6.215/6.255J/15.093J/IDS.200J Optimization Methods

Lecture 5: Duality Theory I

September 23, 2021

Today's Lecture

Outline

- Motivation of duality
- General form of the dual
- Weak and strong duality
- Relations between primal and dual
- Duality - economic interpretation
- Complementary slackness

Motivating Duality

Checking optimality

$$\begin{aligned} \min \quad & x_1 + x_2 + \cdots + x_{300} \\ \text{s.t.} \quad & x_1 + 2x_2 + \cdots + 300x_{300} \geq 1 \\ & 300x_1 + 299x_2 + \cdots + x_{300} \geq 10 \end{aligned}$$

A friend claims to have found an optimal solution $\boldsymbol{x}^\star$ with an optimal value $11/301$. How to check this?

Duality - motivation

An idea from Lagrange

Consider the LOP, called the **primal** with optimal solution $\boldsymbol{x}^*$

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

Duality - motivation

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Consider the LOP, called the **primal** with optimal solution $\boldsymbol{x}^*$

$$\begin{array}{ll} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{array}$$

Relax the equality constraint

$$\begin{array}{rll} g(\boldsymbol{p}) = & \min & \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{p}^T(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) \\ & \text{s.t.} & \boldsymbol{x} \geq 0 \end{array}$$

Duality - motivation

An idea from Lagrange

Consider the LOP, called the **primal** with optimal solution $\boldsymbol{x}^*$

$$\begin{array}{ll} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{array}$$

Relax the equality constraint

$$\begin{array}{rl} g(\boldsymbol{p}) = \min & \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{p}^T(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) \\ \text{s.t.} & \boldsymbol{x} \geq 0 \end{array}$$

$$\Rightarrow g(\boldsymbol{p}) \leq \boldsymbol{c}^T \boldsymbol{x}^* + \boldsymbol{p}^T(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}^*) = \boldsymbol{c}^T \boldsymbol{x}^*$$

Duality - motivation

An idea from Lagrange

Get the tightest lower bound, i.e., $\max g(\boldsymbol{p})$ over all possible $\boldsymbol{p}$.
Let us rewrite $g(\boldsymbol{p})$:

$$\begin{aligned} g(\boldsymbol{p}) &= \min_{\boldsymbol{x}\geq 0}\left[\boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{p}^T(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})\right] \\ &= \boldsymbol{p}^T\boldsymbol{b} + \min_{\boldsymbol{x}\geq 0}(\boldsymbol{c}^T - \boldsymbol{p}^T\boldsymbol{A})\boldsymbol{x} \end{aligned}$$

Duality - motivation

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Note that

$$\min_{\boldsymbol{x}\geq 0}(\boldsymbol{c}^{\boldsymbol{T}} - \boldsymbol{p}^{\boldsymbol{T}}\boldsymbol{A})\boldsymbol{x} = \begin{cases} 0 & \text{if } \boldsymbol{c}^{\boldsymbol{T}} - \boldsymbol{p}^{\boldsymbol{T}}\boldsymbol{A} \geq 0^{\boldsymbol{T}}, \\ -\infty & \text{otherwise.} \end{cases}$$

Duality - motivation

An idea from Lagrange

Get the tightest lower bound, i.e., $\max g(\boldsymbol{p})$ over all possible $\boldsymbol{p}$.
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Note that

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So we have

$$\text{dual}: \quad \max_{\boldsymbol{p}} g(\boldsymbol{p}) \quad \Leftrightarrow \quad \begin{array}{ll} \max & \boldsymbol{p}^T\boldsymbol{b} \\ \text{s.t.} & \boldsymbol{p}^T\boldsymbol{A} \leq \boldsymbol{c}^T \end{array}$$

General form of the dual

Primal

$$
\begin{array}{lll}
\min & \mathbf{c}^T\mathbf{x} & \\
\text{s.t.} & \mathbf{a}_i^T\mathbf{x} \geq b_i & i \in M_1 \\
 & \mathbf{a}_i^T\mathbf{x} \leq b_i & i \in M_2 \\
 & \mathbf{a}_i^T\mathbf{x} = b_i & i \in M_3 \\
 & x_j \geq 0 & j \in N_1 \\
 & x_j \leq 0 & j \in N_2 \\
 & x_j \text{ free} & j \in N_3
\end{array}
$$

Dual

$$
\begin{array}{lll}
\max & \mathbf{p}^T\mathbf{b} & \\
\text{s.t.} & p_i \geq 0 & i \in M_1 \\
 & p_i \leq 0 & i \in M_2 \\
 & p_i \text{ free} & i \in M_3 \\
 & \mathbf{p}^T\mathbf{A}_j \leq c_j & j \in N_1 \\
 & \mathbf{p}^T\mathbf{A}_j \geq c_j & j \in N_2 \\
 & \mathbf{p}^T\mathbf{A}_j = c_j & j \in N_3
\end{array}
$$

General form of the dual

Example

$$
\begin{array}{rl}
\min & x_1 + 2x_2 + 3x_3 \\
\text{s.t.} & -x_1 + 3x_2 = 5 \\
& 2x_1 - x_2 + 3x_3 \geq 6 \\
& x_3 \leq 4 \\
& x_1 \geq 0 \\
& x_2 \leq 0 \\
& x_3 \text{ free},
\end{array}
\qquad
\begin{array}{rl}
\max & 5p_1 + 6p_2 + 4p_3 \\
\text{s.t.} & p_1 \text{ free} \\
& p_2 \geq 0 \\
& p_3 \leq 0 \\
& -p_1 + 2p_2 \leq 1 \\
& 3p_1 - p_2 \geq 2 \\
& 3p_2 + p_3 = 3.
\end{array}
$$

General form of the dual

Summary

primal	min	max	**dual**
constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	variables
variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	constraints

Note: The dual of the dual is the primal

The dual of the dual is the primal

Example

$$
\begin{array}{llllllll}
\min & x_1 & + & 2x_2 & + & 3x_3 & & \\
\text{s.t.} & -x_1 & + & 3x_2 & & & = & 5 \\
 & 2x_1 & - & x_2 & + & 3x_3 & \geq & 6 \\
 & & & & & x_3 & \leq & 4 \\
 & x_1 \geq 0 & & & & & & \\
 & x_2 \leq 0 & & & & & & \\
 & x_3 \text{ free} & & & & & &
\end{array}
$$

$$
\begin{array}{llllllll}
\max & 5p_1 & + & 6p_2 & + & 4p_3 & & \\
\text{s.t.} & p_1 \text{ free} & & & & & & \\
 & p_2 \geq 0 & & & & & & \\
 & p_3 \leq 0 & & & & & & \\
 & -p_1 & + & 2p_2 & & & \leq & 1 \\
 & 3p_1 & - & p_2 & & & \geq & 2 \\
 & & & 3p_2 & + & p_3 & = & 3.
\end{array}
$$

$$
\begin{array}{llllllll}
\min & -5x_1 & - & 6x_2 & - & 4x_3 & & \\
\text{s.t.} & x_1 \text{ free} & & & & & & \\
 & x_2 \geq 0 & & & & & & \\
 & x_3 \leq 0 & & & & & & \\
 & x_1 & - & 2x_2 & & & \geq & -1 \\
 & -3x_1 & + & x_2 & & & \leq & -2 \\
 & & - & 3x_2 & - & x_3 & = & -3
\end{array}
$$

$$
\begin{array}{llllllll}
\max & -p_1 & - & 2p_2 & - & 3p_3 & & \\
\text{s.t.} & p_1 & - & 3p_2 & & & = & -5 \\
 & -2p_1 & + & x_2 & - & 3p_3 & \leq & -6 \\
 & & & & - & p_3 & \geq & -4 \\
 & p_1 \geq 0 & & & & & & \\
 & p_2 \leq 0 & & & & & & \\
 & p_3 \text{ free} & & & & & &
\end{array}
$$

General form of the dual

Compactly, in matrix notation, some pairs of primal-dual

$$\begin{array}{rrcl} \min & \boldsymbol{c}^T\boldsymbol{x} & & \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} & = & \boldsymbol{b} \\ & \boldsymbol{x} & \geq & 0 \end{array} \qquad \begin{array}{rl} \max & \boldsymbol{p}^T\boldsymbol{b} \\ \text{s.t.} & \boldsymbol{p}^T\boldsymbol{A} \leq \boldsymbol{c}^T \end{array}$$

$$\begin{array}{rl} \min & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \end{array} \qquad \begin{array}{rl} \max & \boldsymbol{p}^T\boldsymbol{b} \\ \text{s.t.} & \boldsymbol{p}^T\boldsymbol{A} = \boldsymbol{c}^T \\ & \boldsymbol{p} \geq 0 \end{array}$$

Weak duality

Theorem

If $\mathbf{x}$ is primal feasible and $\mathbf{p}$ is dual feasible then $\mathbf{p}^T\mathbf{b} \leq \mathbf{c}^T\mathbf{x}$

Proof: If the primal is in standard form: $\mathbf{p}^T\mathbf{b} = \mathbf{p}^T\mathbf{A}\mathbf{x} \leq \mathbf{c}^T\mathbf{x}$

Corollary

If $\mathbf{x}$ is primal feasible, $\mathbf{p}$ is dual feasible, and $\mathbf{p}^T\mathbf{b} = \mathbf{c}^T\mathbf{x}$, then $\mathbf{x}$ is optimal in the primal and $\mathbf{p}$ is optimal in the dual.

Strong duality

Theorem

If the primal has an optimal solution, then so does the dual, and the optimal costs are equal.

Proof:

$$\begin{array}{rrcl} \min & \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} & & \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} & = & \boldsymbol{b} \\ & \boldsymbol{x} & \geq & 0 \end{array}$$

Apply (with any anticycling rule) simplex method; get optimal BFS $\boldsymbol{x}$, and corresponding final optimal basis $\boldsymbol{B}$, then:

$$\boldsymbol{c}^{\boldsymbol{T}} - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{A} \geq 0$$

Strong duality

Define $\boldsymbol{p^T} = \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \Rightarrow \boldsymbol{p^T A} \leq \boldsymbol{c^T}$

$\Rightarrow$ $\boldsymbol{p}$ dual feasible for

$$\begin{array}{rl} \max & \boldsymbol{p^T b} \\ \text{s.t.} & \boldsymbol{p^T A} \leq \boldsymbol{c^T} \end{array}$$

Moreover,

$$\boldsymbol{p^T b} = \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{b} = \boldsymbol{c}_B^T \boldsymbol{x}_B = \boldsymbol{c^T x}$$

$\Rightarrow$ $\boldsymbol{x}, \boldsymbol{p}$ are primal and dual optimal

Relations between primal and dual

	Finite opt.	Unbounded	Infeasible
Finite opt.	*		
Unbounded			*
Infeasible		*	*

Duality

Economic interpretation

- Let $\boldsymbol{x}$ optimal nondegenerate solution: $\boldsymbol{B}^{-1}\boldsymbol{b} > 0$
- Suppose $\boldsymbol{b}$ changes to $\boldsymbol{b} + \epsilon$ for some small ϵ
- How is the optimal cost affected?

Duality

Economic interpretation

- Let $\boldsymbol{x}$ optimal nondegenerate solution: $\boldsymbol{B}^{-1}\boldsymbol{b} > 0$
- Suppose $\boldsymbol{b}$ changes to $\boldsymbol{b} + \epsilon$ for some small ϵ
- How is the optimal cost affected?
- For small ϵ:
  - feasibilty unaffected
  - optimality conditions unaffected

- Let $\boldsymbol{x}$ optimal nondegenerate solution: $\boldsymbol{B}^{-1}\boldsymbol{b} > 0$
- Suppose $\boldsymbol{b}$ changes to $\boldsymbol{b} + \boldsymbol{\epsilon}$ for some small $\boldsymbol{\epsilon}$
- How is the optimal cost affected?
- For small $\boldsymbol{\epsilon}$:
  - feasibilty unaffected
  - optimality conditions unaffected
- New cost $\quad \boldsymbol{c}_B^T \boldsymbol{B}^{-1}(\boldsymbol{b} + \boldsymbol{\epsilon}) = \boldsymbol{p}^{\boldsymbol{T}}(\boldsymbol{b} + \boldsymbol{\epsilon})$
- If resource i changes by ϵ_i, cost changes by $p_i \epsilon_i$: "marginal cost"

Complementary slackness

Primal			**Dual**		
min	$\boldsymbol{c}^T\boldsymbol{x}$		max	$\boldsymbol{p}^T\boldsymbol{b}$	
s.t.	$\boldsymbol{a}_i^T\boldsymbol{x} \geq b_i$	$i \in M_1$	s.t.	$p_i \geq 0$	$i \in M_1$
	$\boldsymbol{a}_i^T\boldsymbol{x} \leq b_i$	$i \in M_2$		$p_i \leq 0$	$i \in M_2$
	$\boldsymbol{a}_i^T\boldsymbol{x} = b_i$	$i \in M_3$		p_i free	$i \in M_3$
	$x_j \geq 0$	$j \in N_1$		$\boldsymbol{p}^T\boldsymbol{A}_j \leq c_j$	$j \in N_1$
	$x_j \leq 0$	$j \in N_2$		$\boldsymbol{p}^T\boldsymbol{A}_j \geq c_j$	$j \in N_2$
	x_j free	$j \in N_3$		$\boldsymbol{p}^T\boldsymbol{A}_j = c_j$	$j \in N_3$

Theorem

Let $\boldsymbol{x}$ *primal feasible and* $\boldsymbol{p}$ *dual feasible. Then* $\boldsymbol{x}, \boldsymbol{p}$ *optimal* **if and only if**

$$p_i(\boldsymbol{a}_i^T\boldsymbol{x} - b_i) = 0, \quad \forall i$$

$$(c_j - \boldsymbol{p}^T\boldsymbol{A}_j)x_j = 0, \quad \forall j$$

Complementary slackness

Proof:

- If $\boldsymbol{x}$ primal feasible and $\boldsymbol{p}$ dual feasible, we have $u_i = p_i(\boldsymbol{a}_i^T \boldsymbol{x} - b_i) \geq 0$ and $v_j = (c_j - \boldsymbol{p^T} \boldsymbol{A}_j)x_j \geq 0$ for all i and j.
- Also $\boldsymbol{c^T x} - \boldsymbol{p^T b} = \sum_i u_i + \sum_j v_j$.
- By the strong duality theorem, if $\boldsymbol{x}$ and $\boldsymbol{p}$ are optimal, then $\boldsymbol{c^T x} = \boldsymbol{p^T b} \Rightarrow$ $u_i = v_j = 0$ for all i, j.
- Conversely, if $u_i = v_j = 0$ for all i, j, then $\boldsymbol{c^T x} = \boldsymbol{p^T b}$,
  $\Rightarrow$ $\boldsymbol{x}$ and $\boldsymbol{p}$ are optimal.

Complementary slackness

Example

$$
\begin{array}{rrcrcrcl}
\min & 13x_1 & + & 10x_2 & + & 6x_3 & & \\
\text{s.t.} & 5x_1 & + & x_2 & + & 3x_3 & = & 8 \\
& 3x_1 & + & x_2 & & & = & 3 \\
& x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
\qquad
\begin{array}{rrcrcl}
\max & 8p_1 & + & 3p_2 & & \\
\text{s.t.} & 5p_1 & + & 3p_2 & \leq & 13 \\
& p_1 & + & p_2 & \leq & 10 \\
& 3p_1 & & & \leq & 6
\end{array}
$$

Is $\boldsymbol{x}^* = (1, 0, 1)^T$ optimal?

Complementary slackness

Example

$$\begin{array}{rrcrcrcl}
\min & 13x_1 & + & 10x_2 & + & 6x_3 & & \\
\text{s.t.} & 5x_1 & + & x_2 & + & 3x_3 & = & 8 \\
 & 3x_1 & + & x_2 & & & = & 3 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
\qquad
\begin{array}{rrcrcl}
\max & 8p_1 & + & 3p_2 & & \\
\text{s.t.} & 5p_1 & + & 3p_2 & \leq & 13 \\
 & p_1 & + & p_2 & \leq & 10 \\
 & 3p_1 & & & \leq & 6
\end{array}$$

Is $\boldsymbol{x}^* = (1, 0, 1)^T$ optimal?

$$5p_1 + 3p_2 = 13, \quad 3p_1 = 6$$

$$\Rightarrow p_1 = 2, \quad p_2 = 1$$

It satisfies $p_1 + p_2 \leq 10$, so dual feasible.

Objective=2*8+3*1=19 = 13+6 = 19, so yes $\boldsymbol{x}^*$ optimal.