15.093 Optimization Methods
Practice Questions for Midterm

Problem 1 – True/False Questions: (21 points)

Classify the following statements as true or false. All answers must be justified, or no credit can be given. All LOs are assumed in standard form (min $\mathbf{c}'\mathbf{x}\ :\ \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}$ and $\mathbf{A} \in \mathrm{R}^{m \times n}$ with linearly independent rows).

1. (3 points) An iteration of the simplex method may move the feasible solution by a positive distance while leaving the cost unchanged.
2. (3 points) If there is a nondegenerate optimal basis, then it must be unique.
3. (3 points) If $\mathbf{x}^\star$ is an optimal solution found by the simplex method, no more than m components can be positive.
4. (3 points) The function

$$
\begin{aligned}
-\infty < F(\mathbf{y}) := \min \quad & \mathbf{c}'\mathbf{x} \\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{y} \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}
$$

   is convex.
5. (3 points) Consider a degenerate basic solution $\mathbf{x}$. The solution $\mathbf{x}$ has at least two distinct bases.
6. (3 points) Let $\mathbf{c} \geq 0$. If the primal problem is infeasible, then its dual must be unbounded.
7. (3 points) If the optimal dual variable corresponding to the i-th constraint is $p_i^\star = 0$, than the i-th constraint can not be active at primal optimality.

Problem 2 – Simplex Method: (21 points)

Consider the linear optimization problem

$$
\begin{array}{lrcrl}
\min & -x_1 & - & x_2 & \\
\text{s.t.} & 2x_1 & + & 3x_2 & \leq 6 \\
& 3x_1 & + & 2x_2 & \leq 6 \\
& x_1 & , & x_2 & \geq 0
\end{array}
$$

1. (2 point) Bring the problem in standard form.
2. (8 points) Use the primal simplex method (full tableau) starting from $(x_1, x_2) = (0, 0)$. *Hint : Final* Tableau should look like

		x_1	x_2	s_1	s_2
	12/5	0	0	1/5	1/5
x_2	6/5	0	1	3/5	-2/5
x_1	6/5	1	0	-2/5	3/5

3. (8 points) Suppose we forgot to add the constraint $3x_1 + x_2 + s_3 = 3$, $s_3 \geq 0$. State the new tableau for basis $B = \{x_2, x_1, s_3\}$.
$$\begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 1 & 3 & 1 \end{pmatrix}^{-1} = \frac{1}{5} \cdot \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 3 & -7 & 5 \end{pmatrix}.$$

4. (3 points) Suppose you were to solve the optimization problem with the additional constraint starting from the tableau derived in part 3. Would you use the primal or dual simplex method? Justify your choice.

Problem 3 – Duality/Sensitivity: (21 points)

Consider the linear optimization problem

$$\begin{array}{lrcrcrl} \max & x_1 & & & - & 3x_3 & \\ \text{s.t.} & x_1 & + & 2x_2 & - & 3x_3 & \leq -1 \\ & 2x_1 & - & x_2 & - & x_3 & \leq -2 \\ & x_1 & , & x_2 & , & x_3 & \geq 0 \end{array}$$

1. (3 points) Formulate the dual problem.
2. (5 points) Proof that $\mathbf{x}^\star = (0, 1, 1)$ is an optimal solution using the complementarity conditions.
3. (5 points) Determine the range of all possible objective coefficients of x_3 for which $\mathbf{x}^\star = (0, 1, 1)$ remains optimal. *Hint:*
$$\begin{pmatrix} 2 & -3 \\ -1 & -1 \end{pmatrix}^{-1} = \frac{1}{5} \cdot \begin{pmatrix} 1 & -3 \\ -1 & -2 \end{pmatrix}.$$
4. (5 points) Determine the range of the right-hand side of the first constraint such that the basis $B = \{2, 3\}$ remains optimal.
5. (3 points) Determine the new optimal value if the right-hand side of the first constraint becomes -2.

Solution 1 – True/False Questions

1. False. The rate of change of objective value in an iteration of the simplex method is given by the reduced cost $\bar{c}_i$. Since the simplex method only selects negative reduced cost variables, this implies that movement by a positive distance entails a reduction in the objective.

2. False, there could be multiple bases of which at least one is non-degenerate. For instance, consider the following LP:

$$\begin{aligned} \min \quad & x_2 \\ \text{s.t.} \quad & x_2 = 1, \\ & x_1 + x_3 = 1, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

   Then, $(1, 1, 0)$ and $(0, 1, 1)$ both correspond to non-degenerate optimal BFS's. *Common mistake:* providing a counterexample with multiple optimal solutions but a single optimal basis.

3. True, since simplex pivots between basic feasible solutions, and these solutions have at least $n - m$ binding inequality constraints of the form $x_i \geq 0$, i.e., at most m strictly positive components.

4. True. Since $F(y)$ is assumed to attain a finite value, strong duality holds and we can write

$$F(y) = \max_p \; p^\top By \text{ s.t. } A^\top p \leq c.$$

   This implies that $F(y)$ is the pointwise maximum of a set of linear functions and hence convex.

5. False. Consider the following counterexample:

$$\begin{aligned} \min \quad & x_1 + x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 1, \\ & -x_1 + x_2 + x_3 = 1, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

   Then, the point $(0, 1, 0)$ is certainly degenerate, but it corresponds to a unique basis.

6. True. Since the primal is infeasible, its dual is either unbounded or infeasible. But its dual can be written as follows:

$$\max p^\top b \text{ s.t. } A^\top p \leq c, p \geq 0.$$

   But since $c \geq 0$, we can pick $p = 0$ as a feasible solution. Therefore, the dual must be unbounded.

7. False. The complementary slackness condition reads $p_i^\top(a_i^\top x - b_i) = 0$, but the fact that $p_i^* = 0$ does not imply that $a_i^\top x - b_i \neq 0$. In particular, we can have $p_i^* = 0$ and $a_i^\top x - b_i = 0$ simultaneously.

Solution 2 – Simplex Method:

1. Standard form formulation

$$\begin{array}{lccccccccl} \min & -x_1 & - & x_2 & & & & & & \\ \text{s.t.} & 2x_1 & + & 3x_2 & + & s_1 & & & & = 6 \\ & 3x_1 & + & 2x_2 & & & + & s_2 & & = 6 \\ & x_1 & , & x_2 & , & s_1 & , & s_2 & & \geq 0 \end{array}$$

   Common mistake: The constraints $s_1, s_2 \geq 0$ are crucial.

2. We take as initial basis $B = \{s_1, s_2\}$ with corresponding basic solution $\mathbf{x} = (0, 0, 6, 6)'$. The basic matrix is $\mathbf{B} = I_2$ and the reduced costs is $\bar{\mathbf{c}} = \mathbf{c} - \mathbf{c}'_B \mathbf{B}^{-1}\mathbf{A} = \mathbf{c}$. The initial tableau is hence given as

		x_1	x_2	s_1	s_2
	0	-1	-1	0	0
s_1	6	2	3	1	0
s_2	6	3	2	0	1

Both x_1 and x_2 have a negative reduced cost. There are now two options.

Option 1: We first bring x_1 into the basis in favor of s_2. New tableau:

		x_1	x_2	s_1	s_2
	2	0	-1/3	0	1/3
s_1	2	0	5/3	1	-2/3
x_1	2	1	2/3	0	1/3

As only x_2 has a negative reduced cost, we bring it into the basis in favor of s_1 and reach the final tableau. All reduced costs are positive and hence $(6/5, 6/5, 0, 0)$ is an optimal solution at the optimal basis $B^\star = \{x_2, x_1\}$.

Option 2: We first bring x_2 into the basis in favor of s_1. New tableau:

		x_1	x_2	s_1	s_2
	2	-1/3	0	1/3	0
x_2	2	2/3	1	1/3	0
s_2	2	5/3	0	-2/3	1

As only x_1 has a negative reduced cost, we bring it into the basis in favor of s_2 and reach the final tableau. All reduced costs are positive and hence $(6/5, 6/5, 0, 0)$ is an optimal solution at the optimal basis $B^\star = \{x_2, x_1\}$.

3. The new standard form problem

$$
\begin{array}{rrcrcrcrcrl}
\min & -x_1 & - & x_2 & & & & & & & \\
\text{s.t.} & 2x_1 & + & 3x_2 & + & s_1 & & & & & = 6 \\
 & 3x_1 & + & 2x_2 & & & + & s_2 & & & = 6 \\
 & 3x_1 & + & x_2 & & & & & + & s_3 & = 3 \\
 & x_1 & , & x_2 & , & s_1 & , & s_2 & , & s_3 & \geq 0
\end{array}
$$

includes now the constraint $\mathbf{a}'_3\mathbf{x} + s_3 = b_3$ with $\mathbf{a}_3 = (3, 1, 0, 0)'$ and $b_3 = 3$. The considered basis $\bar{B} = B^\star \cup \{s_3\} = \{x_2, x_1, s_3\}$ has now three elements and has basic matrix

$$
\bar{\mathbf{B}} = \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{a}'_{3,B^\star} & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 1 & 3 & 1 \end{pmatrix} \quad \text{and} \quad \bar{\mathbf{B}}^{-1} = \begin{pmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ -\mathbf{a}'_{3,B^\star}\mathbf{B}^{-1} & 1 \end{pmatrix} = \frac{1}{5} \cdot \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 3 & -7 & 5 \end{pmatrix}.
$$

From Section 5.1 (Sensitivity Analysis) we have that the new reduced costs satisfies $[\mathbf{c}, 0] - c_{\bar{B}}\bar{\mathbf{B}}^{-1}\bar{\mathbf{A}} = [\bar{\mathbf{c}}, 0]$. The new basis feasible solution satisfies $\mathbf{x}_{\bar{B}} = [x_B, b_3 - a'_{3,B^\star}x_B]$. Furthermore, the search directions satisfy

$$
\bar{\mathbf{B}}^{-1}\bar{\mathbf{A}} = \begin{pmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ -\mathbf{a}'_{3,B^\star}\mathbf{B}^{-1} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{a}'_3 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{A} & \mathbf{0} \\ \mathbf{a}'_3 - \mathbf{a}'_{3,B^\star}\mathbf{B}^{-1}\mathbf{A} & 1 \end{pmatrix}.
$$

The new tableau becomes

		x_1	x_2	s_1	s_2	s_3
	12/5	0	0	1/5	1/5	0
x_2	6/5	0	1	3/5	-2/5	0
x_1	6/5	1	0	-2/5	3/5	0
s_3	-9/5	0	0	3/5	-7/5	1

where the blue numbers need to be computed by hand and can not be derived from the final tableau given as a hint in part 2.

4. Note that indeed the basic solution corresponding to the basis $\hat{B}$ is not primal feasible. The dual simplex method should be preferred as the new tableau computed in part 3 corresponds to a dual feasible basic solution.

Solution 3 – Duality/Sensitivity:

1. The dual problem is

$$\begin{array}{lrcrl} \min & -p_1 & - & 2p_2 & \\ \text{s.t.} & p_1 & + & 2p_2 & \geq 1 \\ & 2p_1 & - & p_2 & \geq 0 \\ & -3p_1 & - & p_2 & \geq -3 \\ & p_1 & , & p_2 & \geq 0 \end{array}$$

2. The solution $\mathbf{x}^\star = (0, 1, 1)$ is easily verified to be a feasible solution in the primal problem. Complementarity guarantees that if we can find a $\mathbf{p}$ which is feasible in the dual problem and they satisfy the complementarity system

$$\mathbf{p}_i(\mathbf{a}_i'\mathbf{x} - b_i) = 0 \qquad \forall i = 1, \ldots, m$$
$$\mathbf{x}_i(\mathbf{p}'A_i - c_i) = 0 \qquad \forall i = 1, \ldots, m$$

than $\mathbf{x}$ and $\mathbf{p}$ are optimal in the primal and dual problems, respectively. Notice that the first set of constraints $\mathbf{p}_i(\mathbf{a}_i'\mathbf{x} - b_i) = 0$ is satisfied as $(\mathbf{a}_i'\mathbf{x} - b_i) = 0$. The second set of constraints reduces to the linear system

$$\begin{aligned} 2p_1 - p_2 &= 0, \\ -3p_1 - p_2 &= -3, \end{aligned}$$

with unique solution $\mathbf{p} = (p_1, p_2) = (3/5, 6/5)$. Hence, $\mathbf{x}$ and $\mathbf{p}$ satisfy the complementarity system. It remains to be verified whether $\mathbf{p}$ is dual feasible. Indeed, $p_1 + 2p_2 = 3 \geq 1$ and $p_1, p_2 \geq 0$.

Common mistake: You need to explicitly verify that the dual vector p is indeed feasible in the dual problem derived in part 1 and x is feasible in the primal problem!

3. The cost vector changes as $\mathbf{c} = (1, 0, -3 + \Delta)$.

Option 1: Use the complementarity theorem as in part 1. The dual solution corresponding to the basis $B = \{2, 3\}$ is now given as

$$\begin{aligned} 2p_1 - p_2 &= 0, \\ -3p_1 - p_2 &= -3 + \Delta. \end{aligned}$$

The unique solution is $\mathbf{p} = (p_1, p_2) = ((3 - \Delta)/5, (6 - 2\Delta)/5)$. From the complementarity theorem we know that as long as $\mathbf{p}$ is feasible in the dual problem the solution $(0, 1, 1)$ remains optimal. We need

$$\begin{aligned} p_1 \geq 0 &\iff \Delta \leq 3 \\ p_2 \geq 0 &\iff \Delta \leq 3 \\ p_1 + 2p_2 \geq 1 &\iff \Delta \leq 2 \end{aligned}$$

Hence as long as $\mathbf{c} \in \{(1, 0, r) \ : \ r \in (-\infty, -1]\}$ the solution $(0, 1, 1)$ remains optimal. *Common mistake:* forgetting to check the condition $p_1 + 2p_2 \geq 1$.

Option 2: We may transform the primal problem into standard form

$$
\begin{array}{lccccccccccl}
\min & -x_1 & & & + & (3+\Delta)x_3 & & & & & & \\
\text{s.t.} & x_1 & + & 2x_2 & - & 3x_3 & + & s_1 & & & \leq & -1 \\
 & 2x_1 & - & x_2 & - & x_3 & & & + & s_2 & \leq & -2 \\
 & x_1 & , & x_2 & , & x_3 & , & s_1 & , & s_2 & \geq & 0
\end{array}
$$

Changing the cost vector has no influence on the feasibility of the basic solution $\mathbf{x} = (0, 1, 1)$ corresponding to the basis $B = \{x_2, x_3\}$. The reduced cost is

$$
\begin{aligned}
\bar{\mathbf{c}} &= \mathbf{c} - \mathbf{c}_B' \mathbf{B}^{-1}\mathbf{A} \\
&= [-1, 0, 3+\Delta, 0, 0] - [0, 3+\Delta]' \begin{pmatrix} 2 & -3 \\ -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & -3 & 1 & 0 \\ 2 & -1 & -1 & 0 & 1 \end{pmatrix} \\
&= [-1, 0, 3+\Delta, 0, 0] + \tfrac{1}{5}[3+\Delta, 6+2\Delta]' \begin{pmatrix} 1 & 2 & -3 & 1 & 0 \\ 2 & -1 & -1 & 0 & 1 \end{pmatrix} \\
&= [2+\Delta, 0, 0, (3+\Delta)/5, (6+2\Delta)/5]
\end{aligned}
$$

Sanity check: the reduced cost of the basic variables is indeed always zero. We need the reduced costs $\bar{c}_1, \bar{c}_4, \bar{c}_5$ to be positive for $(0, 1, 1)$ to be optimal. We need

$$
\begin{aligned}
\bar{c}_1 \geq 0 &\iff \Delta \geq -2 \\
\bar{c}_4 \geq 0 &\iff \Delta \geq -3 \\
\bar{c}_5 \geq 1 &\iff \Delta \geq -3
\end{aligned}
$$

Hence as long as $\mathbf{c} \in \{(1, 0, r) \ : \ r \in (-\infty, -1]\}$ the solution $(0, 1, 1)$ remains optimal in the original maximization problem.

4. Changing the budget vector $\mathbf{b}$ has no influence on the reduced costs of the basis $B = \{2, 3\}$. The corresponding basic solution becomes

$$
\begin{aligned}
\mathbf{x}_B &= \mathbf{B}^{-1}\mathbf{b} \\
&= \tfrac{1}{5} \cdot \begin{pmatrix} 1 & -3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -1+\Delta \\ -2 \end{pmatrix} \\
&= \tfrac{1}{5} \cdot \begin{pmatrix} 5+\Delta \\ 5-\Delta \end{pmatrix}
\end{aligned}
$$

Hence, as long as $-5 \leq \Delta \leq 5$ or $b_1 \in [-6, 4]$ the basis $B = \{2, 3\}$ remains optimal.

5. The basis $B = \{2, 3\}$ is optimal as per part 4 with $\Delta = -1$. The new optimal value can be computed as

$$
\mathbf{c}^\star + \Delta \cdot p_1 = -3 - 3/5 = -18/5.
$$