Assignment 2

2017-05-10

Question 1: Inference for the Poisson parameter λ:

(a) Given a general Gamma(a,b) prior, derive the posterior distribution of λ, given data $(x_1, \ldots, x_n)$, followed by the posterior predictive distribution of $(z_1, \ldots, z_m)$.

The Likelihood is given by:

$$L(\lambda|x) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod_{i=1}^{n}(x_i!)}$$

The Prior is a $Gamma(a, b)$:

$$p(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}, \lambda > 0$$

So the posterior is given by:

$$p(\lambda|x) \propto \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\sum x_i+\alpha-1}e^{-(n+\beta)\lambda}, \lambda > 0,$$

which is equivalent to a $Gamma(\sum x_i + \alpha, n + \beta)$, or $Gamma(n\bar{x} + \alpha, n + \beta)$.

The *Posterior Predictive Distribution* for a Poisson parameter λ with data $(z_1, \ldots, z_m)$ is given by:

$$\begin{aligned}
p(z|X) &= \int_0^\infty p(z|\lambda)p(\lambda|X)d\lambda \\
&= \int_0^\infty Poisson(z|\lambda) \cdot Gamma(\sum x_i + \alpha, n + \beta)d\lambda \\
&= \int_0^\infty \frac{e^{-\lambda}\lambda^z}{z!} \cdot \frac{(n+\beta)^{\sum x_i+\alpha}}{\Gamma(\sum x_i + \alpha)}\lambda^{\sum x_i+\alpha-1}e^{-(n+\beta)\lambda}d\lambda \\
&= \frac{(n+\beta)^{\sum x_i+\alpha}}{\Gamma(\sum x_i + \alpha)\Gamma(z+1)} \int_0^\infty \lambda^{z+\sum x_i+\alpha-1}e^{-(n+\beta+1)\lambda}d\lambda \\
&= \frac{(n+\beta)^{\sum x_i+\alpha}}{\Gamma(\sum x_i + \alpha)\Gamma(z+1)} \cdot \frac{\Gamma(z + \sum x_i + \alpha)}{(n+\beta+1)^{z\sum x_i+\alpha}} \\
&= \frac{\Gamma(z + \sum x_i + \alpha)}{\Gamma(\sum x_i + \alpha)\Gamma(z+1)} \left(\frac{n+\beta}{n+\beta+1}\right)^{\sum x_i+\alpha} \left(\frac{1}{n+\beta+1}\right)^z
\end{aligned}$$

(b) Assuming Gamma(a,b) priors for two Poisson parameters λ_1 and λ_2, derive the posterior for $\phi = \lambda_1/\lambda_2$. (Hint: use nuisance parameter $\mu = \lambda_2$).

(b) Assuming Gamma(a,b) priors for two Poisson parameters λ_1 and λ_2, derive the posterior for $\phi = \lambda_1/\lambda_2$. (Hint: use nuisance parameter $\mu = \lambda_2$).

Firstly, the likelihood function for two Poissons is:

$$p(x_1, x_2|\lambda_1, \lambda_2) = p(x_1|\lambda_1)p(x_2|\lambda_2) = \frac{e^{-\lambda_1}\lambda_1^{x_1}}{x_1!} \cdot \frac{e^{-\lambda_2}\lambda_2^{x_2}}{x_2!}.$$

Reparameterising by $\phi = \frac{\lambda_1}{\lambda_2}$, we get:

$$p(x_1, x_2|\phi, \lambda_2) = \frac{e^{-\phi\lambda_2}(\phi\lambda_2)^{x_1}}{x_1!} \cdot \frac{e^{-\lambda_2}\lambda_2^{x_2}}{x_2!}.$$

We then compute the Fisher Information Matrix:

$$\begin{aligned}
I(\theta)_{ij} &= E\left(-\frac{\partial^2 l}{\partial\theta_i\partial\theta_j}\right) \\
\implies F(\phi, \lambda_2) &= \begin{bmatrix} \frac{\lambda_2}{\phi} & 1 \\ 1 & \frac{1+\phi}{\lambda_2} \end{bmatrix} \\
\implies S(\phi, \lambda_2) = F^{-1}(\phi, \lambda_2) &= \begin{bmatrix} \frac{\phi(1+\phi)}{\lambda_2} & -\phi \\ -\phi & \lambda_2 \end{bmatrix}
\end{aligned}$$

Following the algorithm laid out by Barnardo we can define the marginal and conditional asympototic posteriors for ϕ.

$$\begin{aligned}
d_0(\phi, \lambda_2) &= \left[\frac{\phi(1+\phi)}{\lambda_2}\right]^{1/2} \\
d_1(\phi, \lambda_2) &= \left(\frac{\lambda_2}{1+\phi}\right)^{1/2}
\end{aligned}$$

According to Corollary 1 of Proposition 2 in Barnardo's paper, because the nuisance parameter space $\Lambda(\phi) = \Lambda$ is independent of ϕ, we can factorise the above equations as

$$\begin{aligned}
d_0^{-1}(\phi, \lambda_2) &= \frac{1}{\sqrt{\phi(1+\phi)}} \cdot \sqrt{\lambda_2} = a_0(\phi)b_0(\lambda_2) \\
d_1^{-1}(\phi, \lambda_2) &= \sqrt{\phi(1+\phi)} \cdot \frac{1}{\sqrt{\lambda_2}} = a_1(\phi)b_1(\lambda_2)
\end{aligned}$$

which implies that the marginal and conditional reference priors are

$$\begin{aligned}
\pi(\phi) \propto a_0(\phi) &= \frac{1}{\sqrt{\phi(1+\phi)}} \\
\pi(\lambda|\phi) \propto b_1(\lambda_2) &= \frac{1}{\sqrt{\lambda_2}}
\end{aligned}$$

The joint posterior can be derived with the likelihood and joint prior $(d_1^{-1}(\phi, \lambda_2))$ previously derived.

$$\begin{aligned}
\pi(\phi, \lambda_2|x_1, x_2) &\propto \pi(x_1, x_2|\phi, \lambda_2) \cdot \pi(\phi, \lambda_2) \\
&\propto e^{-(\phi+1)\lambda_2} \cdot \phi^{x_1-1/2}\,(1+\phi)^{1/2} \cdot \lambda_2^{x_1+x_2-1/2}
\end{aligned}$$

which, I'm pretty sure, can be factored as

$$\pi(\phi, \lambda_2|x_1, x_2) \propto Gamma(\lambda_2|x_1 + 1/2, 1) \cdot Gamma\left(\phi|x_1, \frac{1}{\lambda_2}\right) \cdot Beta\left(\frac{\phi}{1+\phi}|3/2, 1\right).$$

(c) Derive the Jeffreys prior for (ϕ, μ), and the corresponding marginal posterior for ϕ.

The Jeffreys principle states that the Jeffreys prior is

$$\pi_\phi(\phi) \propto det(I(\phi))^{1/2} = \left(\frac{\lambda_2(1+\phi)}{\phi} - 1\right)^{-1/2} = \phi^{-1/2}$$

(d) Derive the reference prior for (ϕ, μ), and the corresponding marginal posterior for ϕ.

(e) Based on (c), consider the posterior for ϕ based on uniform priors for λ_1 and λ_2, and comment on when inference based on this posterior could be quite different from that based on the reference posterior from (e). Give a data example, in terms of resulting intervals. (Hint: the above posteriors are related to known pdfs, and by transformation may be simplified even further.).

Question 2: Inference for variance components:

(a) Use e.g. SAS (PROC VARCOMP) to perform a classical analysis of the data in Table 5.1.4 of Box & Tiao (1973), based on finding point estimates only.

(b) Use WinBUGS for a Bayesian analysis of (a), and find reasonable point and interval estimates for σ_1^2 and σ_2^2. Include graphs, including one of the joint posterior. [6 marks]

(c) Box & Tiao also studied a 3-component model (Table 5.3.1).

  i. Derive central credible intervals, for the 3 individual components, based on Table 5.3.3.
  ii. Use WinBUGS to do the same, and include graphs.
  iii. While not going as far as Box & Tiao's Figure 5.3.2, produce a graph of the joint posterior of σ_2^2 and σ_3^2, and one of σ_2^2/σ_3^2

(d) Box, Hunter & Hunter (1976, Chapter 17.3) studied a pigment paste example with three components, focusing on point estimates only. Use WinBUGS again to perform a Bayesian analysis. Include graphs.