6.215/6.255J/15.093J/IDS.200J Optimization Methods

Lecture 13: Discrete Optimization I

October 21, 2021

- Modeling with integer variables
- What is a good formulation?
- Theme: The power of formulations

The class of Mixed Integer Optimization (MIO) problems:

$$\begin{array}{ll} \max & \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} \le \boldsymbol{b} \\ & \boldsymbol{x} \in Z_+^n \quad (\boldsymbol{x} \ge 0, \boldsymbol{x} \text{ integer}) \\ & \boldsymbol{y} \in R_+^n \quad (\boldsymbol{y} \ge 0) \end{array} \tag{MIO}$$

Remarks:

- entries of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ usually integers
- $\boldsymbol{x}$: integer-valued (discrete) variables
- $\boldsymbol{y}$: real-valued (continuous) variables

Integer (Linear) Optimization

Special cases

Integer Optimization (IO):

$$\begin{aligned} \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \in Z_+^n \end{aligned} \tag{IO}$$

Binary Optimization (BO):

$$\begin{aligned} \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \in \{0,1\}^n \end{aligned} \tag{BO}$$

- Can use 0/1 variables to model binary choices
- For instance, events that may or may not occur:

$$x_j \in \begin{cases} 1, & \text{if event } j \text{ occurs} \\ 0, & \text{otherwise} \end{cases}$$

Modeling with Binary Variables

Binary choice

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- Example: BO formulation of the "knapsack" problem

n : projects, total budget b
a_j : cost of project j
r_j : revenue of project j

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$$x_j = \begin{cases} 1, & \text{if project } j \text{ is selected.} \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{array}{ll} \max & \sum_{j=1}^{n} r_j x_j \\ \text{s.t.} & \sum_{j=1}^{n} a_j x_j \leq b \\ & x_j \in \{0, 1\} \end{array}$$

- At most one event in S can occur

$$\sum_{j \in S} x_j \leq 1$$

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- Neither or both events i and j can occur

$$x_j - x_i = 0$$

Modeling with Binary Variables

Modeling relations

- At most one event in S can occur

$$\sum_{j \in S} x_j \leq 1$$

- Neither or both events i and j can occur

$$x_j - x_i = 0$$

- If event j occurs, then event i occurs

$$x_j \leq x_i$$

Modeling with Binary Variables

Modeling relations

- At most one event in S can occur

$$\sum_{j \in S} x_j \leq 1$$

- Neither or both events i and j can occur

$$x_j - x_i = 0$$

- If event j occurs, then event i occurs

$$x_j \leq x_i$$

- If event i does't occur, then $y = 0$; otherwise y is unconstrained

$$0 \leq y \leq Mx_i,$$

with M a "large" constant.

Modeling with Binary Variables

An assignment problem

- Instance of the problem

$$\begin{array}{rl} m & \text{jobs} \\ n \geq m & \text{people} \\ c_{ij}: & \text{cost of assigning job } i \text{ to person } j \end{array}$$

Modeling with Binary Variables

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- Decision variable

$$x_{ij} = \begin{cases} 1 & \text{job } i \text{ is assigned to person } j \\ 0 & \text{otherwise} \end{cases}$$

Modeling with Binary Variables

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- Decision variable

$$x_{ij} = \begin{cases} 1 & \text{job } i \text{ is assigned to person } j \\ 0 & \text{otherwise} \end{cases}$$

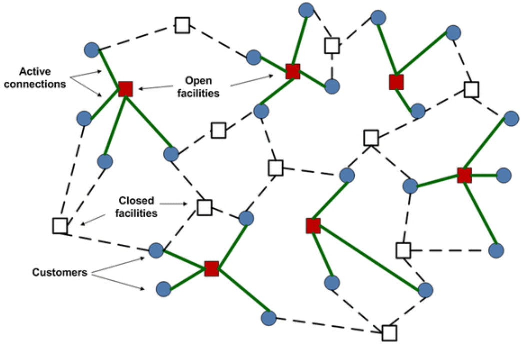
- Formulation

$$\begin{array}{ll} \min & \sum c_{ij} x_{ij} \\ \text{s.t.} & \sum_{j=1}^{n} x_{ij} = 1 \quad \text{each job } i \text{ is assigned, } i = 1, \dots, m \\ & \sum_{i=1}^{m} x_{ij} \leq 1 \quad \text{each person } j \text{ can do at most one job, } j = 1, \dots, n \\ & x_{ij} \in \{0, 1\} \end{array}$$

- Here we could replace $x_{ij} \in \{0, 1\}$ by $0 \leq x_{ij} \leq 1$, why?

What is a Good Formulation?

Facility location problem



- Instance

$$
\begin{array}{ll}
N = \{1 \ldots n\} & \text{potential facility locations} \\
I = \{1 \ldots m\} & \text{set of clients}
\end{array}
$$

c_j : cost of opening a facility at location j

d_{ij} : cost of serving client i from facility j

What is a Good Formulation?

Facility location problem

- Decision variables

$$x_j = \begin{cases} 1, & \text{facility is placed at location } j \\ 0, & \text{otherwise} \end{cases}$$

$$y_{ij} = \begin{cases} 1, & \text{client } i \text{ is served by facility } j \\ 0, & \text{otherwise} \end{cases}$$

- Facility location (FL) formulation:

$$\begin{aligned} IZ_{FL} = \min \quad & \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} y_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^{n} y_{ij} = 1 \\ & y_{ij} \leq x_j \\ & x_j, y_{ij} \in \{0, 1\} \end{aligned}$$

What is a Good Formulation?

Facility location problem

- Aggregate facility location (AFL) formulation

$$
\begin{aligned}
IZ_{AFL} = \min \quad & \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} y_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{n} y_{ij} = 1 \\
& \sum_{i=1}^{m} y_{ij} \leq m \cdot x_j \\
& x_j, y_{ij} \in \{0, 1\}
\end{aligned}
$$

- This is also a valid formulation. Why?
- Which one is preferable?

What is a Good Formulation?

Observations

- $IZ_{FL} = IZ_{AFL}$, since the integer points in both formulations are the same.
- Relaxing the integrality restrictions, consider the two polyhedra:

$$P_{FL} = \{(\boldsymbol{x}, \boldsymbol{y}) : \sum_{j=1}^{n} y_{ij} = 1, y_{ij} \leq x_j, 0 \leq x_j \leq 1, 0 \leq y_{ij} \leq 1\}$$

$$P_{AFL} = \{(\boldsymbol{x}, \boldsymbol{y}) : \sum_{j=1}^{n} y_{ij} = 1, \sum_{i=1}^{m} y_{ij} \leq m \cdot x_j, 0 \leq x_j \leq 1, 0 \leq y_{ij} \leq 1\}$$

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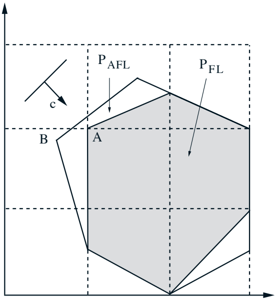
- Now, let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in P_{FL}$:
  - then, $\bar{y}_{ij} \leq \bar{x}_j$ for all i and j
  - summing over i gives $\sum_{i=1}^{m} \bar{y}_{ij} \leq \bar{x}_j \cdot m$
  - hence, $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in P_{FL} \implies (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in P_{AFL}$, that is $P_{FL} \subseteq P_{AFL}$
  - can show that the reverse is not always true
  - so $P_{FL} \subset P_{AFL}$

- Consider the linear optimization relaxations:

$$Z_{FL} = \begin{array}{c} \min \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y}, \\ (\boldsymbol{x}, \boldsymbol{y}) \in P_{FL} \end{array} \qquad Z_{AFL} = \begin{array}{c} \min \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y} \\ (\boldsymbol{x}, \boldsymbol{y}) \in P_{AFL} \end{array}$$

- We have $Z_{AFL} \leq Z_{FL} \leq IZ_{FL} = IZ_{AFL}$



What is a Good Formulation?

Implications

- Finding $IZ_{FL} = IZ_{AFL}$ is difficult.
- Solving to find Z_{FL}, Z_{AFL} are both easy (linear optimization relaxations).
- Since Z_{FL} is closer to IZ_{FL} several methods (to be seen later in the class) would work better (actually much better).
- In conclusion, the (FL) formulation <u>is better</u> than the (AFL) formulation, despite the fact that it has a larger number of constraints
- Can we define a general criterion?

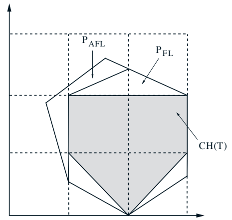
What is a Good Formulation?

Ideal formulations

- Let P be the polyhedron of a linear optimization (LO) relaxation of an integer optimization (IO) problem. Assume P is bounded.
- Let $T = \{\boldsymbol{x} : \boldsymbol{x} \in Z_+^n\} \cap P$, be the finite set of feasible integer solutions
- Consider the **Convex Hull of** T, defined as:

$$CH(T) = \{\boldsymbol{x} : \boldsymbol{x} = \sum_i \lambda_i x^i, \sum_i \lambda_i = 1, \lambda_i \geq 0, x^i \in T\}$$

- $CH(T)$ is a polyhedron
- Example

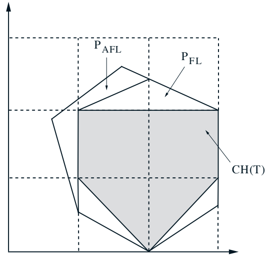


What is a Good Formulation?

Ideal formulations

- Note that the extreme points of the polyhedron $CH(T)$ have integer-valued coordinates.
- So, if we knew $CH(T)$ explicitly, i.e., say, represented as $CH(T) = \{\boldsymbol{x} : \boldsymbol{Dx} \leq d\}$, then by solving the LO $\min_{\boldsymbol{x} \in CH(T)} \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x}$, we would solve the initial IO problem.
- Message: Quality of formulation is judged by closeness to $CH(T)$.
- Example: in the facility location problem

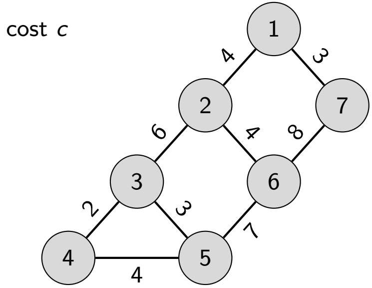
$$CH(T) \subseteq P_{FL} \subseteq P_{AFL}$$



Minimum Spanning Tree (MST)

A binary optimization formulation

- Given a graph $G = (N, E)$ undirected and costs c_e, $e \in E$.
- Find a tree of minimum cost spanning all the nodes.

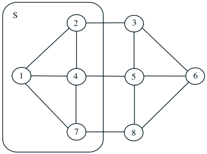


- Decision variables $x_e = \begin{cases} 1, & \text{if edge } e \text{ is included in the tree} \\ 0, & \text{otherwise} \end{cases}$

Minimum Spanning Tree (MST)

A binary optimization formulation

- The tree should be connected. How can you model this requirement?
  - Let S be a set of nodes. Then S and $N \setminus S$ should be connected
  - Define the cutset $\delta(S)$ as $\{e = (i,j) \in E : i \in S \text{ and } j \in N \setminus S\}$



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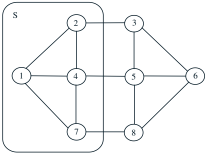

$$\Rightarrow \sum_{e \in \delta(S)} x_e \geq 1$$

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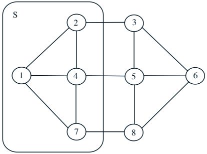

$$\Rightarrow \sum_{e \in \delta(S)} x_e \geq 1$$

- The number of edges in a spanning tree is $n - 1$

$$\Rightarrow \sum_{e \in E} x_e = n - 1$$

Minimum Spanning Tree (MST)

A binary optimization formulation

- Cutset Formulation. Let $\delta(S) = \{e = (i,j) \in E : i \in S, j \in N \setminus S\}$:

$$\begin{array}{ll} \min & \sum_{e \in E} c_e x_e \\ \text{s.t.} & \\ T & \left\{ \begin{array}{ll} \sum_{e \in \delta(S)} x_e \geq 1 & \forall\, S \subseteq N, S \neq \emptyset, N \\ \sum_{e \in E} x_e = n - 1 & \\ x_e \in \{0,1\} & \end{array} \right. \end{array}$$

- This formulation has $2^n - 2 + 1$ constraints (beyond the binary constraints on x)
- Is this a good formulation?

Minimum Spanning Tree (MST)

A binary optimization formulation

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- This formulation has $2^n - 2 + 1$ constraints (beyond the binary constraints on $\boldsymbol{x}$)
- Is this a good formulation?
- Consider the polyhedron of its linear optimization relaxation:

$$P_{cut} = \{\boldsymbol{x} : 0 \leq x_e \leq 1 \ \forall e; \sum_{e \in E} x_e = n - 1; \sum_{e \in \delta(S)} x_e \geq 1 \ \ \forall\, S \subseteq N, S \neq \emptyset, N\}$$

- Is P_{cut} the $CH(T)$? ...no

Minimum Spanning Tree (MST)

Another formulation

- Subtour (cycle) elimination formulation. Let $E(S) = \{e = (i,j) \in E : i,j \in S\}$:

$$
\begin{array}{ll}
\min & \sum_{e \in E} c_e x_e \\
\text{s.t.} & \\
T & \left\{ \begin{array}{ll} \sum_{e \in E(S)} x_e \leq |S| - 1 & \forall\, S \subseteq N,\ S \neq \emptyset, N \\ \sum_{e \in E} x_e = n - 1 & \\ x_e \in \{0,1\}. & \end{array} \right.
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- This is a valid IP formulation. Why?
- Same number of constraints as the previous one. Is this a good formulation?

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- This is a valid IP formulation. Why?
- Same number of constraints as the previous one. Is this a good formulation?
- Consider again the polyhedron of its linear optimization relaxation:

$$P_{sub} = \{\boldsymbol{x} : 0 \leq x_e \leq 1\ \forall e; \sum_{e \in E} x_e = n - 1;\ \sum_{e \in E(S)} x_e \leq |S| - 1\ \ \forall\, S \subset N,\ S \neq \emptyset, N\}$$

- Theorem: $P_{sub} = CH(T) \subset P_{cut} \Rightarrow P_{sub}$ is the best possible formulation.
- Also note that good formulations can have an exponential number of constraints.

The Traveling Salesman Problem

Formulation I

- Problem: Given an undirected graph $G = (N, E)$ and edge costs $c_e \; \forall \; e \in E$, find a tour (a cycle that visits all nodes) of minimum cost.

- Decision variables:

$$x_e = \begin{cases} 1, & \text{if edge } e \text{ is included in the tour.} \\ 0, & \text{otherwise.} \end{cases}$$

- Cutset formulation. Let $\delta(S) = \{e = (i, j) \in E : i \in S, j \in N \setminus S\}$:

$$\begin{array}{ll} \min & \sum_{e \in E} c_e x_e \\ \text{s.t.} & \\ T & \left\{ \begin{array}{ll} \sum_{e \in \delta(S)} x_e \geq 2 & \forall \; S \subseteq N, S \neq \emptyset, N \\ \sum_{e \in \delta(i)} x_e = 2 & \forall \; i \in N \\ x_e \in \{0, 1\} & \end{array} \right. \end{array}$$

The Traveling Salesman Problem

Formulation II

- Decision variables, as before:

$$x_e = \begin{cases} 1, & \text{if edge } e \text{ is included in the tour.} \\ 0, & \text{otherwise.} \end{cases}$$

- Subtour elimination formulation. Let $E(S) = \{e = (i,j) \in E : i,j \in S\}$:

$$\begin{array}{ll} \min & \sum c_e x_e \\ \text{s.t.} & \\ T & \left\{ \begin{array}{ll} \sum_{e \in E(S)} x_e \leq |S| - 1 & \forall\, S \subseteq N, S \neq \emptyset, N \\ \sum_{e \in \delta(i)} x_e = 2 & \forall\, i \in N \\ x_e \in \{0,1\} & \end{array} \right. \end{array}$$

- Consider the polyhedra of their linear optimization relaxations:

$$P_{cut}^{TSP} = \{\mathbf{x} : 0 \le x_e \le 1 \ \forall e; \sum_{e \in \delta(i)} x_e = 2 \ \forall \ i \in N; \sum_{e \in \delta(S)} x_e \ge 2 \ \forall S \subset N, S \neq \emptyset\}$$

$$P_{sub}^{TSP} = \{\mathbf{x} : 0 \le x_e \le 1 \ \forall e; \sum_{e \in \delta(i)} x_e = 2 \ \forall \ i \in N; \sum_{e \in E(S)} x_e \le |S| - 1 \forall S \subset N, S \neq \emptyset\}$$

- Theorem: $P_{cut}^{TSP} = P_{sub}^{TSP} \not\supseteq CH(T)$
- Nobody knows $CH(T)$ for the TSP

Final Observations

- For the MST there are in fact many efficient algorithms. And $CH(T)$ is known explicitly (subtour elimination formulation).
- For TSP there are no known efficient algorithm. TSP is an NP-hard problem. Its $CH(T)$ is not known explicitly.
- Conjecture: The convex hull of IO problems that are polynomially solvable are explicitly known.