6.215/6.255J/15.093J/IDS.200J Optimization Methods

Lecture 9: Network Optimization I

October 7, 2021

Today's Lecture

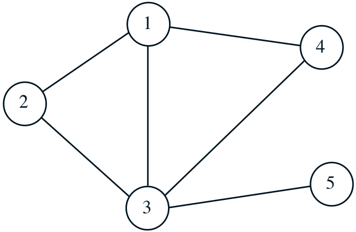
Outline

- Networks - introduction
- Network flow problems
  - Shortest path problem and applications
  - Maximum flow problem and applications
  - Minimum cost flow problem and applications

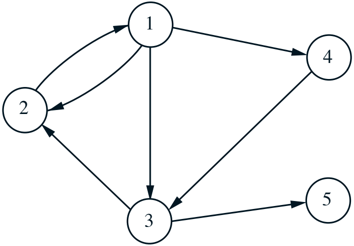
Networks

What are they?

Formally networks are (undirected or directed) graphs with additional information on nodes and/or on edges or arcs.



undirected $G = (N, E)$



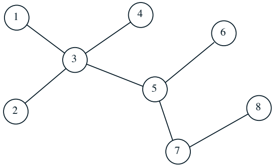
directed $G = (N, A)$

In our contexts, "self-loops" or "self-arcs" will not be considered.

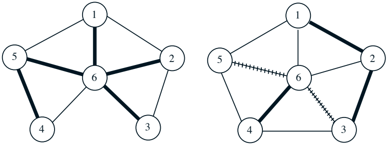
Networks

Some special undirected graphs

- A *tree* is an undirected graph that is connected and has no cycles.



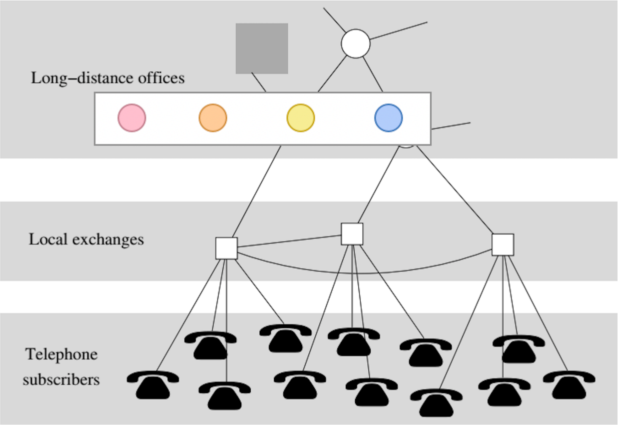
- A *spanning tree* of a graph G is a subgraph that is a tree and contains all nodes of G.



(a) (b)

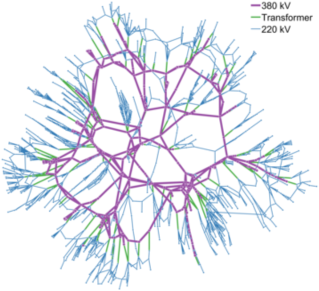
Networks

Illustrations - (three-tiers) telephone networks



Networks

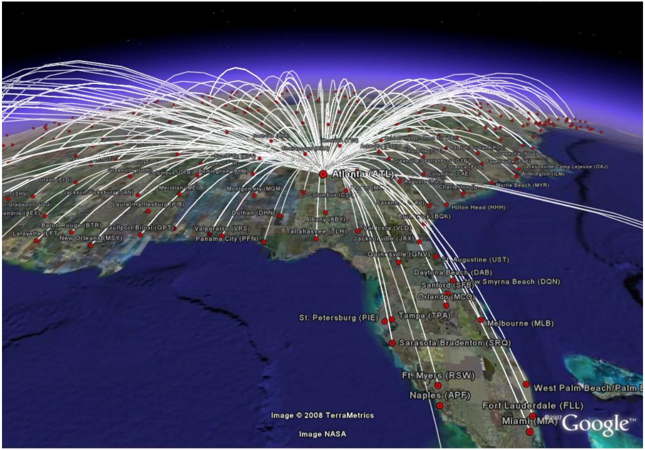
Illustrations - electrical & power networks



Networks

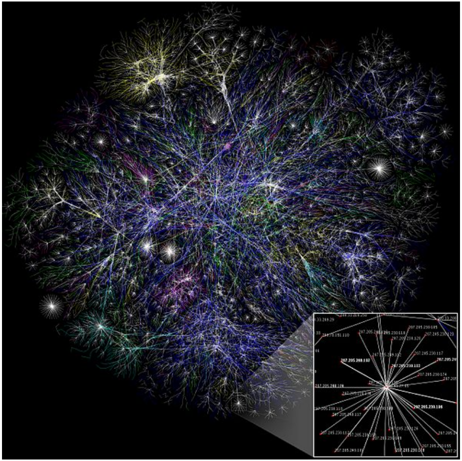
Illustrations - road networks

Illustrations - airline routes



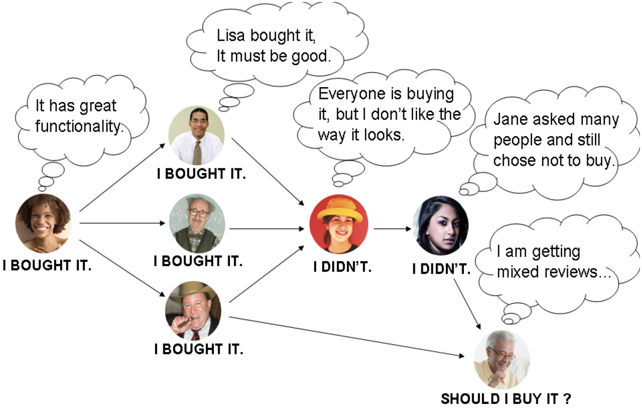
Networks

Illustrations - internet backbone



Networks

Illustrations - social networks



Network optimization

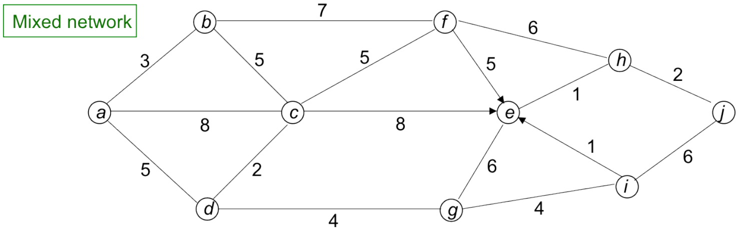
Common thrust

Move some entity (electricity, a consumer product, a person, a vehicle, a message, etc.) from one point to another in the underlying network, as efficiently as possible.

- Note: Many problems are network problems in disguise!
- Today: Learn how to model application settings as network flow problems.
- Next lecture: Study ways to solve the resulting models.

Shortest Path

Description



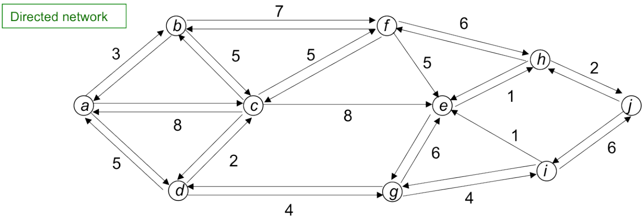
Problem: Identify a shortest path from a given source node s to a given sink node t.

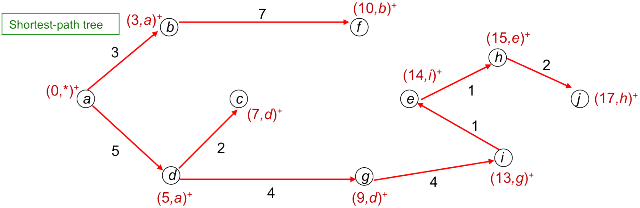
Examples:

- a path of minimum length.
- a path taking minimum time.
- a path of maximum reliability.

Shortest Path

Shortest path tree solution





Shortest Path

Linear optimization formulation

- Graph $G = (N, A)$, source node s, sink node t
- Arc lengths $c : A \to \mathcal{Z}$

$$
\begin{array}{llcll}
\min & \displaystyle\sum_{(i,j)\in A} c_{ij}x_{ij} & & & \\
\text{s.t.} & \displaystyle\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} & = & 0 & \text{for all } i \in N \setminus \{s,t\} \\
& \displaystyle\sum_{j:(s,j)\in A} x_{sj} & = & 1 & \\
& \displaystyle\sum_{j:(j,t)\in A} x_{jt} & = & 1 & \\
& x_{ij} & \geq & 0 & \text{for all } (i,j) \in A
\end{array}
$$

Illustration 1 - Interword spacing in LaTeX

The spacing between words and characters is normally set automatically by LaTeX. Interword spacing within one line is uniform. LaTeX also attempts to keep the word spacing for different lines as nearly the same as possible.

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Shortest Path Formulations

Illustration 1 - Interword spacing in LaTeX

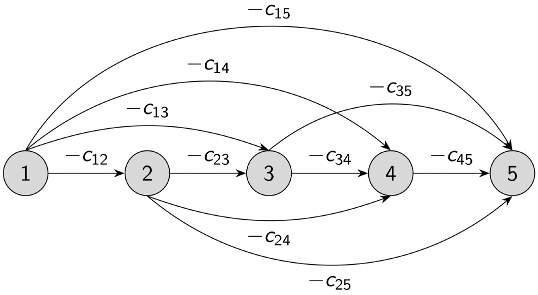
- The paragraph consists of n words, indexed by $1, 2, \ldots, n$.
- c_{ij} is the attractiveness of a line if it begins with i and ends with $j-1$.
- LaTeX uses a (complicated) formula to compute the value of each c_{ij}. For instance,

$$\begin{aligned} c_{12} &= -10,000 \qquad & c_{13} &= -1,000 \\ c_{14} &= -100 \qquad & c_{1,37} &= -100,000 \end{aligned}$$

$$\ldots$$

- The problem of decomposing a paragraph into several lines of text to maximize total attractiveness can be formulated as a shortest path problem.
- Nodes? Arcs? Costs?

- Introduce a node for each word / syllable (and one terminal node).
- For all $i < j$ we introduce an edge between word / syllable i and word / syllable j with length $-c_{ij}$.



Shortest Path Formulations

Illustration 2 - Project management

- A project consists of a set of jobs and a set of precedence relations
- In particular, we are given a set A of job pairs (i, j) indicating that job i cannot start before job j is completed.
- c_i duration of job i
- Find the least possible duration of the project

Shortest Path Formulations

Illustration 2 - Project management

Formulation

- Introduce two artificial jobs s and t, of zero duration, that signify the beginning and the completion of the project
- Add (s, i) and (i, t) to A
- p_i: time that job i begins
- $(i, j) \in A$: $p_j \geq p_i + c_i$
- Project duration: $p_t - p_s$

- Linear optimization formulation:

$$\begin{array}{rll} \min & p_t - p_s & \\ \text{s.t} & p_j - p_i \geq c_i, & \forall\ (i,j) \in A. \end{array}$$

- Dual:

$$\begin{array}{rl} \max & \displaystyle\sum_{(i,j)\in A} c_i x_{ij} \\ \text{s.t.} & \displaystyle\sum_{\{j|(j,i)\in A\}} x_{ji} - \sum_{\{j|(i,j)\in A\}} x_{ij} = b_i \\ & x_{ij} \geq 0 \end{array}$$

  where $b_s = -1$, $b_t = 1$, and $b_i = 0$ for $i \neq s, t$.

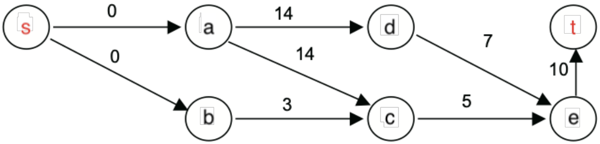
- Shortest path problem, where each precedence relation $(i,j) \in A$ corresponds to an arc with cost of $-c_i$.

Shortest Path Formulations

Illustration 2 - Project management

Example:

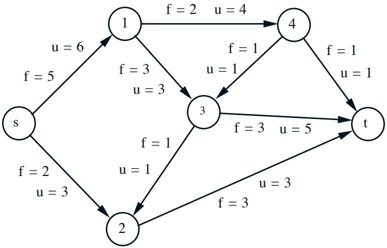
Activity	Immediate Predecessor	Time (c_i)
s		0
a	s	14
b	s	3
c	a,b	5
d	a	7
e	c,d	10
t	e	0



Problem: Determine the maximum flow that can be sent from a given source node s to a given sink node t in a capacitated network.

Examples: Find maximum steady-state flow of

- petroleum products in a pipeline network
- cars in a road network
- messages in a telecommunication network
- electricity in an electrical network



Maximum Flow

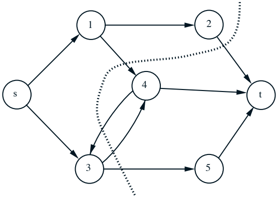
Linear optimization formulation

- Network $G = (N, A)$, source node s, sink node t
- Arc capacities $u : A \to \mathcal{N}$

$$
\begin{array}{lrcll}
\max & \sum_{j:(j,t)\in A} x_{jt} & & & \\
\text{s.t.} & \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} & = & 0 & \text{for all } i \in N \setminus \{s,t\} \\
& x_{ij} & \leq & u_{ij} & \text{for all } (i,j) \in A \\
& x_{ij} & \geq & 0 & \text{for all } (i,j) \in A
\end{array}
$$

An (s,t)-cut in a network $G = (N, A)$ is a partition of N into two disjoint subsets S and T such that $s \in S$ and $t \in T$.

The capacity of an (s,t)-cut (S,T) is $\sum_{i \in S} \sum_{j \in T} u_{ij}$.



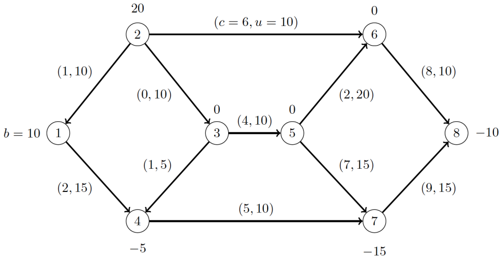
Theorem

The value of a maximum (s,t)-flow = the capacity of a minimum (s,t)-cut.

Problem: Determine a least cost shipment of a commodity through a network in order to satisfy demands at certain nodes from available supplies at other nodes. Arcs have capacities and cost associated with them

Examples:

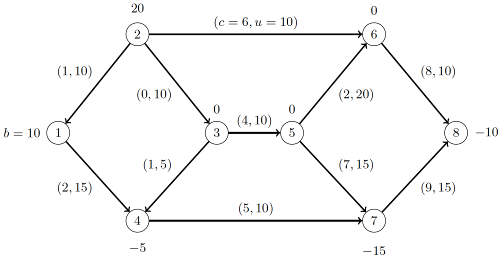
- Distribution of products
- Flow of items in a production line
- Routing of cars through street networks
- Routing of telephone calls



Min-Cost Flow

Linear optimization formulation

- Network $G = (N, A)$.
- Arc costs $c : A \to \mathcal{Z}$.
- Arc capacities $u : A \to \mathcal{N}$.
- Node balances $b : N \to \mathcal{Z}$.

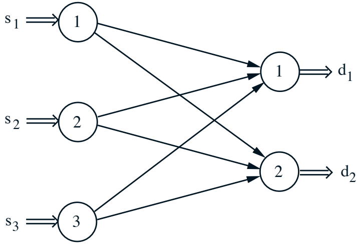


$$
\begin{array}{llcll}
\min & \displaystyle\sum_{(i,j)\in A} c_{ij} x_{ij} & & & \\
\text{s.t.} & \displaystyle\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} & = & b_i & \text{for all } i \in N \\
& x_{ij} & \leq & u_{ij} & \text{for all } (i,j) \in A \\
& x_{ij} & \geq & 0 & \text{for all } (i,j) \in A
\end{array}
$$

Min-Cost Flow

Some special cases

Transportation problem:



Assignment problem: A transportation problem where:

- number of suppliers = number of customers
- each supplier has unit supply
- each customer has unit demand

Min-Cost Flow Formulation

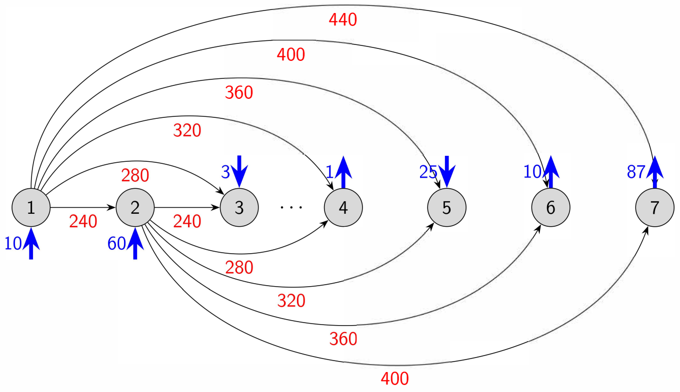
Illustration: Passenger routing

- United Airlines has seven daily flights from BOS to SFO, every two hours, starting at 7am.
- Capacities are 100, 100, 100, 150, 150, 150, and ∞.
- Passengers suffering from overbooking are diverted to later flights.
- Delayed passengers get \$200 plus \$20 for every hour of delay.
- Suppose that today the first six flighs have 110, 160, 103, 149, 175, and 140 confirmed reservations.

Determine the most economical passenger routing strategy!

Min-Cost Flow Formulation

Illustration: Passenger routing



Further Network Problems

- Minimum spanning tree problems,
- Matching problems,
- Postman problems,
- Generalized flow problems,
- Multicommodity flow problems,
- Constrained shortest path problems,
- Unsplittable flow problems,
- Network design problems,
- ...