Scattering amplitude

$$\langle \vec{p}_1+\vec{Q}, \vec{p}_2-\vec{Q} \mid \hat{T}^{(1)} \mid \vec{p}_1, \vec{p}_2 \rangle$$

$$= \frac{1}{2L^3} \sum_{\vec{k}_1 \vec{k}_2 \vec{q}} V(\vec{q}) \langle 0| a_{\vec{p}_2-\vec{Q}}\, a_{\vec{p}_1+\vec{Q}}\, a^+_{\vec{k}_1+\vec{q}}\, a^+_{\vec{k}_2-\vec{q}}\, a_{\vec{k}_2} a_{\vec{k}_1}\, a^+_{\vec{p}_1} a^+_{\vec{p}_2} |0\rangle$$

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$$= \frac{1}{2L^3} \sum_{\vec{q}} V(\vec{q}) \langle 0| a_{\vec{p}_2-\vec{Q}}\, a_{\vec{p}_1+\vec{Q}} \Big(a^+_{\vec{p}_2+\vec{q}}\, a^+_{\vec{p}_1-\vec{q}}\, a_{\vec{p}_1} a_{\vec{p}_2} a^+_{\vec{p}_1} a^+_{\vec{p}_2}$$
$$+ a^+_{\vec{p}_1+\vec{q}}\, a^+_{\vec{p}_2-\vec{q}}\, a_{\vec{p}_2} a_{\vec{p}_1} a^+_{\vec{p}_1} a^+_{\vec{p}_2} \Big) |0\rangle$$

Fermions

$$a_{\vec{p}_1} a_{\vec{p}_2} a^+_{\vec{p}_1} a^+_{\vec{p}_2} |0\rangle = -a_{\vec{p}_1} a^+_{\vec{p}_1} a_{\vec{p}_2} a^+_{\vec{p}_2} |0\rangle = -|0\rangle$$

$$a_{\vec{p}_2} a_{\vec{p}_1} a^+_{\vec{p}_1} a^+_{\vec{p}_2} |0\rangle = a_{\vec{p}_1} a^+_{\vec{p}_1} a_{\vec{p}_2} a^+_{\vec{p}_2} |0\rangle = |0\rangle$$

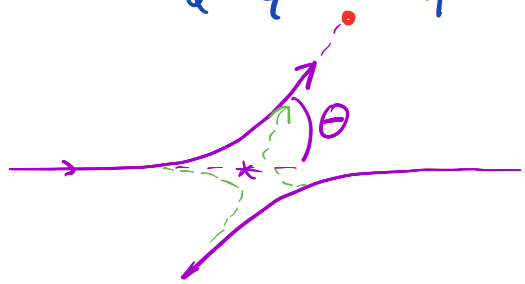
$$\langle \vec{p}_1+\vec{Q}, \vec{p}_2-\vec{Q} \mid \hat{T}^{(1)} \mid \vec{p}_1, \vec{p}_2 \rangle = \frac{1}{2L^3} \sum_{\vec{q}} V(\vec{q}) \langle 0| a_{\vec{p}_2-\vec{Q}}\, a_{\vec{p}_1+\vec{Q}}$$
$$\left(-a^+_{\vec{p}_2+\vec{q}}\, a^+_{\vec{p}_1-\vec{q}} + a^+_{\vec{p}_1+\vec{q}}\, a^+_{\vec{p}_2-\vec{q}} \right) |0\rangle$$

$$\bar q = \bar p_1 - \bar p_2 - \bar Q \qquad \bar p_1 + \bar Q = \bar p_2 + \bar q$$

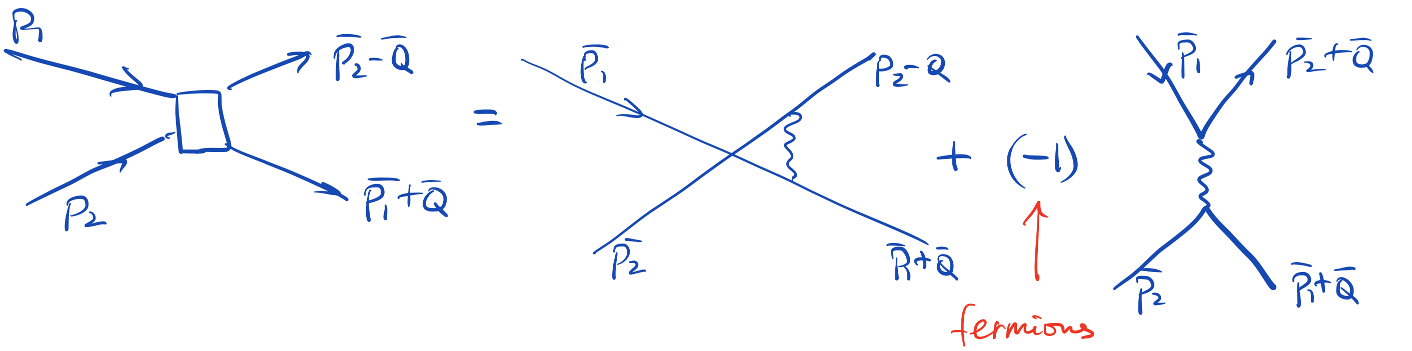
$$= \frac{1}{2L^3} \sum_{\bar q} V(\vec q) \langle 0 | \left(a_{\bar p_2 - \bar Q}\, a_{\bar p_1 + \bar Q}\, a^+_{\bar p_1 + \bar q}\, a^+_{\bar p_2 - \bar q} - a_{\bar p_2 - \bar Q}\, a_{\bar p_1 + \bar Q}\, a^+_{\bar p_2 + \bar q}\, a^+_{\bar p_1 - \bar q} \right) |0\rangle$$

$$\bar Q = -\bar q \qquad \bar q = -\bar Q$$

$$= \frac{1}{L^3} \left(V_{\bar Q} - V_{\bar p_2 - \bar p_1 - \bar Q} \right)$$



cf.: $\quad f_{\text{ferm}} = \underset{\text{distinguishable}}{f(\theta)} - \underset{\text{dist.}}{f(\pi - \theta)}$



3.6. Time evolution of observables, equations of motion

3.6.a Heisenberg and Schrödinger reps.:

$$\langle \hat F \rangle = \mathrm{Tr}\{ \hat\rho\, \hat F(t) \},$$

where $\quad i\hbar \dfrac{d\hat F}{dt} = [\hat F, \hat{\mathcal H}] \qquad$ (Heisenberg rep.)

(no explicit dep. of $\hat F$ on t)

For t-independent $\hat{\mathcal H}$, the formal solution of the eq. of motion for $\hat F$:

$$\hat F(t) = e^{i\hat{\mathcal H}t}\, \hat F(0)\, e^{-i\hat{\mathcal H}t}$$

$$\langle F(t) \rangle = \mathrm{Tr}\left\{ \hat\rho\, e^{i\hat{\mathcal H}t} \hat F(0) e^{-i\hat{\mathcal H}t} \right\} = \mathrm{Tr}\left\{ e^{-i\hat{\mathcal H}t} \hat\rho\, e^{i\hat{\mathcal H}t} F(0) \right\}$$

$$= \mathrm{Tr}\{ \hat\rho(t)\, \hat F(0) \}$$

$\hat{p}(t) = e^{-i\hat{\mathcal{H}}t}\hat{p}e^{i\hat{\mathcal{H}}t}$ satisfies eq. $i\hbar\frac{d\hat{p}}{dt} = [\hat{\mathcal{H}}, \hat{p}]$

(Schrödinger rep.)

3.6.b. Equations of motion for creation and annihilation operators

Consider $\hat{F}^{(1)} = \sum_{ik} f_{ik}^{(1)} \hat{a}_i^+ \hat{a}_k$. Use $\hat{F}(t) = e^{i\hat{\mathcal{H}}t}\hat{F}(0)e^{-i\hat{\mathcal{H}}t}$ to convert to Heisenberg rep. here:

$$\hat{F}^{(1)}(t) = e^{i\hat{\mathcal{H}}t}\hat{F}^{(1)}e^{-i\hat{\mathcal{H}}t} = \sum_{ik} f_{ik}^{(1)} e^{i\hat{\mathcal{H}}t}\hat{a}_i^+\hat{a}_k e^{-i\hat{\mathcal{H}}t}$$

$$= \sum_{ik} f_{ik}^{(1)} e^{i\hat{\mathcal{H}}t}\hat{a}_i^+ \underbrace{e^{-i\hat{\mathcal{H}}t}\cdot e^{i\hat{\mathcal{H}}t}}_{\hat{I}}\hat{a}_k e^{-i\hat{\mathcal{H}}t} \underset{\text{def. of } a^+(t),\, a(t)}{\equiv} \sum_{ik} f_{ik}^{(1)}\hat{a}_i^+(t)\hat{a}_k(t).$$

[Unlike Sakurai's inconvenient convention (2.2.41) we keep one particle basis vectors fixed]

$$\Rightarrow \begin{cases} a_i^+(t) = e^{i\hat{\mathcal{H}}t}\hat{a}_i^+e^{-i\hat{\mathcal{H}}t} \\ \hat{a}_k(t) = e^{i\hat{\mathcal{H}}t}\hat{a}_k e^{-i\hat{\mathcal{H}}t}\end{cases} \Rightarrow \boxed{i\hbar\frac{d\hat{a}_i^+}{dt} = [\hat{a}_i^+, \hat{\mathcal{H}}];\ i\hbar\frac{d\hat{a}_k}{dt} = [\hat{a}_k, \hat{\mathcal{H}}]}$$

Works for any creation, annihilation operators $\Rightarrow$ represent evolution of any observable, $F^{(1)}$, $F^{(2)}$, ...

3.6.c. Particle current density operator

$$i\hbar\frac{d\hat{n}}{dt} = [\hat{n}, \hat{\mathcal{H}}], \quad \hat{n}(\vec{r}) = \hat{\psi}^+(\vec{r})\hat{\psi}(\vec{r}), \quad \hat{\mathcal{H}} = \hat{T} + \hat{\mathcal{U}} + \hat{V}_{ext}$$

$$\hat{\mathcal{U}} = \int d\vec{r}_1\, \underbrace{\hat{\psi}^+(\vec{r}_1)U(\vec{r}_1)\hat{\psi}(\vec{r}_1)}_{U(r_1)\,\hat{n}(r_1)} + \frac{1}{2}\int dr_1 dr_2\, \underbrace{\hat{\psi}^+(r_1)\hat{\psi}^+(r_2)V(r_1,r_2)\hat{\psi}(r_2)\hat{\psi}(r_1)}_{\hat{n}(r_1)\hat{n}(r_2) - \dots \hat{n}(r_1)\delta(r_1-r_2)}$$

(re-order)

Direct check $[\hat{n}(\vec{r}), \hat{\mathcal{U}}] = 0$

$$i\hbar\frac{d\hat{n}}{dt} = [\hat{n}, \hat{\mathcal{H}}] = [\hat{n}, \hat{T}] + \underbrace{[\hat{n}, \hat{\mathcal{U}}]}_{=0} = [\hat{n}, \hat{T}]$$

$$\hat{T} = -\frac{\hbar^2}{2m}\int d\vec{r}_1\, \hat{\psi}^+(\vec{r}_1)\nabla_{r_1}^2\hat{\psi}(\vec{r}_1)$$

$$[\hat{n}, \hat{T}] = -\frac{\hbar^2}{2m}\int d\vec{r}_1 \left[\hat{\psi}^+(\vec{r})\hat{\psi}(\vec{r}), \hat{\psi}^+(\vec{r}_1)\nabla^2_{\vec{r}_1}\hat{\psi}(\vec{r}_1)\right]$$

perform commutation

$$= -\frac{\hbar^2}{2m}\int d\vec{r}_1 \left(\hat{\psi}^+(\vec{r})\hat{\psi}(\vec{r})\hat{\psi}^+(\vec{r}_1)\nabla^2_{\vec{r}_1}\hat{\psi}(\vec{r}_1) - \hat{\psi}^+(\vec{r}_1)\nabla^2_{\vec{r}_1}\hat{\psi}(\vec{r}_1)\hat{\psi}^+(\vec{r})\hat{\psi}(\vec{r})\right)$$

① use commut. relations for $\hat{\psi}(\vec{r}), \psi^+(\vec{r}_1)$

② integrate by parts to move around $\vec{\nabla}_{\vec{r}_1}$

③ use $(\nabla^2\psi^+)\psi - (\psi^+\nabla^2\psi) = -\vec{\nabla}(\psi^+\nabla\psi - (\nabla\psi^+)\psi)$

to show that $i\hbar\frac{d\hat{n}}{dt} = [\hat{n}, \hat{T}] \Rightarrow \frac{d\hat{n}}{dt} + \mathrm{div}\,\hat{j} = 0$

with $$\boxed{\hat{\vec{j}} = \frac{\hbar}{2mi}\left(\hat{\psi}^+(\vec{r})\vec{\nabla}\hat{\psi}(\vec{r}) - (\nabla\hat{\psi}^+(r))\hat{\psi}(r)\right)}$$

This is the operator of current density.

3.6.d. Momentum representation of the operator of current density.

$$\hat{\psi}^+(\vec{r}) = \frac{1}{L^{3/2}}\sum_{\vec{k}} e^{-i\vec{k}\vec{r}} a^+_{\vec{k}}, \qquad \hat{\psi}(\vec{r}) = \frac{1}{L^{3/2}}\sum_{\vec{k}} e^{i\vec{k}\vec{r}} a_{\vec{k}}$$

(periodic b.c., $L \to \infty$ at the end)

Reminder:

$$n(\vec{r}) = \frac{1}{L^3}\sum_{\vec{k}_1\vec{k}_2} e^{-i\vec{k}_1\vec{r}}\cdot e^{i\vec{k}_2\vec{r}} a^+_{\vec{k}_1}a_{\vec{k}_2} = \frac{1}{L^3}\sum_{\vec{q}} e^{i\vec{q}\vec{r}}\sum_{\vec{k}_1} a^+_{\vec{k}_1}a_{\vec{k}_1+\vec{q}}, \quad \vec{k}_2 = \vec{k}_1+\vec{q}.$$

$$n(\vec{r}) = \frac{1}{L^3}\sum_{q} e^{i\vec{q}\vec{r}}\hat{n}(\vec{q}), \qquad \hat{n}(\vec{q}) = \sum_{\vec{k}} a^+_{\vec{k}}a_{\vec{k}+\vec{q}} \Rightarrow n_\sigma(\vec{q}) = \sum_{\vec{k}} a^+_{\vec{k}\sigma}a_{\vec{k}+\vec{q}\sigma}$$

Particle current density (for particles with spin projection σ):

$$\hat{j}_\sigma = \frac{\hbar}{2mi}\cdot\frac{1}{L^3}\sum_{\vec{k}_1\vec{k}_2} e^{-i\vec{k}_1\vec{r}}\cdot e^{i\vec{k}_2\vec{r}}\left(i\vec{k}_2 + i\vec{k}_1\right)a^+_{\vec{k}_1\sigma}a_{\vec{k}_2\sigma}$$

$\vec{k}_2 = \vec{k}_1 + \vec{q}$; $k_1 \to k$

$$= \frac{\hbar}{2m}\cdot\frac{1}{L^3}\sum_{k_1k_2} e^{i(\vec{k}_2-\vec{k}_1)\vec{r}}(\vec{k}_1+\vec{k}_2)a^+_{\vec{k}_1\sigma}a_{\vec{k}_2\sigma} = \frac{1}{L^3}\sum_{q} e^{i\vec{q}\vec{r}}\frac{\hbar}{m}\sum_{\vec{k}}\left(\vec{k}+\frac{\vec{q}}{2}\right)a^+_{\vec{k}\sigma}a_{\vec{k}+\vec{q}\sigma}$$

Fourier comp. of the current density operator (diag. in spin components):

$$\boxed{\hat{\vec{j}}_\sigma(\vec{q}) = \frac{\hbar}{m}\sum_{\vec{k}}\left(\vec{k}+\frac{\vec{q}}{2}\right) a^+_{\vec{k}\sigma}\, a_{\vec{k}+\vec{q}\,\sigma}}$$

02.14.23

4. Free fermions

4.1. Two point correlation function of density in an ideal Fermi gas.

Single-species fermions, transl.-invariant system

Ground state

$$|G\rangle = \prod_{\substack{\vec{k}\\ |\vec{k}|\le k_F}} a^+_{\vec{k}}|0\rangle \;;$$

Fermi distr. function $(T=0)$:

$$n_{\vec{k}} \equiv \langle G| a^+_{\vec{k}} a_{\vec{k}} |G\rangle = \Theta(k_F - |\vec{k}|)$$

Density

$$n = \int \frac{d^d\vec{k}}{(2\pi)^d}\,\theta(k_F - k) = \begin{cases} k_F/\pi & 1D \\ k_F^2/4\pi & 2D \\ k_F^3/6\pi^2 & 3D \end{cases}$$

Notation: $\langle \ldots \rangle \equiv \mathrm{Tr}\{\hat{\rho}\ldots\}$

$$\hat{n}(\vec{r}) = \sum_a \delta(\vec{r}-\hat{\vec{r}}_a)\;; \qquad \langle \hat{n}(\vec{r}_1)\,\hat{n}(\vec{r}_2)\rangle = \langle \sum_a \delta(\vec{r}_1-\hat{\vec{r}}_a)\sum_b \delta(\vec{r}_2-\hat{\vec{r}}_b)\rangle$$

We want to exclude $a=b$: subtract $\delta(\hat{\vec{r}}_1-\hat{\vec{r}}_a)\delta(\vec{r}_2-\hat{\vec{r}}_b)\big|_{a=b}$

$$\delta(\hat{\vec{r}}_1-\hat{\vec{r}}_a)\delta(\vec{r}_2-\hat{\vec{r}}_b)\big|_{a=b} = \delta(\vec{r}_1-\vec{r}_2)\,\delta(\vec{r}_1-\hat{\vec{r}}_a)$$

Def. of corr. function:

$$g(\vec{r}_1,\vec{r}_2) \equiv \langle \hat{n}(\vec{r}_1)\,\hat{n}(\vec{r}_2)\rangle - \langle \hat{n}(\vec{r}_1)\rangle\delta(\vec{r}_1-\vec{r}_2) = \langle \psi^+(\vec{r}_1)\psi(\vec{r}_1)\psi^+(\vec{r}_2)\psi(\vec{r}_2)\rangle$$

$$- \langle \psi^+(\vec{r}_1)\psi(\vec{r}_1)\rangle\delta(\vec{r}_1-\vec{r}_2) = \langle \psi^+(\vec{r}_1)\,\psi^+(\vec{r}_2)\,\psi(\vec{r}_2)\,\psi(\vec{r}_1)\rangle$$

$$g(\vec{r}_1,\vec{r}_2) = \langle \hat{g}(\vec{r}_1,\vec{r}_2)\rangle\;; \quad \hat{g}(\vec{r}_1,\vec{r}_2) = \psi^+(\vec{r}_1)\psi^+(\vec{r}_2)\psi(\vec{r}_2)\psi(\vec{r}_1)$$

(joint probability density operator)

Transl. invariance $\Rightarrow$ plane wave expansion for $\psi(\bar r)$, $\psi^+(\bar r)$:

$$g(\bar r_1,\bar r_2) = \left(\frac{1}{L^{3/2}}\right)^4 \sum_{\bar k_1\bar k_2\bar k_3\bar k_4} e^{-i\bar k_1\bar r_1}\cdot e^{-i\bar k_2\bar r_2}\, e^{i\bar k_3\bar r_2}\, e^{i\bar k_4\bar r_1} \langle a^+_{\bar k_1} a^+_{\bar k_2} a_{\bar k_3} a_{k_4}\rangle$$

Consider (for definiteness) $T=0 \Rightarrow \langle\hat\rho\ldots\rangle = \langle G|\ldots|G\rangle$

$$g(\bar r_1,\bar r_2)\Big|_{T=0} = \left(\frac{1}{L^{3/2}}\right)^4 \sum_{\bar k_1\bar k_2\bar k_3\bar k_4} e^{-i\bar k_1\bar r_1}\cdot e^{-i\bar k_2\bar r_2}\, e^{i\bar k_3\bar r_2}\, e^{i\bar k_4\bar r_1} \langle G| a^+_{\bar k_1} a^+_{\bar k_2} a_{\bar k_3} a_{k_4}|G\rangle$$

$$= \frac{1}{L^6}\sum_{\bar k_1\bar k_2\bar k_3\bar k_4} e^{i(\bar k_4-\bar k_1)r_1}\cdot e^{i(\bar k_3-\bar k_2)r_2} \langle G| a^+_{\bar k_1} a^+_{\bar k_2} a_{\bar k_3} a_{\bar k_4}|G\rangle \left(\delta_{\bar k_1\bar k_4}\delta_{\bar k_2\bar k_3} + \delta_{\bar k_1\bar k_3}\delta_{\bar k_2\bar k_4}\right)$$

$$= \frac{1}{L^6}\sum_{k_1k_2}\left\{\langle G|a^+_{k_1} a^+_{k_2} a_{k_2} a_{k_1}|G\rangle + e^{i(\bar k_2-\bar k_1)(\bar r_1-\bar r_2)}\langle G|a^+_{\bar k_1}a^+_{\bar k_2} a_{\bar k_1} a_{\bar k_2}|G\rangle\right\}$$

$$= \frac{1}{L^6}\sum_{k_1k_2}\Big\{\left(\langle G|a^+_{k_1}a_{k_1}a^+_{k_2}a_{k_2}|G\rangle - \cancel{\langle G|a^+_{k_1}a_{k_2}|G\rangle\delta_{\bar k_1 k_2}}\right)$$

$$- e^{i(\bar k_2-\bar k_1)(\bar r_1-\bar r_2)}\left(\langle G|a^+_{\bar k_1}a_{\bar k_1}a^+_{\bar k_2}a_{\bar k_2}|G\rangle - \cancel{\langle G|a^+_{\bar k_1}a_{\bar k_2}|G\rangle\delta_{\bar k_1 k_2}}\right)\Big\}$$

cancel

$$= \frac{1}{L^6}\sum_{k_1k_2}\left\{\langle G|\hat n_{\bar k_1}\hat n_{\bar k_2}|G\rangle - e^{i(\bar k_2-\bar k_1)(\bar r_1-\bar r_2)}\langle G|\hat n_{\bar k_1}\hat n_{\bar k_2}|G\rangle\right\};\quad \hat n_k \equiv a^+_k a_k$$

$$\langle G|\hat n_{\bar k_1}\hat n_{\bar k_2}|G\rangle = \langle G|\hat n_{k_1}|G\rangle\cdot\langle G|\hat n_{k_2}|G\rangle = n_{\bar k_1} n_{\bar k_2}$$

At $T\neq 0$ (equilibrium with $T>0$):

$$\langle G|\ldots|G\rangle \to \mathrm{Tr}\left\{\frac{1}{Z}e^{-\beta\hat{\mathcal{H}}_0}\ldots\right\} = \frac{1}{Z}\mathrm{Tr}\left\{\exp\left(-\beta\sum_k \varepsilon_k a^+_k a_k\right)\ldots\right\}$$

(replaced by)

$$\langle G|\hat n_{\bar k_1}\hat n_{\bar k_2}|G\rangle \to \frac{1}{Z}\mathrm{Tr}\left\{\exp\left(-\beta\sum_k\varepsilon_k a^+_k a_k\right)\hat n_{k_1}\hat n_{k_2}\right\}$$

$$= \frac{1}{Z}\mathrm{Tr}\left\{\exp\left(-\beta\sum_k\varepsilon_k\hat n_k\right)n_{k_1}n_{k_2}\right\} = \frac{1}{Z}\prod_{k\neq k_1k_2}\mathrm{Tr}\left(e^{-\beta\varepsilon_k\hat n_k}\right)$$

$$\times\,\mathrm{Tr}\left(e^{-\beta\varepsilon_{k_1}\hat n_{k_1}}\hat n_{k_1}\right)\times\mathrm{Tr}\left(e^{-\beta\varepsilon_{k_2}\hat n_{k_2}}\hat n_{k_2}\right) = \langle\hat n_{k_1}\rangle\langle n_{k_2}\rangle = n_F(\varepsilon_{k_1})\,n_F(\varepsilon_{k_2})$$

Fermi functions

$$g(\bar r_1,\bar r_2) = \frac{1}{L^6}\sum_{\bar k_1\bar k_2}\left(1-e^{i(\bar k_2-\bar k_1)(\bar r_1-\bar r_2)}\right) n_F(\varepsilon_{\bar k_1})\, n_F(\varepsilon_{\bar k_2})$$

(any T)

$$= \frac{1}{L^3}\sum_{k_1} n_F(\varepsilon_{\bar k_1})\cdot\frac{1}{L^3}\sum_{k_1} n_F(\varepsilon_{\bar k_2}) - \frac{1}{L^3}\sum_{k_1} e^{-ik_1(\bar r_1-\bar r_2)} n_F(\varepsilon_{\xi})\cdot\frac{1}{L^3}\sum_{k_2} e^{ik_2(\bar r_1-\bar r_2)} n_F(\varepsilon_{\xi})$$

$$= n^2 - \left|\frac{1}{L^3}\sum_{\bar k} e^{i\bar k(\bar r_1-\bar r_2)} n_F(\varepsilon_{\bar k})\right|^2 = n^2\left\{1-\left|\frac{1}{n}\int\frac{d^dk}{(2\pi)^d} e^{i\bar k(\bar r_1-\bar r_2)} n_F(\varepsilon_{\bar k})\right|^2\right\}$$

(we took $L\to\infty$ limit: $\frac{1}{L^3}\sum_{\vec k}\ldots \to \int\frac{d^d\vec k}{(2\pi)^d}\ldots$)

$$\boxed{g(\bar r_1,\bar r_2) = n^2\left\{1-\left|\frac{1}{n}\int\frac{d^dk}{(2\pi)^d} e^{i\bar k(\bar r_1-\bar r_2)} n_F(\varepsilon_{\bar k})\right|^2\right\}}$$

$g(\bar r_1-\bar r_2)$ is a function of $\bar r=\bar r_1-\bar r_2$; $g(\bar r_1,\bar r_1)=0$

$$g(\bar r_1,\bar r_2) = n^2\left(1-g_{exch}(\bar r)\right);\quad g_{exch}(r) = \left|\frac{1}{n}\int\frac{d^dk}{(2\pi)^d} e^{i\bar k\bar r} n_F(\varepsilon_{\bar k})\right|^2$$

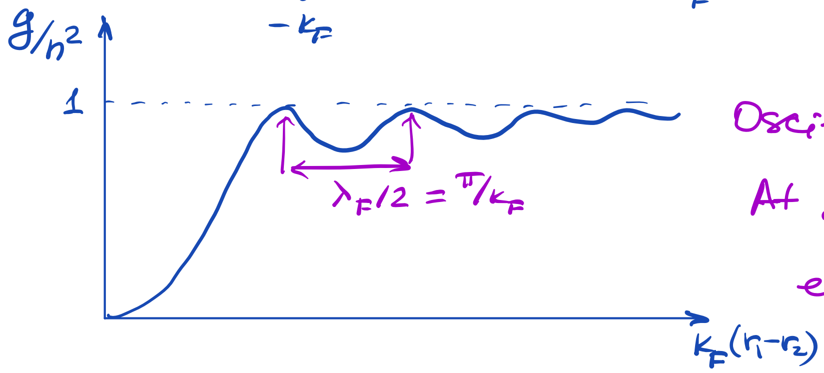
$\bar r=\bar r_1-\bar r_2$

$$T=0 \Rightarrow n_F(\varepsilon_k) = \Theta(k_F-k)$$

$$\int\frac{d^dk}{(2\pi)^d} e^{i\bar k\bar r} n_F(\varepsilon_k) = \int\frac{d^d\bar k}{(2\pi)^d} e^{i\bar k\bar r}\,\Theta(k_F-k)$$

1d case: $\int_{-k_F}^{k_F}\frac{dk}{2\pi} e^{ikr} = n\frac{\sin k_F r}{k_F r}$; $\quad g(r_1-r_2) = n^2\left\{1-\frac{\sin^2(k_F r)}{(k_F r)^2}\right\}$

$r=r_1-r_2$



Oscillations decay as $1/r^d$ at $T=0$. At finite T, decay instead becomes exponential, $\propto \exp\left\{-\frac{r}{v/\pi T}\right\}$

A remark regarding $L\to\infty$ limit: note that each of the terms $\sim\frac{1}{L^6}\sum_{\bar k_1\bar k_2}\langle G|a^+_{k_1}a_{k_2}|G\rangle\delta_{\bar k_1\bar k_2}$ that cancelled out in the derivation would not survive the limit: $\frac{1}{L^6}\sum_{k_1k_2}\delta_{k_1k_2}\ldots = \frac{1}{L^6}\sum_{k_1}\ldots \to \frac{1}{L^3}\int\frac{d^3k}{(2\pi)^3}\ldots \to 0$

02.16.23

Average value of the two-particle interaction energy over the free-fermion state:

$$\langle G|\hat{V}|G\rangle = \frac{1}{2}\langle G|\int d\vec{r}_1\, d\vec{r}_2\; V(\vec{r}_1-\vec{r}_2)\,\hat{g}(\vec{r}_1,\vec{r}_2)\,|G\rangle$$

(see Section 3.3)

$$= \frac{1}{2}L^3\int d\vec{r}\; V(\vec{r})\,g(r) = \frac{1}{2}N\Big\{\underbrace{n\int d\vec{r}\;V(\vec{r})}_{\text{"direct" interaction compensated by interact. with ions in a metal (jellium model)}} - \underbrace{n\int d\vec{r}\;V(\vec{r})\,g_{exch}(\vec{r})}_{\text{exchange term, contributes to cohesion energy of a metal}}\Big\}$$

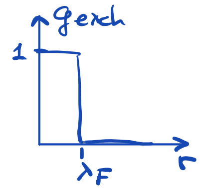
$$\frac{1}{2}\int dr_1 dr_2\; V_{ii}(r_1-r_2)\,n_{ion}(r_1)\,n_{ion}(r_2) + \frac{1}{2}\int dr_1 dr_2\; V_{ee}(r_1-r_2)\,n_{el}(r_1)\,n_{el}(r_2)$$

$$+\int dr_1 dr_2\; V_{ei}(r_1-r_2)\,n_{ion}(r_1)\,n_{el}(r_2)\Big|_{n_{ion}=n_{el}\equiv n} = \frac{1}{2}\int dr_1\, dr_2\,\big(V_{ii}(r_1-r_2)+V_{ee}(r_1-r_2)+2V_{ei}(r_1-r_2)\big)\,n^2$$

$= 0$, as $V_{ii} = V_{ee} = -V_{ei}$ (Coulomb interaction)

Crude estimate of the exchange contribution (no numerical factors)

$$E_{ex} \sim n\int d\vec{r}\;V(\vec{r})\,g_{exch}(\vec{r}) \sim n\int_0^{\lambda_F} r^2 dr\,\frac{e^2}{r}$$

(we replaced $g_{exch}(r)$ by: )

$$\sim n\cdot\lambda_F^3\cdot\frac{e^2}{\lambda_F} \sim e^2\cdot n^{1/3}$$

(this is exchange energy per one electron; we used relation $n\sim\lambda_F^{-3}$)

More reading: D. Pines, Elementary Excitations in Solids, Perseus Books Publ. 1999, p.p. 78-83

4.2. Notion of Wick's theorem.

Analyze the steps of derivation $g(\vec{r}_1,\vec{r}_2)$: (1) we "paired" in $\langle a^+_{k_1} a^+_{k_2} a_{k_3} a_{k_4}\rangle$ operators with equal state labels; (2) we permuted operators to form $\hat{n}_{k_1}\hat{n}_{k_2}$; (3) we factorized $\mathrm{Tr}(\rho_0\, n_{k_1} n_{k_2})$

$= \mathrm{Tr}\, \rho_0 \hat{n}_{k_1} \mathrm{Tr}\, \rho_0 \hat{n}_{k_2} = \langle \hat{n}_{k_1} \rangle \langle \hat{n}_{k_2} \rangle$; (4) we noticed that lower-order $\langle a^+_{k_1} a_{k_2} \rangle$ averages stemming from commutation cancel out

We may generalize to an <u>arbitrary</u> system of non-interacting identical particles (i.e. we may allow any external potential).
Let $\psi_i(\vec{r})$, $\psi_k(r)$... be eigenstates of the single-particle Hamiltonian

$$\langle \hat{g}(\vec{r}_1, \vec{r}_2) \rangle = \langle \hat{\psi}^+(r_1) \hat{\psi}^+(r_2) \hat{\psi}(r_2) \hat{\psi}(r_1) \rangle = \sum_{i,k,l,m} \psi_i^*(r_1) \psi_k^*(r_2) \psi_l(r_2) \psi_m(r_1)$$

$$\times \langle a_i^+ a_k^+ a_l a_m \rangle = \sum_{i,k,l,m} \psi_i^*(r_1) \psi_k^*(r_2) \psi_l(r_2) \psi_m(r_1) \langle a_i^+ a_k^+ a_l a_m \rangle (\delta_{im} \delta_{kl}$$

$$+ \delta_{km} \delta_{il}) = \sum_{ik} \psi_i^*(r_1) \psi_i(r_1) \psi_k^*(r_2) \psi_k(r_2) \langle a_i^+ a_k^+ a_k a_i \rangle$$

$$+ \sum_{ik} \psi_i^*(r_1) \psi_i(r_2) \psi_k^*(r_2) \psi_k(r_1) \langle a_i^+ a_k^+ a_i a_k \rangle$$

$$= \sum_{ik} \psi_i^*(r_1) \psi_i(r_1) \psi_k^*(r_2) \psi_k(r_2) \langle a_i^+ a_i a_k^+ a_k \rangle - \cancel{\sum_i \psi_i^*(r_1) \psi_i(r_1) \psi_i^*(r_2) \psi_i(r_2)}$$

$$\times \cancel{\langle a_i^+ a_i \rangle}$$

$$- \sum_{ik} \psi_i^*(r_1) \psi_i(r_2) \psi_k^*(r_2) \psi_k(r_1) \langle a_i^+ a_i a_k^+ a_k \rangle + \cancel{\sum_i \psi_i^*(r_1) \psi_i(r_2) \psi_i^*(r_2) \psi_i(r_1)}$$

$$\times \cancel{\langle a_i^+ a_i \rangle}$$

$$= \sum_{ik} \psi_i^*(r_1) \psi_i(r_1) \psi_k^*(r_2) \psi_k(r_2) \langle a_i^+ a_i a_k^+ a_k \rangle$$

$$- \sum_{ik} \psi_i^*(r_1) \psi_i(r_2) \psi_k^*(r_2) \psi_k(r_1) \langle a_i^+ a_i a_k^+ a_k \rangle$$

$$= \sum_{ik} \psi_i^*(r_1) \psi_i(r_1) \psi_k^*(r_2) \psi_k(r_2) \langle a_i^+ a_i \rangle \langle a_k^+ a_k \rangle$$

$$- \sum_{ik} \psi_i^*(r_1) \psi_i(r_2) \psi_k^*(r_2) \psi_k(r_1) \langle a_i^+ a_i \rangle \langle a_k^+ a_k \rangle$$

recall that $\hat{\psi}(r) = \sum_i \psi_i(r) a_i$, $\psi^+(r) = \sum_l \psi_l^*(r) a_l^+$, use

$\langle a_i^+ a_\ell \rangle \propto \delta_{i\ell}$ to revert the last two lines to the field-operator form:

$$= \langle \hat{\psi}^+(r_1)\hat{\psi}(r_1)\rangle\langle\hat{\psi}^+(r_2)\hat{\psi}(r_2)\rangle - \langle\hat{\psi}^+(r_1)\hat{\psi}(r_2)\rangle\langle\hat{\psi}^+(r_2)\hat{\psi}(r_1)\rangle$$

A similar calculation of $g(r_1, r_2)$ can be performed for bosons. There are two caveats:

(1) the lower-order terms $\sim \sum_i \psi_i^*(r_1)\psi_i(r_2)\psi_i^*(r_2)\psi_i(r_1)\langle a_i^+ a_i\rangle$ do not cancel out; (2) special care should be also taken of terms $\sim \sum_i |\psi_i(r_1)|^2 |\psi_i(r_2)|^2 \langle a_i^+ a_i^+ a_i a_i\rangle$ (which were non-existent in the case of fermions).

However, there two types of terms survive thermodynamic limit ($L\to\infty$, $N\to\infty$ at fixed $n = N/L^d$) only if a macroscopic (i.e. order-of-N) number of bosons occupy a single state. In the absence of such condensation of bosons, the results for fermions and bosons are similar to each other:

$$\boxed{\begin{aligned}&\langle\hat{\psi}^+(r_1)\hat{\psi}^+(r_2)\hat{\psi}(r_2)\hat{\psi}(r_1)\rangle\\ &= \langle \hat{\psi}^+(r_1)\hat{\psi}(r_1)\rangle\langle\hat{\psi}^+(r_2)\hat{\psi}(r_2)\rangle \underset{B}{\overset{F}{\mp}} \langle\hat{\psi}^+(r_1)\hat{\psi}(r_2)\rangle\langle\hat{\psi}^+(r_2)\hat{\psi}(r_1)\rangle\end{aligned}}$$

The result is basis-independent!

We used only the assumption of non-interacting particles and $\hat{\rho} = \frac{1}{Z} e^{-\beta \mathcal{H}_0}$.

$$\left(\mathcal{H}_0 = \sum_k \varepsilon_k \hat{n}_k \Rightarrow \rho = \frac{1}{Z}\prod_k e^{-\beta\varepsilon_k \hat{n}_k}\right)$$

We stumbled upon an illustration of Wick's theorem – it allows to abbreviate the eval. of products $\langle \hat{O}_1 \hat{O}_2 \cdots \rangle$ of creation and annihilation operators $\hat{O}_1, \hat{O}_2, \ldots$ to evaluation of pair product averages $\langle \hat{O}_i \hat{O}_j \rangle$. For example, for 4 fermion fields

$$\langle \hat{O}_1 \hat{O}_2 \hat{O}_3 \hat{O}_4 \rangle = \langle \hat{O}_1 \hat{O}_2 \rangle \langle \hat{O}_3 \hat{O}_4 \rangle - \langle \hat{O}_1 \hat{O}_3 \rangle \langle \hat{O}_2 \hat{O}_4 \rangle + \langle \hat{O}_1 \hat{O}_4 \rangle \langle \hat{O}_2 \hat{O}_3 \rangle$$

(all permutations followed by pairing). Fermions: include $(-1)^P$ for the # of permutations.

Similar for bosons (with above mentioned caveats) and no $(-1)^P$.

Good for any bilinear $\mathcal{H}_0$ (does not have to conserve # of particles, i.e. may include terms $\sim \psi_\ell \psi_m$ and $\psi^+_\ell \psi^+_m$)

Also works for operators taken at different times, i.e.

$$\hat{O}_i(t_1) = e^{i\mathcal{H}_0 t_1} \hat{O}_i e^{-i\mathcal{H}_0 t_1}, \quad \hat{O}_j(t_2) = e^{i\mathcal{H}_0 t_2} \hat{O}_j e^{-i\mathcal{H}_0 t_2}$$

Reading:

X.-G. Wen, Quantum field theory of Many-Body Systems, Oxford U. Press, p. 70, 71.

Negele, Orland, Sec. 2.3 and problem 2.8 to that section;

Doniach, Sondheimer, after eq. (3.1.25)