6.215/6.255J/15.093J/IDS.200J: Optimization Methods

Problem Set 5

Due: December 2, 2021 1:00 PM

Problem 1: (10 points) Consider the quadratic optimization problem:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T A x + b^T x + c,$$

where A is a positive definite matrix.

(Hint: what are the gradient and minimum of this function?)

(a) From an arbitrary starting position x_0, how many steps will Newton's method take to converge. Justify your answer with a formal proof.

(b) Under what conditions on x_0 will gradient descent (steepest descent) converge to the optimal solution x^* in one step? Justify your answer with a formal proof. (Hint: You might find it helpful to use eigenvalues and eigenvectors.)

Problem 2: (10 points) Classify the following statements as true or false. All answers must be well-justified, either through a short explanation, or a counterexample. If you think a question is ambiguous or not clear, please explain your assumptions in detail.

(a) For a nonlinear optimization problem, if Newtons method converges, then it converges to a local minimum.

(b) The sequence $x_{k+1} = x_k - (\nabla^2 f(x_k))^{-1} \nabla f(x_k)$ generated by Newtons method, when applied to the function $f(x) = x^4$, converges quadratically to zero. ~~method, i.e.,~~ $\cancel{f(x_{k+1}) \le f(x_k)}$~~.~~

Problem 3: (10 points) Consider a set of n points $\{(x_1, y_1), ..., (x_n, y_n)\}$ in the plane. We want to find a point (x, y) such that the sum of squares of Euclidean distances from this point to all the other n points is minimized.

(a) Give an nonlinear optimization formulation of this problem.

(b) Is the objective function differentiable? Is this a convex optimization problem?

(c) Write the corresponding optimality conditions.

(d) Give a closed-form expression for (x, y).

Problem 4: (10 points)

(a) Consider the optimization problem

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & x_1^{-1}x_2^2x_3^3 \\ \text{s.t.} \quad & x_1^{11}x_2^{-12}x_3^{13} \le 14 \\ & x_1^{15}x_2^{16}x_3^{-17} \le 18 \\ & x_1, x_2, x_3 \ge 1 \end{aligned} \tag{1}$$

Show that this is not a convex optimization problem as written (without any re-formulations).

(b) Explain how to use linear programming to compute both the optimal value of (1) and an optimal solution. (Hint: change variables via $x_i = e^{z_i}$, and use properties of log functions.)

(c) Reformulate the non-convex optimization problem

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & x_1^{-1}x_2^2x_3^3 + 5x_1^4x_2^5x_3^{-6} \\ \text{s.t.} \quad & x_1^{11}x_2^{-12}x_3^{13} \le 14 \\ & x_1^{15}x_2^{16}x_3^{-17} + 7x_1^{18}x_2^{-19}x_3^{20} \le 21 \\ & x_1, x_2, x_3 \ge 1 \end{aligned} \tag{2}$$

as a convex optimization problem. Clearly explain why your reformulation works. (Hint: you may use without proof that the function $y \mapsto \log(\sum_{i=1}^k e^{y_i})$ on $\mathbb{R}^k$ is convex.)