Week8

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Questions which review material from Week 6:

1. Consider the operator $T \in \mathcal{L}(\mathbb{C}^4)$, whose matrix representation with respect to the standard basis $\mathcal{B}$ of $\mathbb{C}^4$ is:

$$M(T, \mathcal{B}) = \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 3 & 3 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Define the vectors $v_1 = (1, 0, 0, 0), v_2 = (0, -3, 1, 0), v_3 = (0, 2, 0, 0)$ and $v_4 = (2, 0, 0, 1)$, and form the subspaces $U_1 = \text{span}(v_1, v_2)$ and $U_2 = span(v_3, v_4)$. Let $\lambda_1 = 2$ and $\lambda_2 = 3$.

(a) Show that v_1 and v_2 are generalised eigenvectors corresponding to the eigenvalue λ_1. Show that v_3 and v_4 are generalised eigenvectors corresponding to the eigenvalue λ_2.

Answer: First, let's find $(M - \lambda I)$ for each eigenvalue and then solve for the corresponding eigenvector. For $\lambda_1 = 2$, we get:

$$(M - 2I) = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We can row-reduce this to:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which implies that $x_2 = x_3 = x_4 = 0$ Therefore, the eigenvector corresponding to $\lambda_1 = 2$ is $(1, 0, 0, 0) = v_1$

For $\lambda_2 = 3$, we get:

$$(M - 3I) = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We can row-reduce this to:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which implies that $x_1 = x_3 = x_4 = 0$ Therefore, the eigenvector corresponding to $\lambda_2 = 3$ is $(0, 1, 0, 0) = \frac{1}{2}v_3$

Next, let's calculate the generalised eigenvector for $\lambda_1 = 2$. We can do this by solving:

$$\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

This system of equations shows that $x_4 = 0 \implies x_3 = 1 \implies x_2 = -3x_3 = -3$. Therefore, the generalised eigenvector for $\lambda_1 = 2$ is $(0, -3, 1, 0) = v_2$ Repeating the process for $\lambda_2 = 3$, we get:

$$\begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

This system of equations shows that $x_3 = 0 \implies 4x_4 = 2 \implies x_4 = \frac{1}{2} \implies x_1 = 2x_2 = 1$. Therefore, the generalised eigenvector for $\lambda_2 = 3$ is $(1, 0, 0, \frac{1}{2}) = \frac{1}{2}v_4$.

(b) Show that $\mathcal{B}' = (v_1, v_2, v_3, v_4)$ is a basis of $\mathbb{C}^4$ consisting of generalised eigenvectors of T .

Answer: We define:

$$M(\mathcal{B}') = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3^* = R_3 + 1/3 * R_2} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2/3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1^* = R_1 - 2 * R_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2/3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\implies \xrightarrow{R_3^* = R_3 - R_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2^* = R_2 - 3/2 R_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

From here, we can easily reduce to the standard basis of $\mathbb{C}^4$. Therefore, $\mathcal{B}'$ is a basis of $\mathbb{C}^4$.

(c) Calculate the matrix representation of T with respect to the basis $\mathcal{B}'$.

Answer:

$$\begin{aligned} Tv_1 &= T(1, 0, 0, 0) = (2, 0, 0, 0) = 2v_1 \\ Tv_2 &= T(0, -3, 1, 0) = (1, -6, 1, 0) = v_1 + v_3 - 3/2\, v_2 \\ Tv_3 &= T(0, 2, 0, 0) = (0, 6, 0, 0) = 3v_3 \\ Tv_4 &= T(2, 0, 0, 1) = (5, 4, 0, 3) = 2v_3 + 3v_4 - v_1 \end{aligned}$$

Therefore,

$$M(T, \mathcal{B}') = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 0 & -3/2 & 0 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \xrightarrow{R_3^* = R_3 + 2/3 R_2} \begin{pmatrix} 2 & 1 & 0 & -1 \\ 0 & -3/2 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \xrightarrow{R_1^* = R_1 + 1/3 R_4} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & -3/2 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

2. Let $T \in \mathcal{L}(\mathbb{C}^4)$ be the operator from Question 1, and define the same vectors, subspaces and bases as before.

(a) The matrix representation of T with respect to the basis $\mathcal{B}'$ that you calculated in Question 1(c) should be a block diagonal matrix of the form

$$M(T, \mathcal{B}') = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where the block matrices A_1 and A_2 are upper-triangular. State A_1 and A_2.

Answer: We have:

$$A_1 = \begin{pmatrix} 2 & 1 \\ 0 & -3/2 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$$

(b) The block matrix A_1 corresponds to the matrix representation of the operator $T|_{U_1}$ with respect to the basis (v_1, v_2) of U_1. Similarly, the block matrix A_2 corresponds to the matrix representation of the operator $T|_{U_2}$ with respect to the basis (v_3, v_4) of U_2.
Define the operators N_1 and N_2 to be $(T - \lambda_1 I)|_{U_1}$ and $(T - \lambda_2 I)|_{U_2}$ respectively. Write down the matrix representation of N_1 with respect to the basis (v_1, v_2) of U_1, and the matrix representation of N_2 with respect to the basis (v_3, v_4) of U_2.

Answer: We have:

$$N_1 = (T - 2I)|_{U_1} = \begin{pmatrix} 0 & 1 \\ 0 & -7/2 \end{pmatrix} \xrightarrow{R_2^* = R_2 + 7/2 R_1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$N_2 = (T - 3I)|_{U_2} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

(c) Verify that the operators N_1 and N_2 are nilpotent.

Answer: It is easy to see that:

$$N_1^2 = N_2^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Therefore, N_1 and N_2 are nilpotent.

Introductory level question relating to Week 7:

3. Let $a, b, c \in \mathbb{C}$ be nonzero scalars with $a \neq b$. Consider the operator $T \in \mathcal{L}(\mathbb{C}^5)$, whose matrix representation with respect to some basis of $\mathbb{C}^5$ is:

$$M(T) = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & b & c \\ 0 & 0 & 0 & 0 & b \end{pmatrix}$$

(a) What is the characteristic polynomial, $q(z)$, of the operator T?

Answer: Because this matrix is upper-triangular, we can basically read the characteristic polynomial off of the main diagonal. In this case $q(z) = (a - z)^3(b - z)^2$.

(b) List all the monic polynomials (of degree at least 1) that divide $q(z)$.

Answer: $(a-z)^3, (a-z)^2, (a-z), (b-z)^2$ and $(b-z)$ are all monic polynomials that divide $q(z)$. So are $(a-z)^3(b-z), (a-z)^3(b-z)^2, (a-z)^2(b-z)^2, (a-z)(b-z)$ and $(a-z)(b-z)^2$.

(c) Which of your polynomials from (b) have roots at each distinct eigenvalue of T?

Answer: $(a-z)^3, (a-z)^2, (a-z)$ and $(a-z)^3(b-z), (a-z)^2(b-z), (a-z)^2(b-z)^2, (a-z)(b-z)$ all have a root at $\lambda_1 = a$. $(b-z)^2, (b-z)$; $(a-z)^3(b-z), (a-z)^2(b-z), (a-z)^2(b-z)^2, (a-z)(b-z)$ and $(b-z)^2, (b-z)$ all have a root at $\lambda_2 = b$.

(d) Hence determine the minimal polynomial, $p(z)$, of the operator T by showing $p(T) = 0$.

Answer: The minimimal polynomial is therefore $p(z) = (a-z)(b-z)$.

Submitting your assignment (due 5pm Tuesday 13th September 2016)

Submit your assignment in hardcopy in your Demonstrator's pigeonhole in the Assignment boxes near the Maths Clinic, on the opposite wall to the Maths Clinic, left of the door to v09. We also ask that you scan your written work and submit it on the MATH2320 UoNline/Blackboard site as a backup and proof of submission, not as a substitute. ***Note that we still require the hardcopy submitted in the Assignment Box for your Demonstrator to mark, even if you have submitted a backup on Blackboard.***