Week 1: Formal Languages

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Chapter 1: 1, 4, 6, 10, 22, 29, 34, 38, 42, 46

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1. (a) {0,1,2,3,4,6}

(b) {2,4}

(c) {1,3}

(d) {0,6}

(e) {∅, {1},{2},{3},{4},{1,2},{2,3},{3,4},{2,4},{1,3},{1,4}, {1,2,3},{1,3,4},{1,2,4},{2,3,4},{1,2,3,4}}

4. Proof that:

$$\begin{aligned} X &= \{n^3 + 3n^2 + 3n \mid n \geq 0\} \\ &= \\ Y &= \{n^3 - 1 \mid n > 0\} \end{aligned}$$

Let $a \epsilon Y$, then $a = n^3 - 1$, and $n > 0$, and $n \epsilon \mathbb{N}$

$$a = n^3 - 1$$

Let $n =$ a natural number greater than 0 $(m + 1)$

$$\begin{aligned} a &= (m + 1)^3 - 1 \\ &= m^3 + 3m^2 + 3m + 1 - 1 \\ &= m^3 + 3m^2 + 3m \end{aligned}$$

So if $n > 0$, and $a = n^3 + 3n^2 + 3n$, then $a \epsilon X$, and $Y \subseteq X$.

Now we prove the opposite:
Let $b \epsilon X$, then $b = n^3 + 3n^2 + 3n$, and $n \geq 0$

$$\begin{aligned} b &= n^3 + 3n^2 + 3n \\ &= n^3 + 3n^2 + 3n + 1 - 1 \\ &= (n + 1)^3 - 1 \end{aligned}$$

Therefore: b is the cube of a natural number $> 0 - 1$ $(n^3 - 1 | n > 0)$ and $b \epsilon Y$ and $X \subseteq Y$
Since $X \subseteq Y$ and $Y \subseteq X$: $X = Y$

6. (a) $f(x) = 2x$

(b) $f(x) = \lfloor \frac{1}{2}x \rfloor$

(c) $f(x) = x + 1$ if $x\%2 = 0$, else if $x\%2 = 1$, $x - 1$

(d) $f(x) = \lceil \frac{1}{x} \rceil - 1$

10. Binary relations are equivalence relations if they satisfy reflexivity, symmetry, and transitivity.

Let $n = m$, and $n, m \epsilon \mathbb{N}$

Reflexivity:

$$n = n$$
$$m = m$$

Reflexivity is true.
Symmetry:

$$\text{if } n = m, \text{then } m = n$$

Symmetry is true.
Transitivity:

$$\text{if } n = a \text{ and } a = m \text{ then } n = m$$

Transitivity is true.
$\equiv$ is therefore an equivalence relation.
The equivalence classes are:

$$[n]_{\equiv} = \{m \epsilon \mathbb{N} | n \equiv m\}$$
$$[m]_{\equiv} = \{n \epsilon \mathbb{N} | m \equiv n\}$$

22. First, assume the monotone increasing functions from N x N are countable. Let $f_n(x)$ be all monotone increasing functions from N x N.

Now consider the monotone increasing function: $g(n) = f_n(n) + 1$. This must be in the list of all monotone increasing functions by definition of the list, however the function cannot match anything in the list as they differ when $f_n(n)$. Since there is a contradiction, the monotone increasing functions cannot be countable.

29. Basis: if $n, m = 0$, then $n = m$
Inductive Step: $s(n) = s(m)$
Closure: $n = m$ only if this equality can be obtained from a finite application of the inductive step.

34. Basis: $s(0) - 1 = 0$
Inductive step: $m = s(m) - 1$
Closure: m can only be obtained through finite application of the inductive step.

38. Basis: when n = 1

$$\sum_{i=1}^{1} 3i - 1 = 2 = \frac{1(3(1)+1)}{2}$$

Inductive Hypothesis: for all $k = 1, 2...n$:

$$\sum_{i=1}^{k} 3i - 1 = \frac{k(3k+1)}{2}$$

Inductive step: the goal is to prove using the inductive hypothesis that

$$\begin{aligned}\sum_{i=1}^{n+1} 3i - 1 &= \frac{(n+1)(3(n+1)+1)}{2} \\ &= \frac{(n+1)(3n+4)}{2}\end{aligned}$$

First, break the summation into 2 parts:

$$\sum_{i=1}^{n+1} 3i - 1 = \sum_{i=1}^{n} 3i - 1 + (3(n+1) - 1)$$

then using the inductive hypothesis:

$$\begin{aligned}&= \frac{(n)(3(n)+1)}{2} + (3(n+1) - 1) \\ &= \frac{3n^2+n}{2} + 3n + 2 \\ &= \frac{3n^2+n+6n+4}{2} \\ &= \frac{3n^2+7n+4}{2} \\ &= \frac{(n+1)(3n+4)}{2}\end{aligned}$$

This proves the inductive hypothesis is true.

42. (a) $E_0 = \{A, B\}$
$E_1 = \{(A \wedge B), (A \vee B)\}$
$E_2 = \{((A \wedge B) \vee (A \vee B)), ((A \wedge B) \wedge (A \vee B)), ((A \wedge B) \wedge A)((A \wedge B) \vee A), ((A \vee B) \wedge A)((A \vee B) \vee A),$
$((A \wedge B) \wedge B)((A \wedge B) \vee B), ((A \vee B) \wedge B), ((A \vee B) \vee B)\}$

(b) Basis:

$$\begin{aligned}n_p((u \vee v)) &= n_p((u \wedge v)) \\ n_p((u \vee v)) &= 2 \\ n_o((u \vee v)) &= 1\end{aligned}$$

So, in the base case, $n_p(u) = n_o(u) + 1$
Inductive hypothesis: assume for all expressions $k = u_i, i = 1, 2...$

$$n_p(k) = n_o(k) + 1$$

Inductive step: prove that

$$n_p((u \vee v) \vee w)) = n_o((u \vee v) \vee w)) + 1$$

for all w.

(c) Base case: $i = 0$
$n_L(u), n_R(u)$ means number of right and left parenthesis in an expression u
$n_L(E_0) = 0 = n_R(E_0)$ We see in the base case: $\{A, B\}$, there are no parentheses therefore the number of parentheses is the same.

Hypothesis: $k = 1, 2...$

$$n_L(E_k) = n_R(E_k)$$

Prove:

$$n_L(E_{i+1}) = n_R(E_{i+1})$$

There are 2 cases when applying the recursive step, either you add 1 left parenthesis and 1 right parenthesis by forming $(u \vee v)$ or you create $(u \wedge v)$ and also add 1 parenthesis on both sides. So,

$$n_L(E_{i+1}) = n_L(E_i) + 1 = n_R(E_{i+1}) = n_R(E_i) + 1$$

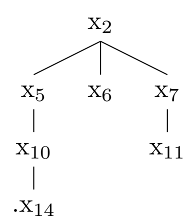
Therefore the number of parentheses will remain the same.

46. (a) 4

(b) x_{11}, x_7, x_2, x_1

(c) x_2, x_1

(d)



(e) $x_{14}, x_6, x_{11}, x_3, x_8, x_{12}, x_{15}, x_{16}$