MATH2320 Assignment 3

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Due: 5pm Tuesday 16th August 2016.

If you have any questions about this assignment, please see or email your lecturer for this course, Dr Daniel Sutherland Daniel.Sutherland@newcastle.edu.au.

Questions which review material from Week 2:

1. Let $P_3(\mathbb{C})$ be the vector space of complex polynomials with degree less than or equal to 3.

a) Show that the list of vectors $B = (x-1, (x-1)^2)$ is linearly independent.
b) Show that B does not span $P_3(\mathbb{C})$ by giving an example of a vector in $P_3(\mathbb{C})$ that does not belong to the span of B.
c) Denote the vector that you found in the previous part by p, and create a new list of vectors $B' = (x-1, (x-1)^2, p)$. Is B' linearly independent? Is B' a basis for $P_3(\mathbb{C})$? *Why/Why not?*

Answer First, let's rewrite B in terms of the coeffiecients a_n of the x^n terms. We can express B as

$$B := \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Nothing we can do to the first vector will give us 1 as the x^2 coefficient. Therefore, the two vectors are independent. We cannot create a new vector $p = (0,0,0,1)$ from any combination of the two vectors in B. Therefore, the vectors in B do not span $P_3(\mathbb{C})$. If we add this vector p to the set B, the resulting set B' must still be linearly independent, because we have added a vector that was not in the span of B, whose vectors we have previously shown to be linearly independent. However, B' still does not span $P_3(\mathbb{C})$ because there is not a combination of vectors in B' that will give us the vector $v = (1,0,0,0)$. Therefore B' is not a basis for $P_3(\mathbb{C})$.

2. Consider the following subspaces of $\mathbb{F}_4$:

$$V_1 = span\left((1,-1,0,0),(1,0,0,-1)\right) \text{ and } V_2 = span\left((0,1,0,-1),(0,-1,0,1)\right)$$

a) What are the dimensions of these subspaces?

b) Show that V_1 spans V_2. Hence, what is $V_1 \cap V_2$?

c) Use the below theorem together with your previous answers to determine $\dim(V_1 + V_2)$.

Theorem 2.18: *If U_1 and U_2 are subspaces of a nite-dimensional vector space, then*

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

Answer We can calculate the dimension of each subsapce by finding the reduced-row echelon form of the matrices formed by each collection of vectors. For V_1:

$$V_1 := \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \xrightarrow{R_2^*=R_2+R_1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \xrightarrow{R_4^*=R_4+R_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1^*=R_1-R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The dimension of V_1 is equal to the number of pivot variables, therefore $\dim V_1 = 2$. Now for V_2:

$$V_2 := \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix} \xrightarrow{swap\ R_1\ and\ R_2} \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \\ -1 & 1 \end{bmatrix} \xrightarrow{R_4^*=R_4+R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, $\dim V_2 = 1$

We can rewrite the vectors in V_2 using the vectors in V_1:

$$\begin{aligned} (0,1,0,-1) &= (1,0,0,-1) - (1,-1,0,0) \\ (0,-1,0,1) &= (1,-1,0,0) - (1,0,0,-1) \end{aligned}.$$

Therefore, V_1 spans V_2 and $V_1 \cap V_2 = V_2$. Using this answer, we can say that:

$$\begin{aligned} \dim(V_1 + V_2) &= \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2) \\ &= \dim V_1 + \dim V_2 - \dim(V_2) \\ &= \dim V_1 \\ &= 2 \end{aligned}$$

Introductory level question relating to Week 3:

3. Consider the map $T : \mathbb{R}^2 \implies \mathbb{R}$ given by the rule $T(x, y) = 2x + 3y$.

(a) Show that T is a linear map.

(b) Find the null space of T , and sketch this space.

(c) Is the linear map T injective? *Why/Why not?*

Hint: Is there a result from the textbook that might help you?

Answer: We need to show that T is closed under additivity and homogeneity. First, Define two vectors $u := (x_1, y_1)$ and $v := (x_2, y_2)$. Then:

$$\begin{aligned} T(u) + T(v) &= 2x_1 + 3y_1 + 2x_2 + 3y_2 \\ &= 2(x_1 + x_2) + 3(y_1 + y_2) \\ &= T(u + v) \end{aligned}$$

Therefore, T is closed under additivity. Next, define a vector $w := (x, y) \implies \forall a \in \mathbb{F}, aw := (ax, ay)$. Then,

$$\begin{aligned} T(aw) &= 2ax + 3ay \\ &= a(2x + 3y). \\ &= aT(w) \end{aligned}$$

Therefore, T is closed under homogeity. Next we define T as a matrix A such that

$$A := \begin{bmatrix} 2 \\ 3 \end{bmatrix} \implies \forall v := (x, y) \in \mathbb{R}^2, T(v) = Av = (x, y) \times \begin{bmatrix} 2 \\ 3 \end{bmatrix} = (2x + 3y) \in \mathbb{R}.$$

We know that in order for a vector v to be an element of *null* T, $Av = 0$. It is obvious that the only vector $v = (x, y)$ that satisfies this is the zero vector. Therefore, *null* $T = \{\vec{0}\}$. Lastly - by Proposition 3.2 in the set text - because the kernel of T only contains $\vec{0}$, T must be injective.

Extension question - will not be marked, just for interest!

4. Define a collection of "forward shift" maps $T_\lambda : \mathbb{F}^\infty \implies \mathbb{F}^\infty$ for $\lambda \in \mathbb{F}$ by the rule

$$T_\lambda(x_1, x_2, x_3, ...) = (\lambda, x_1, x_2, ...).$$

For what value(s) of λ is T_λ a linear map? For these value(s) of λ, is T_λ injective? surjective?