Lecture 15: Solving Laplace's Equation and Capacitance

ECE221: Electric and Magnetic Fields

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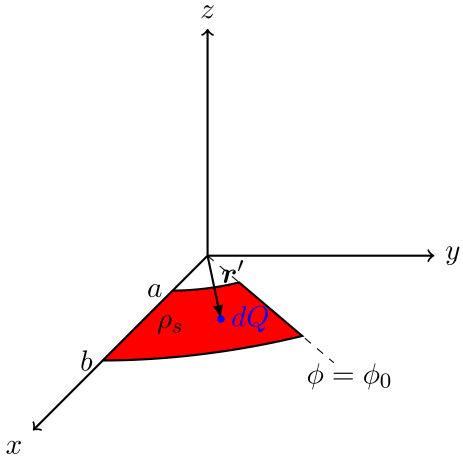
Midterm Example from Office Hours



Find $\boldsymbol{D}(0,0,0)$.

$$\rho_s = \rho_{s0}\rho$$

$$dQ = \rho_{s0}\rho' \cdot \rho' d\rho' d\phi'$$

$$d\boldsymbol{E} = \frac{dQ(\boldsymbol{r} - \boldsymbol{r}')}{4\pi\epsilon_0 |\boldsymbol{r} - \boldsymbol{r}|^3}$$

$$\boldsymbol{r} - \boldsymbol{r'} = 0 - \rho' \hat{\boldsymbol{\rho}}'$$

$$d\boldsymbol{E} = -\frac{\rho_{s0}(\rho')^3 \hat{\boldsymbol{\rho}}'}{4\pi\epsilon_0 (\rho')^3} d\rho' d\phi'$$

$$\boldsymbol{E} = -\int_0^{\phi_0}\int_a^b \frac{\rho_{s0}\hat{\boldsymbol{\rho}}'}{4\pi\epsilon_0} d\rho' d\phi'$$

$$\hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi$$

Outline

$$E = \frac{-(b-a)}{4\pi\varepsilon_0} \int_0^{\phi_0} \ell' \, d\phi'$$

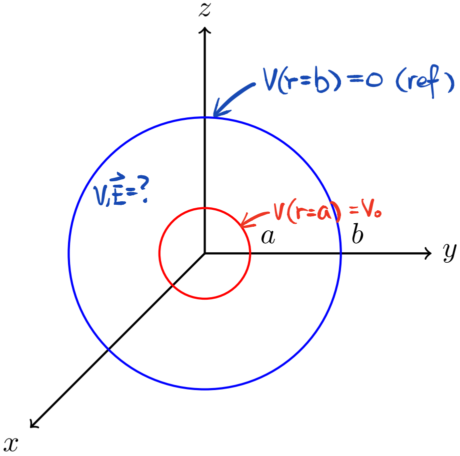
$$= \frac{-(b-a)}{4\pi\varepsilon_0} \left[\hat{x} \sin\phi \Big|_0^{\phi_0} + \hat{y} \cos\phi \Big|_{\phi_0}^{0} \right]$$

$$E = \frac{-(b-a)}{4\pi\varepsilon_0} \left[\hat{x} \sin\phi_0 + \hat{y} (1 - \cos\phi_0) \right]$$

1. Examples of Solutions to Laplace's Equation
2. Capacitance
3. Capacitance Examples

Voltage Between Two Spheres

Find the voltage and electric field in the region $a \leq r \leq b$.



angle invariant.

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \cancel{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}}$$

$$r^2\frac{\partial V}{\partial r} = A \rightarrow \frac{\partial V}{\partial r} = \frac{A}{r^2}$$

$$V = -\frac{A}{r} + B$$

Apply B.C.s

$$V(b) = \frac{-A}{b} + B = 0 \rightarrow B = \frac{A}{b}$$

$$V(a) = \frac{-A}{a} + B = V_0 = \frac{-A}{a} + \frac{A}{b}$$

$$= -V_0\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$A = -V_0\left(\frac{1}{a} - \frac{1}{b}\right)^{-1}$$

Voltage Between Infinite Cone and Infinite Ground Plane

$V(r)=\frac{-1}{r}\left(-V_0\left[\frac{1}{a}-\frac{1}{b}\right]^{-1}\right)+\left(-A\left[\frac{1}{a}-\frac{1}{b}\right]^{-1}\right)/b$

$$\boxed{V(r)=V_0\frac{\frac{1}{r}-\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}}$$

$V(\theta=\alpha)=V_0$

$V(\theta=\pi/2)=0$

Find the voltage and electric field in the region $\alpha \leq \theta \leq \pi/2$, if there is an infinite ground plane located in the plane $\theta = \pi/2$.

xy plane.

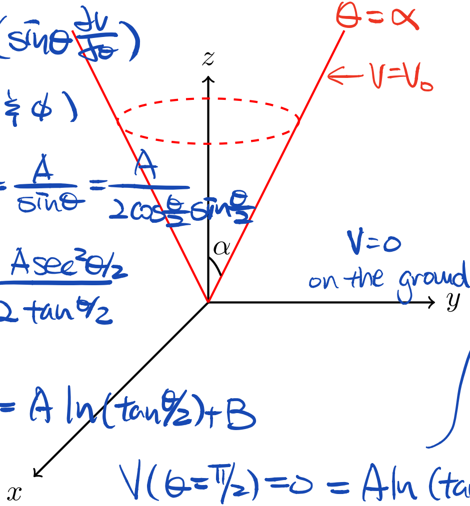
$\theta=\pi/2$

$\vec{E}=-\nabla V \qquad \nabla V = \frac{dV}{dr}\hat{r}+\frac{1}{r}\frac{dV}{d\theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{dV}{d\phi}\hat{\phi}$

$$\boxed{\vec{E}=\hat{r}\,\frac{V_0}{r^2}\left(\frac{1}{\frac{1}{a}-\frac{1}{b}}\right)}$$

$\nabla^2 V = 0 = \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right)$

(no variance in r & ϕ)



$A=\sin\theta\frac{dV}{d\theta} \qquad \frac{dV}{d\theta}=\frac{A}{\sin\theta}=\frac{A}{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}$

上下除 $\cos^2\theta/2$ $\quad = \frac{A\sec^2\theta/2}{2\tan\theta/2}$

$V=0$ on the ground.

$V=A\int\frac{d\tan\theta/2}{\tan\theta/2}=A\ln(\tan\theta/2)+B$

$V(\theta=\alpha)=V_0=A\ln\tan\alpha/2 \rightarrow A=\frac{V_0}{\ln\tan\alpha/2}$

$V(\theta)=\frac{V_0}{\ln\tan(\alpha/2)}\ln\tan\theta/2$

$V(\theta=\pi/2)=0=A\ln(\tan\pi/4)+B \rightarrow B=0$

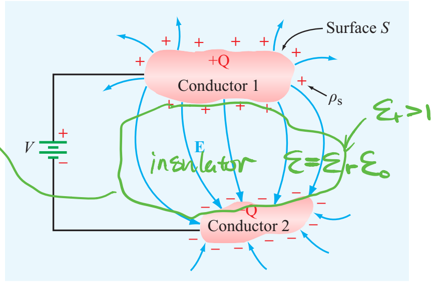
$\vec{E}=-\nabla V=-\frac{1}{r}\frac{dV}{d\theta}\hat{\theta}$

Capacitance

$$= \frac{V_0}{\ln\tan(\theta/2)} \frac{1}{r\tan\theta/2} \frac{1}{2}\sec^2\theta/2$$

- Capacitance is a measure of energy storage capacity in electrical devices.
- Consider two conductors embedded in a dielectric material with permittivity ϵ.
- Each conductor carries equal and opposite charges; the net charge is zero.
- Capacitance is defined as

$$C = \frac{Q}{V}$$



Capacitance

- More explicitly,

$$Q = \int_s \vec{D} \cdot \vec{ds}$$

$$C = \frac{\overbrace{\oint_S \epsilon \boldsymbol{E} \cdot d\boldsymbol{s}}^{}}{-\int_C \boldsymbol{E} \cdot d\boldsymbol{\ell}} \equiv \frac{Q}{V} \equiv [C/V] \equiv [F]$$

- Therefore, capacitance is **only** a function of the dielectrics and the geometry! You may assume charge and find voltage or vice versa to solve capacitance problems.
- Units of capacitance are **Farads** (F) or Coulombs/Volt (C/V).
  - $1\ \mu\text{F} = 10^{-6}$ F
  - 1 nF $= 10^{-9}$ F
  - 1 pF $= 10^{-12}$ F

Parallel Plate Capacitor

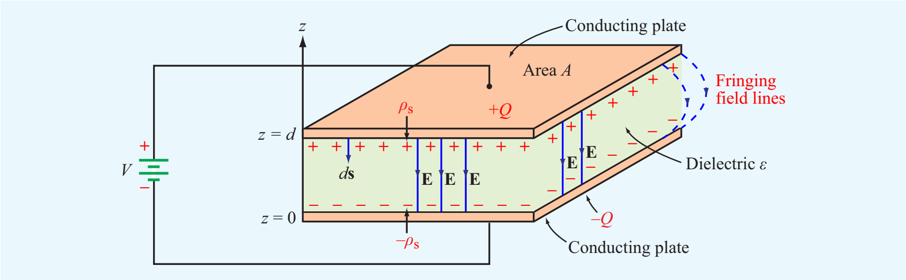


Image Credit: Ulaby and Ravaioli

Coaxial Capacitor

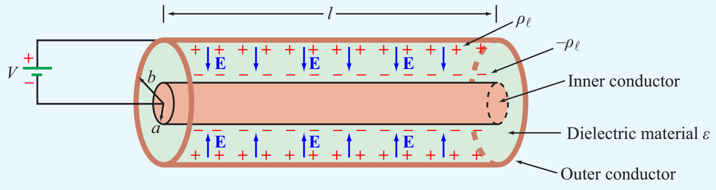
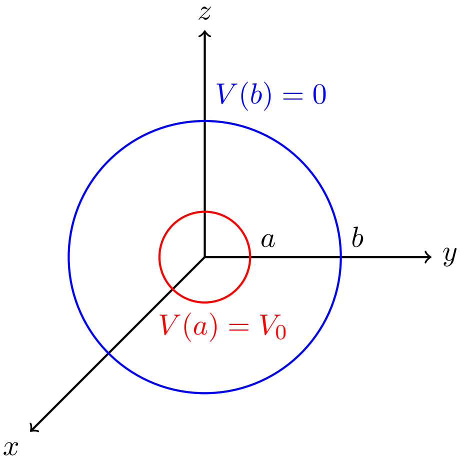


Figure 4-25 Coaxial capacitor filled with insulating material of permittivity ε (Example 4-12).

Image Credit: Ulaby and Ravaioli

Spherical Capacitor



Parallel Plate Capacitor with Multiple Stacked Dielectrics