Lecture 16: Capacitance and Resistance

ECE221: Electric and Magnetic Fields

Prof. Sean V. Hum

Winter 2019

Capacitance

- More explicitly,

$$C = \frac{\oint_S \epsilon \boldsymbol{E} \cdot d\boldsymbol{s}}{-\int_C \boldsymbol{E} \cdot d\boldsymbol{\ell}}$$

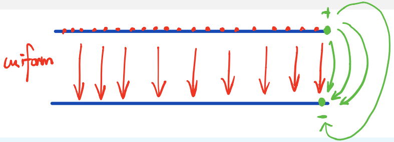
- Therefore, capacitance is **only** a function of the dielectrics and the geometry! You may assume charge and find voltage or vice versa to solve capacitance problems.
- Units of capacitance are **Farads** (F) or Coulombs/Volt (C/V).
  - $1\ \mu\mathrm{F} = 10^{-6}\ \mathrm{F}$
  - $1\ \mathrm{nF} = 10^{-9}\ \mathrm{F}$
  - $1\ \mathrm{pF} = 10^{-12}\ \mathrm{F}$

Reminder: Resistance

$$R = \frac{V}{I} = \frac{-\int_C \boldsymbol{E} \cdot d\boldsymbol{l}}{\iint_S \sigma \boldsymbol{E} \cdot d\boldsymbol{s}}$$

Parallel Plate Capacitor

Assume $A \gg d$



$\vec{E}$ @ the edge

fringing field

(ignore)

field produced by a plate is the same as an infinite one

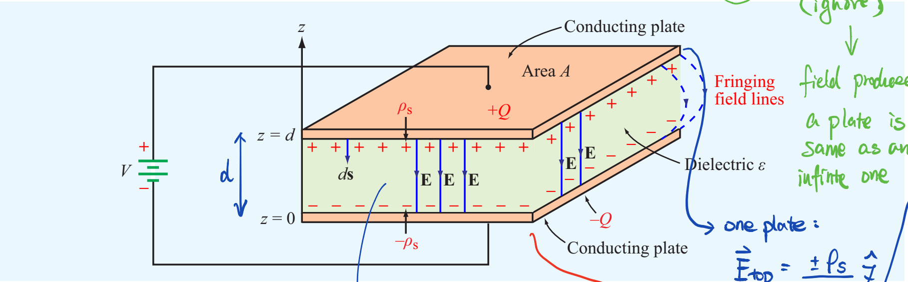


Image Credit: Ulaby and Ravaioli

one plate:

$\vec{E}_{top} = \frac{\pm \rho_s}{2\varepsilon_0}\hat{z}$

$\vec{E}_{bottom} = \frac{\mp \rho_s}{2\varepsilon_0}\hat{z}$

$\vec{E} = \vec{E}_{top} + \vec{E}_{bottom}$

$\vec{E} = \begin{cases} -\rho_s/\varepsilon_0\,\hat{z} & \text{(between the plates)} \\ 0 & \text{(above or below plates)} \end{cases}$

$V = -\int_{z=0}^{d} \vec{E}\cdot d\vec{l} = -\int \left(-\frac{\rho_s}{\varepsilon_0}\hat{z}\right)\cdot\hat{z}\,dz$

$= \frac{\rho_s}{\varepsilon_0} d$

$Q = \rho_s \cdot A$

$C = \frac{Q}{V} = \frac{\rho_s A}{\frac{\rho_s}{\varepsilon_0} d}$

plate area

$\boxed{C = \frac{A\varepsilon}{d}}$

plate distance (cannot be too small → break down)

Coaxial Capacitor

$\vec{E}$ (btw the cylinders):

$$= \frac{-\rho_l}{2\pi\varepsilon\rho}\hat{\rho}$$

$$\vec{V} = -\int_a^b \frac{-\rho_l}{2\pi\varepsilon\rho}\hat{\rho}\cdot\hat{\rho}\,d\rho$$

$$= \frac{\rho_l}{2\pi\varepsilon}\ln(b/a)$$

$$Q = \rho_l l$$

$$C = \frac{Q}{V} = \frac{\rho_l l}{\frac{\rho_l}{2\pi\varepsilon}\ln(b/a)}$$

$$\boxed{C = \frac{2\pi\varepsilon l}{\ln(b/a)}}$$

$$W_e = \frac{1}{2}\iiint_V \varepsilon|E|^2\,dV = \frac{1}{2}\left(\frac{V_0}{d}\right)^2\varepsilon A d$$

$$= \frac{1}{2}V_0^2\frac{\varepsilon A}{d}$$

$$= \frac{1}{2}CV_0^2$$

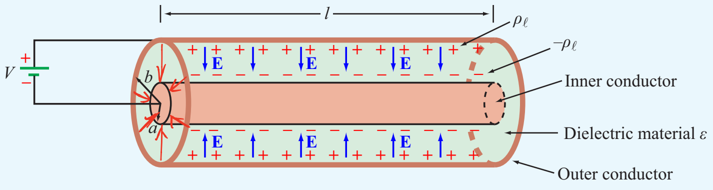


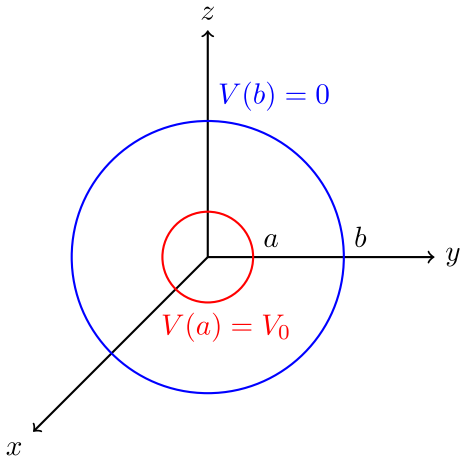
Figure 4-25 Coaxial capacitor filled with insulating material of permittivity ε (Example 4-12).

Image Credit: Ulaby and Ravaioli

Capacitance per unit length: $= C/l = \frac{2\pi\varepsilon}{\ln(b/a)} = C'$

Spherical Capacitor

$V_{ba} = V_a - V_b = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$

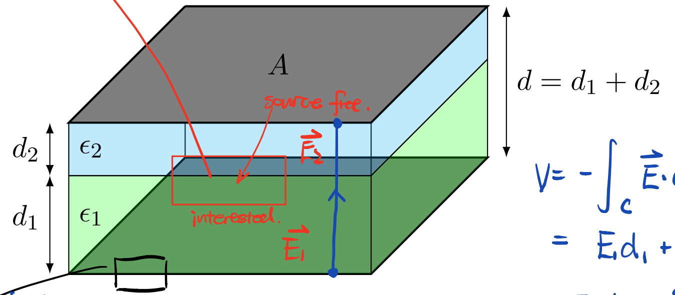


$C = \frac{Q}{V} = 4\pi\varepsilon \Big/ \left(\frac{1}{a} - \frac{1}{b}\right)$

Parallel Plate Capacitor with Multiple Stacked Dielectrics

Electrostatic B.C.s require: $D_{n_1} = D_{n_2} \Rightarrow \varepsilon_1 E_{n_1} = \varepsilon_2 E_{n_2}$

ignore fringing fields



$$V = -\int_C \vec{E}\cdot d\vec{l} = -\int_{①} - \int_{②}$$

$$= E_1 d_1 + E_2 d_2$$

$$= E_1 d_1 + \left(\frac{\varepsilon_1}{\varepsilon_2}\right) E_{n_1} d_2$$

$$E_1 = V \Big/ \left(d + \frac{\varepsilon_1}{\varepsilon_2} d_2\right)$$

$$= V \Big/ \frac{d_1}{\ldots} + \frac{d_2}{\ldots}$$

At bottom plate: (B.C.s)

$$\rho_s = D_{n_1} = \varepsilon_1 E_{n_1} = \varepsilon_1 E_1$$

$$Q = \rho_s A = \frac{VA}{d_1/\varepsilon_1 + d_2/\varepsilon_2}$$

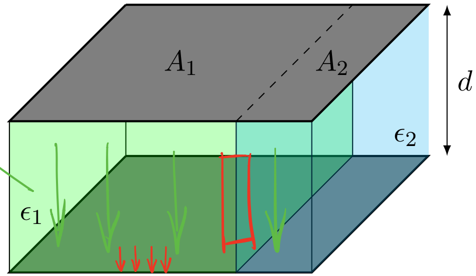
Parallel Plate Capacitor with Side-by-Side Dielectrics

$$\boxed{C = \frac{Q}{V} = \frac{A}{\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}}}$$

$$= \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

series combo of C_1 & C_2

$$\vec{E}_1 = -V_0/d \ \hat{z}$$



$\rho_{s_1} = D_1$

$$Q = \rho_{s_1} A_1 + \rho_{s_2} A_2$$

$$= \frac{\varepsilon_1 V A_1}{d} + \frac{\varepsilon_2 V A_2}{d}$$

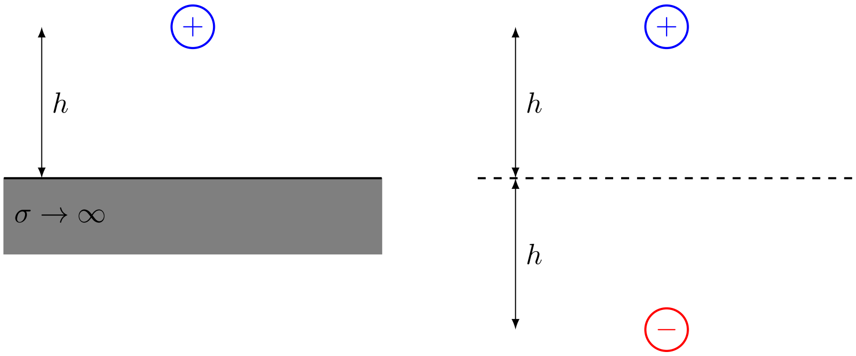
$$\rightarrow C = \frac{Q}{V} = \frac{\varepsilon_1 A_1}{d} + \frac{\varepsilon_2 A_2}{d} = C_1 + C_2$$

Apply tangential BCs ($E_{t_1} = E_{t_2}$)

$$D_{t_1} = \varepsilon_1 E_1 = \varepsilon_1 \frac{V}{d} = D_1$$

$$D_{t_2} = \varepsilon_2 E_2 = \varepsilon_2 \frac{V}{d} = D_2$$

Method of Images



Method of Images