15.093 Optimization Methods
Midterm Fall 2020 – Solutions

October 14, 2020

Problem 1 – True/False Questions: (16 points)

Classify the following statements as true or false. All answers must be briefly justified, or no credit can be given. A standard form linear problem is

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

with $\mathbf{A} \in \mathbf{R}^{m \times n}$, $\mathbf{b} \in \mathbf{R}^m$, and $\mathbf{c} \in \mathbf{R}^n$. It is assumed that $\mathbf{rank}(\mathbf{A}) = m$.

1. The reduced cost of a basic variable is always zero. (1 point)

2. If $x_j > 0$ at some optimal solution to the primal problem, then every optimal solution $\mathbf{p}$ to the dual problem satisfies $\mathbf{p}'\mathbf{A}_j = c_j$. (2 points)

3. If $\mathbf{x}$ is an degenerate optimal basic solution of a linear optimization problem in standard form, no more than $m - 1$ components can be nonzero. (2 points)

4. If $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ has a solution than the constraint $\mathbf{p}'\mathbf{b} \geq 0$ is redundant in the polyhedron

$$\begin{aligned} \mathbf{p}'\mathbf{A} &= \mathbf{0}', \\ \mathbf{p} &\geq \mathbf{0}, \\ \mathbf{p}'\mathbf{b} &\geq 0. \end{aligned}$$

   (3 points)

5. A degenerate basic solution has at least two associated basis. (2 points)

6. If there are two different optimal solutions $\mathbf{x}_1^\star, \mathbf{x}_2^\star$ to a linear optimization problem, then any linear combination of those two solutions, *i.e.*, $\alpha\mathbf{x}_1^\star + \beta\mathbf{x}_2^\star$ where $\alpha, \beta \in \mathbf{R}$, will also be an optimal solution. (2 points)

7. It is possible to formulate the nonlinear constraint $\min_{i=1,\ldots,k}\{\mathbf{f}_i'\mathbf{x} + g_i\} \geq h$ using linear constraints. (2 points)

8. If a linear optimization problem has a degenerate basic feasible point, there is at least one redundant constraint in the formulation of the problem. (2 points)

Solution 1 – True/False Questions: (16 points)

1. True. Consider indeed a basis variable j in a basis B with basis matrix $\mathbf{B}$. Let $\mathbf{e}_j$ be the jth unit vector in R^m such that $\mathbf{B}\mathbf{e}_j = \mathbf{A}_j \iff \mathbf{e}_j = \mathbf{B}^{-1}\mathbf{A}_j$. Clearly we have that its reduced cost satisfies

$$\begin{aligned} \bar{c}_j = &\ c_j - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{A}_j \\ = &\ c_j - \mathbf{c}_B'\mathbf{e}_j = c_j - c_j = 0. \end{aligned}$$

   Grading. Saying by definition of basic variable is not a sufficient justification. 0.5/1 is given in this case. An answer with no justification is given 0.

2. True. The complementary slackness theorem states that if $\mathbf{x}$ is optimal in the primal problem and $\mathbf{p}$ is optimal in the dual problem, then for all $j = \{1, \dots, m\}$, $x_j(c_j - p'\mathbf{A}_j) = 0$. Hence, for all $j = \{1, \dots, m\}$, if $x_j > 0$, we have $c_j - \mathbf{p}'\mathbf{A}_j = 0$ which implies $c_j = \mathbf{p}'\mathbf{A}_j$.

   Grading. Saying "by complimentary slackness" is not a sufficient justification. The students have to write the conditions and show how to use the fact that $x_j > 0$. An answer with no justification is given 0.

3. True. The optimal point $\mathbf{x}^\star$ is degenerate if at least $n+1$ constraints are active. Any optimal solution $\mathbf{x}^\star$ must be feasible and hence the m constraints $\mathbf{A}\mathbf{x}^\star = \mathbf{b}$ are always active. That means that at least $n-m+1$ components of $x^\star$ must be zero. Equivalently, at most $m-1$ components of $\mathbf{x}^\star$ are nonzero.

4. True. We have that $\mathbf{p}'\mathbf{A} = \mathbf{0}$ and hence also $\mathbf{p}'\mathbf{A}\mathbf{x} = 0$ with $\mathbf{A}\mathbf{x} \leq \mathbf{b}$. Consequently, as $\mathbf{p} \geq \mathbf{0}$ we have $0 = \mathbf{p}'\mathbf{A}\mathbf{x} \leq \mathbf{p}'\mathbf{b}$ and hence the last constraint is redundant.

5. False. Consider

$$\begin{aligned} \min\ & x_1 + x_2 \\ \text{s.t. } & x_1 = 0, \\ & x_2 = 0, \\ & x_1,\ x_2 \geq 0. \end{aligned}$$

   where the degenerate BFS is $(x_1, x_2) = (0, 0)$ with unique basis $B = (x_1, x_2)$.

6. False. Consider minimizing $\mathbf{0}'\mathbf{x}$ subject to $\mathbf{x} \in P$, such that P is bounded and contains more than one point. Any point in P will be an optimal solution, take two of them $\mathbf{x}_1^\star \neq 0$ and $\mathbf{x}_2^\star \neq 0$, and set $\beta = 0$. By setting α arbitrarily large we can make the linear combination $\overline{\mathbf{x}} = \alpha\mathbf{x}_1^\star + \beta\mathbf{x}_2^\star = \alpha\mathbf{x}_1^\star$ go outside of P, thus making $\overline{\mathbf{x}}$ infeasible, hence not optimal.

7. True. We have that

$$\min_{i=1,\dots,k}\{\mathbf{f}_i'\mathbf{x} + g_i\} \geq h \iff \mathbf{f}_i'\mathbf{x} + g_i \geq h \quad \forall i = 1, \dots, k.$$

8. False. For a counter example see Figure 2.9(a) in the textbook.

Problem 2 – Simplex Method (12 points)

While solving a standard form problem, we arrive at the following tableau:

-10	u	-3	0	γ
α	1	3	0	η
3	0	-2	v	β

In questions 4, 5 and 6 we ask for **some** values verifying the desired proprieties, not all the possible values. At each of these questions, justify your choice.

1. If $\eta = -1$, explain briefly why we necessarily have $u = 0$ and $v = 1$. (2 points)

   Solutions. The tableau has two rows, therefore we must have two basic variables. The 2×2 square matrix corresponding to these basic variables in the tableau must be the identity. The only matrix than can be the identity is the one composed of the second and third column of the tableau, therefore $v = 1$. The reduced cost of basic variables is 0, therefore $u = 1$. The second varible can not be basic as the column is $(3, -2)$ and fourth variable can not be basic as $\eta = -1$ so the corresponding column can not be $(1, 0)$ or $(0, 1)$.

   Grading. *The student must mention that there are* 2 *basic variables and basic variables correspond to columns with* 0 *reduced cost and a columns vector* $(0, 1)$ *or* $(1, 0)$ *(1/2pts). To get fell credit, the student must mention that* x_2 *and* x_4 *can not be basic and why (1pts).*

   From now on, we fix $u = 0$ and $v = 1$.

2. What are the basic variables? What is the current solution? What is the current cost? (3 points)

   Solutions. The basic variables are x_1 and x_3. The current solution is $(\alpha, 0, 3, 0)$. The current cost is 10. *You must give the values of* x_1, x_2, x_3 *and* x_4*, and not only* x_1 *and* x_3 *to get full points.*

   Grading. *The student must provide the cost (1/3pt), mention what are the basic variables (1/3pt) and give* x_1, x_2, x_3, x_4 *(1/3pts). Not giving* x_2 *and* x_4 *get the student to lose* $0.5/3$ *pts.*

3. If x_1, x_2, x_3, x_4 is some feasible solution, what equality constraint does the first row of the tableau imply on x_1, x_2, x_3, x_4? (1.5 points)

   Solutions. $\alpha = x_1 + 3x_2 + \eta x_4$.

4. Find some parameter values $\alpha, \beta, \gamma, \eta$ such that the problem is unfeasible. (1.5 points)

   Solutions. It suffice to take $\alpha < 0$ and $\eta \geq 0$. A feasible solution verifies necessarily $x_1, \ldots, x_4 \geq 0$ as this is a standard form problem. Such a solution can not verify the constraint of the previous question with $\alpha < 0$ and $\eta \geq 0$, as the left hand side is negative, and the right hand side is non negative.

   Grading. *The objective is to have the problem not feasible (ie the linear programming) not just the current solution. Saying* $\alpha < 0$ *does not answer the question and give the student* $0/1.5$ *pts. Some answer with the dual tableau being unbounded and therefore the primal is infeasible were very good and got full credit. Non justified choice of parameters gets no credit in this question.*

   From now on, we fix $\eta = -1$. We still also have $u = 0$ and $v = 1$.

5. Find some parameter values α, β, γ such that the current solution is feasible, but **not** optimal. (2 points)

   Solutions. It suffices to have $\alpha > 0$. $\alpha \geq 0$ guarantees that the solution is feasible. The reduced cost of x_2 is negative so there a direction that reduces the cost and $\alpha > 0$ imply that we can move in by a non 0 distance in this direction. Therefore the solution is not optimal.

   Grading. *You must mention that* $\alpha > 0$ *allows us to move with a non zero distance to get full points. Having just a negative reduced cost does not guarantee non optimality as we can then move with a 0 distance and make this reduced cost* ≥ 0*. In general, if the students choses* $\alpha > 0$ *but fails to mention that we can move a positive distance, loses the student* $0.5 - 1/2$ *pts. I usually mark this as "*> 0 *dist??". Other arguments like non degeneracy were accepted. Saying just* $\alpha \geq 0$ *(this is wrong as it should be* > 0*) and that a reduced cost is negative gives the student* $1/2pts$*. Correct but not justified choice of parameters get* $0.5/2$ *pts.*

6. Find some parameter values α, β, γ such that the current tableau is unbounded. (2 points)

   Solutions. We can chose $\alpha \geq 0$, $\beta \leq 0$, and $\gamma < 0$. With this choice we can move indefinitely in the fourth basic direction without breaking the feasibility (solution ≥ 0) while improving the cost.

   Grading. *If the student mentions that his choice of params allows to move infinitely in a direction reducing cost while having a feasible solution, the student gets full points. Correct but not justified choice of parameters get* $0.5/2$ *pts.*

Problem 3 – Integer Optimization Relaxations (10 points)

Consider the following **m**ixed-**i**nteger **p**rogramming (MIP) problem

$$\begin{array}{ll} \min & \mathbf{c}'\mathbf{x} + \mathbf{d}'\mathbf{y} \\ \text{s.t.} & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} = \mathbf{b} \\ & \mathbf{D}\mathbf{x} + \mathbf{G}\mathbf{y} \le \mathbf{f} \\ & \mathbf{x} \in \mathbf{R}^n, \mathbf{y} \in \mathbf{Z}^p, \end{array} \tag{MIP}$$

where $\mathbf{x}, \mathbf{y}$ are the decision variables and everything else is problem data. Note that the components of the decision variable $\mathbf{y}$ are forced to take integer values, *i.e.*, $y_i \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ for all $i = 1, 2, \ldots, p$. Due to presence of the constraint $\mathbf{y} \in \mathbf{Z}^p$, problem (MIP) is not a linear program and very difficult to solve in general.

However, by solving a **r**elated **LP**, we can extract valuable information about this hard problem. Consider the following *relaxation* of (MIP), where we drop the constraint $\mathbf{y} \in \mathbf{Z}^p$ and replace it with $\mathbf{y} \in \mathbf{R}^p$:

$$\begin{array}{ll} \min & \mathbf{c}'x + \mathbf{d}'\mathbf{y} \\ \text{s.t.} & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} = \mathbf{b} \\ & \mathbf{D}\mathbf{x} + \mathbf{G}\mathbf{y} \le \mathbf{f} \\ & \mathbf{x} \in \mathbf{R}^n, \mathbf{y} \in \mathbf{R}^p, \end{array} \tag{RLP}$$

which is a linear program now, and it is much easier to solve.

1. What is the relationship between the optimal value of (RLP) and optimal value of (MIP)? Justify your answer briefly (no proof required). (2 points)
2. Write down the dual of (RLP). For (RLP) and its dual, write down the complementary slackness condition. (3 points)
3. Suppose we solve the dual of (RLP) for a given dataset and find that the optimal value of the dual is $+\infty$. What can we say about (RLP) for this dataset? What can we say about the harder problem (MIP) for this dataset? (2.5 points)
4. Suppose we solve (RLP) for a given dataset. In this case, we find that the corresponding optimal value is finite, and the optimal solution found, denoted by $(\mathbf{x}^\star, \mathbf{y}^\star)$, satisfies $\mathbf{y}^\star \in \mathbf{Z}^p$. For this dataset, what would be the relationship between the optimal value of (RLP) and the optimal value of (MIP)? In this situation, how does $(\mathbf{x}^\star, \mathbf{y}^\star)$ relate to (MIP)? (2.5 points)

Solution. For convenience, assume the following dimension on the problem data $\mathbf{c} \in \mathbf{R}^n, \mathbf{d} \in \mathbf{R}^p, \mathbf{A} \in \mathbf{R}^{m_1 \times n}, \mathbf{B} \in \mathbf{R}^{m_1 \times p}, \mathbf{D} \in \mathbf{R}^{m_2 \times n}, \mathbf{G} \in \mathbf{R}^{m_2 \times p}, \mathbf{b} \in \mathbf{R}^{m_1}, \mathbf{f} \in \mathbf{R}^{m_2}$ problem data.

1. Problem (RLP) is a relaxation of (MIP), **i.e.**, if the underlying constraint sets of (RLP) and (MIP) are P_{easy} and P_{hard} respectively, then $P_{\text{hard}} \subseteq P_{\text{easy}}$. In words, feasible set of (RLP) contains the feasible set of (MIP). Hence, the optimal value of (RLP) is less than or equal to the optimal value of (MIP).
2. The dual problem is:

$$\begin{array}{ll} \text{maximize} & \mathbf{b}'\mathbf{q} + \mathbf{f}'\mathbf{p} \\ \text{subject to} & \mathbf{A}'\mathbf{q} + \mathbf{D}'\mathbf{p} = \mathbf{c} \\ & \mathbf{B}'\mathbf{q} + \mathbf{G}'\mathbf{p} = \mathbf{d} \\ & \mathbf{p} \le \mathbf{0} \\ & \mathbf{q} : \text{ free}, \end{array} \tag{1}$$

where $\mathbf{p} \in \mathbf{R}^{m_2}, \mathbf{q} \in \mathbf{R}^{m_1}$ are the decision variables.

For primal feasible $\mathbf{x}, \mathbf{y}$ and dual feasible $\mathbf{p}, \mathbf{q}$ the complementary slackness condition involving inequality constraints will be:

$$\begin{aligned} \mathbf{p}_i \left(\mathbf{d}_i'\mathbf{x} + \mathbf{g}_i'\mathbf{y} - f_i\right) &= 0, \quad i = 1, \ldots, m_2 \\ \mathbf{q}_i \left(\mathbf{a}_i\mathbf{x}' + \mathbf{b}_i'\mathbf{y} - b_i\right) &= 0, \quad i = 1, \ldots, m_1 \\ \mathbf{x}_i \left(\mathbf{A}_i'\mathbf{q} + \mathbf{D}_i'\mathbf{p} - c_i\right) &= 0, \quad i = 1, \ldots, n \\ \mathbf{y}_i \left(\mathbf{B}_i'\mathbf{q} + \mathbf{G}_i'\mathbf{p} - d_i\right) &= 0, \quad i = 1, \ldots, p \end{aligned}$$

where we have used the following notation used in the textbook. The notation $\mathbf{h}_i, \mathbf{H}_j$ represent ith row and jth column of a matrix $\mathbf{H}$.

Common mistake. Many students wrote the complementary slackness as multiplication of matrices which is not correct, and in many of those cases, even the dimensions of the matrices do not match for multiplying them. Complementary slackness is component-wise multiplication of the underlying matrices.

3. If (1) is unbounded, then due to strong duality, the primal problem (RLP) will be infeasible **i.e.**, the feasible set of (RLP) is empty. As feasible set of (RLP) contains the feasible set of (MIP), this means that (MIP) will have an empty feasible set and be infeasible (MIP) for the given dataset.

4. In this situation, the optimal solution to (RLP) will be an optimal solution to (MIP), and both the optimal values will be equal. The reason is that we have already found a solution to (RLP) that is feasible to (MIP), and we know from (a) that this solution would attain the minimum value for (MIP).

Problem 4 – Linear Systems and Duality (10 Points)

Consider the system of linear equations, $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in \mathbf{R}^{m \times n}, \mathbf{b} \in \mathbf{R}^m$ but no other assumptions about $\mathbf{A}$ and $\mathbf{b}$ are made. Using duality, show that the following two statements are equivalent:

1. The system $\mathbf{Ax} = \mathbf{b}$ has a solution.
2. There exists **no** $\mathbf{y} \in \mathbf{R}^m$ such that $\mathbf{A}'\mathbf{y} = 0$ and $\mathbf{b}'\mathbf{y} \neq 0$.

Solution. Before we start, note that (1) means

$$(\exists_{\mathbf{x}} \quad \mathbf{Ax} = \mathbf{b}),$$

and (2) means

$$\neg\left(\exists_{\mathbf{y}:\mathbf{A}'\mathbf{y}=\mathbf{0}} \quad \mathbf{b}'\mathbf{y} \neq 0\right) \equiv \left(\forall_{\mathbf{y}:\mathbf{A}'\mathbf{y}=\mathbf{0}} \quad \mathbf{b}'\mathbf{y} = 0\right),$$

where $\neg$ means logical negation.

Now we start. Consider the primal problem

$$p^\star = \left(\begin{array}{ll} \text{minimize}_{\mathbf{x}} & \mathbf{0}'\mathbf{x} \\ \text{subject to} & \mathbf{Ax} = \mathbf{b} \end{array}\right), \tag{PRIMAL}$$

and its dual

$$d^\star = \left(\begin{array}{ll} \text{maximize}_{\mathbf{y}} & \mathbf{b}'\mathbf{y} \\ \text{subject to} & \mathbf{A}'\mathbf{y} = \mathbf{0} \end{array}\right). \tag{DUAL}$$

- (1) $\Rightarrow$ (2) : If the system $\mathbf{Ax} = \mathbf{b}$ has a solution, then (PRIMAL) is feasible, and it has optimal value $p^\star = 0$. By strong duality, (DUAL) is also feasible and has optimal value $d^\star = 0$. For the sake a reaching a contradiction, let us assume that there is some $\bar{\mathbf{y}}$ such that $\mathbf{A}'\bar{\mathbf{y}} = \mathbf{0}$ and $\mathbf{b}'\bar{\mathbf{y}} \neq 0$. If $\mathbf{b}'\bar{\mathbf{y}} > 0$, then it contradicts $d^\star = 0$, because we have found a feasible solution $\bar{\mathbf{y}}$with a higher objective value. If $\mathbf{b}'\bar{\mathbf{y}} < 0$, then set $\tilde{\mathbf{y}} = -\bar{\mathbf{y}}$, then $\mathbf{A}'\tilde{\mathbf{y}} = \mathbf{0}$ and $\mathbf{b}'\tilde{\mathbf{y}} > 0$ thus again contradicting $d^\star = 0$, because we have found a feasible solution $\tilde{\mathbf{y}}$ with a higher objective value.

  (*shorter proof of* (1) $\Rightarrow$ (2)) This direction could also be proven very concisely as follows. If $\mathbf{Ax} = \mathbf{b}$ has a solution, then there exists some $\bar{\mathbf{x}}$ such that $\mathbf{A}\bar{\mathbf{x}} = \mathbf{b}$. Take any $\mathbf{y}$ that satisfies $\mathbf{A}'\mathbf{y} = \mathbf{0}$, and then,

$$\mathbf{b}'\mathbf{y} = (\mathbf{A}\bar{\mathbf{x}})'\,\mathbf{y} = \bar{\mathbf{x}}'\,\overbrace{\mathbf{A}'\mathbf{y}}^{=\mathbf{0}} = \mathbf{0}.$$

- (2) $\Rightarrow$ (1) : First note that the dual is always feasible as $\mathbf{y} = \mathbf{0}$ is a feasible point for (DUAL). Now (2) says that for all $\mathbf{y} \in \mathbf{R}^m$ that satisfies $\mathbf{A}'\mathbf{y} = \mathbf{0}$ we have $\mathbf{b}'\mathbf{y} = 0$. So the maximum attainable value in (DUAL) will be $d^\star = 0$. Due to strong duality, we will have $p^\star = d^\star = 0$, which means we have a feasible solution to (PRIMAL). This implies that the system $\mathbf{Ax} = \mathbf{b}$ has a solution.

Common mistake. Many students tried to prove the same direction twice, first by proving (1) $\Rightarrow$ (2) and then by proving (not (2)) $\Rightarrow$ (not (1)), but both are the same. Another common mistake was in proving (1) $\Rightarrow$ (2), many just showed that there exists one $\mathbf{y}$ such that $\mathbf{A}'\mathbf{y} = \mathbf{0}$ and $\mathbf{b}'\mathbf{y} = 0$, which is not complete, because the goal is to show that all $\mathbf{y}$ with $\mathbf{A}'\mathbf{y} = \mathbf{0}$ satisfies $\mathbf{b}'\mathbf{y} = 0$.

Problem 5 – Phase I vs Big-M Method (10 points)

Let $\mathbf{A} \in \mathbf{R}^{m\times n}$, $\mathbf{b} \in \mathbf{R}^m$ and $\mathbf{c} \in \mathbf{R}^n$. An alternative to the Phase I algorithm to find an initial BFS in a standard form linear optimization problem

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq 0 \end{aligned} \tag{SFP}$$

is the big-M method in which we solve instead the auxiliary problem

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} + \textstyle\sum_{i=1}^m y_i \cdot M \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{I}_m\mathbf{y} = \mathbf{b} \\ & \mathbf{x} \geq 0,\ \mathbf{y} \geq 0. \end{aligned} \tag{BMP}$$

1. Find a BFS in the auxiliary optimization problem (BMP). *Hint: You can use the fact that without loss of generality it can be assumed* $\mathbf{b} \geq \mathbf{0}$. (2 points)
2. State the optimality conditions for a basis matrix $\mathbf{B}$ to be a primal and dual feasible basis in problem (SFP). (2 points)
3. Prove that if we take M large enough, i.e., $M\mathbf{1}' > \mathbf{c}_B'\mathbf{B}^{-1}$ with $\mathbf{B}$ an optimal basis in problem (SFP), then the basis $\mathbf{B}$ is optimal in problem (BMP) as well and therefore the optimal $\mathbf{x}^\star$ in problem (SFP) is optimal (BMP) too. *Hint: Use sensitivity analysis by considering the variables y_i as a potential new activities.* (6 points)

Solutions.

1. In problem (SFP) we can represent any of the constraints $\mathbf{a}_i'\mathbf{x} = b_i$ with $b_i < 0$ equivalently as $-\mathbf{a}_i'\mathbf{x} = -b_i$. Hence, without loss of generality we may assume that the budget vector $\mathbf{b} \geq 0$. Consequently, a BFS in (BMP) for $\mathbf{b} \geq 0$ is $(\mathbf{x}, \mathbf{y}) = (\mathbf{0}, \mathbf{b})$. That is, all original variables $\mathbf{x}$ are nonbasic while the auxiliary variables $\mathbf{y}$ are basic with basic matrix $\mathbf{B} = \mathbf{I}_m$.

   Grading. A complete answer has to provide the value of $\mathbf{y}$ and the value of $\mathbf{x}$. Only the value of $\mathbf{y}$ is not sufficient and gives only partial credits. Another possibility to have full points is to say that y_i are the basic variables, which implies $\mathbf{x} = 0$.

2. A basis B is optimal if and only if the associated basic primal solution $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \geq 0$ is primal feasible, while its associated basic dual solution $\mathbf{p}' = \mathbf{c}_B'\mathbf{B}^{-1}$ is dual feasible. The dual of the standard form problem (SFP) is

   $$\begin{aligned} \max \quad & \mathbf{b}'\mathbf{p} \\ \text{s.t.} \quad & \mathbf{p} \text{ free}, \\ & \mathbf{p}'\mathbf{A} \leq \mathbf{c}'. \end{aligned}$$

   Dual feasibility is equivalent to having all reduced costs $\mathbf{c}' - \mathbf{p}'\mathbf{A} = \mathbf{c}' - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{A} = \bar{\mathbf{c}}' \geq 0$ positive.

   Grading. This question is about the conditions that a <u>basis</u> is primal and dual feasible, not a solution $\mathbf{x}$. Providing the conditions for $\mathbf{x}$ to be optimal (like slackness conditions for eg) does not answer the question and no points are provided for that. $\mathbf{B}^{-1}b \geq 0$ gives 1pt and $\mathbf{c}' - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{A} = \bar{\mathbf{c}}' \geq 0$ gives 1pt.

3. The basis matrix $\mathbf{B}$ corresponding to the basis B satisfies $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \geq 0$ and $\mathbf{c}' - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{A} = \bar{\mathbf{c}}' \geq \mathbf{0}'$ as it is. Evidently, the basis B is primal feasible in problem (BMP) as well. It remains to verify whether it is dual feasible too. Dual feasibility requires the reduced costs associated with the basis B in problem (BMP) to satisfy

$$\mathbf{c}' = [\mathbf{c}', M \cdot \mathbf{1}] - \mathbf{c}_B\mathbf{B}^{-1}[\mathbf{A}, \mathbf{I}_m] = [\mathbf{c}' - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}, M \cdot \mathbf{1} - \mathbf{c}_B'\mathbf{B}^{-1}] \geq \mathbf{0}'.$$

As we already have that $\mathbf{c}' - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} \geq 0$ we only need $M \cdot \mathbf{1} \geq \mathbf{c}_B'\mathbf{B}^{-1}$.

Grading. Two key things have to be proven here: primal feasibility ($\mathbf{B}^{-1}\mathbf{b} \geq 0$) and dual feasibility ($\bar{c} \geq 0$) which then guarantees optimality. 2pts are given for primal feasiblity and 4pts for dual feasibility. Saying that the solution is still feasible without justification gives 1/2. The hardest part is to show the dual feasibility (4pts). The student has to write a formal proof. Giving only intuition and informal analysis on why it should work (like saying M is large and so y is costly) is given up to $0 - 1.5/4$ depending on how good the analysis is. To show that $\bar{\mathbf{c}} \geq 0$ one has to look at the components corresponding to $\mathbf{x}$ and then the ones corresponding to $\mathbf{y}$. The first one correspond to 1/4 and the second one to 3/4. One common mistake is to consider this as adding just one variable to the system and writing the conditions as seen in the slides ($c_{n+1}\ldots$). This is not the case as we add multiple variables. Also, just writing the sensitivity analysis results in the lecture without analysis or linking to the problem is not given any points.