The Berry phase: theory of transport and holonomy in physics

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1. INTRODUCTION AND HISTORY

In 1984, physicists were astonished that such a simple yet powerful observation by Berry[1] had until then gone unappreciated. Previous examples of the notion of a geometric phase had been known in physics for years. Most notably, Pancharatnum, working in relative isolation, had already described the geometric phase arising in polarized slight after a cyclic series of changes in polarization in 1956[2]. After its introduction, what might best be described as a mad rush ensued to derive, extend, and explain the Berry phase in every manner possible. One observes that Berry himself was a prolific and effective messenger for this cause, probing the wide-ranging implications and in many cases providing the most straightforward and digestible explanations of his eponymous phase. Although Hannay and Pancharatnum were the originators of the classical and optical geometric phase respectively, it is a testament to Berry that readers are often directed to his writings to learn about these subjects. Contemporary mathematical developments in the physics of gauge fields, most notably by Chern, had primed the community to be excited by Berry's discovery of gauge physics arising in simple quantum systems. It was Barry Simon[3] who made the connection between the mathematical formalism underlying gauge theory and the geometric phase. This link is the reason we use the words "Berry connection" or "Berry curvature." In addition to confusing undergraduates, this terminology reflects the fact that the Berry phase is an example of the previously understood concept of holonomy. It has been said that the theory of holonomy and the topological classification of fiber bundles, the objects which give rise to holonomy, is one of the greatest mathematical achievements of the 20^{th} century[4].

In this review, we aim to first demystify the gauge-theoretic foundations for the geometric phase. We will describe parallel transport with a simple example, and show how the Berry connection is a natural object to consider outside of a quantum context. We then explore historically important extensions of the Berry phase, including the Aharonov-Anandan phase[5], the non-abelian Wilcek-Zee phase[6], and the classical Hannay angles[7]. For the reader who is interested in learning more, we refer to Wilczek et al.[6], which is a compendium of central works in the area of geometric phases with minimal annotations. For the mathematical development in the next section, Bohm et al.[4] as been an invaluable resource. For contemporary perspectives on the topic, the reviews by Berry[8] and Anandan[9] are excellent resources.

2. GAUGE THEORY, HOLONOMY, AND TRANSPORT

In order to appreciate the universality of the Berry phase and its relationship to fundamental physics, we need to introduce the foundation of gauge theory. Gauge theory describes redundancies in our description of physical systems, and remarkably, this description has incredibly rich physical consequences. The basic object which encodes this redundancy is a principal bundle, which we will now define, following Bohm et al.[4]

Definition 1 (Fiber Bundle)**.** *Given two smooth manifolds M and F, a smooth fiber bundle over M with typical fiber F is a smooth manifold E with a smooth, surjective projection map $\pi : E \to M$ such that for each $p \in M$ there exists a neighborhood $U \ni p$ such that there is a diffeomorphism $\phi : \pi^{-1}(U) \to U \times F$ called a local trivializing map.*

The above definition describes a space E which is *locally* diffeomorphic (smoothly equivalent) to the Cartesian product $M \times F$. The bundle $M \times F$ is called the trivial bundle, hence the name "local trivializing map." The bundle E may have a nontrivial topology which prevents it from being globally homeomorphic to the trivial bundle.

Definition 2 (Principal Bundle)**.** *If E is a fiber bundle over a manifold M with typical fiber G where G is a group, then E is called a $G-$principal bundle. In this case, G is called the structure group (by mathematicians) or the gauge group (by physicists) of E, and the fibers of E are equipped with a natural right action of G by right multiplication. The set of smooth functions $\Omega : M \to G$ are called gauge transformations.*

This definition may seem oblique, and we will spend the rest of the review giving concrete examples. The example most relevant to the Berry phase is the following:

Example 1. *Let $\mathcal{H}$ be a Hilbert space. We call $\mathcal{P}(\mathcal{H}) = \{\frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} : |\psi\rangle \in \mathcal{H}\}$, the set of equivalence classes of normalized states up to a phase, the projective Hilbert space. The group of complex phases with the group operation of multiplication is called $U(1)$. Quantum states become elements in the principal $U(1)-$bundle over $\mathcal{P}(\mathcal{H})$.*

We can imagine $\mathcal{P}(\mathcal{H})$ as a manifold with a circle "attached" at every point representing the phase of the state. The inability to define a single-valued phase as a state traverses this manifold is equivalent to the nontrivial structure of the bundle.

Definition 3. *The fiber bundle has the structure of a smooth manifold. Curves passing through $p \in E$ that* *lie entirely along the fiber $\pi^{-1(x)}$ give rise to the vertical subspace V_p of the tangent space at p. Then the ambient tangent space is decomposed as $H_pE \oplus V_pE$. However, this decomposition is not unique; although V_pE has now been uniquely specified by π, there is not enough data to uniquely specify H_pE.*

The inability to uniquely specify the horizontal subspaces is equivalent to saying that given a curve $\gamma : \mathbb{R} \to M$, there is not a unique choice of $\Gamma : \mathbb{R} \to E$ such that $\pi \circ \Gamma = \gamma$. The data we need to make this determination is known as a connection, or gauge potential:

Definition 4 (Gauge potential)**.** *A connection, or gauge potential ω is defined as a smooth one-form valued in the Lie algebra $\mathfrak{g}$ of G such that if $X \in T_pE$ such that $X = (p \cdot c)'(0)$ where $c : \mathbb{R} \to G$ is a curve in G, then $\omega(X) = c'(0) \in \mathfrak{g}$ and ω satisfies the transformation rule $(R_g^*\omega_{ug})(X) = (dL_{g^{-1}}dR_g)(\omega_u(X))$ for any $g \in G$.*

Loosely speaking, integration of ω will tell us which way to "go" along the fibers as we traverse a path through E, achieving our goal of uniquely lifting γ to Γ. The transformation rule will make this well-behaved under gauge transformations, i.e. that it is appropriately indifferent to where we lie in each fiber. Now let's make this precise. Locally, we can compare E to a trivial bundle, where the solution is straightforward. This is accomplished by a local choice of gauge: let $p \in M$ and (U, ϕ) be a chart containing p, and define the local section $s(x) = \phi^{-1}((x, e))$, which yields $\phi \circ s : \pi(U) \to U \times \{e\}$. In this way, we can pull back ω to a local differential form $A = s^*\omega$ on $\pi(U) \subseteq M$. Given a path $\gamma_M : \mathbb{R} \to \pi(U)$, we can define a path γ_G such that $\gamma_G'(t) = (\gamma_G(t)g_t)'(0)$, where we have introduced $g_t : \mathbb{R} \to G$ such that $g_t'(0) = A_{\gamma_M(t)}$. Unpacking the definitions, γ_G is defined by the differential equation $\gamma_G'(t) = dR_{\Gamma_G(t)}A_{\gamma_M(t)}$. If G is a matrix group (which will be assumed for the remainder of the review), then we can identify $dR_gX = Xg$ for $X \in \mathfrak{g}$, and we have the simpler equation $\gamma_G'(t) = A_{\gamma_M(t)}\gamma_G(t)$. The formal solution to this equation is

$$\gamma_G(t) = \mathcal{P}\exp\left(-\int_{\gamma_G(0)}^{\gamma_G(t)} A(\gamma_M'(s))ds\right) \tag{1}$$

where $\mathcal{P}$ means path-ordering. Then, we use the transformation rule to show that a path constructed this way is well-defined. The most important ingredient is the following transformation rule for relating local sections:

Proposition 1. *Let s_1 and s_2 be local sections on U such that $s_2(p) = s_1(p)\Omega(p)$, where $\Omega : M \to G$ is a local gauge transformation. Then let $A_1 = s_1^*\omega$ and $A_2 = s_2^*\omega$. Then these are related by*

$$A_2 = \Omega^{-1}A_1\Omega + \Omega^{-1}d\Omega \tag{2}$$

Proof.

$$(ds_2)_p = dR_{\Omega(p)}(ds_1)_p + s_1(p)\Omega^{-1}(p)(d\Omega)_p \tag{3}$$

The second term is a vector field Y such that $Y = (s_1(p) \cdot \Omega(p)\Omega^{-1})'$, so by the first transformation property of ω, if $X \in \mathfrak{g}$ is arbitrary then we have

$$\begin{aligned} A_2(X) &= (s_2^*\omega)(X) = \omega(ds_2(X)) &(4)\\ &= dR_{\Omega(p)}(ds_1)_p + \omega(Y) &(5)\\ &= \Omega^{-1}(p)A_1\Omega(p) + \Omega(p)(d\Omega^{-1})_p &(6) \end{aligned}$$

where assuming g is a matrix group allowed us to write $d(\Omega(p)\Omega^{-1}) = \Omega(p)d\Omega^{-1}$ and $dL_{\Omega^{-1}(p)}dR_{\Omega(p)}A_1 = \Omega^{-1}(p)A_1\Omega(p)$, and we used the second transformation property required of ω. □

This "gluing" rule for local sections can then be used to show that the path we obtain is globally well-defined, because the integral in Eq. 1 differs by a gauge transformation and a total derivative which vanishes when the path is closed. It also allows us to specify a connection ω on E by locally defining smooth $\mathfrak{g}-$valued 1-forms A in each chart, which is the way this problem is approached practically in physics. For further details, we refer to Sec. 10 in the textbook by Nakahara[10]. If C is the set of closed paths in M, the integral in Eq. 1 naturally defines a gauge-invariant mapping $C \to G$ which is called the *holonomy* of the connection ω.

3. THE BERRY PHASE

We can now give an abbreviated derivation of the Berry phase as shown in[3], and illustrate how this implements the theory developed in the previous section. Let $H(X)$ be a Hamiltonian which depends on the control parameters $X : \mathcal{S}^1 \to M$, where M is a manifold. We will write $H(t) = H(X(t))$ for short. The Schrodinger equation reads $i\,|d\psi\rangle = dtH(t)\,|\psi\rangle$. Without loss of generality, let $|\psi(0)\rangle$ be an eigenstate of $H(0)$ with $E = 0$. This gives $i\,\langle\psi|d\psi\rangle = dt\,\langle\psi|\,H(t)\,|\psi\rangle \approx 0$, where in the last step we apply the adiabatic approximation. This implies $\langle\psi|d\psi\rangle \approx 0$. Then let $|n(t)\rangle$ be an instantaneous eigenstate of $H(t)$ with eigenvalue zero which is *smoothly defined* on $X(S^1)$. We have $|n\rangle = e^{i\alpha(t)}\,|\psi\rangle$, wherein $\langle n|dn\rangle = id\alpha\,\langle\psi|\psi\rangle + \langle\psi|d\psi\rangle = id\alpha$. This shows that $e^{i\alpha} = \exp(\oint\langle n|dn\rangle)$. We recognize this an example of holonomy with the connection $\langle n|dn\rangle$, also known as the Berry connection. The principal bundle here is the $U(1)-$bundle over M. This shows how the adiabatic approximation functions to remove the Hilbert space from consideration, because now the geometry of the space is determined by M and $U(1)$, with H inducing the connection. Since $|n(X)\rangle$ is smoothly defined, it returns to itself along a closed path through M, and thus α is independent of the choice of $|n(X)\rangle$. This choice is the choice of local section from the previous discussion in disguise. We have used the shorthand $\langle n|dn\rangle = \langle n|\,\frac{\partial}{\partial X^i}\,|n\rangle\,dX^i$, where the summation is implicit. The dX^i makes this a 1-form, and it takes values in $i\mathbb{R}$, which is the Lie algebra of $U(1)$. Under the gauge transformation $|\psi\rangle \to \Omega(X)|\psi\rangle = e^{i\omega(X)}|\psi\rangle$, we find that $\langle n|dn\rangle = id\alpha - id\omega = d\alpha + \Omega d\Omega^{-1}$. This reproduces the transformation rule from the previous section.

3.1. Example: classical transport on a sphere

We follow the development due to Berry[11] to illustrate how the Berry phase arises as a result of transport along a non-trivial line bundle. Consider a vector v which is always tangent to the sphere and transported around a closed curve C. We ask that the vector not twist about $\mathbf{r}$, so $dv \cdot v' = 0$, where $v' = v \times r$. This immediately implies that $\langle \psi, d\psi\rangle = 0$, where $\psi = \frac{1}{\sqrt{2}}(v + iv')$. We notice that this is exactly the Berry connection from the previous section in an entirely classical context. In this scenario, the connection is equivalent to an earlier prescription by Levi-Civita for parallel transport, wherein a vector is transported over the sphere and projected into the tangent space at every step in the path. We notice that v, v' form an orthonormal frame for the tangent space at r, and under a rotation of this frame about $\hat{r}$ by an angle θ we have $\psi \to e^{i\theta}\psi$. If we fix some reference frame n which is smoothly defined along C such that $n = e^{i\alpha(r)}\psi$, then we have $\langle n, dn\rangle = id\alpha$, where we used the fact that $\langle \psi, d\psi\rangle = 0$, and so we find $i\alpha(C) = \oint_{\partial C}\langle n, dn\rangle$. The choice of a smooth frame along C is always possible. In the language of bundles, we can see that the orientation of the $\mathbf{v}, \mathbf{v}'$ frame defines another $U(1)$ phase with respect to the vector ψ, and so we have a $U(1)-$principal bundle over S^2. Our connection is $\langle n, dn\rangle$. The holonomy is a mapping from the closed path C to the $U(1)$ elements $e^{i\alpha(C)} = \exp\left(\oint_{C\partial}\langle n, dn\rangle\right)$, the same as in the quantum case.

4. AHARONOV-ANDAN NON-ADIABATIC PHASE

Aharonov and Anandan realized that the geometric phase also arises in non-adiabatic contexts[5]. Let $|\psi\rangle$ be a solution to the Schrodinger equation which is closed in projective space, i.e. $|\psi(\tau)\rangle\langle\psi(\tau)| = |\psi(0)\rangle\langle\psi(0)|$ and let $\alpha_d = \langle\psi| H |\psi\rangle$ be the dynamical phase. Consider a lift $|\phi\rangle$ which is periodic on $\mathcal{H}$, and define the Aharonov-Anandan lift via $|\eta\rangle = e^{i\alpha_d}|\psi\rangle$. Let $|\phi\rangle$ and $|\eta\rangle$ be related by $|\eta\rangle = e^{i\alpha_g}|\phi\rangle$. Then we observe that $\langle\eta|d\eta\rangle = \langle\psi|d\psi\rangle + id\alpha_d\langle\psi|\psi\rangle = -idt\langle\psi| H|\psi\rangle + id\alpha_d = 0$ by the definition of α_d. Thus $i\langle\phi|d\phi\rangle = id\alpha_g$, and we have rediscovered the Berry connection, in this case called the Aharonov-Anandan connection. Working back through the definition, we have $|\psi\rangle = e^{-i\alpha_d}|\eta\rangle = e^{-i\alpha_d}e^{i\alpha_g}|\phi\rangle$. Since $|\phi(\tau)\rangle = |\phi(0)\rangle = |\psi(0)\rangle$, we find $|\psi(\tau)\rangle = e^{-i\alpha_d}e^{i\alpha_g}|\psi(0)\rangle$. While this may seem like a somewhat straightforward extension of the Berry phase, the important difference is that now the base manifold is the Hilbert space itself instead of the control space. The AA connection never managed to subsume the Berry connection in the physics literature, but it is of significant interests to mathematicians, who term the AA bundle the *universal $U(1)$ principal bundle*[4] (Chapter 6.3), reflecting the utility of this connection in the classification of all $U(1)$ principal bundles. This harkens back to the work of Chern and Bott, who studied the same connection in the context of parallel transport.

To make this a more useful calculational tool, Moore and Stedman[12] showed how Floquet theory can be used to obtain $|\phi\rangle$ directly when the period of evolution coincides with the period of the Hamiltonian. If $H(t)$ has a period τ, then we may express (or closely approximate) $U(t) = \mathcal{T}\exp\left(-i\int_0^t H(t')dt\right)$ by the operator $U(t) = Z(t)e^{-iMt}$, where M is Hermitian and $Z(t)$ is $\tau-$periodic. Choosing an eigenbasis $|\psi_m(t)\rangle$ for M with eigenvalues ϵ_m, we have $U^\dagger HUdt = iU^\dagger dU = ie^{itM}Z^\dagger dZe^{-itM} - Mdt$, so the AA lift is given by the phase

$$\begin{aligned}\langle\psi_m(t)| H|\psi_m(t)\rangle\, dt &= \langle\psi_m| U^\dagger HU|\psi\rangle && (7)\\ &= i\langle\psi_m| Z^\dagger dZ|\psi_m\rangle - \epsilon_m dt && (8)\end{aligned}$$

The term on the right is simply the dynamical phase, so the geometric phase is determined by the connection $\langle\phi_m|d\phi_m\rangle$, where $|\phi_m\rangle = Z|\psi_m\rangle$. The advantage of this approach is that we have an explicit formula for the closed path in $\mathcal{P}(\mathcal{H})$ lifting $|\psi\rangle$.

5. NON-ABELIAN PHASES: THE WILCEK-ZEE PHASE

Wilcek and Zee[6] were the first to realize that Berry's phase is immediately generalized to a non-abelian gauge group. Let the Hamiltonian H depend on the control parameters X. Suppose that the zero-energy subspace (without loss of generality) has degeneracy $k \geq 2$. Let $|n_1(X)\rangle, \ldots, |n_k(X)\rangle$ be a locally defined smooth frame for this subspace. Define a curve $\gamma : S^1 \to M$, where M is the control space. Now consider a frame $\{|\psi_i\rangle\}$ which are solutions to the Schrodinger equation beginning at $t = 0$. Notice that by the adiabatic approximation, $\langle\psi_i|d\psi_j\rangle = i\langle\psi_i| H|\psi_j\rangle\, dt = \mathcal{O}(dt^2) \approx 0$. Then as $X(\gamma(t))$ is varied sufficiently slowly around the control path such that the adiabatic approximation applies, we may write $U|\psi_i\rangle = |n_i\rangle$, where U is a unitary operator. This means that our state space takes the form of an SU(k)-principal bundle over the control space M. We may find a Lie algebra element M for each X such that $dU = U(X)M(X)dX$, and so we have

$$\begin{aligned}\langle n_i|dn_j\rangle &= \langle\psi_i| U^\dagger dU|\psi_j\rangle + \langle\psi_i|d\psi_j\rangle && (9)\\ &= \langle\psi_i| M|\psi_j\rangle\, dX = M_{ij}(X)dX && (10)\end{aligned}$$

Thus $A = M(X)dX = \langle\vec{n}|d\vec{n}\rangle$ is our required Lie-algebra-valued 1-form, giving us a non-Abelian connection which generalizes the Berry connection. The holonomy for a non-abelian connection is given by $U = \mathcal{P}\exp\left(\oint_\gamma A\right)$. If we choose another local frame which is related to the original by a unitary operator V, then our new unitary becomes VU, and then by substitution we have $A = (U^\dagger)dU = V^\dagger A V + V^\dagger dV$. If M is the lie algebra element such that $dV = VdM$, then $V^\dagger dV = dM$, which is total derivative. Since U was defined with respect to the local basis, this makes the holonomy independent of the choice of local basis, as we expect.

For a practical example of a non-Abelian connection, consider a Hamiltonian H_0 with a $k-$degenerate zero-energy eigenspace and 1 excited state. Let the operator $R(\theta_1, \ldots, \theta_k)$ be defined by $R = \prod_n^{\leftarrow} e^{i\theta_n T_n}$, where T_n is the Lie algebra element that interchanges $|n\rangle$ with $|k+1\rangle$ and leaves the rest unaffected. Then let $H = RHR^\dagger$, with R depending on time through the path $\Gamma : S^1 \to S^k$. The natural choice of local frame is $|n(t)\rangle = R\,|n\rangle$, where $\{|n\rangle\}$ forms a basis for the zero eigenspace. In this basis, the connection becomes $\langle \vec{n}|d\vec{m}\rangle = R^{-1}\frac{\partial R}{\partial \theta_i} d\theta_i$, where i is implicitly summed. While a bit tedious, this expression can be evaluated directly using the Pauli-Euler identity to obtain an explicit form for the connection in terms of the angles θ_i.

Non-abelian gauge theory, as describe in generality in Sec. 2, is the language describing elementary particle physics: the standard model is a $U(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ gauge theory. At the time, the mathematics of line bundles and principle bundles was still in the process of development, most notably by Chern, and this parallel is one reason why the Berry phase quickly gained so much notoriety. The non-abelian generalization of Berry's phase has many applications. Among them, it forms the basis for the SO(5) theory of superconductivity[13] and it has been suggested as a foundation for holonomic quantum computing[14].

6. HANNAY'S ANGLE

Since the Solvay conference of 1911, introducing quantum notions such as adiabaticity into classical physics has been of central importance. In 1985, Hannay described an analogy to the geometric phase appearing in classically integrable systems[15], and then Berry showed how this notion appears as the semiclassical limit of the quantum Berry phase[16]. Understandably, this was an exciting development, because it showed that certain topological effects in quantum physics could be understood semiclassically.

6.1. Geometric phase in action-angle variables

To understand the Hannay angles, we need to develop some basic theory of the action-angle variables. Consider a general Hamiltonian $H(p,q)$ and a new Hamiltonian $K(P,Q) = 0$. By Hamilton's principle, the corresponding Lagrangians must differ by a total derivative. Using a generating function $G(q,P,t)$, we have

$$\dot{Q}P - K = \dot{q}p - H - \frac{\mathrm{d}}{\mathrm{d}t}(QP - G) \tag{11}$$

$$= \dot{q}p - H - \frac{\partial G}{\partial q}\dot{q} - \frac{\partial G}{\partial P}\dot{P} - \frac{\partial G}{\partial t} + \dot{Q}P + \dot{P}Q \tag{12}$$

Comparing both sides, this gives $K = H + \frac{\partial G}{\partial t} = 0$, $Q = \frac{\partial S}{\partial P}$ and $\frac{\partial G}{\partial q} = p$. This is called the Hamilton-Jacobi equation. Crucially, by Hamilton's equations, the new momenta P are constant, so

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial q}\dot{q} = p\dot{q} - H = \mathcal{L} \tag{13}$$

Thus $G = S$ is just the action. Since H is time-independent, we can write the solution to the HJ equation as $S(q,P,t) = W(q,P) - Et$. Here W is known as Hamilton's characteristic function. Since energy is conserved along classical trajectories, we have $S = \int p\dot{q}dt - Et$, and comparing with the expression for S above, we have $W = \int pdq$, from which it follows that $p_i = \frac{\partial W}{\partial q_i}$. Now we apply two assumptions: periodicity and separability. Separability means that we may write $W(q,P) = \sum_i W_i(q_i, P)$. We may now introduce the action variables $J_i = \oint p_i dq_i$, where the integral is taken over a classical trajectory. Since $p_i = \frac{\partial W_i}{\partial q_i}$ and we integrate out q_i, it is clear that J_i depends only on P which are constants of motion, making J constants of motion. We may then write W in terms of q and J, and interpret W as another generating function. The angle variables θ_i are defined as the conjugates to J_i, i.e. $\theta_i = \frac{\partial W}{\partial J_i}$. In particular, $Q_i = \frac{\partial S}{\partial J_i} = \frac{\partial W}{\partial J_i} - \frac{\partial E}{\partial J_i}t = \theta_i - \frac{\partial H}{\partial J_i}$. Since $\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0$, we see that $\theta_i = Q_i + \nu_i t$, where $\nu_i = \frac{\partial H}{\partial J_i}$ is the frequency. We have identified H with E, and since the J_i are constants of motion which define the classical trajectory, we may write H as a function of J_i alone. Furthermore,

$$d\theta_i = \frac{\partial \theta}{\partial q}dq = \frac{\partial}{\partial J_i}\left(\frac{\partial W}{\partial q}dq\right) = \frac{\partial}{\partial J_i}pdq \tag{14}$$

Therefore over one transversal of the trajectory, we have

$$\nu_i t = \oint d\theta_i = \frac{\partial}{\partial J_i}\left(\oint pdq\right) = \frac{\partial}{\partial J_i}(J_i) = 1 \tag{15}$$

This shows that ν_i is in fact the frequency of transversal of the classical trajectory. Now, if X is varied, then $\frac{\partial W}{\partial t} = \dot{X}\frac{\partial W}{\partial X}$. Since we defined $H = E - \frac{\partial W}{\partial t}$, then the Hamiltonian H becomes $H_0 + \dot{X}\frac{\partial W}{\partial X}$, where H_0 is the Hamiltonian corresponding to constant X. From Hamilton's equations, we obtain

$$\omega_i = \frac{\partial H_0}{\partial J_i} + \frac{\partial}{\partial J_i}\frac{\partial W_i}{\partial X}\dot{X} \tag{16}$$

Here the situation becomes more clear; the first term is the dynamical contribution which arises from constant

X, and the second term is the geometric contribution. Integrating around an entire loop, we obtain the Hannay angle:

$$\theta_i^{(H)} = \frac{\partial}{\partial J_i} \oint \frac{\partial W}{\partial X} dX \tag{17}$$

This integral is close to the desired form, but the integrand is not a total derivative. Classically, the adiabatic theorem says that the actions J_i remain constant despite the variation of X. Therefore, we can average out θ_i over one period of the motion, and the resulting average $\langle W_i \rangle$ is only a function of X. This gives our purely geometric change in the angle. However, we can achieve a simpler form using the argument due to Berry. We write $I(\theta, J, X) = W(q(\theta, J, X), I, X)$ to obtain $\frac{\partial I}{\partial X} = \frac{\partial W}{\partial X} + \frac{\partial W}{\partial q}^t \frac{\partial q}{\partial X} = \frac{\partial W}{\partial X} + p^t \frac{\partial q}{\partial X}$. Substituting this into the above expression, we then average out θ to obtain

$$\theta_i^{(H)} = \frac{\partial}{\partial J_i} \oint \left(\frac{\partial \langle I \rangle_\theta}{\partial X} - \langle p^t \frac{\partial q}{\partial X} \rangle_\theta \right) dX = -\frac{\partial}{\partial J_i} \oint \langle p^t dq \rangle_\theta \tag{18}$$

6.2. Example: Bilinear Hamiltonians

This discussion is fairly abstract, so we will follow Kariyado et al.[15] and specialize to the case of a bilinear Hamiltonian taking the form $H = \frac{1}{2}p^2 + \frac{1}{2}q^t M q$, where p and q are both vectors of the conjugate position and momentum coordinates and M is symmetric and positive-semidefinite. Let $m = \sqrt{M}$, and consider the function $F(Q, q) = -\frac{i}{2}(q - a_+ Q)^t m (q - a_- Q)$. Using again the canonical transformation $P\dot{Q} - K = p\dot{q} - H - \frac{\mathrm{d}}{\mathrm{d}t}F$, we see that $P = \frac{\partial F}{\partial Q} = -imq + i\sqrt{2}mQ$ and $p = -\frac{\partial F}{\partial q} = -i\sqrt{2}mq + imQ$, where $a_\pm = \sqrt{2} \pm 1$. Solving for p and q in terms of P and Q, the Hamiltonian becomes $K = iP^t m Q$. Then Hamilton's equations yield $\dot{Q} = imQ$ and $\dot{P} = -imP$, which are both Schrodinger equations for P and Q. We then proceed to introduce normal coordinates γ, η which diagonalize m, and a matrix U such that $P = U\eta$ and $Q = U\gamma$. We suggestively refer to the diagonal elements of m as $2\pi\nu_i$. These coordinates then have the equations of motion $\gamma_i = \gamma_i(0)e^{i\omega_i t}, \eta_i = \eta_i(0)e^{-i\omega_i t}$, whence we find $J_i = \oint p_i dq_i = 2\pi i \nu_i \gamma_i(0)\eta_i(0)$. The characteristic function is similarly evaluated to $2\pi i \sum_i \nu_i \gamma_i(0)\eta_i(0)t = 2\pi i \nu^t J t$, so the coordinate conjugate to J_i is $\frac{\partial W}{\partial J_i} = \nu_i t$. The Hannay angle can be explicitly evaluated

$$\theta_i^{(H)} = -\frac{\partial}{\partial J_i} \oint \oint \frac{dt}{2\pi} p^t dq \tag{19}$$

$$= \frac{\partial}{\partial \gamma_i(0)\eta_i(0)} \oint \oint \frac{d\theta}{2\pi} (\eta^\dagger(0) U^\dagger dU \gamma(0)) \tag{20}$$

$$= i \oint (U^\dagger dU)_{ii} \tag{21}$$

Now we step back to appreciate the meaning of this expression; we have allowed the normal mode matrix M to depend on X. The matrix O which diagonalizes m is then responsible for the connection $A_i = i(U^\dagger dU)_{ii}$, which is nearly identical to the quantum case, where the matrix V that related us back to a smoothly defined local frame induced the connection.

7. CONCLUSION

This completes our absurdly brief discussion of classical transport and the role of holonomy in classical and quantum contexts. Given the high ambitions for this project, there was much material left beyond the scope of this review. In particular, this project came out of a fascination with the fact that Stoke's theorem quantizes the first Chern number, and indirectly provides information about the representations of the control manifold. This can be seen by writing $d\langle n|dn\rangle = \langle dn| \wedge |dn\rangle$, which is known as the Berry curvature. If the parameter space is a compact manifold M, then there is an ambiguity in choosing the interior and exterior of the curve, which is what leads to this quantization. A direct implication is the Dirac charge quantization and a restriction on the allowed representations of SU(2). This relationship is explored in generality by Mostafazadeh[17], who establishes a remarkable relationship between the Berry phase and the Borel-Weil-Bott theorem, which constructs all the finite dimensional irreps of a semisimple compact Lie group from the irreps of their maximal tori. This is a topic for future reading. This project was also inspired by an excellent talk given at Yale University by Prof. Jack Harris and Prof. Nicholas Read on a highly accurate characterization of spectral flow and eigenvalue braiding in dissipative oscillators[18]. The discussion in this paper does not directly connect the spectral flow to a generalized Berry phase by specifying a principal bundle or a connection, and this seems to be an interesting topic for further exploration. Lastly, higher Berry phase resulting from the gauge-invariance in matrix product states has been explored by Ohyama et al.[19], which is the project topic of my colleague Ben Bobell; hopefully this review lays a solid foundation for understanding his exploration of the higher Berry phase.

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