6.255 Optimization Methods

Problem Set 4

Due: November 18, 2021

1 Due exercises

Problem 1: Lagrangean dual – based on Bertsimas & Tsitsiklis, Exercise 11.1. (10 points) Consider the integer programming (IP) problem:

$$\begin{array}{ll} \text{maximize} & 3x_1 + 2x_2 \\ \text{subject to} & 4x_1 + 2x_2 \leq 17 \\ & -2x_1 + 4x_2 \leq 9 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \text{ integer.} \end{array}$$

Using figures, justify the answers to the following questions. Highly or write in a diffrent color the answer for each question.

1. What is the optimal cost of the linear programming relaxation? What is the optimal cost of the integer programming problem?
2. Draw the convex hull of the set of all solutions to the integer programming problem. How does the solution to the IP compare to the solution when the feasibility region is replaced by this convex hull?
3. Give the first Gomory cut for this problem. (Write the problem in standard form, specify the constraints where you are performing the cut and the resulting constraint after the cut.)
4. Suppose you dualize the constraint $4x_1 + 2x_2 \leq 17$ (this means X in your Lagrangian dual is composed of all the other constraints except $4x_1 + 2x_2 \leq 17$). What is the optimal value Z_D of your Lagrangian dual?
5. Suppose you dualize the constraint $-2x_1 + 4x_2 \leq 9$ (this means X in your Lagrangian dual is composed of all the other constraints except $-2x_1 + 4x_2 \leq 9$). What is the optimal value Z_D of your Lagrangian dual?

Problem 2: Dynamic programming. (15 points) Consider a set S of n non-negative integers $\{s_1, s_2, \ldots, s_n\}$. We want to find, if it exists, a partition of the set S into three subsets A, B, C, in such a way that the sum of the numbers in each partition is the same.

For instance, if $n = 8$ and $S = \{1, 1, 1, 2, 2, 4, 5, 5\}$, then a possible partition could be $A = \{1, 2, 4\}$, $B = \{2, 5\}$, and $C = \{1, 1, 5\}$, each of which adds up to 7.

1. What necessary condition must the data $\{s_1, \ldots, s_n\}$ satisfy for the problem to have a feasible solution?
2. Give a dynamic programming algorithm to solve the partition problem. Explain how to compute the partition (if it exists) from your solution.
3. Assume that we modify the problem to incorporate an objective function, where we want to find a solution that maximizes the cardinality of the smallest subset. Give an integer programming formulation for this problem.

Problem 3: Comparisons of relaxations for an assignment problem with a side constraint – Bertsimas & Tsitsiklis, Exercise 11.12. (15 points) We would like to assign n machines to n jobs in order to minimize the total cost of the assignment (it costs c_{ij} to assign machine i to job j. In addition, there is a value d_{ij} if machine i is assigned to job j. We would like the total value from the assignment to be above a threshold b. We formulate the problem as follows:

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{1 \le i,j \le n} c_{ij} x_{ij} & \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} x_{ij} = 1, & 1 \le j \le n, \\ & \displaystyle\sum_{j=1}^{n} x_{ij} = 1, & 1 \le i \le n, \\ & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} x_{ij} \ge b, & \\ & x_{ij} \in \{0,1\} & 1 \le i,j \le n. \end{array}$$

We consider the following alternative relaxations

1. Relax the third constraint.
2. Relax the first and second sets of constraints.
3. Relax the first set of constraint.
4. Relax the first set of constraints and the third constraint.

Let Z_{Di} be the value of the corresponding Lagrangean dual problem, $i = 1, \ldots, 4$. Let Z_{LP} be the optimal cost of the linear programming relaxation. Prove that these relaxations are ordered as follows:

$$Z_{LP} = Z_{D1} = Z_{D4} \le Z_{D2} \le Z_{D3} \le Z_{IP}.$$

Problem 4: An approximation algorithm for maximum satisfiability – Bertsimas & Tsitsiklis, Exercise 11.16. (15 points) This exercise shows the use of randomization in constructing approximation algorithms for the following problem in logic called the maximum satisfiability problem (MAXSAT).

Given a collection $\mathcal{C} = \{C_1, \ldots, C_m\}$ of boolean clauses, where each clause is a disjunction of literals (a literal is either a boolean variable x or its negation $\bar{x}$ from a set of variables $\{x_1, \ldots, x_n\}$, and positive weights w_i for each clause C_i, the goal in MAXSAT is to assign truth values to the variables $x_1, \ldots, x_n$ in order to maximize the sum of weights of the satisfied clauses.

Let us formulate the problem. Let $y_i = 1$ if we set x_i to be true, and $y_i = 0$ otherwise. Let $z_j = 1$ if clause C_j is satisfied. Let I_j^+, (resp. I_j^-) be the set of literals that are not (resp. are) negated in clause C_j. Then, MAXSAT can be formulated as follows:

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{j=1}^{m} w_j z_j \\ \text{subject to} & \displaystyle\sum_{i \in I_j^+} y_i + \sum_{i \in I_j^-} (1 - y_i) \ge z_j, \quad C_j \in \mathcal{C}, \\ & y_i, z_j \in \{0,1\}. \end{array}$$

We denote the optimal cost by Z_{IP} and the optimal cost of the linear programming relaxation by Z_{LP}. Consider the following heuristic:

(a) Solve the linear programming relaxation and find optimal values y_i^*, z_j^*.

(b) Interpret the numbers y_i^* as probabilities. Set $\tilde{y}_i$ to 0 or 1, randomly and independently, with probability

$$\mathbb{P}(\tilde{y}_i = 1) = y_i^*.$$

(c) Set the values $\tilde{z}_j$ to be 0 or 1, with preference given to 1 when possible, so that the resulting solution is feasible i.e. $\tilde{z}_j = 1$ if $\sum_{i \in I_j^+} \tilde{y}_i + \sum_{i \in I_j^-} (1 - \tilde{y}_i) \geq 1$ and $\tilde{z}_j = 0$ if $\sum_{i \in I_j^+} \tilde{y}_i + \sum_{i \in I_j^-} (1 - \tilde{y}_i) < 1$.

The resulting solution is always feasible, but its value is a random variable. Let Z_H be the value of the solution produced by the heuristic and let $\mathbb{E}[Z_H]$ be its expected value.

1. For a given clause C_j, show that

$$\log \mathbb{P}[\tilde{z}_j = 0] \leq -\sum_{i \in I_j^+} y_i^* - \sum_{i \in I_j^-} (1 - y_i^*)$$

2. Show that for any clause $C_j \in \mathcal{C}$,

$$\mathbb{E}[\tilde{z}_j] \geq \frac{e-1}{e} z_j^*$$

3. Prove that

$$Z_{LP} \geq Z_{IP} \geq \mathbb{E}[Z_H] \geq \frac{e-1}{e} Z_{LP},$$

where $e = 2.71..$ is the base of the natural logarithm.

Hint: $\log(1-x) \leq -x$ *for all* $0 \leq x \leq 1$. *Also,* $e^x \leq 1 + x(1 - 1/e)$ *for all* $-1 \leq x \leq 0$. *(Both inequalities are convexity inequalities.)*

2 Practice exercises

Please do not submit answers to these problems, they will not be graded.

Problem 5: MIP vs Linear relaxation (Practice) Bertsimas & Tsitsiklis, Exercise 11.2.

Problem 6: Typography. (Practice) Bertsimas & Tsitsiklis, Exercise 11.6.

Problem 7: A 1/2−approximation algorithm for TSP. (Practice) Formalize the proof seen in class. Bertsimas & Tsitsiklis, Exercise 11.14.