$\hat{Q}\,|B\rangle$asics

Activity 2: Quantum states and measurements

Complex numbers

Euler's Formula

Define $f(\theta) = e^{i\theta}$ and $g(\theta) = \cos(\theta) + i\sin(\theta)$. Show that $f'(\theta) = if(\theta)$ and $g'(\theta) = ig(\theta)$, then show that $f(0) = g(0) = 1$. This proves that $f(\theta) = g(\theta)$, which is Euler's formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Now show that $(e^{i\theta})^* = e^{-i\theta}$, and find $|e^{i\theta}|^2$.

Solution

Taking the derivative,

$$f'(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}e^{i\theta} = ie^{i\theta} = if(\theta)$$

$$g'(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\cos(\theta) + i\frac{\mathrm{d}}{\mathrm{d}\theta}\sin(\theta) = -\sin(\theta) + i\cos(\theta) = i(\cos(\theta) + i\sin(\theta)) = ig(\theta)$$

Then we check $f(0) = e^0 = 1$ and $g(0) = \cos(0) + i\sin(0) = 1$. Since f' and g' are both continuous, f and g both satisfy the same differential equation, and they have the same initial condition, they must be equal for all values of θ. We can use this to show that

$$(e^{i\theta})^* = (\cos(\theta) + i\sin(\theta))^* = \cos(\theta) - i\sin(\theta) = e^{-i\theta}$$

Lastly, we have

$$|e^{i\theta}|^2 = \cos^2(\theta) + \sin^2(\theta) = 1$$

where the magnitude of a complex number $a + bi$ is defined to be $|a + bi|^2 = (a + bi)(a - bi) = a^2 + b^2$.

Magnitudes

Let z_1, z_2 be complex numbers. Show that $|z_1 z_2|^2 = |z_1|^2|z_2|^2$. (You can do this quickly using $|z_1|^2 = z_1^* z_1$)

Solution

We note that given two complex numbers $a + bi$ and $c + di$, we have

$$[(a + bi)(c + di)]^* = [ac - bd + i(ad + bc)]^* = ac - bd - i(ad + bc) = (a - bi)(c - di) = (a + bi)^*(c + di)^*$$

Using $|z_1|^2 = z_1^* z_1$, we have

$$|z_1 z_2|^2 = z_1 z_2 (z_1 z_2)^* = z_1 z_1^* z_2 z_2^* = |z_1|^2 |z_2|^2$$

Change of basis

The choice of basis $|0\rangle, |1\rangle$ was completely arbitrary, and any orthonormal basis will do the trick. We define

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad\qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \tag{1}$$

Orthonormality

Prove that $\{|+\rangle, |-\rangle\}$ form an orthonormal set. Try working this out using Dirac's bra-ket notation, remembering that $\{|0\rangle, |1\rangle\}$ form an orthonormal set.

Solution

An orthonormal collection $\{|n\rangle\}$ satisfies $\langle n|m\rangle = \delta_{nm}$, where

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

We can check the inner products directly using distributivity and the orthonormality of $|0\rangle, |1\rangle$. Checking explicitly:

$$\begin{aligned}
\langle +|+\rangle &= \frac{1}{2}(\langle 0| + \langle 1|)(|0\rangle + |1\rangle) = \frac{1}{2}(\langle 0|0\rangle + \langle 0|1\rangle + \langle 1|0\rangle + \langle 1|1\rangle) = 1 \\
\langle +|-\rangle &= \frac{1}{2}(\langle 0| + \langle 1|)(|0\rangle - |1\rangle) = \frac{1}{2}(\langle 0|0\rangle - \langle 0|1\rangle + \langle 1|0\rangle - \langle 1|1\rangle) = 0 \\
\langle -|-\rangle &= \frac{1}{2}(\langle 0| - \langle 1|)(|0\rangle - |1\rangle) = \frac{1}{2}(\langle 0|0\rangle - \langle 0|1\rangle - \langle 1|0\rangle + \langle 1|1\rangle) = 1
\end{aligned}$$

Note: taking the complex conjugate of an inner product switches the order, so $\langle -|+\rangle = (\langle +|-\rangle)^* = 0$. This is reflective of the properties of the adjoint operation:

$$(\langle +|-\rangle)^* = (\langle +|-\rangle)^\dagger = |-\rangle^\dagger \langle +|^\dagger = \langle -|+\rangle$$

The adjoint switches the order just like taking the transpose. This is also a general property of inner products.

Measuring in another basis

Suppose we have the general single-qubit state

$$|\psi\rangle = \alpha\,|0\rangle + \beta|1\rangle$$

The Borne rule holds with respect to any basis. Asking the system whether it is in the state $|0\rangle$ or $|1\rangle$ is called a $Z-$basis measurement, and the outcome is $|0\rangle$ with probability $|\alpha|^2$ and $|1\rangle$ with probability $|\beta|^2$. We can also ask whether the system is in the $|+\rangle$ state or $|-\rangle$ state, and we will find $|+\rangle$ with probability $|\langle +|\psi\rangle|^2$ and $|-\rangle$ with probability $|\langle -|\psi\rangle|^2$. This is known as an $X-$basis measurement.

- Work out the probability for the system to be in the state $|+\rangle$ and $|-\rangle$ in terms of α and β.
- Once a measurement is made, the system collapses into the state consistent with the measurement result. For instance, after measuring $|+\rangle$, the system is left in $|+\rangle$. After making an $X-$basis measurement of $|\psi\rangle$, what is the probability of measuring $|0\rangle$ or $|1\rangle$?

Solution

There are two ways to go about this. The first way would be to invert the transformation above, finding

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}} \qquad\qquad |1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

Then the state can be re-written in this basis:

$$|\psi\rangle = \alpha\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right) + \beta\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right) = \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|+\rangle + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|-\rangle$$

Then the probability of measuring $|+\rangle$ is $|\langle +|\psi\rangle|^2 = \frac{1}{2}|\alpha+\beta|^2$ and the probability of measuring $|-\rangle$ is $\frac{1}{2}|\alpha-\beta|^2$. The second way to go about this would be to compute the inner products of the basis vectors directly:

$$\langle 0|+\rangle = \left(\frac{\langle 0|0\rangle + \langle 0|1\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \qquad\qquad \langle 0|-\rangle = \left(\frac{\langle 0|0\rangle - \langle 0|1\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

$$\langle 1|+\rangle = \left(\frac{\langle 1|0\rangle + \langle 1|1\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \qquad\qquad \langle 1|-\rangle = \left(\frac{\langle 1|0\rangle - \langle 1|1\rangle}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}$$

Using this, the Born amplitudes are

$$|\langle +|\psi\rangle|^2 = |\alpha\langle +|0\rangle + \beta\langle +|1\rangle|^2 = \frac{1}{2}|\alpha+\beta|^2$$
$$|\langle -|\psi\rangle|^2 = |\alpha\langle -|0\rangle + \beta\langle -|1\rangle|^2 = \frac{1}{2}|\alpha-\beta|^2$$

Lastly, if the system is measured in the X basis, then it collapses into $|+\rangle$ or $|-\rangle$ with the probabilities given above. If it collapses to $|+\rangle$, then the probability of a successive measurement yielding $|0\rangle$ is $|\langle 0|+\rangle|^2 = \frac{1}{2}$, and $|1\rangle$ with complimentary probability $\frac{1}{2}$. If the first measurement instead leaves the system in $|-\rangle$, then the probability of measuring $|0\rangle$ is again $|\langle 0|-\rangle|^2 = \frac{1}{2}$. Independent of the first measurement outcome, a second measurement in the Z basis yields completely random results.

Mutually unbiased bases

The property described above categorizes the $X-$ and $Z-$ bases as mutually unbiased. We can also define the $Y-$basis

$$|+i\rangle = \frac{|0\rangle + i\,|1\rangle}{\sqrt{2}} \qquad |-i\rangle = \frac{|0\rangle - i\,|1\rangle}{\sqrt{2}} \tag{2}$$

Orthonormality

Show that this basis is orthonormal. Then prove that the inner product between either of $\{|+i\rangle, |-i\rangle\}$ with *any* of $|0\rangle, |1\rangle, |+\rangle$ or $|-\rangle$ has magnitude $\frac{1}{\sqrt{2}}$. What does this mean about successive measurements in different bases?

Solution

$$\langle 0|+i\rangle = \left(\frac{\langle 0|0\rangle + i\langle 0|1\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \qquad \langle 0|-i\rangle = \left(\frac{\langle 0|0\rangle - i\langle 0|1\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$
$$\langle 1|+i\rangle = \left(\frac{\langle 1|0\rangle + i\langle 1|1\rangle}{\sqrt{2}}\right) = \frac{i}{\sqrt{2}} \qquad \langle 1|-i\rangle = \left(\frac{\langle 1|0\rangle - i\langle 1|1\rangle}{\sqrt{2}}\right) = -\frac{i}{\sqrt{2}}$$

$$\langle +|+i\rangle = \left(\frac{\langle +|0\rangle + i\langle +|1\rangle}{\sqrt{2}}\right) = \frac{1+i}{2} \qquad \langle +|-i\rangle = \left(\frac{\langle +|0\rangle - i\langle +|1\rangle}{\sqrt{2}}\right) = \frac{1-i}{2}$$
$$\langle -|+i\rangle = \left(\frac{\langle -|0\rangle + i\langle -|1\rangle}{\sqrt{2}}\right) = \frac{1-i}{2} \qquad \langle -|-i\rangle = \left(\frac{\langle -|0\rangle - i\langle -|1\rangle}{\sqrt{2}}\right) = \frac{1+i}{2}$$

We can check that the magnitude of each is $\frac{1}{\sqrt{2}}$

$$\left|\frac{1\pm i}{2}\right| = \frac{\sqrt{1+1}}{2} = \frac{1}{\sqrt{2}}$$

If one measurement is made in the X, Y, or Z basis, then a second measurement in a different one of these bases will yeild a completely random result.

How many unbiased bases?

Show that there are no more unbiased bases besides these three.

Solution

Suppose that there were another vector $|v\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ besides the six listed whose inner product with any of the X, Y or Z basis kets has magnitude $\frac{1}{\sqrt{2}}$. Then we have

$$|\langle 0|v\rangle|^2 = |\alpha|^2 = \frac{1}{2} \qquad\qquad |\langle 1|v\rangle|^2 = |\beta|^2 = \frac{1}{2}$$

We also find

$$|\langle +|v\rangle|^2 = \frac{1}{2}|\alpha+\beta|^2 = \frac{1}{2}$$

This gives us $|\alpha+\beta|^2 = |\alpha|^2 + |\beta|^2 + 2\,\mathrm{Re}(\alpha^*\beta) = 1$, so $\mathrm{Re}(\alpha^*\beta) = 0$. Next, we take the inner product with a vector from the Y basis:

$$|\langle +i|v\rangle|^2 = |\,\langle +i|0\rangle + \langle +i|1\rangle\,|^2 = \left|\frac{\alpha}{\sqrt{2}} - \frac{i\beta}{\sqrt{2}}\right|^2 = \frac{1}{2}|\alpha - i\beta|^2 = \frac{1}{2}$$

This gives $|\alpha - i\beta|^2 = |\alpha|^2 + |\beta|^2 - 2\,\mathrm{Im}(\alpha^*\beta) = 1$, so $\mathrm{Im}(\alpha^*\beta) = 0$. But combining this with the above result, we have $\alpha^*\beta = 0$, which is only possible if $\alpha = \beta = 0$. This shows that no such vector $|v\rangle$ exists. It turns out that the X, Y and Z bases, as the labels suggest, are related to the $\hat{x}, \hat{y}$, and $\hat{z}$ axes, and the fact that there are only three mutually unbiased bases reflects the three orthogonal axes in $\mathbb{R}^3$.