Recitation 4: Duality

6.255 Optimization Methods

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Agenda

- Duality: intuition and practice.
- How to take the dual of a problem.
- Using duality to prove a result.

Intuition

Primal:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned}$$

Dual:

$$\begin{aligned} \max \quad & \mathbf{p}'\mathbf{b} \\ \text{s.t.} \quad & \mathbf{p}'\mathbf{A} = \mathbf{c}' \\ & \mathbf{p} \leq \mathbf{0} \end{aligned}$$

- $\mathbf{x}$ is a primal solution, $\mathbf{p}$ is a dual solution.
- Dual is the adversary problem of the primal.
- Primal wants to minimize and dual wants to maximize, they meet at the same optimal.

In practice

Primal:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b} \end{aligned}$$

Dual:

$$\begin{aligned} \max \quad & \mathbf{p}'\mathbf{b} \\ \text{s.t.} \quad & \mathbf{p}'\mathbf{A} = \mathbf{c}' \\ & \mathbf{p} \le \mathbf{0} \end{aligned}$$

- **Weak duality:** If $\mathbf{x}$ is primal feasible and $\mathbf{p}$ dual feasible then $\mathbf{p}'\mathbf{b} \le \mathbf{c}'\mathbf{x}$.
- **Strong duality:** If $\mathbf{x}$ is primal optimal and $\mathbf{p}$ dual optimal then $\mathbf{p}'\mathbf{b} = \mathbf{c}'\mathbf{x}$.

In practice

- **Complementary slackness:** Let $\mathbf{x}$ primal feasible and $\mathbf{p}$ dual feasible. Then $\mathbf{x}, \mathbf{p}$ optimal **if and only if**

$$p_i(\mathbf{a}_i'\mathbf{x} - b_i) = 0, \quad \forall i$$

$$x_j(c_j - \mathbf{p}'\mathbf{A}_j) = 0, \quad \forall j$$

- Gives you a free bonus propriety on $\mathbf{x}$,$\mathbf{p}$ when you know they are optimal.
- Can be used to check optimality when given an $\mathbf{x}$ and $\mathbf{p}$.

Relations Between Primal and Dual

	Finite opt.	**Unbounded**	**Infeasible**
Finite opt.	*		
Unbounded			*
Infeasible		*	*

Computing the dual

"Constraint in primal $\leftrightarrow$ Variable in dual"
"Variable in primal $\leftrightarrow$ Constraint in dual"

Primal

$$\begin{array}{lll} \min & \mathbf{c}'\mathbf{x} & \\ \text{s.t.} & \mathbf{a}_i'\mathbf{x} \geq b_i & i \in M_1 \\ & \mathbf{a}_i'\mathbf{x} \leq b_i & i \in M_2 \\ & \mathbf{a}_i'\mathbf{x} = b_i & i \in M_3 \\ & x_j \geq 0 & j \in N_1 \\ & x_j \leq 0 & j \in N_2 \\ & x_j \text{ free} & j \in N_3 \end{array}$$

Dual

$$\begin{array}{lll} \max & \mathbf{p}'\mathbf{b} & \\ \text{s.t.} & p_i \geq 0 & i \in M_1 \\ & p_i \leq 0 & i \in M_2 \\ & p_i \text{ free} & i \in M_3 \\ & \mathbf{p}'\mathbf{A}_j \leq c_j & j \in N_1 \\ & \mathbf{p}'\mathbf{A}_j \geq c_j & j \in N_2 \\ & \mathbf{p}'\mathbf{A}_j = c_j & j \in N_3 \end{array}$$

Computing the dual

Primal	min	max	**dual**
constraints	$\geq b_i$	≥ 0	variables
	$\leq b_i$	≤ 0	
	$= b_i$	free	
variables	≥ 0	$\leq c_j$	constraints
	≤ 0	$\geq c_j$	
	free	$= c_j$	

Exercise 1

$$
\begin{aligned}
\text{minimize} \quad & 8x_1 + 4x_2 + 2x_3 \\
\text{s.t.} \quad & 2x_1 + x_2 \geq 2 \\
& x_1 + x_3 \geq 2 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

1. Formulate the dual problem.
2. Graph the feasible region of the dual problem. Identify the optimal basic feasible solution.
3. Using complementary slackness, show that there is a unique optimal basic feasible solution for the primal problem.

Exercise 1: dual

$$\begin{aligned} \text{minimize} \quad & 8x_1 + 4x_2 + 2x_3 \\ \text{s.t.} \quad & 2x_1 + x_2 \geq 2 \\ & x_1 + x_3 \geq 2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Exercise 1: slackness

$$
\begin{array}{ll|ll}
\text{minimize} & 8x_1 + 4x_2 + 2x_3 & \text{maximize} & 2p_1 + 2p_2 \\
\text{s.t.} & 2x_1 + x_2 \geq 2 & \text{s.t.} & 2p_1 + p_2 \leq 8 \\
 & x_1 + x_3 \geq 2 & & p_1 \leq 4 \\
 & x_1, x_2, x_3 \geq 0 & & p_2 \leq 2 \\
 & & & p_1, p_2 \geq 0
\end{array}
$$

$$
p_i(\mathbf{a}_i'\mathbf{x} - b_i) = 0, \ \forall i; \quad x_j(c_j - \mathbf{p}'\mathbf{A}_j) = 0, \ \forall j
$$

Bonus: Exercice 2

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{D}\mathbf{x} = \mathbf{h} \\ & \mathbf{x} \geq 0 \end{aligned}$$

1. Formulate the dual problem.

$$\begin{aligned} \max \quad & \mathbf{p}'\mathbf{b} + \mathbf{q}'\mathbf{h} \\ \text{s.t.} \quad & \mathbf{A}'\mathbf{p} + \mathbf{D}'\mathbf{q} \leq \mathbf{c} \\ & \mathbf{p} \leq 0 \end{aligned}$$

Exercise 3

Let $\mathbf{A}$ be a given matrix. Show that exactly one the following alternatives must hold.

1. There exists some $\mathbf{x} \neq 0$ such that $\mathbf{Ax} = 0$, $\mathbf{x} \geq 0$.
2. There exists some $\mathbf{p}$ such that $\mathbf{p}^\top \mathbf{A} > \mathbf{0}^\top$.

Exercise 3: solution

In order to show that we need to show both of the following:

- If 1 is true then 2 is false.
- If 1 is false then 2 is true.

Let us show the first point. Suppose 1 is true. Let $\mathbf{x} \neq 0$ such that $\mathbf{A}\mathbf{x} = 0$ and $\mathbf{x} \geq 0$. Then, $\mathbf{A}(\theta\mathbf{x}) = 0$ and $\theta\mathbf{x} \geq 0$ for all $\theta \geq 0$. Thus, the following LP is unbounded:

$$\begin{aligned} \text{min} \quad & -\mathbf{e}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = 0 \\ & \mathbf{x} \geq 0 \end{aligned}$$

By weak duality, this implies that the dual problem is infeasible.

$$\begin{aligned} \text{max} \quad & 0 \\ \text{s.t.} \quad & \mathbf{p}^\top\mathbf{A}\mathbf{x} \leq -\mathbf{e} \end{aligned}$$

Exercise 3: solution

Since there is no $\mathbf{p}$ that satisfies $\mathbf{p}^\top \mathbf{A} \leq -\mathbf{e}$, then there is no $\mathbf{p}$ that satisfies $\mathbf{p}^\top \mathbf{A} \geq \mathbf{e}$, which implies there is no $\mathbf{p}$ that satisfies $\mathbf{p}^\top \mathbf{A} \geq \theta \mathbf{e}$ for any $\theta > 0$. Hence, there is no such $\mathbf{p}$ that satisfies $\mathbf{p}^\top \mathbf{A} > 0$.
Let us now show the second point. Suppose that 1 is false, ie, there exists no $\mathbf{x} \neq 0$ such that $\mathbf{A}\mathbf{x} = 0$, $\mathbf{x} \geq 0$. Then, the only solution to $\mathbf{A}\mathbf{x} = 0$, $\mathbf{x} \geq 0$ is $\mathbf{x} = 0$. Hence, the following LP has objective value of 0:

$$\begin{array}{rl} \min & -\mathbf{e}'\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = 0 \\ & \mathbf{x} \geq 0 \end{array}$$

This implies that the dual is feasible. Therefore, there exists a $\mathbf{p}$ such that $\mathbf{p}^\top \mathbf{A} \leq -\mathbf{e}$. Hence there exists a $\mathbf{p}$ such that $\mathbf{p}^\top \mathbf{A} \geq \mathbf{e} > 0$.

Bonus - Exercise 4: Exercise 4.6 of [BT]

Let $A \in \mathbb{R}^{m \times n}$ be a matrix and let $b \in \mathbb{R}^m$ be a vector. We consider the problem of minimizing $||Ax - b||_\infty$ over all $x \in \mathbb{R}^n$. Here, $|| \cdot ||_\infty$ is the vector norm defined by $||y||_\infty := \max_i |y_i|$. Let ν be the optimal cost of this regression problem.

1. Let p be any vector in $\mathbb{R}^m$: $\sum_{i=1}^m |p_i| = 1,\ A^\top p = 0$. Show that $p^\top b \leq \nu$.

Exercise 4: Exercise 4.6 of [BT]

Let $A \in \mathbb{R}^{m \times n}$ be a matrix and let $b \in \mathbb{R}^m$ be a vector. We consider the problem of minimizing $||Ax - b||_\infty$ over all $x \in \mathbb{R}^n$. Here, $|| \cdot ||_\infty$ is the vector norm defined by $||y||_\infty := \max_i |y_i|$. Let ν be the optimal cost of this regression problem.

1. Let p be any vector in $\mathbb{R}^m$: $\sum_{i=1}^m |p_i| = 1,\ A^\top p = 0$. Show that $p^\top b \leq \nu$.

Hint: To show that a cost is bounded from above by some optimal cost, we need weak duality.

Exercise 4: Exercise 4.6 of [BT]: Solution part 1

Let $A \in \mathbb{R}^{m \times n}$ be a matrix and let $b \in \mathbb{R}^m$ be a vector. We consider the problem of minimizing $||Ax - b||_\infty$ over all $x \in \mathbb{R}^n$. Here, $||\cdot||_\infty$ is the vector norm defined by $||y||_\infty := \max_i |y_i|$. Let ν be the optimal cost of this regression problem.

1. Let p be any vector in $\mathbb{R}^m : \sum_{i=1}^m |p_i| = 1,\ A^\top p = 0$. Show that $p^\top b \leq \nu$.

Our primal is:

$$\min \quad z, \text{ s.t. } Ax - ez \leq b,\ Ax + ez \geq b,$$

And its dual is:

$$\begin{aligned} \max \quad & b^\top (p+q) \\ \text{s.t.} \quad & A^\top (p+q) = 0,\ e^\top (q-p) = 1, \\ & q \geq 0,\ p \leq 0. \end{aligned}$$

Therefore, result follows from weak duality.

Exercise 4: Exercise 4.6 of [BT]: Part 2

Let $A \in \mathbb{R}^{m \times n}$ be a matrix and let $b \in \mathbb{R}^m$ be a vector. We consider the problem of minimizing $||Ax - b||_\infty$ over all $x \in \mathbb{R}^n$. Here, $|| \cdot ||_\infty$ is the vector norm defined by $||y||_\infty := \max_i |y_i|$. Let ν be the optimal cost of this regression problem.

1. Let p be any vector in $\mathbb{R}^m$: $\sum_{i=1}^m |p_i| = 1,\ A^\top p = 0$. Show that $p^\top b \leq \nu$.
    - Use weak duality.
2. Show that the optimal cost of your dual problem is ν.

Exercise 4: Exercise 4.6 of [BT]: Part 2

Let $A \in \mathbb{R}^{m \times n}$ be a matrix and let $b \in \mathbb{R}^m$ be a vector. We consider the problem of minimizing $||Ax - b||_\infty$ over all $x \in \mathbb{R}^n$. Here, $|| \cdot ||_\infty$ is the vector norm defined by $||y||_\infty := \max_i |y_i|$. Let ν be the optimal cost of this regression problem.

1. Let p be any vector in $\mathbb{R}^m : \sum_{i=1}^m |p_i| = 1, \; A^\top p = 0$. Show that $p^\top b \leq \nu$.
   - Use weak duality.
2. Show that the optimal cost of your dual problem is ν.
   - Follows from strong duality.