Recitation 3

15.093 Optimization Methods

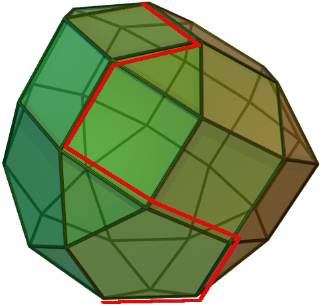
09/24/2021

Agenda

- Summary of simplex algorithm
- True/False
- Exercise on tableau manipulation

Simplex algorithm: geometric view

Simplex algorithm: a path on the edges of the polyhedron graph

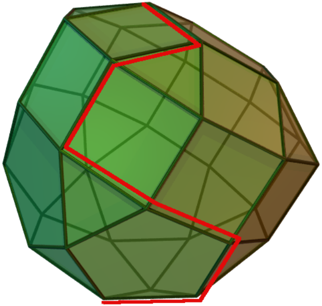


Edges directed to decrease objective cost: **monotone** path

Simplex algorithm: geometric view

A step corresponds to a change of solution by pivot:

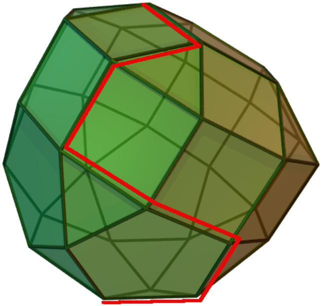
- leave a facet of the polyhedron (entering variable)
- follow the edge: $n-1$ facets kept active (non-basic variables remain null except for entering variable)
- hit another facet of polyhedron (leaving variable)



Simplex algorithm: geometric view

What if we hit at least 2 planes at same time? Degenerate vertex

- we can choose which variable to leave the basis
- several basis will represent the same geometric point



Simplex algorithm: algebraic view

- Initial basis $\boldsymbol{B}$, and solution $\boldsymbol{x}$: $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$ and $\mathbf{x}_N = 0$.
- compute reduced costs: $\tilde{\boldsymbol{c}}^\top = \boldsymbol{c}^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{A}$
  - if $\tilde{\boldsymbol{c}} \geq 0$: optimality
  - otherwise select entering variable $\tilde{c}_j < 0$.
- compute basic direction $\boldsymbol{d}_B = -\boldsymbol{B}^{-1}\boldsymbol{A}_j$
  - if $\boldsymbol{d}_B \geq 0$: **unbounded problem**
  - otherwise select entering variable $B(l)$ in $\arg\min_i \frac{x_{B(i)}}{-d_{B(i)}}$
- compute solution for new basis

To find initial BFS

Assuming $\boldsymbol{b} \geq \mathbf{0}$, solve the problem

$$\begin{aligned} \min \quad & y_1 + \dots y_m \\ s.t. \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b} \\ & \boldsymbol{x}, \boldsymbol{y} \geq \mathbf{0}. \end{aligned}$$

- if positive cost: **infeasible problem**
- otherwise get a BFS by pushing the y_i out of basis

Pivot rules and complexity

Which entering variable to choose at each step?

- minimal index: Bland's minimal rule
- most negative reduced cost: Dantzig rule
- most improvement in objective cost
- steepest edge
- shadow vertex rule
- rules on dual
- ...

Pivot rules and complexity

- Open question: does there exist a pivot rule for which number of simplex steps is polynomial in m and n?
- Worst case is exponential for all presented pivot rules.

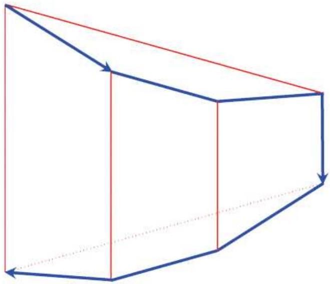


Figure: Klee-Minty cube

Exercise 1: True/False

1. Every polyhedron $P \subset \mathbb{R}^n$ can be written in standard form $P = \{\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0\}$.
2. Suppose a LP has finite cost. The set of optimal solutions is a polyhedron.
3. At an optimal solution of an LP in $\mathbb{R}^n$ there are at least n active constraints
4. If there exists a vector $\boldsymbol{q} \neq 0$ for which $\boldsymbol{A}\boldsymbol{q} = 0$, then the polyhedron $\{\boldsymbol{x}, \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$ doesn't have any vertex.
5. Suppose $\{\boldsymbol{x}, \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$ is non-empty and bounded, then $\boldsymbol{x} = 0$ is the only vector for which $\boldsymbol{A}\boldsymbol{x} = 0$.
6. Consider the LP $\min \boldsymbol{c}^\top \boldsymbol{x}$ s.t. $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$. If we increase some component of $\boldsymbol{b}$ then the optimal cost cannot increase.

Simplex tableau

Allows to do the pivots easily "by hand"

$-\boldsymbol{c}_B^\top \boldsymbol{x}_B$	$\tilde{\boldsymbol{c}} = \boldsymbol{c} - \boldsymbol{c}_B^\top \boldsymbol{A}$
$\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$	$\boldsymbol{B}^{-1}\boldsymbol{A}$

Exercise 2: Simplex tableau

Consider a LP in standard form with $\boldsymbol{A} = \begin{bmatrix} \star & \star & 1 & 0 & \star \\ \star & \star & 0 & 1 & \star \end{bmatrix}$. After some iterations we get the tableau

$$
\begin{array}{c|ccccc}
\star & 0 & 0 & 2 & \gamma & 3 \\
\hline
3 & 1 & 0 & -2 & -3 & -2 \\
\beta & 0 & 1 & 0 & \alpha & 0
\end{array}
$$

a) Suppose $\beta > 0$. Find necessary and sufficient condition for current solution to be optimal.

b) If $\beta \geq 0$ and problem is bounded, what can we say on α?

c) Show that $x_5 = 0$ at an optimal solution.

d) Suppose $\alpha = \beta = \gamma = 1$. If $\boldsymbol{B}$ is the current basis matrix, what is $\boldsymbol{B}^{-1}$.

e) Suppose the current BFS is nondegenerate, does there exist a solution for which $x_2 = x_4 = 0$?

Exercise 2: Simplex tableau

Consider a LP in standard form with $\boldsymbol{A} = \begin{bmatrix} \star & \star & 1 & 0 & \star \\ \star & \star & 0 & 1 & \star \end{bmatrix}$. After some iterations we get the tableau

$$\begin{array}{c|ccccc} \star & 0 & 0 & 2 & \gamma & 3 \\ \hline 3 & 1 & 0 & -2 & -3 & -2 \\ \beta & 0 & 1 & 0 & \alpha & 0 \end{array}$$

f) Suppose $\beta = 1$ and that current solution is optimal. Find necessary and sufficient solution for having multiple optimal solutions.
g) Suppose $\beta = 1$. Find necessary and sufficient condition to terminate simplex after 1 additional iteration, with indication that the problem is unbounded.
h) Suppose $\beta = -1$. Find necessary and sufficient condition for the problem to be infeasible.