Assignment 2

2017-05-04

Question 1: Inference for the Poisson parameter λ:

(a) Given a general Gamma(a,b) prior, derive the posterior distribution of λ, given data $(x_1, \ldots, x_n)$, followed by the posterior predictive distribution of $(z_1, \ldots, z_m)$.

The Likelihood is given by:

$$L(\lambda|x) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod_{i=1}^{n}(x_i!)}$$

The Prior is a $Gamma(a,b)$:

$$p(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}, \lambda > 0$$

So the posterior is given by:

$$p(\lambda|x) \propto \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\sum x_i+\alpha-1}e^{-(n+\beta)\lambda}, \lambda > 0,$$

which is equivalent to a $Gamma(\sum x_i + \alpha, n+\beta)$, or $Gamma(n\bar{x} + \alpha, n+\beta)$.

The *Posterior Predictive Distribution* for a Poisson parameter λ with data $(z_1, \ldots, z_m)$ is given by:

$$\begin{aligned}
p(z|X) &= \int_0^\infty p(z|\lambda)p(\lambda|X)d\lambda \\
&= \int_0^\infty Poisson(z|\lambda) \cdot Gamma(\sum x_i + \alpha, n+\beta)d\lambda \\
&= \int_0^\infty \frac{e^{-\lambda}\lambda^z}{z!} \cdot \frac{(n+\beta)^{\sum x_i+\alpha}}{\Gamma(\sum x_i+\alpha)}\lambda^{\sum x_i+\alpha-1}e^{-(n+\beta)\lambda}d\lambda \\
&= \frac{(n+\beta)^{\sum x_i+\alpha}}{\Gamma(\sum x_i+\alpha)\Gamma(z+1)}\int_0^\infty \lambda^{z+\sum x_i+\alpha-1}e^{-(n+\beta+1)\lambda}d\lambda \\
&= \frac{(n+\beta)^{\sum x_i+\alpha}}{\Gamma(\sum x_i+\alpha)\Gamma(z+1)} \cdot \frac{\Gamma(z+\sum x_i+\alpha)}{(n+\beta+1)^{z\sum x_i+\alpha}} \\
&= \frac{\Gamma(z+\sum x_i+\alpha)}{\Gamma(\sum x_i+\alpha)\Gamma(z+1)}\left(\frac{n+\beta}{n+\beta+1}\right)^{\sum x_i+\alpha}\left(\frac{1}{n+\beta+1}\right)^z
\end{aligned}$$

```r
# Gelman, q. 13 page 69-70
# Poisson with gamma prior
deaths <- c(734,516,754,877,814,362,764,809,223,1066)
lambda.range <- seq(0,3000)
# prior
prior <- dgamma(lambda.range, shape=7, scale=1/.01)
# log likelihood (likelihood values are too large)
log.likelihood <- sum(deaths)*log(lambda.range)-length(deaths)*lambda.range
# posterior
alpha.star <- 7+sum(deaths)
```

```r
beta.star <- .01+length(deaths)
post <- dgamma(lambda.range, shape=alpha.star, scale=1/beta.star)
plot(lambda.range, prior, type = "l", lwd = 1.5, xlab = expression(lambda),
ylab = "Density", cex.axis = 1.3, cex.lab = 1.3, bty = "L", ylim = c(0,0.05))
lines(lambda.range, post)
par(new = TRUE)
plot(lambda.range, log.likelihood, type = "l", lty = 2, lwd = 1.5,
axes = FALSE, xlab = "", ylab = "")
axis(side = 4, cex.axis = 1.3)
```

(b) Assuming Gamma(a,b) priors for two Poisson parameters λ_1 and λ_2, derive the posterior for $\phi = \lambda_1/\lambda_2$. (Hint: use nuisance parameter $\mu = \lambda_2$).

For

(c) Derive the Jeffreys prior for (ϕ, μ), and the corresponding marginal posterior for ϕ.

(d) Derive the reference prior for (ϕ, μ), and the corresponding marginal posterior for ϕ.

(e) Based on (c), consider the posterior for ϕ based on uniform priors for λ_1 and λ_2, and comment on when inference based on this posterior could be quite different from that based on the reference posterior from (e). Give a data example, in terms of resulting intervals. (Hint: the above posteriors are related to known pdfs, and by transformation may be simplified even further.).

Question 2: Inference for variance components:

(a) Use e.g. SAS (PROC VARCOMP) to perform a classical analysis of the data in Table 5.1.4 of Box & Tiao (1973), based on finding point estimates only.

(b) Use WinBUGS for a Bayesian analysis of (a), and find reasonable point and interval estimates for σ_1^2 and σ_2^2. Include graphs, including one of the joint posterior. [6 marks]

(c) Box & Tiao also studied a 3-component model (Table 5.3.1).

  i. Derive central credible intervals, for the 3 individual components, based on Table 5.3.3.
  ii. Use WinBUGS to do the same, and include graphs.
  iii. While not going as far as Box & Tiao's Figure 5.3.2, produce a graph of the joint posterior of σ_2^2 and σ_3^2, and one of σ_2^2/σ_3^2

(d) Box, Hunter & Hunter (1976, Chapter 17.3) studied a pigment paste example with three components, focusing on point estimates only. Use WinBUGS again to perform a Bayesian analysis. Include graphs.