6.255 Optimization Methods
Midterm Fall 2021

October 15, 2021

This midterm has **4 Problems**. You can answer them and their questions in the order you want.

- **Problem 1** – Modeling a Market (22 points)
- **Problem 2** – Duality (13 points)
- **Problem 3** – Simplex (17 points)
- **Problem 4** – Ising Model and Network Flows (19 points)

At the end of the exam, you are asked to scan and submit your solutions through Canvas, assignment section, in 1PDF file. Please scan it in the order of the problems (1 then 2 then...). You still need to turn on your paper solutions at the end.
The midterm length is **1hour 55min**. After that, **5min** will be dedicated to the submission. In this 5min, and only in this 5min, you can take out your phone and computer.

First Name: ____

Last Name: ____

Problem 1 – Market problem (22 points)

We consider a market consisting of a certain number of providers $i = 1, \dots, n_1$ and demanders $j = 1, \dots, n_2$ of a commodity and a network of routes between the providers and the demanders along which the commodity can be shipped from the providers to the demanders. In particular, between each provider i and demander j, there exists a route. We denote by c_{ij} the unit shipment cost on the route between i and j. We denote by s_i the available supply at the provider i and by d_j the demand at the demander j. The goal is determine how much commodity to ship from i to j such that all the demand is satisfied at minimum total shipment cost.

1. (3pts) Formulate the problem as a linear optimization problem. Use the variable x_{ij} to denote the quantities of the commodity that is shipped from i to j.

2. (2pts) Under what condition on $(s_i)_{i=1\dots n_1}$ and $(d_j)_{j=1\dots n_2}$ is the problem feasible?

3. (3pts) Formulate the problem as a network flow problem. You only need to draw the corresponding network flow, detail what each node represents and explain briefly why it models correctly the problem.

4. (3pts) Formulate the dual of the problem.

5. (2pts) Give a *brief* interpretation of the dual problem. Interpret in particular the cost function and constraints of the dual.
   Hint: It can be useful to first make all your dual variables positive if they're not. You can then interpret a certain set of dual variables as supply price and the other set as the demand market price in a competitive market.

6. (2pts) Write the complementary slackness conditions for the problem. (1pts) What does $\sum_i x_{ij}$ and $\sum_j x_{ij}$ represents for each i and j? (2pts) Give a *brief* interpretation of each of the slackness conditions.

7. (2pts) Suppose you were hired at Amazon to optimize their delivery plan from warehouses to neighborhoods using this optimization problem. Amazon has already some implemented solution (how much they already ship from each warehouse i to neighborhood j). They do not want to perturb too much the current implemented solution which we denote $\bar{x}_{ij}$. Hence they want you to optimize the problem while keeping the new solution x_{ij} within a small distance $\delta > 0$ from the current solution $\bar{x}_{ij}$, ie, $|x_{ij} - \bar{x}_{ij}| \leq \delta$. Write a new optimization problem satisfying this condition and reformulate it as a linear optimization problem.

8. (2pts) Suppose now that instead of minimizing the total shipment cost, the company wants to minimize the maximum shipment cost among all the routs $i - j$. Formulate a new optimization problem modeling this case, and reformulate it as a linear optimization problem.
   Hint: recall that shipment cost in route $i - j$ is $c_{ij}x_{ij}$.

Solutions:

1.

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{m} \sum_{i=1}^{n} c_{ij} x_{ij} \\ \text{subject to} \quad & \sum_{i=1}^{n} x_{ij} \leq s_j \quad \text{for } j = 1, 2, \dots, n_2 \\ & \sum_{j=1}^{n} x_{ij} \geq d_i \quad \text{for } i = 1, 2, \dots, n_1 \\ & x_{ij} \geq 0 \quad \text{for } i = 1, 2, \dots, n_1,\ j = 1, \dots, n_2 \end{aligned}$$

Equality in the second constraint (demand) is ok, but not in the first one (supply).

2. Total supply must be larger than total demand: $\sum_{j=1}^{n_2} s_j \geq \sum_{i=1}^{n_1} d_i$.
   Common mistakes. Saying $s > d$ which doesn't make sense, these are vectors of different dimension. Also, even if they are of the same dimension, you don't need the supply of each i to be bigger than the demand of location i. You just need to look at total supply and total demand.

3. See
   Common mistakes.

   - You need to take into account extra supply (if $\sum_i d_i > \sum_j s_j$). If not taken into account, $1pt$ is taken away.
   - Drawing the flow without the supply capacity might require you to have $\sum_i d_i = \sum_j s_j$ which is not assumed here.
   - You need to mention what are the capacities and outer flows, or the solution is not complete.

4. We introduce the dual variables p_i, for $i = 1, 2, \ldots, n_1$ and q_j for $j = 1, 2, \ldots, n_2$.

$$\begin{aligned} \text{Maximize} \quad & \sum_{j=1}^{n_2} d_j p_j + \sum_{i=1}^{n_1} s_i q_i \\ \text{subject to} \quad & p_j + q_i \leq c_{ij} \quad \text{for } j = 1, 2, \ldots, n_2 \text{ and } i = 1, 2, \ldots, n_1 \\ & p_j \geq 0 \quad \text{for } j = 1, 2, \ldots, n_2 \\ & q_i \leq 0 \quad \text{for } i = 1, 2, \ldots, n_1 \end{aligned}$$

   All other equivalent formulation are also correct. For example one can have in the objective $\sum_{j=1}^{n_2} d_j p_j - \sum_{i=1}^{n_1} s_i q_i$ and have $q_i \geq 0$ and $p_j - q_j \leq c_{ij}$ instead.

5. We can write the dual problem as

$$\begin{aligned} \text{Maximize} \quad & \sum_{j=1}^{n_2} d_j p_j - \sum_{i=1}^{n_1} s_i q_i \\ \text{subject to} \quad & p_j + q_i \leq c_{ij} \quad \text{for } j = 1, 2, \ldots, n_2 \text{ and } i = 1, 2, \ldots, n_1 \\ & p_j \geq 0 \quad \text{for } j = 1, 2, \ldots, n_2 \\ & q_i \geq 0 \quad \text{for } i = 1, 2, \ldots, n_1 \end{aligned}$$

   We can interpret p_i as the unit supply price at location i and q_j as the market selling price at location j. The dual problem is therefore maximizing the gap between the total demand selling price $\sum_{j=1}^{n_2} d_j p_j$ (i.e. what we earn by satisfying the demand) and the total supply buying price $\sum_{i=1}^{n_1} s_i q_i$ (i.e. what will cost us to buy the supply). This makes sens as the dual problem is "adversary". We also expect from this interpretation that at optimal, this gap will be equal to the shipment cost, which is exactly the primal objective.

   The constraint $p_i + c_{ij} \geq q_j$ means that the supply price at i + shipment cost c_{ij} cannot be lower than the demand selling price at j. This condition models the equilibrium of a competitive market: if we had $p_i + c_{ij} < q_j$ then supplier i will increase its production of i which downs the market price q_i.

6.

$$x_{ij}(p_i + c_{ij} - q_j) = 0, \quad \forall i, j \tag{1}$$

$$p_i(s_i - \sum_j x_{ij})) = 0, \quad \forall i \tag{2}$$

$$q_i(d_j - \sum_i x_{ij})) = 0, \quad \forall j \tag{3}$$

   (1): If the cost of delivery c_{ij} strictly exceeds the market price, ie $p_i + c_{ij} - q_j > 0$, then nothing is shipped from i to j because doing so incures a loss.

(2): In case $s_i > \sum_j x_{ij}$, there is an excessive supply at i. In a competitive market, the provider is not willing to pay for more supply, because he is already over-supplied, and hence $p_i = 0$.
(3): In case $d_j > \sum_i x_{ij}$ which means that the supply exceeds the demand. Hence, the demander is not willing to pay for more goods, i.e., $q_j = 0$.

7. We add to the previous optimization problem the constraint $|x_{ij} - \bar{x}_{ij}| \le \delta$. This constraint can be transformed into a linear constraint by introducing a variable z and the constraints

$$t \ge x_{ij} - \bar{x}_{ij}$$
$$t \ge -(x_{ij} - \bar{x}_{ij})$$

8. We replace the objective by minimize $\max_{ij} c_{ij} x_{ij}$. This can be reformulated as linear problem by introducting a variable z, replacing the objective by minimize$z$ and adding the constraints $z \ge c_{ij} x_{ij}$ for all i, j.

Problem 2 – Duality (13 points)

1. (3pts) Compute the dual of the following optimization problem

$$\begin{array}{llllllll}
\max & 13x_1 & + & 9x_2 & - & 10x_3 & & \\
s.t. & 4x_1 & + & 2x_2 & - & x_3 & = & 18 \\
 & 3x_1 & + & 3x_2 & - & 3x_3 & \le & 18 \\
 & -x_1 & - & 3x_2 & + & 6x_3 & \ge & -18 \\
 & & & & & x_2 & \ge & 0 \\
 & & & & & x_3 & \le & 0
\end{array}$$

2. (4pts) Suppose your friend tells you that the optimal solution is $x^* = (4, 0, -2)$. Assuming that your friend is correct, use complementary slackness to find the optimal solution to the dual problem without resolving

3. (3pts) Using complementary slackness, prove that the solution your friend gave you is correct.

4. (3pts) Suppose that the right-hand side of the primal optimization problem changes such that the problem becomes infeasible. Does the new dual problem have a finite optimum, is it unbounded, is it infeasible, or is more information required? Explain your answer.

Solution:

1. $$\begin{array}{llllllll}
\min & 18p_1 & + & 18p_2 & - & 18p_3 & & \\
s.t. & 4p_1 & + & 3p_2 & - & p_3 & = & 13 \\
 & 2p_1 & + & 3p_2 & - & 3p_3 & \ge & 9 \\
 & -p_1 & - & 3p_2 & + & 6p_3 & \le & -10 \\
 & & & & & p_2 & \ge & 0 \\
 & & & & & p_3 & \le & 0
\end{array}$$

   Many people converted the primal into a different form before taking the dual. I accepted this as long as the conversion in the primal was shown. However, make sure that going forward, you know how to take the dual of a LP no matter what form it is in originally.

2. $p^* = (1, 3, 0)$. To compute this, we can first note that $p_3^* = 0$ because the third constraint in the primal is not active. Then, we see that the first and third constraints in the dual must be active. Using this, we can solve a 2x2 system of linear equations for p_1^* and p_2^*.

$$4p_1 + 3p_2 = 13$$

$$-p_1 - 3p_2 = -10$$

3. Our first step is to confirm that both the primal and dual solutions are feasible. Next, we can show optimality using one of two steps. You can show optimality using complementary slackness by testing all 6 constraints. Additionally, you can show this using strong duality by showing that both the primal and dual have the same objective (72). **Common mistake**: Many people forgot to check feasibility of the primal and/or dual. Without checking this, we cannot confirm that the solution our friend gave us is optimal.

4. It becomes unbounded. The primal becomes infeasible meaning that the dual will either become unbounded or infeasible. However, we note that the dual constraints do not change because the only thing that changed was the right-hand side of the primal. We previously found that there is a feasible solution so this means that the problem becomes unbounded. **Common mistake**: Many people got that it was unbounded or infeasible. However, the trick to this question is to notice that if only the right-hand side changes, the feasibility of the dual problem will not change.

Problem 3 – Simplex Method (17 points)

Consider the following linear optimization problem.

$$
\begin{array}{llllllll}
\min & 4x_1 & + & 2x_2 & + & x_3 & & \\
s.t. & x_1 & - & 2x_2 & - & 6x_3 & & = -2 \\
 & x_1 & + & 2x_2 & + & 3x_3 & & = 3 \\
 & x_1 & + & 10x_2 & + & 21x_3 & & = 13 \\
 & x_1 & + & 2x_2 & + & 6x_3 & + \; x_4 & = 4 \\
 & & & & & & x_1, x_2, x_3, x_4 & \geq 0
\end{array}
$$

1. (4pts) Formulate the auxiliary problem (the phase 1 of linear optimization), which is to find a basic feasible solution, by introducing artificial variables. Besides, provide the initial tableau based on the formulation.

   Hint: Introduce $\mathbf{y}$ such that $\mathbf{Ax} + \mathbf{y} = \mathbf{b}$ with $\mathbf{x}, \mathbf{y} \geq 0$ and modify the objective function.

2. (4pts) After performing simplex method to the auxiliary problem, we get the following tableau:

0	0	0	0	0	3	4	0	1
$\frac{5}{4}$	0	1	$\frac{9}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	0	0
$\frac{1}{2}$	1	0	-$\frac{3}{2}$	0	-$\frac{1}{2}$	$\frac{1}{2}$	0	0
0	0	0	0	0	-2	-3	1	0
1	0	0	3	1	1	-2	0	1

   What is the solution based on the tableau? Is there any redundant row in the original problem?

3. (3pts) Based on the tableau in 2., form the tableau of the original problem, perform simplex method to the original problem and find the optimal solution.

   Hint:

$$
\begin{pmatrix} 2 & -1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/4 & 1/4 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & -1 & 1 \end{pmatrix}
$$

4. (3pts) What is the optimal solution if the objective becomes $\min\ 2x_1 + 4x_2 + x_3$?

5. (3pts) What is the optimal solution if the second constraint becomes $x_1 + 2x_2 + 3x_3 = 1$ (b is changed from 3 to 1) and the third constraint $x_1 + 10x_2 + 21x_3 = 13$ is removed.

   Hint: We can reuse the result from 3. and perform primal or dual simplex method.

Solution:

1. (4pts) For each constraint, we introduce a variable y_i and minimize the summation of y_i. The auxiliary problem is as follows: (2pts)

$$
\begin{array}{llllllllllllll}
\min & y_1 & + & y_2 & + & y_3 & + & y_4 & & & & & & \\
s.t. & -x_1 & + & 2x_2 & + & 6x_3 & & & + & y_1 & & & = & 2 \\
 & x_1 & + & 2x_2 & + & 3x_3 & & & & + \; y_2 & & & = & 3 \\
 & x_1 & + & 10x_2 & + & 21x_3 & & & & & + \; y_3 & & = & 13 \\
 & x_1 & + & 2x_2 & + & 6x_3 & + & x_4 & & & & + \; y_4 & = & 4 \\
 & & & & & & & & & & & x_1, x_2, x_3, x_4 & \geq & 0 \\
 & & & & & & & & & & & y_1, y_2, y_3, y_4 & \geq & 0
\end{array}
$$

We obtain the initial tableau by setting all x_i to 0 and y to b. (2pts)

-22	-2	-16	-36	-1	0	0	0	0
2	-1	2	6	0	1	0	0	0
3	1	2	3	0	0	1	0	0
13	1	10	21	0	0	0	1	0
4	1	2	6	1	0	0	0	1

2. (4pts)

0	0	0	0	0	3	4	0	1
$\frac{5}{4}$	0	1	$\frac{9}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	0	0
$\frac{1}{2}$	1	0	-$\frac{3}{2}$	0	-$\frac{1}{2}$	$\frac{1}{2}$	0	0
0	0	0	0	0	-2	-3	1	0
1	0	0	3	1	1	-2	0	1

Based on the above tableau, $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (0.5, 1.25, 0, 1, 0, 0, 0, 0)$. (2pts)

The third row is redundant since all the coefficients associated with x_i are 0. (2pts)

3. (3pts) From 2., the basic variables are x_2, x_1, x_4.

The basis matrix B and its inverse are:

$$B = \begin{pmatrix} 2 & -1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 1/4 & 1/4 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

We can use $B^{-1}A$ and $(c^T - c_B^T B^{-1} A)$ to construct a tableau:

$\frac{-9}{2}$	0	0	$\frac{5}{2}$	0
$\frac{5}{4}$	0	1	$\frac{9}{4}$	0
$\frac{1}{2}$	1	0	$\frac{-3}{2}$	0
1	0	0	3	1

The initial BFS $(x_1, x_2, x_3, x_4) = (0.5, 1.25, 0, 1)$ is optimal, and the optimal cost is 4.5.

4. (3pts) The initial tableau is:

-6	0	0	-5	0
$\frac{5}{4}$	0	1	$\frac{9}{4}$	0
$\frac{1}{2}$	1	0	$\frac{-3}{2}$	0
1	0	0	3^*	1

After one iteration of simplex method, it becomes:

$-\frac{13}{3}$	0	0	0	$\frac{5}{3}$
$\frac{1}{2}$	0	1	0	$\frac{-3}{4}$
1	1	0	0	$\frac{1}{2}$
$\frac{1}{3}$	0	0	1	$\frac{1}{3}$

The optimal solution is $(x_1, x_2, x_3, x_4) = (1, \frac{1}{2}, \frac{1}{3}, 0)$, and the optimal cost is 4.

5. (3pts) We use the basis matrix B from 3. and calculate x_B:

$$x_B = B^{-1}b = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{4} \\ \frac{-1}{2} \\ 3 \end{pmatrix}$$

The tableau becomes:

$\frac{1}{2}$	0	0	$\frac{5}{2}$	0
$x_2 = \frac{3}{4}$	0	1	$\frac{9}{4}$	0
$x_1 = \frac{-1}{2}$	1	0	$\frac{-3}{2}^*$	0
$x_4 = 3$	0	0	3	1

After one iteration of dual simplex method, it becomes:

$\frac{-1}{3}$	$\frac{-5}{2}$	0	0	0
$x_2 = 0$	$\frac{3}{2}$	1	0	0
$x_3 = \frac{1}{3}$	$\frac{-2}{3}$	0	1	0
$x_4 = 2$	2	0	0	1

The optimal solution is $(x_1, x_2, x_3, x_4) = (0, 0, \frac{1}{3}, 2)$ and the optimal cost is $\frac{1}{3}$.

Problem 4 – Ising Model and Network Flows (19 points)

In this problem, we efficiently solve an interaction problem – so-called Ising model – using network flows.

The goal is to denoise a deteriorated black and white image. We model the problem by a graph $\mathcal{G} = (N, A)$ where $N = \{1, \dots, n\}$ is the set of pixels and A is the set of arcs through which pixels can interact e.g. 2-dimensional lattice: a pixel interacts with its 4 direct neighbors. To each pixel we associate a binary variable $\sigma_i \in \{-1, +1\}$. We denote by $\alpha_i \in \{\pm 1\}$ the pixel values for the initial image. In the Ising model, we try to minimize the "energy" of the image σ, defined as

$$J(\sigma) := -\sum_{i \in N} \alpha_i \sigma_i - \sum_{(i,j) \in A} \gamma_{i,j} \sigma_i \sigma_j,$$

where $\gamma_{i,j} \geq 0$ (interaction stsrength between i and j) are given. The denoised image is the image minimizing the energy $J(\sigma)$.

1. (3pts) Show that without loss of generality we can suppose that the graph is complete (all arcs are present in A) and that $\gamma_{i,j} = \gamma_{j,i}$ for any $i \neq j \in \{1, \dots, n\}$. *Hint: What edge interaction $\gamma_{i,j}$ can we set for all edges not initially present in the graph to keep the energy unchanged?*

2. (2pts) We will now make these assumptions and set the capacity $u_{i,j}$ of each edge (i,j) to $\gamma_{i,j}$. To each image σ we associate a partition of the vertices $\{1, \dots, n\}$ in two sets as follows

$$T^+ = \{i, \text{ s.t. } \sigma_i = 1\}, \quad T^- = \{i, \text{ s.t. } \sigma_i = -1\}.$$

Show that for any $1 \leq i, j \leq n$,

$$\frac{1 - \sigma_i \sigma_j}{2} = \begin{cases} 0, & \text{if } i, j \in T^+ \\ 0, & \text{if } i, j \in T^- \\ 1, & \text{if } i \in T^+ \text{ and } j \in T^-, \text{ or } i \in T^- \text{ and } j \in T^+. \end{cases}$$

3. (4pts) Show that the value $C(T^+, T^-)$ of the cut (T^+, T^-) satisfies

$$\sum_{1 \leq i,j \leq n} \gamma_{i,j} \frac{1 - \sigma_i \sigma_j}{2} = 2C(T^+, T^-).$$

We recall that the value of cut $C(T^+, T^-)$ is defined as

$$C(T^+, T^-) := \sum_{(i,j) \in A, i \in T^+, j \in T^-} u_{i,j},$$

where $u_{i,j}$ is the capacity of edge (i,j). By convention, the value of the cut is 0 if one of T^+ or T^- is empty.

4. (5pts) We now add to the network a source node s and a sink node t and add new edges for each $i = 1, \dots, n$ as follows:

- If $\alpha_i = 1$, add edge (s, i) with capacity 1/2: $u_{s,i} = 1/2$.
- if $\alpha_i = -1$, add edge (i, t) with capacity 1/2: $u_{i,t} = 1/2$.

Show that we can interpret $J(\sigma)$ as the value of a cut up to additive or multiplicative terms. Conclude that minimizing the energy J is equivalent to finding the minimum cut separating the source and the sink in the constructed network.

5. (3pts) Argue that the problem of denoising the initial image can be solved using linear programming. *Hint: MAX-FLOW is a linear programming problem.*

6. (2pts) Our current analysis works only when $\gamma_{i,j} \geq 0$ for all $(i,j) \in A$. Why?
In general we cannot solve efficiently the Ising model when $\gamma_{i,j}$ are allowed to be negative. Show that if $\gamma_{i,j} \leq 0$ for all $(i,j) \in A$, the Ising model is equivalent to a MAX-CUT problem.

MAX-CUT is known to be NP-hard: this argument shows that solving general Ising models is NP-hard.

Solution.

1. Consider the new graph $\tilde{G} = (N, \tilde{A})$ where $A = \{(i,j), i \neq j \in N\}$, i.e. the complete graph. To each edge we assign a new interaction strength as follows

$$\tilde{\gamma}_{i,j} = \begin{cases} \frac{\gamma_{i,j} + \gamma_{j,i}}{2} & \text{if } (i,j), (j,i) \in A \\ \frac{\gamma_{i,j}}{2} & \text{if } (i,j) \in A \text{ and } (j,i) \notin A \\ \frac{\gamma_{j,i}}{2} & \text{if } (i,j) \notin A \text{ and } (j,i) \in A \\ 0 & \text{if } (i,j) \notin A \text{ and } (j,i) \notin A \end{cases}$$

Any image $\sigma = (\sigma_i)_{i\in N}$ has same energy in the first graph and in the second graph, therefore solving the Ising model on the old or the new graph is equivalent. Note that the new graph is complete and symmetric: $\tilde{\gamma}_{i,j} = \tilde{\gamma}_{j,i}$ for any $i \neq j \in N$.

Common mistakes: Assigning $\gamma_{i,j}$ to non-existing arcs (without ensuring symmetry $\gamma_{i,j} = \gamma_{j,i}$).

2. If $i \in T^+$, then $\sigma_i = 1$ and if $i \in T^-$, then $\sigma_i = -1$. Idem for σ_j. The question results from direct computations.

3. Using 2,

$$\begin{aligned}\sum_{1\leq i,j\leq n} \gamma_{i,j}\frac{1-\sigma_i\sigma_j}{2} &= \sum_{i\in T^+, j\in T^-} \gamma_{i,j} + \sum_{i\in T^-, j\in T^+} \gamma_{i,j} \\ &= \sum_{i\in T^+, j\in T^-} (\gamma_{i,j} + \gamma_{j,i}) \\ &= 2\sum_{i\in T^+, j\in T^-} \gamma_{i,j} \\ &= 2C(T^+, T^-).\end{aligned}$$

where in the third equality we used the fact that $\gamma_{i,j} = \gamma_{j,i}$.

Common mistakes: Missing factor 2 due to symmetry $\gamma_{i,j} = \gamma_{j,i}$.

4. For any σ we define the sets T^+_σ and T^-_σ as in 2. We will relate the value of the cut $C(T^+\cup\{s\}, T^-\cup\{t\})$ to $J(\sigma)$. We denote by A the new set of arcs in our network flow.

$$\begin{aligned}C(T^+\cup\{s\}, T^-\cup\{t\}) &= \sum_{i\in T^+\cup\{s\}, j\in T^-\cup\{t\}, (i,j)\in A} u_{i,j} \\ &= \sum_{i\in T^+, j\in T^-} u_{i,j} + \sum_{j\in T^-, s.t. (s,j)\in A} u_{s,j} + \sum_{i\in T^+, s.t. (i,t)\in A} u_{i,t} \\ &= \sum_{i\in T^+, j\in T^-} \gamma_{i,j} + \sum_{j\in T^-, s.t. \alpha_j = 1} 1/2 + \sum_{i\in T^+, s.t. \alpha_i=-1} 1/2 \\ &= \frac{1}{2}\sum_{1\leq i,j\leq n} \gamma_{i,j}\frac{1-\sigma_i\sigma_j}{2} + \sum_{j\in T^-}\frac{1-\alpha_i\sigma_i}{4} + \sum_{i\in T^+}\frac{1-\alpha_i\sigma_i}{4} \\ &= \frac{1}{4}\sum_{1\leq i,j\leq n}\gamma_{i,j} - \frac{1}{4}\sum_{1\leq i,j\leq n}\gamma_{i,j}\sigma_i\sigma_j + \frac{n}{4} - \frac{1}{4}\sum_{i\in N}\alpha_i\sigma_i \\ &= \frac{1}{4}J(\sigma) + \frac{n}{4} + \frac{1}{4}\sum_{1\leq i,j\leq n}\gamma_{i,j}.\end{aligned}$$

Therefore, we can write $C(T^+\cup\{s\}, T^-\cup\{t\}) = \alpha J(\sigma) + \beta$ where $\alpha = 1/4$ and $\beta = \frac{n}{4} + \frac{1}{4}\sum_{1\leq i,j\leq n}\gamma_{i,j}$ are two constants. Minimizing the value of the cut $C(T^+\cup\{s\}, T^-\cup\{t\})$ is equivalent to minimizing the energy $J(\sigma)$. Further, one can see that there is a one-to-one mapping between $s-t$ cuts in the graph and images: given an image we have the cut $(T^+\cup\{s\}, T^-\cup\{t\})$ and given a $s-t$ cut (S,T) we have the image σ with

$$\sigma_i = \begin{cases} 1 \text{ if } i\in S \\ 0 \text{ if } i \in T\end{cases}$$

for any $i \in N$. Finally, this proves that minimizing $J(\sigma)$ over images σ is equivalent to finding the optimal $s-t$ cut of our constructed network flow.

5. We know that $MAX - FLOW = MIN - CUT$ and that the two problems are dual. To solve the denoising problem, it suffices to solve the $s-t$ min-cut problem described above. To do so, we solve

its dual which is the $s-t$ min-cut problem on the defined network flow (from the optimal flow we can get the optimal cut by solving the slackness conditions equations) which is an LP.

Common mistakes: Inverting the order of arguments. There is no reason a priori why solving max-flow on this network gives a solution to the denoising problem. We first prove that minimizing the energy is the same as finding the min-cut, which can then be solved with max-flow.

6. We defined our capacities as $u_{i,j} = \gamma_{i,j}$. For this to define a network flow we need to have non-negative capacities $\gamma_{i,j} \geq 0$.

   If all $\gamma_{i,j} \neq 0$, we can construct the same network flow problem with capacities $u_{i,j} = -\gamma_{i,j}$, and with the negative image $\tilde{\alpha} = -\alpha$ instead. The value of a cut now becomes:

$$
\begin{aligned}
C(T^+ \cup \{s\}, T^- \cup \{t\}) &= -\frac{1}{4} \sum_{1 \leq i,j \leq n} \gamma_{i,j} + \frac{1}{4} \sum_{1 \leq i,j \leq n} \gamma_{i,j} \sigma_i \sigma_j + \frac{n}{4} + \frac{1}{4} \sum_{i \in N} \alpha_i \sigma_i \\
&= -\frac{1}{4} J(\sigma) + \frac{n}{4} - \frac{1}{4} \sum_{1 \leq i,j \leq n} \gamma_{i,j}.
\end{aligned}
$$

   Now, maximizing the value of the cut is equivalent to minimizing the energy $J(\sigma)$. With the same reasoning as in 3, we show that solving the $s-t$ max-cut problem on this new network flow is equivalent to solving the denoising problem.

   Common mistakes: Negative capacities always lead to infeasible problem: by definition, a feasible flow satisfies non-negativity constraints. If some capacities are negative, the network flow polyhedron is empty, not unbounded.