# 6.215/6.255J/15.093J/IDS.200J: Optimization Methods

Problem Set 1

Due: September 28, 2021

**A few questions will require content from the simplex lectures. They will be specified**

**Problem 1: Rocket control problem. (15 points)** Consider a rocket that travels along a straight path. Let $x_t, v_t$, and $a_t$ be the position, velocity, and acceleration of the rocket at time $t$, respectively. By discretizing time and by taking the time increment to be unity, we obtain an approximate discrete-time model of the form:

$$x_{t+1} = x_t + v_t$$
$$v_{t+1} = v_t + a_t.$$

We assume that the acceleration is under our control, which is controlled by the rocket thrust. In a rough model, the magnitude $|a_t|$ of the acceleration is proportional to the rate of fuel consumption at time $t$.

Suppose that the rocket is initially at rest at the origin, *i.e.*, $x_0 = 0$ and $v_0 = 0$. We wish the rocket to take off and "land softly" at distance $d$ unit after $T$ time units, *i.e.*, $x_T = d$ and $v_T = 0$. The total fuel consumption of the rocket, given by $\sum_{t=0}^{T-1} c_t|a_t|$ (where $c_1, \ldots, c_{T-1}$ are positive numbers known to us), cannot be more than available amount of fuel $f$. To ensure a smooth trajectory, we want to ensure that the acceleration of the rocket does not change too abruptly, *i.e.*, $|a_{t+1} - a_t|$ is always less than or equal to some known value $\delta$.

Now, we want to control the rocket in a manner to minimize the maximum thrust required, which is $\max_{t\in\{0,\ldots,T-1\}} |a_t|$, subject to the preceding constraints.

(a) Provide a linear programming formulation for this rocket control problem.

(b) Formulate and solve the model in `Julia` for $T = 100$, $d = 50$, $\delta = 10^{-3}$, $f = 1000$, and $c_0 = \ldots = c_{T-1} = 1$. Plot acceleration, velocity, and position of the rocket vs time. Attach the code.

**Problem 2: Reformulation as a linear programming problem. (12 points)** (a) Consider the problem

$$\begin{array}{ll} \text{minimize} & c_1x_1 + c_2|x_2 - 10| \\ \text{subject to} & c_3|x_1 + 2| + c_4|x_2| \leq 5, \end{array}$$

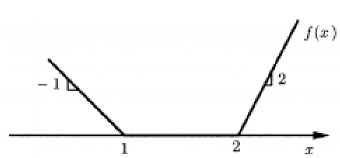


**Figure 1:** The function $f$ of Problem 2(b)

where $x_1, x_2$ are the decision variables, and $c_1, \ldots, c_4$ are problem data. Provide the range of values for $c_1, \ldots, c_4$ so that we can formulate the problem above as a linear programming problem, and provide that formulation.

(b) Consider the problem of minimizing a cost function of the form $c'x + f(d'x)$, subject to the linear constraints $Ax \geq b$, where $c, d \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, and $A \in \mathbf{R}^{m \times n}$ are the problem data, and $x \in \mathbf{R}^n$ is the decision variable. The function $f : \mathbf{R} \to \mathbf{R}$ is specified in Figure 1. Provide a linear programming formulation for the problem.

**Problem 3: Range of a matrix for non-negative inputs. (10 points)** Recall that range of a matrix $A \in \mathbf{R}^{m \times n}$ is defined as, $\mathbf{ran}(A) = \{Ax \mid x \in \mathbf{R}^n\}$. One interpretation of $\mathbf{ran}(A)$ is that it is the set of vectors that can be "hit" by the linear mapping $y = f(x) = \sum_{i=1}^n x_i A_i$, where $A_i$ is the $i$th column of $A$. The range of a matrix for nonnegative inputs can be defined as:

$$\mathbf{ran}_+(A) = \{Ax \mid x \in \mathbf{R}^n, x \geq 0\}.$$

Show that any element of $\mathbf{ran}_+(A)$ can be expressed in the form $\sum_{i=1}^n x_i A_i$,with $x_i \geq 0$, and at most $m$ of the coefficients $x_i$ being nonzero.

*Hint: consider the polyhedron:*

$$C = \{x \in \mathbf{R}^n \mid y = Ax, x \geq 0\}.$$

**Problem 4: True or false? (18 points)** Consider the standard form polyhedron $P = \{x \mid Ax = b, x \geq 0\}$. Suppose that the matrix $A$ has dimensions $m \times n$ and that its rows are linearly independent. For each of the statements below, state whether it is true or false. If true, provide an informal justification (no formal proof required), else, provide a counter example.

(a) If $n = m + 1$, then $P$ has at most two basic feasible solutions.

(b) The set of all optimal solutions is bounded.

(c) At every optimal solution, no more than $m$ variables can be positive.

(d) If there is more than one optimal solution, then there are uncountable many optimal solution.

(e) If there are several optimal solutions, then there exist at least two basic feasible solutions that are optimal.

(f) Consider the problem of minimizing $\max\{c'x, d'x\}$, over the set $P$. If this problem has an optimal solution, it must have an optimal solution which is an extreme point of $P$.

**Problem 5: Find the unknown parameters. (10 points) Will require content from simplex lectures** While solving a standard form problem, we arrive at the following tableau with $x_3, x_4$, and $x_5$ being the basic variables.

| | | | | | |
|---|---|---|---|---|---|
| -10 | $\delta$ | -2 | 0 | 0 | 0 |
| 4 | -1 | $\eta$ | 1 | 0 | 0 |
| 1 | $\alpha$ | -4 | 0 | 1 | 0 |
| $\beta$ | $\gamma$ | 3 | 0 | 0 | 1 |

The entries $\alpha, \beta, \gamma, \delta, \eta$ in the tableau are unknown parameters. For each of the following statements, find some parameter values that will make the statement true.

(a) The current solution is optimal and there are multiple optimal solutions.

(b) The optimal cost is $-\infty$.

(c) The current solution is feasible but not optimal.

**Problem 6: Simple simplex. (15 points)** **Will require content from simplex lectures** Consider the problem

$$\begin{array}{ll} \text{minimize} & -2x_1 - x_2 \\ \text{subject to} & x_1 - x_2 \leq 2 \\ & x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0, \end{array}$$

where $x_1, x_2$ are the decision variables.

(a) Convert the problem into standard form and construct a basic feasible solution at which $(x_1, x_2) = (0, 0)$.

(b) Carry out full tableau implementation of the simplex method, starting with the basic feasible solution of part (a).