MATH2320 Assignment 4

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Due: 5pm Tuesday 23rd August 2016.

If you have any questions about this assignment, please see or email your lecturer for this course, Dr Daniel Sutherland Daniel.Sutherland@newcastle.edu.au.

Questions which review material from Week 3:

1. Consider the system of simultaneous linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad\qquad\qquad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

for constants a_{ij} and unknowns x_j, in the case where $m < n$ (i.e. where there are more unknowns than equations).

Use some results from the course so far to prove that there must exist in nitely many solutions to this system of equations.

Hint: There's a vague hint hidden in one of the lecture recordings!

Answer: The system of equations defined above are **homogenous**: therefore, there exists *at least one* solution to the system. Next, if we define an $m \times n$ matrix $\mathcal{A}$ as the matrix consisting of the of all the coefficients $a_{ij} : 1 \leq i \leq m, 1 \leq j \leq n, m < n$, then by definition $\mathcal{A}$ is not a square matrix. Therefore, $det\mathcal{A} = 0$. The determinant of some matrix $\mathcal{M}$ defined as the coefficient matrix of some system of linear equations *determines* whether or not there exists a unique solution to that system: if $det\mathcal{M} \neq 0$ then there exists a unique solution to the system; if $det\mathcal{M} = 0$ then there are either *infinitely* many solutions or *none* whatsoever. We have already shown that at least one solution exists for the system given. Therefore, because $det\mathcal{A} = 0$, there must be infinitely many solutions.

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2. Recall that for $T \in \mathcal{L}(V, W)$, where V has basis $B = (v_1, \cdots, v_n)$ and where W has basis $B' = (w_1, \cdots, w_m)$, that the matrix representation of T is the matrix of column vectors:

$$\mathcal{M}(T) = ([T_{v1}]'_B \quad [T_{v2}]'_B \ \cdots \ [T_{vn}]'_B)$$

Define the linear map $T \in \mathcal{L}(\mathcal{P}_1(\mathbb{R}), \mathbb{R})$ by

$$T(p(x)) = \int_0^1 p(x)dx$$

to be definite integration over the interval $[0, 1]$.

(a) Find $\mathcal{M}(T)$ with respect to the standard bases $\mathcal{B} = (1, x)$ and $\mathcal{B}' = (1)$ of $\mathcal{P}_1(\mathbb{R})$ and $\mathbb{R}$.

Answer: Suppose we have some vector $\vec{b} = [c_1\ c_2]^T \in span\, B$. The linear map $\mathcal{M}(T)$ that returns the definite integral of b over the range $[0, 1]$ is:

$$\mathcal{M}(T) := \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} * \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 + \frac{c_2}{2} \end{bmatrix}$$

(b) Use $\mathcal{M}(T)$ to find $\int_0^1 (3 + 5x)dx$ through matrix multiplication. Afterwards, check that your result agrees with what you get by directly doing the integration.

Answer: We can verify this by computing the multiplication of the vector $\vec{x} = (3, 5)$ and the matrix $\mathcal{M}(T)$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} * \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 + \frac{5}{2} \end{bmatrix} = \frac{11}{2}$$

which is what we would expect to see when evaluating the definite integral. We can confirm this analytically:

$$T((3 + 5x)) = \int_0^1 (3 + 5x)dx = \left[3x + \frac{5}{2}x^2\right]_0^1 = \left[3 + \frac{5}{2} - 0\right] = \frac{11}{2}$$

Introductory level question relating to Week 4:

3. Consider the linear operator $T \in \mathcal{L}(\mathbb{R}^2)$ defined by

$$T(x, y) = (3x + 2y, 3x - 2y).$$

(a) Either show that this operator is injective or surjective – your choice!

Answer: First, let's rewrite the operator as the combination of a square (2×2) matrix and a vector:

$$T(x, y) = \mathcal{A}\vec{x} = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x + 2y \\ 3x - 2y \end{bmatrix}$$

From this we can see that $det\mathcal{A} = -12 \neq 0$. Therefore, the matrix $\mathcal{A}$ is invertible. Now, Proposition 3.17 in the course notes states that if a matrix is invertible iff it is bijective. Therefore, because $\mathcal{A}$ in invertible, it is injective and surjective – your choice.

(b) Use a result from the course to conclude that T must be an invertible operator.

Answer: See above.

(c) Show that the one-dimensional subspace $U = span((1, -3))$ is invariant under T.

Answer: First, note that $U = span((1, -3)) \equiv \{au : a \in \mathbb{R}, u = (1, -3) \in U\}$. This gives us:

$$\begin{aligned} T(a, -3a) &= (3 * (a) + 2 * (-3a), 3 * (a) - 2 * (-3a)) \\ &= (3a + (-6a), 3a - (-6a)) \\ &= (-3a, 9a) \\ &= -3a(1, -3) \\ &\in span((1, -3)) \end{aligned}$$

which shows that $\forall\, au : a \in \mathbb{R}, u \in U\, \{T(au) = (-3)au\}$, i.e. a scalar multiple of au. Therefore, U is invariant under T.

Extension question - will not be marked, just for interest!

3. (d) Find another one-dimensional subspace $V \neq U$ which is also invariant under T. Sketch both subspaces U and V, and show that $U \oplus V = \mathbb{R}^2$.
   (e) For every non-zero choice of $u \in U$ and $v \in V$, explain why $B = (u, v)$ is a basis of $\mathbb{R}^2$.
   (f) Pick non-zero $u \in U$ and $v \in V$. Find $\mathcal{M}(T)$ with respect to the bases $B = B' = (u, v)$.

Submitting your assignment (due 5pm Tuesday 23rd August 2016) *Submit your assignment in hardcopy in your Demonstrator's pigeonhole in the Assignment boxes near the Maths Clinic, on the opposite wall to the Maths Clinic, left of the door to V109. We also ask that you scan your written work and submit it on the MATH2320 UoNline/Blackboard site as a backup and proof of submission, not as a substitute.* ***Note that we still require the hardcopy submitted in the Assignment Box for your Demonstrator to mark, even if you have submitted a backup on Blackboard.***