Lie Algebra Cohomology
Lie Groups and Rep Theory

Ben McDonough
Dr. Surya Raghavendran

Introduction

Homology arose in the late 19^{th} century, and was used in the early 1900's by Poincare to describe holes in simplicial complexes. Cohomology can be viewed as an extension of homology to assign a richer set of invariants to topological spaces. Lie algebra cohomology was introduced in 1929 by Cartan to relate the methods in de Rham cohomology to properties of Lie algebras. This was later generalized by Chevalley and Eilenberg to Lie algebra cohomology with coefficients in an arbitrary Lie module. In a Lie group, the de Rham cohomology induces a cohomology on the exterior algebra of the Lie algebra. A compact, simply-connected Lie group is determined by its Lie algebra, so it is unsurprising that this determines the cohomology of the manifold. Remarkably, this effectively reduces a topological problem into a linear algebra one, although this does not necessarily hold for noncompact Lie groups. We will first define the basic tools of cohomology in a general setting. We will define the de Rham cohomology and prove that it is a topological invariant. Then we will give an explicit formula for computing the Lie algebra cohomology with coefficients in an arbitrary module using the Chevalley-Eilenberg complex. We will show how the first few cohomology groups encode important properties of the Lie group, and then we will use cohomology to give a proof for several important theorems, namely the Whitehead lemmas, the Levi-Malcev theorem, and Weyl reducibility.[1]

Homological Algebra

Def 1. *Cochain complexes, cocycles, coboundaries, and differentials*

Given a $\mathsf{k}-$module C and a $\mathsf{k}-$linear map d that $\mathsf{d}^2 = 0$, we call (C, d) a **differential complex** and d the **differential**. Elements in the kernel of d are called **cocyles**, and elements in the image of d are called **coboundaries**. Differential complexes are graded, with $\mathsf{C} = \bigoplus_n \mathsf{C}^n$ and $\mathsf{d}_n : \mathsf{C}^n \to \mathsf{C}^{n+1}$ being a degree-1 map. This gives a sequence:

$$0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \dots$$

The cohomology of (C, d) is then defined as $\mathsf{H}^*(\mathsf{C}, \mathsf{d}) = \bigoplus_n \mathsf{H}^n(\mathsf{C}, \mathsf{d})$, where $\mathsf{H}^n(\mathsf{C}, \mathsf{d}) = \ker(\mathsf{d}_n)/\mathrm{im}(\mathsf{d}_{n-1})$, or the cocycles quotient by the coboundaries. Notice that the cohomology measures the failure of the above sequence to be exact, i.e. $\ker(\mathsf{d}_n) \subsetneq \ker(\mathsf{d}_{n-1})$.

Cohomology can typically be computed simply by specifying a convenient choice of d. For associative algebras, and by extension groups and Lie algebras, there is a unified way to define a cohomology which reveals the connection to many useful properties of these objects. The most general setting for homological algebra is that of Abelian categories.

Def 2. ***AB****, additive, and Abelian categories. Also, kernels, cokernels, monics, and epis.*

A category $\mathcal{A}$ is called an **AB**-category if $\mathrm{Hom}_{\mathcal{A}}(\mathsf{A}, \mathsf{B})$ for each A, B is an Abelian group with respect to an addition operation $(+)$, and morphism composition distributes over this addition. An **AB**-category is called **additive** if there is a zero object and a product $\mathsf{A} \times \mathsf{B}$ between objects. We note that the existence of a zero morphism $0_{\mathsf{AB}} : \mathsf{A} \to \mathsf{B}$ is implied by the Abelian structure of $\mathrm{Hom}_{\mathcal{A}}(\mathsf{A}, \mathsf{B})$. Abelian categories are a particular kind of additive category. The existence of zero-morphisms allows us to define the notion

[1] I fell short of this last goal–I did not finish the Levi-Malcev theorem or Whitehead lemmas.

of kernels and cokernels. Given a morphism $f : B \to C$, the **kernel** of f is a map $\ker(f) = i : A \to B$ such that $f \circ i = 0$ and i is universal with respect to this property. The **cokernel** $\mathrm{coker}(f) = e$ is similarly universal with respect to $e \circ f = 0$.

$$
\begin{tikzcd}
 & A \arrow[d, "i"] \arrow[dr, "0_{AC}"] \arrow[ddl, "k"'] & \\
 & B \arrow[r, "f"] & C \\
B \arrow[ur, "i'"] \arrow[urr, "f"'] & &
\end{tikzcd}
$$

The diagram above defines the universal property of $i = \ker(f)$: if some i' is chosen as above, then there exists a unique morphism k such that the diagram commutes. f is called **monic** if $fg = 0$ implies $g = 0$ and **epi** if $gf = 0$ implies $g = 0$ for all maps g. An additive category is called an **Abelian category** if every map has a kernel and a co-kernel, every monic is the kernel of its co-kernel, and every epi is the cokernel of its kernel. The **image** of f is defined to be $\ker(\mathrm{coker}(f))$. In particular, this is equivalent to requiring that every morphism $f : B \to C$ factors as

$$
B \xrightarrow{e} \mathrm{im}(f) \xhookrightarrow{m} C
$$

where e is an epimorphism, i.e. has a right-inverse, and m is a monomorphism, i.e. it has a left-inverse. Abelian categories are necessary for defining the notion of exact sequences.

Ex 1. *R-**mod***

The prototypical example of an Abelian category is the category R-**mod** of modules over a ring R. Morphisms are $R-$linear maps, which inherit their addition operation from pointwise addition in the underlying ring, and the compatibility with morphism composition comes from R-linearity. If $f : A \to B$ is a morphism, then the set-theoretic definition of $\ker(f)$ is $\{a \in A | f(a) = 0\}$. This is clearly an $R-$module, and thus the projection π onto this set is a morphism. Combining this with the inclusion map ι gives $f \circ (\iota \circ \pi) = 0$. Similarly, the set-theoretic cokernel of f is $B/\mathrm{im}(f)$. This quotient is an object in R-mod, and projection onto the quotient is then the cokernel of f in the category-theoretic sense. This shows additionally that every such f has a kernel and a cokernel. In R-mod, the notion of monic is the same as injective, and epi is the same as surjective.

Def 3. *Additive functor, left/right exact functor, and morphisms of chain complexes*

A functor $F : \mathcal{A} \to \mathcal{B}$ between two **AB**-categories is called **additive** if the map $\mathrm{Hom}_{\mathcal{A}}(A', A) \to \mathrm{Hom}_{\mathcal{B}}(FA', FA)$ is a homomorphism of Abelian groups. F is called **exact** if it preserves exact sequences. F is called **left exact** if for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact in $\mathcal{B}$. F is called **right exact** if this holds instead for $F(A) \to F(B) \to F(C) \to 0$. A **morphism of a chain complexes** is a set of maps $u_n : C_n \to D_n$ such that $u_{n+1} d_n = d_n u_n$.

Def 4. *Projectives, covariant hom functor, and projective resolutions*

An object $P \in \mathcal{A}$ is called **projective** if for any surjection $g : B \to C$ and a map $\gamma : P \to C$, there exists a map $\beta : B \to P$ such that $\alpha = g \circ \beta$.

$$
\begin{tikzcd}
 & P \arrow[dl, "\exists\beta"'] \arrow[d, "\gamma"] \\
B \arrow[r, two heads] & C
\end{tikzcd}
$$

Now we define an important functor: to every object $A \in \mathcal{A}$ there corresponds a functor $\mathrm{Hom}_{\mathcal{A}}(A, -)$ called the covariant hom-functor. First, we show that this functor is left-exact. Consider the exact sequence $0 \to X \xrightarrow{h} Y \xrightarrow{g} Z \to 0$. This induces a sequence $0 \to \mathrm{Hom}(A, X) \xrightarrow{h} \mathrm{Hom}(A, Y) \xrightarrow{g} \mathrm{Hom}(A, Z) \to 0$. It is clear that h^* is injective and that $\mathrm{im} h \leqslant \ker g$. To show that $\ker g \leqslant \mathrm{im} h$, consider $g \circ f = 0$. Then the image of f lies in the kernel of g, which is the image of h by assumption. Since h is injective by assumption, $h \circ h^{-1} \circ f = f$. This shows left-exactness. What is required for the covariant hom-functor to be right-exact? We need $f = g \circ k$ for all maps $f : A \to Z$ and some $k : Y \to Z$. This is exactly the condition for A to be projective. Thus we have shown that A is projective iff $B \mapsto \mathrm{Hom}(A, B)$ is exact.

We then introduce the idea of a projective resolution. A left-resolution of an object A is a complex P_i and a map $\epsilon : P_0 \to M$ such that the augmented complex

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

is exact. If each P_i is projective, then this is called a projective resolution. In R-mod, one way to construct projective resolutions is through free resolutions, which are similarly defined by left-resolutions consisting of free modules. We have the following theorem:

Prop 1. *An R-module is projective iff it is a direct summand of a free R-module.*

This follows immediately from the definition. If A is an R-module and $F(A)$ is the free module generated by A, then we get a surjection $F(A) \to A$, and conversely, applying the universal lifting property to the identity map on a projective module A embeds A as a direct summand of $F(A)$.

The dual notion to projectives is that of injectives:

Def 5. *Injectives, contravariant hom functor, enough injectives, and injective resolutions*

An object $I \in \mathcal{A}$ is called **injective** if for any injection $A \to B$ and a map $\alpha : A \to I$, there exists a map $\beta : B \to I$ such that $\alpha = \beta \circ f$.

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "\alpha"] & B \arrow[dl, "\exists\beta"] \\ I & \end{tikzcd}$$

Prop 2 (Baer's Criterion)**.** *A right $R-$module E is injective if and only if for every right ideal J of R, every map $J \to E$ can be extended to a map $R \to E$. For a proof, see Weibel, page 39 [5].*

We can similarly define the contravariant hom-functor $B \mapsto \mathrm{Hom}(B, -)$. Similar to the proof for projectivity, we can view this functor as a covariant functor from $\mathcal{A}^{op}$ to the category of Abelian groups, and in this view it is left-exact. The injectivity condition is the same as requiring that this functor is right-exact.

If for every A there exists an injective I and an injection $A \to I$, then $\mathcal{A}$ is said to **have enough injectives**. An injective resolution of an object M is a cochain complex C and a map $M \to C^0$ such that the augmented complex $0 \to M \to C^0 \to C^1 \to \dots$ is exact and each C^i is injective.

Lem 1. *If $\mathcal{A}$ has enough injectives, then every $A \in \mathcal{A}$ has an injective resolution.*

To prove this, let A be an object in an Abelian category with enough injectives. Then we have an injective I and an injection $\alpha : A \to I$. Since the category is Abelian, this is completed to an exact sequence $0 \to A \to I \to \mathrm{coker}\alpha \to 0$. Since the category has enough injectives, we get another injection $\mathrm{coker}\alpha \to I_1$, and composing this with the projection $I \to \mathrm{coker}(\alpha)$ extends the sequence, and repeating this process gives the desired injective resolution.

Prop 3. *The category R-mod has enough injectives.*

This is necessary to make sense of cohomology.

We would like to measure of the *failure* of an object in the category to be injective. A sensible approach, given what we have just developed, is to choose an injective resolution to come up with an exact sequence, pass this chain complex through the covariant hom-functor, and then take the cohomology of the resulting chain complex. This idea leads us to the notion of right-derived functors:

Def 6. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between two abelian categories. If $\mathcal{A}$ has enough injectives, then the right-derived functors of F are defined as follows: If $A \in \mathcal{A}$, choose an injective resolution $A \to C$ and define $R^i F(A) = H^i(F(C))$.*

One might wonder how the definition of the right-derived functor depends on the choice of resolution. This ambiguity is taken care of by the Comparison Lemma:

Thm 1 (Comparison Lemma)**.** *Let $N \to I$ be an injective resolution of N and $f'M \to N$ a morphism. Then for every resolution $M \to E$ there is a cochain map $F : E \to I$ lifting f', and this map is unique up to cochain homotopy equivalence.*

Now we introduce the culmination of these defintions:

Def 7. *The right-derived functors of the contravariant hom-functor are called are called* Ext *groups:*

$$\mathrm{Ext}_R(A, B) = R^i \mathrm{Hom}_R(A, -)(B)$$

As shown in Weibel [5], we can equivalently define $\mathrm{Ext}_R(A, B)$ by dualizing these notions, such that $\mathrm{Ext}_R(A, B) = R^i \mathrm{Hom}_R(,B)(A)$, where an injective resolution of A in $\mathcal{A}^{op}$ becomes a projective resolution in $\mathcal{A}$. This will become important later, as it will be much more convenient to find projective resolutions.

de Rham Cohomology

Given an $n-$dimensional manifold M, the tangent space T_pM at p is the vector space of linear derivations of $C^\infty(M)$ at p. Then $(T_pM)^*$ denotes the cotangent space at p. By T^kT^*M we denote the bundle of linear maps on $(T_pM)^{\otimes k} \to \mathbb{R}$, and we define $\Lambda^k T^*M$ to be the smooth subbundle consisting of alternating maps. Then by $\Gamma(M, \Lambda^k T^*M)$ we denote the vector space of smooth sections of the canonical projection map $\Lambda^k T^*M \to M$, and we call this vector space $\Omega^k(M)$, whose elements are the smooth differential $k-$forms on M.

Let V be an arbitary vector space, and denote the subspace of $V^{\otimes k}$ consisting of alternating $k-$tensors as $\Lambda^k(V)$. Let Alt_k be a projector onto this subspace. We may also write this space as $(V^{\otimes k} \otimes \mathrm{sgn})^{S_k}$, where $\sigma \in S_k$ acts on $V^{\otimes k}$ by permuting the factors, and σ acts on sgn as $\mathrm{sgn}(\sigma)$. We obtain an explicit form for Alt_k by averaging the action of S_n on $V^{\otimes k} \otimes \mathrm{sgn}$. We then define the exterior product as the bilinear map $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ via $\omega \wedge \eta = \frac{k+l!}{k!l!} \mathrm{Alt}(\omega \otimes \eta)$, where the coefficient is chosen so that $\varepsilon^I \wedge \varepsilon^J = \varepsilon^I \otimes \varepsilon^J \Big| \Lambda^{k+l}(V)$, where $\{\varepsilon^i\}$ is a basis for V^* with I, J increasing multi-indices.

Since T_pM is $n-$dimensional, it is clear that $\Omega^{k>n}(M) = 0$. We let $\Omega^*(M) = \bigoplus_{j=0}^n \Omega^j(M)$ be the exterior algebra. We see that $\wedge$ makes $\Omega^*(M)$ into a graded algebra.

We are now in a position to define the exterior derivative as a map $d : \Omega^*(M) \to \Omega^*(M)$. We identify $\Omega^0(M)$ with smooth real-valued functions. For any such f, we define $df \in \Omega^1(M)$ as the cotangent field $df(X) = Xf$. We extend this to general k with $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^*(M)$. In other words, d fulfills the graded Leibniz rule:

Def 8. *If $d(ab) = (da)b + (-1)^{|a|} a(db)$, where $|a| = k$ if $a \in C^k$ is the degree of a, then d is said to fulfil the graded Leibniz rule.*

In essence, this construction gives the map which agrees with the notion of a differential on functions and makes the exterior algebra into a graded differential algebra. In addition, the following property makes $\Omega^*(M)$ into a cochain complex:

Prop 4. $d^2 = 0$

If $\omega \in \Omega^1(M)$, then ω can locally be written as $u dv$ for $u, v \in \Omega^0(M)$, the properties of the exterior derivative show that $d\omega(X \otimes Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$. Applying this to df, we have $d(df)(X \otimes Y) = XYf - YXf - [X, Y]f = 0$. Inductively, this shows that $d^2 = 0$. From the previous definitions, it is clear that d induces a cochain complex in $\Omega^*(M)$. The cohomology of this cochain is called the de Rham cohomology.

Prop 5 (Coordinate-free definition of Exterior Derivative)**.** *If $\omega \in \Omega^k(M)$, then for smooth vector fields $X_1, \ldots, X_{k+1}$, we ahve*

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) = &\sum_{1 \leqslant i \leqslant k+1} (-1)^{i-1} X_i(\omega(X_1, \ldots, \widehat{X}_i, \ldots, X_{k+1})) \\ &+ \sum_{1 \leqslant i < j \leqslant k+1} (-1)^{i+j} \omega([X_i, X_j], x_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1}) \end{aligned}$$

The proof of this uses the properties above, and can be found on page 370 of [4].

The exterior derivative is natural in the sense that it commutes with pullbacks of differential forms by smooth functions. If $\omega \in \Omega^k(M)$ and $F : M \to N$ is smooth, then we define the pullback of ω by F as $F^*\omega_p = \omega_{F(p)} \circ (DF)_p^{\otimes k}$, where ω_p is the restriction of ω to $T^k(T_pM)^*$ and $DF_p : T_pM \to T_pN$ via $[(DF)_pX]f = X(f \circ F)$. From this definition it is clear that F^* is a linear map on $\Omega^*(M)$, and that $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$ for all $\omega, \eta \in \Omega^*(M)$. We can also see that $d(F^*\omega) = F^*(d\omega)$.

Def 9. *Define the de Rham cohomology group in degree p to be the quotient vector space*

$$H^p_{dR}(M) = \{\textit{closed } p \textit{ forms}\}/\{\textit{exact } p \textit{ forms}\}$$

where the closed forms are the p-cocycles and the exact forms are the p-coboundaries. The first property to note is the diffeomorphic invariance of the de Rham cohomology. If $F : M \to N$ and ω is a closed $p-$form then $d(F^*\omega) = F^*(d\omega) = 0$. If η is an exact $p-$form then $\eta = d\beta$ for some $p-1$-form β, and so $d(F^*\beta) = F^*\eta$. This shows that F^* maps closed forms to closed forms and exact forms to exact forms,so F descends to a linear map map $F : H^p_{dR}(M) \to H^p_{dR}(N)$. If F is additionally a diffeomorphism, then $(F^*)^{-1} = (F^{-1})^*$, and thus F^* is an isomorphism and $H^p_{dR}(M) \cong H^p_{dR}(N)$. This establishes that the de Rham cohomology groups are diffeomorphic invariants.

Corr 1. *The assignment $M \mapsto H^p_{dR}(M)$ and $F \mapsto F^*$ is a contravariant functor from the category of smooth manifolds to the category of real vector spaces.*

It turns out that through the Whitney approximation theorem, the diffeomorphic invariance extends to a much stronger topological invariance:

Prop 6. *The de Rham cohomology is a topological invariant.*

If F and G are homotopic to each other, then $F^* = G^*$ as maps from $H^p(M) \to H^p(N)$ (this requires a small amount of work to show, see [4] for details). If $F : M \to N$ is a homotopy equivalence, then there exists G such that $F \circ G$ is homotopic to the identity map on M. The Whitney approximation theorem allows us to construct a smooth homotopy $F \cong \widetilde{F}$ and $G \cong \widetilde{G}$ where $\widetilde{F}$ and $\widetilde{G}$ are smooth. Applying the previous result shows that F^* is an isomorphism. Since every homeomorphism is a homotopic equivalence, this implies that the de Rham cohomology is also a topological invariant. To further illustrate the power of this, notice that M is locally contractible, so for every $x \in M$ there is a neighborhood U of x such that $H^p(U) = 0$. This shows that every closed form is locally exact, and so $H^p(M)$ is a nonlocal property.

Now we specialize further to a Lie group G. If $g \in G$ is arbitrary and $\omega \in \Omega^k(G)$ is a left-invariant form, then $[L_g^*(d\omega)]_{gh} = d[(L_g^*\omega)]_h = (d\omega)_h$. Let $\Omega^*_L(G)$ be the vector space of left-invariant forms, and the previous argument shows that $\Omega^*_L(G)$ is a subcomplex of $\Omega^*(G)$. Since $\omega_g = (L_g^*\omega)_e$ and L_g is a diffeomorphism, we identify the space of left-invariant differential $k-$forms with $\Lambda^k\mathfrak{g}$. This shows that left-invariant forms have a unique simplifying property; namely that their value on G is determined by their value at the identity. The exterior derivative $d : \Omega^k(G) \to \Omega^{k+1}(G)$ induces a map $d : \Lambda^k(G) \to \Lambda^{k+1}(G)$, and this is called the Lie cohomology of $\mathfrak{g}$ with coefficients in $\mathbb{R}$, considered as the trivial $\mathfrak{g}$-module.

Prop 7. *If G is connected and compact, then the cohomology of left-invariant vector fields is isomorphic to the de Rham cohomology.*

Since G is compact, we can fix a normalized bivariant volume form on G, and define $\int_G dg$ to be integration with respect to this form. We define a map $\tau : \Omega^*(M) \to \Omega^*_L(M)$ by averaging ω over G:

$$\tau(\omega) = \int_G (L_g)^*\omega dg \tag{1}$$

and we can immediately see that $\tau(\omega)$ is left-invariant (by u-sub,) $\tau(\omega) = \omega$ if $\omega \in \Omega^*_L(M)$, and τ commutes with the exterior derivative. With some effort, we can show that τ is homotopic to the identity, which proves the claim. For details, refer to [3].

We can generalize this definition to coefficients in an arbitrary $\mathfrak{g}$-module M by the following cochain complex: We define the exterior derivative on $\Lambda^k(\mathfrak{g}^*)$ exactly as above. For $m \in M$, we let $(dm)(X) = Xm$ (just as d acts on $C^\infty(G)$). In addition, we make $\bigoplus_k^n \Lambda^k\mathfrak{g}^* \otimes M$ into a graded differential algebra by requiring that d satisfies the graded Leibniz rule, i.e, $d(\omega \otimes m) = d\omega \otimes m + (-1)^k\omega \wedge dm$. Identifying $\Lambda^k(\mathfrak{g}^*, M) \cong \mathrm{Hom}(\Lambda^k\mathfrak{g}, M)$, we have specified a cochain complex called the Chevalley-Eilenberg complex with coefficients in M, and correspondingly the Lie algebra cohomology of $\mathfrak{g}$ with values in M.

Corr 2. *The above definition when $M = k$ is the trivial $\mathfrak{g}$-module agrees with the de Rham subcomplex $\Omega_L(M)$ of left-invariant forms.*

With the explicit construction of the CE complex, we find that we can obtain $\Omega^*_L(M)$ just through properties of the Lie algebra. Furthermore, when G is compact and connected, $\Omega^*_L(G) = \Omega^*(G)$.

Universal enveloping algebra

Def 10. *If $\mathfrak{g}$ is a Lie algebra over a field k, then a $\mathfrak{g}-$module M is a $k-$module with a $k-$linear product $\mathfrak{g} \otimes_k M \to M$ such that $[x, y]m = x(ym) - y(xm)$.*

Given an associative algebra A over k, we can define the commutator bracket $[x, y] = xy - yx$, and the fact that $[-, -]$ obeys Jacobi's identity follows from the associativity of A. This defines a Lie algebra $\mathrm{Lie}(A)$ by forgetting the algebraic structure and only retaining the bracket, making Lie a functor from $k-\mathrm{alg}$ to Lie. It is then natural to ask if this functor has a left-adjoint.

Def 11. *If $\mathfrak{g}$ is a Lie algebra over k, then the simplest way to construct an associative algebra A such that Lie$(A) = \mathfrak{g}$ is to consider the tensor algebra $T(\mathfrak{g}) = \bigoplus_n \mathfrak{g}^{\otimes_k n}$ and quotient by the relationship $x \otimes y - y \otimes x = [x, y]$. This is called the universal enveloping algebra $U\mathfrak{g}$.*

U is the left-adjoint of Lie, in that

$$\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Lie}(A)) \cong \mathrm{Hom}_{k-\mathrm{alg}}(U\mathfrak{g}, A)$$

This adjointness has the following important consequence:

Thm 2. *There is a natural bijection between $\mathfrak{g}$-modules and $U\mathfrak{g}$-modules.*

A Lie algebra morphism $\mathfrak{g} \to \mathrm{Lie}(\mathrm{End}_k M)$ making M into a $\mathfrak{g}-$module clearly induces a map $\mathfrak{g} \otimes M \to M$. Since $\mathrm{End}_k M$ is associative, the reverse is also true. By adjointness, every Lie morphism $\mathfrak{g} \to \mathrm{Lie}(\mathrm{End}_k M)$ induces a morphism of associative $k-$algebras $U\mathfrak{g} \to \mathrm{End}_k M$, which proves the claim.

Lie algebra cohomology

Let M be a $\mathfrak{g}$-module, and $M^{\mathfrak{g}}$ the subspace of M annihilated by $\mathfrak{g}$. This induces a functor $-^{\mathfrak{g}} : \mathfrak{g} \to M^{\mathfrak{g}}$ from $\mathfrak{g}$-mod to k-mod which is left adjoint to the trivial $\mathfrak{g}$-module functor. Right-adjoint additive functors are left-exact, following from the Yoneda embedding, so $-^{\mathfrak{g}}$ is left-exact.

Def 12. *We write* $H^*(\mathfrak{g}, M)$ *for the right-derived functors* $R^*(-^{\mathfrak{g}})(M)$ *and call them the cohomology groups of* $\mathfrak{g}$ *with coefficients in* M.

Corr 3.

$$H^*(\mathfrak{g}, M) \cong \mathrm{Ext}^*_{U\mathfrak{g}}(k, M)$$

This follows from $\mathrm{Hom}_{U\mathfrak{g}}(k, M) = \mathrm{Hom}_{\mathfrak{g}}(k, M) = M^{\mathfrak{g}}$.

All that is left is to show that the CE cohomology is isomorphic to $H^*(\mathfrak{g}, M)$:

Thm 3. *The above definitions of* $H^*(\mathfrak{g}, M)$ *agree.*

Note that $\mathrm{Hom}_k(\wedge^* \mathfrak{g}, M) = \mathrm{Hom}_{\mathfrak{g}}(\mathcal{U}\mathfrak{g} \otimes_k \wedge^* \mathfrak{g}, M)$. Unspooling the definitions, we have $H^*(\mathfrak{g}, M) = R^i \mathrm{Hom}_{\mathfrak{g}}(k, -)(M)$, so we need $\mathcal{U}\mathfrak{g} \otimes_k \wedge^* \mathfrak{g}$ to be a projective resolution of k. However, we can show that $\mathcal{U}\mathfrak{g} \otimes_k \wedge^* \mathfrak{g}$ is in fact a free $\mathfrak{g}$-module, so by 1, each object is projective. [*to be completed*] We can look at the image of the CE differential defined previously through the isomorphism $\mathrm{Hom}_{\mathfrak{g}}(\mathcal{U}\mathfrak{g} \otimes_k \wedge^* \mathfrak{g}, M) \cong \wedge^* \mathfrak{g} \otimes_k M$ to show that this is indeed a left-resolution.

Results

We have seen that Lie algebra cohomology has the potential to encode many useful properties, and we will now back up that claim:

Lem 2. $H^1(\mathfrak{g}, M) \cong \mathit{Der}(\mathfrak{g}, M)/\mathit{Der}_{Inn}(\mathfrak{g}, M)$

This says that we can obtain information about the outer derivations from $\mathfrak{g} \to M$ from $H^1(\mathfrak{g}, M)$, and in particular, $H^1(\mathfrak{g}, M) \cong \mathrm{Der}(\mathfrak{g}, M)$ if M is the trivial module. First, we define a derivation:

Def 13. *The derivations* $\mathit{Der}(\mathfrak{g}, M)$ *from* $\mathfrak{g} \to M$ *are* $k-$*linear maps statisfying the Leibniz formula:*

$$D([x, y]) = xD(y) - yD(x)$$

This generalizes the construction of $\mathrm{Der}(A)$ for an arbitrary algebra, which satisfy the more familiar Leibniz formula$D(ab) = D(a)b + aD(b)$, and are a Lie algebra under the commutator bracket. Since M is a $\mathfrak{g}-$module, we have $[x, y]m = x(ym) - y(xm)$, and so $D_m(x) = xm$ is a well-defined derivation from $\mathfrak{g} \to M$ called an inner derivation. Furthermore, if $f \in \mathrm{Hom}_{\mathfrak{g}}(U\mathfrak{g}, M)$ then $f(1) = m$ for some m and therefore $f(x) = xm$, so $D_f \in \mathrm{Der}_{Inn}(\mathfrak{g})$.

Def 14. *The adjunction of the zero map* $\mathfrak{g} \to \mathit{Lie}(k)$ *leads to a map* $\varepsilon : U\mathfrak{g} \to k$. *The augmentation ideal* $\mathcal{I}$ *is defined as the kernel of* ε, *which is the ideal generated by* $\mathfrak{g}$.

If $\psi : \mathcal{I} \to M$ is a $\mathfrak{g}-$map, then ϕ defines a derivation from $\mathfrak{g}$ to M. Furthermore, all derivations arise this way:

Prop 8. *The map* $\phi \mapsto D_\phi$ *is an isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}(\mathcal{I}, M) \cong \mathit{Der}(\mathfrak{g}, M)$$

Since $\mathcal{I} = (U\mathfrak{g})\mathfrak{g}$, the map $\pi : u \otimes \mathfrak{g} \mapsto u\mathfrak{g}$ is a surjection, and $\pi(u \otimes [x, y] + (uy) \otimes y - (ux) \otimes y) = u([x, y] + x \otimes y - y \otimes x) \sim 0$ by the construction of $U\mathfrak{g}$. Any derivation D from $\mathfrak{g} \to M$ induces a map $f : u \otimes x \mapsto u(Dx)$ which factors through this kernel, and so it lifts to a map $\widetilde{f} : \mathcal{I} \to M$ which is a $\mathfrak{g}$-map from $\mathcal{I}$ to M. This map is inverse to the map in the proposition, proving the isomorphism.

Taking the following free resolution of M

$$0 \to \mathcal{I} \to U\mathfrak{g} \to M \to 0$$

then

$$H^1(\mathfrak{g}, M) \cong \mathrm{Hom}_{\mathfrak{g}}(\mathcal{I}, M)\backslash \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, M) \cong \mathrm{Der}(\mathfrak{g})\backslash \mathrm{Der}_{Inn}(\mathfrak{g})$$

If M is the trivial $\mathfrak{g}$—module then all inner derivations are trivial, so $H^1(\mathfrak{g}, M) \cong \mathrm{Der}(\mathfrak{g}, M)$. In particular, all maps in $\mathrm{Der}(\mathfrak{g}, M)$ are just Lie algebra homomorphisms from $\mathfrak{g} \to M$ when M is considered as an Abelian Lie algebra, so $H^1(\mathfrak{g}, M) \cong \mathrm{Hom}_k(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], M)$, because all such homomorphisms factor through the Abelianization.

There is also an interpretation of H^1 in terms of the semidirect product $\mathfrak{g} \ltimes M$, with the bracket defined by $[(g, m), (h, n)] = ([g, h], gn - hm)$. Clearly $\mathfrak{g} \ltimes M/M \cong \mathfrak{g}$, so this is particular Abelian extension of $\mathfrak{g}$ by M:

$$0 \to M \to \mathfrak{g} \ltimes M \to \mathfrak{g} \to 0$$

Moreover, we say that $\sigma \in \mathrm{Aut}(\mathfrak{g} \ltimes M)$ stabilizes M and $\mathfrak{g}$ if there is a commutative diagram of Lie algebras

$$\begin{tikzcd} & \mathfrak{g} \ltimes M \arrow[dr] & \\ M \arrow[ur] \arrow[dr] & & \mathfrak{g} \\ & \mathfrak{g} \ltimes M \arrow[uu, "\sigma"] \arrow[ur] & \end{tikzcd}$$

This is equivalent to requiring σ stabilize $M \times \{0\}$ and cosets of $M \times \{0\}$.

Prop 9. *The map* $D \mapsto \sigma_D$ *is an isomorphism* $Der(\mathfrak{g}, M) \to Aut(\mathfrak{g} \ltimes M)$*, where*

$$\sigma_D(m, g) = (m + D(g), g)$$

We observe that

$$[(m+Dg, g), (n+Dh, h)] = (g(n+Dh)-h(m+Dg), [g, h]) = (gn-hm+D[g, h], [g, h]) = \sigma_D([(m, g), (n, h)])$$

It is clear to see that $\sigma_D(m, 0) = (m, 0)$ and $\sigma_D(m, \mathfrak{g}) = (m, 0) + \sigma_D(0, \mathfrak{g})$, so $\sigma \in \mathrm{Aut}(\mathfrak{g} \ltimes M)$. Furthermore, the map $D_\sigma(g) = \pi_M \sigma(0, \mathfrak{g})$ defines a derivation, so $\sigma \mapsto D_\sigma$ is the inverse. Thus $\mathrm{Der}(\mathfrak{g}, M)$ is isomorphic to a subgroup of $\mathrm{Aut}(\mathfrak{g} \ltimes M)$, and $H^1(\mathfrak{g}, M)$ is a subquotient.

Lem 3. $H^2(\mathfrak{g}, k)$ *is ...*

The extension problem:

Def 15. *A short exact sequence*

$$0 \to M \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \to 0$$

is called an extension of Lie algebras. If $\ker(\pi)$ *is Abelian, then this is an Abelian extension. If* $[M, \mathfrak{e}] = 0$*, then the extension is called central.*

Suppose M is Abelian. Since $\operatorname{im}\iota = \ker \pi$ is a Lie ideal, we can define an action of $\mathfrak{g}$ on M by lifting the adjoint action to $\iota(M) \subset \mathfrak{e}$. Since M is Abelian, this map is well-defined on cosets.

Ex 2. *For example, consider the extension*

$$0 \to M \to \mathfrak{g} \ltimes M \to \mathfrak{g} \to 0$$

Choosing a representative $(0, g)$ *of* $g \in \mathfrak{g}$*, we have defined* $g.m = \iota^{-1}([(0, g), (m, 0)]) = \iota^{-1}((gm, 0)) = gm$*. This shows that the definition agrees with the original action of* $\mathfrak{g}$ *on* M*.*

We say that an extension splits if there exitsts a Lie algebra section of π. If $\mathfrak{e} = \mathfrak{g} \ltimes M$, then clearly $\sigma(g) = (0, g)$ is the desired section. If such σ exists, then consider the map $\psi : (g, m) \mapsto i(m) + \sigma(g)$, and we will show that this is the desired isomorphism. First, we have $e \cong \operatorname{im}\iota \oplus e \backslash \ker \pi$, and $(m, e + \ker \pi) \mapsto (m, \pi(e))$ is well-defined and inverse to ψ. Lastly, we find that

$$\begin{aligned}
[\iota(m_1) + g_1, \iota(m_2) + g_2] &= \iota([m_1, m_2]) + [\sigma(g_1), \iota(m_2)] - [\sigma(g_2), \iota(m_1)] + \sigma([g_1, g_2] \\
&= \iota(g_1 m_2 - g_2 m_1) + \sigma([g_1, g_2]) \\
&= \psi([(m_1, g_1), (m_2, g_2)])
\end{aligned}$$

This shows that the extension splits iff it is isomorphic to the semidirect product, and under this isomorphism π is the canonical projection.

In general, the following theorem establishes the relationship between the second cohomology group and the nonisomorphic extensions of a Lie algebra for which the action of the Lie algebra on M by the extension agrees with the original action of $\mathfrak{g}$ on M:

Thm 4. *There is a 1-1 correspondence between* $H^2(\mathfrak{g}, M)$ *and the isomorphism classes of such extensions.*

Since π is surjective, let σ be a section of π. Notice that for any $e \in \mathfrak{e}$, we may write $e = \sigma(\pi(e)) + i^{-1}(e - \sigma(\pi(e))) = \sigma(g) + i(m)$ where $g \in \mathfrak{g}$ and $m \in M$. Since i is injective, this decomposition is unique. Therefore

$$\begin{aligned}
[e_1, e_2] &= [\sigma(g_1), \sigma(g_2)] + [\sigma(g_1), m_2] - [\sigma(g_2), m_1] + [m_1, m_2] && (2) \\
&= \sigma([g_1, g_2]) + \chi(g_1, g_2) + \psi(g_1)m_2 - \psi(g_2)m_1 + [m_1, m_2] && (3) \\
&\mapsto ([g_1, g_2], \chi(g_1, g_2) + \psi(g_1)m_2 - \psi(g_2)m_1 + [m_1, m_2]) && (4)
\end{aligned}$$

where $\psi \in \operatorname{Der}(\mathfrak{g}, M)$ and $\chi : \mathfrak{g} \times \mathfrak{g} \to M$ via $\chi(g_1, g_2) = [\sigma(b_1), \sigma(b_2)] - \sigma([g_1, g_2])$ is a bilinear form. The problem reduces to finding (ψ, χ) that produce a valid Lie algebra structure on $\mathfrak{e}$. This is known as a factor system.

We will not give a complete proof of the theorem, but we will show how Lie algebra cohomology arises when considering central extensions by the trivial module k. In this case, $\psi = 0$ and $[m_1, m_2] = 0$, and the Lie bracket is given by $[(g_1, m_1), (g_2, m_2)] = ([g_1, g_2], \chi(g_1, g_2))$. We must find the condition on χ such that it satisfies the Jacobi identity, i.e.

$$\chi([g_1, g_2], g_3) + \chi([g_2, g_3], g_1) + \chi([g_3, g_1], g_2) = 0$$

Going back to the CE complex with coefficients in the trivial module, we can use the formula in 5 to compute $d\omega$. Since the action of the Lie group of k is trivial, the middle terms vanish, and we are left with

$$d\omega(g_1, g_2, g_3) = -\omega([g_1, g_2], g_3) + \omega([g_1, g_3], g_2) - \omega([g_2, g_3], g_1) = 0 \qquad (5)$$

We can see that this is exactly the condition placed above on χ.

Thm 5 (Whitehead Theorem)**.** *Let M be a nontrivial simple module over a semi-simple Lie algebra $\mathfrak{g}$. Then $H^n(\mathfrak{g}, M) = 0$ for all $n \geqslant 0$.*

[To be completed]

Lem 4 (First Whitehead Lemma)**.** *Let $\mathfrak{g}$ be a finite dimensional semi-simple Lie algebra and let A be a finite dimensional $\mathfrak{g}$-module. Then $H^1(\mathfrak{g}, A) = 0$.*

[To be completed]

Thm 6 (Weyl reducibility)**.** *All finite-dimensional modules over semisimple Lie algebras are completely reducible.*

Proof. Let V be a finite-dimensional module over a semisimple Lie algebra $\mathfrak{g}$, and consider a submodule $U \leqslant V$. This gives a short exact sequence

$$0 \to U \to V \to V/U \to 0$$

Which induces another exact sequence

$$0 \leftarrow \mathrm{Hom}_k(U, U) \leftarrow \mathrm{Hom}_k(V, U) \leftarrow \mathrm{Hom}_k(V/U, U) \leftarrow 0$$

We note that $\mathrm{Hom}_k(V, U)$ is a $\mathfrak{g}$-module with $(xf)(b) = xf(b) - f(bx)$. From the above sequence, we get a long exact cohomology sequence

$$0 \to H^0(\mathfrak{g}, \mathrm{Hom}_k(V/U, U)) \to H^0(\mathfrak{g}, \mathrm{Hom}_k(V, U)) \to H^0(\mathfrak{g}, \mathrm{Hom}_k(U, U)) \to H^1(\mathfrak{g}, \mathrm{Hom}_k(U/V, U))) \to \dots$$

By the first Whitehead lemma, $H^1(\mathfrak{g}, \mathrm{Hom}_k(V/U, U))$ is trivial, so we have a surjection $H^0(\mathfrak{g}, \mathrm{Hom}_k(V, U)) \to H^0(\mathfrak{g}, \mathrm{Hom}_K(U, U))$. Since $H^0(\mathfrak{g}, M) = M^{\mathfrak{g}}$ for any $\mathfrak{g}-$module M, we have $\mathrm{Hom}_{\mathfrak{g}}(V, U) \to \mathrm{End}_{\mathfrak{g}}(U)$ is surjective, so we can pull the identity map on U back to $\pi \in \mathrm{Hom}_{\mathfrak{g}}(V, U)$, which is a $\mathfrak{g}-$equivariant projection from V to U. ☐

Thm 7 (Levi - Malcev)**.** *Any finite-dimensional lLie algebra over a field of characteristic 0 is the semidirect product of its radical and a semisimple subalgebra called the Levi subalgebra.*

[To be completed]

References

[1] Juan Camilo Fiallo R. Lie algebra cohomology, 2013.

[2] José Figueroa-O'Farrill. Brst cohomology. *Lecture notes*, 2006.

[3] Werner Greub, Stephen Halperin, and Ray Vanstone. *Connections, Curvature, and Cohomology V1: De Rham cohomology of manifolds and vector bundles.* Academic press, 1972.

[4] John M Lee and John M Lee. *Smooth manifolds.* Springer, 2012.

[5] Charles A Weibel. *An introduction to homological algebra.* Number 38. Cambridge university press, 1994.