(*) Let $\{a_k\}$ be the FS coefficients of the periodic signal $x(t)$ with period T. Find the FS coefficient of:

a) $x(t-t_0) + x(t+t_0)$

b) $\mathcal{E}\{x(t)\}$

c) $Re\{x(t)\}$

d) $\frac{d^2x(t)}{dt^2}$

a) Time shift property:

$$x(t-t_0) \xleftrightarrow{fs} e^{-jk(2\pi/T)t_0} a_k$$

$$x(t+t_0) \xleftrightarrow{fs} e^{+jk(2\pi/T)t_0} a_k$$

linearity: $A x(t) + B y(t) \xleftrightarrow{fs} A a_k + B b_k$

$$x(t-t_0) + x(t+t_0) \xleftrightarrow{fs} e^{-jk\omega_0 t_0} a_k + e^{+jk\omega_0 t_0} a_k$$

$$= a_k \left(e^{-jk\omega_0 t_0} + e^{jk\omega_0 t_0}\right)$$

$$= 2a_k \cos(k\omega_0 t_0)$$

$\omega_0 = 2\pi/T$

b) $Even\{x(t)\} = x_e(t) = \frac{x(t) + x(-t)}{2}$

Time Reversal property: $\begin{cases} x(-t) \xleftrightarrow{fs} a_{-k} \\ x(t) \xleftrightarrow{fs} a_k \end{cases}$

$$\frac{x(t) + x(-t)}{2} \xleftrightarrow{fs} \frac{a_k + a_{-k}}{2} = \text{Even}\{x(t)\}$$

c) $Re\{x(t)\}$

$$x(t) \xleftrightarrow{fs} a_k$$

$$x^*(t) \xleftrightarrow{fs} a^*_{-k}$$

$$Re\{x(t)\} = \frac{x(t) + x^*(t)}{2} \xleftrightarrow{fs} \frac{a_k + a^*_{-k}}{2}$$

d) $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$, $\omega_0 = 2\pi/T$ use synthesis.

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left(\sum_{-\infty}^{+\infty} a_k e^{jk\omega_0 t}\right) = \sum_{-\infty}^{+\infty} (a_k jk\omega_0) e^{jk\omega_0 t}$$

new a_k (1^{st} deri.)

$$\frac{d^2x(t)}{dt^2} = \frac{d}{dt}\left(\sum_{-\infty}^{+\infty} a_k (jk\omega_0) e^{jk\omega_0 t}\right)$$

$$= \sum_{k=-\infty}^{+\infty} (-a_k k^2 \omega_0^2) e^{jk\omega_0 t}$$

$\rightarrow b_k = -a_k k^2 \omega_0^2$, $\omega_0 = 2\pi/T$

final answer.

Conjugation property. revisit.

$$x(t) \xleftrightarrow{fs} a_k$$

$$x^*(t) \xleftrightarrow{fs} a^*_{-k}$$

if $x(t)$ is real, $a_k = a^*_{-k}$

$$a_{-k} = a^*_k$$

Rectangular : $\begin{cases} Re\{a_{-k}\} = Re\{a_k^*\} = Re\{a_k\} \\ Im\{a_{-k}\} = Im\{a^*_k\} = -Im\{a_k\} \end{cases}$

Polar : $|a_{-k}| = |a^*_{-k}| = |a_k|$

$$\measuredangle a_{-k} = \measuredangle a_k^* = -\measuredangle a_k$$

$(x+jy)^* = x-jy \qquad (re^{j\theta})^* = re^{-j\theta}$

✱ If $x(t)$ is a real & even signal, show $a_k \in \mathbb{R}$, $k \in \mathbb{Z}$

If $x(t) \in \mathbb{R}$ $\quad a_k = a^*_{-k} \quad \underline{a_{-k} = a_k^*}$ ①

If $x(t)$ is even. $\quad x(t) \xleftrightarrow{fs} a_k$

$(x(t) = x(-t)) \quad x(-t) \xleftrightarrow{fs} a_{-k}$ $\Big\} \rightarrow a_k = a_{-k}$ ②

① & ② $\quad a_k = a_{-k} = a_k^*$

since $a_k = a_k^*$, $\therefore a_k \in \mathbb{R}$ //

(Im part is 0)

✱ If $x(t)$ is a real and odd function, show a_k is purely imaginary.

If $x(t) \in \mathbb{R}$ $\quad a_k = a^*_{-k} \quad a_{-k} = a_k^*$

If $x(t)$ is odd. $\quad x(t) \xleftrightarrow{fs} a_k \quad \Big\} \rightarrow -a_k = a_{-k}$

$(x(t) = -x(-t))$ $\quad x(-t) \xleftrightarrow{fs} a_{-k}$ }

$$\underbrace{-a_k = a_k^*}\ = a_{-k}$$

multi. by -1, but still equal, so no real part.

$-a_k$ 把 Re 和 Im 都 $* -1$

a_k^* 只把 Im $* -1$ } ⇒ 但相等 ∴

If $x(t)$ is a real signal, show:

$$fs\{x_e(t)\} = Re\{a_k\}$$

$$fs\{x_o(t)\} = j\,Im\{a_k\}$$