Week 1: Types

Ben Klein

October 2, 2018

1. (a) To prove these functions are extensionally equal, we must prove they produce the same output $f(n)$ for all n

   (b) Base case: $n = 1$

$$\sum_{i=1}^{1} i = \frac{1(1+1)}{2} \;= 1$$

Inductive hypothesis: assume

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Recursive step: Prove that

$$\sum_{i=1}^{n+1} i = \frac{(n+1)((n+1)+1)}{2}$$

Separating the summation term:

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i \; + (n+1)$$

Using the inductive hypothesis:

$$\begin{aligned} &= \frac{n(n+1)}{2} \; + (n+1) \\ &= \frac{n(n+1)+2(n+1)}{2} \\ &= \frac{(n+1)((n+1)+1)}{2} \end{aligned}$$

2. (a) To prove that strong induction is equivalent to the principle of induction you must prove they are extensionally equal.

3. Pierce 2.2.6
   $R' = R \cup \; \{(s,s)|s\epsilon S\}$ The definition of reflexive closure R'' is the smallest reflexive relation R on $s\epsilon S$. In other words $R'' = R \cup \; \{(s,s)|s\epsilon S\}$ Since $R'' = R'$, R' is the smallest possible relation that fits the definition.

4. Pierce 2.2.7
   Using induction to prove the rules given will produce a transitive closure on R:

Base case: a set $R_0 = \{s, t, u, (s,t), (s,u)\}$
after one recursive step, R_1:
$(s,u) \epsilon R_1, R_0 \cup R_1 =$
$R^+ = \{s, t, u, (s,t), (t,u), (s,u)\}$
This is the smallest transitive relation R.
Inductive hypothesis:

$$R^+ \text{is the smallest transitive relation of } \bigcup_i R_i$$

Recursive case: prove that

$$\bigcup_i R_{i+1} = \bigcup_i R_i$$

Because $\bigcup_i R_i$ is the smallest transitive closure, adding any transitive relation to the set will redundant because $\bigcup_i R_i$ contains all pairs that can be obtained from 1 step of transitivity. It is therefore the transitive closure.

5. Pierce 2.2.8
R^* is the smallest reflexive and transitive closure on a set S. A predicate, P is defined by a one place relation on a set S if $s \epsilon S$, and $s \epsilon P$. If P(S) is true after the binary relation R, then to check if P is preserved by R*, we must check if P(s=s) and P(s=u) is true. (By definition of reflexive and transitive) Since these are equivalence relations on the parameter of P, they must be true. $(s = s) \epsilon P, (s = u) \epsilon P$