$\hat{Q}\,|B\rangle$asics

Activity 4: Characterizing quantum states

Ramsey Fringe

Suppose you are given a quantum gate $U(\theta)$ for some unknown $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ (**correction: should say* $0 \leq \theta < \pi$) which maps $|0\rangle$ to $|\theta\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}|1\rangle)$. You can run the quantum circuit as many times as you want. Using a Hadamard gate and $Z-$basis measurements, how can you determine θ? Try it out in Qiskit.

Solution

In rough terms, a Hadamard gate converts relative phase between the two basis states $|0\rangle$ and $|1\rangle$ in superposition into a difference in amplitudes. This corresponds to switching the measurement basis from Z to X. Recall that $H|0\rangle = |+\rangle$ and $H|1\rangle = |-\rangle$, where

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad\qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

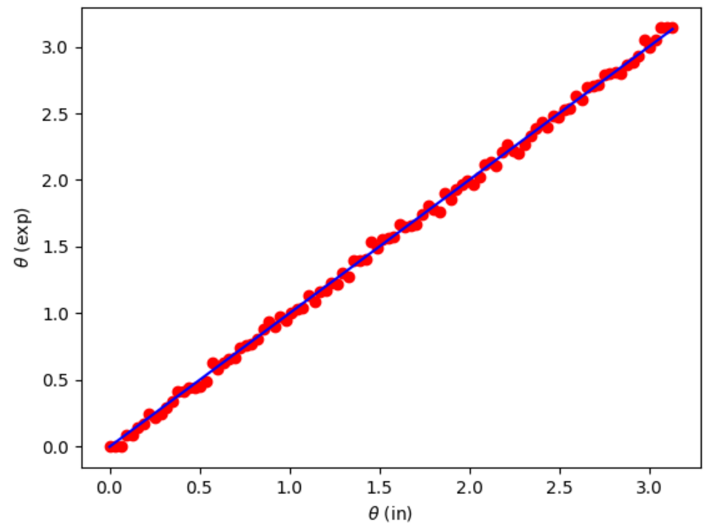
We find

$$\begin{aligned}
H|\theta\rangle &= \frac{|+\rangle + e^{i\theta}|-\rangle}{\sqrt{2}} \\
&= \frac{1}{2}\big((1+e^{i\theta})|0\rangle + (1-e^{i\theta})|1\rangle\big) \\
&= e^{i\theta/2}\left(\left(\frac{e^{i\theta/2}+e^{-i\theta/2}}{2}\right)|0\rangle + \left(\frac{e^{i\theta/2}-e^{-i\theta/2}}{2}\right)|1\rangle\right) \\
&= e^{i\theta/2}(\cos(\theta/2)|0\rangle + i\sin(\theta/2)|1\rangle)
\end{aligned}$$

Therefore we measure $|0\rangle$ with probability $\cos^2(\theta/2) = \frac{\cos(\theta)+1}{2}$. In order to determine θ experimentally, we run the circuit many times and compute the fraction of time $|0\rangle$ is measured and solve for θ. This recovers θ uniquely if $0 \leq \theta < \pi$. In order to fully recover theta, we need another set of measurements, which the next question will establish. Here is the code to run this in Qiskit:

```python
from qiskit import QuantumCircuit, execute
from qiskit.providers.aer import AerSimulator
from matplotlib import pyplot as plt
import numpy as np
sim = AerSimulator()
def prep(theta):
    qc = QuantumCircuit(1)
    #prepare |theta>
    qc.h(0)
    qc.rz(theta, 0)
    return qc
thetas = np.linspace(0, np.pi-.01, 100)
theta_guesses = []
```

```python
shots = 1000 #precision
for theta in thetas:
    qc = prep(theta)
    qc.h(0)
    qc.measure_all()
    counts = execute(qc, sim, shots = shots).result().get_counts().get("0",0)
    expfreq = counts/shots
    theta_guesses.append(np.arccos(2*expfreq - 1))
np.nan_to_num(theta_guesses);
```



Solution 2

With a little more machinery, there is a more illuminating way to solve this problem. By the Pauli-Euler identity,

$$|\theta\rangle = e^{i\theta\frac{Z+1}{2}}|+\rangle = e^{i\frac{\theta}{2}}e^{i\frac{\theta}{2}Z}|+\rangle = e^{i\frac{\theta}{2}}e^{i\frac{\theta}{2}Z}H|0\rangle$$

Since $HZH = X$, we have $He^{i\frac{\theta}{2}Z}H = e^{i\frac{\theta}{2}X}$ (which you can see by considering the Taylor series), and so

$$H|\theta\rangle = e^{i\frac{\theta}{2}}He^{i\frac{\theta}{2}Z}H|0\rangle = e^{i\frac{\theta}{2}}e^{i\frac{\theta}{2}X}|0\rangle$$

Using the Pauli-Euler identity again,

$$e^{i\frac{\theta}{2}}e^{i\frac{\theta}{2}X}|0\rangle = e^{i\frac{\theta}{2}}(\cos\left(\frac{\theta}{2}\right)|0\rangle + i\sin\left(\frac{\theta}{2}\right)|1\rangle)$$

which gives the same result as above.

The Bloch Sphere

The X, Y, and Z operators are related to the $\hat{x}$, $\hat{y}$, $\hat{z}$ axes, and now we are in a position to explore this connection. Let

$$|\theta,\phi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$$

As a reminder, the outer product definitions of X, Y, and Z are

$$X = |0\rangle\langle 1| + |1\rangle\langle 0| \qquad Y = -i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0| \qquad Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Part a

Show that

$$\begin{aligned}\langle X\rangle &= \sin(\theta)\cos(\phi)\\ \langle Y\rangle &= \sin(\theta)\sin(\phi)\\ \langle Z\rangle &= \cos(\theta)\end{aligned}$$

This is the polar representation of a vector on the unit sphere. In this representation, the $X-$eigenkets $|+\rangle, |-\rangle$ (corresponding to $\theta = \pm\pi/2$ and $\phi = 0$) lie on the $\hat{x}-$axis, the $Y-$eigenkets $|i\rangle, |-i\rangle$ (corresponding to $\theta = \pm\pi/2$ and $\phi = \pi/2$) lie on the $\hat{y}-$axis, and the $Z-$eigenkets $|0\rangle, |1\rangle$ (corresponding to $\theta = 0, \pi$) lie on the $\hat{z}-$axis.

Solution

We have shown previously that if $|\psi\rangle = \alpha\,|0\rangle + \beta\,|0\rangle$ then

$$\begin{aligned}\langle Z\rangle &= |\alpha|^2 - |\beta|^2\\ \langle Y\rangle &= 2\,\mathrm{Im}(\alpha^*\beta)\\ \langle X\rangle &= 2\,\mathrm{Re}(\alpha^*\beta)\end{aligned}$$

Applying this to $|\theta,\phi\rangle$, we find

$$\begin{aligned}\langle Z\rangle &= \cos^2(\theta/2) - \sin^2(\theta/2) = \cos(\theta)\\ \langle Y\rangle &= 2\,\mathrm{Im}(\cos(\theta/2)\sin(\theta/2)e^{i\phi}) = 2\cos(\theta/2)\sin(\theta/2)\,\mathrm{Im}(e^{i\phi}) = \sin(\theta)\sin(\phi)\\ \langle X\rangle &= 2\,\mathrm{Re}(\cos(\theta/2)\sin(\theta/2)e^{i\phi}) = 2\cos(\theta/2)\sin(\theta/2)\,\mathrm{Re}(e^{i\phi}) = \sin(\theta)\cos(\phi)\end{aligned}$$

Part b

Now let $\hat{n} = (\langle X\rangle, \langle Y\rangle, \langle Z\rangle)$. Show that

$$|\theta,\phi\rangle\langle\theta,\phi| = \frac{I + n_x X + n_y Y + n_z Z}{2}$$

The vector $\hat{n}$ is often called the Bloch vector of the state $|\theta,\phi\rangle$.

Solution

Foiling out the outer product,

$$\begin{aligned}|\theta,\phi\rangle\langle\theta,\phi| &= \begin{pmatrix}\cos(\theta/2)\\ e^{i\phi}\sin(\theta/2)\end{pmatrix}\begin{pmatrix}\cos(\theta/2) & e^{-i\phi}\sin(\theta/2)\end{pmatrix}\\ &= \begin{pmatrix}\cos(\theta/2)\cos(\theta/2) & e^{-i\phi}\cos(\theta/2)\sin(\theta/2)\\ e^{i\phi}\cos(\theta/2)\sin(\theta/2) & \sin(\theta/2)\sin(\theta/2)\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}1+\cos(\theta) & \sin(\theta)e^{-i\phi}\\ \sin(\theta)e^{i\phi} & 1-\cos(\theta)\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}1+n_z & n_x - in_y\\ n_x + in_y & 1-n_z\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{n_x}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{n_y}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \frac{n_z}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\end{aligned}$$

This gives the desired result. The outer product $|\theta, \phi\rangle\langle\theta, \phi|$ is commonly known as the density matrix.