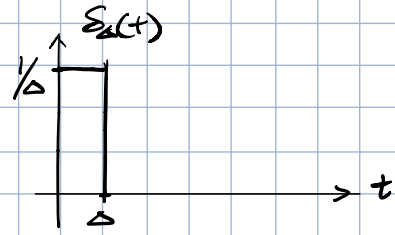


Convolution Integral for CT LTI systems.

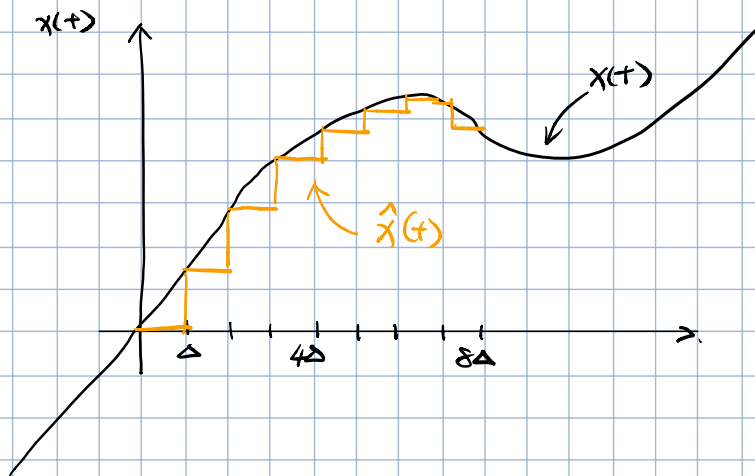
$$\delta_{\Delta}(t) = \begin{cases} \frac{1}{\Delta} & 0 < t < \Delta \\ 0 & \text{otherwise.} \end{cases}$$



$$\delta(t) = \lim_{\Delta \rightarrow 0} \delta_{\Delta}(t)$$

We can use  $\delta_{\Delta}(t)$  to find an approximation of a given CT signal.

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta) \delta_{\Delta}(t - k\Delta) \Delta \quad k \in \mathbb{Z}$$



$$x(t) = t^3 \quad \Delta = 0.1$$

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(0.1k) \delta_{0.1}(t - 0.1k) 0.1$$

$$t = 0.47 \quad \hat{x}(0.47) = \sum_{k=-\infty}^{+\infty} x(0.1k) \delta_{0.1}(0.47 - 0.1k) 0.1$$

The only non-zero term under this sum occurs

$$\text{for } 0 < 0.47 - 0.1k < \underbrace{0.1}_{\Delta}$$

$k=4$  for this inequality to be true.

$$\hat{x}(0.47) = x(0.1 \times 4) \delta_{0.1}(0.47 - 0.1 \times 4) 0.1$$

$$= x(0.4) \delta_{0.1}(0.47 - 0.4) 0.1$$

$$= x(0.4)$$

The smaller is  $\Delta$ , the more closely  $\hat{x}(t)$  approximates  $x(t)$

$$\Delta = 0.1 \quad \hat{x}(0.47) = x(0.4)$$

$$\Delta = 0.05 \quad \hat{x}(0.47) = x(0.45)$$

$$x(t) = \lim_{\Delta \rightarrow 0} \hat{x}(t) = \lim_{\Delta \rightarrow 0} \sum_{k=-\infty}^{+\infty} x(k\Delta) \delta(t - k\Delta) \Delta$$

$$\rightarrow x(t) = \int_{-\infty}^{+\infty} x(\tau) \delta(t - \tau) d\tau$$

Remark: For a CT LTI system.

$$\delta_{\Delta}(t) \rightarrow \boxed{\text{LTI}} \rightarrow h_{\Delta}(t)$$

$$\underbrace{\sum_{k=-\infty}^{+\infty} x(k\Delta) \delta_{\Delta}(t - k\Delta) \Delta}_{\hat{x}(t)} \rightarrow \boxed{\text{LTI}} \rightarrow \sum_{k=-\infty}^{+\infty} x(k\Delta) h_{\Delta}(t - k\Delta) \Delta$$

If  $\Delta \rightarrow 0$

$$\underbrace{\int_{-\infty}^{+\infty} x(\tau) \delta(t - \tau) d\tau}_{x(t)} \rightarrow \boxed{\text{LTI}}_{h(t)} \rightarrow \underbrace{\int_{-\infty}^{+\infty} x(\tau) h(t - \tau) d\tau}_{y(t)}$$

→ For an CT LTI system, the impulse response,  $h(t)$ , is the only information needed to determine the system output for any given input.

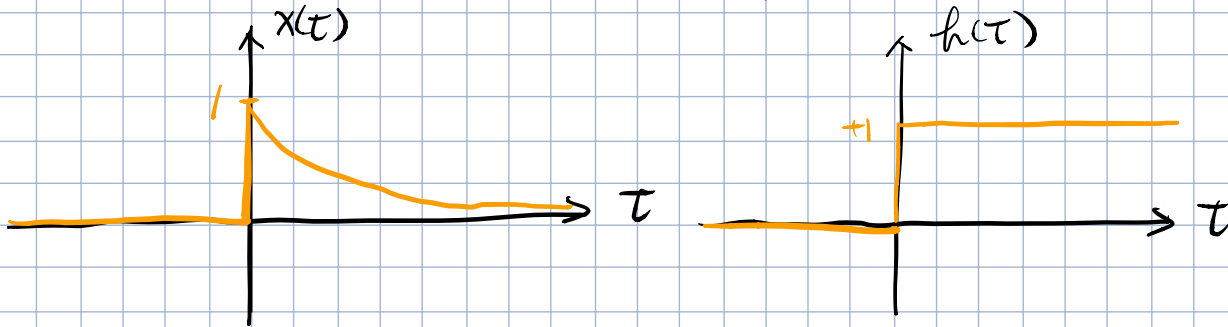
$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t - \tau) d\tau = x(t) * h(t)$$

## Convolution integral of $x(t)$ and $h(t)$

(\*) For a CT LTI system.

$h(t) = u(t)$  What is the output for  $x(t) = e^{-at} u(t)$   $a > 0$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$$



$t < 0$ :

$$y(t) = \int_{-\infty}^{+\infty} 0 d\tau = 0.$$

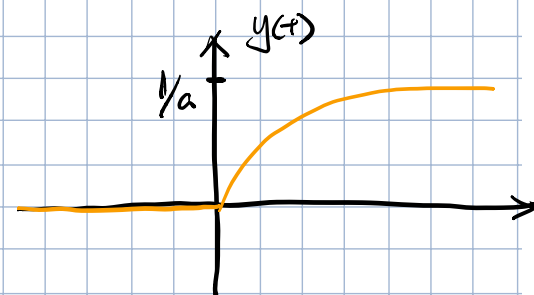
A plot of  $h(t-\tau)$  vs  $\tau$  for  $t < 0$ . The step function is shifted to the left of the vertical axis, so it is zero for all  $\tau$ . The label  $h(t-\tau)$  is written in red above the plot.

$t > 0$ :

$$y(t) = \int_0^t e^{-a\tau} \cdot 1 d\tau$$
$$= -\frac{1}{a} e^{-a\tau} \Big|_0^t = \frac{1}{a} (1 - e^{-at})$$

A plot of  $h(t-\tau)$  vs  $\tau$  for  $t > 0$ . The step function is shifted to the right of the vertical axis, starting at  $\tau = t$  and ending at  $\tau = 0$ . The label  $h(t-\tau)$  is written in red above the plot.

$$\therefore y(t) = \frac{1}{a} (1 - e^{-at}) u(t)$$



Properties of LTI systems. :

$$\text{DT: } y[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k]$$

$$\text{CT: } y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$$

① Commutative property  $x(t) * h(t) = h(t) * x(t)$

proof:  $x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$

$$t - \tau = m$$

$$\tau = t - m$$

$$d\tau = -dm$$

$$= \int_{+\infty}^{-\infty} x(t-m) h(m) -dm$$

This property means:

$$\begin{array}{c} x(t) \\ \longrightarrow \end{array} \boxed{\text{LTI}} \begin{array}{c} h(t) \\ \text{below box} \end{array} \longrightarrow y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} h(m) x(t-m) dm$$

$$\begin{array}{c} h(t) \\ \longrightarrow \end{array} \boxed{\text{LTI}} \begin{array}{c} x(t) \\ \text{below box} \end{array} \longrightarrow y(t) = h(t) * x(t)$$