6.215/6.255J/15.093J/IDS.200J Optimization Methods

Lecture 12: Data Uncertainty in Linear Optimization

October 19, 2021

Today's Lecture

Outline

- dealing with data uncertainty; motivation
- stochastic approach
- robust approach

Linear optimization under data uncertainty

Motivation (example from Ben-Tal 2009)

- Consider PILOT4, a linear optimization problem from the NETLIB benchmark library (http://www.netlib.org/lp/data/) with $n = 1000$ variables and $m = 410$ constraints
- One of the constraints (#372) from PILOT4:

$$
\begin{aligned}
\boldsymbol{a}_{372}^T \boldsymbol{x} = & -15.79081 x_{826} - 8.598819 x_{827} - 1.88789 x_{828} - 1.362417 x_{829} - 1.526049 x_{830} \\
& - 0.031883 x_{849} - 28.725555 x_{850} - 10.792065 x_{851} - 0.19004 x_{852} - 2.757176 x_{853} \\
& - 12.290832 x_{854} + 717.562256 x_{855} - 0.057865 x_{856} - 3.785417 x_{857} - 78.30661 x_{858} \\
& - 122.163055 x_{859} - 6.46609 x_{860} - 0.48371 x_{861} - 0.615264 x_{862} - 1.353783 x_{863} \\
& - 84.644257 x_{864} - 122.459045 x_{865} - 43.15593 x_{866} - 1.712592 x_{870} - 0.401597 x_{871} \\
& + 1 \cdot x_{880} - 0.946049 x_{898} - 0.946049 x_{916} \geq b_{372} = 23.387405
\end{aligned}
$$

Linear optimization under data uncertainty

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\end{aligned}$$

- Many numbers here such as -8.598819 are potentially inaccurate
- Numbers such as 1 are probably exact

Linear optimization under data uncertainty

Motivation (example from Ben-Tal 2009)

- The related *nonzero* optimal variables reported by the solver CPLEX are:

$$
\begin{array}{llll}
x^*_{826} & = 255.6112787181108, & x^*_{827} & = 6240.488912232100, \\
x^*_{828} & = 3624.613324098961, & x^*_{829} & = 18.20205065283259, \\
x^*_{849} & = 174397.0389573037, & x^*_{870} & = 14250.00176680900, \\
x^*_{871} & = 25910.00731692178, & x^*_{880} & = 104958.3199274139.
\end{array}
$$

- Within machine precision x^* makes the previous constraints active (an equality).
- Suppose the coefficients in $\boldsymbol{a}_{372}$ are 0.1% accurate approximations of their true values. Will the above solution still be feasible?

Linear optimization under data uncertainty

Motivation (example from Ben-Tal 2009)

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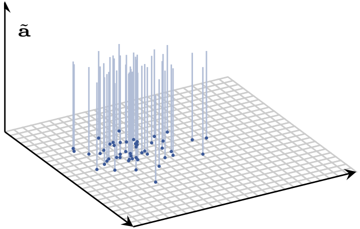
- Within machine precision x^* makes the previous constraints active (an equality).
- Suppose the coefficients in $\boldsymbol{a}_{372}$ are 0.1% accurate approximations of their true values. Will the above solution still be feasible?
- Worst-case infeasibility : $\tilde{\boldsymbol{a}}^T_{372}\boldsymbol{x}^* - b_{372} = -104.9 \ll 0$
  A violation of 448.5% of the right hand side!
- Small perturbations make our "optimal" solution $\boldsymbol{x}^*$ practically meaningless.

Linear optimization under data uncertainty

Philosophical considerations

- probability distributions are used to model uncertainty but are rarely observed in practice
- data is however observed in practice



- can this provide alternate model of uncertainty with tractable formulations?

Linear optimization under data uncertainty

Describing uncertainty

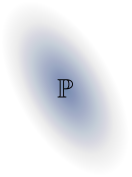
Two fundamentally different approaches to describe an **uncertain quantity** $\tilde{\boldsymbol{a}}$.

(1) **probabilistic description**

- a distribution $\mathbb{P}$ describes all possible outcomes

$$\tilde{\boldsymbol{a}} \sim \mathbb{P}$$

- directly related to *stochastic optimization*

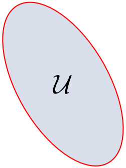


(2) **set description**

- a set $\mathcal{U}$ describes all possible outcomes

$$\tilde{\boldsymbol{a}} \in \mathcal{U}$$

- directly related to *robust optimization*



Linear optimization under data uncertainty

A (very short) overview!

- deterministic linear optimization:
  $$\min_{\boldsymbol{x}\in\mathcal{X}} \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} \;\; \text{s.t. } \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$
- linear optimization under uncertainty $\tilde{\boldsymbol{z}} \in \mathcal{W}$:
  $$\min_{\boldsymbol{x}\in\mathcal{X}} \boldsymbol{c}(\tilde{\boldsymbol{z}})^{\boldsymbol{T}}\boldsymbol{x} \;\; \text{s.t. } \boldsymbol{A}(\tilde{\boldsymbol{z}})\boldsymbol{x} \leq \boldsymbol{b}(\tilde{\boldsymbol{z}})$$
  - without loss of generality, we can always assume that data uncertainty affects only $\boldsymbol{A}$ and $\boldsymbol{c}$, but not the vector $\boldsymbol{b}$.
  - probability distributions available: stochastic optimization (Prékopa 1995; Birge and Louveaux 1997; etc. ...)
  - distributions unavailable:
    - robust optimization (Ben-Tal and Nemirovski 1999; Bertsimas and Sim 2004; Ben-Tal 2009; etc. ...)
    - distributionally robust optimization (Delage and Ye 2010; Wiesemann et al. 2014; etc. ...)

Linear optimization under data uncertainty

The "easy case", uncertainty in $\boldsymbol{c}$ only

- linear optimization under uncertainty $\tilde{\boldsymbol{z}} \in \mathcal{W}$: $\min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{c}(\tilde{\boldsymbol{z}})^{\boldsymbol{T}} \boldsymbol{x}$ s.t. $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$
  - probability distributions available:
    replace $\boldsymbol{c}(\tilde{\boldsymbol{z}})$ by its expected value, or a linear combination of its expected value and standard deviation, ...
  - distributions unavailable:
    - use lower or upper bounds of the ranges taken by $\boldsymbol{c}(\tilde{\boldsymbol{z}})$, ...
- in general, relatively well understood, close to sensitivity analysis, feasibility of a solution is maintained ...

- linear optimization under uncertainty $\tilde{\boldsymbol{z}} \in \mathcal{W}$: $\min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{c}^T \boldsymbol{x}$ s.t. $\boldsymbol{A}(\tilde{\boldsymbol{z}})\boldsymbol{x} \leq \boldsymbol{b}$
- probability distributions available:
  probabilistic feasibility $\Rightarrow$ chance-constrained optimization

$$\begin{array}{ll} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & \mathbb{P}\left(\boldsymbol{A}(\tilde{\boldsymbol{z}})\boldsymbol{x} \leq \boldsymbol{b}\right) \geq \delta \\ & \boldsymbol{x} \in \mathcal{X} \end{array}$$

- comments:
  - need to specify a given $\mathbb{P}$
  - need to specify a confidence level $\delta \in [0, 1]$
  - in most cases, nonlinear constraints, hard to solve exactly

Linear optimization under data uncertainty

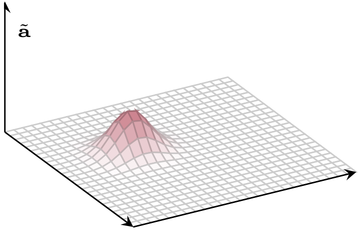
Uncertainty in $\boldsymbol{A}$ only

- linear optimization under uncertainty $\tilde{\boldsymbol{z}} \in \mathcal{W}$: $\min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{c}^T \boldsymbol{x} \;\; \text{s.t. } \boldsymbol{A}(\tilde{\boldsymbol{z}})\boldsymbol{x} \leq \boldsymbol{b}$
- probability distributions unavailable:
  worst-case feasibility $\Rightarrow$ robust optimization

$$\begin{array}{ll} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}(\boldsymbol{z})\boldsymbol{x} \leq \boldsymbol{b} \;\; \forall \boldsymbol{z} \in \mathcal{U}(\Gamma) \\ & \boldsymbol{x} \in \mathcal{X} \end{array}$$

- comments:
  - need to specify a family of *uncertainty sets*, $\mathcal{U}(\Gamma)$
  - need to specify a "budget of uncertainty" $\Gamma \in \mathbb{R}_+$, corresponding to the level of uncertainty that must be tolerated.
  - in some important cases, lead to tractable formulations

- how feasible is $\boldsymbol{x}$ in $\tilde{\boldsymbol{a}}^T\boldsymbol{x} \leq b$ if the vector $\tilde{\boldsymbol{a}}$ is uncertain?
- probabilistic feasibility $\Rightarrow \mathbb{P}\left(\tilde{\boldsymbol{a}}^T\boldsymbol{x} \leq b\right) \geq \delta$ with $\delta \in [0,1]$.
- assume that $\tilde{\boldsymbol{a}} \sim \boldsymbol{N}(\boldsymbol{a}, \boldsymbol{\Sigma})$ is a normal random vector.



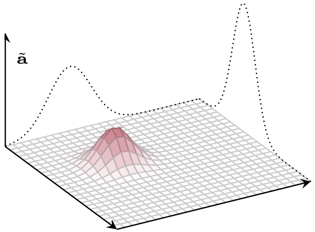
  - mean vector: $\boldsymbol{a} \in \Re^n$
  - variance-covariance matrix: $\boldsymbol{\Sigma} \in \Re^{n\times n}$.

Linear optimization under data uncertainty

Uncertainty in $\boldsymbol{A}$ - probabilistic feasibility example

- normal distributions satisfy a projection property:

$$\tilde{\boldsymbol{a}} \sim \boldsymbol{N}(\boldsymbol{a}, \boldsymbol{\Sigma}) \implies \tilde{\boldsymbol{a}}^T \boldsymbol{x} \sim N(\boldsymbol{a}^T \boldsymbol{x}, \boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x})$$

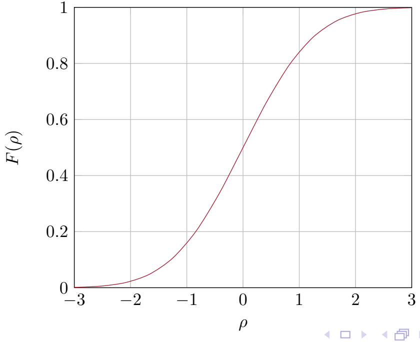


- a specific property of the multivariate normal distribution

- computing the probability for a *single* constraint is then "easy":

$$\mathbb{P}\left(\tilde{\boldsymbol{a}}^T\boldsymbol{x} \le b\right) = \mathbb{P}\left(\boldsymbol{N} \le b\right) = F\left(\frac{b - \boldsymbol{a}^T\boldsymbol{x}}{\sqrt{\boldsymbol{x}^T\boldsymbol{\Sigma}\boldsymbol{x}}}\right)$$

where $\boldsymbol{N}$ is a normal random variable with mean $\boldsymbol{a}^T\boldsymbol{x}$ and variance $\boldsymbol{x}^T\boldsymbol{\Sigma}\boldsymbol{x}$, and where $F(\cdot)$ is its cumulative distribution function.



Linear optimization under data uncertainty

Uncertainty in $\boldsymbol{A}$ - probabilistic feasibility example

- stochastic optimization

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x} \in \mathcal{X} \\ & \mathbb{P}\left(\tilde{\boldsymbol{a}}^T \boldsymbol{x} \leq b\right) \geq \delta \end{aligned}$$

- equivalent deterministic formulation

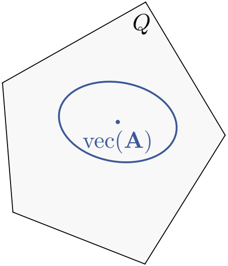
$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x} \in \mathcal{X} \\ & F\left(\frac{b - \boldsymbol{a}^T \boldsymbol{x}}{\sqrt{\boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x}}}\right) \geq \delta \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x} \in \mathcal{X} \\ & b - \boldsymbol{a}^T \boldsymbol{x} \geq F^{-1}(\delta) \cdot \sqrt{\boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x}} \end{aligned}$$

- can be solved using (slower but convex) optimization methods.

Linear optimization under data uncertainty

Uncertainty in $\boldsymbol{A}$ - probabilistic feasibility example

- multivariate extension would be $\mathbb{P}(\tilde{\boldsymbol{A}}\boldsymbol{x} \leq \boldsymbol{b}) \geq \delta$ with $\boldsymbol{vec}(\tilde{\boldsymbol{A}}) \in \Re^{m \cdot n}$ a normal random vector.
- verifying feasibility would require us to compute $\mathbb{P}(\boldsymbol{vec}(\tilde{\boldsymbol{A}}) \in Q$ where $Q := \{\boldsymbol{vec}(\boldsymbol{A}) \in \Re^{m \cdot n} : \ \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\})$



- integrating in high dimensions is very hard!

- even for simple distributions and small problems, calculating the probability of violation of multiple constraints is often difficult.
- simulation is only an approximation and may be very computationally expensive.
- problems get worse as number of variables and constraints grow.
- optimization in these environment is even harder.

- in practice *robustness* is often as important as *optimality* (sometimes more).
- *motivation 1:* create solutions that are immune to implementation errors and data uncertainty
- *motivation 2:* develop a theory of performance analysis and optimization under uncertainty via optimization that is tractable in high dimensions.
- a remark on *tractability:* we do not mean polynomial solvability; rather the ability to solve problems of the size and in times that are appropriate for the application.

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \tilde{\boldsymbol{a}}_1^T \boldsymbol{x} \leq b_1 \\ & \dots \\ & \tilde{\boldsymbol{a}}_m^T \boldsymbol{x} \leq b_m \end{aligned}$$

- uncertain vectors $(\tilde{\boldsymbol{a}}_1, \dots, \tilde{\boldsymbol{a}}_m) \in \mathcal{U}$.
- **robust problem formulation**

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \tilde{\boldsymbol{a}}_1^T \boldsymbol{x} \leq b_1 \qquad \forall (\tilde{\boldsymbol{a}}_1, \dots, \tilde{\boldsymbol{a}}_m) \in \mathcal{U} \\ & \dots \\ & \tilde{\boldsymbol{a}}_m^T \boldsymbol{x} \leq b_m \end{aligned}$$

problem:

- variable x_i : complete products $i = [1, 2, 3, 4]$ (e.g., chairs, desks, tables, beds).
- profit: $50x_1 + 40x_2 + 60x_3 + 30x_4$.
- time resource constraint:

$$\tilde{\boldsymbol{a}}^T \boldsymbol{x} \leq b \iff 120x_1 + 100x_2 + 180x_3 + 140x_4 \leq 5000$$

robust formulation:

$$\begin{aligned} \max \quad & 50x_1 + 40x_2 + 60x_3 + 30x_4 \\ \text{s.t.} \quad & \tilde{\boldsymbol{a}}^T \boldsymbol{x} \leq b \end{aligned}$$

problem:

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$$\tilde{\boldsymbol{a}}^T \boldsymbol{x} \le b \iff 120x_1 + 100x_2 + 180x_3 + 140x_4 \le 5000$$

the coefficient vector $\tilde{\boldsymbol{a}} \in \mathcal{U}$ is typically uncertain for new products!

robust formulation:

$$\begin{aligned} \max \quad & 50x_1 + 40x_2 + 60x_3 + 30x_4 \\ \text{s.t.} \quad & \tilde{\boldsymbol{a}}^T \boldsymbol{x} \le b \quad \forall \tilde{\boldsymbol{a}} \in \mathcal{U} \end{aligned}$$

uncertainty set:

- rule of thumb : we know coefficients up to Γ% inaccuracy
- uncertainty set

$$\mathcal{U} = \left\{ \tilde{\boldsymbol{a}} \in \Re^4 : \quad \begin{array}{c} (1-\Gamma)\cdot 120 \leq \tilde{a}_1 \leq (1+\Gamma)\cdot 120, \\ (1-\Gamma)\cdot 100 \leq \tilde{a}_2 \leq (1+\Gamma)\cdot 100, \\ (1-\Gamma)\cdot 180 \leq \tilde{a}_3 \leq (1+\Gamma)\cdot 180, \\ (1-\Gamma)\cdot 140 \leq \tilde{a}_4 \leq (1+\Gamma)\cdot 140. \end{array} \right\}$$

- notice that we insist the solution be feasible for any value of $\tilde{\boldsymbol{a}}$ in that range. (in particular, the **worst** values.)
- by varying the value of $\Gamma \geq 0$ we can control level of robustness.

- a fair criticism of the robust approach is that it is "unlikely" that all coefficients take their worst value at once.
- is there a more reasonable uncertainty set?
- suggestions?

Linear optimization under data uncertainty

Uncertainty in $\boldsymbol{A}$ - robust optimization - using probability !!

- if coefficients $\tilde{a}_i$ were random variables independent with mean μ and variance σ^2
- central limit theorem: for large n

$$\frac{\tilde{a}_1 + \cdots + \tilde{a}_n - n\mu}{\sigma \cdot \sqrt{n}} \to N(0,1).$$

- intuitively, this tells us that sums of **independent** random variables tend to be close to their mean.
- how might we use this to construct an uncertainty set?

$$\mathcal{U} = \left\{ \tilde{\mathrm{a}} : \; -\Gamma\sqrt{n} \leq \sum_{i=1}^{n} \frac{\tilde{a}_i - a_i}{\sigma_i} \leq \Gamma\sqrt{n} \right\}$$

- naive

$$\mathcal{U} = \left\{ \tilde{\boldsymbol{a}} : \ \left| \frac{\tilde{a}_i - a_i}{\sigma_i} \right| \leq \Gamma \quad \forall i \in 1, \ldots, n \right\}$$

- CLT

$$\mathcal{U} = \left\{ \tilde{\boldsymbol{a}} : \ -\Gamma\sqrt{n} \leq \sum_{i=1}^{n} \frac{\tilde{a}_i - a_i}{\sigma_i} \leq \Gamma\sqrt{n} \right\}$$

- bounded CLT

$$\mathcal{U} = \left\{ \tilde{\boldsymbol{a}} : \ \sum_{i=1}^{n} \left| \frac{\tilde{a}_i - a_i}{\sigma_i} \right| \leq \Gamma\sqrt{n} \right\}$$

- ellipsoidal sets

$$\mathcal{U} = \left\{ \tilde{\boldsymbol{a}} : \ \sum_{i=1}^{n} \frac{(\tilde{a}_i - a_i)^2}{\sigma_i^2} \leq \Gamma\sqrt{n} \right\}$$

- ...

Linear optimization under data uncertainty

Uncertainty in $\boldsymbol{A}$ - robust optimization - tractability

- consider a special case: robust optimization with row-wise uncertainty

$$
\begin{aligned}
\min \quad & \boldsymbol{c}^T \boldsymbol{x} \\
\text{s.t.} \quad & \tilde{\boldsymbol{a}}_i^T \boldsymbol{x} \leq \boldsymbol{b}_i \quad \forall \tilde{\boldsymbol{a}}_i \in U_i, \; i \in 1, \ldots, m.
\end{aligned}
$$

- primitives: uncertainty sets U_i, $i \in 1, \ldots, m$
- problem could be regarded as a linear optimization problem with (possibly infinitely) many constraints $\Rightarrow$ impractical.
- reformulation:

$$
\begin{aligned}
\min \quad & \boldsymbol{c}^T \boldsymbol{x} \\
\text{s.t.} \quad & \max_{\tilde{\boldsymbol{a}}_i \in U_i} \{\tilde{\boldsymbol{a}}_i^T \boldsymbol{x}\} \leq \boldsymbol{b}_i \quad \forall i \in 1, \ldots, m.
\end{aligned}
$$

- suppose that U_i, $i \in 1, \ldots, m$ are polyhedral sets.
- given a fixed $\boldsymbol{x}$, we can efficiently solve $\forall i \in 1, \ldots, m$ the *feasibility problems*:
  - verify whether
$$\max_{\tilde{\boldsymbol{a}}_i \in U_i} \{\tilde{\boldsymbol{a}}_i^T \boldsymbol{x}\} \leq b_i,$$
  - or find $\bar{\boldsymbol{a}}_i \in U_i$ such that
$$\bar{\boldsymbol{a}}_i^T \boldsymbol{x} > b_i.$$
- how should we then solve the robust optimization problem ?

Linear optimization under data uncertainty

Uncertainty in $\boldsymbol{A}$ - robust optimization - robust counterpart

- robust optimization with row-wise uncertainty

$$\begin{array}{ll} \min & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \max\limits_{\tilde{\boldsymbol{a}}_i \in U_i} \{\tilde{\boldsymbol{a}}_i^T \boldsymbol{x}\} \leq \boldsymbol{b}_i \quad \forall i \in 1, \ldots, m. \end{array}$$

- nonempty uncertainty sets $U_i = \{\tilde{\boldsymbol{a}}_i : \ \boldsymbol{D}_i \tilde{\boldsymbol{a}}_i \leq \boldsymbol{d}_i\}$ with $\boldsymbol{D}_i \in \Re^{k_i \times n}$.

- consider the dual of inner maximization problem

$$\begin{array}{lllll} \max & \tilde{\boldsymbol{a}}_i^T \boldsymbol{x} & = & \min & \boldsymbol{p}_i^T \boldsymbol{d}_i \\ \text{s.t.} & \boldsymbol{D}_i \tilde{\boldsymbol{a}}_i \leq \boldsymbol{d}_i, & & \text{s.t.} & \boldsymbol{p}_i \geq 0, \\ & \tilde{\boldsymbol{a}}_i \ \text{free} & & & \boldsymbol{p}_i^T \boldsymbol{D}_i = \boldsymbol{x}^T. \end{array}$$

- we have strong duality! (note: If primal is unbounded than dual is infeasible and we interpret $+\infty = +\infty$.)

so robust optimization is a linear optimization itself!

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{p}_i^T \boldsymbol{d}_i \leq \boldsymbol{b}_i \quad \forall i \in 1, \ldots, m \\ & \boldsymbol{p}_i^T \boldsymbol{D}_i = \boldsymbol{x}^T \quad \forall i \in 1, \ldots, m \\ & \boldsymbol{p}_i \geq 0 \quad \forall i \in 1, \ldots, m \end{aligned}$$

- nominal problem : n variables with m constraints.
- robust problem : $n + \sum_{i=1}^m k_i$ variables with $m + n \cdot m + \sum_{i=1}^m k_i$ constraints.