6.215/6.255J/15.093J/IDS.200J Optimization Methods

Lecture 6: Duality Theory II

September 28, 2021

Today's Lecture

Outline

- Recap on duality
- Geometry view of duality
- The dual simplex algorithm
- Duality and degeneracy
- Farkas lemma
- Duality revisited (bonus)

Recap ... duality

An idea from Lagrange

Consider the linear optimization problem, called the **primal** with optimal solution $\boldsymbol{x}^*$

$$\begin{array}{ll} \min & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{array}$$

Relax the equality constraint

$$\begin{array}{rl} g(\boldsymbol{p}) = \min & \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{p}^T(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) \\ \text{s.t.} & \boldsymbol{x} \geq 0 \end{array}$$

For any $\boldsymbol{p}$ we have

$$g(\boldsymbol{p}) \leq \boldsymbol{c}^T\boldsymbol{x}^* + \boldsymbol{p}^T(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}^*) = \boldsymbol{c}^T\boldsymbol{x}^*$$
$$\Rightarrow \max_{\boldsymbol{p}} g(\boldsymbol{p}) \leq \boldsymbol{c}^T\boldsymbol{x}^*$$

$$g(\boldsymbol{p}) = \min_{\boldsymbol{x}\geq 0}\left[\boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{p}^T(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})\right] = \boldsymbol{p}^T\boldsymbol{b} + \min_{\boldsymbol{x}\geq 0}(\boldsymbol{c}^T - \boldsymbol{p}^T\boldsymbol{A})\boldsymbol{x} = \begin{cases} \boldsymbol{p}^T\boldsymbol{b} & \text{if } \boldsymbol{c}^T - \boldsymbol{p}^T\boldsymbol{A} \geq 0, \\ -\infty & \text{otherwise.} \end{cases}$$

So $\max_{\boldsymbol{p}} g(\boldsymbol{p})$ is equivalent to the following problem, called the **dual**:

$$\begin{array}{ll} \max & \boldsymbol{p}^T\boldsymbol{b} \\ \text{s.t.} & \boldsymbol{p}^T\boldsymbol{A} \leq \boldsymbol{c}^T \end{array}$$

Recap ... general form of the dual

primal:

$$
\begin{array}{lll}
\min & \boldsymbol{c}^T \boldsymbol{x} & \\
\text{s.t.} & \boldsymbol{a}_i^T \boldsymbol{x} \geq b_i & i \in M_1 \\
 & \boldsymbol{a}_i^T \boldsymbol{x} \leq b_i & i \in M_2 \\
 & \boldsymbol{a}_i^T \boldsymbol{x} = b_i & i \in M_3 \\
 & x_j \geq 0 & j \in N_1 \\
 & x_j \leq 0 & j \in N_2 \\
 & x_j \text{ free} & j \in N_3
\end{array}
$$

dual:

$$
\begin{array}{lll}
\max & \boldsymbol{p}^T \boldsymbol{b} & \\
\text{s.t.} & p_i \geq 0 & i \in M_1 \\
 & p_i \leq 0 & i \in M_2 \\
 & p_i \text{ free} & i \in M_3 \\
 & \boldsymbol{p}^T \boldsymbol{A}_j \leq c_j & j \in N_1 \\
 & \boldsymbol{p}^T \boldsymbol{A}_j \geq c_j & j \in N_2 \\
 & \boldsymbol{p}^T \boldsymbol{A}_j = c_j & j \in N_3
\end{array}
$$

Note: The dual of the dual is the primal

Recap ... theorems

Theorem (weak duality)

If $\boldsymbol{x}$ is primal feasible and $\boldsymbol{p}$ is dual feasible then $\boldsymbol{p}^T\boldsymbol{b} \leq \boldsymbol{c}^T\boldsymbol{x}$

Theorem (strong duality)

If the primal has an optimal solution, then so does the dual, and the optimal costs are equal.

	Finite opt.	Unbounded	Infeasible
Finite opt.	*		
Unbounded			*
Infeasible		*	*

Recap ... complementary slackness

primal:

$$\begin{array}{lll} \min & \boldsymbol{c}^T\boldsymbol{x} & \\ \text{s.t.} & \boldsymbol{a}_i^T\boldsymbol{x} \geq b_i & i \in M_1 \\ & \boldsymbol{a}_i^T\boldsymbol{x} \leq b_i & i \in M_2 \\ & \boldsymbol{a}_i^T\boldsymbol{x} = b_i & i \in M_3 \\ & x_j \geq 0 & j \in N_1 \\ & x_j \leq 0 & j \in N_2 \\ & x_j \text{ free} & j \in N_3 \end{array}$$

dual:

$$\begin{array}{lll} \max & \boldsymbol{p}^T\boldsymbol{b} & \\ \text{s.t.} & p_i \geq 0 & i \in M_1 \\ & p_i \leq 0 & i \in M_2 \\ & p_i \text{ free} & i \in M_3 \\ & \boldsymbol{p}^T\boldsymbol{A}_j \leq c_j & j \in N_1 \\ & \boldsymbol{p}^T\boldsymbol{A}_j \geq c_j & j \in N_2 \\ & \boldsymbol{p}^T\boldsymbol{A}_j = c_j & j \in N_3 \end{array}$$

Theorem

Let $\boldsymbol{x}$ primal feasible and $\boldsymbol{p}$ dual feasible. Then $\boldsymbol{x}, \boldsymbol{p}$ optimal **if and only if**

$$p_i(\boldsymbol{a}_i^T\boldsymbol{x} - b_i) = 0, \quad \forall i$$
$$(c_j - \boldsymbol{p}^T\boldsymbol{A}_j)x_j = 0, \quad \forall j$$

Quick check in ...

Q: What is the dual? primal/dual optimal solutions, and optimal value?

$$
\begin{array}{lrl}
\max & 6x_1 + 2x_2 & \\
\text{s.t.} & 3x_1 + x_2 & \geq 3 \\
& x_1 + 7x_2 & \leq 8 \\
& x_1 - x_2 & = 0 \\
& x_1 & \geq 0 \\
& x_2 & \text{free}
\end{array}
$$

The geometry of duality

Geometry using complementary slackness

$$\begin{array}{ll} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{a}_i^T \boldsymbol{x} \geq b_i, \;\; i = 1, \ldots, m \end{array} \qquad\qquad \begin{array}{ll} \max & \boldsymbol{p}^T \boldsymbol{b} \\ \text{s.t.} & \sum_{i=1}^m p_i \boldsymbol{a}_i = \boldsymbol{c} \\ & \boldsymbol{p} \geq 0 \end{array}$$

Equivalent to solving:

- Primal feasibility: $\boldsymbol{a}_i^T \boldsymbol{x}^\star \geq b_i$ for all $i = 1, \ldots, m$
- Dual feasibility: $\boldsymbol{p}^\star \geq 0$ and $\sum_{i=1}^m p_i^\star \boldsymbol{a}_i = \boldsymbol{c}$
- Complementary slackness : $p_i^\star(\boldsymbol{a}_i^T \boldsymbol{x}^\star - b_i) = 0$ for all $i = 1, \ldots, m$

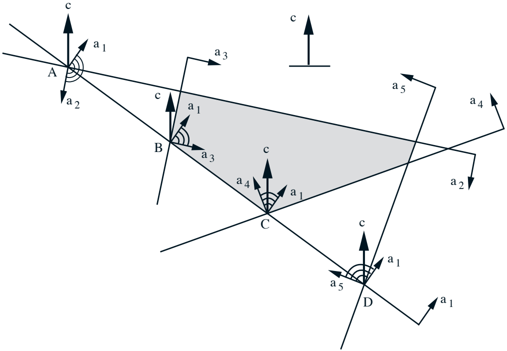
Definitions

- Primal point $\boldsymbol{x}$ is primal feasible if $\boldsymbol{a}_i^T \boldsymbol{x} \geq b_i$ for all $i = 1, \ldots, m$
- Primal point $\boldsymbol{x}$ is dual feasible if there exists $\boldsymbol{p} \geq 0$ such that

$$p_i(\boldsymbol{a}_i^T \boldsymbol{x} - b_i) = 0 \quad \forall i = 1, \ldots, m \qquad \text{and} \qquad \sum_{i=1}^m p_i \boldsymbol{a}_i = \boldsymbol{c}$$

The geometry of duality

Visualization without drawing dual feasible set



Point x	Primal Feasible?	Dual Feasible?
A	no	no
B	yes	no
C	yes	yes
D	no	yes

Dual simplex algorithm

Motivation

$$\begin{array}{llrcl} \min & \boldsymbol{c}^T\boldsymbol{x} & & & \\ \text{s.t.} & & \boldsymbol{A}\boldsymbol{x} & = & \boldsymbol{b} \\ & & \boldsymbol{x} & \geq & 0 \end{array} \qquad\qquad \begin{array}{ll} \max & \boldsymbol{p}^T\boldsymbol{b} \\ \text{s.t.} & \boldsymbol{p}^T\boldsymbol{A} \leq \boldsymbol{c}^T \end{array}$$

- Primal problem in standard form.
- Simplex method feasibility $\boldsymbol{B}^{-1}\boldsymbol{b} \geq 0$
- **Primal optimality condition**

$$\boldsymbol{c}^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A} \geq 0$$

  which is the same as **dual feasibility** for $\boldsymbol{p}^T = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}$ in dual problem
- Simplex is a **primal algorithm**: maintains **primal feasibility** and works towards **dual feasibility**
- **Dual algorithm**: maintains **dual feasibility** and works towards **primal feasibility**

Dual simplex algorithm

Mechanics

- Tableau as before

$-\boldsymbol{c}_B^T \boldsymbol{x}_B$	$\bar{c}_1$	$\dots$	$\bar{c}_n$
$x_{B(1)}$	$\mid$		$\mid$
$\vdots$	$\boldsymbol{B}^{-1}\boldsymbol{A}_1$	$\dots$	$\boldsymbol{B}^{-1}\boldsymbol{A}_n$
$x_{B(m)}$	$\mid$		$\mid$

- Do not require $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} \geq 0$ (a basic solution but not necessarily a BFS)
- Require $\bar{\boldsymbol{c}} := \boldsymbol{c}^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{A} := \boldsymbol{c}^T - \boldsymbol{p}^T\boldsymbol{A} \geq 0$ (dual feasibility)
- Dual cost is

$$\boldsymbol{p}^T\boldsymbol{b} = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{c}_B^T \boldsymbol{x}_B$$

- If $\boldsymbol{B}^{-1}\boldsymbol{b} \geq 0$ then both dual feasibility and primal feasibility, and also same cost $\Rightarrow$ **optimality**
- Otherwise, change basis

Dual simplex algorithm

An iteration

$-\mathbf{c}_B^T \mathbf{x}_B$	$\bar{c}_1$	$\dots$	$\bar{c}_n$
$x_{B(1)}$	$\vert$		$\vert$
$\vdots$	$\mathbf{B}^{-1}\mathbf{A}_1$	$\dots$	$\mathbf{B}^{-1}\mathbf{A}_n$
$x_{B(m)}$	$\vert$		$\vert$

1. Start with basis matrix $\mathbf{B}$ and all reduced costs ≥ 0.
2. If $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \geq 0$ optimal solution found; else, choose ℓ s.t. $x_{B(\ell)} < 0$.
3. Consider the ℓth row (pivot row) $x_{B(\ell)}, v_1, \dots, v_n$. If $\forall i \; v_i \geq 0$ then dual optimal cost $= +\infty$ and algorithm terminates (this implies that the primal problem is infeasible).
4. Else, let j s.t.
$$\frac{\bar{c}_j}{|v_j|} = \min_{\{i|v_i<0\}} \frac{\bar{c}_i}{|v_i|}$$
5. Pivot element v_j: $\mathbf{A}_j$ enters the basis and $\mathbf{A}_{B(\ell)}$ exits.

Primal

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \geq 2 \\ & x_1 \geq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \max \quad & 2p_1 + p_2 \\ \text{s.t.} \quad & p_1 + p_2 \leq 1 \\ & 2p_1 \leq 1 \\ & p_1, p_2 \geq 0 \end{aligned}$$

Primal problem in standard for:

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 - x_3 = 2 \\ & x_1 - x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & -x_1 - 2x_2 + x_3 = -2 \\ & -x_1 + x_4 = -1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Dual simplex algorithm

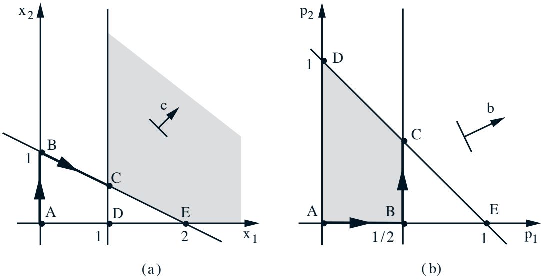
Geometries of the primal and dual

Primal

$$\begin{array}{ll} \min & x_1 + x_2 \\ \text{s.t.} & x_1 + 2x_2 \geq 2 \\ & x_1 \geq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

Dual

$$\begin{array}{ll} \max & 2p_1 + p_2 \\ \text{s.t.} & p_1 + p_2 \leq 1 \\ & 2p_1 \leq 1 \\ & p_1, p_2 \geq 0 \end{array}$$

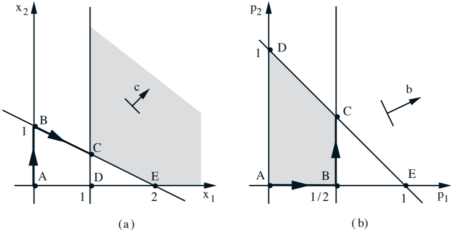


Dual simplex algorithm

Initial tableau

Primal

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & -x_1 - 2x_2 + x_3 = -2 \\ & -x_1 + x_4 = -1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$



Initial tableau associated with the dual BFS point A:

		x_1	x_2	x_3	x_4
	0	1	1	0	0
$x_3 =$	-2	-1	-2^*	1	0
$x_4 =$	-1	-1	0	0	1

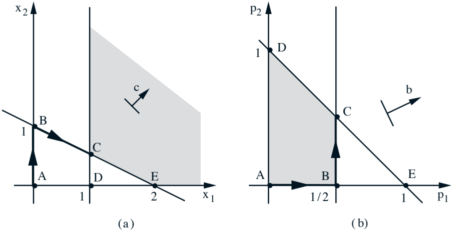
Dual simplex algorithm

An example

Two iterations of the dual simplex method:

		x_1	x_2	x_3	x_4
	-1	$1/2$	0	$1/2$	0
$x_2 =$	1	$1/2$	1	$-1/2$	0
$x_4 =$	-1	-1^*	0	0	1

		x_1	x_2	x_3	x_4
	$-3/2$	0	0	$1/2$	$1/2$
$x_2 =$	$1/2$	0	1	$-1/2$	$1/2$
$x_1 =$	1	1	0	0	-1



Duality and degeneracy

$$\begin{array}{llll} \min & \boldsymbol{c}^T \boldsymbol{x} & & \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} & = & \boldsymbol{b} \\ & \boldsymbol{x} & \geq & 0 \end{array} \qquad \begin{array}{ll} \max & \boldsymbol{p}^T \boldsymbol{b} \\ \text{s.t.} & \boldsymbol{p}^T \boldsymbol{A} \leq \boldsymbol{c}^T \end{array}$$

- Any basis matrix $\boldsymbol{B}$ leads to dual basic solution $\boldsymbol{p}^T = \boldsymbol{c_B}^T \boldsymbol{B}^{-1}$.
- The dual constraint $\boldsymbol{p}^T \boldsymbol{A}_j \leq c_j$ is active (i.e., $\boldsymbol{p}^T \boldsymbol{A}_j = c_j$) if and only if the reduced cost $\bar{c}_j = c_j - \boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{A}_j$ is zero.
- Since $\boldsymbol{p}$ is m-dimensional, dual degeneracy (more than m active constraints) implies more than m reduced costs that are zero.
- Since all reduced costs of basic variables in the primal must be zero, dual degeneracy is obtained whenever there exists a nonbasic variable (in the primal) whose reduced cost is zero.

Degeneracy

Relation between primal and dual basic solutions: Example

$\boxed{\text{Primal}}$

$$\begin{array}{rl} \min & 3x_1 + x_2 \\ \text{s.t.} & x_1 + x_2 - x_3 = 2 \\ & 2x_1 - x_2 - x_4 = 0 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$\boxed{\text{Dual}}$

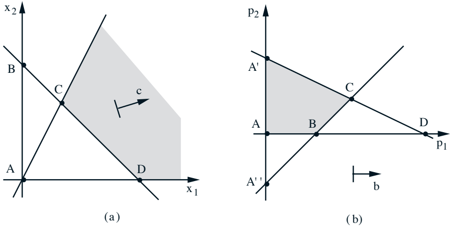
$$\begin{array}{rl} \max & 2p_1 \\ \text{s.t.} & p_1 + 2p_2 \leq 3 \\ & p_1 - p_2 \leq 1 \\ & p_1, p_2 \geq 0 \end{array}$$

(equivalent primal problem - easier to visualize in two dimensions)

$$\begin{array}{rrcrcl} \min & 3x_1 & + & x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \geq & 2 \\ & 2x_1 & - & x_2 & \geq & 0 \\ & & & \multicolumn{3}{l}{x_1, x_2 \geq 0} \end{array}$$

Degeneracy

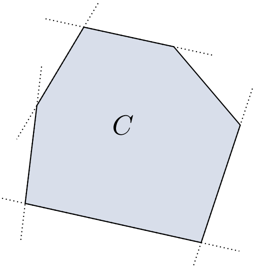
Relation between primal and dual basic solutions: Example



- Four basic solutions in primal: A, B, C, D.
- Six distinct basic solutions in dual: A, A', A'', B, C, D.
- Different bases may lead to the same basic solution for the primal, but to different basic solutions for the dual. Some are feasible and some are infeasible.

Suppose we have the *polyhedral* set

$$C = \left\{ \boldsymbol{x} \in \Re^n_+ : \ \boldsymbol{a}_i^T \boldsymbol{x} = b_i, \quad i \in [1, \ldots, m] \right\}$$



How to show that $C \neq \emptyset$?

- Any element $\bar{x} \in C$ serves as a *certificate*

How to show that $C = \emptyset$?

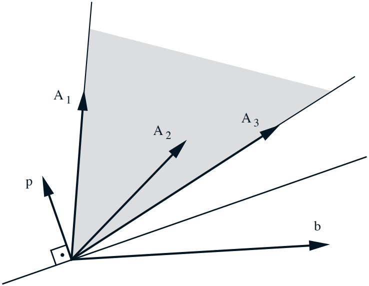
- Certifying nonexistence seems hard ...

Farkas Lemma

Theorem

Exactly one of the following two alternatives hold:

1. $\exists \boldsymbol{x} \geq 0$ *s.t.* $\boldsymbol{Ax} = \boldsymbol{b}$.
2. $\exists \boldsymbol{p}$ *s.t.* $\boldsymbol{p}^T \boldsymbol{A} \geq 0^T$ *and* $\boldsymbol{p}^T \boldsymbol{b} < 0$.



Farkas Lemma

Proof using duality

$\Rightarrow$ (Easy) If $\exists \boldsymbol{x} \geq 0$ s.t. $\boldsymbol{Ax} = \boldsymbol{b}$, and if $\boldsymbol{p}^T\boldsymbol{A} \geq 0^T$, then $\boldsymbol{p}^T\boldsymbol{b} = \boldsymbol{p}^T\boldsymbol{Ax} \geq 0$

$\Leftarrow$ (Harder) Assume there is no $\boldsymbol{x} \geq 0$ s.t. $\boldsymbol{Ax} = \boldsymbol{b}$

$$\begin{array}{llrcl} (P) & \max & 0^T\boldsymbol{x} & & \\ & \text{s.t.} & \boldsymbol{Ax} & = & \boldsymbol{b} \\ & & \boldsymbol{x} & \geq & 0 \end{array} \qquad \begin{array}{lll} (D) & \min & \boldsymbol{p}^T\boldsymbol{b} \\ & \text{s.t.} & \boldsymbol{p}^T\boldsymbol{A} \geq 0^T \end{array}$$

(P) infeasible $\Rightarrow$ (D) either unbounded or infeasible

Since $\boldsymbol{p} = 0$ is feasible $\Rightarrow$ (D) unbounded
$\Rightarrow \exists \boldsymbol{p}: \quad \boldsymbol{p}^T\boldsymbol{A} \geq 0^T$ and $\boldsymbol{p}^T\boldsymbol{b} < 0$

Duality revisited

- So far: Simplex $\longrightarrow$ Duality $\longrightarrow$ Farkas lemma

  - specialized to LP, relied on a particular algorithm

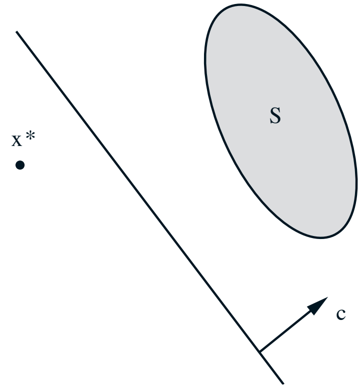
- Alternative: Separation theorem (a geometric property) $\longrightarrow$ Farkas lemma $\longrightarrow$ Duality

  - purely geometric, generalizes to general nonlinear problems, more fundamental

Separating hyperplane theorem

Theorem

Let S be a nonempty closed convex subset of $\Re^n$ and let $\mathbf{x}^* \in \Re^n$ such that $\mathbf{x}^* \notin S$. Then there exists a vector $\mathbf{c} \in \Re^n$ such that $\mathbf{c}^T\mathbf{x}^* < \mathbf{c}^T\mathbf{x} \quad \forall \mathbf{x} \in S$.



Farkas' lemma revisited

Consider only the hard part of the lemma:

Theorem

If $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq 0$ is infeasible, then there exists a vector $\boldsymbol{p}$ such that $\boldsymbol{p}^T\boldsymbol{A} \geq 0^T$ and $\boldsymbol{p}^T\boldsymbol{b} < 0$.

- let $S = \left\{ \boldsymbol{y} \;\middle|\; \exists \boldsymbol{x} \text{ such that } \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x},\ \boldsymbol{x} \geq 0 \right\}$ and assume that $\boldsymbol{b} \notin S$
- S is convex; nonempty; closed (it is indeed the projection of $\{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x},\ \boldsymbol{x} \geq 0\}$ onto the $\boldsymbol{y}$ coordinates, so also a polyhedron, and therefore closed)
- $\boldsymbol{b} \notin S$: $\exists \boldsymbol{p}$ such that $\boldsymbol{p}^T\boldsymbol{b} < \boldsymbol{p}^T\boldsymbol{y}$ for every $\boldsymbol{y} \in S$
- since $0 \in S$, we must have $\boldsymbol{p}^T\boldsymbol{b} < 0$
- $\forall \boldsymbol{A}_i$ and $\forall \lambda > 0$, $\lambda \boldsymbol{A}_i \in S$ and $\boldsymbol{p}^T\boldsymbol{b} < \lambda \boldsymbol{p}^T\boldsymbol{A}_i$
- divide by λ and then take limit as λ tends to infinity: $\boldsymbol{p}^T\boldsymbol{A}_i \geq 0 \ \forall i \Rightarrow \boldsymbol{p}^T\boldsymbol{A} \geq 0^T$

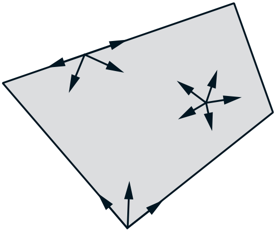
Duality theorem revisited

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \end{aligned} \qquad\qquad \begin{aligned} \max \quad & \boldsymbol{p}^T \boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{p}^T \boldsymbol{A} = \boldsymbol{c}^T \\ & \boldsymbol{p} \geq 0 \end{aligned}$$

Assume that the primal has an optimal solution $\boldsymbol{x}^*$. We will show that the dual problem also has a feasible solution with the same cost. Strong duality follows then from weak duality.

Duality theorem revisited

- $I(\boldsymbol{x}^*) = \{i \mid \boldsymbol{a}_i^T \boldsymbol{x}^* = b_i\}$ set of indices of the active constraints
- feasible directions at $\boldsymbol{x}^*$ are $\left\{\boldsymbol{d} : \ \boldsymbol{a}_i^T \boldsymbol{d} \geq 0 \quad \forall i \in I(\boldsymbol{x}^*)\right\}$



- we first show that if $\boldsymbol{a}_i^T \boldsymbol{d} \geq 0$ for every $i \in I(\boldsymbol{x}^*)$, then $\boldsymbol{c}^T \boldsymbol{d} \geq 0$:
  - consider such a $\boldsymbol{d}$, then $\boldsymbol{a}_i^T(\boldsymbol{x}^* + \epsilon \boldsymbol{d}) \geq \boldsymbol{a}_i \boldsymbol{x}^* = b_i$ for all $i \in I(\boldsymbol{x}^*)$
  - if $i \notin I(\boldsymbol{x}^*)$, we have $\boldsymbol{a}_i^T \boldsymbol{x}^* > b_i \Rightarrow \boldsymbol{a}_i^T(\boldsymbol{x}^* + \epsilon \boldsymbol{d}) > b_i$ for any ϵ small enough
  - $\boldsymbol{x}^* + \epsilon \boldsymbol{d}$ is feasible for any ϵ small enough
  - by optimality of $\boldsymbol{x}^*$, $\boldsymbol{c}^T \boldsymbol{d} \geq 0$

Duality theorem revisited

- by Farkas' lemma, $\exists\ p_i \geq 0\ \forall i \in I(\boldsymbol{x}^*)$ such that:

$$\boldsymbol{c} = \sum_{i \in I} p_i \boldsymbol{a}_i$$

- for $i \notin I(\boldsymbol{x}^*)$, define $p_i = 0$, so $\boldsymbol{p}^T \boldsymbol{A} = \boldsymbol{c}^T$
- in conclusion:

$$\boldsymbol{p}^T \boldsymbol{b} = \sum_{i \in I(\boldsymbol{x}^*)} p_i b_i = \sum_{i \in I(\boldsymbol{x}^*)} p_i \boldsymbol{a}_i^T \boldsymbol{x}^* = \boldsymbol{c}^T \boldsymbol{x}^*$$