Čech Cohomology: categorical limits, axioms, and continuity

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1 Introduction

The fundamental problem in topology is the classification of topological spaces. By the late 1920s, finite complexes were reasonably well understood. The remaining challenge was then to approximate poorly behaved spaces with these 'nice' spaces. Of several approximation methods being developed around the time, Alexandroff introduced the idea of associating a finite abstract simplicial complex $\mathsf{N}(\mathcal{U})$ to each finite open cover $\mathcal{U}$ of a space which encodes the combinatorial data of the intersections in the cover, called its nerve. Alexandroff made the key realization that if $\mathcal{V}$ is a refinement of $\mathcal{U}$, then there is a collection of projections $\pi : \mathsf{N}(\mathcal{V}) \to \mathsf{N}(\mathcal{U})$. These maps are not unique, but remarkably, it turns out that any such projection induces a *unique* projection in the simplicial (co-)homology of the nerves. In the following years, this idea was formalized in the language of direct and inverse limits, culimating in what is now called the Čech cohomology around 1935. For an in-depth history of algebraic topology, we refer the reader to [2].

This review will serve as an introduction to Čech cohomology. First, we will present a primer on directed systems, paying particular attention to their category-theoretic definition, the nature of their morphisms, and induced maps on the limit. This development mostly follows [3]. Next, we will introduce the nerve of an open cover and show how to construct the Čech cohomology using the machinery of directed systems previously developed. In particular, we will show how continuous maps induce maps on the Čech groups, and prove that this is functorial. Then, we will establish the Eilenberg-Steenrod axioms, going throught the proof of homotopy-invariance in particular detail to illustrate the style of proofs. For this, we will follow [4].

Lastly, we will conclude by addressing the property of continuity which makes the Čech theory unique among cohomology theories, and present several theorems which relate the Čech cohomology to other commonly studied theories in restricted contexts.

2 Definition of Čech cohomology

2.1 Direct limits

Definition 1 (Directed set)**.** *A directed set is a nonempty set* I *with a reflexive and transitive binary relation* $\leqslant$*, called a quasi-ordering, with the property that any pair of elements has an upper bound, i.e. for each* $a, b \in I$ *there exists* $c \in I$ *with* $a \leqslant c$ *and* $b \leqslant c$*. The set* I *forms a category with* $a \to b$ *whenever* $a \leqslant b$*.*

Definition 2 (Cofinal subset)**.** *A subset* $I' \leqslant I$ *is cofinal if for all* $i \in I$ *there exists* $i' \in I'$ *with* $i \leqslant i'$*.*

The notion of cofinality generalizes the notion of a subsequence, and a similar result to subsequential limits will apply to the limit of a direct sequence.

Definition 3 (Direct systems, direct limits)**.** *Let* $(I, \leqslant)$ *be a directed set and let* $\mathcal{C}$ *be a category. A direct* I *system over* I *is a functor* $\mathsf{D} : I \to \mathcal{C}$*.*

In other words, D assigns to each $i \in I$ an object $\mathsf{D}_i \in \mathcal{C}$ and to each $i \leqslant j$ a map $\mathsf{D}_{ij} : \mathsf{D}_i \to \mathsf{D}_j$, where morphism composition respects $\mathsf{D}_{ij}\mathsf{D}_{jk} = \mathsf{D}_{ik}$ for any $i \leqslant j \leqslant k$.

Proposition 1. *Direct systems in* $\mathcal{C}$ *form a category* $\mathrm{Ind}(\mathcal{C})$*. A morphism between two directed systems* $\mathsf{D} : I \to \mathcal{C}$ *and* $\mathsf{D}' : I' \to \mathcal{C}$ *consists of a functor* $\phi : I \to I'$ *and a natural transformation* $\mathsf{D} \to \mathsf{D}' \circ \phi$*.*

More concretely, ϕ is a morphism of direct systems $\mathsf{D} \to \mathsf{D}'$ if for each $i \leqslant j$ in I, then the following diagram commutes:

$$\begin{array}{ccc} \mathsf{D}_i & \xrightarrow{\phi} & \mathsf{D}'_{\phi(i)} \\ \downarrow{\scriptstyle \mathsf{D}_{ij}} & & \downarrow{\scriptstyle \mathsf{D}'_{\phi(i)\phi(j)}} \\ \mathsf{D}_j & \xrightarrow{\phi} & \mathsf{D}'_{\phi(j)} \end{array}$$

Definition 4. *Let* $\Delta : \mathcal{C} \to \mathcal{C}^I$ *be the diagonal functor which maps* $\mathsf{K} \in \mathcal{C}$ *to the direct system* $[\Delta(\mathsf{K})]_i = \mathsf{K}$ *for all* $i \in I$*. Let a map of directed systems* $\mathsf{D} \to \Delta(\mathsf{K})$ *for any* K *be called a transformation of* D*.*

Definition 5 (Direct limit)**.** *Given a direct system* D*, a morphism of direct systems* $\mathsf{u} : \mathsf{D} \to \Delta(\mathsf{L})$ *is called a universal transformation if it satisfies the following universal property: for every morphism of direct systems* $\phi : \mathsf{D} \to \Delta(\mathsf{K})$*,* $\mathsf{K} \in \mathcal{C}$ *there exists a unique morphism* $\psi : \mathsf{L} \to \mathsf{K}$ *such that* $\Delta(\psi) \circ \mathsf{u} = \phi$*.*

To illutrate the point, consider the diagrams in Fig. 2.1. The first diagram shows the universality property in the

$$\begin{tikzcd} \mathsf{D} \arrow[r, "\mathsf{u}"] \arrow[dr, "\phi"'] & \mathsf{F(L)} \arrow[d, "\mathsf{F}(\psi)"] \\ & \mathsf{F(K)} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{D_i} \arrow[dr, "\mathsf{u_i}"] \arrow[dd, "\mathsf{D_{ij}}"'] \arrow[drr, bend left, "\phi_i"] & & \\ & \mathsf{L} \arrow[r, "\psi"] & \mathsf{K} \\ \mathsf{D_j} \arrow[ur, "\mathsf{u_j}"'] \arrow[urr, bend right, "\phi_j"'] & & \end{tikzcd}$$

category Ind($\mathcal{C}$) and the second shows the equivalent diagram within $\mathcal{K}$. The uniqueness is manifestly unique: Given two limits L and L', we have unique morphisms $\psi : \mathsf{L} \to \mathsf{L}$ and $\psi' : \mathsf{L}' \to \mathsf{L}$. Since $\psi \circ \psi'$ are a morphism $\mathsf{L} \to \mathsf{L}$ that makes the diagram commute with L, K are both taken to be L, it must be the unique morphism with this property, but of course Id_{L} is also such a morphism, so the two limits must be isomorphic. The second diagram also illutrates a more general category-theoretic construction of a colimit, and this diagram goes by the special name of a universal co-cone. The universal property satisfied by L makes it into what is called a universal repelling target.

Proposition 2. *If* $\mathrm{I}' \subseteq \mathrm{I}$ *is cofinal, then the directed subystem* D' *obtained from the inclusion* $\mathrm{I}' \hookrightarrow \mathrm{I}$ *satisfies* $\varinjlim \mathsf{D}' = \varinjlim \mathsf{D}$.

We will not prove this in general, but for Abelian groups it follows easily from the next proposition.

Proposition 3. *in the category of Abelian groups, modules, or complexes, the direct limit* $\varinjlim \mathsf{D}$ *of a direct system* D *is the object* $\varinjlim \mathsf{D} = \bigoplus_{i \in \mathrm{I}} \mathsf{D_i}/\{\mathsf{u_i} - \mathsf{D_{ij}u_i} : \mathsf{i} \leqslant \mathsf{j}, \mathsf{u_i} \in \mathsf{D_i}\}$.

In other words, the direct limit is the set of all point that 'eventually' become equivalent under the direct system, in that each $\mathsf{x_i}$ is equal to $\mathsf{D_{ij}x_i}$ for all $\mathsf{i} \leqslant \mathsf{j}$.

Proof. We will show that the transformation

$$\mathsf{D_i} \hookrightarrow \bigoplus_{i \in \mathrm{I}} \mathsf{D_i} \longrightarrow \bigoplus_{j \in \mathrm{I}} \mathsf{D_j}/\{\mathsf{u_i} - \mathsf{D_{ij}u_i} : \mathsf{i} \leqslant \mathsf{j}, \mathsf{u_i} \in \mathsf{D_i}\} \equiv \mathsf{L} \tag{1}$$

is universal. Suppose that $\phi : \mathsf{D} \to \Delta(\mathsf{K})$ is a morphism of direct systems. By the universal property of direct sums, there is a unique morphism $\psi' : \bigoplus_{i \in \mathrm{I}} \mathsf{D_i} \to \mathsf{K}$ extending ϕ. Since $\psi'(\mathsf{x_i} - \mathsf{D_{ij}x_j}) = \phi_i \mathsf{x_i} - \phi_j \mathsf{D_{ij}x_j} = \phi_i \mathsf{x_i} - \phi_i \mathsf{x_i} = 0$ (simply because $\mathsf{F(K)}$ is constant), this map factors uniquely through the quotient. ☐

Definition 6. *A map* $\mathsf{d} : \mathsf{D_i} \to \mathsf{D}'_{\mathsf{d(i)}}$ *for all* $\mathsf{i} \in \mathrm{I}$ *(not necessarily a morphism of direct systems) passes to the limit if for any* $\mathsf{K} \in \mathcal{K}$ *and a transformation* $\phi' : \mathsf{D}' \to \Delta(\mathsf{K})$*, the composition* $\phi' \circ \mathsf{d}$ *is a transformation* $\mathsf{D} \to \Delta(\mathsf{K})$*. In this case,* d *induces a unique morphism* $\varinjlim \mathsf{d} : \varinjlim \mathsf{D} \to \varinjlim \mathsf{D}'$ *via* $(\varinjlim \mathsf{d})\mathsf{u_i} = \mathsf{u}'_{\mathsf{d(i)}}\mathsf{d_i}$*.*

Proof. Consider the following diagram:

$$\begin{tikzcd} \mathsf{D} \arrow[r, "\mathsf{d}"] \arrow[d, "\mathsf{u}"] & \mathsf{D}' \arrow[r, "\mathsf{u}'"] & \varinjlim \mathsf{D}' \\ \varinjlim \mathsf{D} \arrow[urr, "\varinjlim \mathsf{d}"'] & & \end{tikzcd}$$

The existence and uniqueness of $\varinjlim \mathsf{d}$ follow from the universal property of the limit. ☐

Proposition 4. *If* $\mathsf{D}, \mathsf{D}', \mathsf{D}''$ *are direct systems and* $\mathsf{d} : \mathsf{D} \to \mathsf{D}'$*,* $\mathsf{d}' : \mathsf{D}' \to \mathsf{D}''$ *pass to the limit, then* $\mathsf{d}' \circ \mathsf{d}$ *passes to the limit and* $\varinjlim \mathsf{d}'' \circ \mathsf{d}' = (\varinjlim \mathsf{d}') \circ (\varinjlim \mathsf{d})$*.*

Proof. If ϕ' is a transformation then $\phi' \circ \mathsf{d}'$ is a transformation, and then $(\phi' \circ \mathsf{d}') \circ \mathsf{d}''$ is a transformation. Thus $\mathsf{d}'' \circ \mathsf{d}'$ passes to the limit. Next, if $\mathsf{u} : \mathsf{D} \to \mathsf{L}$ and $\mathsf{u} : \mathsf{D}'' \to \mathsf{L}''$ are universal transformations, then

$$\lim_{\to}(\mathsf{d}' \circ \mathsf{d}) = \mathsf{u}'' \circ (\mathsf{d}' \circ \mathsf{d}) = (\mathsf{u}'' \circ \mathsf{d}') \circ \mathsf{d} = (\lim_{\to} \mathsf{d}') \circ (\mathsf{u}' \circ \mathsf{d}) = (\lim_{\to} \mathsf{d}') \circ (\lim_{\to} \mathsf{d}') \circ \mathsf{u}$$

by the uniqueness of the limit map, this implies the claim. ☐

Thus we can view the direct limit as a functor from the category of direct systems and maps that pass to the limit.

Proposition 5. *If each sequence in a direct system in the category of chain complexes over Abelian groups is exact, then the limit sequence is also exact.*

Proof. Let D be a direct system of exact sequences. Without loss of generality, we may take the sequences to have length 3; for $i \in I$ we have $D_i = D_i^A \xrightarrow{f_i} D_i^B \xrightarrow{g_i} D_i^C$. By exactness, we have $f_i \circ g_i = 0$. Since $f_i \circ g_i$ is a morphism of direct systems $D^A \to D^C$, it clearly passes to the limit, and since the limit is unique, we have $\varinjlim f \circ g = \varinjlim f \circ \varinjlim g = 0$. This shows that the limit sequence is a chain complex.

Now suppose that $(\varinjlim g)(x) = 0$. Then there exists a x_i with $u_i(x_i) = x$ (where $u : D \hookrightarrow \bigoplus_i D_i \twoheadrightarrow \bigoplus_i D_i/\{x_i - D_{ij}x_j\}$ is the map we constructed previously). Then $u_i(g_i(x_i)) = (\varinjlim g)(u_i(x_i)) = 0$. Therefore there exists j such that $D_{ij}(g_i(x_i)) = 0$. Since $D_{ij} \circ g_i = g_j \circ D_{ij}$ (because D_{ij} is a map of chain complexes), this gives $g_j(D_{ij}(x_i)) = 0$. By exactness, we can find y_j such that $f_j(y_j) = D_{ij}(x_i)$, and then

$$\varinjlim f(u_j(y_j)) = u_j(f_j(y_j)) = u_j(D_{ij}x_i) = u_i(x_i) = x$$

This shows that the limit sequence is exact, as desired. □

2.2 Cohomology groups

2.2.1 The Nerve

Definition 7 (Abstract simplicial complex)**.** *A abstract simplicial complex $\mathcal{K}$ over a set of vertices V is a set of nonempty finite subsets of V called simplices, where each $v \in V$ is a simplex and if $\Delta \subset V$ is a simplex then any subset of Δ is also a simplex.*

Definition 8. *Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an indexed family of open sets in X. If $X = \bigcup U_i$ then $\mathcal{U}$ is an open covering of X. The set of open coverings is denoted* Cov(X)*. Let X, A be a topological pair. If $I^A \subseteq I$ such that $A \subset \bigcup_{i \in I^A} U_i$, then $\mathcal{U}$ is called a covering of the pair (X, A) with indexing set (I, I^A). The set of all open coverings of (X, A) is denoted* Cov(X, A).

Definition 9. *Given simplicial complexes K, K', two maps $f, g : K \to K'$ are called contiguous if for any $\Delta \in K$, $f(\Delta)$ and $g(\Delta)$ are faces of a common simplex in K'.*

Proposition 6. *If f, g are contiguous as above, then the maps $f^\bullet, g^\bullet : H_\Delta^\bullet(K'; G) \to H_\Delta^\bullet(K; G)$ induced in the simplicial cohomology are equal.*

The proof is a standard result in simplicial cohomology, and may be found in e.g. [4]. Essentially, contiguity plays the role of homotopy equivalence with abstract simplicial complexes.

Definition 10. *Let $\mathcal{U} \in$ Cov(X) with indexing set I. Let $S_\mathcal{U}$ be be the simplicial complex of all simplices whose vertices are elements of I. If $\Delta \in S_\mathcal{U}$ is a simplex, then let the carrier of Δ, denoted* Car(Δ)*, be the intersection $\bigcup_{i \in \Delta} U_i$. The nerve of $\mathcal{U}$, denoted by $X_\mathcal{U}$, is the simplicial subcomplex of $S_\mathcal{U}$ consisting of all simplices with nonempty carriers. If $\mathcal{U} \in$ Cov(X, A) with indexing set (I, I^A), then let $A_\mathcal{U}$ be the subcomplex consisting of all simplices $\Delta \in X_\mathcal{U}$ with vertices in I^A such that $A \cap$ Car$(\Delta) \neq \emptyset$. In this case the pair $(X_\mathcal{U}, A_\mathcal{U})$ will be called the nerve of $\mathcal{U}$, denoted $N(\mathcal{U})$.*

Proposition 7. $N(\mathcal{U})$ *is a simplicial complex.*

Proof. If $\Delta^m \subset \Delta^n$ where Δ^n is a simplex in $N(\mathcal{U})$, then since the vertices of Δ^n have a non-trivial intersection, it follows that the vertices of Δ^m also intersect non-trivially. Therefore Δ^m has a non-trivial carrier, and so it belongs to $N(\mathcal{U})$. Since every open set $U \in \mathcal{U}$ intersects non-trivially with itself, U is a simplex. □

This would lead us naturally to consider the simplicial homology of this complex, which will be the foundation of the Čech group.

2.2.2 Refinements and projections

Definition 11. *A cover $\mathcal{V}$ is a called a refinement of $\mathcal{U}$, denoted $\mathcal{U} \leqslant \mathcal{V}$, if for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ with $V \subseteq U$.*

Proposition 8. *The relationship $\leqslant$ makes* Cov(X, A) *into a directed set.*

Proof. Reflexivity and transitivity are clear. For the upper bound property, for any two covers $\mathcal{U}, \mathcal{V} \in$ Cov(X, A) just take pairwise intersections of all the sets in the cover. □

Definition 12. *If* $f : (X, A) \to (Y, B)$ *is continuous map and* $\mathcal{V}$ *is a covering of* (Y, B) *then* $\widetilde{f} : \mathcal{V} \mapsto f^{-1}\mathcal{V}$ *(taken element-wise) is a map* $\widetilde{f} : \mathrm{Cov}(Y, B) \to \mathrm{Cov}(X, A)$*. If* $\mathcal{V}_1 \leqslant \mathcal{V}_2$ *then* $\widetilde{f}(V_1) \leqslant \widetilde{f}(V_2)$*, so* $\widetilde{f}$ *is order-preserving on the directed set of coverings. For an* $\mathcal{U} \in \mathrm{Cov}(X, A)$*, then* $N(f^{-1}\mathcal{U})$ *is a subcomplex of* $\mathcal{U}$*, and the inclusion map is denoted* $f_{\mathcal{U}}$*.*

Definition 13. *If* $\mathcal{U} \leqslant \mathcal{V}$ *with indexing sets* $(I_U, I_U^A), (I_V, I_V^A)$*, then a map* $\pi : (I_V, I_V^A) \to (I_U, I_U^A)$ *is called a projection if* $V_i \subseteq U_{\pi(i)}$ *for each* $I \in I_V$*.*

Notice that such a projection clearly exists: If $\mathcal{U} \leqslant \mathcal{V}$, then for each $V_i \in \mathcal{V}$ there is some $U_j \in \mathcal{U}$ such that $V_i \subseteq U_j$, so the map $\pi(i) = j$ would be such a projection. However, this projection is not uniquely defined.

Lemma 1. *If* $\pi, \pi' : \mathcal{V} \to \mathcal{U}$ *(implicit:* $\mathcal{U} \leqslant \mathcal{V}$*) are two projections, then they induce maps* $\pi, \pi' : N(\mathcal{V}) \to N(\mathcal{U})$ *which are contiguous. In particular, this means the induced map* $H^*_\Delta(N(\mathcal{U}); G) \to H^*_\Delta(N(\mathcal{V}); G)$ *is unique.*

Proof. It suffices to show that the induced simplicial maps are continguous. Let Δ be a simplex of $N(\mathcal{V})$. Since π, π' are both projections, the simplex constructed from the images of the vertices of Δ under π and π' will have a non-empty intersection, so $\pi(\Delta)$ and $\pi'(\Delta)$ are both faces of a simplex in $N(\mathcal{U})$, and therefore they are contiguous. □

Notice that projections on coverings are ordered with respect to inclusion, but the induced map on the cohomology respect reverse inclusion, which follows the ordering of refinement.

Proposition 9. *These projections lead to a direct system* $D : \mathrm{Cov}(X, A) \xrightarrow{N} \mathrm{Simp} \xrightarrow{H^q_\Delta} \mathrm{Ab}$ *called the* q^{th} *Čech cohomology system of* (X, A)*.*

Proposition 10. *The inclusion map of simplicial complexes* $f : N(f^{-1}\mathcal{V}) \hookrightarrow N(\mathcal{V})$ *induced by a continuous map* $f : (X, A) \to (Y, B)$ *induces is a morphism of directed systems.*

Proof. Let $f : (X, A) \to (Y, B)$ and consider $\mathcal{U} \leqslant \mathcal{V} \in \mathrm{Cov}(Y, B)$. Let $\pi : N(\mathcal{V}) \to N(\mathcal{U})$ be any projection map. Consider the map π' obtained by restricting π to act on $N(f^{-1}\mathcal{V})$. If $V_i \subseteq U_{\pi(i)}$ then $f^{-1}(V_i) \subseteq f^{-1}(U_{\pi(i)})$, so $\pi' : N(f^{-1}V) \to N(f^{-1}\mathcal{U})$ is a projection map. Therefore commutativity holds in the left diagram below, and thus it also holds in the right diagram. This shows that f induces a morphism of direceted sequences. □

$$
\begin{tikzcd}
N(\mathcal{V}) \arrow[r, "\pi"] \arrow[d] & N(\mathcal{U}) \arrow[d] \\
N(f^{-1}\mathcal{V}) \arrow[r, "\pi'"] & N(f^{-1}\mathcal{U})
\end{tikzcd}
\qquad
\begin{tikzcd}
H^q_\Delta(N(\mathcal{V}); G) & H^q_\Delta(N(\mathcal{U}); G) \arrow[l, "\pi^q"'] \\
H^q_\Delta(N(f^{-1}\mathcal{V}); G) \arrow[u] & H^q_\Delta(N(f^{-1}\mathcal{U}); G) \arrow[l, "\pi'^q"'] \arrow[u]
\end{tikzcd}
$$

Definition 14. *The directed limit of the* q^{th} *Cech system is called the* q^{th} *Cech cohomology of* (X, A) *with coefficients in* G*, denoted* $\check{H}^q(X, A; G)$*.*

We should note here that from Prop. 3 defining the directed limit in the category of Abelian groups, we can see that an object only exists in the category of finite Abelian groups if the directed system is finite; This means that the Cech cohomology with coefficients in a finite group can only be defined over a compact space, which by definition has a cofinal set of finite open covers.

Theorem 1 (Functoriality)**.** *The Cech cohomology is a functor from the category of topological spaces to the category of abelian groups.*

Proof. We have already shown that a continuous map f of topological pairs induces a map of directed systems; this necessarily means that f passes to the limit, and the result follows. □

3 Axioms

Theorem 2 (Dimension axiom)**.** *If* $\{*\}$ *is a point, then* $\check{H}^k(\{*\}; G) = 0$ *for* $k > 0$ *and* $\check{H}^0(\{*\}; G) \cong G$*.*

Proof. Since $\{*\}$ is the only possible open cover, the direct limit is clearly the simplicial cohomology of a single $0-$simplex. The result follows by the dimension axiom for simplicial cohomology. □

Theorem 3 (Additivity)**.** *If* $X = \bigsqcup_\alpha X_\alpha$*, then* $\check{H}^q(X; G) \cong \bigoplus_\alpha \check{H}^q(X_\alpha; G)$*.*

Proof. By definition, each X_α is open. Taking any open cover $\mathcal{U} \in \mathrm{Cov}(X)$ and intersecting with X_α for each α given a refinement of $\mathcal{U}$ which is a disjoint union of covers $\mathcal{U}_\alpha$, so such covers are a cofinal subset of $\mathrm{Cov}(X)$. Thus $H^q_\Delta(N(\bigsqcup_\alpha \mathcal{U}_\alpha); G) \cong \bigoplus_\alpha H^q_\Delta(N(\mathcal{U}_\alpha); G)$ by the additivty property of simplicial cohomology, and this isomorphism passes to the limit. □

3.1 Homotopy axiom

Theorem 4 (Homotopy Axiom)**.** *Let g_0, g_1 be homotopic maps. Then $g_0^\bullet = g_1^\bullet$.*

Proposition 11 (Alternative homotopy axiom)**.** *If $g_0, g_1 : (X, A) \to (X, A) \times I$ are defined by $g_0(x) = (x, 0)$ and $g_1(x) = (x, 1)$, then $g_0^\bullet = g_1^\bullet$.*

Corollary 1. *The second homotopy axiom implies the first.*

Proof. If h is a homotopy of maps $f_0, f_1 : (X, A) \to (Y, B)$, then $f_0 = hg_0$ and $f_1 = hg_1$, so by the functoriality of $\check{H}^\bullet$,

$$f_0^\bullet = g_0^\bullet h^\bullet = g_1^\bullet h^\bullet = f_1^\bullet \tag{2}$$

□

The proof of the homotopy axiom is completed in several lemmas:

Lemma 2. *If $\mathcal{U}$ is a finite open covering of an interval I by connected sets for which no proper containment holds, then $N(\mathcal{U})$ is nullhomotopic (in the sense that the identity map is continguous to the constant map).*

Note that the set of covers for which proper containment holds is cofinal, so we may restrict to these covers.

Proof. Sort the open sets in the cover so that U_i has enpoints l_i, r_i, where

$$0 = l_0 \leqslant l_1 < l_2 < \cdots < l_n \qquad\qquad r_0 < r_1 < \cdots < r_{n-1} \leqslant r_n = 1 \tag{3}$$

□

Then consider the simplicial maps on the nerves f_i defined $f_i(\nu_j) = \nu_j$ if $j \leqslant i$ and $f_i(\nu_j) = \nu_i$ otherwise. Consider the maps f_i and f_{i+1} for some fixed i. Let Δ be a simplex, and since $f_i(\Delta) = f_{i+1}(\Delta)$ if all the vertices in Δ are $\nu_{j \leqslant i}$, we take Δ to contain some verices $\nu_{j>i}$. Then $f_i(\Delta) = \Delta'\nu_i$ where Δ' has vertices less than i and $f_{i+1}(\Delta) = \Delta'\nu_i\nu_{i+1}$ or just $f_{i+1}(\Delta) = \Delta'\nu_{i+1}$. In either case, we can see (draw a picture) that $\Delta'\nu_i\nu_{i+1}$ is a simplex containing the simplices $f_i(\Delta)$ and $f_{i+1}(\Delta)$.

Thus, we have that f_i is continguous to f_{i+1}. Since f_0 is the constant map and f_n is the identity map (which follows since I is a finite cover, but this is fine because I is compact and thus finite open covers are cofinal), this shows that $N(I)$ is nullhomotopic.

Definition 15. *A finite covering of an interval I with n elements is called regular if*

(1) I_0 *and* I_n *are the only sets containing the endpoints* 0 *and* 1 *respectively*

(2) I *intersects nontrivially with* I_{i+1}

(3) I *intersects trivially with* $I_{j<i-1}$

Proposition 12. *The regular coverings are a cofinal subset of* $\mathrm{Cov}(I)$.

Proof. Notice that a regular covering, up to reordering, is just a finite cover which has no proper subcover. We may always refine a cover of a compact interval to have these properties, so the set of regular coverings is cofinal. □

Definition 16. *Let $\mathcal{U} \in \mathrm{Cov}(X, A)$ indexed by (J, J^A). If to each $j \in J$ there corresponds a regular covering $\mathcal{R}_i$ of the interval I, then the resultant product covering is called a stacked covering.*

Proposition 13. *Stacked coverings form a cofinal subset of* $\mathrm{Cov}(X \times I, A \times I)$.

Proof. Let $\mathcal{W} \in \mathrm{Cov}(X \times I, A \times I)$ be indexed by J, J^A. For each $(x, t) \in X \times I$ pick a rectangular neighborhood $U_{x,t} \times V_{x,t}$ contained within W_i for some i, and for some $i \in J^A$ if $x \in A$. For a fixed x, let $\widetilde{\mathcal{V}}_x = \{V_{x,i}\}_i$ be a regular refinement of $\{V_{x,t}\}_t$. Then there are sets $U_{x,i}$ such that $U_{x,i} \times V_{x,i}$ is contained within W_i for some i, and for some $i \in J^A$ if $x \in A$. Then put $U_x = \bigcap_i U_{x,i}$. The cover $\{U_x \times V_{x,i}\}$ is a stacked open cover and a refinement of $\mathcal{W}$, proving the claim. □

Lemma 3. *If $\mathcal{W}$ is a stacked cover over $\mathcal{U} \in \mathrm{Cov}(X)$, then if $N(\mathcal{U})$ is a finite simplex, $N(\mathcal{W})$ is nullhomotopic.*

Proof. Let $\Delta = \{(\nu_0, i_0), \ldots, (\nu_n, i_n)\} \in N(\mathcal{W})$. Since $N(\mathcal{W})$ is a simplex, $\mathrm{Car}(\nu_0, \ldots, \nu_n)$ is non-empty, so $\mathrm{Car}(\Delta)$ is only empty if $\mathrm{Car}(i_0, \ldots, i_n)$ is. Since $N(\mathcal{W})$ is furthermore finite, we can form the finite cover $\mathcal{V} = \bigcup_{U \in \mathcal{U}} \mathcal{V}_U$, where $\mathcal{V}_U$ is the cover of I stacked over $U \in \mathcal{U}$. Thus we have $N(\mathcal{W}) = N(\mathcal{V})$, which we have already shown is nullhomotopic. □

Lemma 4. *Let $\mathcal{U} \in \mathrm{Cov}(X, A)$ be indexed by J, J^A, and let $\mathcal{W}$ be stacked over $\mathcal{U}$. Consider the simplicial maps $l, u : N(\mathcal{U}) \to N(\mathcal{W})$ (for lower and upper) via $l(v) = (v, 0)$ and $u(v) = (v, n^v)$ (where n^v is the size of the cover of* I *stacked over the vertex v). Then $l^\bullet = u^\bullet$.*

Before we prove this lemma, we need a technical tool which generalizes the notion of continguity, and is proven in [4]:

Definition 17. *If K, K' are simplicial complexes and C is a map which assigns to every $\Delta \in K$ a subcomplex $C(\Delta) \in K'$, then if for any $\Delta' \subset \Delta \in K$, $C(\Delta')$ is a subcomplex of $C(\Delta)$, we call C a carrier function. If $f : K \to K'$ is algebraic, meaning that the $\epsilon(c) = \epsilon(fc)$ for any cycle c where ϵ is the agumentation map, and $c \subseteq \Delta$ implies $f(c) \subseteq C(\Delta)$, then C is called a carrier of f. If furthermore $C(\Delta)$ is nullhomotopic for each $\Delta \in K$, then C is called an acyclic carrier.*

Lemma 5. *Let $f, g : K \to K'$ be algebraic maps with an acyclic carrier. Then $f^\bullet = g^\bullet$.*

Proof of Lemma 4. For every $\Delta \in N(\mathcal{U})$, let $C(\Delta)$ be the subcomplex of $N(\mathcal{W})$ consisting of all simplices with vertices of the form (v, i), where $v \in \Delta$. Since $C(s)$ is a stacked cover over a simplex, it is nullhomotopic. It is quickly verified that C is a carrier for both f and g, and thus $f^\bullet = g^\bullet$. □

Proof of Prop. 11. Let $\mathcal{U} \in \mathrm{Cov}(X, A)$, and let $\mathcal{W}$ be stacked over $\mathcal{U}$. Then the map $u : \mathcal{U} \to \mathcal{W}$ via $u(v) = (v, n^v)$ as above may be written as $g_1 u'$, where g_1 denotes the inclusion map $N(g_1^{-1}\mathcal{W}) \hookrightarrow N(\mathcal{W})$, and $u' : N(\mathcal{U}) \to N(g_1^{-1}\mathcal{W})$ via $u'(v) = (v, n^v)$ is an isomorphism. This follows because $g_1(x) = (x, 1)$, and because the stacks are regular covers, there is only one open set per stack containing $(x, 1)$.

Next, let $U \times V \in \mathcal{W}$. Then $g_1^{-1}(U \times V) = U$ if $1 \in V$ and zero otherwise. Since the stacks are regular, we may find $U \times V'$ with $0 \in V'$, so $g_0^{-1}(U \times V') = U$, and therefore $g_1^{-1}\mathcal{W} \leqslant g_0^{-1}\mathcal{W}$.

We may define a map $\pi : N(g_1^{-1}\mathcal{W}) \to N(g_0^{-1}\mathcal{W})$ via $\pi(v, i) = (v, 0)$, and observe by the same reasoning as above that this map is a projections. We may also write $l = g_0 \pi u'$, where $l(v) = (v, 0)$ is the map defined previously. Therefore

$$l^\bullet = u'^\bullet \pi^\bullet g_0^\bullet \qquad\qquad u^\bullet = u'^\bullet g_1^\bullet \tag{4}$$

Since $l^\bullet = u^\bullet$ by the previousl lemma and $u'^\bullet$ is an isomorphism, we have $g_1^\bullet = \pi^\bullet g_0^\bullet$. Since $\pi^\bullet$ is a projection, when we pass to the direct limit we have $\varinjlim g_1^\bullet = \varinjlim g_0^\bullet$, proving the claim. □

3.2 Excision

Theorem 5 (Excision axiom)**.** *If U is open in X and its closure is contained in the interior of $A \subset X$, then the inclusion map $f : (X - U, A - U) \to (X, A)$ induces an isomorphism $f^\bullet : \check{H}^\bullet_\Delta(X, A; G) \to \check{H}^\bullet_\Delta(X - U, A - U; G)$.*

Lemma 6. *Let D be the subset of $\mathrm{Cov}(X, A)$ consisiting of covers $\mathcal{U}$ indexed over I, I^A such that if $U_i \cap U \neq \emptyset$ then $i \in I^A$ and $U_i \subseteq A$. Then D is cofinal in $\mathrm{Cov}(X, A)$.*

Proof. Consider the covering $\mathcal{U}'$ obtained as the union of $\{U_i - \overline{U}\}$ and $\{U_i \cap \mathring{A}\}$. Since $\overline{U} \subseteq \mathring{A}$, $\mathcal{U}'$ is a covering of (X, A). Clearly $\mathcal{U} \leqslant \mathcal{U}'$ and $\mathcal{U}' \in D$. □

Lemma 7. *Again let $f : (X - U, A - U) \hookrightarrow (X, A)$ be the inclusion map. Then $f^{-1}D$ is cofinal in $\mathrm{Cov}(X - U, A - U)$.*

Proof. If $\mathcal{V}$ is a covering of $(X - U, A - U)$, then we can construct a covering $\mathcal{U} = \{V_i \cup U : i \in I\}$, so that $\mathcal{V} = f^{-1}\mathcal{U}$. Then we may find a refinement $\mathcal{U}' \in D$ of $\mathcal{U}$, and because f^{-1} is order-preserving, we have $f^{-1}U' \leqslant f^{-1}U\mathcal{V}$. □

Lemma 8. *Let $\mathcal{U} \in D$ and $N(\mathcal{U}) = (X_\mathcal{U}, A_\mathcal{U})$ as in def. 10. Then $X_\mathcal{U} = f^{-1}X_\mathcal{U} \cup A_\mathcal{U}$ and $f^{-1}A_\mathcal{U} = f^{-1}X_\mathcal{U} \cap A_\mathcal{U}$.*

Proof. Since $N(f^{-1}\mathcal{U})$ is a subcomplex of $N(\mathcal{U})$, we clearly have $f^{-1}X_\mathcal{U} \cup A_\mathcal{U} \subseteq X_\mathcal{U}$ and $f^{-1}A_\mathcal{U} \subseteq f^{-1}X_{mathcalU} \cap A_\mathcal{U}$, so the difficult part is showing inclusion in the other direction.

Let $\Delta \in X_\mathcal{U}$ and $\Delta \notin f^{-1}X_\mathcal{U}$. Thus $\mathrm{Car}(\Delta) \neq \emptyset$ but $\mathrm{Car}(\Delta) \cap f^{-1}X = \emptyset$. Since $f^{-1}X = X - U$, this means that $U_i \cap U \neq \emptyset$ for $i \in \Delta$, and since $U \subset A$, thus $\Delta \in A_\mathcal{U}$.

For the the second reverse inclusion, let $\Delta \in f^{-1}X_\mathcal{U} \cap A_\mathcal{U}$. Then $\mathrm{Car}_\mathcal{U}(\Delta) \cap f^{-1}X = \mathrm{Car}_{f^{-1}}(\Delta) \neq \emptyset$ and $\mathrm{Car}_\mathcal{U}(\Delta) \cap A \neq \emptyset$. If $\mathrm{Car}_\mathcal{U}(\Delta) \subseteq f^{-1}X$, then

$$\mathrm{Car}_{f^{-1}\mathcal{U}}(\Delta) \cap f^{-1}A = \mathrm{Car}_\mathcal{U}(\Delta) \cap f^{-1}X \cap A = \mathrm{Car}_\mathcal{U}(\Delta) \cap A \neq 0 \tag{5}$$

which implies that $\Delta \in f^{-1}A_\mathcal{U}$. Otherwise, if $\mathrm{Car}_\mathcal{U}(\Delta)$, then $U_i \subseteq A$ for every $i \in \Delta$, because by assumption $\mathcal{U} \in D$. This shows that $\mathrm{Car}_\mathcal{U}(\Delta) \subset A$, so

$$\mathrm{Car}_{f-1\mathcal{U}}(\Delta) \cap f^{-1}A = \mathrm{Car}_\mathcal{U}(\Delta) \cap f^{-1}X \cap A = \mathrm{Car}_\mathcal{U}(\Delta) \cap f^{-1}X \neq 0 \tag{6}$$

this again shows that $\Delta \in f^{-1}A_\mathcal{U}$. This completes the proof. □

Proof of Thm. 5. We have shown that $X_{\mathcal{U}} = f^{-1}X_{\mathcal{U}} \cup A_{\mathcal{U}}$ and $f^{-1}A_{\mathcal{U}} = f^{-1}X_{\mathcal{U}} \cap A_{\mathcal{U}}$. By the excision property of singular cohomology, this gives an isomorphism

$$H^\bullet_\Delta(X_{\mathcal{U}}, A_{\mathcal{U}}) = H^\bullet_\Delta(f^{-1}X_{\mathcal{U}} \cup A_{\mathcal{U}}, A_{\mathcal{U}}) \to H^\bullet_\Delta(f^{-1}X_{\mathcal{U}}, f^{-1}X_{\mathcal{U}} \cap A_{\mathcal{U}}) = H^\bullet_\Delta(f^{-1}X_{\mathcal{U}}, f^{-1}A_{\mathcal{U}}) \tag{7}$$

This proves excision. □

3.3 Exactness

Theorem 6 (Exactness axiom). *Given a topological pair (X, A), the inclusions $A \hookrightarrow X \hookrightarrow (X, A)$ induce a long exact sequence in the homology*

$$\cdots \to \check{H}^{q-1}(A; G) \to \check{H}^q(X, A; G) \to \check{H}^q(X; G) \to \check{H}^q(A; G) \to \dots \tag{8}$$

The essential detail of the proof above is that direct limits, as we have shown, preserve exactness, so the exactness of the Čech cohomology follows directly from the same axiom for simplicial cohomology. This is in fact a proof when A is closed, due to the following lemmas:

Lemma 9. *A covering $\mathcal{U} \in \mathrm{Cov}(X, A)$ indexed by I, I^A is called proper if I^A is the set of all $i \in I$ with $U_i \cap A \neq \emptyset$.*

Lemma 10. *The map $\mathrm{Cov}(X) \to \mathrm{Cov}(X, A)$ via $I \to (I, I^A)$ where $I^A = \{i \in I : U_i \cap A \neq \emptyset\}$ is a 1-1 order-preserving map.*

Lemma 11. *If A is closed, then the set of proper covers of (X, A) is cofinal.*

This shows that we can use $\mathrm{Cov}(X)$ as an indexing set when A is closed. The failure of the inverse limit to preserve exactness is the reason that the corresponding homology theory is nuanced to define.

4 Continuity

The Čech cohomology theory is essentially alone in satifying the axiom of continuity. Continuity is the property that if a compact pair (X, A) are obtained as the inverse limit of compact pairs (X_i, A_i), then the cohomology is the direct limit of the cohomology groups of (X_i, A_i). This leads to a much stronger version of excision which is computationally useful.

Theorem 7. *Let $(X, A) = D$ and let $u : D \to \varinjlim D$ be the projection to the direct limit. Then $\check{H}^q(D)$ is a directed system, and u induces a transformation $u^* : \check{H}^q(D) \to \check{H}^q(\varinjlim D)$. Thus u passes to the limit, defining $l = \varinjlim u : \varinjlim \check{H}^q(D) \to \check{H}^q(\varinjlim D)$. The morphism l is a natural transformation, in the sense that commutativity holds in the following diagrams:*

$$\begin{array}{ccc} \varinjlim \check{H}^q(X_i, A_i) & \xrightarrow{l} & \check{H}^q(\varinjlim(X_i, A_i)) \\ {\scriptstyle \lim \delta}\uparrow & & \uparrow{\scriptstyle \delta} \\ \varinjlim \check{H}^{q-1}(A_i) & \xrightarrow{l} & \check{H}^{q-1}(\varinjlim(A_i)) \end{array} \qquad \begin{array}{ccc} \varinjlim \check{H}^q(X_i, A_i) & \xrightarrow{l} & \check{H}^q(\varinjlim(X_i, A_i)) \\ {\scriptstyle \lim(f^\bullet)}\uparrow & & \uparrow{\scriptstyle (\lim f)^\bullet} \\ \varinjlim \check{H}^q(Y_i, B_i) & \xrightarrow{l} & \check{H}^q(\varinjlim(Y_i, B_i)) \end{array}$$

Definition 18. *A homology theory is said to be continuous if the transformation above is a natural equivalence $l : \varinjlim H^q(X_i, A_i) \cong H^q(\varinjlim(X_i, A_i))$.*

Theorem 8. *The Čech cohomology theory is continuous on the category of compact pairs.*

The continuity property of the Čech is clearly useful for situations in which a compact topological space is constructed as an inverse limit of other compact topological spaces. It also has the important computational consequence of a stronger excision theorem based on relative homeomorphism, which we will outline here.

Definition 19. *A map $f : (X, A) \to (Y, B)$ is called a relative homeomrphism if f maps $X - A$ homeomorphically into $Y - B$.*

We also have a useful equivalent characterization of relative homeomorphisms:

Proposition 14. *If $f : (X, A) \to (Y, B)$ is a map of compact pairs which maps $X - A$ injectively into $Y - B$, then f is a relative homeomorphism.*

Theorem 9. *The Čech cohomology theory on the category of compact pairs are invariant under relative homeomorphisms.*

5 Relation to other Cohomology theories

As one would expect, the Čech cohomoly agrees with other cohomology theories on "nice" spaces. In particular, the Čech construction forms the foundation for the cohomology of sheaves. The first theorem regards triangulable topological pairs:

Theorem 10 (9.3 in [4]). *Let* $T = \{t, (K, L)\}$ *be a triangulation of the pair (X,A). The cohomology sequence of* (K, L) *in the simplicial theory and the cohomology sequence of* (X, A) *in the the Čech theory are isomorphic. The isomorphism is obtained by projecting the cohomology of the nerve of the triangulation to the direct limit (as one would expect!)*

A triangulation gives a covering of a space which is particularly nice. There is a more general situation when this is the case;

Definition 20. *An open cover of a topological space* X *is called good if all finite intersections* $U_1 \cap \cdots \cap U_n$ *are contractible.*

Proposition 15 (Cor. 5.2 in [1]). *The good covers are cofinal in the set of all covers of a smooth manifold* M*.*

Proposition 16 (Thm. 8.9 In [1]). *If* M *is a smooth manifold and* $\mathcal{U}$ *is a good open cover, then*

$$H^{\bullet}(N(\mathcal{U}), \mathbb{R}) \cong H^{\bullet}_{DR}(M) \tag{9}$$

Combining these together and observing that the isomorphism constructed in the previous theorem passes to the limit, we obtain

Proposition 17 (Prop. 10.6 in [1]). *The Čech cohomology on a smooth manifold* M *with values in* $\mathbb{R}$ *is isomorphic to the de Rham cohomology.*

In fact, there is a stronger result, which shows the equivalence of all cohomology theories on paracompact spaces with a cofinal set of good open covers:

Theorem 11 (The Nerve Theorem; Cor. 4G.3 in [5]). *If* $\mathcal{U}$ *is a good open cover of a paracompact space* X*, then* X *is homotopy equivalent to* $N(\mathcal{U})$*.*

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