6.255/15.093 Optimization Methods
MIT, Fall 2013

Midterm Exam

Name: ______________________________

MIT Email: ______________________________

Instructions :

- **Do not open the exam** until instructed to do so.
- You have **two hours** (academic time, 110 minutes) to complete the exam.
- You can use the course textbook, and up to two sheets of paper with formulas (or similar) that you have prepared. Nothing else is allowed (NO homework sets, NO lecture notes, NO past exams, etc).
- You *cannot* use laptops, or access the Web via mobile devices during the exam. If desired, you can use a standard ("dumb") calculator, but it shouldn't be necessary.
- Violating these rules goes against MIT's academic honesty policies. We take these very seriously, and if we become aware of any violation we must enforce them to the maximum extent possible. *Please do not engage in behavior that could potentially be misconstrued as violating these rules.*
- There are **four** problems in the exam, with multiple parts each. Please plan your time carefully to ensure that you will work on all of them.

Problem	P1	P2	P3	P4	Total
Points	30	25	30	15	100
Score					

Problem 1 (30 points) :

Classify the following statements as true or false. All answers must be *fully justified*, or no credit will be given. Unless stated otherwise, all LP problems are in standard form ($\min \mathbf{c}^T\mathbf{x} \,:\, \mathbf{A}\mathbf{x} = \mathbf{b},\, \mathbf{x} \geq \mathbf{0}$).

(a) If a primal variable has zero reduced cost, then it is a basic variable.

(b) Let $\theta \in \mathbb{R}$. The function $F(\theta) := \{\max \mathbf{c}^T\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b_1} - \theta\mathbf{b}_2\}$ is concave.

(c) The set $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1 - x_2| \leq 1 + |x_2 - x_3|\}$ can be modeled in terms of linear programming.

(d) At optimality, complementary slackness implies that if a primal variable is nonzero, then the corresponding inequality in the dual problem must be tight.

(e) If the dual problem is feasible, then so is the primal problem.

(f) The dual simplex method is particularly useful for re-solving a problem after changes in the right-hand side $\mathbf{b}$.

(g) If an LP in standard form has an optimal BFS, then so does the dual.

(h) A nonempty polyhedron in standard form always has finitely many extreme points.

(i) The reduced costs $\bar{\mathbf{c}}_j$ quantify the sensitivity of the optimal cost with respect to variations in the entries of the cost vector $\mathbf{c}$.

(j) When running the primal simplex method on a problem for which all BFS are non-degenerate, the cost of the current solution strictly decreases at every iteration.

Solution:

(a) False. Basic variables have zero reduced cost, but the converse is not necessarily true (e.g., for degenerate problems).

(b) True. This can be easily seen from the dual

$$\min_{\mathbf{p}} \, (\mathbf{b}_1 - \theta\mathbf{b}_2)^T\mathbf{p} \quad \text{s.t.} \quad \mathbf{A}^T\mathbf{p} = \mathbf{c}, \quad \mathbf{p} \geq 0.$$

By replacing the feasible set by its finitely many extreme points $\mathbf{p}_i$, we see that $F(\theta)$ is the pointwise minimum of finitely many affine functions of θ, and thus it is a concave function.

(c) False. This set is not convex, and thus it is not a polyhedron.

(d) True. This is a simple consequence of the complementary slackness condition $\mathbf{x}_i(\mathbf{A}^T\mathbf{p} - \mathbf{c})_i = 0$.

(e) False. The primal can certainly be infeasible, e.g., if the dual is unbounded.

(f) True. Changing $\mathbf{b}$ does not affect dual feasibility, and thus the dual simplex method is appropriate.

(g) True. This is the main result in linear programming, or alternatively, it follows from strong duality.

(h) True. A BFS is given by a selection of m linearly independent columns of A (for which $\mathbf{B}^{-1}\mathbf{b} \geq 0$, and there are at most $\binom{n}{m}$ of these.

(i) False. The reduced costs quantify the sensitivity with respect to changes in the values of the primal variables.

(j) True. If all BFSs are nondegenerate, the reduced costs of nonbasic variables are nonzero and the value of θ^* is strictly positive, and thus the cost strictly decreases at each step.

Problem 2 (25 points) :

Consider the following simplex tableau, corresponding to a linear programming problem in standard form (min $\mathbf{c}^T\mathbf{x}$: $\mathbf{A}\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$). All answers must be fully justified.

10	?	3	0	1	0
α	0	2	0	-1	1
1	1	-2	0	0	0
-3	0	1	?	β	0

(a) What should be the value of the two missing entries (marked with "?")? Why?

(b) What are the variables in the current basis? What is the current solution $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$? Is this solution feasible? What is the current cost?

(c) Give specific values of α and β (if they exist) for this tableau to be optimal.

(d) Give specific values of α and β (if they exist) for which the primal LP is infeasible.

(e) Give specific values of α and β (if they exist) for which the primal LP is unbounded.

(f) Set $\beta = -1$. Compute an iteration of the dual simplex method. Find conditions on α for the new tableau to be optimal. What is the optimal cost? What is the optimal primal solution $\mathbf{x}$?

Solution:

(a) The missing entries should be "0" and "1", respectively.

The first entry corresponds to the reduced cost of a basic variable, and thus should vanish. This can be easily seen from the fact that the fifth, first, and third columns of $\mathbf{B}^{-1}\mathbf{A}$ must be the columns of an identity matrix (this is the only such possibility). As a consequence, the second entry must be equal to 1.

(b) By the argument given above, the basis is (x_5, x_1, x_3). Thus, the current solution is $\mathbf{x} = (1, 0, -3, 0, \alpha)$. This is a basic solution, but it is not a BFS. The current cost is equal to -10.

(c) This tableau cannot be optimal (regardless of the values of α and β), since the current solution is never primal feasible.

(d) For the primal LP to be infeasible, the dual must be either unbounded or infeasible. Since the dual is feasible (current solution works), then the only possibility is for the dual to be unbounded. If $\beta \geq 0$, then pivoting on the last row gives a direction along which the dual cost becomes $+\infty$.

(e) For the primal LP to be unbounded, the dual must be infeasible. But for this problem, the current solution is dual feasible (reduced costs are nonnegative), and thus no such values of α and β exist.

(f) For the given data, the initial tableau is given below. The only possibility is to pivot on the marked element.

10	0	3	0	1	0
α	0	2	0	-1	1
1	1	-2	0	0	0
-3	0	1	1	-1^*	0

After pivoting on the marked element, the new tableau is:

7	0	4	1	0	0
$3+\alpha$	0	1	-1	0	1
1	1	-2	0	0	0
3	0	-1	-1	1	0

For this to be optimal, we need $\alpha \geq -3$. In this case, the new basis is (x_5, x_1, x_4), the current primal solution is $\mathbf{x} = (1, 0, 0, 3, 3 + \alpha)$, and the optimal cost is equal to -7.

Problem 3 (30 points): The coffee retailer "SunDollars Coffee, Inc." is asking for your help in designing and pricing their menu offerings. The company has three main products (brewed coffee, espresso and latte). These products have different requirements in terms of preparation time, raw materials and energy, as well as different revenue per unit sold. These are described in the following table (unspecified units are arbitrary):

	Brewed Coffee	Espresso	Latte	Limit
Time (hours)	1	5	7	240
Materials	1	2	3	140
Energy	2	5	8	400
Retail price (dollars)	2	4	7	

(a) Write an LP formulation for solving the optimal production planning problem, i.e., the product offering that maximizes sales revenue. Assume that demand is infinite, (i.e., every item that is produced will be sold), and ignore integer constraints if they arise.

(b) Write the dual of your linear programming formulation. What is the interpretation of the dual variables?

(c) Consider a primal solution where we produce 65 coffees, no espressos, and 25 lattes. Show that this solution is optimal, and compute the optimal dual solution.

(d) We are offered the option of hiring new baristas (which will increase the number of hours available). The terms of the offer are to pay a lump sum of \$10, for a total increase of 20 hours. Should we accept this offer? Why? Justify your answer.

(e) Compute the allowable range of prices for a latte, if we want the current basis to remain optimal. What happens at the extremes of this range?

(f) If we must produce 12 espressos, how would our sales revenue change? (you can safely assume the current basis remains optimal for this perturbation).

Solution:

(a) The natural LP formulation is the following:

$$\max\ 2B + 4E + 7L \qquad \text{s.t.} \qquad \begin{cases} B + 5E + 7L \leq 240 \\ B + 2E + 3L \leq 140 \\ 2B + 5E + 8L \leq 400 \\ B, E, L \geq 0 \end{cases}$$

(b) The dual problem is

$$\min\ 240p + 140q + 400r \qquad \text{s.t.} \qquad \begin{cases} p + q + 2r \geq 2 \\ 5p + 2q + 5r \geq 4 \\ 7p + 3q + 8r \geq 7 \\ p, q, r \geq 0 \end{cases}$$

(c) The proposed optimal solution is $(B, E, L) = (65, 0, 25)$. The corresponding optimal cost is then equal to 305.

To solve for the dual variables, we use complementary slackness. Notice that the structure of the primal solution tells us that the first and third dual constraints are active. The third primal inequality being loose means that $r = 0$. Thus, solving the resulting linear system

$$p + q = 2, \qquad 7p + 3q = 7,$$

we have $p = 1/4$, $q = 7/4$, i.e., the optimal dual solution is $(p, q, r) = (1/4, 7/4, 0)$. As a sanity check, we have that the optimal dual cost is $(240 + 7 * 140)/4 = 305$.

(d) The shadow price p of the time constraint is 0.25. An increase of 20 hours (from 240 to 260) would increase our sales renevue by only 5 additional dollars. Since this increase is smaller than what have to pay, we should decline the offer.

(e) Changing the latte price can only affect optimality, not feasibility. A simple way to compute this, assuming the structure of the basis is the same, is to solve the same equations as before but where now the price of the latte is θ:

$$p + q = 2, \qquad 7p + 3q = \theta.$$

which yields $p = (\theta - 6)/4$ and $q = (14 - \theta)/4$. For the range $\theta \in [6, 14]$, the first and third reduced costs are zero, and the second reduced cost is nonnegative. Thus, the current solution is optimal as long as the price of the latte is in the interval $[6, 14]$.

(f) Since E is a nonbasic variable, the cost of producing an espresso is the reduced cost of E. Since reduced costs are exactly the slacks of the dual, the second inequality yields $5p + 2q + 5r - 4 = 5 * (1/4) + 2 * (7/4) - 4 = 3/4$. Thus, if we must produce 12 espressos our cost would decrease by 9 dollars, yielding a cost of 296 instead of 305.

Problem 4 (15 points) :

An $n \times n$ matrix is called *row-stochastic* if all its entries are nonnegative and the sum of the entries in each row is equal to 1. In other words, M is row-stochastic if

$$M_{ij} \geq 0 \quad \text{for } i,j = 1,\dots,n \qquad \text{and} \qquad \sum_{j=1}^{n} M_{ij} = 1 \quad \text{for } i = 1,\dots,n.$$

Alternatively, in matrix notation, $M \geq 0$ and $M\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the vector with all entries equal to 1.

(a) Show that if M is row-stochastic, then for any vector $\mathbf{p}$ we have $(M\mathbf{p})_i \leq p_{max}$, where $p_{max} := \max_i \mathbf{p}_i$ is the largest entry of $\mathbf{p}$.

(b) Assume that M is row-stochastic. Show using LP duality that there exists a *nonzero* vector $\mathbf{x}$ with *nonnegative entries*, such that $M^T\mathbf{x} = \mathbf{x}$ (i.e., M has a nonnegative left eigenvector with associated eigenvalue 1).

Hint: Write an LP and its dual, and analyze the entries of the vector $M\mathbf{p}$ using part (a).

Remark: In the case where the matrix M describes a Markov chain, these results prove the existence of a (not necessarily unique) stationary distribution.

Solution:

(a) Since p_{max} is the largest entry of $\mathbf{p}$, we have

$$(M\mathbf{p})_i = \sum_{j=1}^{n} M_{ij}\mathbf{p}_j \leq \sum_{j=1}^{n} M_{ij}\, p_{max} = p_{max},$$

where the inequality follows from $M_{ij} \geq 0$ and the last equation from the condition $\sum_{j=1}^{n} M_{ij} = 1$.

(b) There are a few possible formulations. For instance, consider the linear program

$$\max\ \mathbf{1}^T\mathbf{x} \qquad \text{s.t.} \quad (M^T - I)\mathbf{x} = 0, \quad \mathbf{x} \geq \mathbf{0}.$$

This LP is always feasible ($\mathbf{x} = 0$ is a feasible solution), and is unbounded if and only if a nonzero $\mathbf{x}$ exists (i.e., M has a nonnegative left eigenvector with eigenvalue 1).

The corresponding dual LP is

$$\min\ \mathbf{0}^T\mathbf{p} \qquad \text{s.t.} \quad (M - I)\mathbf{p} \geq \mathbf{1}.$$

The last constraint can also be written as $M\mathbf{p} \geq \mathbf{p} + \mathbf{1}$. But this inequality cannot possibly hold, since it says that the entries of the vector $M\mathbf{p}$ are all strictly larger than those of $\mathbf{p}$, but on the other hand (by (a)) they must be bounded above by p_{max}.

Thus, the dual LP is infeasible, and as a consequence the primal is unbounded, and thus a nonnegative left eigenvector of M always exists.

END OF EXAM