Lecture 6: Gauss' Law

ECE221: Electric and Magnetic Fields

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Outline

1. Electrix Flux Density
2. Gauss' Law
3. Applying Gauss' Law

Fluxes and Flux Densities

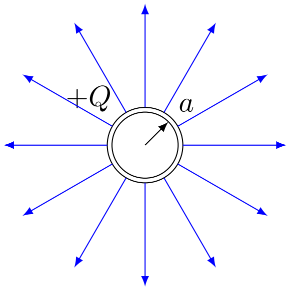
- We now consider a different class of vector fields called **flux density fields**.
- Flux denisities are expressed per unit area. Therefore we expect a unit of the form $\square/\mathrm{m}^2$.

$$\vec{D} = [C/m^2]$$

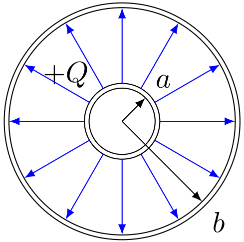
electric flux $\phi = \iint_S \vec{D}(\vec{r}) \cdot d\vec{s} \quad [C]$

if $\vec{D}$ is uniform w/ position, $\phi = DA$ A = area.

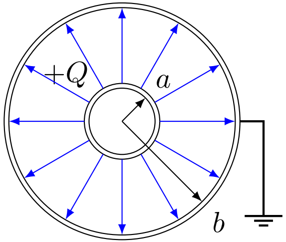
Faraday Experiment



1) Inner Sphere is charged to a known positive charge level Q

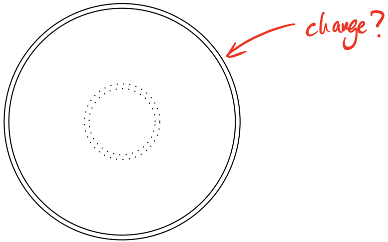


2) An outer conducting sphere, is placed around the inner sphere



3) The outer sphere is discharged by connecting it momentarily to ground

Faraday Experiment

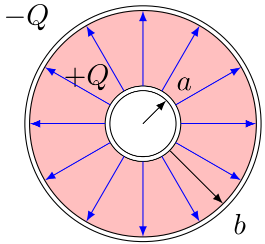


4) The outer sphere is carefully dismanted to remove the inner sphere. What is the charge left on the outer sphere?

Q

Displacement Flux Density $\boldsymbol{D}$

The result is the same even if a perfectly insulating material is placed between the two spheres!



Displacement Flux Density of a Point Charge

Let the radius of the inner sphere $a \to 0$:

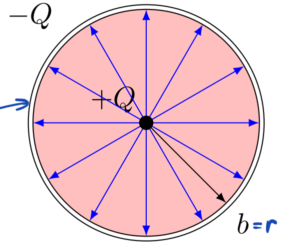
elect. flux $\psi = Q$ [C]

flux density $D = \dfrac{\psi}{A_{sphere}}$

$= \dfrac{\psi}{4\pi b^2}$

$\vec{D} = \dfrac{Q}{4\pi b^2}\hat{r}$

sphere.



$\vec{D} = \varepsilon_0 \vec{E}$

$\vec{E} = \dfrac{1}{\varepsilon_0}\vec{D} \Rightarrow \vec{E} = \dfrac{1}{\varepsilon_0}\dfrac{Q}{4\pi r^2}\hat{r}$

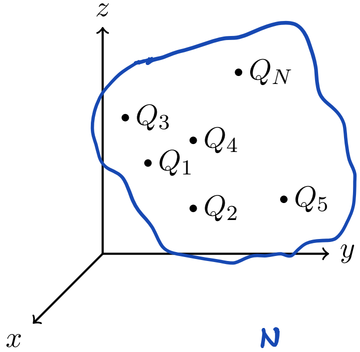
Gauss' Law

Gauss' Law states: *The electruc flux passing through any closed surface is equal to the total charge enclosed by that surface.*

$$\psi = \boxed{\oint\!\!\!\oint_S \vec{D} \cdot \vec{ds}\,' = Q_{encl.}}$$

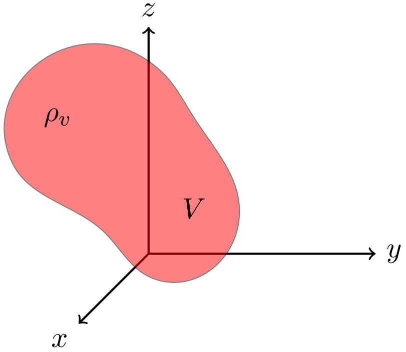
Extending Gauss' Law to Different Charge Distributions

Discrete charge distribution



$$Q_{encl.} = \sum_{n=1}^{N} Q_n$$

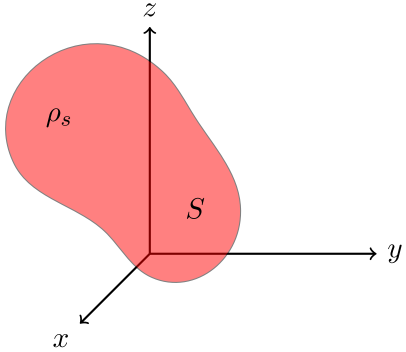
Volumetric charge distribution



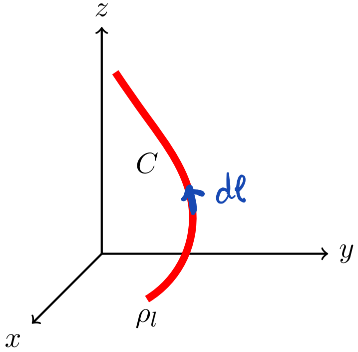
$$Q = \iiint_V \rho_v(\vec{r}\,')\, dv'$$

Extending Gauss' Law to Different Charge Distributions

Surface charge distribution



Line charge distribution



$$Q_{encl} = \iint_S \rho_s(\vec{r}\,')\, ds'$$

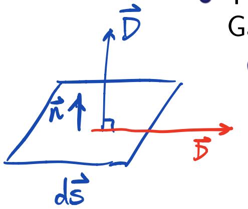
$$Q_{encl} = \int_C \rho_l(\vec{r}\,')\, dl'$$

Applying Gauss' Law

- Instead of doing superposition integrals as we did before, we will not use Gauss' Law to find $\boldsymbol{D}$, and hence $\boldsymbol{E}$.
- The solution is much easier if you chooose the closed surface S in Gauss' Law to satisfy two conditions:

Gaussian surface.

$$\oiint \vec{D} \cdot d\vec{s} = Q_{encl}.$$



1. $\boldsymbol{D}$ is either normal or tangential to the closed surface

a) if D is normal to S, then $\vec{D} \cdot d\vec{s}' = D_n ds$ (scalar multiplication)

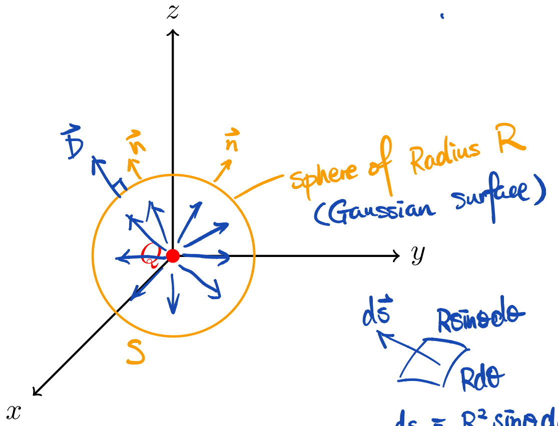
b) if D is tangential to S, then $\vec{D} \cdot d\vec{s}' = 0$

2. On that portion of the close surface for which $\boldsymbol{D} \cdot d\boldsymbol{s}'$ is not zero, $D =$ constant.

$$\int \underbrace{D(\vec{r})}_{\text{constant } D_0} \cdot d\vec{s} = D_0 \int_S d\vec{s} = D_0 A$$

Example: Field of a Point Charge

Find the electric flux density $\boldsymbol{D}$ and electric field $\boldsymbol{E}$ produced by a point charge Q.



$\vec{D}$ is constant over S

$$\oint\!\!\oint \vec{D}\cdot d\vec{s} = Q_{encl} = Q$$

$$\iint D_r \hat{r}\cdot\hat{r}\, R^2 \sin\theta\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi} D_r \hat{r}\cdot\hat{r}\, R^2 \sin\theta\, d\theta\, d\phi$$

$$= 2\pi D_r R^2 \int_0^{\pi} \sin\theta\, d\theta$$

$$= 4\pi R^2 D_r = Q \Rightarrow D_r = \frac{Q}{4\pi R^2}$$

$$ds = R^2 \sin\theta\, d\theta\, d\phi$$

$$d\vec{s} = ds\cdot\hat{n} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$$

Applying Gauss' Law

$\vec{D}_r = \frac{Q}{4\pi R^2}\hat{r}$ $\vec{E} = \vec{D}/\varepsilon_0 = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{r}$ //

matches w/ columb's law.

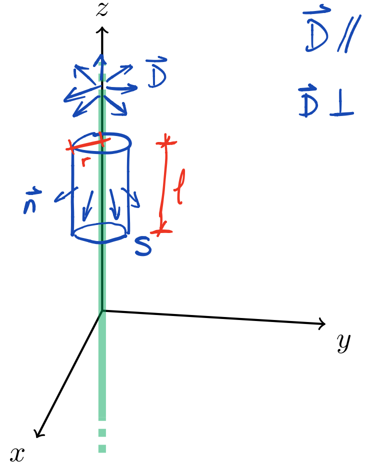
Remember try to answer similar questions to what we did for E-field calculations:

1. Which coordinate system makes the most sense to use, given the charge distribution?
2. With which coordinates does the field $\boldsymbol{D}$ vary?
3. Which components of $\boldsymbol{D}$ are present?

Use the answers to determine the best **Gaussian surface** to use for your problem.

Example: Infinite Line Charge

Determine the electric flux density of an infinitely long and uniform line charge along the z-axis. The line charge density is ρ_l C/m.



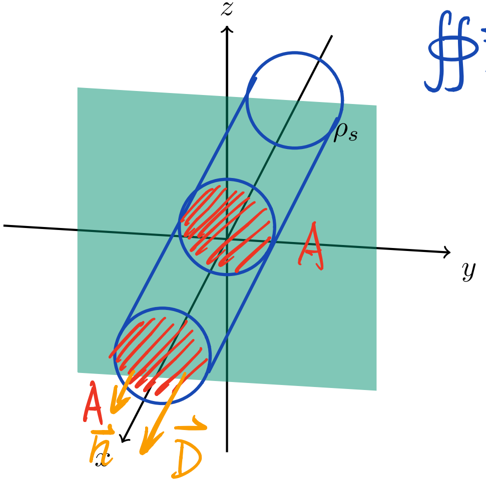
$\vec{D} \parallel d\vec{s}$ over the sides of $S \longrightarrow \vec{D}\cdot d\vec{s} = D\,ds$

$\vec{D} \perp d\vec{s}$ over top & bottom of $S \longrightarrow \vec{D}\cdot d\vec{s} = 0$

$$\oint\!\!\oint_S \vec{D}(\vec{r})\cdot d\vec{s} = \cancel{\iint_{bottom} \vec{D}(\vec{r})\, d\vec{s}\,'}^{\,0} + \iint_{side} \vec{D}(\vec{r})\, d\vec{s}\,' + \cancel{\iint_{top} \vec{D}(\vec{r})\, d\vec{s}\,'}^{\,0} = \int_{z_1}^{z_2}\int_{0}^{2\pi} D\, \underbrace{\hat{\rho}\cdot\hat{\rho}}_{1}\, \rho\, d\phi\, dz$$

Example: Infinite Surface Charge

Determine the electric flux density of an infinite sheet of charge with a uniform charge density ρ_s [C/m^2] placed on the yz plane ($x = 0$).



$$= \rho D 2\pi (z_2 - z_1) = Q_{encl.}$$

$$D = \frac{Q}{2\pi\rho L} = \frac{\rho_l L}{2\pi\rho L} = \frac{\rho_l}{2\pi\rho}$$

$$\vec{D} = \frac{\rho_l}{2\pi\rho}\hat{\rho} \rightarrow \vec{E} = \frac{\rho_l}{2\pi\varepsilon_0\rho}\hat{\rho}$$

$$\oiint \vec{D}\cdot\vec{ds} = \underbrace{\iint}_{\text{front}} + \underbrace{\cancel{\iint}}_{\cancel{\text{side}}} + \underbrace{\iint}_{\text{back}}$$

$$= Q_{encl.} = \rho_s A \; [C]$$

$$D_x A + D_x A = \rho_s A$$

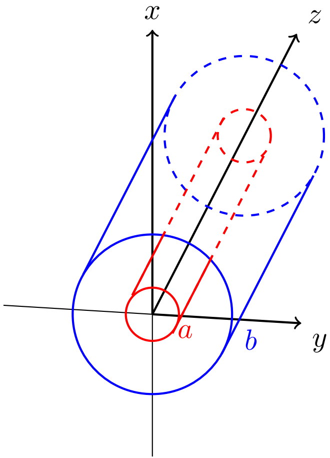
$$2D_x = \rho_s \qquad \vec{D} = \frac{\rho_s}{2}\hat{n}$$

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{n}$$

$2\varepsilon_0$

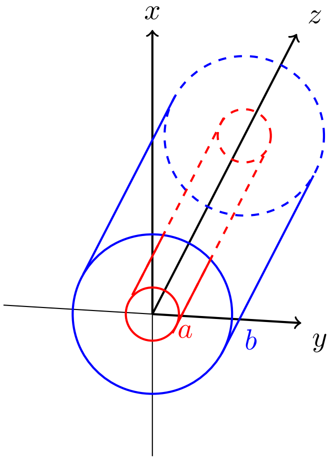
Example: Infinitely-Long Coaxial Cylinders

Determine $\boldsymbol{D}$ for an infinite coaxial arrangement of cylinders. The inner cylinder has a surface charge density of $\rho_s\ \mathrm{C/m^2}$, and the outer one, $-\rho_s\ \mathrm{C/m^2}$.



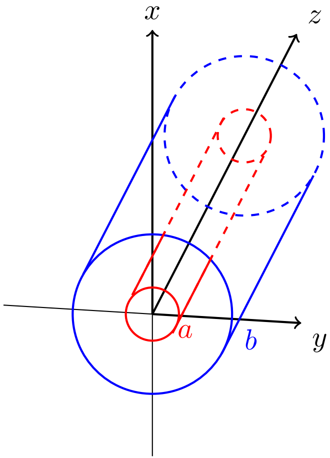
Example: Infinitely-Long Coaxial Cylinders

For $\rho < a$:



Example: Infinitely-Long Coaxial Cylinders

For $a < \rho < b$:



Example: Infinitely-Long Coaxial Cylinders

For $\rho > b$: