Assignment 2

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Question 1

Let $\{X_1, X_2, ..., X_n\}$ be a stationary time series. Verify that the sample autocovariance function $\hat{\gamma}_X(h)$, defined on Slide 39, is a non-negative definite function.

Hint There are two simple elegant solutions to this question in the article

> A.I. McLeod and C. Jimen´ez, Nonnegative Definiteness of the Sample Autocovariance Function, The American Statistician, 38(4):297-298, 1984.

Read those two methods carefully and rewrite one of them in your own wording. If you present a novel method different from those two, you will get bonus mark.

Answer: Working from the first example. If we take the sample mean to be $\bar{x} = 0$ to simplify the problem without losing generalisation, we can define the auto-correlation matrix of X_t by this process:

If we define a vector $Z = [z_1, z_2, z_3, ...]$ as the vector of time-series sample data, then auto-correlation matrix of that data can be written as:

$$C = Z'Z$$

Now, we know that any matrix Y is positive semi-definite if there exists no vector $\vec{u}$ such that $u'Yu < 0$. Now we can prove that C must be non-negative definite by contradiction. Suppose that the auto-correlation matrix C is not non-negative definite. Then $\exists \vec{\alpha} : \alpha' C \alpha < 0$.

But we know we can rewrite the above equation as:

$$(\alpha' C \alpha) = (\alpha' Z' Z \alpha) = (Z\alpha)'(Z\alpha) = \beta_1^2 + \beta_2^2 \cdots$$

, where $\beta = Z\alpha$. Therefore, $Z'Z$ contains the inner-product of all the columns in Z. Also, $\alpha' C \alpha$ must be a sum of squares and therefore $\alpha' C \alpha \geq 0$, which is a contradiction. We can extend this to any matrix that can be written in the form $\alpha' Z' Z \alpha$. So, by contradiction, we have shown that C must be non-negative definite. It follows that C_n must be the sum of {n} non-negative definite matrices, which is trivially non-negative definite. Therefore, sample autocovariance matrix - and by extension the sample auto-covariance function - must be non-negative definite.

Bonus Question It should be clear from the discussion that a strictly stationary, finite variance, time series is also stationary. However, the converse may or may not be true, in general.

A time series $\{X_t; t = 0, \pm 1, \pm 2, ...\}$ is said to be a Gaussian process, if the n-dimensional random vectors $X = (X_{t_1}, X_{t_2}, ..., X_{t_n})$ 0, for every collection of time points $t_1, t_2, ..., t_n$, and every positive integer n, have a Multivariate Normal distribution.

Show that if the time series $\{X_t; t = 0, \pm 1, \pm 2, ...\}$ is a stationary Gaussian process, then it is strictly stationary, as well.

Answer: We are given that the $\{X_t; t = 0, \pm 1, \pm 2, ...\}$ is a weakly stationary time-series. Therefore, we know that $\mu_t = \mu \, \forall t \in \{0, \pm 1, \pm 2, ...\}$. We also know that the autocovariance of the time-series depends only on the lag between any two time-points $\forall s, t \in \mathbb{Z} \implies \gamma(x_s, x_t) = \gamma(\mid s - t \mid)$. Therefore, we know that the mean and autocovariance are not dependent on time, only on the lag between time-points. Now, recall the Multivariate Normal Distribution from Assignment 1.

$$f_X(x) = \frac{1}{\sqrt{(2\pi)^n |\mathbf{\Gamma}|}} \exp\left(-\frac{1}{2}(x - \mu)^T \mathbf{\Gamma}^{-1}(x - \mu)\right)$$

It is clear here that neither μ or the covariance matrix Γ depend on the time t. Therefore, all probability distributions of the Gaussian proces X_t only depend on the lag between time points. By extension the CDFs

$$Pr(X_{t_1} \leq c_1, \cdots, X_{t_n} \leq c_n) = Pr(X_{t_{1+h}} \leq c_1, \cdots, X_{t_{n+h}} \leq c_n),$$

which - using the result on slide 27 of the Chapter 1 Lecture Notes - implies that X_t is *strictly* stationary.