STAT3040 Assignment 1

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Due: 9am Thursday 11th August 2016.

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1. In general, if two random variables are **uncorrelated**, they **may** or **may not** be **independent**.

   Show that if two random variables $X \sim N[\mu_X, \sigma_X^2]$ and $Y \sim N[\mu_Y, \sigma_Y^2]$ are **uncorrelated**, then they are **independent** as well.

   Answer: If two random variables X and Y are uncorrelated, then the following is true:

   $$\rho_{XY}(X,Y) = \frac{COV(X,Y)}{\sigma_X \sigma_Y} = 0$$

   . The joint distribution of two Normal random variables X and Y is given by:

   $$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right)\right\}$$

   If we sub in $\rho = 0$ into this distribution, we get the following:

   $$\begin{aligned} f_{XY}(x,y) &= \frac{1}{2\pi\sigma_X\sigma_Y} \exp\left\{-\frac{1}{2}\left(\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right)\right\} \\ &= \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left\{-\frac{1}{2}\left(\frac{(x-\mu_X)^2}{\sigma_X^2}\right)\right\} + \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left\{-\frac{1}{2}\left(\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right)\right\} \\ &= f_X(x) f_Y(y) \end{aligned}$$

   This shows that the joint pdf of X and Y - $f_{XY}(x,y)$ - is equal to the product of the marginal pdfs of X and Y - $f_X(x)f_Y(y)$. Therefore, the two variables are independent.

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2. Show that the **joint pdf** of a **Multivariate Normal** distribution with $n = 2$ can be simplified to the **joint pdf** of a **Bivariate Normal** distribution.

   Answer: The joint pdf of a Multivariate Normal distribution is given by:

   $$f_X(x) = \frac{1}{\sqrt{(2\pi)^n|\mathbf{\Gamma}|}} \exp\left(-\frac{1}{2}(x-\mu_X)^T\mathbf{\Gamma}^{-1}(x-\mu_X)\right),$$

   where $X := (X_1, X_2, ..., X_n)'$, $x := (x_1, x_2, ..., x_n)'$ and $\mu := (\mu_1, \mu_2, ..., \mu_n)'$. When $n = 2$, the Covariance Matrix Γ is given by:

   $$\begin{aligned} \Gamma &= \begin{bmatrix} \sigma_{X_1}^2 & \rho\sigma_{X_1}\sigma_{X_2} \\ \rho\sigma_{X_1}\sigma_{X_2} & \sigma_{X_2}^2 \end{bmatrix} \\ \implies \Gamma^{-1} &= \frac{1}{\sigma_{X_1}^2\sigma_{X_2}^2(1-\rho^2)} \begin{bmatrix} \sigma_{X_2}^2 & -\rho\sigma_{X_1}\sigma_{X_2} \\ -\rho\sigma_{X_1}\sigma_{X_2} & \sigma_{X_1}^2 \end{bmatrix} \\ &= \frac{1}{(1-\rho^2)} \begin{bmatrix} \frac{1}{\sigma_{X_1}^2} & -\frac{\rho}{\sigma_{X_1}\sigma_{X_2}} \\ -\frac{\rho}{\sigma_{X_1}\sigma_{X_2}} & \frac{1}{\sigma_{X_2}^2} \end{bmatrix} \\ \implies det(\Gamma) &= \sigma_{X_1}^2\sigma_{X_2}^2(1-\rho^2) \end{aligned}$$

If we sub these values into the joint pdf of the Multivariate Normal distribution with $n = 2$, we get:

$$
\begin{aligned}
f_X(x) &= \frac{1}{2\pi\sqrt{\sigma_{X_1}^2\sigma_{X_2}^2(1-\rho^2)}} \\
&\quad\times \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\begin{pmatrix} x_1-\mu_{X_1} \\ x_2-\mu_{X_2}\end{pmatrix}^T \begin{bmatrix} \frac{1}{\sigma_{X_1}^2} & -\frac{\rho}{\sigma_{X_1}\sigma_{X_2}} \\ -\frac{\rho}{\sigma_{X_1}\sigma_{X_2}} & \frac{1}{\sigma_{X_2}^2}\end{bmatrix}\begin{pmatrix} x_1-\mu_{X_1} \\ x_2-\mu_{X_2}\end{pmatrix}\right)\right\} \\
&= \frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_{X_2})^2}{\sigma_{X_1}^2} - \frac{2\rho(x_1-\mu_{X_1})(x_2-\mu_{X_2})}{\sigma_{X_1}\sigma_{X_2}} + \frac{(x_2-\mu_{X_2})^2}{\sigma_{X_2}^2}\right)\right\}
\end{aligned},
$$

which is equivalent to the bivariate Normal Distribution stated in Question 1.

3. **Bonus Question** Consider to random variables X and Y with a completely **linear** relationship, that is

$$Y = aX + b,$$

for constants $a, b \in \mathbb{R}$ and $a \neq 0$. Show that if $a > 0$, then $\rho(X,Y) = 1$ and if $a < 0$, then $\rho(X,Y) = -1$.

Answer. We know that $\rho(X,Y) = \frac{COV(X,Y)}{\sigma_X\sigma_Y}$ and that $COV(X,Y) = E[XY] - E[X]E[Y]$. Now let's determine every value for Y in terms of X via the linear relationship given in the question.

$$
\begin{aligned}
E[Y] &= E[aX+b] = aE[X] + b \\
\implies E[X]E[Y] &= E[X](aE[X]+b) = aE[X]^2 + bE[X] \\
E[XY] &= E[X(aX+b)] = E[aX^2+bX] = aE[X^2] + bE[X] \\
Var(Y) &= E[Y^2] - E[Y]^2 = E[(aX+b)^2] - E[aX+b]^2 \\
&= a^2E[X^2] + 2abE[X] + b^2 - (aE[X]+b)^2 \\
&= a^2E[X^2] + 2abE[X] + b^2 - a^2E[X]^2 - 2abE[X] - b^2 \\
&= a^2\sigma_X^2 \\
\implies \sigma_Y &= \|a\sigma_X\|
\end{aligned}
$$

Now we can rewrite the covariance term as:

$$
\begin{aligned}
COV(X,Y) &= aE[X^2] + bE[X] - \big(aE[X]^2 + bE[X]\big) \\
&= aE[X^2] + bE[X] - aE[X]^2 - bE[X] \\
&= aE[X^2] - aE[X]^2 \\
&= a\sigma_X^2
\end{aligned}
$$

Now, if we sub both of these values into the Correlation equation given earlier, we get:

$$
\begin{aligned}
\rho(X,Y) &= \frac{a\sigma_X^2}{\|a\sigma_X^2\|} \\
&= \frac{a}{\|a\|}
\end{aligned},
$$

which, when $a > 0$ will result in $\rho(X,Y) = 1$ but when $a < 0$, $\rho(X,Y) = -1$.