

Assignment 2

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Question 1

Let $\{X_1, X_2, \dots, X_n\}$ be a stationary time series. Verify that the sample autocovariance function $\hat{\gamma}_X(h)$, defined on Slide 39, is a non-negative definite function.

Hint There are two simple elegant solutions to this question in the article

A.I. McLeod and C. Jimenez, Nonnegative Definiteness of the Sample Autocovariance Function, The American Statistician, 38(4):297-298, 1984.

Read those two methods carefully and rewrite one of them in your own wording. If you present a novel method different from those two, you will get bonus mark.

Answer: Working from the first example. If we take the sample mean to be $\bar{x} = 0$ to simplify the problem without losing generalisation, we can define the auto-correlation matrix of X_t by this process:

If we define a vector $Z = [z_1, z_2, z_3, \dots]$ as the vector of time-series sample data, then auto-correlation matrix of that data can be written as:

$$C = Z'Z$$

Now, we know that any matrix Y is positive semi-definite if there exists no vector \vec{u} such that $u'Yu < 0$. Now we can prove that C must be non-negative definite by contradiction. Suppose that the auto-correlation matrix C is not non-negative definite. Then $\exists \vec{\alpha} : \alpha' C \alpha < 0$.

But we know we can rewrite the above equation as:

$$(\alpha' C \alpha) = (\alpha' Z' Z \alpha) = (Z \alpha)' (Z \alpha) = \beta_1^2 + \beta_2^2 \dots$$

, where $\beta = Z \alpha$. Therefore, $Z'Z$ contains the inner-product of all the columns in Z . Also, $\alpha' C \alpha$ must be a sum of squares and therefore $\alpha' C \alpha \geq 0$, which is a contradiction. We can extend this to any matrix that can be written in the form $\alpha' Z' Z \alpha$. So, by contradiction, we have shown that C must be non-negative definite. It follows that C_n must be the sum of $\{n\}$ non-negative definite matrices, which is trivially non-negative definite. Therefore, sample autocovariance matrix - and by extension the sample auto-covariance function - must be non-negative definite.

Bonus Question It should be clear from the discussion that a strictly stationary, finite variance, time series is also stationary. However, the converse may or may not be true, in general.

A time series $\{X_t; t = 0, \pm 1, \pm 2, \dots\}$ is said to be a Gaussian process, if the n -dimensional random vectors $X = (X_{t_1}, X_{t_2}, \dots, X_{t_n})'$, for every collection of time points t_1, t_2, \dots, t_n , and every positive integer n , have a Multivariate Normal distribution.

Show that if the time series $\{X_t; t = 0, \pm 1, \pm 2, \dots\}$ is a stationary Gaussian process, then it is strictly stationary, as well.

Answer: We are given that the $\{X_t; t = 0, \pm 1, \pm 2, \dots\}$ is a weakly stationary time-series. Therefore, we know that $\mu_t = \mu \forall t \in \{0, \pm 1, \pm 2, \dots\}$. We also know that the autocovariance of the time-series depends only on the lag between any two time-points $\forall s, t \in \mathbb{Z} \implies \gamma(x_s, x_t) = \gamma(|s - t|)$. Therefore, we know that the mean and autocovariance are not dependent on time, only on the lag between time-points. Now, recall the Multivariate Normal Distribution from Assignment 1.

$$f_X(x) = \frac{1}{\sqrt{(2\pi)^n |\mathbf{\Gamma}|}} \exp\left(-\frac{1}{2}(x - \mu)^T \mathbf{\Gamma}^{-1}(x - \mu)\right)$$

It is clear here that neither μ or the covariance matrix $\mathbf{\Gamma}$ depend on the time t . Therefore, all probability distributions of the Gaussian process X_t only depend on the lag between time points. By extension the CDFs

$$Pr(X_{t_1} \leq c_1, \dots, X_{t_n} \leq c_n) = Pr(X_{t_1+h} \leq c_1, \dots, X_{t_n+h} \leq c_n),$$

which - using the result on slide 27 of the Chapter 1 Lecture Notes - implies that X_t is *strictly* stationary.