6.215/6.255J/15.093J/IDS.200J: Optimization Methods

Problem Set 1

Due: September 28, 2021

Problem 1: Rocket control problem. (15 points) Consider a rocket that travels along a straight path. Let x_t, v_t, and a_t be the position, velocity, and acceleration of the rocket at time t, respectively. By discretizing time and by taking the time increment to be unity, we obtain an approximate discrete-time model of the form:

$$x_{t+1} = x_t + v_t$$
$$v_{t+1} = v_t + a_t.$$

We assume that the acceleration is under our control, which is controlled by the rocket thrust. In a rough model, the magnitude $|a_t|$ of the acceleration is proportional to the rate of fuel consumption at time t.

Suppose that the rocket is initially at rest at the origin, *i.e.*, $x_0 = 0$ and $v_0 = 0$. We wish the rocket to take off and "land softly" at distance d unit after T time units, *i.e.*, $x_T = d$ and $v_T = 0$. The total fuel consumption of the rocket, given by $\sum_{t=0}^{T-1} c_t|a_t|$ (where $c_1, \ldots, c_{T-1}$ are positive numbers known to us), cannot be more than available amount of fuel f. To ensure a smooth trajectory, we want to ensure that the acceleration of the rocket does not change too abruptly, *i.e.*, $|a_{t+1} - a_t|$ is always less than or equal to some known value δ.

Now, we want to control the rocket in a manner to minimize the maximum thrust required, which is $\max_{t\in\{0,\ldots,T-1\}} |a_t|$, subject to the preceding constraints.

(a) Provide a linear programming formulation for this rocket control problem.

(b) Formulate and solve the model in `Julia` for $T = 100$, $d = 50$, $\delta = 10^{-3}$, $f = 1000$, and $c_0 = \ldots = c_{T-1} = 1$. Plot acceleration, velocity, and position of the rocket vs time. Attach the code.

Solution See the attached HTML file.

Problem 2: Reformulation as a linear programming problem. (12 points) (a) Consider the problem

$$\begin{array}{ll} \text{minimize} & c_1x_1 + c_2|x_2 - 10| \\ \text{subject to} & c_3|x_1 + 2| + c_4|x_2| \leq 5, \end{array}$$

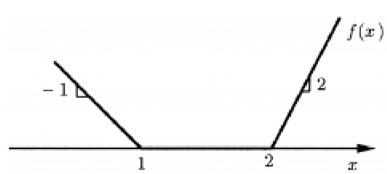

Figure 1: The function f of Problem 2(b)

where x_1, x_2 are the decision variables, and $c_1, \ldots, c_4$ are problem data. Provide the range of values for $c_1, \ldots, c_4$ so that we can formulate the problem above as a linear programming problem, and provide that formulation.

(b) Consider the problem of minimizing a cost function of the form $c'x + f(d'x)$, subject to the linear constraints $Ax \geq b$, where $c, d \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, and $A \in \mathbf{R}^{m \times n}$ are the problem data, and $x \in \mathbf{R}^n$ is the decision variable. The function $f : \mathbf{R} \to \mathbf{R}$ is specified in Figure 1. Provide a linear programming formulation for the problem.

Solution. (a) The following range of variables will allow formulation as linear program: c_1 free (can be any real number), c_2 is nonnegative, c_3, c_4 are {both nonnegative or both nonpositive}.

Let us formulate the problems case by case:

1. c_1 free , c_2 is nonnegative, both c_3, c_4 are nonnegative. The formulation is:

$$\begin{array}{ll} \text{minimize} & c_1 x_1 + c_2 t \\ \text{subject to} & t \geq x_2 - 10, t \geq -(x_2 - 10) \\ & c_3 y + c_4 z \leq 5 \\ & y \geq x_1 + 2, y \geq -(x_1 + 2) \\ & z \geq x_2, z \geq -x_2, \end{array}$$

where the decision variables are t, y, z, x_1, x_2.

2. c_1 free , c_2 is nonnegative, both c_3, c_4 are nonpositive. In this case: the constraint $c_3|x_1 + 2| + c_4|x_2| \leq 5$ will be always satisfied, so not matter what value of x_1, x_2 we pick, so this becomes a redundant constraint that we can remove. The formulation is just:

$$\begin{array}{ll} \text{minimize} & c_1 x_1 + c_2 t \\ \text{subject to} & t \geq x_2 - 10, t \geq -(x_2 - 10), \end{array}$$

where t, x_1, x_2 are the decision variables. Note that depending on the value c_1 the optimal value can be unbounded.

(b) The function in Figure 1 can be written as a piece-wise linear convex function $f(x) = \max(1-x, 0, 2x-4)$. Hence, using the epigraph approach, the given problem can be formulated as the following linear programming problem:

$$\begin{array}{ll} \text{minimize} & c'x + t \\ \text{subject to} & t \geq 1 - d'x \\ & t \geq 0 \\ & t \geq 2d'x - 4 \\ & Ax \geq b, \end{array}$$

where x, t are the decision variables.

Problem 3: Range of a matrix for non-negative inputs. (10 points) Recall that range of a matrix $A \in \mathbf{R}^{m \times n}$ is defined as, $\mathbf{ran}(A) = \{Ax \mid x \in \mathbf{R}^n\}$. One interpretation of $\mathbf{ran}(A)$ is that it is the set of vectors that can be "hit" by the linear mapping $y = f(x) = \sum_{i=1}^n x_i A_i$, where A_i is the ith column of A. The range of a matrix for nonnegative inputs can be defined as:

$$\mathbf{ran}_+(A) = \{Ax \mid x \in \mathbf{R}^n, x \geq 0\}.$$

Show that any element of $\mathbf{ran}_+(A)$ can be expressed in the form $\sum_{i=1}^n x_i A_i$,with $x_i \geq 0$, and at most m of the coefficients x_i being nonzero.

Hint: consider the polyhedron:

$$C = \{x \in \mathbf{R}^n \mid y = Ax, x \geq 0\}.$$

Solution.

- (Case 1: $m > n$) Suppose we have $m > n$ and we have some $y \in \mathbf{ran}_+(A)$, which means that there is some $x \geq 0$ such that $y = Ax$. Now, if A has $m > n$ that means we have more constraints than variables, yet we have a solution. So, some of them must be redundant. So we can remove the redundant constraints and we end up with some $\widetilde{A}x = \widetilde{y}$, where $\widetilde{A} \in \mathbf{R}^{n\times n}$ being invertible and $\widetilde{y} \in \mathbf{R}^n$ with $x \in \mathbf{R}^n$. So, $x = \widetilde{A}^{-1}\widetilde{y} \geq 0$ which has dimension n. So in this case at most n can be nonzero. As $m > n$, we have proven that at most m components of x are nonzero trivially.
- (Case 2: $m \leq n$) In this case, without loss of generality we can assume the matrix A has rank m (see the comment below if the rank is less than m). Define, the standard form polyhedron:

$$C = \{x \in \mathbf{R}^n \mid y = Ax, x \geq 0\}.$$

  As $y \in \mathbf{ran}_+(A)$, we have some $x \geq 0, y = Ax$, *i.e.*, C is nonempty. So, C must have one basic feasible solution. At a basic feasible solution, we must have n active constraints. So, at least $n - m$ of the coefficients x_i will be zero. Therefore, at most m of the x_is will be nonzero.
  - Comment: If $\mathbf{rank}(A) = \bar{m} < m$, there must be some redundant constraints in A. We can remove the redundant constraints and construct a full row rank matrix $\bar{A} \in \mathbf{R}^{\bar{m}\times n}$ with rank $\bar{m} < m \leq n$ and corresponding $\bar{y} \in \mathbf{R}^{\bar{m}}$. Then we consider the standard form polyhedron: $\{x \in \mid \bar{A}x = \bar{y}, x \geq 0\}$, and we proceed like before.

Problem 4: True or false? (18 points) Consider the standard form polyhedron $P = \{x \mid Ax = b, x \geq 0\}$. Suppose that the matrix A has dimensions $m \times n$ and that its rows are linearly independent. For each of the statements below, state whether it is true or false. If true, provide an informal justification (no formal proof required), else, provide a counter example.

(a) If $n = m + 1$, then P has at most two basic feasible solutions.

(b) The set of all optimal solutions is bounded.

(c) At every optimal solution, no more than m variables can be positive.

(d) If there is more than one optimal solution, then there are uncountable many optimal solution.

(e) If there are several optimal solutions, then there exist at least two basic feasible solutions that are optimal.

(f) Consider the problem of minimizing $\max\{c'x, d'x\}$, over the set P. If this problem has an optimal solution, it must have an optimal solution which is an extreme point of P.

Solution. (a) True.

We have $m = n - 1$, and due to the full row rank assumption, rank of A is $n - 1$. Let $\bar{x}$ be a point that satisfies: $A\bar{x} = b$, then the set:

$$\begin{aligned}&\{x \mid Ax = b = Ay \Leftrightarrow A(\underbrace{x - \bar{x}}_{=z(\text{say})}) = 0\}\\ =&\{z + \bar{x} \mid Az = 0\},\end{aligned}$$

so any solution x to the system $Ax = b$ can be written as: $x = z + \bar{x}$, where $z \in \text{nullspace}(A)$. The nullspace has dimension $n - \text{rank}(A) = n - (n-1) = 1$, so it has only one vector (say v) in its basis. So any element $z \in \text{nullspace}(A)$ can be written as $z = \lambda v$, where $\lambda \in \mathbf{R}$. Hence, any solution x to the system $Ax = b$ can be written as $\bar{x} + \lambda v$, which is a line. So P which is intersection of $\{x \mid Ax = b\}$ and $\{x \mid x \geq 0\}$ is part of some line. So it cannot have more than two extreme point (see Definition 2.6).

(b) False.

Consider the standard form optimization problem:

$$\begin{array}{ll} \text{minimize} & x_2 \\ \text{subject to} & x_2 = 0 \\ & x_1, x_2 \geq 0, \end{array}$$

which has the optimal solution set $\{(x_1, 0) \mid x_1 \geq 0\}$, which is unbounded.

(c) False.

Consider the standard form optimization problem:

$$\begin{array}{ll} \text{minimize} & 0'x \\ \text{subject to} & Ax = b \\ & x \geq 0, \end{array}$$

where any feasible x will be an optimal solution, no matter how many components are positive.

(d) True.

If x, y are optimal then any convex combination of them will be in P and have the same optimal value, ***i.e.*, any convex combination of x, y will an optimal solution.**

(e) False.

Consider the optimization problem in (2). This has unbounded solution set, but only one bfs.

(f) False.

Consider the problem

$$\begin{array}{ll} \text{minimize} & |x_1 - \frac{1}{2}| = \max\{x_1 - \frac{1}{2}, -(x_1 - \frac{1}{2})\} \\ \text{subject to} & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0, \end{array}$$

whose optimal solution appears at $(\frac{1}{2}, \frac{1}{2})$ which is not an extreme point of the feasible set.

Problem 5: Find the unknown parameters. (10 points) While solving a standard form problem, we arrive at the following tableau with x_3, x_4, and x_5 being the basic variables.

-10	δ	-2	0	0	0
4	-1	η	1	0	0
1	α	-4	0	1	0
β	γ	3	0	0	1

The entries $\alpha, \beta, \gamma, \delta, \eta$ in the tableau are unknown parameters. For each of the following statements, find some parameter values that will make the statement true.

(a) The current solution is optimal and there are multiple optimal solutions.

(b) The optimal cost is $-\infty$.

(c) The current solution is feasible but not optimal.

Solution. (a) For the current solution to be optimal, it must be degenerate, as the reduced cost has one negative entry (Theorem 3.1(b)). This implies $\beta = 0$. To have multiple solutions, we need $\delta = 0$, because if the reduced cost of every nonbasic variable is positive, then the current optimal solution will be unique. Now we perform a change of basis, where A_2 enters the basis (3 is the pivot element) and A_5 leaves the basis. After one simplex iteration we have

-10	$\frac{2\gamma}{3}$	0	0	0	2/3
x_3 =4	$-1-\frac{\eta\gamma}{3}$	η	1	0	0
x_4 =1	$\alpha+\frac{4\gamma}{3}$	0	0	1	4/3
x_2 =0	$\gamma/3$	1	0	0	1/3

,

and set $\gamma = 0$, which makes the reduced cost $\bar{c} \geq 0$. So, the bfs $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 4, 1, 0)$ is an optimal solution. Putting the values of the parameters the tableau above will look like:

-10	0	0	0	0	2/3
x_3 =4	-1	η	1	0	0
x_4 =1	α	0	0	1	4/3
x_2 =0	0	1	0	0	1/3

,

where we have not assigned any values of α and η yet. To find another optimal solution set $\alpha > 0$, we can perform another change of basis, where A_1 enters the basis (α being the pivot element) and A_4 leaves. This will give us a different bfs that is optimal. Note that we have not assigned any value to η yet, so it can be free.

(b) If we set $\delta < 0$, $\alpha \leq 0, \gamma \leq 0$, then $\bar{c}_1$is negative and all the elements of $u = B^{-1}A_1 \leq 0$, which will make the optimal cost $-\infty$. Rest of the parameters can be made free.

(c) If we set $\beta > 0$, then we have nondegenerate bfs (feasible), but due to the existence of a negative reduced cost $\bar{c}_2 < 0$, we will have suboptimality. Rest of the parameters can be made free.

Problem 6: Simple simplex. (15 points) Consider the problem

$$\begin{array}{ll} \text{minimize} & -2x_1 - x_2 \\ \text{subject to} & x_1 - x_2 \leq 2 \\ & x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0, \end{array}$$

where x_1, x_2 are the decision variables.

(a) Convert the problem into standard form and construct a basic feasible solution at which $(x_1, x_2) = (0, 0)$.

(b) Carry out full tableau implementation of the simplex method, starting with the basic feasible solution of part (a).

Solution. (a) Adding the slack variables $x_3, x_4 \geq 0$, we have the transformed standard form problem:

$$\begin{array}{ll} \text{minimize} & -2x_1 - x_2 + 0x_3 + 0x_4 \\ \text{subject to} & x_1 - x_2 + x_3 = 2 \\ & x_1 + x_2 + x_4 = 6 \\ & x_1, x_2, x_3, x_4 \geq 0, \end{array}$$

where the basic feasible solution will be $(0, 0, 2, 6)$.

(b) The tableau at the beginning:

0	-2	-1	0	0
2	$1^\star$	-1	1	0
6	1	1	0	1

,

after one iteration:

4	0	-3	2	0
2	1	-1	1	0
4	0	$2^\star$	-1	1

,

final step:

10	0	0	0.5	1.5
4	1	0	0.5	0.5
2	0	1	-0.5	0.5

,

where the optimal solution is $(4, 2, 0, 0)$ with optimal value -10.