6.215/6.255J/15.093J/IDS.200J Optimization Methods

Lecture 2: Geometry of Linear Optimization

September 14, 2021

Today's Lecture

Outline

- Few remaining points from Lecture 1
- General Linear Optimization Problems
- Standard Form
- Preliminary Geometric Insights
- Geometric Concepts (Polyhedra, "Corners")
- Equivalence of Algebraic and Geometric Concepts

Another Formulation Example - Scheduling

Data & Constraints; Decision variables?

- Hospital wants to make weekly nightshift for its nurses
- d_j demand for nurses, $j = 1 \dots 7$
- Every nurse works 5 days in a row
- Goal: hire minimum number of nurses

Another Formulation Example - Scheduling

Data & Constraints; Decision variables?

- Hospital wants to make weekly nightshift for its nurses
- d_j demand for nurses, $j = 1 \dots 7$
- Every nurse works 5 days in a row
- Goal: hire minimum number of nurses

Decision Variables

x_j: # nurses starting their week on day j

Scheduling
Formulation

$$
\begin{array}{llllllllll}
\min & \sum_{j=1}^{7} x_j \\
\text{s.t.} & x_1+ & & & x_4+ & x_5+ & x_6+ & x_7 & \geq & d_1 \\
& x_1+ & x_2 & & & x_5+ & x_6+ & x_7 & \geq & d_2 \\
& x_1+ & x_2+ & x_3 & & & x_6+ & x_7 & \geq & d_3 \\
& x_1+ & x_2+ & x_3+ & x_4+ & & & x_7 & \geq & d_4 \\
& x_1+ & x_2+ & x_3+ & x_4+ & x_5 & & & \geq & d_5 \\
& & x_2+ & x_3+ & x_4+ & x_5+ & x_6 & & \geq & d_6 \\
& & & x_3+ & x_4+ & x_5+ & x_6+ & x_7 & \geq & d_7 \\
& x_j \geq 0
\end{array}
$$

Messages

How to formulate?

- Define your decision variables clearly.
- Write constraints and objective function.
- No systematic method available.

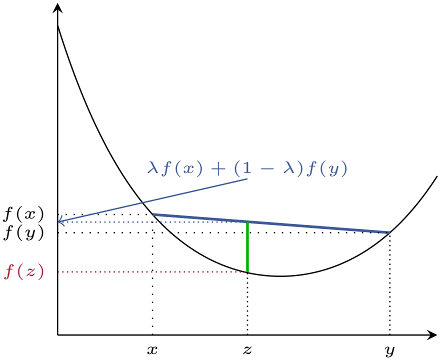
What is a good LO formulation?

A formulation with a small number of variables and constraints, and the matrix $\boldsymbol{A}$ is sparse.

(Nonlinear) Optimization

The general problem

$$
\begin{array}{ll}
\text{min} & f(x_1, \ldots, x_n) \\
\text{s.t.} & g_1(x_1, \ldots, x_n) \le 0 \\
& \vdots \\
& g_m(x_1, \ldots, x_n) \le 0
\end{array}
$$



$$z = \lambda x + (1-\lambda) y$$

- $f : S \to R$ is convex if for all $x, y \in S$, and every $\lambda \in [0,1]$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

- $f(.)$ concave if $-f(.)$ convex.

General Linear Optimization Problems

All possible variants

$$
\begin{array}{rlll}
\min / \max & \boldsymbol{c}^{\boldsymbol{T}} \boldsymbol{x} & & \\
\text{s.t.} & \boldsymbol{a}_{\boldsymbol{i}}^{\boldsymbol{T}} \boldsymbol{x} = b_i & & i \in M_1 \\
& \boldsymbol{a}_{\boldsymbol{i}}^{\boldsymbol{T}} \boldsymbol{x} \leq b_i & & i \in M_2 \\
& \boldsymbol{a}_{\boldsymbol{i}}^{\boldsymbol{T}} \boldsymbol{x} \geq b_i & & i \in M_3 \\
& x_j \geq 0 & & j \in N_1 \\
& x_j \leq 0 & & j \in N_2 \\
& x_j \text{ free} & & j \in N_3
\end{array}
$$

Central Problem

Standard form

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

Characteristics:

- minimization problem
- equality constraints
- non-negative variables

Claim: specialization is "without loss of generality".

Equivalence Transformations I

Two problems are **equivalent** if

- Solution of either problem can be found "easily" once the other is solved
- A formal definition from complexity theory exists but is beyond our scope

Example: From max to min

$$\begin{array}{ll} \max & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{a}_i^T\boldsymbol{x} \leq b_i \\ & \boldsymbol{a}_i^T\boldsymbol{x} \geq b_i \\ & \boldsymbol{x} \text{ free} \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} -\min & -\boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{a}_i^T\boldsymbol{x} \leq b_i \\ & \boldsymbol{a}_i^T\boldsymbol{x} \geq b_i \\ & \boldsymbol{x} \text{ free} \end{array}$$

Equivalence Transformations II

Two problems are **equivalent** if

- Solution of either problem can be found "easily" once the other is solved
- A formal definition from complexity theory exists but is beyond our scope

Example: Slack variables

$$\begin{array}{rl} -\min & -\boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{a}_i^T\boldsymbol{x} \le b_i \\ & \boldsymbol{a}_i^T\boldsymbol{x} \ge b_i \\ & \boldsymbol{x} \text{ free} \end{array} \quad \Leftrightarrow \quad \begin{array}{rl} -\min & -\boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{a}_i^T\boldsymbol{x} + s_i = b_i, \quad s_i \ge 0 \\ & \boldsymbol{a}_i^T\boldsymbol{x} - s_i = b_i, \quad s_i \ge 0 \\ & \boldsymbol{x} \text{ free} \end{array}$$

Equivalence Transformations III

Two problems are **equivalent** if

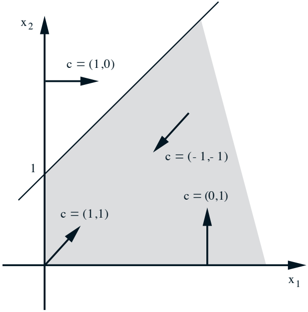
- Solution of either problem can be found "easily" once the other is solved
- A formal definition from complexity theory exists but is beyond our scope

Example: Decomposition as difference

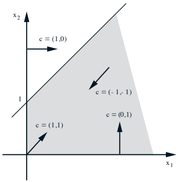
$$
\begin{aligned}
-\min \quad & -\boldsymbol{c}^T\boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{a}_i^T\boldsymbol{x} + s_i = b_i,\ s_i \geq 0 \\
& \boldsymbol{a}_i^T\boldsymbol{x} - s_i = b_i,\ s_i \geq 0 \\
& \boldsymbol{x} \text{ free}
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
-\min \quad & -\boldsymbol{c}^T(\boldsymbol{x}^+ - \boldsymbol{x}^-) \\
\text{s.t.} \quad & \boldsymbol{a}_i^T(\boldsymbol{x}^+ - \boldsymbol{x}^-) + s_i = b_i, \quad s_i \geq 0 \\
& \boldsymbol{a}_i^T(\boldsymbol{x}^+ - \boldsymbol{x}^-) - s_i = b_i, \quad s_i \geq 0 \\
& \boldsymbol{x}^+ \geq 0,\ \boldsymbol{x}^- \geq 0
\end{aligned}
$$

Preliminary Geometric Insights

$$
\begin{array}{lrcl}
\text{minimize} & c_1 x_1 + c_2 x_2 & & \\
\text{subject to} & -x_1 + x_2 & \leq & 1 \\
 & x_1 & \geq & 0 \\
 & x_2 & \geq & 0
\end{array}
$$



- There exists a unique optimal solution.
- There exist multiple optimal solutions; in this case, the set of optimal solutions can be either bounded or unbounded.
- The optimal cost is $-\infty$, and no feasible solution is optimal.



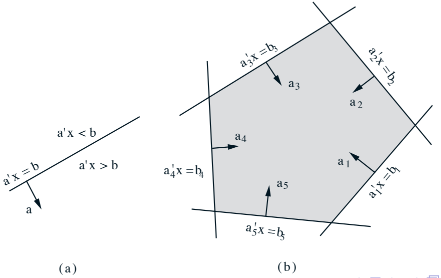
- The feasible set is empty.

Polyhedra

Definitions

- The set $\{\boldsymbol{x} \mid \boldsymbol{a}^{\boldsymbol{T}}\boldsymbol{x} = b\}$ is called a **hyperplane**.
- The set $\{\boldsymbol{x} \mid \boldsymbol{a}^{\boldsymbol{T}}\boldsymbol{x} \geq b\}$ is called a **halfspace**.
- The (finite) intersection of many halfspaces is called a **polyhedron**.
- A polyhedron $P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$ is a convex set, i.e.,

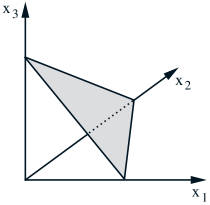
  if $\boldsymbol{x}, \boldsymbol{y} \in P$, then $\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in P$ for any $0 \leq \lambda \leq 1$.



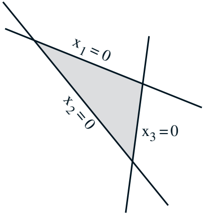
Polyhedra - Standard Form

$$P = \{\boldsymbol{x} \in \Re^n \ : \ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq 0\}$$

- $\boldsymbol{A} \in \Re^{m \times n}$ has full row rank $m < n$
- P lives in R^{n-m} dimensional subspace



(a)



(b)

Geometric vs Standard Representations

Geometric representation : $\{\boldsymbol{x} \ : \ \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$.

- Easier to visualize
- Harder to manipulate algebraically

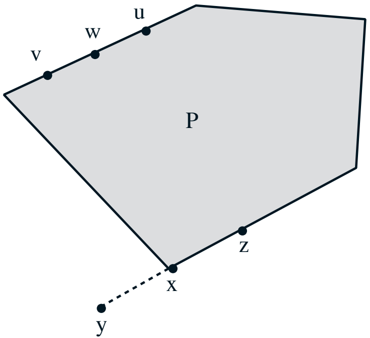
Standard Representation : $\{\boldsymbol{x} \ : \ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq 0\}$.

- Harder to visualize
- Easier to manipulate algebraically

Corners

Extreme Points

- Polyhedron $P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$
- $\boldsymbol{x} \in P$ is **an extreme point** of P if there does not exist $\boldsymbol{y}, \boldsymbol{z} \in P$, both different from $\boldsymbol{x}$, such that:
  $\boldsymbol{x} = \lambda \boldsymbol{y} + (1 - \lambda)\boldsymbol{z}$, $0 < \lambda < 1$

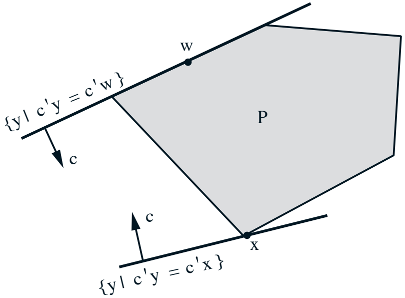


Corners

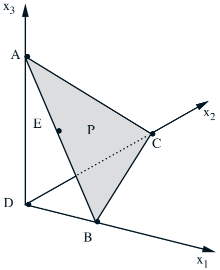
Vertex

- Polyhedron $P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$
- $\boldsymbol{x} \in P$ is **a vertex** of P if $\exists$ $\boldsymbol{c}$ such that $\boldsymbol{x}$ is the unique optimum of

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T\boldsymbol{y} \\ \text{subject to} & \boldsymbol{y} \in P \end{array}$$



$$P = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 1, \quad x_1, x_2, x_3 \geq 0\}$$

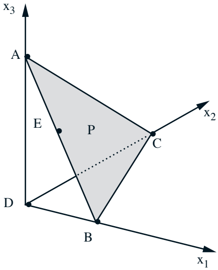


- Points A,B,C,D: 3 constraints active (active = satisfied as an equality)
- Point E: 2 constraints active

Corners

Active Constraints

$$P = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 1, \quad x_1, x_2, x_3 \geq 0\}$$

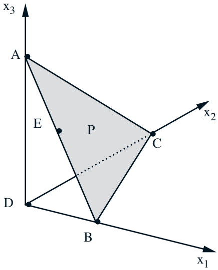


- Points A,B,C,D: 3 constraints active (active = satisfied as an equality)
- Point E: 2 constraints active
- Now, suppose we add $2x_1 + 2x_2 + 2x_3 = 2 \Rightarrow$ 3 hyperplanes are active at point E.

Corners

Active Constraints

$$P = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 1, \quad x_1, x_2, x_3 \geq 0\}$$



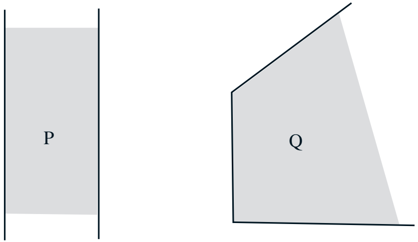
- Points A,B,C,D: 3 constraints active (active = satisfied as an equality)
- Point E: 2 constraints active
- Now, suppose we add $2x_1 + 2x_2 + 2x_3 = 2 \Rightarrow 3$ hyperplanes are active at point E. But these constraints are not linearly independent.

- (Classical Form) Polyhedron $P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$
- **Intuition:** a basic or BFS is a point at which n inequalities are active and corresponding equations are linearly independent.
- Formally:
  - Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{\boldsymbol{m}}$ the rows of $\boldsymbol{A}$, $\boldsymbol{x} \in \Re^n$, and $I = \{i \mid \boldsymbol{a}_{\boldsymbol{i}}^{\boldsymbol{T}}\boldsymbol{x} = b_i\}$.
  - $\boldsymbol{x}$ is a **basic solution** if subspace spanned by $\{\boldsymbol{a}_{\boldsymbol{i}}, i \in I\}$ is $\Re^n$.
  - It is a **basic feasible solution (BFS)** if we also have $\boldsymbol{x} \in P$.

Equivalence: vertex $\Leftrightarrow$ extreme point $\Leftrightarrow$ BFS

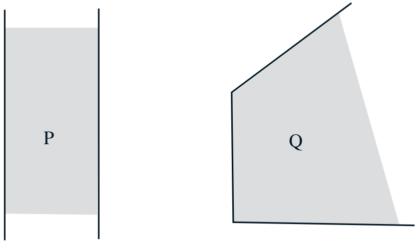
Theorem: $P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$. Let $\boldsymbol{x} \in P$. Then $\boldsymbol{x}$ is a vertex $\Leftrightarrow$ $\boldsymbol{x}$ is an extreme point $\Leftrightarrow$ $\boldsymbol{x}$ is a BFS.

Existence of extreme points



Note that $P = \{(x_1, x_2) : 0 \leq x_1 \leq 1\}$ does not have an extreme point, while $Q = \{(x_1, x_2) : x_1 + 5 \geq x_2, x_1 \geq 0, x_2 \geq 0\}$ has two. Why?

Existence of extreme points



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Definition

A polyhedron $P \subset \Re^n$ **contains a line** if there exists a vector $\boldsymbol{x} \in P$ and a nonzero vector $\boldsymbol{d} \in \Re^n$ such that $\boldsymbol{x} + \lambda \boldsymbol{d} \in P$ for all scalars λ.

Existence of extreme points

Theorem

Suppose that the polyhedron $P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{a_i}^T \boldsymbol{x} \geq b_i,\ i = 1, \ldots, m\}$ *is nonempty. Then, the following are equivalent:*

(a) *The polyhedron P has at least one extreme point.*

(b) *The polyhedron P does not contain a line.*

(c) *There exist n vectors out of the family $\boldsymbol{a}_1, \ldots, \boldsymbol{a_m}$, which are linearly independent.*

Corollary

- *(Nonempty) polyhedra in standard form contain an extreme point.*
- *(Nonempty) bounded polyhedra contain an extreme point.*

Theorem

- *Consider the LO*

$$\begin{array}{ll} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}. \end{array}$$

- *Assume that P has no line and that the LO has an optimal solution.*
- *Then there exists an optimal solution which is an extreme point of P.*

BFS for standard form polyhedra

Basic and BFS

- Standard Form Polyhedron $P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{Ax} = \boldsymbol{b}, \boldsymbol{x} \geq 0\}$
- Assume that the $m \times n$ matrix $\boldsymbol{A}$ has linearly independent rows
- $\boldsymbol{x} \in \Re^n$ is a **basic solution** if and only if $\boldsymbol{Ax} = \boldsymbol{b}$, and there exist indices $B(1), \ldots, B(m)$ such that:
    - The columns $\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(m)}$ are linearly independent
    - If $i \neq B(1), \ldots, B(m)$, then $x_i = 0$
- Note: If we also have $\boldsymbol{x} \geq 0$, then $\boldsymbol{x}$ is a **basic feasible solution**.

BFS for standard form polyhedra

Algebraic construction of BFS

Procedure for constructing basic (feasible) solutions

1. Choose m linearly independent columns $\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$
3. Solve $\boldsymbol{Ax} = \boldsymbol{b}$ for $x_{B(1)}, \ldots, x_{B(m)}$

$$\boldsymbol{Ax} = \boldsymbol{b} \quad \to \quad \boldsymbol{Bx}_B + \boldsymbol{Nx}_N = \boldsymbol{b}$$

$$\boldsymbol{x}_N = 0, \quad \boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$$

4. If we have $\boldsymbol{x}_B \geq 0$, then $\boldsymbol{x}$ is a BFS.

BFS for standard form polyhedra

Example 1

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}$$

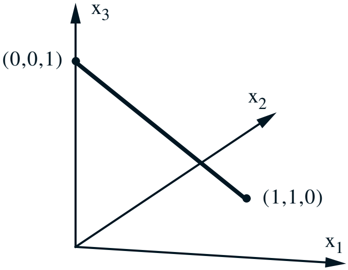
- $\boldsymbol{A}_4, \boldsymbol{A}_5, \boldsymbol{A}_6, \boldsymbol{A}_7$ basic columns
- Solution: $\boldsymbol{x}^{\boldsymbol{T}} = (0, 0, 0, 8, 12, 4, 6)$, a BFS
- Another basis: $\boldsymbol{A}_3, \boldsymbol{A}_5, \boldsymbol{A}_6, \boldsymbol{A}_7$ basic columns.
- Solution: $\boldsymbol{x}^{\boldsymbol{T}} = (0, 0, 4, 0, -12, 4, 6)$, not a BFS

Basic Solution - Degeneracy

With various representations of polyhedra

- Classical: Polyhedron $P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$, $\boldsymbol{A} \in \Re^{m \times n}$.
  - degenerate if more than n of the constraints are active at $\boldsymbol{x}$
- Standard: $P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0\}$, $\boldsymbol{A} \in \Re^{m \times n}$, $\text{rank}(\boldsymbol{A}) = m$.
  - degenerate if it contains more than $n - m$ zeros
- General: P defined by both equality and inequality constraints.
  - degenerate if more than n of the constraints are active at $\boldsymbol{x}$

Degeneracy is representation dependent



- $P = \left\{(x_1, x_2, x_3) \mid x_1 - x_2 = 0,\ x_1 + x_2 + 2x_3 = 2,\ x_1, x_2, x_3 \geq 0\right\}$
- $n = 3,\ m = 2$ and $n - m = 1$
  $\Rightarrow (1, 1, 0)$ is nondegenerate, while (0,0,1) is degenerate.
- $P = \left\{(x_1, x_2, x_3) \mid x_1 - x_2 = 0,\ x_1 + x_2 + 2x_3 = 2,\ x_1 \geq 0,\ x_3 \geq 0\right\}$
  $\Rightarrow (0, 0, 1)$ is now nondegenerate !

Checking ...

Are the points A, B, C, D, E basic solutions?, BFS?, Denegerate?