Qiskit Crash Course Workshop
YuQC Fall Fest

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Linear Algebra Concepts

Spanning set: Let V be a vector space and $\{\vec{s_1}, ..., \vec{s_N}\} = S$ be a set of vectors spanning V. Then if $\vec{v} \in V$,

$$\vec{v} = \sum_{i=1}^{N} c_i \vec{s_i}$$

for some coefficients c_i. Orthogonal: Two vectors $\vec{v_1}$ and $\vec{v_2}$ are called orthogonal if their inner product is zero:

$$\vec{v_1} \cdot \vec{v_2} = 0$$

Normal: A vector $\vec{v}$ is called normal if it has a norm of one:

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = 1$$

Basis: an orthogonal, normal (orthonormal) spanning set is called a basis

Linear Algebra Contd.

A map $T(\vec{v})$ is called linear if

$$T\left(\sum_{i=1} c_i \vec{v}_i\right) = \sum_{i=1} c_i T(\vec{v}_i)$$

If $\vec{v}$ is an arbitrary vector and B is a basis for V, then combining the definition of spanning

$$T(\vec{v}) = \sum_{i}^{N} c_i T(\vec{b}_i)$$

In this way, T is uniquely specified by defining $T(\vec{b}_i)$ for all b_i. A vector $\vec{v}$ is called an eigenvector of T with eigenvalue λ if

$$T(\vec{v}) = \lambda \vec{v}$$

Every linear map has as a representation as a matrix.

From Matrices to Kets

Quantum states are represented in terms of distinguishable basis states

Example

A coin can be in two states: heads or tails. Imagine representing heads with $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and tails with $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then a weighted coin in heads with probability α and tails with probability β could be represented as $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, with $\alpha + \beta = 1$

Another way to represent the basis states in to use bra-ket notation. For instance, say

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \doteq |\text{heads}\rangle \qquad\qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \doteq |\text{tails}\rangle$$

Then the state from before would be $|\text{coin}\rangle = \alpha |\text{heads}\rangle + \beta |\text{tails}\rangle$

Overlaps and Dot Product

We define the overlap of two states as $\langle a|b \rangle$

- Since the states $|\text{heads}\rangle$ and $|\text{tails}\rangle$ are distinguishable, we say they do not overlap. We require that the overlap of a state with itself is always one:

$$\langle \text{tails}|\text{heads}\rangle = 0 \quad \text{and} \quad \langle \text{tails}|\text{tails}\rangle = 1$$

- This means that $|\text{tails}\rangle$ is a linear map such that $\langle \text{tails}| \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$ and $\langle \text{tails}| \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$. In matrix form, the bra is the transpose of the ket:

$$\langle \text{tails}| \doteq \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\top} \doteq |\text{tails}\rangle^{\top}$$

This is also recognizable as the dot product $\langle \text{tails}|\text{heads}\rangle = |\text{tails}\rangle \cdot |\text{heads}\rangle$.

Leap to Quantum

In quantum computing, a single qubit has two possible states, just like the coin: $|0\rangle$ and $|1\rangle$. These states form a basis for the qubit state space.

States belong to a "Hilbert Space", so the scalars are complex numbers

Complex numbers
- Complex numbers are written $z = a + bi$, where $i = \sqrt{-1}$
- The conjugate is defined as $z^* \cong a - bi$

Therefore an arbitrary qubits state, traditionally represented by ψ, can be written

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

With α and β being complex numbers.

Leap to Quantum Contd.

Normalization: The Hilbert space is endowed with the norm:

$$\| |\psi\rangle \| = \sqrt{\langle\psi|\psi\rangle}$$

The normalization condition requires that $\langle\psi|\psi\rangle = 1$. In matrix form, the transpose is replaced by the adjoint:

$$\langle\psi|\psi\rangle = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (|\psi\rangle)^\dagger |\psi\rangle$$

As shown above, the adjoint, or Hermitian conjugate of a matrix is performed by taking the transpose and then conjugating all the matrix elements.

The Borne Rule

One of the fundamental postulates of quantum mechanics is discrete measurement outcomes.

Example

Stern-Gerlach Experiment Silver atoms have a magnetic moment determined by the spin of their outermost electron. When silver atoms are passed through a magnetic field, one would expect them to be deflected in random directions corresponding to random spins. The Stern-Gerlach experiment showed that they were only deflected in two directions, corresponding to discrete measurement outcomes.

Take $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ to be an arbitrary state. Then the probability of measuring $|\psi\rangle$ in the states $|0\rangle$ and $|1\rangle$ are given by

$$p(0) = |\langle 0|\psi\rangle|^2 = |\alpha|^2 \qquad\qquad p(1) = |\langle 1|\psi\rangle|^2 = |\beta|^2$$

Euler's formula

Let $f(\theta) = \frac{\cos(\theta) + i\sin(\theta)}{e^{i\theta}}$ Take the derivative with the product rule:

$$f'(\theta) = \frac{e^{i\theta}(-\sin(theta) - i\cos(\theta)) + ie^{i\theta}(\cos(\theta) + \sin(\theta))}{e^{2i\theta}} = 0$$

We have $f'(\theta) = 0$, and so $f(\theta) = C$. Plugging in $\theta = 0$, we see $f(0) = 1$, so

$$\cos(\theta) + i\sin(\theta) = e^{i\theta}$$

for all θ. This gives a new way to represent complex numbers:

$$z = a + bi = \sqrt{a^2 + b^2}[\underbrace{\frac{a}{\sqrt{a^2 + b^2}}}_{\cos(\theta)} + i\underbrace{\frac{b}{\sqrt{a^2 + b^2}}}_{\sin(\theta)}] = |z|e^{i\theta}$$

Where $|z| = \sqrt{a^2 + b^2}$.

The Bloch Sphere

Multiplying $|\psi\rangle$ by $e^{i\phi'}$ does not affect the measurement outcome, because

$$|\langle\delta|\, e^{i\phi'}\,|\psi\rangle\,|^2 = |e^{i\phi'}|^2|\,\langle\delta|\psi\rangle\,|^2 = |\,\langle\delta|\psi\rangle\,|^2$$

This makes a complex "global phase" undetectable.

Let $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. Through Euler's formula, we can write $\alpha = ae^{i\phi'}$ and $\beta = be^{i\phi''}$, and with this we can factor out the phase to make α real WLOG:

$$|\psi\rangle = a\,|0\rangle + be^{i\phi}\,|1\rangle$$

Where $\phi = \phi' - \phi''$. Then, we require that $\langle\psi|\psi\rangle = a^2 + b^2 = 1$, so we can use the trig identity $\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) = 1$ to rewrite

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}\,|1\rangle$$

Bloch Sphere Contd.

If we imagine a unit sphere with antipodal points corresponding to $|0\rangle$ and $|1\rangle$, then $|\psi\rangle$ can be represented on this unit sphere with polar angle θ and azimuthal angle ϕ (this was the reason for choosing $\frac{\theta}{2}$):
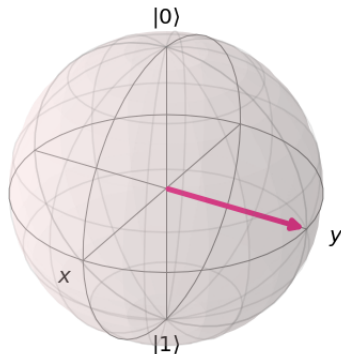

Figure: Bloch plot for $\theta = \pi$, $\phi = \pi$

Quantum Operators

In quantum mechanics, the state changes in time by multiplication by a linear operator U.

> ### note
> U should preserve the norm of states so that probability still makes sense. This means that $\| \, |\delta\rangle \, \|^2 = \langle\delta| \, U^\dagger U \, |\delta\rangle = \langle\delta|\delta\rangle$, so $U^\dagger U = I$. This property is called *unitarity*.

Any single-qubit operator U is defined by $U |0\rangle = c_0 |\psi_0\rangle$ and $U |1\rangle = c_1 |\psi_1\rangle$. Another way to express this is with outer products:

$$U = c_0 |\psi_0\rangle\langle 0| + c_1 |\psi_1\rangle\langle 1|$$

The X gate

The X gate, or Pauli-X gate, is a common single-qubit gate. This is the gate that flips the $|0\rangle \leftrightarrow |1\rangle$ states. Symbolically, this can be written

$$X = |0\rangle\langle 1| + |1\rangle\langle 0| \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The matrix representation can be found in the computational basis by putting the coefficient on $|i\rangle\langle j|$ in the ith row and j^{th} column of the matrix.

The eigenstates of this operator form the $X-$basis, and are even combinations of $|0\rangle$ and $|1\rangle$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = -\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

The Hadamard Gate

Another important gate is the Hadamard gate. This gate takes the $Z-$eigenstates to the $X-$ eigenstates, so

$$H\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad\qquad H\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

If we write the $X-$eigenstates as $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$, then we can write the Hadamard matrix as

$$H = |+\rangle\langle 0| + |-\rangle\langle 1| = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

In matrix representation,

$$H \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

State Collapse

When an operator M is measured, the experimentalist obtains the eigenvalue λ and the system collapses into then corresponding eigenvector $|\lambda\rangle$.

Important!

Like the coin in the box, the system is left in a definite state after observation. But don't let this deceieve you–the coin was in a definite but unknown state before measurement, whereas the qubit was not in any definite state at all.

note

A Hermitian operator is one such that $M^\dagger = M$. Hermitian operators correspond to physical observable quantities because they have strictly real eigenvalues.

Measurement irreversibly destroys quantum information!

Expectation Values

A property of Hermitian operators is that they can be written as "diagonal" in the basis of their eigenvectors. If M is an operator with eigenvectors $|a\rangle$ and $|b\rangle$ with corresponding eigenvalues a and b, then

$$M = a\,|a\rangle\langle a| + b\,|b\rangle\langle b|$$

Given some sort of operator M that we want to find the expectation value of, we can simply compute,

$$\langle M\rangle_\psi = \langle\psi|M|\psi\rangle$$

For a Hermitian operator, this means

$$\langle M\rangle_\psi = \langle\psi|\,a\,|a\rangle\langle a| + b\,|b\rangle\langle b|\,|\psi\rangle = a|\langle a|\psi\rangle|^2 + b|\langle b|\psi\rangle|^2 = ap(a) + bp(b)$$

Entanglement

This can easily be extended to multiple qubits. Two separate qubit states can be combined using the "Kronecker product" symbol,

$$|01\rangle_{AB} = |0\rangle_B \otimes |1\rangle_A$$

When a multi-qubit state can be written as a Kronecker product, it is called a product state. Operators can also take the form of kronecker products, and they act on their respective subsystems

$$(X_B \otimes X_A)|0\rangle_B \otimes |1\rangle_A = X_A|0\rangle_A \otimes X_B|1\rangle_B = |1\rangle_B \otimes |0\rangle_A$$

When states cannot be represented as product states, they are called product states. The canonical example is the first Bell state $|\Phi_0\rangle$:

$$|\Phi_0\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$