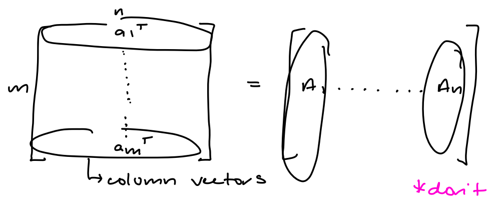
Linear Algebra Review

Matrix



lower case → column
upper case → rows

*don't divide by vector

$$Ax = \sum_i x_i \underbrace{A_i}_{\text{column}}$$

$$= \begin{bmatrix} a_1^T x \\ \vdots \\ a_m^T x \end{bmatrix} \longrightarrow \underbrace{Ax \ge b}_{\sum a_i x_i} \Rightarrow \begin{matrix} a_1^T x \ge b_1 \\ a_2^T x \ge b_2 \\ \vdots \\ a_m^T x > b_m \end{matrix}$$

linear combination: linear combination of $x^1 \dots x^k$ ⇒ vector of the form $\sum_{k=1}^{k} a_k x^k$ where $a_1 \dots a_k \in \mathbb{R}$

linear dependent: exists $a_1 \dots a_k$ st $\sum_{k=1}^{k} a_k x^k = 0$

linear independent: $\sum_{k=1}^{k} a_k x^k \neq 0$ ⇒ cannot be expressed by other vectors (not redundant)

inverse: $AB = BA = I$

Suppose A has 2 inverse B & C:

$AB = I$

$\overset{I}{\cancel{CA}} B = C$

$B = C$

Equivalent statements: (A is square)

- A is invertible
- rows of A are linearly independent
- cols of A are linearly independent
- For some vector b, $Ax = b$ has a unique solution

  $Ax = b \Rightarrow x = A^{-1}b$

- determinant is non zero

Span: $S \subset \mathbb{R}^n$ is a subspace for all $x, y \in S$ and $a, b \in \mathbb{R}$, $ax + by \in S$

*subspace always contains origin

Affine subspace: translated subspace of form $S_0 + x^0 = \{x + x^0 \mid x \in S_0\}$

Basis: set of linearly independent vectors $b^1 \dots b^k$ st any vector s can be expressed as a linear combination of $b^1 \dots b^k$

2D → 2 vectors (3 would mean one is lin dependent)

→ all basis will have same number of elements

dimension → # of vectors in the basis

$\mathbb{R}^n$ has a base of n vectors

→ any n linearly independent vectors form basis of $\mathbb{R}^n$

→ any $k > n$ of $\mathbb{R}^n$ are linearly dep

if matrix is invertible, its rows form a basis of $\mathbb{R}^n$

Ex. Suppose we have $b^1 \dots b^n$ are lin indep in $\mathbb{R}^n$. Express y as a linear combination of $b^1 \dots b^n$. Find an expression for $a_1, \dots a_n$ st $\sum_{j=1}^{n} a_j b^j = y$

$(b^1 \dots b^m, y)$ is lin dep since it has $n+1$ vectors which is more than the dimension n. Therefore, there exists $\alpha_1 \dots \alpha_{n+1} \in \mathbb{R}$ (not 0) st

$$\sum_i^n \alpha_i b^i + \alpha_{n+1} y = 0$$

Therefore:

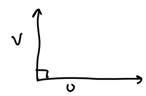
$$\alpha_{n-1} y = \sum_i^n -\alpha_i b^i$$

Suppose $\alpha_{n-1} = 0$ then $\sum_{i=1}^{m} -\alpha_i b^i = 0$ is absurd since b^i are linearly independent

Suppose $\alpha_{n-1} \neq 0$, then $y = \sum_{i=1}^{m} \frac{-\alpha_i}{\alpha_{n+1}} b^i$

orthogonality: a is $\perp$ to subspace S if $a^T x = 0$ for every $x \in S$

$u^T v = 0$



$\rightarrow m \leq n$

- if S is a subspace of $\mathbb{R}^n$ with dim m, then there exists $n-m$ lin independent vectors $\perp$ to S.
- every subspace has an orthogonal basis

col space: span of columns of A $(\mathbb{R}^m)$

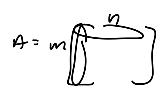
row space: span of rows of A $(\mathbb{R}^n)$

rank (A): dim $(\mathrm{Im}(A))$

↳ number of y can be expressed

Kernel: $\ker(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$ (null space)

$\mathrm{rank}(A) + \dim(\ker(A)) = n$



$Ax = \sum_j x_j \underbrace{A_j}$ column of A

Image (A): span $((A_j)_{j=1}^{n})$

if invertible ⇒ image will be whole basis

if not ⇒ image will be a subspace