# Qiskit Crash Course Workshop

## YuQC Fall Fest

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2022

# Linear Algebra Concepts

Spanning set: Let $V$ be a vector space and $\{\vec{s}_1, ..., \vec{s}_N\} = S$ be a set of vectors spanning $V$. Then if $\vec{v} \in V$,

$$\vec{v} = \sum_{i=1}^{N} c_i \vec{s}_i$$

for some coefficients $c_i$. Orthogonal: Two vectors $\vec{v}_1$ and $\vec{v}_2$ are called orthogonal if their inner product is zero:

$$\vec{v}_1 \cdot \vec{v}_2 = 0$$

Normal: A vector $\vec{v}$ is called normal if it has a norm of one:

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = 1$$

Basis: an orthogonal, normal (orthonormal) spanning set is called a basis

# Linear Algebra Contd.

A map $T(\vec{v})$ is called linear if

$$T\left(\sum_{i=1} c_i \vec{v}_i\right) = \sum_{i=1} c_i T(\vec{v}_i)$$

If $\vec{v}$ is an arbitrary vector and $B$ is a basis for $V$, then combining the definition of spanning

$$T(\vec{v}) = \sum_{i}^{N} c_i T(\vec{b}_i)$$

In this way, $T$ is uniquely specified by defining $T(\vec{b}_i)$ for all $b_i$. A vector $\vec{v}$ is called an eigenvector of $T$ with eigenvalue $\lambda$ if

$$T(\vec{v}) = \lambda \vec{v}$$

Every linear map has as a representation as a matrix.

# From Matrices to Kets

Quantum states are represented in terms of distinguishable basis states

**Example**

A coin can be in two states: heads or tails. Imagine representing heads with $\begin{pmatrix}1\\0\end{pmatrix}$ and tails with $\begin{pmatrix}0\\1\end{pmatrix}$. Then a weighted coin in heads with probability $\alpha$ and tails with probability $\beta$ could be represented as $\begin{pmatrix}\alpha\\\beta\end{pmatrix}$, with $\alpha+\beta=1$

Another way to represent the basis states in to use bra-ket notation. For instance, say

$$\begin{pmatrix}1\\0\end{pmatrix} \doteq |\text{heads}\rangle \qquad \begin{pmatrix}0\\1\end{pmatrix} \doteq |\text{tails}\rangle$$

Then the state from before would be $|\text{coin}\rangle = \alpha\,|\text{heads}\rangle + \beta\,|\text{tails}\rangle$

# Overlaps and Dot Product

We define the overlap of two states as $\langle a|b\rangle$

- Since the states $|\text{heads}\rangle$ and $|\text{tails}\rangle$ are distinguishable, we say they do not overlap. We require that the overlap of a state with itself is always one:

$$\langle \text{tails}|\text{heads}\rangle = 0 \quad \text{and} \quad \langle \text{tails}|\text{tails}\rangle = 1$$

- This means that $|\text{tails}\rangle$ is a linear map such that $\langle \text{tails}| \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$ and $\langle \text{tails}| \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$. In matrix form, the bra is the transpose of the ket:

$$\langle \text{tails}| \doteq \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\top} \doteq |\text{tails}\rangle^{\top}$$

This is also recognizable as the dot product $\langle \text{tails}|\text{heads}\rangle = |\text{tails}\rangle \cdot |\text{heads}\rangle$.

# Leap to Quantum

In quantum computing, a single qubit has two possible states, just like the coin: $|0\rangle$ and $|1\rangle$. These states form a basis for the qubit state space.

States belong to a "Hilbert Space", so the scalars are complex numbers

## Complex numbers

- Complex numbers are written $z = a + bi$, where $i = \sqrt{-1}$
- The conjugate is defined as $z^* \cong a - bi$

Therefore an arbitrary qubits state, traditionally represented by $\psi$, can be written

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

With $\alpha$ and $\beta$ being complex numbers.

# Leap to Quantum Contd.

Normalization: The Hilbert space is endowed with the norm:

$$\| \, |\psi\rangle \, \| = \sqrt{\langle\psi|\psi\rangle}$$

The normalization condition requires that $\langle\psi|\psi\rangle = 1$. In matrix form, the transpose is replaced by the adjoint:

$$\langle\psi|\psi\rangle = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = (|\psi\rangle)^\dagger \, |\psi\rangle$$

As shown above, the adjoint, or Hermitian conjugate of a matrix is performed by taking the transpose and then conjugating all the matrix elements.

# The Borne Rule

One of the fundamental postulates of quantum mechanics is discrete measurement outcomes.

### Example

**Stern-Gerlach Experiment** Silver atoms have a magnetic moment determined by the spin of their outermost electron. When silver atoms are passed through a magnetic field, one would expect them to be deflected in random directions corresponding to random spins. The Stern-Gerlach experiment showed that they were only deflected in two directions, corresponding to discrete measurement outcomes.

Take $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ to be an arbitrary state. Then the probability of measuring $|\psi\rangle$ in the states $|0\rangle$ and $|1\rangle$ are given by

$$p(0) = |\langle 0|\psi\rangle|^2 = |\alpha|^2 \qquad p(1) = |\langle 1|\psi\rangle|^2 = |\beta|^2$$

# Euler's formula

Let $f(\theta) = \frac{cos(\theta)+i\sin(\theta)}{e^{i\theta}}$ Take the derivative with the product rule:

$$f'(\theta) = \frac{e^{i\theta}(-\sin(theta) - i\cos(\theta)) + ie^{i\theta}(\cos(\theta) + \sin(\theta))}{e^{2i\theta}} = 0$$

We have $f'(\theta) = 0$, and so $f(\theta) = C$. Plugging in $\theta = 0$, we see $f(0) = 1$, so

$$\cos(\theta) + i\sin(\theta) = e^{i\theta}$$

for all $\theta$. This gives a new way to represent complex numbers:

$$z = a + bi = \sqrt{a^2+b^2}[\underbrace{\frac{a}{\sqrt{a^2+b^2}}}_{\cos(\theta)} + i\underbrace{\frac{b}{\sqrt{a^2+b^2}}}_{\sin(\theta)}] = |z|e^{i\theta}$$

Where $|z| = \sqrt{a^2+b^2}$.

# The Bloch Sphere

Multiplying $|\psi\rangle$ by $e^{i\phi'}$ does not affect the measurement outcome, because

$$|\langle\delta|\, e^{i\phi'}\, |\psi\rangle\,|^2 = |e^{i\phi'}|^2|\,\langle\delta|\psi\rangle\,|^2 = |\,\langle\delta|\psi\rangle\,|^2$$

This makes a complex "global phase" undetectable.

Let $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. Through Euler's formula, we can write $\alpha = ae^{i\phi'}$ and $\beta = be^{i\phi''}$ , and with this we can factor out the phase to make $\alpha$ real WLOG:

$$|\psi\rangle = a\,|0\rangle + be^{i\phi}\,|1\rangle$$

Where $\phi = \phi' - \phi''$. Then, we require that $\langle\psi|\psi\rangle = a^2 + b^2 = 1$, so we can use the trig identity $\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) = 1$ to rewrite

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}\,|1\rangle$$

# Bloch Sphere Contd.

If we imagine a unit sphere with antipodal points corresponding to $|0\rangle$ and $|1\rangle$, then $|\psi\rangle$ can be represented on this unit sphere with polar angle $\theta$ and azimuthal angle $\phi$ (this was the reason for choosing $\frac{\theta}{2}$):

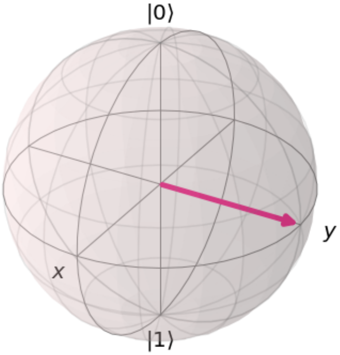


Figure: Bloch plot for $\theta = \pi$, $\phi = \pi$

# Quantum Operators

In quantum mechanics, the state changes in time by multiplication by a linear operator $U$.

**note**

$U$ should preserve the norm of states so that probability still makes sense. This means that $\| \left|\delta\right\rangle \|^2 = \left\langle\delta\right| U^\dagger U \left|\delta\right\rangle = \langle\delta|\delta\rangle$, so $U^\dagger U = I$. This property is called *unitarity*.

Any single-qubit operator $U$ is defined by $U\left|0\right\rangle = c_0 \left|\psi_0\right\rangle$ and $U\left|1\right\rangle = c_1 \left|\psi_1\right\rangle$. Another way to express this is with outer products:

$$U = c_0 \left|\psi_0\right\rangle\!\left\langle 0\right| + c_1 \left|\psi_1\right\rangle\!\left\langle 1\right|$$

# The $X$ gate

The $X$ gate, or Pauli-$X$ gate, is a common single-qubit gate. This is the gate that flips the $|0\rangle \leftrightarrow |1\rangle$ states. Symbolically, this can be written

$$X = |0\rangle\langle 1| + |1\rangle\langle 0| \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The matrix representation can be found in the computational basis by putting the coefficient on $|i\rangle\langle j|$ in the $i^{\text{th}}$ row and $j^{\text{th}}$ column of the matrix.

The eigenstates of this operator form the $X-$basis, and are even combinations of $|0\rangle$ and $|1\rangle$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = - \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

# The Hadamard Gate

Another important gate is the Hadamard gate. This gate takes the $Z-$eigenstates to the $X-$ eigenstates, so

$$H\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}\qquad\qquad H\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}$$

If we write the $X-$eigenstates as $|+\rangle=\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|-\rangle=\frac{|0\rangle-|1\rangle}{\sqrt{2}}$, then we can write the Hadamard matrix as

$$H=|+\rangle\langle 0|+|-\rangle\langle 1|=\frac{1}{\sqrt{2}}(|0\rangle\langle 0|+|0\rangle\langle 1|+|1\rangle\langle 0|-|1\rangle\langle 1|)$$

In matrix representation,

$$H\doteq\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$$

# State Collapse

When an operator $M$ is measured, the experimentalist obtains the eigenvalue $\lambda$ and the system collapses into then corresponding eigenvector $|\lambda\rangle$.

**Important!**

Like the coin in the box, the system is left in a definite state after observation. But don't let this deceieve you–the coin was in a definite but unknown state before measurement, whereas the qubit was not in any definite state at all.

**note**

A Hermitian operator is one such that $M^\dagger = M$. Hermitian operators correspond to physical observable quantities because they have strictly real eigenvalues.

Measurement irreversibly destroys quantum information!

# Expectation Values

A property of Hermitian operators is that they can be written as "diagonal" in the basis of their eigenvectors. If $M$ is an operator with eigenvectors $|a\rangle$ and $|b\rangle$ with corresponding eigenvalues $a$ and $b$, then

$$M = a\,|a\rangle\langle a| + b\,|b\rangle\langle b|$$

Given some sort of operator $M$ that we want to find the expectation value of, we can simply compute,

$$\langle M \rangle_\psi = \langle \psi | M | \psi \rangle$$

For a Hermitian operator, this means

$$\langle M \rangle_\psi = \langle \psi|\, a\,|a\rangle\langle a| + b\,|b\rangle\langle b|\,|\psi\rangle = a|\langle a|\psi\rangle|^2 + b|\langle b|\psi\rangle|^2 = ap(a) + bp(b)$$

# Entanglement

This can easily be extended to multiple qubits. Two separate qubit states can be combined using the "Kronecker product" symbol,

$$|01\rangle_{AB} = |0\rangle_B \otimes |1\rangle_A$$

When a multi-qubit state can be written as a Kronecker product, it is called a product state. Operators can also take the form of kronecker products, and they act on their respective subsystems

$$(X_B \otimes X_A)\,|0\rangle_B \otimes |1\rangle_A = X_A\,|0\rangle_A \otimes X_B\,|1\rangle_B = |1\rangle_B \otimes |0\rangle_A$$

When states cannot be represented as product states, they are called product states. The canonical example is the first Bell state $|\Phi_0\rangle$:

$$|\Phi_0\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$