6.215/6.255J/15.093J/IDS.200J Optimization Methods

Lecture 8: Large Scale Optimization

October 5, 2021

Today's Lecture

Outline

- Large scale optimization:
  - Column generation methods
  - Example: The cutting stock problem
  - Cutting plane methods
- Multi-stage optimization

Large-Scale Problems

- Consider a linear optimization problem

$$
\begin{aligned}
\min \quad & \boldsymbol{c}^T \boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{x} \geq 0
\end{aligned}
$$

- When is such a problem "large scale"?
  - the number of variables n is too large to represent explicitly; and/or
  - the number of constraints m is too large to store in system memory.
- What to do?

Consider the linear optimization problem with n large as the **master** problem

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

Column generation idea:

- consider I a subset of $1, \ldots, n$.
- find optimal basis $\boldsymbol{B}$ (with simplex method) for **restricted** problem

$$\begin{aligned} \min \quad & \textstyle\sum_{i \in I} c_i x_i \\ \text{s.t.} \quad & \textstyle\sum_{i \in I} \boldsymbol{A_i} x_i = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

- find a new column $\boldsymbol{A_j}$, $j \notin I$ with negative reduced cost $\overline{c}_j < 0$. If not possible $\boldsymbol{B}$ is optimal in the master problem.
- update $I := I \cup \{j\}$.

Large-Scale Problems

Meta algorithm: Column generation

A column generation iteration has two main computational tasks

1. solving the restricted problems
2. **dual feasibility problem:** verifying dual feasibility; finding a negative reduced cost

- **task 1:** done using primal simplex method since basis $\boldsymbol{B}$ remains primal feasible in the new restricted problem

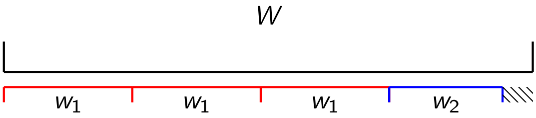
$$
\begin{array}{ll}
\min & \boldsymbol{c}^T \boldsymbol{x} \\
\text{s.t.} & \sum_{i \in I} \boldsymbol{A_i} x_i + \boldsymbol{A}_j x_j = \boldsymbol{b}, \\
 & \boldsymbol{x} \geq 0.
\end{array}
$$

- **task 2:** more critical, the master problem needs to possess some structure which enables us to find negative reduced cost efficiently (without enumerating all columns).

The Cutting Stock Problem

- Company has a supply of large rolls of paper, each of width W.
- b_i rolls of width w_i, $i = 1, \ldots, m$ need to be produced.
- Example: $W = 70$ can be cut in 3 rolls of width $w_1 = 17$ and 1 roll of width $w_2 = 15$, with a waste of

$$70 - (3 \times 17 + 1 \times 15) = 4$$



The Cutting Stock Problem

- Given $w_1, \ldots, w_m$ and W there are many cutting patterns:

  in our previous example $W = 70$, $w_1 = 17$, $w_2 = 15$, here are two patterns: $(3, 1)$ and $(2, 2)$

$$\begin{aligned} 3 \times 17 + 1 \times 15 = 66 &\leq 70 \\ 2 \times 17 + 2 \times 15 = 64 &\leq 70 \end{aligned}$$

- In general a pattern is defined as m integers $(a_1, \ldots, a_m)$ such that

$$\sum_{i=1}^{m} a_i w_i \leq W$$

- But ... there can be a huge number of such patterns. Too many to enumerate.

The Cutting Stock Problem

Problem definition

- Input: Given $w_i,\ b_i,\ i = 1, \ldots, m$ (b_i number of rolls of width w_i demanded by customers), and W (width of large rolls available to the company):

- Problem: Find how to minimize the number of large rolls in order to meet the customers demand.

The Cutting Stock Problem

A concrete example

- $W = 70, w_1 = 20, w_2 = 11, b_1 = 12, b_2 = 17$
- There are 15 feasible patterns:

$$\binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \binom{0}{1}, \binom{1}{1}, \binom{2}{1}, \binom{0}{2}, \binom{1}{2}, \binom{2}{2}, \binom{0}{3}, \binom{1}{3}, \binom{0}{4}, \binom{1}{4}, \binom{0}{5}, \binom{0}{6}$$

- $x_1, \ldots, x_{15} = \#$ of patterns of type $\binom{1}{0}, \ldots, \binom{0}{6}$, respectively

$$\begin{aligned} \min \quad & x_1 + \cdots + x_{15} \\ \text{s.t.} \quad & x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \cdots + x_{15} \begin{pmatrix} 0 \\ 6 \end{pmatrix} \geq \begin{pmatrix} 12 \\ 17 \end{pmatrix} \\ & x_1, \ldots, x_{15} \geq 0 \end{aligned}$$

- rounding or other ad hoc method may be necessary to get integer solution.

The Cutting Stock Problem

A concrete example

$$\begin{aligned} \min \quad & x_1 + \cdots + x_{15} \\ \text{s.t.} \quad & x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \cdots + x_{15} \begin{pmatrix} 0 \\ 6 \end{pmatrix} \geq \begin{pmatrix} 12 \\ 17 \end{pmatrix} \\ & x_1, \ldots, x_{15} \geq 0 \end{aligned}$$

Example:

$$2 \begin{pmatrix} 0 \\ 6 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 5 \end{pmatrix} + 4 \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 17 \end{pmatrix} \quad \text{7 rolls used}$$

$$4 \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 4 \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 17 \end{pmatrix} \quad \text{9 rolls used}$$

The Cutting Stock Problem

Formulation

Decision variables:
x_j = number of rolls cut by pattern j characterized by vector $\boldsymbol{A_j}$.

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} \boldsymbol{A_j} \cdot x_j \geq \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = \boldsymbol{b} \\ & x_j \geq 0 \quad \forall j \quad \text{(integer)} \end{aligned}$$

- Huge number of variables.
- Let us consider the linear optimization problem relaxation (removing the integrality restriction).
- Can we apply column generation, that is generate the patterns $\boldsymbol{A_j}$ on the fly?

The Cutting Stock Problem

Algorithm

Idea: Generate feasible patterns as needed.

1. Start with initial patterns (m of them):

$$\begin{pmatrix} \lfloor \frac{W}{w_1} \rfloor \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \lfloor \frac{W}{w_2} \rfloor \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \lfloor \frac{W}{w_3} \rfloor \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \lfloor \frac{W}{w_m} \rfloor \end{pmatrix}$$

2. Solve:

$$\begin{array}{ll} \min & x_1 + \cdots + x_m \\ \text{s.t.} & x_1 \boldsymbol{A_1} + \cdots + x_m \boldsymbol{A_m} = \boldsymbol{b} \\ & x_i \geq 0, \ 1 \leq i \leq m \end{array}$$

3. Compute reduced costs $\overline{c}_j = 1 - \boldsymbol{p}^T \boldsymbol{A_j}$ for all patterns j

   If $\overline{c}_j \geq 0$ current set of patterns optimal
   If $\overline{c}_s < 0 \Rightarrow x_s$ needs to enter basis

Question: How to compute $\overline{c}_j = 1 - \boldsymbol{p}^T \boldsymbol{A_j}$ for all j? (huge number !!)

The Cutting Stock Problem

Algorithm - computing reduced costs $\overline{c}_j = 1 - \boldsymbol{p}^T \boldsymbol{A_j}$ for all j

Key idea: Solve the following problem (integer knapsack) problem:

$$\begin{aligned} z^* = \max \quad & \sum_{i=1}^{m} p_i a_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} w_i a_i \leq W \\ & a_i \geq 0, \quad \text{integer} \end{aligned}$$

- If $z^* \leq 1 \quad \Rightarrow \quad 1 - \boldsymbol{p}^T \boldsymbol{A_j} > 0 \; \forall j \quad \Rightarrow$ current solution optimal
- If $z^* > 1 \quad \Rightarrow \quad \exists\, s$: $1 - \boldsymbol{p}^T \boldsymbol{A_s} < 0 \quad \Rightarrow$ Variable x_s becomes basic, i.e., a new pattern $\boldsymbol{A_s}$ will enter the basis.
- Perform min-ratio test and update the basis (revised simplex method)

The Cutting Stock Problem

Solving the integer knapsack problem

Using dynamic programming:

$$\begin{aligned} F(u) = \max \quad & p_1 a_1 + \cdots + p_m a_m \\ \text{s.t.} \quad & w_1 a_1 + \cdots + w_m a_m \leq u \\ & a_i \geq 0, \ \text{integer} \end{aligned}$$

- For $u \leq w_{min}$, $F(u) = 0$.
- For $u \geq w_{min}$

$$F(u) = \max_{i=1,\ldots,m} \{p_i + F(u - w_i)\}$$

why ?

The Cutting Stock Problem

Solving the integer knapsack problem - example

$$\begin{array}{ll} \max & 11x_1 + 7x_2 + 5x_3 + x_4 \\ \text{s.t.} & 6x_1 + 4x_2 + 3x_3 + x_4 \leq 25 \\ & x_i \geq 0, \quad x_i \text{ integer} \end{array}$$

- $F(0) = 0$ (There is nothing to cut)
- $F(1) = 1$ (Only w_4 can be cut)
- $F(2) = 1 + F(1) = 2$ (Only w_4 can be cut)
- $F(3) = \max(5 + F(0), 1 + F(2)) = 5$ (Best cut w_3)
- $F(4) = \max(7 + F(0), 5 + F(1), 1 + F(3)) = 7$ (Best cut w_2)
- $F(5) = \max(7 + F(1), 5 + F(2), 1 + F(4)) = 8$ (Cut w_2 or w_4)
- $F(6) = \max(11 + F(0), 7 + F(2), 5 + F(3), 1 + F(5)) = 11$ (Best cut off w_1)
- $F(7) = \max(11 + F(1), 7 + F(3), 5 + F(4), 1 + F(6)) = 12$ (All w_i are equal)
- $F(8) = \max(11 + F(2), 7 + F(4), 5 + F(5), 1 + F(7)) = 14$ (Best cut off w_2)
- $F(9) = \max(11 + F(3), 7 + F(5), 5 + F(6), 1 + F(8)) = 16$ (Cut w_1 or w_3)
- $F(10) = \max(11 + F(4), 7 + F(6), 5 + F(7), 1 + F(9)) = 18$ (Cut w_1 or w_2)
- $F(u) = 11 + F(u - 6) = 16 \qquad u \geq 11$

$\Rightarrow F(25) = 11 + F(19) = 11 + 11 + F(13) = 11 + 11 + 11 + F(7) = 33 + 12 = 45$

$$\boxed{x^* = (4, 0, 0, 1)}$$

Cutting Plane Methods

- Consider the dual of a primal in standard form:

$$\begin{array}{rrcl} \min & \boldsymbol{c}^T\boldsymbol{x} & & \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} & = & \boldsymbol{b} \\ & \boldsymbol{x} & \geq & 0 \end{array} \quad \Rightarrow \quad \begin{array}{rrcl} \max & \boldsymbol{p}^T\boldsymbol{b} & & \\ \text{s.t.} & \boldsymbol{p}^T\boldsymbol{A} & \leq & \boldsymbol{c}^T \end{array}$$

- Large n in primal $\Rightarrow$ large number of constraints in dual
- Let I a subset of $\{1, \ldots, n\}$ and solve the relaxed dual problem:

$$\begin{array}{rl} \max & \boldsymbol{p}^T\boldsymbol{b} \\ \text{s.t.} & \boldsymbol{p}^T\boldsymbol{A_i} \leq c_i, \ i \in I \end{array}$$

Cutting Plane Methods

- Let I a subset of $\{1, \ldots, n\}$ and solve the relaxed dual problem:

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- If optimal solution of relaxed problem $\boldsymbol{p}^*$ satisfies **all** constraints of the original problem, then it is optimal for the original problem
- If optimal solution of relaxed problem is infeasible for the original problem, bring a violated constraint into I
- Method needs:
  - a way to check feasibility
  - a way to identify violated constraints (the "separation problem")
    (one possibility is to solve $\min_i \{c_i - \boldsymbol{p}^{*T} \boldsymbol{A_i}\}$ over all i.)
- Cutting planes for dual = Column generation for primal

Multi-Stage Optimization

Example

Problem:

	Wrenches	Pliers	Cap.
Steel (lbs)	1.5	1.0	27,000
Molding machine (hrs)	1.0	1.0	21,000
Assembly machine (hrs)	0.3	0.5	9,000
Demand limit (tools/day)	15,000	16,000	
Contribution to earnings ($/1000 units)	$125	$100	

Formulation:

W: # wrenches; P # pliers ($\times$1000)

$$\begin{aligned} \text{max} \quad & 125W + 100P \\ \text{s.t.} \quad & W \leq 15 \\ & P \leq 16 \\ & 1.5W + P \leq 27 \\ & W + P \leq 21 \\ & 0.3W + 0.5P \leq 9 \\ & W, P \geq 0 \end{aligned}$$

Problem:

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Steel (lbs)	1.5	1.0	**27,000**
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 & P \leq 16 \\
 & 1.5W + P \leq 27 \\
 & W + P \leq 21 \\
 & 0.3W + 0.5P \leq 9 \\
 & W, P \geq 0
\end{array}
$$

Random data:

- Assembly capacity:
$$\begin{cases} 8000 & \text{with probability } 0.5 \\ 10,000 & \text{with probability } 0.5 \end{cases}$$
- Contribution from wrenches:
$$\begin{cases} 160 & \text{with probability } 0.5 \\ 90 & \text{with probability } 0.5 \end{cases}$$

Multi-Stage Optimization

Time dynamics: A two-stage problem

Decisions

- **Stage 1:** Need to decide steel capacity in the current quarter. Cost \$58/1000lbs.
  Soon after uncertainty (assembly cap. and wrenches contrib.) will be resolved.
- **Stage 2:** Next quarter, company will decide production quantities.

Possible states in the second stage

State	Cap.	W. contr.	Prob.
1	8,000	160	0.25
2	10,000	160	0.25
3	8,000	90	0.25
4	10,000	90	0.25

Decision variables: S: steel capacity,
$P_i, W_i : i = 1, \dots, 4$ production plan under state i.

Formulation

$$
\begin{array}{lll}
\max & & -58S + 0.25Z_1 + 0.25Z_2 + 0.25Z_3 + 0.25Z_4 \\
\text{s.t.} & \text{Ass. 1} & 0.3W_1 + 0.5P_1 \leq 8 \\
& \text{Mol. 1} & W_1 + P_1 \leq 21 \\
& \text{Ste. 1} & -S + 1.5W_1 + P_1 \leq 0 \\
& \text{W.d. 1} & W_1 \leq 15 \\
& \text{P.d. 1} & P_1 \leq 16 \\
& \text{Obj. 1} & -Z_1 + 160W_1 + 100P_1 = 0
\end{array}
$$

and

$$
\begin{array}{lrl}
\text{s.t.} & \text{Ass. 2} & 0.3W_2 + 0.5P_2 \leq 10 \\
& \text{Mol. 2} & W_2 + P_2 \leq 21 \\
& \text{Ste. 2} & -S + 1.5W_2 + P_2 \leq 0 \\
& \text{W.d. 2} & W_2 \leq 15 \\
& \text{P.d. 2} & P_2 \leq 16 \\
& \text{Obj. 2} & -Z_2 + 160W_2 + 100P_2 = 0
\end{array}
$$

and

$$
\begin{array}{lrl}
\text{s.t.} & \text{Ass. } 3 & 0.3W_3 + 0.5P_3 \le 8 \\
 & \text{Mol. } 3 & W_3 + P_3 \le 21 \\
 & \text{Ste. } 3 & -S + 1.5W_3 + P_3 \le 0 \\
 & \text{W.d. } 3 & W_3 \le 15 \\
 & \text{P.d. } 3 & P_3 \le 16 \\
 & \text{Obj. } 3 & -Z_3 + 90W_3 + 100P_3 = 0
\end{array}
$$

and finally

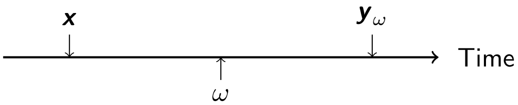
$$
\begin{array}{lrl}
\text{s.t.} & \text{Ass. 4} & 0.3W_4 + 0.5P_4 \leq 10 \\
& \text{Mol. 4} & W_4 + P_4 \leq 21 \\
& \text{Ste. 4} & -S + 1.5W_4 + P_4 \leq 0 \\
& \text{W.d. 4} & W_4 \leq 15 \\
& \text{P.d. 4} & P_4 \leq 16 \\
& \text{Obj. 4} & -Z_4 + 90W_4 + 100P_4 = 0 \\
& & S \geq 0; W_i, P_i \geq 0, 1 \leq i \leq 4
\end{array}
$$

Solution: $S = 27,250$lb.

	W_i	P_i
1	15,000	4,750
2	15,000	4,750
3	12,500	8,500
4	5,000	16,000

Multi-Stage Optimization

Two-stage optimization problems



- First stage decisions: $\boldsymbol{x}$:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \qquad \boldsymbol{x} \geq 0$$

- Random scenarios indexed by $\omega = 1, \ldots, K$.

scenario ω has probability α_ω

- Second stage decisions: $\boldsymbol{y}_\omega$: $\omega = 1, \ldots, K$.

$$\boldsymbol{B}_\omega \boldsymbol{x} + \boldsymbol{D}_\omega \boldsymbol{y}_\omega = \boldsymbol{d}_\omega, \qquad \boldsymbol{y}_\omega \geq 0$$

- Objective: Find $\boldsymbol{x}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_K$ so at to minimize the "expected cost":
- Formulation

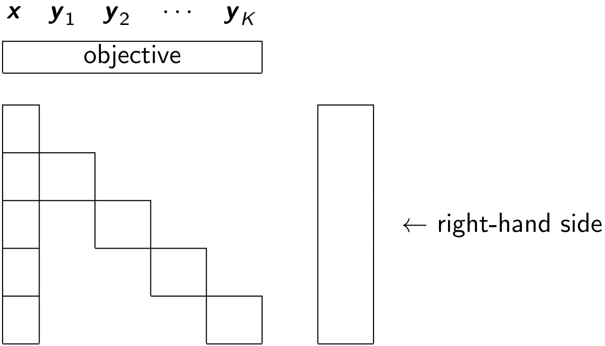
$$
\begin{array}{llllll}
\min & \boldsymbol{c}^T\boldsymbol{x} + & \alpha_1 \boldsymbol{f}_1^T \boldsymbol{y}_1 + & \cdots + & \alpha_k \boldsymbol{f}_K^T \boldsymbol{y}_K & \\
& \boldsymbol{A}\boldsymbol{x} & & & & = \boldsymbol{b} \\
& \boldsymbol{B}_1\boldsymbol{x} + & \boldsymbol{D}_1\boldsymbol{y}_1 & & & = \boldsymbol{d}_1 \\
& \boldsymbol{B}_2\boldsymbol{x} + & & \boldsymbol{D}_2\boldsymbol{y}_2 & & = \boldsymbol{d}_2 \\
& \vdots & & & \ddots & \vdots \\
& \boldsymbol{B}_K\boldsymbol{x} + & & & \boldsymbol{D}_K\boldsymbol{y}_K & = \boldsymbol{d}_K \\
& \boldsymbol{x}, & \boldsymbol{y}_1, & \boldsymbol{y}_2, \ \ldots & \boldsymbol{y}_K & \geq 0.
\end{array}
$$

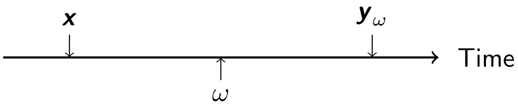
- Note that even if the number of scenarios K is moderate, this formulation can have a lot of variables, but it has a nice structure ...

Multi-Stage Optimization

Two-stage optimization problems

Structure of the formulation





Observation:

- Given a first-stage decision $\boldsymbol{x}$, the second-stage problems can all be solved separately.
- Given $\boldsymbol{x}$ and a realization ω, the optimal second-stage cost is

$$
\begin{aligned}
z_\omega(\boldsymbol{x}) := \min \quad & \boldsymbol{f}^T \boldsymbol{y}_\omega \\
\text{s.t.} \quad & \boldsymbol{D}_\omega \boldsymbol{y}_\omega = \boldsymbol{d}_\omega - \boldsymbol{B}_\omega \boldsymbol{x} \\
& \boldsymbol{y}_\omega \geq 0
\end{aligned}
$$

- To find the optimal first-stage decision $\boldsymbol{x}$ we need to solve

$$
\begin{aligned}
\min \quad & \boldsymbol{c}^T \boldsymbol{x} + \sum_{\omega=1}^{K} \alpha_\omega z_\omega(\boldsymbol{x}) \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{x} \geq 0
\end{aligned}
$$

Use the dual formulation of the second-stage problems:

- Strong duality

$$\begin{aligned} z_\omega(\boldsymbol{x}) = \quad & \max \quad \boldsymbol{p}_\omega^T(\boldsymbol{d}_\omega - \boldsymbol{B}_\omega \boldsymbol{x}) \\ & \text{s.t.} \quad \boldsymbol{p}_\omega^T \boldsymbol{D}_\omega \leq \boldsymbol{f}^T \end{aligned}$$

  holds when dual feasible set $\mathcal{P}_\omega := \left\{\boldsymbol{p} : \ \boldsymbol{p}^T \boldsymbol{D}_\omega \leq \boldsymbol{f}^T\right\}$ is nonempty and bounded.

- In that case, we also know that

$$z_\omega(\boldsymbol{x}) = \max_i \ \boldsymbol{p}_{\omega,i}^T(\boldsymbol{d}_\omega - \boldsymbol{B}_\omega \boldsymbol{x})$$

  where $\boldsymbol{p}_{\omega,i}$ are the corners of the dual polytope P_ω.

Reformulation of the first-stage problem:

$$
\begin{aligned}
\min \quad & \boldsymbol{c}^T \boldsymbol{x} + \textstyle\sum_{\omega=1}^{K} \alpha_\omega z_\omega \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{p}_{\omega,i}^T(\boldsymbol{d}_\omega - \boldsymbol{B}_\omega \boldsymbol{x}) \le z_\omega \quad \forall i, \omega \\
& \boldsymbol{x} \ge 0
\end{aligned}
$$

Observation:

- In this reformulation the number of variables has been reduced significantly.
- Potential huge number of constraints $\Rightarrow$ cutting plane methods.

We can check feasibility $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}})$ very easily by computing the second stage costs

$$
\begin{aligned}
z_\omega(\bar{\boldsymbol{x}}) := \min \quad & \boldsymbol{f}^T \boldsymbol{y}_\omega & = \quad \max \quad & \boldsymbol{p}_\omega^T(\boldsymbol{d}_\omega - \boldsymbol{B}_\omega \bar{\boldsymbol{x}}) \\
\text{s.t.} \quad & \boldsymbol{D}_\omega \boldsymbol{y}_\omega = \boldsymbol{d}_\omega - \boldsymbol{B}_\omega \bar{\boldsymbol{x}} & \text{s.t.} \quad & \boldsymbol{p}_\omega \text{ free} \\
& \boldsymbol{y}_\omega \geq 0 & & \boldsymbol{p}_\omega^T \boldsymbol{D}_\omega \leq \boldsymbol{f}^T
\end{aligned}
$$

to get $z_\omega(\bar{\boldsymbol{x}})$ and corresponding optimal dual $\boldsymbol{p}_\omega^\star$ for all ω.

Possibilities:

- $\bar{z}_\omega \geq z_\omega(\bar{\boldsymbol{x}})$ for all ω; $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}})$ is feasible.
- $\bar{z}_\omega < z_\omega(\bar{\boldsymbol{x}})$ for some ω; We add a violating constraint by demanding

$$z_\omega \geq (\boldsymbol{p}_\omega^\star)^T(\boldsymbol{d}_\omega - \boldsymbol{B}_\omega \bar{\boldsymbol{x}}) \;\; \ell$$

1. Solve relaxed master problem

$$
\begin{aligned}
\min \quad & \boldsymbol{c}^T \boldsymbol{x} + \textstyle\sum_{\omega=1}^{K} \alpha_\omega z_\omega \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{p}_{\omega,i}^T (\boldsymbol{d}_\omega - \boldsymbol{B}_\omega \boldsymbol{x}) \leq z_\omega \quad \forall (i,\omega) \in \mathcal{C}_\ell \\
& \boldsymbol{x} \geq 0
\end{aligned}
$$

The sets $\mathcal{C}_\ell$ contain some but not all constraints. Primal solution $(\boldsymbol{x}_\ell, \boldsymbol{z}_\ell)$.

2. Check feasibility $(\boldsymbol{x}_\ell, \boldsymbol{z}_\ell)$ by checking for all ω that

$$z_{\ell,\omega} \geq z_\omega(\boldsymbol{x}_\ell) := \min \quad \boldsymbol{f}^T \boldsymbol{y}_\omega \\ \text{s.t.} \quad \boldsymbol{D}_\omega \boldsymbol{y}_\omega = \boldsymbol{d}_\omega - \boldsymbol{B}_\omega \boldsymbol{x}_\ell \\ \boldsymbol{y}_\omega \geq 0$$

   with dual optimal basic solution $\boldsymbol{p}_{\ell,\omega}$.
3. If so then solution $(\boldsymbol{x}_\ell, \boldsymbol{z}_\ell)$ is optimal
4. Otherwise select a violating ω_ℓ.
5. Optimal dual solution $\boldsymbol{p}^\star_{\omega_\ell} = \boldsymbol{p}_{\omega_\ell, i_\ell}$ is a corner of the dual feasible set.
   $\Rightarrow$ Add constraint : $\mathcal{C}_{\ell+1} = \mathcal{C}_\ell \cup \{(\omega_\ell, i_\ell)\}$.
6. Resolve updated relaxed master problem