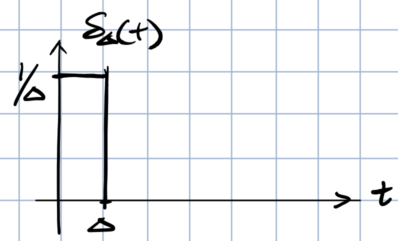
Convolution Integral for CT LTI systems.

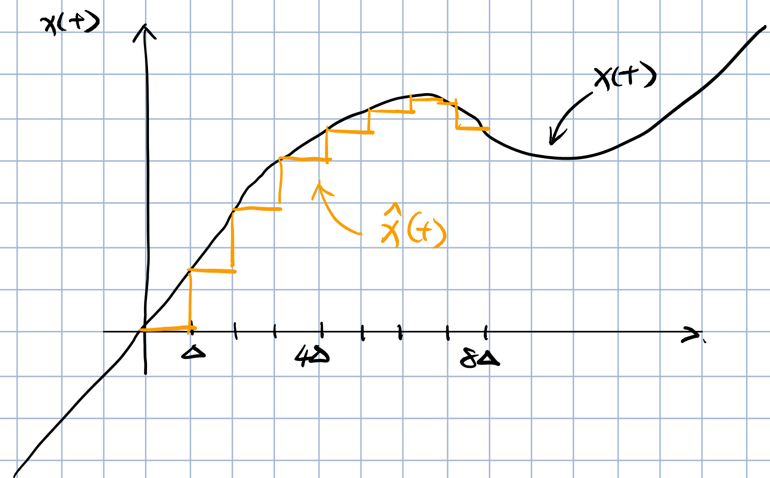
$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & 0<t<\Delta \\ 0 & \text{otherwise.} \end{cases}$$



$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

We can use $\delta_\Delta(t)$ to find an approximation of a given CT signal.

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta \qquad k \in \mathbb{Z}$$



$x(t) = t^3 \qquad \Delta = 0.1$

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(0.1k)\,\delta_{0.1}(t-0.1k)\,0.1$$

$$t = 0.47 \qquad \hat{x}(0.47) = \sum_{k=-\infty}^{+\infty} x(0.1k)\,\delta_{0.1}(0.47-0.1k)\,0.1$$

The only non-zero term under this sum occurs

for $\quad 0 < 0.47 - 0.1k < \underbrace{0.1}_{\Delta}$

$k=4$ for this inequity to be true.

$$\hat{x}(0.47) = x(0.1 \times 4)\,\delta_{0.1}(0.47 - 0.1 \times 4)\,0.1$$

$= x(0.4)\,\delta$

$$= x(0.4)\,\underbrace{\delta_{0.1}(0.07)}_{1/\Delta = \frac{1}{\cancel{0.1}}}\,\cancel{0.1}$$

$$= x(0.4)$$

The smaller is Δ, the more closely $\hat{x}(t)$ approximates $x(t)$

$\Delta = 0.1 \qquad \hat{x}(0.47) = x(0.4)$

$\Delta = 0.05 \qquad \hat{x}(0.47) = x(0.45)$

$$x(t) = \lim_{\Delta \to 0} \hat{x}(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\delta(t-k\Delta)\Delta$$

$$\rightarrow \; x(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

Remark: For a CT LTI system.

$$\delta_\Delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h_\Delta(t)$$

$$\underbrace{\sum_{k=-\infty}^{+\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\Delta}_{\hat{x}(t)} \longrightarrow \boxed{\text{LTI}} \longrightarrow \sum_{-\infty}^{+\infty} x(k\Delta)\,h_\Delta(t-k\Delta)\Delta$$

If $\Delta \to 0$

$$\underbrace{\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau}_{x(t)} \longrightarrow \underset{h(t)}{\boxed{\text{LTI}}} \longrightarrow \underbrace{\int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau}_{y(t)}$$

$\longrightarrow$ For an CT LTI system, the impulse response, $h(t)$, is the only information needed to determine the system output for any given input.

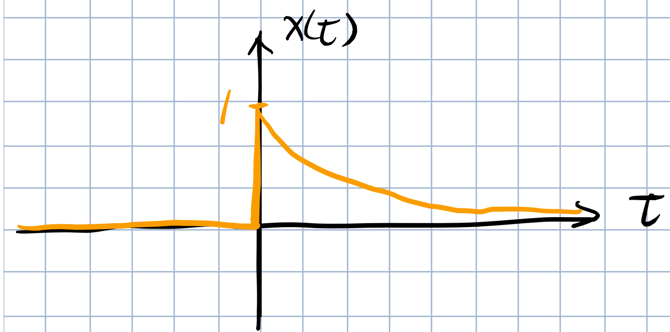
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * h(t)$$

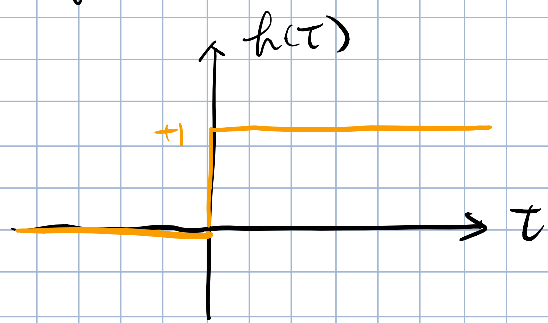
Convolution integral of $x(t)$ and $h(t)$

(*) For a CT LTI system.

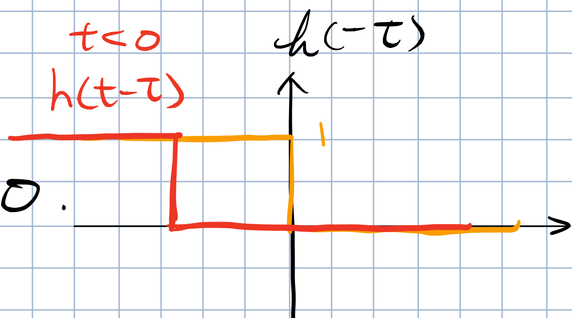
$h(t) = u(t)$ What is the output for $x(t) = e^{-at}u(t)$ $a>0$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau$$



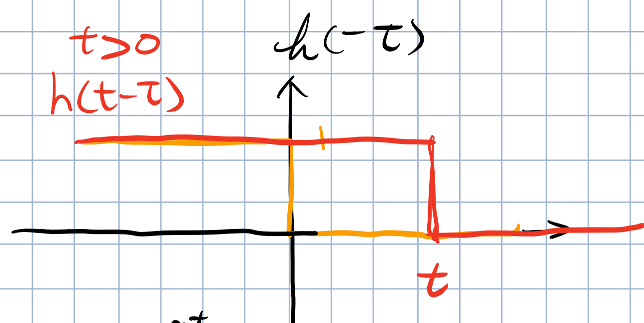


$t<0$: $y(t) = \int_{-\infty}^{+\infty} 0 \, d\tau = 0$.



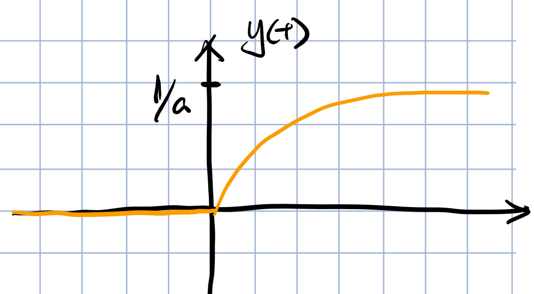
$t>0$:

$$y(t) = \int_0^t e^{-a\tau} \cdot 1$$



$$= -\frac{1}{a} e^{-a\tau}\Big|_0^t = \frac{1}{a}(1-e^{-at})$$

$$\therefore\ y(t) = \frac{1}{a}(1-e^{-at})\,u(t)$$



Properties of LTI systems. :

DT: $$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

CT: $$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

① Commutative property $x(t) * h(t) = h(t) * x(t)$

proof: $$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$t-\tau = m$

$\tau = t-m$

$d\tau = -dm$

$$= \int_{+\infty}^{-\infty} x(t-m)h(m) - dm$$

$$= \int_{-\infty}^{+\infty} h(m)x(t-m)dm$$

This property means:

$$\xrightarrow{x(t)} \underset{h(t)}{\boxed{LTI}} \longrightarrow y(t) = x(t) * h(t)$$

$$\xrightarrow{h(t)} \underset{x(t)}{\boxed{LTI}} \longrightarrow y(t) = h(t) * x(t)$$