6.215/6.255J/15.093J/IDS.200J Optimization Methods

Lecture 14: Discrete Optimization II

October 26, 2021

Today's Lecture
Outline

- Weak and strong relaxations
- Cutting plane methods
- Branch and bound methods

Strong and weak relaxations

- Consider the integer optimization problem (with integer inputs)

$$\begin{array}{lrcl} \min & \boldsymbol{c}^T\boldsymbol{x} & & \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} & = & \boldsymbol{b} \\ & \boldsymbol{x} & \geq & 0 \\ & \boldsymbol{x} & \text{integer} & \end{array}$$

  with $T = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0, \boldsymbol{x} \text{ integer}\}$ its feasible set.

- The LO relaxation is usually weak, except if $\boldsymbol{A}$ is totally unimodular, then the LO relaxation "solves" the IO problem

$$\begin{array}{lrcl} \min & \boldsymbol{c}^T\boldsymbol{x} & & \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} & = & \boldsymbol{b} \\ & \boldsymbol{x} & \geq & 0 \end{array}$$

- Strongest possible linear relaxation (assuming CH(T) is explicitly known):

$$\begin{array}{lrcl} \min & \boldsymbol{c}^T\boldsymbol{x} & & \\ \text{s.t.} & \boldsymbol{x} & \in & CH(T) \end{array}$$

Total unimodularity (supplementary material, fyi)

- **Definitions**: A square, integer matrix $\boldsymbol{B}$ is called unimodular if its determinant ± 1. An integer matrix $\boldsymbol{A}$ is called totally unimodular if every square, nonsingular submatrix of $\boldsymbol{A}$ is unimodular.

- **Theorem**: If $\boldsymbol{A}$ is totally unimodular the following LO problem (with integer input) has integral extreme points

$$\begin{array}{rrcl} \min & \boldsymbol{c}^T\boldsymbol{x} & & \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} & = & \boldsymbol{b} \\ & \boldsymbol{x} & \geq & 0 \end{array}$$

- A special case: $\boldsymbol{A}$ is a "network" matrix, i.e., a node-arc incidence matrix (each of its columns contains exactly one +1 and one -1).

- What to do otherwise if $\boldsymbol{A}$ is not totally unimodular, and if $CH(T)$ is not explicitly known for the problem at hand?

Cutting plane methods

Meta algorithm

1. Solve the LO relaxation. Let $\boldsymbol{x}^*$ be an optimal LO solution.
2. If $\boldsymbol{x}^*$ is integer stop; $\boldsymbol{x}^*$ is an optimal solution to IO.
3. If not, add a linear inequality constraint to LO relaxation that all integer solutions satisfy, but $\boldsymbol{x}^*$ does not; go to Step 1.

Quality of the cuts is very important !

Cutting plane methods

A warming example

- Let

$$\begin{array}{rrcl} \min & \boldsymbol{c}^T\boldsymbol{x} & & \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} & = & \boldsymbol{b} \\ & \boldsymbol{x} & \geq & 0 \end{array}$$

  be the linear optimization relaxation after some iteration.

- Let $\boldsymbol{x}^* = (\boldsymbol{x}_B^*, \boldsymbol{x}_N^*) = (\boldsymbol{x}_B^*, 0)$ be an optimal BFS to that relaxation with at least one fractional basic variable.

Cutting plane methods

A warming example

- Let

$$\begin{array}{rrcl} \min & \boldsymbol{c}^T\boldsymbol{x} & & \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} & = & \boldsymbol{b} \\ & \boldsymbol{x} & \geq & 0 \end{array}$$

  be the linear optimization relaxation after some iteration.

- Let $\boldsymbol{x}^* = (\boldsymbol{x}_B^*, \boldsymbol{x}_N^*) = (\boldsymbol{x}_B^*, 0)$ be an optimal BFS to that relaxation with at least one fractional basic variable.

- Then a valid cut for the original IO problem is:

$$\sum_{j \in N} x_j \geq 1$$

  - by contradiction, if there were an integer feasible solution $\bar{\boldsymbol{x}}$ with $\sum_{j \in N} \bar{x}_j = 0$
  - then $\bar{\boldsymbol{x}}_N = 0 \implies \bar{\boldsymbol{x}} = \boldsymbol{x}^\star$
  - a contradiction since $\boldsymbol{x}^*$ has at least one fractional basic variable

- Let the linear optimization relaxation (at some iteration of the meta algorithm) be:

$$\begin{array}{lrcl} \min & \boldsymbol{c}^T\boldsymbol{x} & & \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} & = & \boldsymbol{b} \\ & \boldsymbol{x} & \geq & 0 \end{array}$$

- Let $\boldsymbol{x}^* = (\boldsymbol{x}_B^*, \boldsymbol{x}_N^*) = (\boldsymbol{x}_B^*, 0)$ be an optimal BFS to that relaxation with at least one fractional basic variable.

- Corresponding optimal tableau:

$-\boldsymbol{c}_B^T\boldsymbol{x}_B^\star$	0	$\bar{\boldsymbol{c}}_N$
$\boldsymbol{x}_B^\star$	$\boldsymbol{I}_{m\times m}$	$\boldsymbol{B}^{-1}\boldsymbol{N}$

($\boldsymbol{d}_j = \boldsymbol{B}^{-1}\boldsymbol{A}_j$ are the columns in the tableau)

- For any (integer) feasible point, we have $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, or equivalently

$$\boldsymbol{x}_B + \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_N = \boldsymbol{x}_B + \sum_{j\in N}\boldsymbol{d}_j x_j = \boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{x}_B^*$$

- Consider a basic index $i \in B$ with $x_i^\star$ fractional
- $\boldsymbol{Ax} = \boldsymbol{b}$ iff $\boldsymbol{x}_B + \sum_{j \in N} \boldsymbol{d}_j \boldsymbol{x}_j = \boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{x}_B^*$, so for any integer feasible solution

$$x_i + \sum_{j \in N} d_{ij} x_j = x_i^\star$$

- Since feasibility requires $x_j \geq 0$ for all j,

$$x_i + \sum_{j \in N} \lfloor d_{ij} \rfloor x_j \leq x_i + \sum_{j \in N} d_{ij} x_j = x_i^\star$$

- Since feasibility requires x_j to be integer for all j, the constraint

$$x_i + \sum_{j \in N} \lfloor d_{ij} \rfloor x_j \leq \lfloor x_i^\star \rfloor$$

is satisfied by all integer feasible solutions. So it is a valid cut.

Cutting plane methods

Example

- Consider the integer optimization problem:

$$\begin{array}{lrcrcl} \min & x_1 & - & 2x_2 & & \\ \text{s.t.} & -4x_1 & + & 6x_2 & \leq & 9 \\ & x_1 & + & x_2 & \leq & 4 \\ & \multicolumn{5}{l}{x_1, x_2 \geq 0} \\ & \multicolumn{5}{l}{x_1, x_2 \text{ integer.}} \end{array}$$

- The linear optimization relaxation problem in standard form:

$$\begin{array}{lrcrcrcrcl} \min & x_1 & - & 2x_2 & & & & & & \\ \text{s.t.} & -4x_1 & + & 6x_2 & + & x_3 & & & = & 9 \\ & x_1 & + & x_2 & & & + & x_4 & = & 4 \\ & \multicolumn{9}{l}{x_1, \ldots, x_4 \geq 0} \end{array}$$

Cutting plane methods

Example

- LO relaxation optimal tableau

$$\begin{array}{|c|cccc|}
\hline
35/10 & 0 & 0 & 3/10 & 2/10 \\
\hline
15/10 & 1 & 0 & -1/10 & 6/10 \\
25/10 & 0 & 1 & 1/10 & 4/10 \\
\hline
\end{array} \Rightarrow \boldsymbol{x}^1 = (x_1^1, x_2^1) = (15/10, 25/10)$$

- From the optimal tableau all feasible solutions satisfy

$$x_2 + \frac{1}{10}x_3 + \frac{4}{10}x_4 = \frac{25}{10}$$

- Gomory cut:

$$x_2 \leq 2$$

- Add constraints $x_2 + x_5 = 2$, $x_5 \geq 0$

Cutting plane methods

Example

- New LO relaxation optimal tableau

13/4	0	0	1/4	0	2/4
3/4	1	0	−1/4	0	6/4
2	0	1	0	0	1
5/4	0	0	1/4	1	−10/4

$\Rightarrow \boldsymbol{x}^2 = (x_1^2, x_2^2) = (3/4, 2)$

- From the new optimal tableau:

$$x_1 - \frac{1}{4}x_3 + \frac{6}{4}x_5 = \frac{3}{4}$$

- New Gomory cut

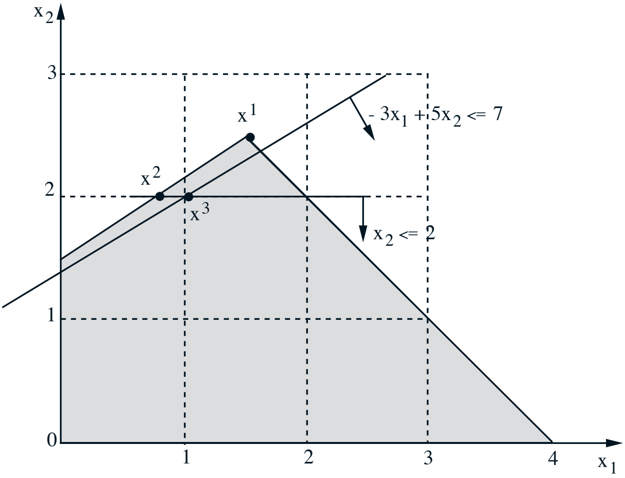
$$x_1 - x_3 + x_5 \leq 0$$

(that cut In terms of original variables is $-3x_1 + 5x_2 \leq 7$)

- Add constraints $x_1 - x_3 + x_5 + x_6 = 0$, $x_6 \geq 0$
- New optimal solution is $\boldsymbol{x}^3 = (x_1^3, x_2^3) = (1, 2)$. This is an optimal solution to the original IO problem.

Cutting plane methods

Example



Branch and bound

Illustrative idea

- Binary optimization problem:

$$\begin{array}{ll} \max & 12x_1 + 12x_2 + 4x_3 + 2x_4 \\ \text{s.t.} & 8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 15 \\ & x_1, x_2, x_3, x_4 \ \text{ binary} \end{array}$$

Feasible solution $x_1, x_2, x_3, x_4 = 0$; Value=0

Branch and bound

Illustrative idea

- Binary optimization problem:

$$\begin{array}{rl} \max & 12x_1 + 12x_2 + 4x_3 + 2x_4 \\ \text{s.t.} & 8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 15 \\ & x_1, x_2, x_3, x_4 \ \text{ binary} \end{array}$$

Feasible solution $x_1, x_2, x_3, x_4 = 0$; Value=0

- Relaxation

$$\begin{array}{rl} \max & 12x_1 + 12x_2 + 4x_3 + 2x_4 \\ \text{s.t.} & 8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 15 \\ & x_1, x_2, x_3, x_4 \leq 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

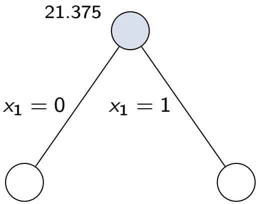
LO solution: $x_1 = 5/8,\ x_2 = 1,\ x_3 = 0,\ x_4 = 0$; Value$=21 + 3/8 = 21.375$

Branch and bound

Illustrative idea, start

$$\begin{array}{ll} \max & 15x_1 + 12x_2 + 4x_3 + 2x_4 \\ \text{s.t.} & 8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10 \\ & x_1, x_2, x_3, x_4 \ \ \text{binary} \end{array}$$

- Best feasible solution:
  $x_1, x_2, x_3, x_4 = 0$
  Value=0
- Relaxed solution:
  $x_1 = 5/8, x_2 = 1, x_3 = 0, x_4 = 0$
  Value=21.375
- Branch:
  Either $x_1 = 0$ or $x_1 = 1$

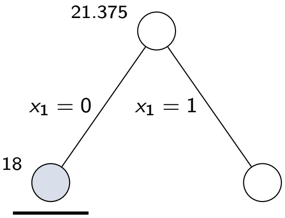


Branch and bound

Illustrative idea, $x_1 = 0$

$$\begin{array}{ll} \max & 12x_1 + 12x_2 + 4x_3 + 2x_4 \\ \text{s.t.} & 8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10 \\ & x_1 = 0 \\ & x_1, x_2, x_3, x_4 \ \text{ binary} \end{array}$$

- Best feasible solution:
  $x_1, x_2, x_3, x_4 = 0$
  Value=0
- Relaxed solution:
  $x_1 = 0, \ x_2 = 1, \ x_3 = 1, \ x_4 = 1$
  Value=18
- Optimal, prune:
  New best feasible solution with value=18

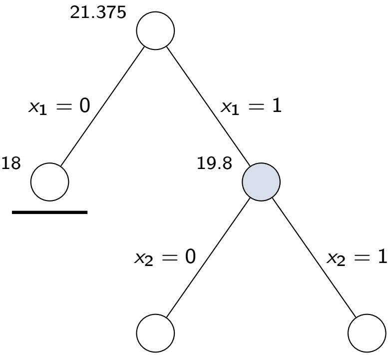


Branch and bound

Illustrative idea, $x_1 = 1$

$$\begin{array}{ll} \max & 12x_1 + 12x_2 + 4x_3 + 2x_4 \\ \text{s.t.} & 8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10 \\ & x_1 = 1 \\ & x_1, x_2, x_3, x_4 \ \ \text{binary} \end{array}$$

- Best feasible solution: $x_1 = 0,\ x_2 = 1,\ x_3 = 1,\ x_4 = 1.$ Value=18
- Relaxed solution: $x_1 = 1,\ x_2 = 2/5,\ x_3 = 0,\ x_4 = 0.$ Value$=19 + 4/5 = 19.8$
- Branch: Either $x_2 = 0$ or $x_2 = 1$

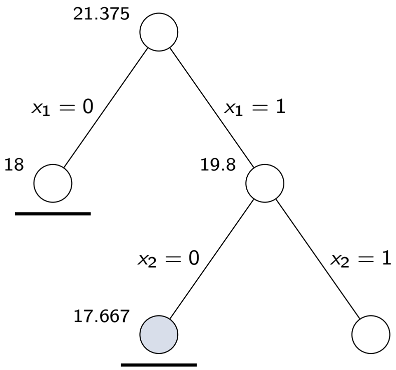


Branch and bound

Illustrative idea, $x_1 = 1,\ x_2 = 0$

$$\begin{array}{ll} \max & 12x_1 + 12x_2 + 4x_3 + 2x_4 \\ \text{s.t.} & 8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10 \\ & x_1 = 1,\ x_2 = 0 \\ & x_1, x_2, x_3, x_4 \ \text{ binary.} \end{array}$$

- Best feasible solution : $x_1 = 0,\ x_2 = 1,\ x_3 = 1,\ x_4 = 1.$ Value=18.
- Relaxed solution : $x_1 = 1,\ x_2 = 0,\ x_3 = 2/3,\ x_4 = 0.$ Value=$17 + 2/3 \approx 17.667$.
- Suboptimal, prune: Lower value than best so far !

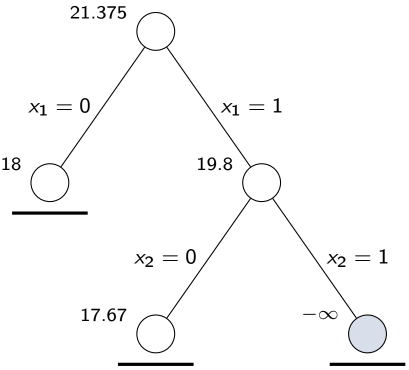


Branch and bound

Illustrative idea, $x_1 = 1,\ x_2 = 1$

$$\begin{array}{ll} \max & 12x_1 + 12x_2 + 4x_3 + 2x_4 \\ \text{s.t.} & 8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10 \\ & x_1 = 1,\ x_2 = 1 \\ & x_1, x_2, x_3, x_4 \ \text{binary.} \end{array}$$

- Best feasible solution: $x_1 = 0,\ x_2 = 1,\ x_3 = 1,\ x_4 = 1.$ Value=18.
- Infeasible, prune.
- Optimal integer solution: $x_1 = 0,\ x_2 = 1,\ x_3 = 1,\ x_4 = 1.$ Value=18.



Branch and bound

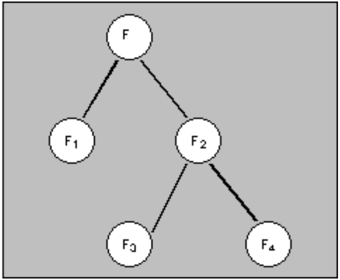
General idea ... "divide and conquer"

- Let $\boldsymbol{F}$ be the set of feasible solution to the problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \boldsymbol{F} \end{array}$$

- Partition $\boldsymbol{F}$ into a finite collection of subsets $\boldsymbol{F}_1, \cdots, \boldsymbol{F}_k$.
- Solve separately each of the k subproblems:

$$\begin{array}{ll} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \boldsymbol{F}_i \end{array}$$



Branch and bound

Meta algorithm

- Branching: Select an active subproblem $\boldsymbol{F}_i$
- Pruning: If the subproblem is infeasible, delete it.
- Bounding: Otherwise, compute a lower bound $b(\boldsymbol{F}_i)$ for the subproblem.
- Pruning: If $b(\boldsymbol{F}_i) \geq U$, the current best upperbound, delete the subproblem.
- Partitioning: If $b(\boldsymbol{F}_i) < U$, either obtain an optimal solution to the subproblem (stop), or break the corresponding problem into further subproblems, which are added to the list of active subproblem.

Branch and bound

LO Based

- Compute the lower bound $b(\boldsymbol{F})$ by solving the LO relaxation of the discrete optimization problem.
- From the LO solution $\boldsymbol{x}^*$, if there is a component x_i^* which is fractional, we create two subproblems by adding either one of the constraints

$$x_i \leq \lfloor x_i^* \rfloor, \text{ or } x_i \geq \lceil x_i^* \rceil.$$

  Note that both constraints are violated by $\boldsymbol{x}^*$.
- If there are more than 2 fractional components, we use selection rules like maximum infeasibility etc. to determine the inequalities to be added to the problem
- Select the active subproblem using either depth-first or breadth-first search strategies.