6.215/6.255J/15.093J/IDS.200J Optimization Methods

Lecture 7: Sensitivity Analysis

September 30, 2021

- Last word on duality (for now)
- Sensitivity analysis: Motivation
- Global sensitivity analysis
- Local sensitivity analysis
    - Changes in $\boldsymbol{b}$
    - Changes in $\boldsymbol{c}$
    - A new variable is added
    - A new constraint is added
    - Changes in $\boldsymbol{A}$
- Detailed example

Duality revisited

- So far: Simplex $\longrightarrow$ Duality $\longrightarrow$ Farkas lemma
  - specialized to LP, relied on a particular algorithm (simplex)
- Alternative: Separation theorem $\longrightarrow$ Farkas lemma $\longrightarrow$ Duality
  - purely geometric, generalizes to some nonlinear problems, more fundamental

Background - elements of real analysis

- **Continuous function:** A function $f : S \to \Re$ is continuous at $\boldsymbol{x} \in S$ iff

$$f(\boldsymbol{x}^i) \to f(\boldsymbol{x})$$

  for all converging sequences $\{\boldsymbol{x}^i\} \in S$ with limit $\boldsymbol{x}$.
- **Closed set:** A set $S \subset \Re^n$ is closed iff when $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots$ is a sequence of elements of S that converges to some $\boldsymbol{x} \in \Re^n$, then $\boldsymbol{x} \in S$.
- **Remarks:**
  - Level sets $\{\boldsymbol{x} : \ f(\boldsymbol{x}) \leq \alpha\}$, $\{\boldsymbol{x} : \ f(\boldsymbol{x}) \geq \alpha\}$, $\{\boldsymbol{x} : \ f(\boldsymbol{x}) = \alpha\}$ of continuous functions are closed sets for any α.
  - Every halfspace is closed.
  - Intersections of closed sets are closed.
  - Every polyhedron is closed.

Weierstrass' theorem

Theorem

If $f : \Re^n \mapsto \Re$ is a continuous function, and if S is a nonempty, closed, and bounded subset of $\Re^n$, then there exists some $\boldsymbol{x}^* \in S$ such that $f(\boldsymbol{x}^*) \leq f(\boldsymbol{x})$ for all $\boldsymbol{x} \in S$. Similarly, there exists some $\boldsymbol{y}^* \in S$ such that $f(\boldsymbol{y}^*) \geq f(\boldsymbol{x})$ for all $\boldsymbol{x} \in S$.

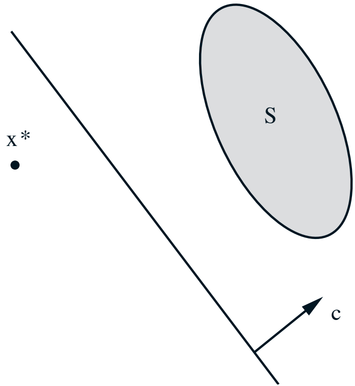
Note: Weierstrass' theorem is not valid if the set S is not closed.

(for example, consider $S = \{x \in \Re \mid 0 < x \leq 1\}$, $f(x) = x$)

Separating hyperplane theorem

Theorem

Let S be a nonempty closed convex subset of $\Re^n$ and let $\mathbf{x}^* \in \Re^n$ such that $\mathbf{x}^* \notin S$. Then there exists a vector $\mathbf{c} \in \Re^n$ such that $\mathbf{c}^T\mathbf{x}^* < \mathbf{c}^T\mathbf{x} \quad \forall \mathbf{x} \in S$.



Sensitivity Analysis - Motivation

Questions

$$\begin{aligned} z = \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

Sensitivity Analysis:

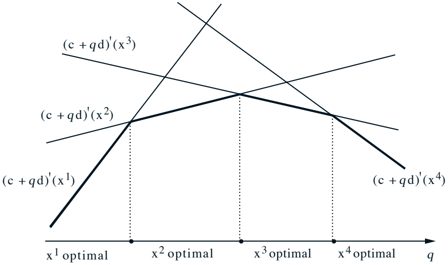
- How does z depend globally on $\boldsymbol{c}$? on $\boldsymbol{b}$?
- How does z change locally if either $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{A}$ change?
- How does z change if we add new constraints, introduce new variables?
- Importance: Insight about linear optimization and practical relevance

Global sensitivity analysis

Dependence on $\boldsymbol{c}$

$$
\begin{aligned}
G(\boldsymbol{c}) = \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{x} \geq 0
\end{aligned}
$$

- let $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^N$ be the BFSs in the feasible set (assumed nonempty)
- $\Rightarrow G(\boldsymbol{c}) = \min_{i=1,\ldots,N} \boldsymbol{c}^T \boldsymbol{x}^i$ is a concave piecewise-linear function of $\boldsymbol{c}$



Easier to see from the dual:

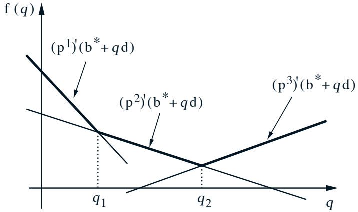
primal

$$F(\boldsymbol{b}) = \min \quad \boldsymbol{c}^T\boldsymbol{x} \quad \text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x} \geq 0$$

dual

$$F(\boldsymbol{b}) = \max \quad \boldsymbol{p}^T\boldsymbol{b} \quad \text{s.t.} \quad \boldsymbol{p}^T\boldsymbol{A} \leq \boldsymbol{c}^T$$

- let $\boldsymbol{p}^1, \boldsymbol{p}^2, \ldots, \boldsymbol{p}^N$ be the extreme points of the dual feasible set
- $\Rightarrow F(\boldsymbol{b}) = \max_{i=1,\ldots,N} \{(\boldsymbol{p}^i)^T\boldsymbol{b}\}$ is a convex piecewise-linear function of $\boldsymbol{b}$



Local sensitivity analysis

- Linear optimization in standard form:

$$\begin{aligned} z = \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

- What does it mean that a basis $\boldsymbol{B}$ is optimal?
  - feasibility conditions: $\boldsymbol{B}^{-1}\boldsymbol{b} \geq 0$
  - optimality conditions: $\boldsymbol{c}^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{A} \geq 0$

- Suppose that there is a change in either $\boldsymbol{b}$ or $\boldsymbol{c}$.

  How do we find whether $\boldsymbol{B}$ is still optimal?
  - check whether the feasibility and optimality conditions are satisfied

Local sensitivity analysis

Changes in $\boldsymbol{b}$

b_i becomes $b_i + \delta$, i.e. $\boldsymbol{b} \to \boldsymbol{b} + \delta \boldsymbol{e_i}$

$$
\begin{array}{lll}
(P) & \min & \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} \\
 & \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
 & & \boldsymbol{x} \geq 0
\end{array}
\quad \to \quad
\begin{array}{lll}
(P') & \min & \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} \\
 & \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} + \delta \boldsymbol{e_i} \\
 & & \boldsymbol{x} \geq 0
\end{array}
$$

- $\boldsymbol{B}$ optimal basis for (P)
- Is $\boldsymbol{B}$ optimal for (P')?

Local sensitivity analysis

Changes in $\boldsymbol{b}$

Need to check:

1. Feasibility: $\boldsymbol{B}^{-1}(\boldsymbol{b} + \delta \boldsymbol{e_i}) \geq 0$
2. Optimality: $\boldsymbol{c}^{\boldsymbol{T}} - \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}} \boldsymbol{B}^{-1} \boldsymbol{A} \geq 0$

Observations:

1. Changes in $\boldsymbol{b}$ affect feasibility
2. Optimality conditions are not affected

Feasibility condition:

$$\boldsymbol{B}^{-1}(\boldsymbol{b} + \delta \boldsymbol{e_i}) \geq 0$$

Define

$$\beta_{ij} = [\boldsymbol{B}^{-1}]_{ij}, \qquad \overline{b}_j = [\boldsymbol{B}^{-1}\boldsymbol{b}]_j$$

Thus,

$$[\boldsymbol{B}^{-1}\boldsymbol{b}]_j + \delta[\boldsymbol{B}^{-1}\boldsymbol{e_i}]_j \geq 0 \quad \Rightarrow \quad \overline{b}_j + \delta\beta_{ji} \geq 0 \quad \Rightarrow$$

$$\max_{\beta_{ji}>0}\left(-\frac{\overline{b}_j}{\beta_{ji}}\right) \leq \delta \leq \min_{\beta_{ji}<0}\left(-\frac{\overline{b}_j}{\beta_{ji}}\right)$$

Local sensitivity analysis

Changes in $\boldsymbol{b}$

$$\underline{\delta} \leq \delta \leq \overline{\delta}$$

Within this range

- Current basis $\boldsymbol{B}$ is optimal
- Optimal cost is affine in δ:

$$z = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}(\boldsymbol{b} + \delta \boldsymbol{e_i}) = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{b} + \delta p_i$$

- What if $\delta > \overline{\delta}$?

Local sensitivity analysis

Changes in $\boldsymbol{b}$

$$\underline{\delta} \leq \delta \leq \overline{\delta}$$

Within this range

- Current basis $\boldsymbol{B}$ is optimal
- Optimal cost is affine in δ:

$$z = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}(\boldsymbol{b} + \delta \boldsymbol{e_i}) = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{b} + \delta p_i$$

- What if $\delta > \overline{\delta}$?

$\Rightarrow$ current solution is primal infeasible, but satisfies optimality conditions (so dual feasible) $\Rightarrow$ use dual simplex method

Local sensitivity analysis

Changes in $\boldsymbol{c}$

After perturbation, c_j becomes $c_j + \delta$, i.e. $\boldsymbol{c} \to \boldsymbol{c} + \delta \boldsymbol{e_j}$.

Is current basis $\boldsymbol{B}$ still optimal?

Need to check:

- Feasibility: $\boldsymbol{B}^{-1}\boldsymbol{b} \geq 0$, unaffected
- Optimality: $\boldsymbol{c}^{\boldsymbol{T}} - \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}} \boldsymbol{B}^{-1} \boldsymbol{A} \geq 0$, affected

There are two cases:

- x_j nonbasic
- x_j basic

Local sensitivity analysis

Changes in $\boldsymbol{c}$; x_j nonbasic

x_j nonbasic:

- Then $\boldsymbol{c_B}$ are unaffected
- Solution remains optimal if

$$(c_j + \delta) - \boldsymbol{c_B^T B^{-1} A}_j \geq 0 \quad \Rightarrow \quad \overline{c}_j + \delta \geq 0$$

so solution still optimal if $\delta \geq -\overline{c}_j$

- What if $\delta < -\overline{c}_j$?

$\Rightarrow$ apply the primal simplex to the current BFS

Local sensitivity analysis

Changes in $\boldsymbol{c}$; x_j basic

x_j basic:

- Then $\boldsymbol{c_B} \to \hat{\boldsymbol{c}}_{\boldsymbol{B}} = \boldsymbol{c_B} + \delta \boldsymbol{e}_j$
- Solution optimal if for all $i \neq j$:

$$[\boldsymbol{c}^{\boldsymbol{T}} - \hat{\boldsymbol{c}}_{\boldsymbol{B}}^{\boldsymbol{T}} \boldsymbol{B}^{-1} \boldsymbol{A}]_i \geq 0 \quad \Rightarrow \quad c_i - [\boldsymbol{c_B} + \delta \boldsymbol{e}_j]^T \boldsymbol{B}^{-1} \boldsymbol{A}_i \geq 0$$

- Defining $\overline{a}_{ji} := [\boldsymbol{B}^{-1} \boldsymbol{A}]_{ji}$

$$\overline{c}_i - \delta \overline{a}_{ji} \geq 0 \quad \Rightarrow \quad \max_{\overline{a}_{ji} < 0} \frac{\overline{c}_i}{\overline{a}_{ji}} \leq \delta \leq \min_{\overline{a}_{ji} > 0} \frac{\overline{c}_i}{\overline{a}_{ji}}$$

- What if δ is outside this range?

$\Rightarrow$ apply primal simplex to current BFS

Local sensitivity analysis

A new variable is added

$$
\begin{array}{ll} \min & \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{array} \quad \rightarrow \quad \begin{array}{ll} \min & \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} + c_{n+1}\boldsymbol{x}_{n+1} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{A}_{n+1}\boldsymbol{x}_{n+1} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{array}
$$

In the new problem is $x_{n+1} = 0$ or $x_{n+1} > 0$? (i.e., is the new activity profitable?)

Current basis $\boldsymbol{B}$. Is solution $\boldsymbol{x} = \boldsymbol{B}^{-1}\boldsymbol{b}, x_{n+1} = 0$ optimal?

- Feasibility conditions are satisfied
- Optimality conditions:

$$c_{n+1} - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{A}_{n+1} \geq 0 \quad \Rightarrow \quad c_{n+1} - \boldsymbol{p}^T \boldsymbol{A}_{n+1} \geq 0?$$

- If yes, solution $\boldsymbol{x} = \boldsymbol{B}^{-1}\boldsymbol{b}, x_{n+1} = 0$ optimal
- Otherwise, use primal simplex

Local sensitivity analysis

A new constraint is added

$$\begin{array}{ll} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{array} \quad \rightarrow \quad \begin{array}{ll} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{a}_{m+1}{}^T \boldsymbol{x} = b_{m+1} \\ & \boldsymbol{x} \geq 0 \end{array}$$

If current solution feasible, it is optimal; otherwise, apply dual simplex

Local sensitivity analysis

Changes in $\boldsymbol{A}$

- Suppose $a_{ij} \to a_{ij} + \delta$
- Assume $\boldsymbol{A}_j$ does not belong in the basis
  - Feasibility conditions:

    $\boldsymbol{B}^{-1}\boldsymbol{b} \geq 0$, unaffected
  - Optimality conditions: $c_\ell - \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{A}_\ell \geq 0$, $\ell \neq j$, unaffected
  - Optimality condition: $c_j - \boldsymbol{p}^{\boldsymbol{T}}(\boldsymbol{A}_j + \delta\boldsymbol{e_i}) \geq 0 \Rightarrow \overline{c}_j - \delta p_i \geq 0$
- What if $\boldsymbol{A}_j$ is basic?

  see BT exercise 5.3

Example

A furniture company

- A furniture company makes desks, tables, chairs
- The production requires wood, finishing labor, carpentry labor

	Desk	Table	Chair	Avail.
Profit	60	30	20	-
Wood (ft)	8	6	1	48
Finish hrs.	4	2	1.5	20
Carpentry hrs.	2	1.5	0.5	8

Example

A furniture company - formulation

Decision variables:
$x_1 = \#$ desks, $x_2 = \#$ tables, $x_3 = \#$ chairs

LO formulation:

$$\begin{array}{lll}
\max & 60x_1 + 30x_2 + 20x_3 & \\
\text{s.t.} & 8x_1 + 6x_2 + x_3 \leq 48 & \text{(Wood)} \\
& 4x_1 + 2x_2 + 1.5x_3 \leq 20 & \text{(Finishing)} \\
& 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 & \text{(Carpentry)} \\
& x_1, x_2, x_3 \geq 0 &
\end{array}$$

Example

A furniture company - formulation standard form

Decision variables:

$x_1 = \#$ desks, $x_2 = \#$ tables, $x_3 = \#$ chairs

$$
\begin{array}{lll}
\max & 60x_1 + 30x_2 + 20x_3 & \\
\text{s.t.} & 8x_1 + 6x_2 + x_3 & \leq 48 \quad \text{(Wood)} \\
 & 4x_1 + 2x_2 + 1.5x_3 & \leq 20 \quad \text{(Finishing)} \\
 & 2x_1 + 1.5x_2 + 0.5x_3 & \leq 8 \quad \text{(Carpentry)} \\
 & x_1, x_2, x_3 & \geq 0
\end{array}
$$

In standard form:

$$
\begin{array}{llll}
z := & \min & -60x_1 - 30x_2 - 20x_3 & \\
 & \text{s.t.} & s_1 + 8x_1 + 6x_2 + x_3 & = 48 \quad \text{(Wood)} \\
 & & s_2 + 4x_1 + 2x_2 + 1.5x_3 & = 20 \quad \text{(Finishing)} \\
 & & s_3 + 2x_1 + 1.5x_2 + 0.5x_3 & = 8 \quad \text{(Carpentry)} \\
 & & s_1, s_2, s_3, x_1, x_2, x_3 & \geq 0
\end{array}
$$

Note: profit $= -z$

Example

A furniture company - simplex tableau(s)

Initial tableau:

		s_1	s_2	s_3	x_1	x_2	x_3
	0	0	0	0	-60	-30	-20
$s_1 =$	48	1	0	0	8	6	1
$s_2 =$	20	0	1	0	4	2	1.5
$s_3 =$	8	0	0	1	2	1.5	0.5

Final tableau:

		s_1	s_2	s_3	x_1	x_2	x_3
	280	0	10	10	0	5	0
$s_1 =$	24	1	2	-8	0	-2	0
$x_3 =$	8	0	2	-4	0	-2	1
$x_1 =$	2	0	-0.5	1.5	1	1.25	0

Example

A furniture company - information in the final tableau

Problem in standard form:

$$\begin{array}{lll} \min & -60x_1 - 30x_2 - 20x_3 & \\ \text{s.t.} & s_1 + 8x_1 + 6x_2 + x_3 & = 48 \\ & s_2 + 4x_1 + 2x_2 + 1.5x_3 & = 20 \\ & s_3 + 2x_1 + 1.5x_2 + 0.5x_3 & = 8 \\ & s_1, s_2, s_3, x_1, x_2, x_3 & \geq 0 \end{array}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 8 & 6 & 1 \\ 0 & 1 & 0 & 4 & 2 & 1.5 \\ 0 & 0 & 1 & 2 & 1.5 & 0.5 \end{pmatrix}$$

Final tableau - What is $\boldsymbol{B}$, $\boldsymbol{B}^{-1}$?

		s_1	s_2	s_3	x_1	x_2	x_3
	280	0	10	10	0	5	0
$s_1 =$	24	1	2	-8	0	-2	0
$x_3 =$	8	0	2	-4	0	-2	1
$x_1 =$	2	0	-0.5	1.5	1	1.25	0

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{pmatrix}$$

$$\boldsymbol{B}^{-1} = \begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{pmatrix}$$

What is the dual optimal solution $\boldsymbol{p}$

$$\boldsymbol{p}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1} = (0, -20, -60) \begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{pmatrix} = (0, -10, -10)$$

Example

Changes in $\boldsymbol{b}$ - furniture company - marginal cost/shadow price

Why is the optimal dual solution $p_2 = -10$ called a marginal cost or shadow price for the finishing hours constraints?

- Suppose that finishing hours become 21 (from 20).
- Currently only desks (x_1) and chairs (x_3) are produced
- Finishing and carpentry hours constraints are tight
- Does this modification (from 20 to 21) leave current basis optimal?

New solution:

$$\begin{aligned} 8x_1 + x_3 + s_1 &= 48 \\ 4x_1 + 1.5x_3 &= 21 \\ 2x_1 + 0.5x_3 &= 8 \end{aligned} \quad \Rightarrow \quad \begin{array}{|l|c} \text{New} & \text{Previous} \\ s_1 = 26 & 24 \\ x_1 = 1.5 & 2 \\ x_3 = 10 & 8 \end{array}$$

Solution change:

$$z_{\text{new}} - z_{\text{old}} = (-60 \cdot 1.5 - 20 \cdot 10) - (-60 \cdot 2 - 20 \cdot 8) = -10$$

An increase profit of 10

Example

Changes in $\boldsymbol{b}$ - furniture company - budget range for finishing hours

- Suppose finishing hours change by δ becoming $(20 + \delta)$
- Basic variables

$$\boldsymbol{x}_B = \boldsymbol{B}^{-1} \begin{pmatrix} 48 \\ 20 + \delta \\ 8 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{pmatrix} \begin{pmatrix} 48 \\ 20 + \delta \\ 8 \end{pmatrix} = \begin{pmatrix} 24 + 2\delta \\ 8 + 2\delta \\ 2 - 0.5\delta \end{pmatrix}$$

- Current basis optimal if and only if $\boldsymbol{x}_B \geq 0$ or

$$-4 \leq \delta \leq 4$$

- Note that even if current basis is optimal, optimal solution values do change:

$$\begin{aligned} s_1 &= 24 + 2\delta \\ x_3 &= 8 + 2\delta \\ x_1 &= 2 - 0.5\delta \\ z &= -60(2 - 0.5\delta) - 20(8 + 2\delta) = -280 - 10\delta \end{aligned}$$

Example

Changes in $\boldsymbol{b}$ - furniture company - reduced cost

- What does it mean that the reduced cost for x_2 is 5?
- Suppose you are forced to produce $x_2 = 1$ (1 table)
- How much will the profit decrease?

$$\begin{aligned} 8x_1 + x_3 + s_1 \quad + 6 \cdot 1 &= 48 \\ 4x_1 + 1.5x_3 \quad + 2 \cdot 1 &= 20 \\ 2x_1 + 0.5x_3 \quad + 1.5 \cdot 1 &= 8 \end{aligned} \quad \Rightarrow \quad \begin{aligned} s_1 &= 26 \\ x_1 &= 0.75 \\ x_3 &= 10 \end{aligned}$$

$$\text{new profit - old profit} = \Delta_{\text{profit}} = -(z_{\text{new}} - z_{\text{old}})$$

$$\Delta_{\text{profit}} = (30 \cdot 1 + 60 \cdot 0.75 + 20 \cdot 10) - (60 \cdot 2 + 20 \cdot 8) = 275 - 280 = -5$$

Example

Changes in $\boldsymbol{b}$ - furniture company - reduced cost

$$\begin{array}{rlll} (-\text{profit}) \leftarrow z := & \min & -60x_1 - 30x_2 - 20x_3 & & \\ & \text{s.t.} & s_1 + 8x_1 + 6x_2 + x_3 & = 48 & \text{(Wood)} \\ & & s_2 + 4x_1 + 2x_2 + 1.5x_3 & = 20 & \text{(Finishing)} \\ & & s_3 + 2x_1 + 1.5x_2 + 0.5x_3 & = 8 & \text{(Carpentry)} \\ & & s_1, s_2, s_3, x_1, x_2, x_3 & \geq 0 & \end{array}$$

Another way to calculate the same thing using optimal dual variables:
$\boldsymbol{p}^T = (0, -10, -10)$
If $x_2 = 1$

Profit effect due to producing one table	$-(-30) = +30$
Profit effect due to decrease wood by -6	$-(-6 * 0) = 0$
Profit effect due to decrease finishing hours by -2	$-(-2 * -10) = -20$
Profit effect due to decrease carpentry hours by -1.5	$-(-1.5 * -10) = -15$
Total profit effect	-5

Suppose profit from tables increases from \$30 to \$34.
Should it be produced? At \$35? At \$36?

Example

Changes in $\boldsymbol{c}$ - furniture company

- Suppose profit from desks becomes $60 + \delta$.
- For what values of δ does current basis remain optimal?
- Optimality conditions

$$c_j - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{A}_j \geq 0 \Longleftrightarrow \boldsymbol{c} - \boldsymbol{p}^T \boldsymbol{A} \geq 0$$

for dual vector

$$\boldsymbol{p}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1} = [0, -20, -(60+\delta)] \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix}$$

$$= [0, -10 + 0.5\delta, -10 - 1.5\delta]$$

- We have $c_i - \boldsymbol{p}^T \boldsymbol{A}_i = 0$ for all basic variables.
- We need to check for all non-basic variables $c_j - \boldsymbol{p}^T \boldsymbol{A}_j \geq 0$.

Example

Changes in $\boldsymbol{c}$ - furniture company

- Variables s_1, x_3, x_1 are basic
- Reduced costs of non-basic variables

$$\begin{bmatrix} \bar{c}_{x_2} \\ \bar{c}_{s_2} \\ \bar{c}_{s_3} \end{bmatrix} = \begin{bmatrix} 30 \\ 0 \\ 0 \end{bmatrix} - [0, -10 + 0.5\delta, -10 - 1.5\delta] \begin{bmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 1.5 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \bar{c}_{x_2} \\ \bar{c}_{s_2} \\ \bar{c}_{s_3} \end{bmatrix} = \begin{bmatrix} 5 + 1.25\delta \\ 10 - 0.5\delta \\ 10 + 1.5\delta \end{bmatrix}$$

- Current basis optimal if $-4 \le \delta \le 20$

Example

A new variable is added - furniture company

- Suppose the company has the opportunity to produce stools
- Profit \$ 15; requires 1 ft of wood, 1 finishing hour, 1 carpentry hour.
- Should the company produce stools?

$$
\begin{array}{lllllllll}
\max & 60x_1 & +30x_2 & +20x_3 & +15x_4 & & & & \\
 & 8x_1 & +6x_2 & +x_3 & +x_4 & +s_1 & & & = 48 \\
 & 4x_1 & +2x_2 & +1.5x_3 & +x_4 & & +s_2 & & = 20 \\
 & 2x_1 & +1.5x_2 & +0.5x_3 & +x_4 & & & +s_3 & = 8 \\
 & x_1, & x_2, & x_3, & x_4, & & & & \geq 0
\end{array}
$$

- Reduced cost $c_4 - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} A_4 = -15 - (0, -10, -10) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 5 \geq 0$
- Current basis still optimal. Do not produce stools