# 6.215/6.255J/15.093J/IDS.200J Optimization Methods

Lecture 4: The Simplex Method II

September 21, 2021

# Today's Lecture

## Outline

- Review - Simplex method
- Dealing with degeneracy
- Revised Simplex method
- The full tableau implementation
- Finding an initial BFS
- The complete algorithm
- Computational efficiency

# Review ...

LO in standard form, $A$ full row rank, Basis, Reduced costs

$$\begin{aligned} \min \quad & \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

$\boldsymbol{x}^{\boldsymbol{T}} = (\boldsymbol{x}_B^T, \boldsymbol{x}_N^T)$, $\boldsymbol{x}_B$ basic variables, $\boldsymbol{x}_N$ non-basic variables

$$\begin{gathered} \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{A} = [\boldsymbol{B}|\boldsymbol{N}] \\ \Rightarrow \boldsymbol{B}\boldsymbol{x}_B + \boldsymbol{N}\boldsymbol{x}_N = \boldsymbol{b} \\ \Rightarrow \boldsymbol{x}_B + \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_N = \boldsymbol{B}^{-1}\boldsymbol{b} \\ \Rightarrow \boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_N \end{gathered}$$

$$\begin{aligned} z \quad & = \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} = \boldsymbol{c}_B^T\boldsymbol{x}_B + \boldsymbol{c}_N^T\boldsymbol{x}_N \\ & = \boldsymbol{c}_B^T(\boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}N\boldsymbol{x}_N) + \boldsymbol{c}_N^T\boldsymbol{x}_N \\ & = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} + (\boldsymbol{c}_N^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{N})\boldsymbol{x}_N \end{aligned}$$

$$\boxed{\bar{c}_j = c_j - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A}_j \quad \forall j \in N \quad \text{the relevant reduced costs}}$$

# Recap ... The Simplex method

1. Start with basis $\boldsymbol{B} = [\boldsymbol{A}_{B(1)}, \dots, \boldsymbol{A}_{B(m)}]$ and a BFS $\boldsymbol{x}$.
2. Compute *reduced costs*: $\bar{c}_j = c_j - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{A}_j$, $\forall j \in N$
   - If $\bar{c}_j \geq 0$, $\forall j \in N$; $\boldsymbol{x}$ optimal; stop.
   - Else select $j : \bar{c}_j < 0$.
3. Compute *basic direction*: $d_j = 1$, $\boldsymbol{d}_B = -\boldsymbol{B}^{-1}\boldsymbol{A}_j$.
   - If $\boldsymbol{d}_B \geq 0 \Rightarrow$ cost unbounded; stop
   - Else
4. Greedy step-size: $\theta^* = \min\limits_{1 \leq i \leq m, d_{B(i)} < 0} \dfrac{x_{B(i)}}{-d_{B(i)}} \doteq \dfrac{x_{B(\ell)}}{-d_{B(\ell)}}$
5. Form a new basis $\bar{\boldsymbol{B}}$ by replacing $\boldsymbol{A}_{B(\ell)}$ with $\boldsymbol{A}_j$.
6. New BFS $\boldsymbol{y} = \boldsymbol{x} + \theta^* \boldsymbol{d}$. $y_j = \theta^*$, $y_{B(i)} = x_{B(i)} + \theta^* d_{B(i)}$, $i \neq \ell$.

**Theorem**

- $P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x} \geq 0\} \neq \emptyset$
- *Assume that every BFS non-degenerate*

  *Then:*
- *Simplex method terminates after a finite number of iterations*
- *At termination, we have an optimal basis B or we have a direction* $\boldsymbol{d} : \boldsymbol{A}\boldsymbol{d} = 0, \boldsymbol{d} \geq 0, \boldsymbol{c}^T\boldsymbol{d} < 0$ *and optimal cost is* $-\infty$.

- $\theta^*$ can equal zero (why?)

  $\Rightarrow \boldsymbol{y} = \boldsymbol{x}$, although $\bar{\boldsymbol{B}} \neq \boldsymbol{B}$.

- Even if $\theta^* > 0$, there might be a tie for

$$\min_{1 \le i \le m, d_{B(i)} < 0} \frac{x_{B(i)}}{-d_{B(i)}}$$

  $\Rightarrow$ next BFS degenerate.

- Conclusion: Finite termination not guaranteed; cycling is possible.

# The Simplex method

## Avoiding cycling

- Cycling can be avoided by carefully selecting which variables enter and exit the basis.

- One example:
  - among all variables $\bar{c}_j < 0$, pick the smallest subscript;
  - among all variables eligible to exit the basis, pick the one with the smallest subscript.

1. Start with (feasible) basis $\boldsymbol{B} = [\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(m)}]$ and $\boldsymbol{B}^{-1}$
2. Compute $\boldsymbol{p}^{\boldsymbol{T}} = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}$, $\bar{c}_j = c_j - \boldsymbol{p}^{\boldsymbol{T}} \boldsymbol{A}_j$ for all (nonbasic indices) $j$.
   - If $\bar{c}_j \geq 0$ for all $j$; $\boldsymbol{x}$ optimal; Stop.
   - Else select $j : \bar{c}_j < 0$.
3. Compute $\boldsymbol{u} \doteq -\boldsymbol{d}_B = \boldsymbol{B}^{-1} \boldsymbol{A}_j$. (Note this is just $u_i = -d_B(i)$)
   - If $\boldsymbol{u} \leq 0 \Rightarrow$ cost unbounded; Stop
   - Else
4. $\theta^* = \min\limits_{1 \leq i \leq m, u_i > 0} \dfrac{x_{B(i)}}{u_i} \doteq \dfrac{x_{B(\ell)}}{u_\ell}$
5. Form a new basis $\bar{\boldsymbol{B}}$ by replacing $\boldsymbol{A}_{B(\ell)}$ with $\boldsymbol{A}_j$.
6. $y_j = \theta^*$, $y_{B(i)} = x_{B(i)} - \theta^* u_i$, $i \neq \ell$.
7. Efficiently compute $\bar{\boldsymbol{B}}^{-1}$ by transforming $[\boldsymbol{B}^{-1} | \boldsymbol{u}]$ into $[\bar{\boldsymbol{B}}^{-1} | \boldsymbol{e}_\ell]$ (where $\boldsymbol{e}_\ell$ is the unit vector in $\Re^m$ with a 1 in its $\ell^{th}$ row)

# Revised Simplex

## Step 7: Updating the inverse of a matrix - how?

- Suppose that we start at a BFS with basic indices $B$ and basis matrix

$$\boldsymbol{B} = [\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(m)}] \quad \text{with inverse} \quad \boldsymbol{B}^{-1}.$$

- We have a simplex iteration in which $B(\ell)$ leaves in favor of $j \notin B$.

- New basic indices $\bar{B} = (B(1), \ldots, B(\ell-1), j, B(\ell+1), \ldots, B(m))$ with basis matrix

$$\bar{\boldsymbol{B}} = [\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(\ell-1)}, \boldsymbol{A}_j, \boldsymbol{A}_{B(\ell+1)}, \ldots, \boldsymbol{A}_{B(m)}].$$

- How do we compute the inverse of $\bar{\boldsymbol{B}}$?

# Revised Simplex

## Updating the inverse of a matrix - explanation

- It turns out that $\boldsymbol{B}^{-1}$ is a close approximation to $\bar{\boldsymbol{B}}^{-1}$:

$$\boldsymbol{B}^{-1}\bar{\boldsymbol{B}} = [\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{\ell-1}, \boldsymbol{u}, \boldsymbol{e}_{\ell+1}, \ldots, \boldsymbol{e}_m]$$

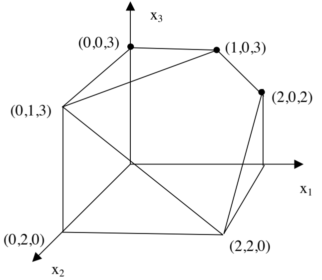
- At most $m$ "row operations" necessary to transform $\boldsymbol{B}^{-1}\bar{\boldsymbol{B}}$ into an identity matrix $\boldsymbol{I}$ (in matrix form this corresponds to finding $\boldsymbol{Q}$ so that $\boldsymbol{Q}\boldsymbol{B}^{-1}\bar{\boldsymbol{B}} = \boldsymbol{I}$, which then implies that $\bar{\boldsymbol{B}}^{-1} = \boldsymbol{Q}\boldsymbol{B}^{-1}$)

- The same row operations convert $[\boldsymbol{B}^{-1}|\boldsymbol{u}]$ into $[\bar{\boldsymbol{B}}^{-1}|\boldsymbol{e}_\ell]$

# Revised Simplex

## Back to our example

$$
\begin{array}{llllll}
\min & x_1 + & 5x_2 & -2x_3 & & \\
\text{s.t.} & x_1 + & x_2 + & x_3 & \leq 4 \\
 & x_1 & & & \leq 2 \\
 & & & x_3 & \leq 3 \\
 & & 3x_2 + & x_3 & \leq 6 \\
 & x_1, & x_2, & x_3 & \geq 0
\end{array}
$$



# Revised Simplex

## Back to our example

$\mathrm{B} = \{\boldsymbol{A}_1, \boldsymbol{A}_3, \boldsymbol{A}_6, \boldsymbol{A}_7\}$, BFS: $\boldsymbol{x} = (2,0,2,0,0,1,4)^T$

$$\boldsymbol{B} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{B}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

$\bar{\boldsymbol{c}}^T = (0,7,0,2,-3,0,0)$

$(u_1, u_2, u_3, u_4)^T = \boldsymbol{B}^{-1}\boldsymbol{A}_5 = (1,-1,1,1)^T$

$\theta^* = \min\left(\frac{2}{1}, \frac{1}{1}, \frac{4}{1}\right) = 1$

$\Rightarrow \boldsymbol{A}_6$ exits the basis ($\ell = 3$, $B(3) = 6$)

# Revised Simplex

## Back to our example

"...Efficiently compute $\bar{\boldsymbol{B}}^{-1}$ by transforming $[\boldsymbol{B}^{-1}|\boldsymbol{u}]$ into $[\bar{\boldsymbol{B}}^{-1}|\boldsymbol{e}_\ell]$..."

$$[\boldsymbol{B}^{-1}|\boldsymbol{u}] = \left[\begin{array}{rrrr|r} 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & -1 \\ -1 & 1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \bar{\boldsymbol{B}}^{-1} = \left[\begin{array}{rrrr} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

# Revised Simplex

## Updating the inverse of a matrix - practical issues

- **Numerical Stability**
  $\boldsymbol{B}^{-1}$ needs to be computed from scratch once in a while, as errors accumulate

- **Sparsity**
  $\boldsymbol{B}^{-1}$ is represented in terms of sparse triangular matrices (LU decomposition).

# Full tableau implementation

Instead of simply maintaining and updating $\boldsymbol{B}^{-1}$, we maintain and update the $m \times (n+1)$ matrix $\boldsymbol{B}^{-1}[\boldsymbol{b}|\boldsymbol{A}]$, called the simplex tableau.

Augmenting it with a top row (the *zeroth row*), we have:

| | |
|---|---|
| $-\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b}$ | $\boldsymbol{c}^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A}$ |
| $\boldsymbol{B}^{-1}\boldsymbol{b}$ | $\boldsymbol{B}^{-1}\boldsymbol{A}$ |

or, in more detail,

$$
\begin{array}{c|ccc}
-\boldsymbol{c}_B^T\boldsymbol{x}_B & \bar{c}_1 & \ldots & \bar{c}_n \\
\hline
x_{B(1)} & | & & | \\
\vdots & \boldsymbol{B}^{-1}\boldsymbol{A}_1 & \ldots & \boldsymbol{B}^{-1}\boldsymbol{A}_n \\
x_{B(m)} & | & & |
\end{array}
$$

# Full tableau implementation

## Example 3.5

$$\begin{array}{lrcrcrcl} \min & -10x_1 & - & 12x_2 & - & 12x_3 & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & 2x_3 & \leq & 20 \\ & 2x_1 & + & x_2 & + & 2x_3 & \leq & 20 \\ & 2x_1 & + & 2x_2 & + & x_3 & \leq & 20 \\ & \multicolumn{7}{l}{x_1, x_2, x_3 \geq 0} \end{array}$$

$$\begin{array}{lrcrcrcrcrcrcl} \min & -10x_1 & - & 12x_2 & - & 12x_3 & & & & & & & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & & & & & = & 20 \\ & 2x_1 & + & x_2 & + & 2x_3 & & & + & x_5 & & & = & 20 \\ & 2x_1 & + & 2x_2 & + & x_3 & & & & & + & x_6 & = & 20 \\ & \multicolumn{13}{l}{x_1, \ldots, x_6 \geq 0} \end{array}$$

BFS: $\boldsymbol{x} = (0, 0, 0, 20, 20, 20)^T$
$\boldsymbol{B}=[\boldsymbol{A}_4, \boldsymbol{A}_5, \boldsymbol{A}_6]$

# Full tableau implementation

## Example 3.5

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|---|
| | 0 | $-10$ | $-12$ | $-12$ | 0 | 0 | 0 |
| $x_4 =$ | 20 | 1 | 2 | 2 | 1 | 0 | 0 |
| $x_5 =$ | 20 | 2* | 1 | 2 | 0 | 1 | 0 |
| $x_6 =$ | 20 | 2 | 2 | 1 | 0 | 0 | 1 |

$$\bar{\mathbf{c}}^T = \mathbf{c}^T - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A} = (-10, -12, -12, 0, 0, 0)$$

# Full tableau implementation

## Example 3.5

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|---|
| | 100 | 0 | $-7$ | $-2$ | 0 | 5 | 0 |
| $x_4 =$ | 10 | 0 | 1.5 | $1^*$ | 1 | $-0.5$ | 0 |
| $x_1 =$ | 10 | 1 | 0.5 | 1 | 0 | 0.5 | 0 |
| $x_6 =$ | 0 | 0 | 1 | $-1$ | 0 | $-1$ | 1 |

# Full tableau implementation

## Example 3.5

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|---|
| | 120 | 0 | $-4$ | 0 | 2 | 4 | 0 |
| $x_3 =$ | 10 | 0 | 1.5 | 1 | 1 | $-0.5$ | 0 |
| $x_1 =$ | 0 | 1 | $-1$ | 0 | $-1$ | 1 | 0 |
| $x_6 =$ | 10 | 0 | 2.5* | 0 | 1 | $-1.5$ | 1 |

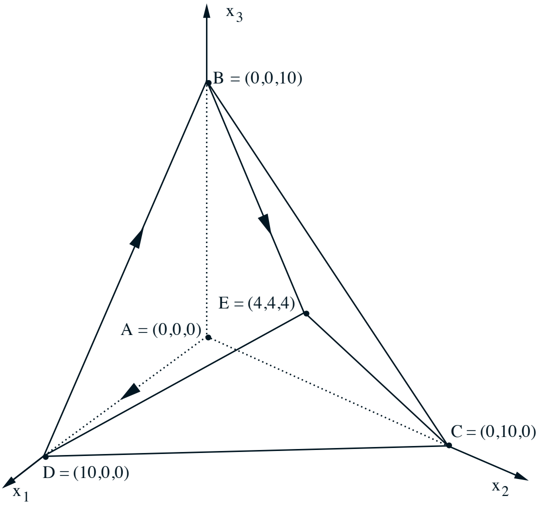
# Full tableau implementation

## Example 3.5

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|---|
| | 136 | 0 | 0 | 0 | 3.6 | 1.6 | 1.6 |
| $x_3 =$ | 4 | 0 | 0 | 1 | 0.4 | 0.4 | $-0.6$ |
| $x_1 =$ | 4 | 1 | 0 | 0 | $-0.6$ | 0.4 | 0.4 |
| $x_2 =$ | 4 | 0 | 1 | 0 | 0.4 | $-0.6$ | 0.4 |

# Full tableau implementation

## Example 3.5



# Comparison of implementations

| | Full tableau | Revised simplex |
|---|---|---|
| **Memory** | $O(mn)$ | $O(m^2)$ |
| **Worst-case time** | $O(mn)$ | $O(mn)$ |
| **Best-case time** | $O(mn)$ | $O(m^2)$ |

# "Back to Square 1": Finding an initial BFS

- **Goal:** Obtain a BFS of $\boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{x} \geq 0$
  or decide that the problem is infeasible.

- Special case: $\boldsymbol{b} \geq 0,\ \boldsymbol{Ax} \leq \boldsymbol{b},\ \boldsymbol{x} \geq 0$

$$\Rightarrow \boldsymbol{Ax} + \boldsymbol{s} = \boldsymbol{b},\ \boldsymbol{x}, \boldsymbol{s} \geq 0$$

$$\boldsymbol{s} = \boldsymbol{b},\ \boldsymbol{x} = 0$$

# Finding an initial BFS

## Artificial variables

$$\boldsymbol{Ax} = \boldsymbol{b}, \ \ \boldsymbol{x} \geq 0$$

1. Multiply some rows with $-1$ to insure $\boldsymbol{b} \geq 0$.
2. Introduce artificial variables $\boldsymbol{y}$, start with initial $\boldsymbol{y} = \boldsymbol{b}$, $\boldsymbol{x} = 0$, and apply simplex to auxiliary problem
$$\begin{array}{ll} \min & y_1 + y_2 + \ldots + y_m \\ \text{s.t.} & \boldsymbol{Ax} + \boldsymbol{y} = \boldsymbol{b} \\ & \boldsymbol{x}, \boldsymbol{y} \geq 0 \end{array}$$
3. If cost $> 0 \ \Rightarrow$ **problem infeasible**; stop.
4. If cost $= 0$ and no artificial variable is in the basis, then a BFS was found.
5. Else, all $y_i^* = 0$, but some are still in the basis. Say we have $\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(k)}$ in basis $k < m$. There are $m - k$ additional columns of $\boldsymbol{A}$ to form a basis.

# Finding an initial BFS

## Artificial variables

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq 0$$

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$$\begin{array}{ll} \min & y_1 + y_2 + \ldots + y_m \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b} \\ & \boldsymbol{x}, \boldsymbol{y} \geq 0 \end{array}$$
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5. Else, all $y_i^* = 0$, but some are still in the basis. Say we have $\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(k)}$ in basis $k < m$. There are $m - k$ additional columns of $\boldsymbol{A}$ to form a basis.
6. Drive artificial variables out of the basis: If $\ell$th basic variable is artificial examine $\ell$th row of $\boldsymbol{B}^{-1}\boldsymbol{A}$. If all elements $= 0 \Rightarrow$ row redundant. Otherwise pivot with $\neq 0$ element.

# Finding an initial BFS

## Example 3.8

$$
\begin{array}{lrcrcrcrcl}
\min & x_1 & + & x_2 & + & x_3 & & & & \\
s.t. & x_1 & + & 2x_2 & + & 3x_3 & & & = & 3 \\
 & -x_1 & + & 2x_2 & + & 6x_3 & & & = & 2 \\
 & & & 4x_2 & + & 9x_3 & & & = & 5 \\
 & & & & & 3x_3 & + & x_4 & = & 1 \\
 & \multicolumn{9}{l}{x_1, \ldots, x_4 \geq 0.}
\end{array}
$$

# Finding an initial BFS

## Example 3.8

$$
\begin{array}{rrcrcrcrcl}
\min & x_1 & + & x_2 & + & x_3 & & & & \\
s.t. & x_1 & + & 2x_2 & + & 3x_3 & & & = & 3 \\
 & -x_1 & + & 2x_2 & + & 6x_3 & & & = & 2 \\
 & & & 4x_2 & + & 9x_3 & & & = & 5 \\
 & & & & & 3x_3 & + & x_4 & = & 1 \\
\end{array}
$$

$$x_1, \ldots, x_4 \geq 0.$$

$$
\begin{array}{rrcrcrcrcrcrcrcrcl}
\min & & & & & & & & & x_5 & + & x_6 & + & x_7 & + & x_8 & & \\
s.t. & x_1 & + & 2x_2 & + & 3x_3 & & & + & x_5 & & & & & & & = & 3 \\
 & -x_1 & + & 2x_2 & + & 6x_3 & & & & & + & x_6 & & & & & = & 2 \\
 & & & 4x_2 & + & 9x_3 & & & & & & & + & x_7 & & & = & 5 \\
 & & & & & 3x_3 & + & x_4 & & & & & & & + & x_8 & = & 1 \\
\end{array}
$$

$$x_1, \ldots, x_8 \geq 0.$$

# Finding an initial BFS

## Example 3.8

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
|---|---|---|---|---|---|---|---|---|---|
| | $-11$ | 0 | $-8$ | $-21$ | $-1$ | 0 | 0 | 0 | 0 |
| $x_5 =$ | 3 | 1 | 2 | 3 | 0 | 1 | 0 | 0 | 0 |
| $x_6 =$ | 2 | $-1$ | 2 | 6 | 0 | 0 | 1 | 0 | 0 |
| $x_7 =$ | 5 | 0 | 4 | 9 | 0 | 0 | 0 | 1 | 0 |
| $x_8 =$ | 1 | 0 | 0 | 3 | 1* | 0 | 0 | 0 | 1 |

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
|---|---|---|---|---|---|---|---|---|---|
| | $-10$ | 0 | $-8$ | $-18$ | 0 | 0 | 0 | 0 | 1 |
| $x_5 =$ | 3 | 1 | 2 | 3 | 0 | 1 | 0 | 0 | 0 |
| $x_6 =$ | 2 | $-1$ | 2 | 6 | 0 | 0 | 1 | 0 | 0 |
| $x_7 =$ | 5 | 0 | 4 | 9 | 0 | 0 | 0 | 1 | 0 |
| $x_4 =$ | 1 | 0 | 0 | 3* | 1 | 0 | 0 | 0 | 1 |

# Finding an initial BFS

## Example 3.8

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
|---|---|---|---|---|---|---|---|---|---|
| | $-4$ | 0 | $-8$ | 0 | 6 | 0 | 0 | 0 | 7 |
| $x_5 =$ | 2 | 1 | 2 | 0 | $-1$ | 1 | 0 | 0 | $-1$ |
| $x_6 =$ | 0 | $-1$ | $2^*$ | 0 | $-2$ | 0 | 1 | 0 | $-2$ |
| $x_7 =$ | 2 | 0 | 4 | 0 | $-3$ | 0 | 0 | 1 | $-3$ |
| $x_3 =$ | $1/3$ | 0 | 0 | 1 | $1/3$ | 0 | 0 | 0 | $1/3$ |

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
|---|---|---|---|---|---|---|---|---|---|
| | $-4$ | $-4$ | 0 | 0 | $-2$ | 0 | 4 | 0 | $-1$ |
| $x_5 =$ | 2 | $2^*$ | 0 | 0 | 1 | 1 | $-1$ | 0 | 1 |
| $x_2 =$ | 0 | $-1/2$ | 1 | 0 | $-1$ | 0 | $1/2$ | 0 | $-1$ |
| $x_7 =$ | 2 | 2 | 0 | 0 | 1 | 0 | $-2$ | 1 | 1 |
| $x_3 =$ | $1/3$ | 0 | 0 | 1 | $1/3$ | 0 | 0 | 0 | $1/3$ |

# Finding an initial BFS

## Example 3.8

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 0 | 1 |
| $x_1 =$ | 1 | 1 | 0 | 0 | $1/2$ | $1/2$ | $-1/2$ | 0 | $1/2$ |
| $x_2 =$ | $1/2$ | 0 | 1 | 0 | $-3/4$ | $1/4$ | $1/4$ | 0 | $-3/4$ |
| $x_7 =$ | 0 | 0 | 0 | 0 | 0 | $-1$ | $-1$ | 1 | 0 |
| $x_3 =$ | $1/3$ | 0 | 0 | 1 | $1/3$ | 0 | 0 | 0 | $1/3$ |

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|---|
| | $*$ | $*$ | $*$ | $*$ | $*$ |
| $x_1 =$ | 1 | 1 | 0 | 0 | $1/2$ |
| $x_2 =$ | $1/2$ | 0 | 1 | 0 | $-3/4$ |
| $x_3 =$ | $1/3$ | 0 | 0 | 1 | $1/3$ |

# A complete algorithm for LO

**Phase I:**

1. By multiplying some of the constraints by $-1$, change the problem so that $\boldsymbol{b} \geq 0$.
2. Introduce $y_1, \dots, y_m$, if necessary, and apply the simplex method to $\min \sum_{i=1}^{m} y_i$.
3. If cost$> 0$, original problem is infeasible; STOP.
4. If cost$= 0$, a feasible solution to the original problem has been found.
5. Drive artificial variables out of the basis, potentially eliminating redundant rows.

# A complete algorithm for LO

**Phase II:**

1. Let the final basis and tableau obtained from Phase I be the initial basis and tableau for Phase II.
2. Compute the reduced costs of all variables for this initial basis, using the cost coefficients of the original problem.
3. Apply the simplex method to the original problem.

# A complete algorithm for LO

## Possible outcomes

1. Infeasible: Detected at Phase I.
2. $\boldsymbol{A}$ has linearly dependent rows: Detected at Phase I, eliminate redundant rows.
3. Unbounded (cost$= -\infty$): detected at Phase II.
4. Optimal solution: Terminate at Phase II in optimality check.

# The big-$M$ method

## An alternative method

- Similar but with a different cost function to start with ... combines the two phases into one:

$$\begin{aligned} \text{min} \quad & \sum_{j=1}^{n} c_j x_j + M \sum_{i=1}^{m} y_i \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b} \\ & \boldsymbol{x}, \boldsymbol{y} \geq 0 \end{aligned}$$

- $M$ needs to be chosen carefully ...

# Computational efficiency of the simplex method

Exceptional practical behavior: linear in $m$ or $n$

Worst case?

# Computational efficiency of the simplex method

Exceptional practical behavior: linear in $m$ or $n$

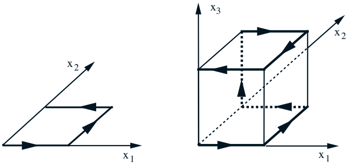
Worst case?

Consider

$$\begin{array}{ll} \max & x_n \\ \text{s.t.} & \epsilon \leq x_1 \leq 1 \\ & \epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \qquad i = 2, \ldots, n \end{array}$$

# Computational efficiency

## worst case




(a) (b)

# Computational efficiency

Theorem

- *The feasible set has* $2^n$ *vertices*
- *The vertices can be ordered so that each one is adjacent to and has lower cost than the previous one.*
- *There exists a pivoting rule under which the simplex method requires* $2^n - 1$ *changes of basis before it terminates.*