Exercise 7.9 Consider the uncapacitated network flow problem shown in Figure 7.37. The label next to each arc is its cost.

(a) What is the matrix **A** corresponding to this problem?

(b) Solve the problem using the network simplex algorithm. Start with the tree indicated by the dashed arcs in the figure.

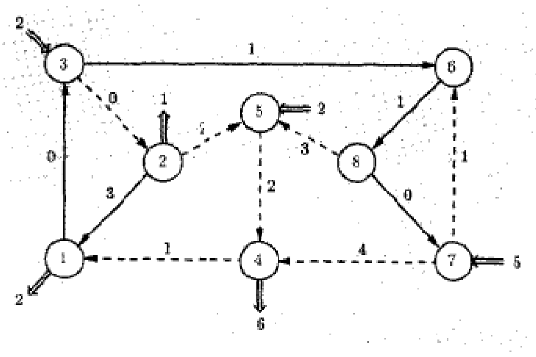


Figure 7.37: The network flow problem in Exercise 7.9.

$\mathcal{N} = \{1, \dots, 8\} \Rightarrow |\mathcal{N}| = n = 8.$

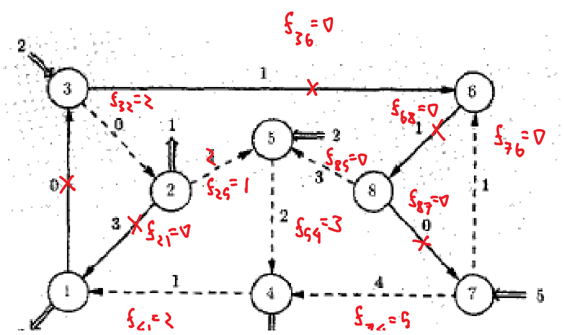
$\mathcal{A} = \{ (2,1), (2,5), (3,2), (4,1), (1,3), (3,6), (7,4),$

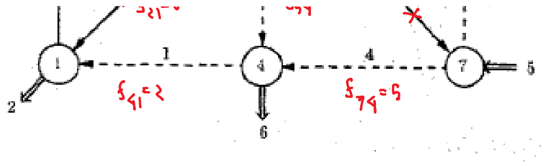
$(8,7), (6,7), (6,8), (8,5), (5,4) \} \Rightarrow |\mathcal{A}| = m = 12$

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \quad \text{node} / \text{arc} \\
A = & \begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0
\end{bmatrix}
\begin{array}{c} (2,1) \\ (2,5) \\ (3,2) \\ (4,1) \\ (1,3) \\ (3,6) \\ (7,4) \\ (8,7) \\ (6,7) \\ (6,8) \\ (8,5) \\ (5,4) \end{array}
\end{array}
$$

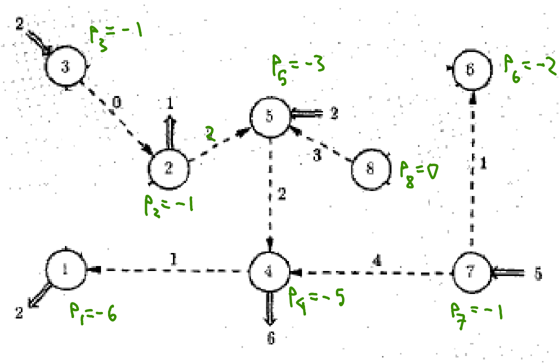
(b) Network simplex algorithm:

Step 1: Find the arc flows

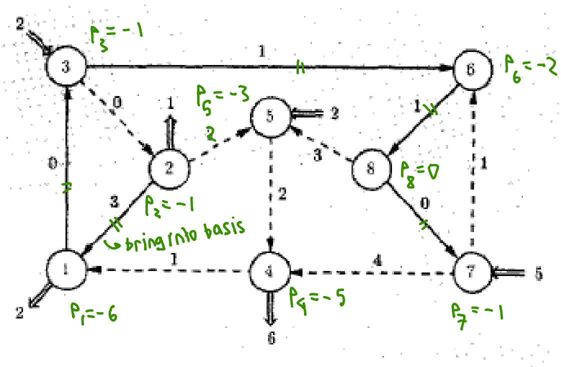




Step 2: Finding the dual variable (reduced cost)



Step 3: Compute reduced cost.



reduced cost:

All arcs in the tree has reduced cost = 0

For the nonbasic arcs (nontree)

$\bar{c}_{36} = c_{36} - (p_3 - p_4) = 1 - (-1+2) = 1 - 1 = 0$

$\bar{c}_{68} = c_{68} - (p_6 - p_8) = 1 - (-2-0) = 3$

$\bar{c}_{87} = c_{87} - (p_8 - p_7) = 0 - (0+1) = 1$

$\bar{c}_{21} = c_{21} - (p_2 - p_1) = 3 - (-1+6) = 3 - 5 = -2 < 0$ ⇒ bring 21 into basis

$\bar{c}_{13} = c_{13} - (p_1 - p_3) = 0 - (1+1) = 0$

Step 4: (2,1) brought into basis;

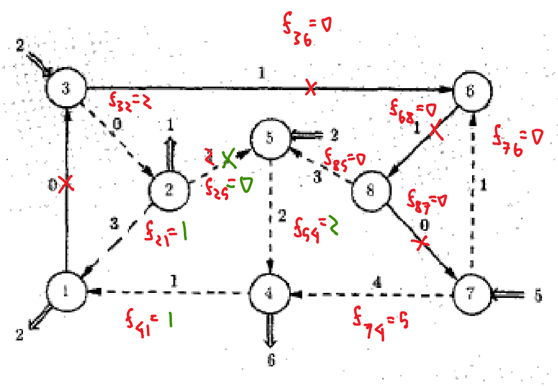
Formed unique cycle: $(2,1), \underbrace{(4,1),(5,4),(2,5)}$

$= B$ (opposite to (2,1))

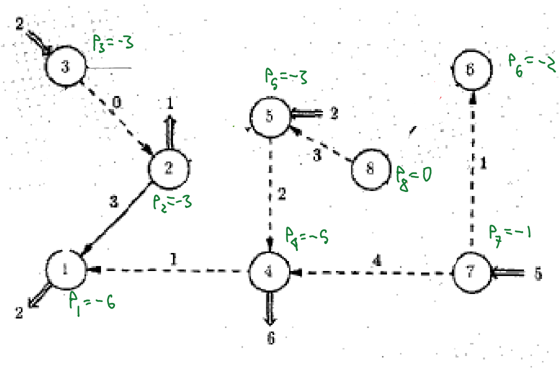
$\theta^* = \min_{(k,l)\in B} f_{kl} = \min\{1,3,2\} = 1$

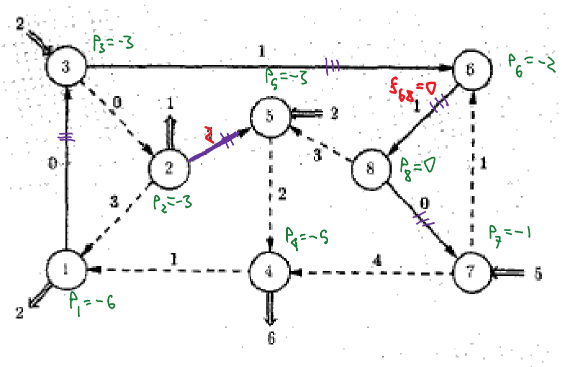
Push 1 unit of flow around the cycle and

arc (2,5) leaves the tree.



Computing dual variable:





reduced cost .

All arcs in the tree has reduced cost=0

for the nonbasic (nontree) solution:

$\bar{c}_{36} = c_{36} - (p_3 - p_6) = 1 - (-3+2) = 1 - (-1) = 2$

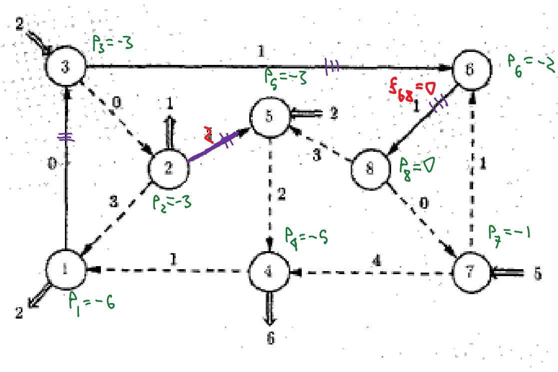
$\bar{c}_{68} = c_{68} - (p_6 - p_8) = 1 - (-2+0) = 3$

$\bar{c}_{87} = c_{87} - (p_8 - p_7) = 0 - (0+1) = -1 < 0$

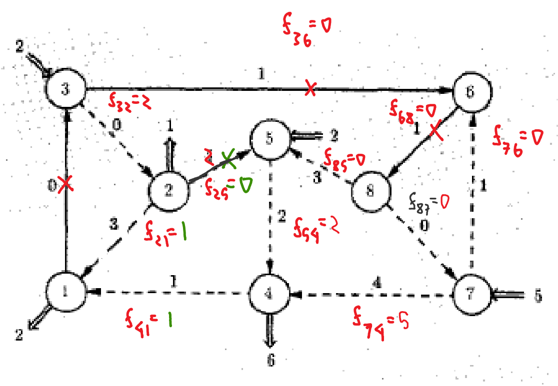
$\bar{c}_{13} = c_{13} - (p_1 - p_3) = 0 - (-6+3) = 3$

$\bar{c}_{25} = c_{25} - (p_2 - p_5) = 2 - (-3+3) = 0$

(8,7) enters the tree.

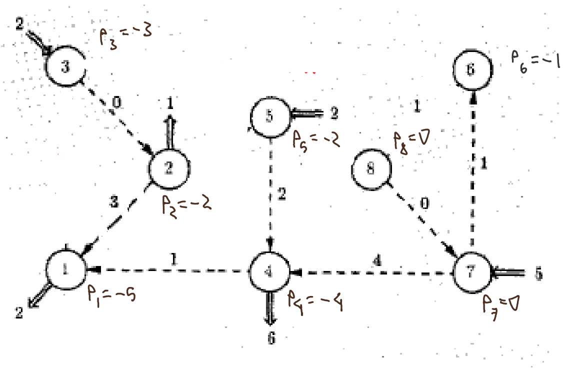


$\theta^* = \min_{(k,l)\in B} f_{kl} = 0$ units of flow pushed around the cycle.

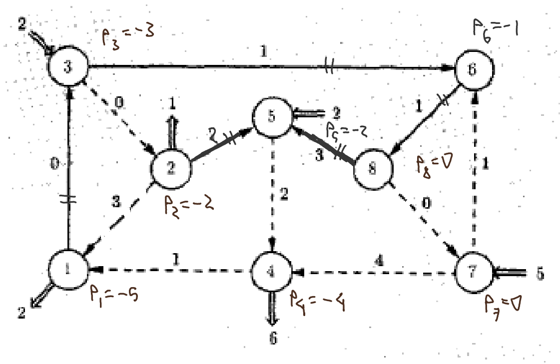


this is a degenerate solution, we make (8,5) leave the tree,

Dual variable.



Reduced cost:



reduced cost:

All arcs in the tree has reduced cost = 0

For the nonbasic arcs (nontree)

$\bar{c}_{36} = c_{36} - (p_3 - p_6) = 1 - (-3+1) = 1 - (-2) = 3$

$\bar{c}_{68} = c_{68} - (p_6 - p_8) = 1 - (-1 - 0) = 2$

$\bar{c}_{85} = c_{85} - (p_8 - p_5) = 3 - (0 + 2) = 1$

$\bar{c}_{25} = c_{25} - (p_2 - p_5) = 2 - (-2+2) = 2$

$\bar{c}_{13} = c_{13} - (p_1 - p_3) = 0 - (-5+3) = -(-2) = 2.$

reduced cost ≥ 0

optimal solution reached.