Faculty of Engineering

Department of Computer Science

Dual numbers & automatic differentiation

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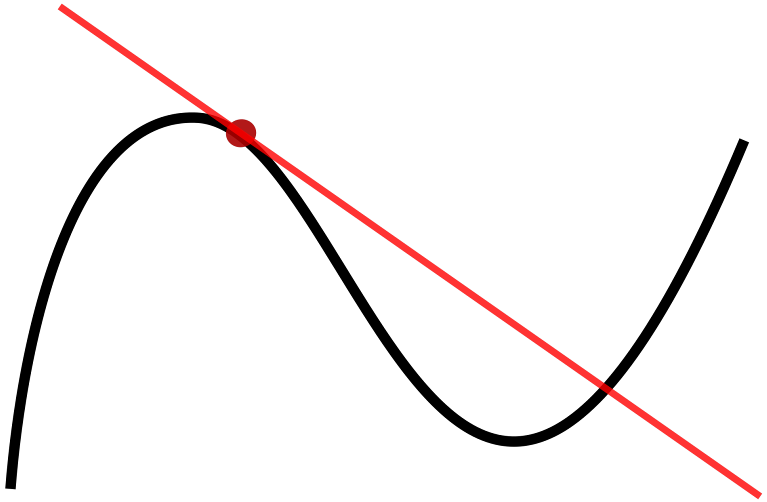
Training deep neural networks involves computing the slopes (or derivatives) of complicated functions at individual points

$$f(w_1, \ldots, w_l) = \frac{1}{2}\left(\tanh\left(w_l \tanh\left(w_{l-1} \tanh\left(\cdots w_2 \tanh(w_1 x)\right)\cdots\right)\right) - y\right)^2$$



Today we learn how to do this with the help of dual numbers

Dual Numbers

What are dual numbers?

Natural numbers

(0,) 1, 2, 3, 4, 5, 6, …

Integers

…, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, …

Rational numbers

Integers and fractions of integers e.g. 1/3, -5/11, 0, 4/2 (= 2), etc.

Real numbers

Rational numbers and irrational numbers, e.g. $\sqrt{2}$

What are dual numbers?

Real numbers

Rational numbers and irrational numbers, e.g. $\sqrt{2}$

Complex numbers

Real numbers and imaginary numbers

Complex number

$$x = a + i\,b$$

Real numbers

Imaginary unit with $i^2 = -1$

What are dual numbers?

$$x = a + i\,b$$

Complex number

Real numbers

Imaginary unit with $i^2 = -1$

Dual numbers

$$x = a + \varepsilon\,b$$

Dual number

Real numbers

Dual unit with $\varepsilon^2 = 0$, but not $\varepsilon = 0$

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Dual unit with $\varepsilon^2 = 0$, but not $\varepsilon = 0$

Matrix representation

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$ because $$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

What are dual numbers?

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Dual unit with $\varepsilon^2 = 0$, but not $\varepsilon = 0$

Example

$$xy = (a + \varepsilon b)(c + \varepsilon d) = ac + \varepsilon(ad + bc) + \underbrace{\varepsilon^2 bd}_{=0} = ac + \varepsilon(ad + bc)$$

What are dual numbers good for?

Suppose we look at the polynomial

$$3x^3 - 2x^2 + 1$$

for the dual number $x = a + \varepsilon b$. We observe

$$3x^3 - 2x^2 + 1 = 3(a + \varepsilon b)^3 - 2(a + \varepsilon b)^2 + 1$$

$$= 3(a + \varepsilon b)(a^2 + 2ab\varepsilon + \varepsilon^2 b^2) - 2(a^2 + 2ab\varepsilon + \varepsilon^2 b^2) + 1$$

$$= 3(a + \varepsilon b)(a^2 + 2ab\varepsilon) - 2(a^2 + 2ab\varepsilon) + 1$$

What are dual numbers good for?

$$
\begin{aligned}
3x^3 - 2x^2 + 1 &= 3(a + \varepsilon b)(a^2 + 2ab\varepsilon) - 2(a^2 + 2ab\varepsilon) + 1 \\
&= 3a^3 + 6a^2b\varepsilon + 3\varepsilon ba^2 + 6ab^2\varepsilon^2 - 2a^2 - 4ab\varepsilon + 1 \\
&= 3a^3 - 2a^2 + 1 + (6a^2 + 3a^2 - 4a)\varepsilon b \\
&= \underbrace{3a^3 - 2a^2 + 1}_{} + \underbrace{(9a^2 - 4a)\varepsilon b}_{}
\end{aligned}
$$

This is the function itself, but in a

What about this function?

What are dual numbers good for?

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What about this function?

The derivative(/slope) of $f(x) = 3x^2 - 2x^2 + 1$ is

$$\boxed{f'(x) = 9x^2 - 4x}$$

Is it a coincidence that $(9a^2 - 4a)b\varepsilon$ is the same as $f'(a)b\varepsilon$?

What are dual numbers good for?

The derivative(/slope) of $f(x) = 3x^2 - 2x^2 + 1$ is

$$\boxed{f'(x) = 9x^2 - 4x}$$

Is it a coincidence that $(9a^2 - 4a)b\varepsilon$ is the same as $f'(a)b\varepsilon$?

The answer is: No!

Maybe you will eventually learn this in Calculus, but there is this concept known as Taylor series, with which we can approximate differentiable functions as follows:

$$f(x) = f(y) + f'(y)(x-y) + \frac{f''(y)}{2}(x-y)^2 + \frac{f'''(y)}{6}(x-y)^3 + \cdots$$

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Maybe you will eventually learn this in Calculus, but there is this concept known as Taylor series, with which we can approximate differentiable functions as follows:

$$f(x) = f(y) + f'(y)(x-y) + \frac{f''(y)}{2}(x-y)^2 + \frac{f'''(y)}{6}(x-y)^3 + \cdots$$

If we choose $x = a + \varepsilon b$ and $y = a$, we observe

$$f(a+\varepsilon b) = f(a) + f'(a)\varepsilon b + \frac{f''(a)}{2}\varepsilon^2 b^2 + \frac{f'''(a)}{6}\varepsilon^3 b^3 + \cdots$$

$$= f(a) + \varepsilon\, b\, f'(a)$$ because $\varepsilon^2 = 0$

We can use this to teach a computer to compute derivatives(/slopes) of functions

Live demo!