6.215/6.255J/15.093J/IDS.200J Optimization Methods

Lecture 3: The Simplex Method I

September 16, 2021

Today's Lecture
Outline

- Recap from Lecture 2
- Linear Optimization in Standard Form
  - Feasible Directions; Reduced Costs
  - Optimality Conditions
  - Improving the Cost
  - Unboundness
  - Moving from one basis to another: Example
- The Simplex Algorithm
- The Simplex Algorithm on Degenerate Problems

Polyhedra

Recap: Geometric vs Standard Representation

Geometric representation : $\{\boldsymbol{x} \ : \ \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$.

- Easier to visualize
- Harder to manipulate algebraically

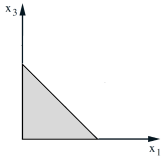
Standard Representation : $\{\boldsymbol{x} \ : \ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq 0\}$.

- Harder to visualize
- Easier to manipulate algebraically

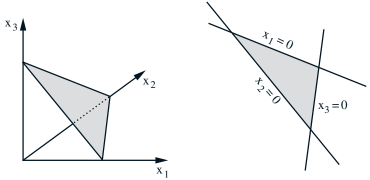
Polyhedra

Recap: Geometric vs Standard Representation, Example

Geometric representation : $\{(x_1, x_3) \mid x_1 + x_3 \leq 1, \quad x_1, x_3 \geq 0\}$



Standard representation : $\{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 1, \quad x_1, x_2, x_3 \geq 0\}$



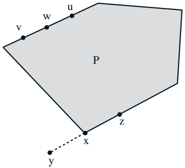
(a) (b)

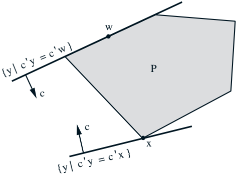
Linear Optimization (LO)

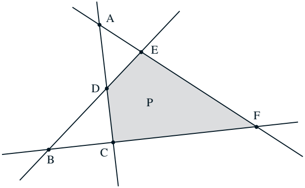
Recap: What we have learned so far

1. Polyhedra P in geometric or standard form

2. Extreme point $\iff$ Vertex $\iff$ BFS



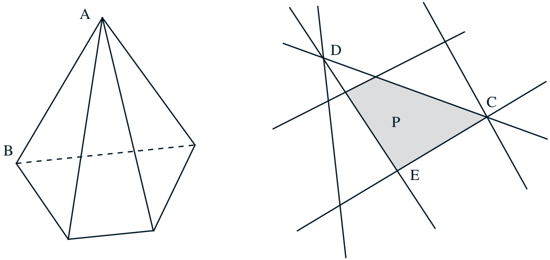




3. Degenerate B(F)S: More than n constraints are active.

… checking again …

Are the points A, B, C, D, E extreme points, basic solutions?, BFS?, Degenerate?



(a) (b)

Linear Optimization (LO)

Recap: What we have learned so far, cont.

4. Linear Optimization

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x} \in P \end{aligned}$$

   Possibilities:

   - There exists a unique optimal solution.
   - There exist multiple optimal solutions; in this case, the set of optimal solutions can be either bounded or unbounded.
   - The optimal cost is $-\infty$, and no feasible solution is optimal.
   - The feasible set is empty.

5. **Theorem:** Suppose P has at least one extreme point. Either optimal cost is $-\infty$ or there exists an extreme point which is optimal.

Linear Optimization in Standard Form

The simplex method expects input problems in the standard form

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

with $\boldsymbol{A} \in \Re^{m \times n}$, $\boldsymbol{b} \in \Re^m$ and $\boldsymbol{c} \in \Re^n$.

Its expected outcome is either:

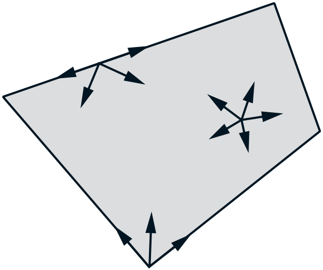
- Empty feasible set.
- Optimal cost is $-\infty$.
- An optimal BFS.

Linear Optimization in Standard Form

Feasible directions

- We are at $\boldsymbol{x} \in P$ and we contemplate moving away from $\boldsymbol{x}$, in some direction.
- Consider directions that do not immediately take us outside the feasible set.
- A vector $\boldsymbol{d} \in \Re^n$ is said to be a **feasible direction** at $\boldsymbol{x}$, if there exists a positive scalar $\theta > 0$ for which $\boldsymbol{x} + \theta \boldsymbol{d} \in P$.

  $\Rightarrow$ the set of such feasible directions is the polytope $\{\boldsymbol{d} \in \Re^n \ : \ \boldsymbol{A}\boldsymbol{d} = 0, d_i \geq 0 \text{ if } x_i = 0\}$



Linear Optimization in Standard Form

Points of particular interest: BFS's

A point $\boldsymbol{x}$ is a basic feasible solution (BFS) in a standard form polyhedron if $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq 0$ and there are *basic indices* $B(1), \ldots, B(m)$ such that

1. The columns of $\boldsymbol{B} = [\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(m)}]$ are independent
2. $i \notin [B(1), \ldots, B(m)] \implies x_i = 0$.

Linear Optimization in Standard Form

Feasible directions of interest

- Let $\boldsymbol{x}$ be a BFS to the standard form problem with basic indices B.
  - $x_i = 0,\ i \notin B$, $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$.
- We consider moving away from $\boldsymbol{x}$, to a new vector $\boldsymbol{x} + \theta \boldsymbol{d}$, by selecting a nonbasic variable x_j and increasing it to a positive value θ, while keeping the remaining nonbasic variables at zero.
  - Algebraically, $d_j = 1$, and $d_i = 0$ for every nonbasic index $i \neq j$.
  - The vector $\boldsymbol{x}_B$ of basic variables changes to $\boldsymbol{x}_B + \theta \boldsymbol{d}_B$.

Linear Optimization in Standard Form

Feasible directions of interest

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  - The vector $\boldsymbol{x}_B$ of basic variables changes to $\boldsymbol{x}_B + \theta\boldsymbol{d}_B$.
- To remain feasible: $\boldsymbol{A}(\boldsymbol{x} + \theta\boldsymbol{d}) = \boldsymbol{b} \ \Rightarrow\ \boldsymbol{A}\boldsymbol{d} = 0$.
  - $0 = \boldsymbol{A}\boldsymbol{d} = \sum_{i=1}^{n} \boldsymbol{A}_i d_i = \sum_{i=1}^{m} \boldsymbol{A}_{B(i)} d_{B(i)} + \boldsymbol{A}_j = \boldsymbol{B}\boldsymbol{d}_B + \boldsymbol{A}_j$
    $\Rightarrow \boldsymbol{d}_B = -\boldsymbol{B}^{-1}\boldsymbol{A}_j$.

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    $\Rightarrow \boldsymbol{d}_B = -\boldsymbol{B}^{-1}\boldsymbol{A}_j$.
- Nonnegativity constraints?
  - If $\boldsymbol{x}$ nondegenerate, $\boldsymbol{x}_B > 0$; thus $\boldsymbol{x}_B + \theta\boldsymbol{d}_B \geq 0$ for θ small.
  - If $\boldsymbol{x}$ degenerate, then $\boldsymbol{d}$ is not always a feasible direction. Why?

Linear Optimization in Standard Form

Feasible directions of interest

- Let $\boldsymbol{x}$ be a BFS to the standard form problem with basic indices B.
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- Nonnegativity constraints?
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  - If $\boldsymbol{x}$ degenerate, then $\boldsymbol{d}$ is not always a feasible direction. Why?
- Effects in cost?
  - Cost change: $\boldsymbol{c}^{\boldsymbol{T}}(\boldsymbol{x} + \theta\boldsymbol{d}) - \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} = \theta\boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{d} = \theta(c_j - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A}_j)$
  - $\overline{c}_j = c_j - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A}_j$ is called the **reduced cost** of the variable x_j.

Theorem

Consider $\boldsymbol{x}$ a BFS associated with basis $\boldsymbol{B}$, and let $\overline{\boldsymbol{c}}$ be the corresponding reduced cost, that is $\overline{\boldsymbol{c}} = (\overline{c}_1, \ldots, \overline{c}_n)^T$, where $\overline{c}_j = c_j - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{A}_j$, $\forall j$. Then

(a) *If $\overline{\boldsymbol{c}} \geq 0 \Rightarrow \boldsymbol{x}$ optimal*

(b) *If $\boldsymbol{x}$ optimal and non-degenerate $\Rightarrow \overline{\boldsymbol{c}} \geq 0$*

Proof of (a)

- Let $\boldsymbol{y}$ arbitrary feasible solution and $\boldsymbol{d} = \boldsymbol{y} - \boldsymbol{x}$.
- $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{y} = \boldsymbol{b} \Rightarrow \boldsymbol{A}\boldsymbol{d} = 0$
- $\Rightarrow \boldsymbol{B}\boldsymbol{d}_B + \sum_{i \in N} \boldsymbol{A}_i d_i = 0 \Rightarrow \boldsymbol{d}_B = -\sum_{i \in N} \boldsymbol{B}^{-1} \boldsymbol{A}_i d_i$
- $\Rightarrow \boldsymbol{c}^T \boldsymbol{d} = \boldsymbol{c}_B^T \boldsymbol{d}_B + \sum_{i \in N} c_i d_i = \sum_{i \in N} (c_i - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{A}_i) d_i = \sum_{i \in N} \overline{c}_i d_i$
- Since $y_i \geq 0$ and $x_i = 0, i \in N$, then $d_i = y_i - x_i \geq 0, i \in N$
- $\boldsymbol{c}^T \boldsymbol{d} = \boldsymbol{c}^T (\boldsymbol{y} - \boldsymbol{x}) \geq 0 \;\Rightarrow \boldsymbol{c}^T \boldsymbol{y} \geq \boldsymbol{c}^T \boldsymbol{x} \Rightarrow \boldsymbol{x}$ optimal

Again ... slowing down a bit ... matrix view

LO in standard form, A full row rank

$$
\begin{array}{ll}
\min & \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} \\
\text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
 & \boldsymbol{x} \geq 0
\end{array}
$$

$\boldsymbol{x}^{\boldsymbol{T}} = (\boldsymbol{x}_B^T, \boldsymbol{x}_N^T)$

$\boldsymbol{x}_B$ basic variables
$\boldsymbol{x}_N$ non-basic variables

$$
\begin{aligned}
&\boldsymbol{A} = [\boldsymbol{B}, \boldsymbol{N}] \\
&\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \Rightarrow \quad \boldsymbol{B}\boldsymbol{x}_B + \boldsymbol{N}\boldsymbol{x}_N = \boldsymbol{b} \\
&\Rightarrow \boldsymbol{x}_B + \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_N = \boldsymbol{B}^{-1}\boldsymbol{b} \\
&\Rightarrow \boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_N
\end{aligned}
$$

Again ... slowing down a bit ...

Reduced Costs

$$\begin{aligned} z &= \boldsymbol{c}_B^T \boldsymbol{x}_B + \boldsymbol{c}_N^T \boldsymbol{x}_N \\ &= \boldsymbol{c}_B^T (\boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1} N \boldsymbol{x}_N) + \boldsymbol{c}_N^T \boldsymbol{x}_N \\ &= \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{b} + (\boldsymbol{c}_N^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{N}) \boldsymbol{x}_N \end{aligned}$$

$$\boxed{\overline{c}_j = c_j - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{A}_j \quad \forall j \in N \quad \text{the relevant reduced costs}}$$

Again ... slowing down a bit ...

Recall ... Optimality Conditions

Theorem

- *$\boldsymbol{x}$ BFS associated with basis B*
- *$\overline{\boldsymbol{c}}$ reduced costs*

  Then
- *If $\overline{\boldsymbol{c}} \geq 0 \Rightarrow \boldsymbol{x}$ optimal*
- *$\boldsymbol{x}$ optimal and non-degenerate $\Rightarrow \overline{\boldsymbol{c}} \geq 0$*

Improving the Cost

When does this happen?

- Let $\boldsymbol{d}_B = -\boldsymbol{B}^{-1}\boldsymbol{A}_j$
  $d_j = 1,\ d_i = 0,\ i \neq B(1), \ldots, B(m), j.$
- Let $\boldsymbol{y} = \boldsymbol{x} + \theta\boldsymbol{d}, \quad \theta > 0$ scalar

$$\begin{aligned} \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{y} - \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} &= \theta\boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{d} \\ &= \theta(\boldsymbol{c}_B^T\boldsymbol{d}_B + c_j d_j) \\ &= \theta(c_j - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A}_j) \\ &= \theta\overline{c}_j \end{aligned}$$

Thus, if $\overline{c}_j < 0$ cost will decrease.

Unboundness

When does this happen?

- Is $\boldsymbol{y} = \boldsymbol{x} + \theta \boldsymbol{d}$ feasible?
  Since $\boldsymbol{Ad} = 0 \Rightarrow \boldsymbol{Ay} = \boldsymbol{Ax} = \boldsymbol{b}$
- $\boldsymbol{y} \geq 0$?
  If $\boldsymbol{d} \geq 0 \Rightarrow \boldsymbol{x} + \theta \boldsymbol{d} \geq 0 \quad \forall\, \theta \geq 0$
  $\Rightarrow$ objective unbounded.

Step-size

How big can we make the step length θ

If $d_i < 0$, then

$$x_i + \theta d_i \geq 0 \Rightarrow \theta \leq -\frac{x_i}{d_i}$$

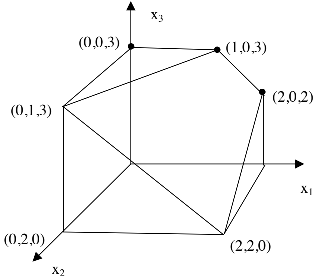
$$\Rightarrow \theta^* = \min_{\{i | d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$$

$$\Rightarrow \theta^* = \min_{\{i=1,\ldots,m | d_{B(i)} < 0\}} \left(-\frac{x_{B(i)}}{d_{B(i)}} \right)$$

Moving from one basis to another

Example

$$
\begin{array}{llllll}
\min & x_1 + & 5x_2 & -2x_3 & & \\
\text{s.t.} & x_1 + & x_2 + & x_3 & \leq 4 \\
& x_1 & & & \leq 2 \\
& & & x_3 & \leq 3 \\
& & 3x_2 + & x_3 & \leq 6 \\
& x_1, & x_2, & x_3 & \geq 0
\end{array}
$$



Moving from one basis to another

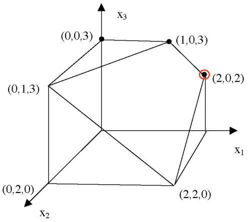
Example

$$\begin{array}{ccccccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \\ 6 \end{pmatrix}$$

$B = [\boldsymbol{A}_1, \boldsymbol{A}_3, \boldsymbol{A}_6, \boldsymbol{A}_7]$

BFS: $\boldsymbol{x} = (2, 0, 2, 0, 0, 1, 4)^T$



Moving from one basis to another

Example

BFS: $\boldsymbol{x}^{\boldsymbol{T}} = (2, 0, 2, 0, 0, 1, 4)$

$$\boldsymbol{B} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{B}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

$$\overline{\boldsymbol{c}}^{T} = (0, 7, 0, 2, -3, 0, 0) \qquad (= \boldsymbol{c}^{\boldsymbol{T}} - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{A})$$

$$d_5 = 1, d_2 = d_4 = 0, \quad \boldsymbol{d}_B = \begin{pmatrix} d_1 \\ d_3 \\ d_6 \\ d_7 \end{pmatrix} = -\boldsymbol{B}^{-1} \boldsymbol{A}_5 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\boldsymbol{d}^{\boldsymbol{T}} = (-1, 0, 1, 0, 1, -1, -1)$$

$$\boldsymbol{y}^{\boldsymbol{T}} = \boldsymbol{x}^{\boldsymbol{T}} + \theta \boldsymbol{d}^{\boldsymbol{T}} = (2 - \theta, 0, 2 + \theta, 0, \theta, 1 - \theta, 4 - \theta)$$

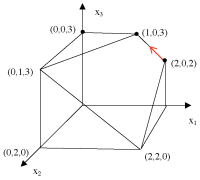
Moving from one basis to another

Example

$$\boldsymbol{y}^{\boldsymbol{T}} = \boldsymbol{x}^{\boldsymbol{T}} + \theta \boldsymbol{d}^{\boldsymbol{T}} = (2-\theta, 0, 2+\theta, 0, \theta, 1-\theta, 4-\theta)$$

What happens as θ increases?

$$\theta^* = \min_{\{i=1,\dots,m | d_{B(i)} < \mathbf{0}\}} \left(-\frac{x_{B(i)}}{d_i}\right) = \min\left(-\frac{2}{(-1)}, -\frac{1}{(-1)}, -\frac{4}{(-1)}\right) = 1.$$



$\Rightarrow \boldsymbol{A}_6$ exits the basis.

New solution $\boldsymbol{y} = (1, 0, 3, 0, 1, 0, 3)^T$

New basis $\overline{\boldsymbol{B}} = (\boldsymbol{A}_1, \boldsymbol{A}_3, \boldsymbol{A}_5, \boldsymbol{A}_7)$

Moving from one basis to another

Example

New basis $\overline{\boldsymbol{B}} = (\boldsymbol{A}_1, \boldsymbol{A}_3, \boldsymbol{A}_5, \boldsymbol{A}_7)$

$$\overline{\boldsymbol{B}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \overline{B}^{-1} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\overline{\boldsymbol{c}}^T = \boldsymbol{c}^{\boldsymbol{T}} - \boldsymbol{c}_{\overline{B}}^T \, \overline{\boldsymbol{B}}^{-1} \boldsymbol{A} = (0, 4, 0, -1, 0, 3, 0)$$

... need to continue, column $\boldsymbol{A}_4$ enters the basis ...

The Simplex Algorithm

1. Start with basis $\boldsymbol{B} = [\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(m)}]$ and a BFS $\boldsymbol{x}$.
2. Compute $\overline{c}_j = c_j - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{A}_j$
   - If $\overline{c}_j \geq 0$; $\boldsymbol{x}$ optimal; stop.
   - Else select $j : \overline{c}_j < 0$.
3. Compute $\boldsymbol{d} = -\boldsymbol{B}^{-1} \boldsymbol{A}_j$.
   - If $\boldsymbol{d} \geq 0 \Rightarrow$ cost unbounded; stop
   - Else
4. $\theta^* = \min_{1 \leq i \leq m, d_i < 0} \dfrac{x_{B(i)}}{-d_i} \doteq \dfrac{x_{B(\ell)}}{-d_\ell}$
5. Form a new basis by replacing $\boldsymbol{A}_{B(\ell)}$ with $\boldsymbol{A}_j$.
6. $y_j = \theta^*$, $y_{B(i)} = x_{B(i)} + \theta^* d_i$, $i \neq \ell$.

Theorem

- $P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x} \geq 0\} \neq \emptyset$
- *Every BFS non-degenerate*
  Then
- *Simplex method terminates after a finite number of iterations*
- *At termination, we have an optimal basis B or we have a direction* $\boldsymbol{d} : \boldsymbol{A}\boldsymbol{d} = 0, \boldsymbol{d} \geq 0, \boldsymbol{c}^T\boldsymbol{d} < 0$ *and optimal cost is* $-\infty$.

- θ^* can equal zero (why?)
  $\Rightarrow \boldsymbol{y} = \boldsymbol{x}$, although $\overline{\boldsymbol{B}} \neq \boldsymbol{B}$.
- Even if $\theta^* > 0$, there might be a tie for

$$\min_{1 \leq i \leq m, u_i > 0} \frac{x_{B(i)}}{u_i}$$

  $\Rightarrow$ next BFS degenerate.
- Conclusion: Finite termination not guaranteed; cycling is possible.

- Cycling can be avoided by carefully selecting which variables enter and exit the basis.
- Example: among all variables $\overline{c}_j < 0$, pick the smallest subscript; among all variables eligible to exit the basis, pick the one with the smallest subscript.