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Homological algebra final paper
December 8, 2025

Topological notion of group cohomology

In the course of our introduction to Homological Algebra, we have been presented the notions of homology and cohomology from a purely algebraic point of view. This view is slightly ahistorical; homology arose largely from the work of Riemann, Betti, and Poincaré in the latter half of the 19^{th} century. These topologists developed the tools of homology—and notions such as the genus and Betti numbers—to assign homotopic invariants to topological spaces. In 1925, Emmy Noether shifted the focus to developing algebraic techniques for computing homology groups, and in 1936, Hurewicz showed that the homotopy type of an aspherical space is determined by its fundamental group, which led to the homology and cohomology of these spaces being considered as properties of the fundamental group itself. The major revolution which established the familiar tools of homological algebra came from Cartan and Eilenberg in 1956, systematically using projective and injective resolutions of modules to unify different theories of homology [5].

In this short review, we will go backward in time; starting from our modern perspective of modules, complexes, and resolutions, we will develop the basic notions of topology that allow us to understand the connection between the topology of certain spaces and the algebraic properties of their fundamental group.

1 The combinatorics of topology

In this section we define an abstract simplicial set, cell complexes, and singular and cellular (co)homology. We show that a simplicial set admits a geometrical realization as a topological space, which connects the combinatorial notion of the (co)homology of a simplicial set to the geometric properties of a topological space.

Definition 1.1. *Let S be a collection of sets. S is called a **simplicial complex** if for every set $X \in S$ and every subset $Y \subset X$ we have $Y \in S$.*

Definition 1.2. *Let Δ be the category with objects being finite totally ordered sets and morphisms being order-preseving maps. Every object is isomorphic to an object labeled by $[n] = \{0, 1, 2, \dots n\}$ and the morphisms are products of the elementary **face** and **degeneracy** maps $\partial_{n,i} : [n-1] \to [n]$ and $\delta^{n,i} : [n+1] \to [n]$ defined by*

$$\partial_{n,i}(k) = \begin{cases} k & k \le i \\ k+1 & k > i \end{cases} \qquad \delta^{n,i}(k) = \begin{cases} k & k < i \\ k-1 & k \ge i \end{cases} \tag{1}$$

*Δ is called the **simplex category**. A **simplicial set** is a contravariant functor $X : \Delta \to$ **Set**. Objects isomorphic to $X([n])$ are called n-**simplices of** X.*

Definition 1.3 (Standard geometrical simplex)**.** *The standard geometrical n-simplex, Δ_n is the convex hull (or set of convex combinations) of the first $n+1$ standard basis vectors $e_0, \dots, e_n$, denoted $[e_0, \dots, e_n]$.*

Definition 1.4 (CW complex)**.** *A CW complex X is defined inductively as follows. Let X^0 be a set of vertices with the discrete topology. Given X^{k-1} constructed, let $\{e^k_\alpha\}_\alpha$ be a set of topological spaces homeomorphic to the closed unit disk D^k. For each α, let $g^k_\alpha : \partial e^k_\alpha \to X^{k-1}$ be a continuous map. Then define $X^k \equiv X^{k-1} \sqcup \{e^k_\alpha\}_\alpha / \sim$, where the equivalence relation identifies each $x \in \partial e^k_\alpha$ with $g^k_\alpha(x) \in X^{k-1}$.*

Definition 1.5 (Geometrical realization)**.** *Given a simplicial set $X : \Delta \to$ **Set**, associate an $n-$cell Δ^n to each object isomorphic to $X([n])$, and define the gluing maps as determined by the degeneracy maps. The resulting CW complex is called the geometrical realization of X.*

Definition 1.6 (Simplicial complex)**.** *Let S be a collection of sets. S is called a* ***simplicial complex*** *if for every set $X \in S$ and every subset $Y \subset X$ we have $Y \in S$. Sets $X \in S$ containing $n+1$ elements are called n simplices. The sets $X \in S$ containing only one element are called vertices. By definition every simplex is the union of vertices.*

Definition 1.7 (Simplicial homology)**.** *Given a simplicial complex X we can construct a simplicial chain complex $C^{simp}_\bullet(X)$ where $C^{simp}_n(X)$ is defined as the free $\mathbb{Z}$ module over oriented $n-$simplices of X. Let $\sigma = \{v_0 \dots v_n\} \in S$ Be an ordered $n-$simplex. By orientation we mean $\sigma' = -\sigma$ if σ and σ' differ by an odd permutation. Differentials $\partial_n : C^{simp}_n(X) \to C^{simp}_{n-1}(X)$ are defined on generators σ as follows via degeneracy maps.*

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i d_{n-1,i}(\sigma)$$

One can straight-forwardly calculate that $\partial^2 = 0$ producing a valid chain complex. If $|X|$ is the geometric realization of X then we also say $C^{simp}_\bullet(X)$ is the simplicial chain complex for $|X|$. For the rest of the paper we will not distinguish between simplicial sets X and their geometric realizations $|X|$.

The geometrical realization which is most important for our discussion is the following:

Definition 1.8. *Let G be a discrete group, and let EG be the geometrical realization of the poset of G. Let G act on EG by $g[g_0, \dots, g_n] = [gg_0, \dots, gg_n]$. Define $BG \equiv EG/G$.*

We will see that for sufficiently nice spaces, topological data is encoded entirely through the combinatorial data of the cell complex.

2 Computing cohomology of topological spaces

The rigid structure of a cell complex or a simplicial complex is useful for explicit computations, but the more general properties of the homology of topological spaces are encapsulated by the singular homology, which we will now define.

Definition 2.1 (Singular chain complex)**.** *Given a topological space X, denote by $C_k(X)$ the free Abelian group on the set of continuous maps $\sigma : \Delta^n \to X$.*

Definition 2.2 (Boundary maps)**.** *Consider the face maps $\partial_{n,i}$ from Def. 1.2. Define the singular boundary maps $\partial : C_k(X) \to C_{k-1}(X)$ via*

$$\partial(\sigma) = \sum_{i=0}^{n} \sigma \circ \partial_{n,i} \tag{2}$$

We also note that $\partial^2 = 0$, so ∂ makes $C_k(X)$ into a chain complex.

Definition 2.3 (Singular (co)homology)**.** *Define the singular homology to be the homology of the singular chain complex. Given an Abelian group M, define the singular cohomology to be the homology of the cochain complex* $\mathrm{Hom}(C_\bullet(X), M)$*.*

Now we will connect this general notion of the homology of a topological space to the more concrete computation of cellular homology. Let X be a CW complex, and let X^k denote the k-skeleton.

Definition 2.4. *Given a map $f : S^n \to S^n$, define the degree of f to be*

$$f_* : H_n(S^n) \mapsto \deg(f) H_n(S^{n-1}) \tag{3}$$

Definition 2.5. *Let $H_n(X_n, X_{n-1})$ be the free abelian group generated by the n-cells in X. The cellular chain complex is defined*

$$\cdots \to H_{n+1}(X_{n+1}, X_n) \to H_n(X_n, X_{n-1}) \to \dots H(X_1, \emptyset) \to 0 \tag{4}$$

To define the boundary maps, let e^n_α be an n-cell, let $g^n_\alpha : \partial e^n_\alpha \to X^{n-1}$ be the attaching map, and let $\Phi^n_\alpha : e^n_\alpha \to S^n$ be the characteristic map. Define $\chi_{\alpha\beta}$ as the composition

$$S^{n-1} \xrightarrow{g^n_\alpha} X^{n-1} \longrightarrow X^{n-1}/(X^{n-1} - e^{n-1}_\beta) \xrightarrow{(\Phi^{n-1}_\beta)^{-1}} S^{n-1}$$

i.e. collapsing all of X^{n-1} except for e^{n-1}_β to a point and then identifying e^{n-1}_β with S^{n-1} via the characteristic map. Then the cellular boundary maps are defined

$$\partial = \sum_{\beta \in X^{n-1}} d(\chi_{\alpha\beta}) e^{n-1}_\beta \tag{5}$$

Proposition 2.6. *$H_n(X_n, X_{n-1})$ is the relative singular homology, and the cellular homology of a CW complex X is isomorphic to the singular homology. The proof may be found in standard references such as [3].*

Thus we have a simpler way of computing homology for a CW complex with the same functorial properties. In view of the correspondence that we are demonstrating between the cohomology of topological spaces and that of groups, here are some important tools and results for computing singular cohomology:

Proposition 2.7 (Induced chain map)**.** *Let $f : X \to Y$ be a map between topological spaces. Then f descends to a chain map $f_* C_\bullet(X) \to C_\bullet(Y)$ on the singular chains.*

Proposition 2.8. *Given topological spaces X, Y, maps $f, g : X \to Y$ and a homotopy $H_t : f \implies g$, H induces a chain homotopy on the singular chains.*

Proof. Define $p_i : \Delta^{n+1} \to \Delta^n \times I$ via

$$(t_0, \dots, t_n) \mapsto (t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1}, t_{i+1} + \cdots + t_{n+1}) \tag{6}$$

Then given a simplex σ in X, define $P_i(\sigma) = (\sigma \times \mathrm{Id}) \circ p_i$. Lastly, define $j_k : X \to X \times I$ via $j_k : x \mapsto (x, k)$.

We may verify that $\partial_0 P_0 = (j_1)_*$, $\partial_{n+1} P_n = (j_0)_*$, and $\partial_j P_i = P_i \partial_{j-1}$ if $i \leq j-1$ and $i \leq n$ and $\partial_j P_i = P_{i-1}\partial_j$ if $j+1 \leq i \leq n$ through direct computation. In particular, defining the prism

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i P_i(\sigma) \tag{7}$$

we can show that P is a chain homotopy between the induced maps $(j_1)_*$ and $(j_0)_*$. In particular, H induces a chain map $H_* : C_k(X \times I) \to C_k(Y)$. Thus

$$\partial(H_*P) + (H_*P)\partial = H_*(\partial P + P\partial) = H_*[(j_1)_* - (j_0)_*] = (Hj_1)_* - (Hj_0)_* = g_* - f_* \tag{8}$$

□

Corollary 2.9. *If X is a contractible topological space then it has trivial cellular homology.*

Proof. Since cellular homology is homotopy invariant, then any contractible topological space has the homology of a point. The point contains only one $0-$cell as a CW complex, so has cellular chain complex

$$0 \ldots \longrightarrow 0 \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} 0$$

Which gives us homology $H_\bullet(X) = \mathbb{Z}$ in degree 0 and 0 otherwise. □

Proposition 2.10 (Excision)**.** *If X is a topological space and $B \subset A \subset X$ such that $\overline{B} \subseteq A^o$, Then $H_n(X - B, A - B) \cong H_n(X, A)$.*

Definition 2.11 (Good pairs)**.** *If $A \subset X$ is closed with an open neighborhood $U \supset A$ such that U deformation retracts onto A then (X, A) is called a good pair.*

Proposition 2.12. *If X is a CW complex and $Y \subset X$ is a subcomplex, then (X, Y) is a good pair.*

Proposition 2.13. *If (X, A) is a good pair, then $H(X, A) \cong H(X/A)$.*

Proposition 2.14 (Mayer-Vietoris Sequence)**.** *Given $X = A \cup B$, let $C_n(A + B) \subseteq C_n(X)$ be the subgroup consisting of sums of chains in A and B. Then the following sequence is exact:*

$$0 \to C_n(A \cap B) \to C_n(A) \oplus C_n(B) \to C_n(A + B) \to 0 \tag{9}$$

The long exact sequence induced in the (co)homology is called a Mayer-Vietoris sequence.

Proposition 2.15 (Universal coefficients)**.** *If C is a chain complex, then there is a split exact sequence*

$$0 \to \mathrm{Ext}^1(H_{n-1}(C), G) \to H^n(C; G) \to \mathrm{Hom}(H_n(C), G) \to 0 \tag{10}$$

3 Topological spaces with group actions

As we will show in Section 4, the fundamental group of a topological space X naturally acts on its universal cover. This group action then descends to an action on the cellular chains, which is what provides the algebraic connection to G-modules. In this section, we will define the notion of a group action on a CW complex and its properties.

Definition 3.1 (G-complex). *Let G be a group, a **G-complex** is a CW complex X equipped with a G action that permutes cells. That is, for each $g \in G$ we have that $g : X \to X$ is homeomorphism and if $\sigma \subset X$ is an $n-$cell then $g(\sigma)$ is another $n-$cell. We say X is a **free G-complex** if G acts freely on the cells of X. That is if σ is an n-cell then $g(\sigma) = \sigma$ only if $g = e$.*

Since the action of G on X maps $n-$cells to $n-$cells, there is an induced action of G on $C_n(X)$ making $C_n(X)$ into a G-module. If X is a free G-complex then $C_n(X)$ has a basis which is freely permuted by G.

Lemma 3.2. *Let M be a free $\mathbb{Z}$-Module equipped with a free G action. Then M has the structure of a free $\mathbb{Z}G$-module.*

Proof. Let $M_i \subset M$ denote the i^{th} orbit of G on the $\mathbb{Z}$ basis for M. For each M_i select a representative $m_i \in M_i$. We claim the set $\{m_i\}$ is a $\mathbb{Z}G$ basis for M. To see that this is a generating set, note that $\{M_i\}$ is a partition of the $\mathbb{Z}$ basis for M and each $m \in M_i$ can be expressed as $gm_i = m$ by construction. We find $\{m_i\}$ spans the $\mathbb{Z}$ basis for M is therefore a $\mathbb{Z}G$ spanning set. Let $\sum z_i g_i m_i = 0$. Each $g_i m_i$ is a $\mathbb{Z}$ basis element of M, so $\sum z_i m_i g_i = 0$ can only be true if $g_i m_i = g_j m_j$, but this can't be true since m_i and m_j are in different G orbits. We find this spanning set is linearly independent. □

Corollary 3.3. *Let X be a $G-$complex, then $C_\bullet(X)$ is a free $\mathbb{Z}G$-module.*

Definition 3.4. *The **canoncial augmentation** map $\epsilon : C_0(X) \to \mathbb{Z}$ map for a $G-$complex X is defined on basis elements $v \in C_0(X)$ by $\epsilon(v) = 1$. The augemented chain complex is defined as*

$$\ldots \longrightarrow C_n(X) \xrightarrow{\partial} C_{n-1}(X) \xrightarrow{\partial} \ldots \longrightarrow C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

Although we have defined $G-$complexes to be CW spaces, it often more practical to consider a topological realization of G through a simplicial complex.

Definition 3.5 (Ordered Chain Complex). *Let G be a group and consider ordered $n+1$ tuples $(g_0, g_1 \ldots g_n)$ of elements of G. Let such tuples be $n-$ simplices with a G action given by*

$$g \cdot (g_0, g_1 \ldots g_{n+1}) = (gg_0, gg_1 \ldots gg_n)$$

The ordered chain complex $C'_n(G)$ is the simplex complex of simplices of G. Notice that the left action of G gives $C'_n(G)$ the structure of a free $\mathbb{Z}G$ module over $n-$simplices

Remark 3.6. *By construction the ordered chain complex of G computes the simplicial chains of the geometric realization the simplices of G considered as a simplicial set. In fact for any simplicial complex X on which G acts simplicially, the simplicial chain complex $C_\bullet^{simp}(X)$ acquires the structure of a complex of free $\mathbb{Z}G$ modules.*

Proposition 3.7 (4.1 in [2]). *Let X be a contractible free G-complex. Then the agumented cellular chain complex of X is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

Proof. By Cor. 2, if X is contractible as a topological space, then X has the homotopy type of a point. Therefore the cohomology of X is that of a point, i.e. $H^k(X) = 0$ if $k \geq 0$ and $H^0(X) = \mathbb{Z}$. This implies that the following sequence is exact:

$$\ldots \longrightarrow C_n(X) \xrightarrow{\partial} C_{n-1}(X) \xrightarrow{\partial} \ldots \longrightarrow C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

Combining this with Cor. 3.3 which says that $C_k(X)$ is a free G-module, this makes the agumented cellcular chain complex a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. $\square$

4 Elementary notions from topology

In the next section, we define precisely the manner in which the fundamental group of a topological space acts on its open cover, and introduce tools that will be useful later on for constructing classifying spaces.

Definition 4.1 (Covering). *A covering space of a space X is a space Y and a map $p : Y \to X$ such that there is an open cover $\{U_\alpha\}$ of X with $p^{-1}(U_\alpha)$ a union of disjoint open sets each homeomorphic to U_α. Such a neighborhood U_α is called balanced.*

Definition 4.2 (Isomorphism of covering spaces). *If $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ are two covering spaces, then an isomorphism $\phi : Y_1 \to Y_2$ is a homeomorphism such that $p_1 = p_2\phi$.*

Definition 4.3 (Universal covering). *A simply connected covering space of a path connected, locally path-connected space X is called the universal cover.*

Definition 4.4 (Lift). *Given a covering space $p : \widetilde{X} \to X$ and a map $f : Y \to X$, a lift of f is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $p\widetilde{f} = f$.*

Proposition 4.5 (Lifting property). *Given a covering space $p : \widetilde{X} \to X$, a homotopy $f_t : Y \to X$, and a lift $\widetilde{f_0}$ of f_0, there exists a unique homotopy $\widetilde{f_t} : \widetilde{X} \to Y$ lifting f_t.*

Proof. Fix a point $y \in Y$. Since f is continuous and I is compact, we can find a neighborhood $N \ni y$ and a partition $0 = t_1 < \cdots < t_N = 1$ such that $f([t_i, t_{i+1}] \times N) \subset U_i$, where U_i is a balanced neighborhood. We assume inductively that $\widetilde{F}$ has been constructed on $N \times [0, t_i]$, and we further take N to be small enough such that $\widetilde{F}(N \times \{t_i\}) \subset \widetilde{U}_i$, where $\widetilde{U}_i$ projects homeomorphically onto U_i. Then put $\widetilde{F}(N \times [t_i, t_{i+1}]) \equiv p^{-1}(F(N \times [t_i, t_{i+1}]))$. This defines $\widetilde{F}$ on $y \ni N \times I$ for each y.

To prove uniqueness, choose a partition $0 = t_1 < \cdots < t_M = 1$ such that $\widetilde{F}(N \times [t_i, t_{i+1}])$ and $\widetilde{F}'(N \times [t_i, t_{i+1}])$ both lie in $\widetilde{U}_i$. Assume inductively that $\widetilde{F}(N \times \{t_i\}) = \widetilde{F}'(N \times \{t_i\})$. Since p is a homeomorphism on $\widetilde{U}_i$ and $p\widetilde{F} = p\widetilde{F}'$, this shows that $\widetilde{F} = \widetilde{F}'$ everywhere. $\square$

Corollary 4.6 (Path lifting property). *Thinking of a path as a homotopy of constant maps, there is a unique lift of a path $\gamma : I \to X$ for each $\widetilde{x}_0 \in p^{-1}(x_0)$.*

Proposition 4.7 (Lifting criterion). *Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering space. Suppose (Y, y_0) is a path-connected and locally path-connected space, and let $f : (Y, y_0) \to (X, x_0)$. Then there exists a lift $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ iff $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.*

Proof. If $\widetilde{f}$ is a lift of f, then $f_* = p_*\widetilde{f}_*$, so $\operatorname{im} f_* \subseteq \operatorname{im} p_*$. For the other direction, let $y \in Y$ and let γ be a path in Y from y_0 to y. Then by Prop. 4.5, there is a unique lift of $f\gamma$, $\widetilde{f\gamma}$. Define the lift $\widetilde{f}$ by $\widetilde{f}(y) \equiv \widetilde{f\gamma}(1)$. This map is well-defined; if γ' is a path from y_0 to y, then $f\gamma' * \overline{f\gamma} = h_0$ is a loop based at x_0 with $[h_0] \in \operatorname{im} f_*$. Since $\operatorname{im} f_* \subseteq \operatorname{im} p_*$, this means that h_0 is homotopic to a loop h_1 which lifts to a loop $\widetilde{h}_1$ in $\widetilde{X}$. By the homotopy lifting property 4.5, $\widetilde{h}_1$ is homotopic to a loop $\widetilde{h}_0$ in $\widetilde{X}_0$ which lifts h_0, and therefore $\widetilde{h}_0$ is in fact a loop in $\widetilde{X}$ (which is not guaranteed). Since this lift is unique, $\widetilde{h}_0 = \widetilde{f\gamma'} * \widetilde{\overline{f\gamma}}$, which shows that $\widetilde{f\gamma'}(1) = \widetilde{f\gamma}(1)$. □

Proposition 4.8 (Classifying covering spaces)**.** *Suppose X is path-connected and locally path-connected. Then two covering spaces $p_1 : (\widetilde{X}, \widetilde{x}_1) \to (X, x_0)$, $p_2 : (\widetilde{X}, \widetilde{x}_2) \to (X, x_2)$ are isomorphic if* $\operatorname{im} p_{1*} = \operatorname{im} p_{2*}$.

Proof. By Prop. 4.7, we may lift p_1 to $\widetilde{p}_1$ such that $p_2\widetilde{p}_1 = p_1$. By the same token, we also have a lift $\widetilde{p}_2$ such that $p_1\widetilde{p}_2 = p_2$. Then $\widetilde{p}_1\widetilde{p}_2$ lifts the identity on $\widetilde{X}_2$ and $\widetilde{p}_2\widetilde{p}_1$ lifts the identity on X_1, so by the uniqueness of lifts, both are the identity map, making $\widetilde{p}_1, \widetilde{p}_2$ mutually inverse isomorphisms. □

Corollary 4.9. *A universal cover is unique up to isomorphism.*

Definition 4.10 (Deck transformations)**.** *Given a covering space Y of X, the set of isomorphisms $Y \to Y$ are called deck transformations, and they form a group $G(Y)$.*

Proposition 4.11 (Uniqueness of lifts)**.** *Given a covering space $p : \widetilde{X} \to X$ and a map $f : Y \to X$ with two lifts $\widetilde{f}_1, \widetilde{f}_2 : Y \to \widetilde{X}$ which agree at one point of Y, then if Y is connected, these two lifts must agree on Y.*

Proof. If $\widetilde{f}_1(y) = \widetilde{f}_2(y)$, then $\widetilde{f}_1$ and $\widetilde{f}_2$ agree on a balanced neighborhood of Y. Similarly, if $\widetilde{f}_1(y) \neq \widetilde{f}_2(y)$, then there is a balanced neighborhood of y on which the two lifts disagree. Therefore the set $\{y : \widetilde{f}_1(y) = \widetilde{f}_2(y)\}$ is both open and closed in Y, and if Y is connected, then it equals all of y. □

Definition 4.12 (Covering space action)**.** *An action of a group G on a topological space Y is a homomorphism $\rho : G \to$ Homeo(Y). If each $y \in Y$ has a neighborhood U such that $gU \cap hU = \emptyset$ for each $g \neq h$, then the action is called a covering space action.*

Proposition 4.13. *If Y is path connected and the action of G on Y is a covering space action, then $p : Y \to Y/G$ is a covering space and G is the group of deck transformations of this covering space.*

Proof. Since the action of G is a covering space action, every point Y has a balanced neighborhood $U \ni y$, i.e. gU and hU are disjoint for every $h \neq g$. By definition, an action is a map $G \to$ Homeo(Y), so U is homeomorphic to gU. This shows that p is a covering space. Clearly $G \subseteq G(Y)$. If $\phi \in G(Y)$ is arbitrary, then $p\phi = p$, which implies that $\phi(y)$ and y are in the same G-orbit. Thus, $\phi(y) = g(y)$, and by Prop. 4.11, $\phi = g$ everywhere. □

5 Eilenberg-Maclane complexes

With all of the required machinery laid out, we are in a position to introduce the main object of study: $K(G, 1)$ complexes. We will show that these are the appropriate generalization of the geometrical realization of of BG as constructed previously. We will then illustrate some properties of these spaces, in particular the uniqueness of their homotopy type.

Definition 5.1 (Eilenberg-MacLane complex)**.** *Let X be a connected CW complex with a contractible universal cover such that $\pi_1 X = G$. Then X is called an Eilenberg-MacLane complex of type $(G, 1)$, denoted $K(G, 1)$.*

Proposition 5.2. *If $\widetilde{X}$ is the universal cover of X, then $\pi_1(X) \cong G(\widetilde{X})$.*

Proof. Let $\gamma \in \pi_1(X, x_0)$. Then γ lifts to a path $\widetilde{\gamma}$ between $\widetilde{\gamma}(0) = \widetilde{x}_0$ and $\widetilde{\gamma}(1) = \widetilde{x}_1$. Let $p_1 : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ and $p_2 : (\widetilde{X}, \widetilde{x}_1) \to (X, x_0)$ be covering maps. Since $(p_1)_* \pi_1(\widetilde{X}, \widetilde{x}_0) = (p_2)_* \pi_1(\widetilde{X}, \widetilde{x}_1) = \{1\}$, by Prop. 4.8, then there is a unique isomorphism $(\widetilde{X}, \widetilde{x}_0) \cong (\widetilde{X}, \widetilde{x}_1)$ sending $\widetilde{x}_0 \mapsto \widetilde{x}_1$. Furthermore, since any $\widetilde{x}_0, \widetilde{x}_1$ lifting x_0 are connected by a path, all deck transformations arise this way. Lastly, it is clear that this is a morphism of groups. □

Corollary 5.3. *If X is a $K(G, 1)$-complex then the universal cover $\widetilde{X}$ is a free G-complex.*

Proof. Consider the CW structure on X, which consists of cells e^k_α, homeomorphic characteristic maps $\Phi^k_\alpha : D^k \to e^k_\alpha$, and continuous gluing maps $g^k_\alpha : \partial e^k_\alpha$. To obtain a CW structure on $\widetilde{X}$, we note that D^k is connected and simply connected, so it satisfies the unique lifting criterion (Prop. 4.7 and Prop. 4.11). Thus, for each $x \in X$, we have a set of preimages $\widetilde{x}_1, \cdots \in p^{-1}(x)$. By the unique lifting criterion, each preimage $\widetilde{x}_i$ induces unique lifts $\widetilde{g}^k_\alpha$ and $\widetilde{\Phi}^k_\alpha$, where $\widetilde{g}^k_\alpha$ is continuous and $\widetilde{\Phi}^k_\alpha$ is a homeomorphism, which is seen by lifting the inverse map. Since these lifts must be disjoint for a given cell e^k_α, this specifies the CW structure on $\widetilde{X}$. Furthermore, consider a deck transformation ϕ and the following commuting diagram:

$$\begin{tikzcd}
D^k \arrow[r, "\widetilde{\Phi}^k_\alpha"] \arrow[rr, bend left, "\widetilde{\Phi}'"] \arrow[dr, bend right, "\Phi^k_\alpha"'] & \widetilde{X} \arrow[r, "\phi"] \arrow[d, "p"] & \widetilde{X} \arrow[dl, "p"] \\
& X &
\end{tikzcd}$$

We can see that $\widetilde{\Phi}'$ and $\phi\widetilde{\Phi}^k_\alpha$ are both lifts of Φ^k_α, so by uniqueness, ϕ permutes the cells in each preimage. Since ϕ is a homeomorphism, it clearly acts freely on the cells. □

Corollary 5.4. *Let X be a $K(G, 1)$ with universal cover $\widetilde{X}$ and M a trivial G-module. Then*

$$H^i(X; M) \cong H^i(G; M) \tag{11}$$

where on the left we have the cohomology of a cell complex and on the right we have the group cohomology of G with coefficients in M.

Proof. First, we recall the definition of cellular cohomology;

$$H^i(X; M) \equiv H^i \operatorname{Hom}_{\mathbb{Z}}(C_\bullet(X), M) \tag{12}$$

From Def. 2.5, $C_k(X)$ is the free Abelian group generated by the k-cells in X. Since the action of $G \cong \pi_1(X) \cong G(\widetilde{X})$ on the cells of $\widetilde{X}$ is free and cellular, this means that $C_k(X) \cong C_k(\widetilde{X})/G$. Since M is a trivial G-module, for each $\phi \in \operatorname{Hom}_{\mathbb{Z}G}(C(\widetilde{X}), M)$ we have $g\phi(\sigma) = \phi(g\sigma) = \phi(\sigma)$, therefore ϕ is constant on $G-$orbits. Thus,

$$\operatorname{Hom}_{\mathbb{Z}}(C_\bullet(X), M) \cong \operatorname{Hom}_{\mathbb{Z}G}(C_\bullet(\widetilde{X})/G, M) \cong \operatorname{Hom}_{\mathbb{Z}G}(C_\bullet(\widetilde{X}), M) \tag{13}$$

where these are isomorphisms of chain complexes. Then Prop. 3.7 tells us that $\cdots \to C_1(\widetilde{X}) \to C_0(\widetilde{X}) \to \mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, and therefore

$$H^i H_{\mathbb{Z}G}(C_\bullet(\widetilde{X}), M) \cong \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, M) \equiv H^i(G; M) \tag{14}$$

This completes the proof. □

Proposition 5.5. *EG is $G-$complex.*

Proof. The action of G permutes the cells of EG by construction. Since g acts on $x = \sum_i t_i g_i \in EG$ by $gx = \sum_i t_i g g_i$, the action of G on EG is continuous. By the same token, the inverse g^{-1} is continuous, so G acts by homeomorphisms. □

Proposition 5.6. *BG is a $K(G, 1)$.*

Proof. First, we will show that EG is contractible. As a topological space, the n-simplex $[g_0, \ldots, g_n]$ is the set of convex linear combinations $\sum_i t_i[g_i]$ where $\sum_i t_i = 1$. Consider the homotopy $H_t(\sum_i t_i[g_i]) = t[e] + (1-t)\sum_i t_i[g_i]$. Since $t[e] + (1-t)\sum_i t_i[g_i] \in [e, g_0, \ldots, g_n]$ which by definition is included in EG, this exhibits a continuous map $I \times EG \to EG$ such that $H_0 = \mathrm{Id}$ and $H_1 : EG \to [e]$.

To show that the action of G on BG is a covering space action, given any point $x \in BG$, take a barycentric subdivision until $x \in U$ such that U contains the barycenter of some simplex and no other. Then gU is the barycenter of a different simplex and contains the barycenter of no other simplex, thus U is the desired balanced neighborhood. Since EG is constructed as the quotient by this action, $EG \to BG$ is a covering space. Since EG is contractible, it is simply connected. By Cor. 4.9, it is the universal cover of BG. By Prop. 4.13, $\pi_1(BG) \cong G$, which shows that BG is a $K(G, 1)$. □

Definition 5.7. *Every $n-$simplex of EG can be written uniquely in the form*

$$[g_0, g_0g_1, g_0g_1g_2, \ldots, g_0g_1 \ldots g_n] = g_0[e, g_1, g_1g_2, \ldots, g_1 \ldots g_n] \tag{15}$$

Therefore BG has a CW-complex structure, and the cells of BG can be written as

$$[g_1| \ldots |g_n] \equiv G[e, g_1, g_1g_2, \ldots, g_1 \ldots g_n] \ . \tag{16}$$

The construction above produces very large spaces. One reason for using this construction is its functoriality:

Lemma 5.8. *$B : G \to BG$ is a functor from* **Grp** *to* **Top**.

There are different ways of building $K(G, 1)$ complexes that produce spaces which are easier to work with. A good example is the following:

Definition 5.9. *Let G have the presentation $\langle g_\alpha \mid r_\beta \rangle$. Then construct the wedge sum of circles $\bigvee_\alpha S_\alpha$. The fundamental group of this wedge sum is the free group on the generators g_α. For each relation $g_{i_1}^{o_1} \ldots g_{i_n}^{o_n}$ generating r_β, attach a 2-cell to the edges $i_1, \ldots, i_n$, with orientation determined by o_i. Then for every nontrivial element of $\pi_2(X)$, attach a 3-cell to make it trivial, and lastly, take the limit of this process.*

While the construction above can produce nicer spaces, these are often still very large. For instance, we have the following lemma:

Lemma 5.10. *If G has torsion elements, then any $K(G,1)$ complex is infinite-dimensional.*

Proof. Let X be a $K(G,1)$ complex and $g \in G$ with $g^n = 1$. Since the action of G on the universal cover $\widetilde{X}$ is free, $\widetilde{X}/\langle g\rangle$ has the structure of a cell complex and $\widetilde{X} \to \widetilde{X}/\langle g\rangle$ is a covering map, making $\widetilde{X}/\langle g\rangle$ a $K(\langle g\rangle, 1)$. Therefore we can reduce our claim to G being a finite cyclic group.

Let $G \cong C_n$ and let X be a $K(G,1)$ complex. From class, we know that $H^k(C_n;\mathbb{Z})$ is nonvanishing in all degrees. But by Prop. 5.4, $H^k(C_n;\mathbb{Z}) \cong H^k(X;\mathbb{Z})$. But a finite CW complex has vanishing cohomology in high dimension. Therefore, the dimension of X must be infinite. $\square$

As the contrapositive, we see that any finite-dimensional acyclic CW complex has a torision-free fundamental group, which like would have been difficult to demsontrate directly!

Proposition 5.11. *Let X be a connected CW complex and let Y be a $K(G,1)$. Then every homomorphism $\pi_1(X,x_0) \to G$ is induced by a map $(X,x_0) \to (Y,y_0)$ that is unique up to pointed homotopy. For a proof, see Prop. 1B.9 in [3].*

The above proposition shows that $K(G,1)$ is a unique homotopy class. Indeed, if X, Y are both $K(G,1)$'s, then the identity maps $\pi_1(X,x_0) \to \pi_1(Y,y_0)$ and $\pi_1(Y,y_0) \to \pi_1(X,x_0)$ are induced by maps f, g, and $fg = \mathrm{Id}_G$ and $gf = \mathrm{Id}_G$, so both compositions are homotopic to the identity on their respective spaces.

Corollary 5.12. *Let G be an Abelian group, and let $\langle X, Y\rangle$ denote pointed homotopy classes of maps of pointed spaces. If X is a connected CW complex and G is an Abelian group and Y is a $K(G,1)$, then*

$$H^1(X;G) \cong \langle X, Y\rangle \tag{17}$$

Proof. First, we have the Hurewicz theorem, which says that $H_1(Y) \cong \pi_1(G)^{ab}$, where $\pi_1(G)^{ab}$ denotes the Abelianization of $\pi_1(X)$. Since $\pi_1(Y) = G$ is Abelian by assumption, we have

$$H_1(Y) \cong \pi_1(Y) = G \tag{18}$$

Let $f \in \langle X, Y\rangle$. Then consider the map $f_* : H_1(X) \to H_1(Y) \cong G$. Since $H_0(X) = \mathbb{Z}$ is free, we have $\mathrm{Ext}^1(H_0(X), G) = 0$, and so by the Universal Coefficients Theorem, $H^1(X) \cong \mathrm{Hom}(H_1(X), G)$, so we can identify f_* with an element of $H^1(X)$. That this map is an isomorphism is a direct consequence of Prop. 5.11. $\square$

6 Example classifying spaces

Example 6.1. *If $\mathbb{F}_n$ is a free group on n generators, then $X = \bigvee_{i=1}^n S_i$ is a $K(\mathbb{F}_n, 1)$.*

For the example above, the universal cover of $\bigvee_{i=1}^n S_i$ is a regular tree of degree n, which is contractible. By Van Kampen's theorem, the fundamental group of X is $\mathbb{F}_n$, with a loop around each of the circles in the wedge sum generating $\pi_1(X)$.

To compute the cohomology of this space, consider $C_0(X) \cong \mathbb{Z}$, $C_1(X) \cong \mathbb{Z}^n$, and $C_k(X) = 0$ for $k \geq 2$. The formula for the 1-boundary is $\partial_1(e^1_\alpha) = v_f - v_i$, where e^1_α begins at v_i and ends at v_f. Therefore all of the boundary maps are zero, and so the homology groups can be readily computed:

$$H_0(X) \cong \mathbb{Z} \qquad H_1(X) \cong \mathbb{Z}_n \qquad H_{k\geq 2}(X) \cong 0 \tag{19}$$

All of the homology groups are free abelian; therefore, $\mathrm{Ext}^1(H_{n-1}(X), G)$ vanishes, so $H^k(X;M) \cong \mathrm{Hom}(H_n(X), M) \cong H_n(X)$ from the universal coefficients theorem.

Example 6.2. *If $\mathbb{Z}^n$ is a free abelian group of rank n, then the n-torus T^n is a $K(\mathbb{Z}^n, 1)$. We compute $H^k(\mathbb{Z}^n; M) \cong M^{\binom{n}{k}}$.*

We construct the n-torus as $T^n \cong \mathbb{R}^n/\mathbb{Z}^n$, and $\mathbb{Z}^n$ acts on $\mathbb{R}^n$ by translations, which is a covering space action (Def. 4.12). Therefore, T^n inherits a cell complex structure and $\mathbb{R}^n$ is the universal cover. $\mathbb{Z}^n$ is the group of deck transformations of this cover by Prop. 4.13, so $\pi_1(T^n) \cong \mathbb{Z}^n$. To compute the cohomology of T^n, we first compute the number of cells in each dimension. We notice that each k-cell is a hypercube in $\mathbb{R}^n$ which descends to a cube with bottom corner at the origin and lying along the coordinate axes. Thus there are $\binom{n}{k}$ such cells. By the same token, we see that the attaching maps have degree $+1$ in one face and degree -1 in the opposite face, and the remaining k-cells get send to a point. The boundary maps all therefore zero in all degrees, and so $H_k(T^n) \cong \mathbb{Z}^{\binom{n}{k}}$. These groups are free, and so we have $H^k(T^n; M) \cong \mathrm{Hom}(H_k(T^n), M) \cong M^{\binom{n}{k}}$.

Lemma 6.3. *A CW complex X is called aspherical if $\pi_k(X) = 0$ for $k \geq 2$. Any aspherical CW complex X is a $K(\pi_1(X), 1)$.*

Proof. Let $\widetilde{X}$ be the universal cover of X. By lifting the CW structure on X, we ensure that $\widetilde{X}$ is a CW complex. By the homotopy lifting property, every map $S^n \to \widetilde{X}$ is nullhomotopic, so we again get an acyclic resolution from the universal cover (it turns out that acyclic CW complexes are contractible by the Whitehead theorem). Thus, X is a $K(\pi_1(X), 1)$. □

Proposition 6.4. *Let $S^\infty \subset \mathbb{C}^\infty$ be the unit sphere. If $\mathbb{Z}_m$ acts on S^∞ by component-wise scalar multiplication by the m^{th} roots of unity, then the lense space $L_m \cong S^\infty/\mathbb{Z}_m$ is a $K(\mathbb{Z}_m, 1)$.*

Proof. First, we will show that S^∞ is contractible. Let $T : S^\infty \to S^\infty$ via $T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$. We define the straight-line homotopy between Id and T and normalize it so that it remains in S^∞, which we can do since $t\,\mathrm{Id} + (1-t)T$ never passes through the origin. The image of T does not contain the point $(1, 0, 0, \dots)$, so we can contract the image of T to this point by again taking a straight-line homotopy and normalizing. This shows that S^∞ is contractible, so it is the universal cover of L_m. □

Example 6.5. *$\mathbb{R}P^\infty$ is a $K(\mathbb{Z}_2, 1)$.*

One can compute the homology of $\mathbb{R}P^\infty$ by noticing that $\mathbb{Z}_2$ acts by the antipode map $a : S^1 \to S^1$. The CW structure on S^k can be constructed inductively by putting two k-cells on each hemisphere, with S^{k-1} joining them at the equator. The projection $S^\infty \to \mathbb{R}P^\infty$ identifies these two k-cells in each degree, so the CW structure on $\mathbb{R}P^\infty$ has one k-cell in each degree. Furthermore, the preimage of a point x in the image of the gluing map g^k consists of two points, $\{x, ax\}$. By the local degree theorem, the degree of this map is then $\deg(g^k) = \deg(\mathrm{Id}) + \deg(a)$. Since $\deg(a) = (-1)^k$, we see that $\partial e^k = 1 + (-1)^k$, from which the cellular chain complex is

$$0 \longleftarrow \mathbb{Z} \xleftarrow[\times 0]{} \mathbb{Z} \xleftarrow[\times 2]{} \mathbb{Z} \xleftarrow[\times 0]{} \mathbb{Z} \longleftarrow \dots$$

From this, we can read off the homology groups: $H_0(\mathbb{R}P^\infty) = \mathbb{Z}$, $H_k(\mathbb{R}P^\infty) = \mathbb{Z}_2$ for even $k \geq 2$, and zero otherwise. We can then easily compute the cohomology with coefficients in $\mathbb{Z}_2$. If k is even, $\mathrm{Ext}^1(H_{k-1}(X), \mathbb{Z}_2)$ vanishes, so $H^k(\mathbb{Z}_2; \mathbb{Z}_2) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Proposition 6.6. *$K(G, 1) \times K(H, 1)$ is a $K(G \times H, 1)$.*

Proposition 6.7. *If X is a $K(G, 1)$ with universal cover $\widetilde{X}$ and $H \leq G$, then $\widetilde{X}/H$ (with the action by deck transformations) is a $K(H, 1)$.*

Proof. This follows from our previous remarks. Since G acts by a covering space action, the map $\widetilde{X} \mapsto \widetilde{X}/H$ is a covering space. The deck transformations of this cover are given by elements of H, because any covering space isomorphism must preserve cosets of H, and each deck transformation is uniquely determined by where it sends a point by virtue of Prop. 5.2. We have shown that $G(\widetilde{X}) \cong \pi_1(\widetilde{X}/H)$ in Prop. 4.13. Together, this shows that $\widetilde{X}/H$ is a $K(G,1)$. $\square$

7 Coefficients in an arbitrary module

As we say in the previous section, the identification $\mathrm{Hom}_{\mathbb{Z}G}(C(\widetilde{X}), M) \cong \mathrm{Hom}_{\mathbb{Z}}(C(X), M)$ only works when M is the trivial module. When M is a non-trivial module, we need additional structure; an action $G \cong \pi_1(X, x_0) \mapsto \mathrm{Aut}(M)$. This additional data is captured by a local system.

Definition 7.1 (Presheaf)**.** *Consider a topological space X as a category with objects being the open sets of X and morphisms being inclusion of open sets. A presheaf on X is a contravariant functor $\mathcal{F} : X \to$ **Set**. For an open set $U \subset X$ an element $s \in \mathcal{F}(U)$ is called a section of U. Furthermore, given an inclusion $i : V \hookrightarrow U$, if s is a section of U then we will denote $\mathcal{F}(i) : \mathcal{F}(U) \to \mathcal{F}(V)$ via $s \mapsto s|_V$.*

Definition 7.2 (Sheaf)**.** *Let U be an open set and $\{U_i\}_{i\in I}$ be an open cover of U with $U_i \subset U$. A sheaf $\mathcal{F}$ on a topoligical space X is a presheaf satisfying the following condition: Let $\{s_i \in \mathcal{F}(U_i)\}_{i\in I}$ be a family of sections such that $s_i|_{U_i\cap U_j} = s_i|_{U_i\cap U_j}$ for all $i, j \in I$. Then there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.*

Example 7.3. *Let X, Y be a topological spaces and $\mathcal{F}(X)$ be the set of continuous functions from $X \to Y$. Then $\mathcal{F}(X)$ is a sheaf on X. As a functor $\mathcal{F}(U)$ will send open sets of X to continuous functions defined on those open sets. $\mathcal{F}$ sends inclusion of open sets (morphisms in X) to restrictions of continuous functions which one can see is contravariant. The locality conditions follows trivially from the fact that functions of topological spaces are defined pointwise, while the gluing condition follows from the gluing property of continuous functions.*

Sheaves are in a sense generalizations of continuous functions on topological spaces; they "continuously" assign data to topological spaces. One may notice that while continuous functions are defined pointwise, sheaves are only defined on open sets. We may still be interested in the "value" of section of a point x in some topoligical space, which requires the notion of a **stalk** at x.

Definition 7.4 (Stalk)**.** *Let X be a topological space and $\mathcal{F}$ be a sheaf on X. Let $x \in X$ and consider two open neighborhoods U, V of x and sections $f \in \mathcal{F}(U), s \in \mathcal{F}(V)$. We say $(U, f) \sim (V, s)$ if there exists an open neighborhood $W \subset U \cap V$ of x such that $f|_W = s|_W$. The* ***stalk*** *$\mathcal{F}_x$ of x is the set of pairs (U, f) under this equivalence relation.*

While a point $x \in X$ may not be open, a stalk of x allows us study pointwise data of a topological space by considering a decreasing sequence of open neighborhoods of x.

Remark 7.5 (Constant Sheaf)**.** *Let X be a topological space, A a set, and $\mathcal{F} : X \to A$ be the constants sheaf. The sections $s \in \mathcal{F}(U)$ can be thought of as continuous functions $s : U \to A$ with A given the discrete topology. If U is connected then the sections are constant functions.*

Definition 7.6 (Local system)**.** *A sheaf $\mathcal{F}(X)$ is a local system if for each $x \in X$ there is an open neighborhood U of x such that the restriction $\mathcal{F}(U)$ is the constant sheaf.*

Proposition 7.7. *Let X be $\mathcal{F}(X)$ be a local system. Let U be an open set such that $\mathcal{F}(U)$ be constant. Then $\mathcal{F}_x \cong \mathcal{F}(U)$ for every $x \in U$.*

Proof. Let $\phi : \mathcal{F}(U) \to \mathcal{F}_x$ by $s \mapsto [U, s]$. Let $\psi : \mathcal{F}_x \to \mathcal{F}(U)$ by $[V, f] \mapsto f|_{U\cap V}$. ψ is well defined since $V \cap U \subset U \implies \mathcal{F}(V \cap U) = \mathcal{F}(U)$ since $\mathcal{F}(U)$ is constant. We show ϕ and ψ are inverses:

$$\phi \circ \psi([V, f]) = [U, f|_{U\cap V}] = [V, f]$$
$$\psi \circ \phi(s) = \psi([U, s]) = s|_U = s$$

This shows that ϕ and ψ are isomorphisms, as desired. □

Proposition 7.8. *Local systems on path connected spaces have isomorphic stalks at every point.*

Proof. Let $\mathcal{F}$ be a local system of X and $x, y \in X$. Now let $\gamma : [0, 1] \to X$ be a path connecting x and y. For each $t \in [0, 1]$ there exists an open neighborhood U_t of $\gamma(t)$ such that $\mathcal{F}(U_t)$ is constant since $\mathcal{F}$ is a local system. For each t there exists an ϵ such that $\gamma(t+\epsilon) \in U_t$. Both $\mathcal{F}(U_t)$ and $\mathcal{F}(U_{t+\epsilon})$ are the constant sheaf so we have $\mathcal{F}(U_t \cap U_{t+\epsilon}) = \mathcal{F}(U_t) = \mathcal{F}(U_{t+\epsilon})$. The previous propsition gives bijections

$$\mathcal{F}_t \cong \mathcal{F}(U_t) = \mathcal{F}(U_{t+\epsilon}) \cong \mathcal{F}_{t+\epsilon} \ ,$$

which gives a bijection between $\mathcal{F}_t$ and $\mathcal{F}_{t+\epsilon}$. Therefore every point along the path has an isomorphic stalk. Moreover we have just shown that paths in X provide canonical automorphisms of the stalk of $\mathcal{F}$. □

Remark 7.9. *Let X be a path connected topological space and $\mathcal{F} : X \to$ **R-mod** be a local system. We associate the module M to every point $x \in X$, and paths $\gamma : [0, 1] \to X$ will give automorphisms of M.*

Proposition 7.10. *Let X be path connected and $\mathcal{L}$ be a local system with stalk L. There is a bijective correspondence between local systems and mondromy representations $\rho : \pi_1(X, x_0) \to$* $\mathrm{Aut}(L)$.

Proof. Let $[\gamma] \in \pi_1(X, x_0)$. γ provides a canonical automorphism on L as in prop 7.8. To show that such an automorphism is homotopy-invariant, we consider a homotopy $H : \gamma_1 \Rightarrow \gamma_2$ as a map $H : I^2 \to X$. We may divide I^2 into small boxes B_i such that $H(B_i) \subset U$, where $\mathcal{L}$ is constant on U. Then for any $x \in U$ and $[V, s] \in \mathcal{L}_x$, by the definition of a sheaf there exists a unique $\widetilde{s}$ extending s on U. Therefore the isomorphism $\mathcal{L}_{\gamma(0)} \cong \mathcal{L}_{\gamma(1)}$ constructed in Prop. 7.7 depends only on the restriction of $\widetilde{s}$, and it is independent of the path.

Therefore, for each $[\gamma]$, we get an automorphism on L which gives the desired monodromy representation $\rho_{\mathcal{L}}$ for any given local system. Conversely let $\rho : \pi_1(X, x_0) \to \mathrm{Aut}(L)$ be a monodromy representation. Let $p : \widetilde{X} \to X$ be the universal cover and consider the constant sheaf $\tilde{\mathcal{L}}(U) = L$ for $U \subset X$. By Prop. 5.2, $\pi_1(X, x_0)$ acts on $\widetilde{X}$ by deck transformations. For $U \subset X$ let $\mathcal{L}_\rho(U)$ be the set of ρ equivariant sections

$$\mathcal{L}_\rho(U) = \{s \in \tilde{\mathcal{L}}(p^{-1}(U)) \mid \rho_g s(\tilde{y}) = s(g\tilde{y}) \text{ for all } g \in \pi_1(X, x_0), \tilde{y} \in p^{-1}(U)\} \tag{20}$$

where we view s as a continuous function $s : p^{-1}(U) \to L$ which is constant on connected components of $p^{-1}(U)$ as in Rmk. 7.5. Now we will show that $\mathcal{L}_\rho$ is a local system on X. To see this consider a connected open neighborhood U of X such that $p^{-1}(U) = \bigsqcup_\alpha U_\alpha$ with $U_\alpha \cong U$. Any section $s \in \mathcal{L}(p^{-1}U)$ will be constant on each U_α so we find $s(\tilde{y}) = l_\alpha$ for $\tilde{y} \in U_\alpha$. If $s \in \mathcal{L}_\rho(U)$

then $\rho_g s(\tilde{y}) = s(g\tilde{y})$ and we find $s(g\tilde{y}) = \rho_g l_\alpha$. But each element $\tilde{x} \in p^{-1}(x)$ can be written as $g\tilde{x}_0$ for some distinguished point $\tilde{x}_0 \in p^{-1}(x)$. This means we need only specify a single value l_α, after which $s(\tilde{x})$ is completely determined by the action of ρ. This gives an identification between $s \in \mathcal{L}_\rho(U)$ and $l \in L$. We can think of $s(U) = l$ as the constant function and we find $L_\rho(U)$ is a constant sheaf. One can show that the identifications $\rho \to \mathcal{L}_\rho$ and $\mathcal{L} \to \rho_\mathcal{L}$ are inverse, completing the proof. □

One way to define the cohomology of X with coefficients in a local system $\mathcal{L}$ is the following:

Definition 7.11 (Cohomology with coefficients in a local system)**.** *Let X be a path connected topological space with universal cover $\widetilde{X}$. $\pi_1(X, x_0)$ acts on $\widetilde{X}$ via deck transformation giving an induced action on the singular chains of $C_n(\widetilde{X})$ by post-composition. Let L be an abelian group with action by $\pi_1(X, x_0)$ and form the cochain complex of $\pi_1(X, x_0)$ equivariant maps*

$$Hom_{\pi_1(X,x_0)}(C_n(\widetilde{X}), L) \tag{21}$$

The cohomology of this complex is defined as $H^n(X, L)$ called the homology of X with local coefficients L.

Under suitable assumptions, this definition agrees with other notions of sheaf cohomology, such as the Čech cohomology. As a consequence of this definition, we find agreement between the group cohomology of G with coefficients in L and the cohomology of a $K(G, 1)$:

Corollary 7.12. *Let X be a $K(G, 1)$. Then*

$$H^\bullet(X, \mathcal{L}) \cong H^\bullet(G, L) \tag{22}$$

where on the left, we have the cohomology of X with coefficients in a local system $\mathcal{L}$, and on the right we have the group cohomology with coefficients in the G-module L.

This completes our discussion of the topological underpinnnings of group cohomology. Appealingly, the concept of a free resolution was presented to us by the topological properties of aspherical CW complexes, without appealing to prior knowledge of the underlying algebraic structure. For future reading, we would like to further explore the cohomology of sheaves, in particular the Čech cohomology, which makes the connection to the geometry of the space more obvious. Furthermore, there are fascinating applications of these ideas in physics, where mondromy representations in an Abelian group correspond to flat connections in an Abelian gauge theory.

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