Problem Set #1

Bennett Hellman & Benjamin Siegel
15.093 - Optimization Methods
MIT

September 27, 2021

Problem 1: Rocket control problem

Let $z = \max_{t\in\{0,\ldots,T-1\}} |a_t|$ We can reformulate this as a linear programming problem:

$$\begin{aligned}
\min \quad & z \\
\text{s.t.} \quad & a_t \leq z \\
& -a_t \leq z \\
& a_{t+1} - a_t \leq \delta \\
& a_t - a_{t+1} \leq \delta \\
& a_t \leq \hat{a}_t \\
& -a_t \leq \hat{a}_t \\
& \sum_{t=0}^{T-1} c_t * \hat{a}_t \leq f \\
& c_t \geq 0 \\
& x_0 = 0 \\
& v_0 = 0 \\
& x_T = d \\
& v_T = 0 \\
& v_{t+1} = v_t + a_t \\
& x_{t+1} = x_t + v_t
\end{aligned} \tag{1}$$

Where in this situation we have the constants $T = 100$, $\delta = .001$, $f = 1000$, $d = 50$, and $c_0 = \ldots = c_{T-1} = 1$. Running this model in Julia (see attached code), we end up with the following plots:

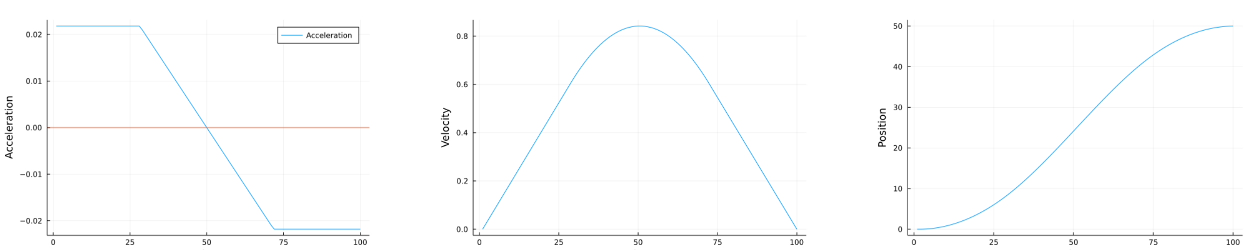


Figure 1: Acceleration, velocity, and position of rocket versus time.

Problem 2: Reformulation as a linear programming problem.

(a) The variables c_1 and c_2 are free because any combination of x_1 and $|x_2 - 10|$ creates a piecewise linear convex objective function, which can be cast into a linear programming problem. It is important to note that $c_1 = c_2 = 0$ creates a trivial objective function. In general, c_3 and c_4 can take on any value as long as the signs of both variables are not opposing. For example, both values being strictly positive produces a convex feasible region, as demonstrated in Figure 2:

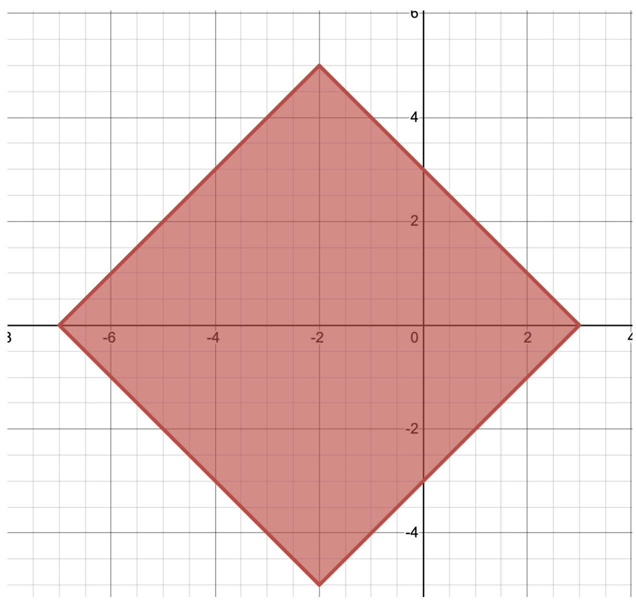


Figure 2: Convex set with $c_3, c_4 > 0$

Both values being strictly negative produces a feasible region over the entire $\mathbb{R}^2$ space. Both values being zero creates an unconstrained optimization problem. Having only one value be zero creates an unbounded optimization problem.

To show that these are the only feasible values for c_3 and c_4, we will make a geometric argument by cases. First, consider the case where $c_3 < 0$ and $c_4 > 0$:

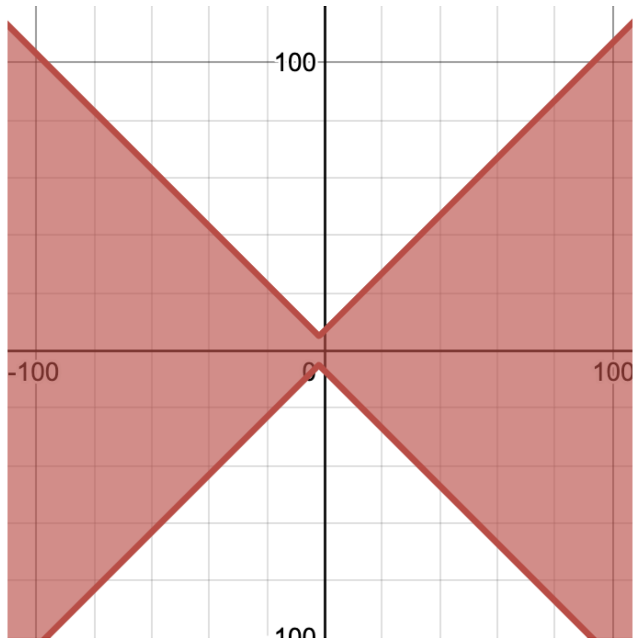


Figure 3: Feasible region with $c_3 < 0, c_4 > 0$

As seen in Figure 3, this combination of constraints produces a non-convex feasible region.

Second, consider the case where $c_3 > 0$ and $c_4 < 0$:

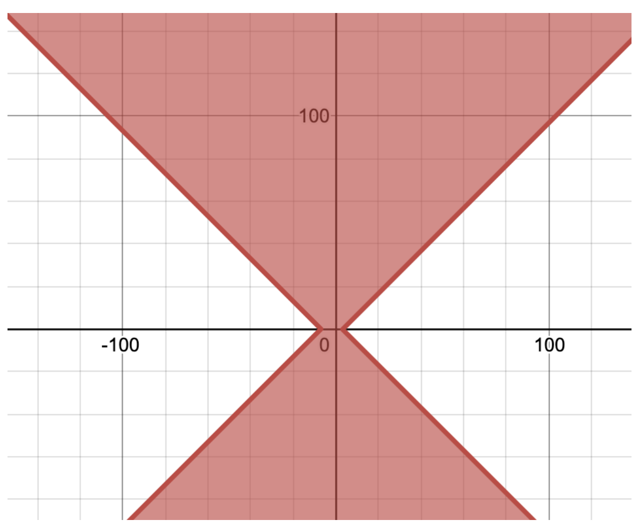


Figure 4: Feasible region with $c_3 > 0, c_4 < 0$

As seen in Figure 4, this combination of constraints also produces a non-convex feasible region.

(b) The function $f(d'x)$ can be written as: $\max\{-d'x+1, 0, 2d'x-4\}$. We will convert the problem to epigraph form and let $z = f(d'x) = \max\{-d'x+1, 0, 2d'x-4\}$:

$$\begin{aligned} \min_{x,z} \quad & c'x + z \\ \text{s.t.} \quad & Ax \geq b \\ & z \geq -\text{d'}x + 1 \\ & z \geq 0 \\ & z \geq 2\text{d'}x - 4 \end{aligned} \tag{2}$$

Problem 3: Range of a matrix for non-negative inputs

Assume an element $y \in \text{ran}_+(A)$. This implies $y = Ax, x \in R^n, x \geq 0$, where A is an $m \times n$ matrix with $k \leq m$ linearly independent rows. Any x^* vector that contributes to $\text{ran}_+(A)$ is an element of the polyhedron $C = \{x \in R^n : y = Ax, x \geq 0\}$. Since $y = Ax^*$, this provides us with k linearly independent constraints in the rows of A. It follows that A also has k linearly independent columns that span R^k with indices $A_1, ..., A_k$. Since the matrix A spans R^k we know that any element $y \in \text{ran}_+(A) = \{Ax : x \in R^n, x \geq 0\}$ can be written as the linear combination $A_1x_1 + ... + A_kx_k$ where $x_i \geq 0$. This is the same as $\sum_{i=1}^{k} x_iA_i$ with $x_i \geq 0$. Since $k \leq m$ we have at most m of the coefficients x_i being nonzero and $\text{ran}_+(A) = \sum_{i=1}^{n} x_iA_i$ with $x_i \geq 0$ and $x_i = 0 \ \forall i \notin [k]$.

Problem 4: True or false?

(a) If $n = m + 1$, then P has at most two basic feasible solutions.
True
Since the polyhedron, P, lives in R^{n-m}, this polyhedron lives in the space $R^{(m+1-m)}$ or R^1. This means, at most, P can have two basic feasible solution since P itself is a line.

(b) The set of all optimal solutions is bounded.
False
In the trivial formulation of *min* $c'x$ s.t. $x \geq 0$ where $c = 0$, any non-negative value of x is an optimal solution, making the set of optimal solutions unbounded.

(c) At every optimal solution, no more than m variables can be positive.
False
Using the same objective function as part b, any number of variables can be positive since none have nonzero coefficients in the objective function.

(d) If there is more than one optimal solution, then there are uncountable many optimal solution.
True
Since the standard form set is convex, we know a solution lies at an extreme point/vertex. If there is more than one optimal solution, then we know they are both at extreme points. Since they are contained in a convex set, we know that we can draw a line between the two points, by the definition of convexity. This implies the line drawn between the two is orthogonal to the objective function, and any of the infinitely many combinations of points on that line is therefore optimal.

(e) If there are several optimal solutions, then there exist at least two basic feasible solutions that are optimal.
False
This follows partially from part b. Consider a problem with only one BFS, yet an unbounded feasible space and uncountable many optimal solution: $min\ x$ s.t. $x \geq 0$. In a two dimensional case, the origin is the only BFS that is optimum, yet [0,∞) for either x would be the solution (depending on which one you intend on minimizing).

(f) Consider the problem of minimizing $\max\{c'x, d'x\}$ over the set P. If this problem has an optimal solution, it must have an optimal solution which is an extreme point of P.
False
Although this statement is true for a linear programming problem, it is not true in general. Consider the example of minimizing $\max\{x_1 - 1, -x_1 + 1\}$ with constraints $x_1 > 0$, $x_2 > 0$, $x_1 + x_2 \leq 2$, $x_1 - x_2 \geq 2$. The objective function becomes $|x_1 - 1|$ subject to the constraints defined above. The feasible region and objective function is show in Figure 5:

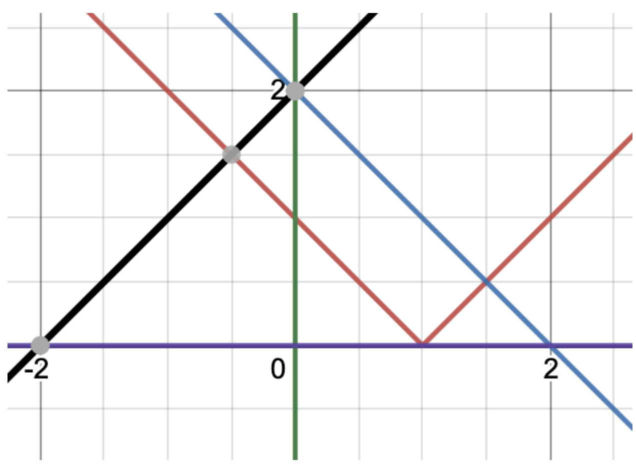


Figure 5: Part (f)

In this example the minimum feasible solution is $(x_1, x_2) = (1, 0)$, which is not an extreme point since it is not at a vertex of the polyhedron.

Problem 5: Find the unknown parameters

(a)
$(\alpha, \beta, \gamma, \delta, \eta) = (1, 0, 0, 0, -3)$
To make this solution optimal with many solutions we need a zero reduced cost for a nonbasic variable, $\delta = 0$, $\beta = 0$ so that it will not change the optimal cost, $\gamma = 0$ so that the reduced cost will remain at 0, $\alpha > 0$ so that u_1 is not all negative causing an unbounded solution, and $\eta < 0$ to force the pivot on third row.
(b)
$(\alpha, \beta, \gamma, \delta, \eta) = (-1, 3, -2, -4, 1)$
p. 100
We need a nonbasic variable with a negative reduced cost, hence $\delta < 0$. We also need that pivot column to have every entry be negative, $\alpha, \gamma < 0$. Lastly, we needed a positive β to ensure the solution is basic.
(c)
$(\alpha, \beta, \gamma, \delta, \eta) = (1, 1, 1, 1, 1)$
The only needed feasibility condition is $\beta > 0$. Since a reduced cost is already negative, δ is free to be any variable and the solution will still not be optimal. We also have to ensure that no columns are linearly dependent.

Problem 6: Simple simplex.

(a)

$$\begin{aligned} \min \quad & -2x_1 - x_2 \\ \text{s.t.} \quad & x_1 - x_2 \leq 2 \\ & x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{3}$$

Reformulated into standard form as:

$$\begin{aligned} \min \quad & -2x_1 - x_2 \\ \text{s.t.} \quad & x_1 - x_2 + s_1 \qquad = 2 \\ & x_1 + x_2 \qquad + s_2 = 6 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned} \tag{4}$$

At a basic feasible solution $(x_1, x_2) = (0, 0)$, it follows that $(s_1, s_2) = (2, 6)$ to satisfy the constraints. Therefore, s_1 & s_2 are basic variables while x_1 & x_2 are nonbasic. This forms the following basis for the initial basic feasible solution:
$A_B = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$
(b)
Using the initial BFS from part a, we get the resulting initial tableau:

		x_1	x_2	s_1	s_2
	0	-2	-1	0	0
s_1	2	1^*	-1	1	0
s_2	6	1	1	0	1

Pivoting in order for x_1 to enter the basis we obtain:

		x_1	x_2	s_1	s_2
	4	0	-3	2	1
x_1	2	1	-1	1	0
s_2	4	0	2^*	-1	1

Pivoting in order for x_2 to enter the basis we obtain:

		x_1	x_2	s_1	s_2
	10	0	0	.5	1.5
x_1	4	1	0	.5	.5
x_2	2	0	1	$-.5$	.5

The final basic feasible solution is $x_1 = 4$ and $x_2 = 2$ with an objective cost of -10.

In [1]:

```
using JuMP, Clp, LinearAlgebra, Plots
```

In [2]:

```
## Constants
T = 100;
delta = .001;
f = 1000;
d = 50;
c = 1;

## Model
rocket_model = Model(Clp.Optimizer)

## Variables
@variables(
    rocket_model,
    begin
        x[1:T]
        v[1:T]
        a[1:T]
        a_hat[1:T]
        z
    end
)

## Objective
@objective(rocket_model,Min,z);

## Constraints
@constraints(
    rocket_model,
    begin
        [t = 1:T-1], v[t+1] == v[t] + a[t]
        [t = 1:T-1], x[t+1] == x[t] + v[t]
        x[1] == 0
        v[1] == 0
        x[T] == d
        v[T] == 0
        [t = 1:T], a[t] <= a_hat[t]
        [t = 1:T], -a[t] <= a_hat[t]
        sum(c*a_hat[t] for t in 1:T) <= f
        [t = 1:T-1], a[t+1]-a[t] <= delta
        [t = 1:T-1], a[t]-a[t+1] <= delta
        [t = 1:T], a[t] <= z
        [t = 1:T], -a[t] <= z
    end
)
```

```
## Optimize
optimize!(rocket_model)
```

```
Coin0506I Presolve 643 (-158) rows, 243 (-158) columns and 1577 (-317) elements
Clp0006I 0  Obj 0 Primal inf 1229.1373 (7) Dual inf 0.0082850669 (1) w.o. free dual inf (0)
Clp0006I 68  Obj 1.7221316e-16 Primal inf 3.8630718 (17)
Clp0006I 130  Obj 2.5920038e-16 Primal inf 47.298084 (10)
Clp0006I 202  Obj 0.044947993 Primal inf 2.3096975 (69)
Clp0006I 258  Obj 0.41221429 Primal inf 3.5135943 (19)
Clp0006I 264  Obj 3.115931 Primal inf 8.1475538 (22)
Clp0006I 310  Obj 44.056 Primal inf 135.3529 (3)
Clp0006I 310  Obj 44.056 Primal inf 135.3529 (3)
Clp0006I 384  Obj -4.7249994e+12 Primal inf 6.6448068e+14 (93) Dual inf 5.3766826e-13 (15)
Clp0006I 384  Obj 8.5464688e+13 Primal inf 5.9032471e+14 (93) Dual inf 2.9042729e+15 (58) w.o. free dual inf (30)
Clp0006I 417  Obj 0 Primal inf 8.7165031 (138) Dual inf 1.1913143e+16 (90)
Clp0006I 486  Obj 0.032907302 Primal inf 0.29593831 (22) Dual inf 2.2396909e+15 (75)
Clp0006I 562  Obj 0.029404301 Dual inf 14.16514 (51)
Clp0006I 611  Obj 0.023533578 Dual inf 0.29510251 (7)
Clp0006I 636  Obj 0.02181992
Clp0000I Optimal - objective value 0.02181992
Coin0511I After Postsolve, objective 0.02181992, infeasibilities - dual 0 (0), primal 0 (0)
Clp0032I Optimal objective 0.02181991952 - 636 iterations time 0.072, Presolve 0.01
```

In [3]:

```
## Optimal objective value
p_star = objective_value(rocket_model)
```

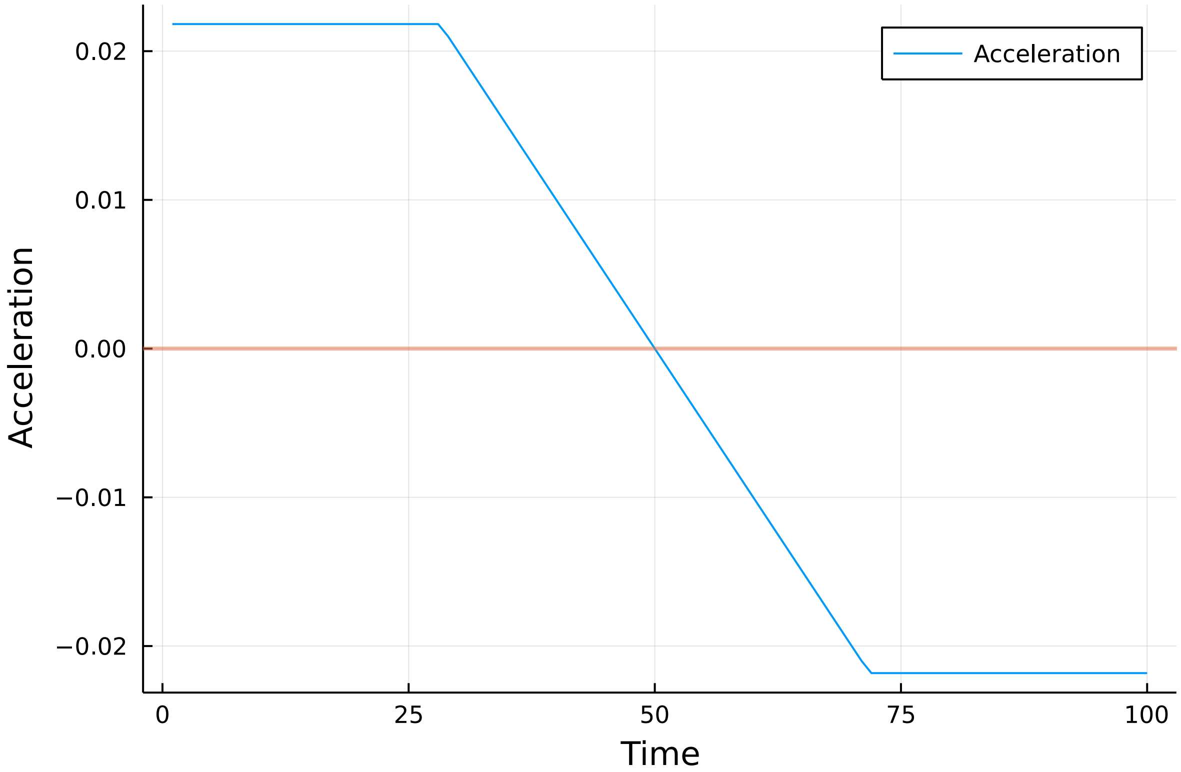
Out[3]:

```
0.021819919516018697
```

In [4]:

```
plot([1:T],value.(a),label = "Acceleration",xlabel = "Time",ylabel = "Acceleration")
hline!([0],width = 2,alpha = 0.5,label = "")
```

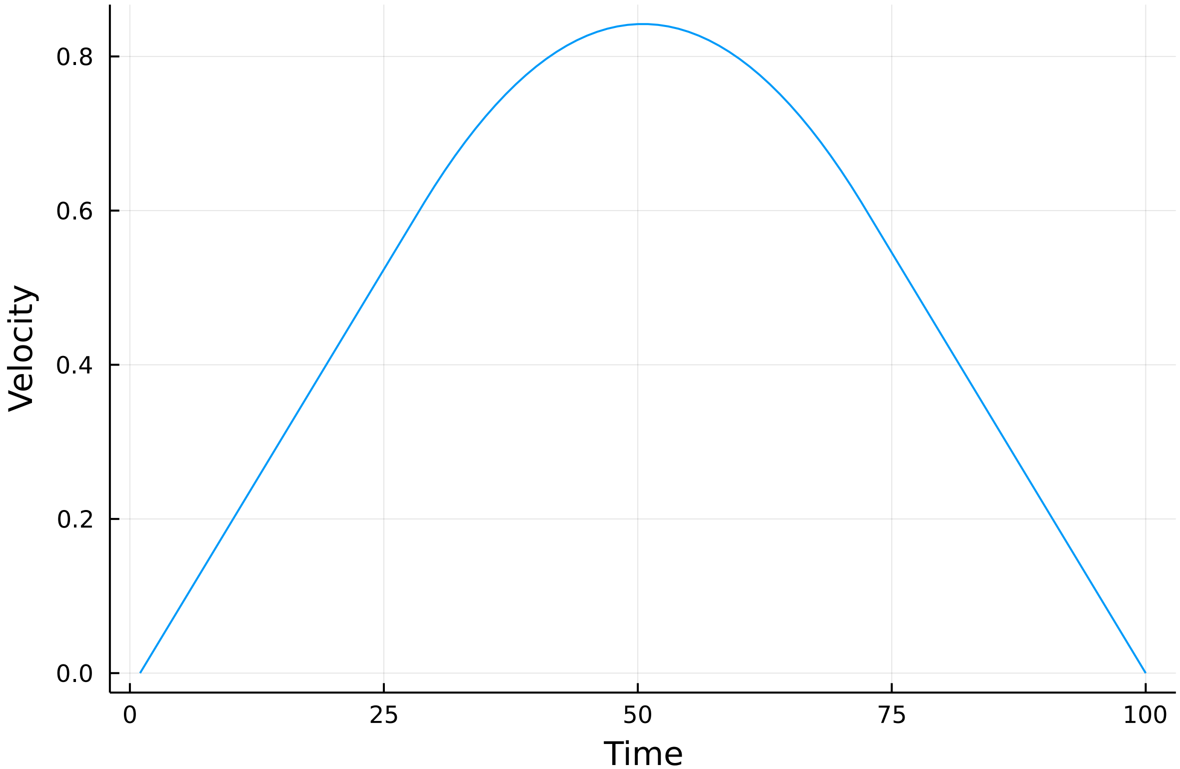
Out[4]:



In [5]:

```
plot([1:T],value.(v),label = "",xlabel = "Time",ylabel = "Velocity")
```

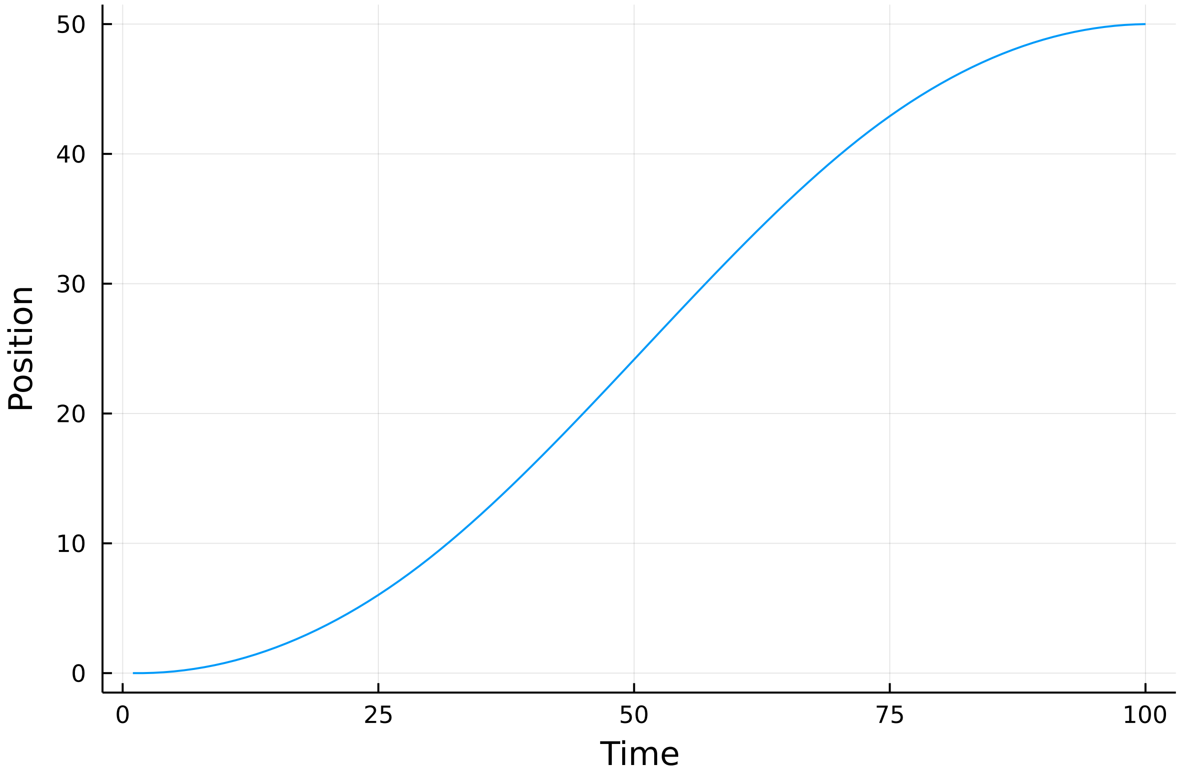
Out[5]:



In [6]:

```
plot([1:T],value.(x),label = "",xlabel = "Time",ylabel = "Position")
```

Out[6]:



Position
Time
0
10
20
30
40
50
0
25
50
75
100