot\frac{\xi^2}{E_F}}$$ (Coulomb int.)