15.093/6.215, Fall 2017

Midterm 1, Solutions

Name:
MIT email:

Instructions :

- Do not open the exam until instructed to do so.
- You have 80 minutes to complete the exam.
- Yo may use the formula sheet given in the last page. No additional material is allowed (including course book, assignments, personal formula sheets etc.).
- You cannot use laptops, or access the Web via mobile devices during the exam. If desired, you can use a standard ("dumb") calculator, but it shouldn't be necessary.
- Violating these rules goes against MIT's academic honesty policies. We take these very seriously, and if we become aware of any violation we must enforce them to the maximum extent possible. Please do not engage in behavior that could potentially be misconstrued as violating these rules.
- There are three problems in the exam, with multiple parts each. Please plan your time carefully to ensure that you will work on all of them.

1. (32 points) Classify the following statements as true or false and justify your answer. Unless stated otherwise, all LP problems are in standard form. If the answer is False, then provide a counter-example (if relevant, a graphical picture will suffice).

   (a) The reduced cost of a basic variable is always zero.

   (b) Two different basic solutions correspond to two different bases.

   (c) If $x_j > 0$ at some optimal solution to the primal problem, then every optimal solution $\mathbf{p}$ to the dual problem satisfies $\mathbf{p}^T\mathbf{A}_j = c_j$.

   (d) Let $\mathbf{x}$ be a basic feasible solution. If x_j is zero, then it is a nonbasic variable.

   (e) If there is more than one optimal solution, then there are at least two basic feasible solutions that are optimal.

   (f) If the primal problem has an unbounded cost, then the dual problem is infeasible.

   (g) The dual of the auxiliary primal problem considered in the Phase I of the simplex method is always feasible.

   (h) Consider the following optimization problem.

$$\begin{array}{ll} \text{minimize} & \max\{-\mathbf{c}_1^T\mathbf{x}, -\mathbf{c}_2^T\mathbf{x}\} \\ \text{subject to} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \end{array}$$

   This problem can be solved as a single linear optimization problem. (If the answer is true, write down the linear optimization problem that you would solve (doesn't need to be in standard form), and explain why it is equivalent).

Solution:

(a) True: If j is the k-th basic variable, then $\mathbf{B}^{-1}\mathbf{A}_j = \mathbf{e}_k$, where $\mathbf{e}_k$ is the k-th column of the identity matrix. Thus, the reduced cost is $\bar{c}_j = c_j - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{A}_j = c_j - c_{B(k)} = c_j - c_j = 0$.

(b) True: Each basis corresponds to a basic solution of the form $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$. If two basic solutions are not equal, then they must be associated with different bases.

(c) True: This follows directly from complementary slackness for the dual constraints.

(d) False: Consider the following problem with one basic solution.

$$\begin{array}{ll} \text{minimize} & c_1x_1 + c_2x_2 \\ \text{subject to} & 1x_1 + 0x_2 = 0 \\ & 0x_1 + 1x_2 = 0 \\ & x_1, x_2 \geq 0. \end{array}$$

The only basic solution is $\mathbf{x} = (0, 0)$.

(e) False: Consider the example

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & 1x_1 + 0x_2 = 0 \\ & x_1, x_2 \geq 0. \end{array}$$

The only BFS is $\mathbf{x} = (0, 0)$, but there are infinitely many optimal solutions.

(f) True: This follows directly from weak duality.

(g) True: Phase I is always feasible and has a finite cost (since it can never be less than 0). Thus, by strong duality, the dual to Phase I always is feasible and has a finite cost.

(h) True: The formulation is as follows.

$$\begin{aligned} \text{minimize} \quad & v \\ \text{subject to} \quad & v \geq -\mathbf{c}_1^T \mathbf{x} \\ & v \geq -\mathbf{c}_2^T \mathbf{x} \\ & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & x_1, x_2 \geq 0. \end{aligned}$$

2. (34 points) Consider a school district with I neighborhoods and J school. We want to assign students to schools. Our constraints are as follows.

   1. Each school $j \in J$ can be assigned at most C_j students.
   2. The student population in neighborhood $i \in I$ is S_i, and each student must be assigned to a school.
   3. Each neighborhood $i \in I$ cannot send more than half of its students to the same school. (This is to ensure that a neighborhood's student population is not concentrated at any one school.)
   4. In each school $j \in J$, the number of students assigned from each neighborhood $i \in I$ cannot exceed half of the total number of students assigned to the school. (This is to ensure diversity in each school.)

   Finally, the distance of school j from neighborhood i is d_{ij}. Formulate a linear optimization problem (doesn't need to be in standard form) whose objective is to assign all students to schools, while minimizing the total distance traveled by all students. (You may ignore the the fact that numbers of students must be integer).

Solution: Let x_{ij} denote the number of students from neighborhood $i \in I$ that are assigned to school $j \in J$. We want to minimize total distance, hence our objective function is

$$\text{minimize} \quad \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij}.$$

The number of students from neighborhood $i \in I$ assigned to school $j \in J$ cannot be negative, hence

$$x_{ij} \geq 0, \quad \forall i \in I, j \in J.$$

For each school $j \in J$, we can assign *at most* C_j students of grade g:

$$\sum_{i \in I} x_{ij} \leq C_j, \quad \forall j \in J.$$

For each neighborhood $i \in I$, we have to assign *each* student of each grade to some school.

$$\sum_{j \in J} x_{ij} = S_i, \quad \forall i \in I.$$

For each neighborhood $i \in I$ and school $j \in J$, at most $\frac{S_i}{2}$ students from neighborhood i are assigned to school j.

$$x_{ij} \leq \frac{S_i}{2}, \quad \forall i \in I, j \in J.$$

Finally, for each neighborhood $i \in I$ and school $j \in J$, at most $\frac{\sum_{i \in I} x_{ij}}{2}$ students from neighborhood i are assigned to school j.

$$x_{ij} \leq \frac{\sum_{l \in I} x_{lj}}{2}, \quad \forall i \in I, j \in J.$$

Combining these results, our optimization problem is the following.

$$\begin{aligned}
\text{minimize} \quad & \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} \\
\text{subject to} \quad & \sum_{i \in I} x_{ij} \leq C_j, \quad \forall j \in J \\
& \sum_{j \in J} x_{ij} = S_i, \quad \forall i \in I \\
& x_{ij} \leq \frac{S_i}{2}, \quad \forall i \in I, j \in J \\
& x_{ij} \leq \frac{\sum_{l \in I} x_{lj}}{2}, \quad \forall i \in I, j \in J \\
& x_{ij} \geq 0, \quad \forall i \in I, j \in J
\end{aligned}$$

Common mistakes:

- Not introducing a new index $l \in I$ in the last constraint, and instead using the index i in both places. That is, writing the last constraint as $x_{ij} \leq \frac{\sum_{i \in I} x_{ij}}{2}$.
- Forgetting the nonnegativity constraints.
- Writing Constraint 4 as $x_{ij} \leq \frac{C_j}{2}$; these are not equivalent when the school has fewer than C_j students.

3. (34 points) Consider the following optimization problem

$$\begin{array}{lllll}
\min & -2x_1 & -2x_2 & -2x_3 & \\
\text{s.t.} & x_1 & +3x_2 & +3x_3 & \leq 5 \\
& x_1 & & +3x_3 & \leq 4 \\
& x_1, & x_2, & x_3 & \geq 0.
\end{array}$$

with the following final tableau (x_4 and x_5 are the slack variables)

	x_1	x_2	x_3	x_4	x_5
α	δ	0	4	γ	$\frac{4}{3}$
$\frac{1}{3}$	0	1	0	$\frac{1}{3}$	$-\frac{1}{3}$
4	1	β	3	0	1

(a) What are the values of α, β, and δ. Explain your answer.

(b) What is the matrix $\mathbf{B}$ of the current basis, and what is the value of γ.

(c) Suppose the RHS vector $\mathbf{b}$ changed from $(5, 4)^T$ to $(5 + \Delta, 4 + \Delta)^T$. What is the range of Δ for which the current basis remains optimal?

(d) The constraint $x_1 + x_2 \leq \frac{10}{3}$ is added to the problem. Find the new optimal solution and its value.

Note: part (d) does not require the solution to part (c).

Solution:

(a) δ is associated with the reduced cost of a basic variables and so it is zero. Since x_2 is a basic variable its column is a unit vector, and therefore, $\beta = 0$. α is minus the objective function value of the solution and hence is $\alpha = -\mathbf{c}_B \mathbf{x}_B = -(1/3 * (-2) + 4 * (-2)) = 26/3$.

(b) $\mathbf{B} = [\mathbf{A}_2, \mathbf{A}_1] = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$. Its inverse can be found under the slack variables $B^{-1} = \begin{bmatrix} 1/3 & -1/3 \\ 0 & 1 \end{bmatrix}$ and thus the reduced cost associated with variable x_4 is given by

$$\gamma = \bar{c}_4 = c_4 - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A}_4 = -(-2, -2) \begin{bmatrix} 1/3 & -1/3 \\ 0 & 1 \end{bmatrix} (1, 0)^T = (2, 2)^T \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} = 2/3$$

(c) A change in $\mathbf{b}$ will only affect feasibility, therefore we will check when the current basis is still non-negative

$$\mathbf{x}_B = \mathbf{B}^{-1} \tilde{\mathbf{b}} = \mathbf{B}^{-1} \mathbf{b} + \mathbf{B}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Delta = \begin{bmatrix} 1/3 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta \geq 0$$

Thus, for $\Delta \geq -4$ the current basis remains optimal.

(d) We first notice that the current solution does not satisfy the constraint. We therefore change the constraint to an equality by adding a slack variable x_6, so the new constraint is $x_1 + x_2 + x_6 = \frac{10}{3}$. We add this constraint as an additional row in the tableau with associated basic solution x_6.

We now must compute the new tableau. Below are two ways this can be done.

- **Approach 1**: We first calculate the new basis $\mathbf{B} = \begin{bmatrix} \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_6 \end{bmatrix}$ and its inverse.

$$\mathbf{B} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix}$$

  Using $\mathbf{B}^{-1}$, we obtain the new tableau via calculating $\mathbf{B}^{-1}\mathbf{b}$, $\mathbf{B}^{-1}\mathbf{A}$, $\mathbf{c}^T - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A}$, and $-\mathbf{c}_B^T \mathbf{x}$. This new tableau is

	x_1	x_2	x_3	x_4	x_5	x_6
$\frac{26}{3}$	0	0	4	$\frac{2}{3}$	$\frac{4}{3}$	0
$\frac{1}{3}$	0	1	0	$\frac{1}{3}$	$-\frac{1}{3}$	0
4	1	0	3	0	1	0
-1	0	0	-3	$-\frac{1}{3}$	$-\frac{2}{3}$	1

- **Approach 2**: We first add the new constraint directly onto the tableau.

	x_1	x_2	x_3	x_4	x_5	x_6
$\frac{26}{3}$	0	0	4	$\frac{2}{3}$	$\frac{4}{3}$	0
$\frac{1}{3}$	0	1	0	$\frac{1}{3}$	$-\frac{1}{3}$	0
4	1	0	3	0	1	0
$\frac{10}{3}$	1	1	0	0	0	1

  This tableau is not valid, since we do not have an identity matrix. Thus, we can subtract the first and second rows from the third row in order to obtain a valid tableau.

	x_1	x_2	x_3	x_4	x_5	x_6
$\frac{26}{3}$	0	0	4	$\frac{2}{3}$	$\frac{4}{3}$	0
$\frac{1}{3}$	0	1	0	$\frac{1}{3}$	$-\frac{1}{3}$	0
4	1	0	3	0	1	0
-1	0	0	-3	$-\frac{1}{3}$	$-\frac{2}{3}$	1

We will conduct a dual simplex iteration taking x_6 out of the basis. Since x_3 obtains the minimum of the fraction $\min\{\frac{\bar{c}_3}{|v_3|}, \frac{\bar{c}_4}{|v_4|}, \frac{\bar{c}_5}{|v_5|}\} = \min\{\frac{4}{3}, 2, 2\}$ it enters the basis.

	x_1	x_2	x_3	x_4	x_5	x_6
$\frac{22}{3}$	0	0	0	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{4}{3}$
$\frac{1}{3}$	0	1	0	$\frac{1}{3}$	$-\frac{1}{3}$	0
3	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	1
$\frac{1}{3}$	0	0	1	$\frac{1}{9}$	$\frac{2}{9}$	$-\frac{1}{3}$

The new solution is therefore $(3, \frac{1}{3}, \frac{1}{3})$ with value $-\frac{22}{3}$.

Formula Shteet

General form simplex tableau:

$-\mathbf{c}'_B\mathbf{B}^{-1}\mathbf{b}$	$\mathbf{c}' - \mathbf{c}'_B\mathbf{B}^{-1}\mathbf{A}$
$\mathbf{B}^{-1}\mathbf{b}$	$\mathbf{B}^{-1}\mathbf{A}$

Duality:

	min	max	
constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 $\gtrless 0$	variables
variables	≥ 0 ≤ 0 $\gtrless 0$	$\leq c_j$ $\geq c_j$ $= c_j$	constraints