# Lecture 23: Magnetic Flux Density, Magnetic Potentials

ECE221: Electric and Magnetic Fields

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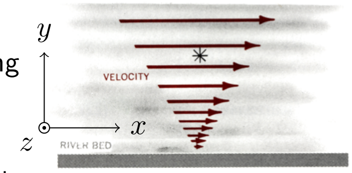
# Outline

# Physical Interpretation (3)

$$\nabla \times \boldsymbol{H} = \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \hat{\boldsymbol{x}} + \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \hat{\boldsymbol{y}} + \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \hat{\boldsymbol{z}} = \boldsymbol{J}$$

- **The point is:** we need **three** orientations of the paddle wheel to complete describe three components of the velocity (magnetic) field.
- Each orientation only tells us something about the components of **two** of $\boldsymbol{H}$ if $\boldsymbol{J}$ is known.
- There is insufficient information in that component to solve for $\boldsymbol{H}$.
- Curl gives us three equations in three unknowns



# Calculation of Curl

- Remember that curl is an *operator formed from gradient* $\boldsymbol{\nabla}$!
- Employ definition cross product definition to find curl rather than resorting to aid sheets:

$$\boldsymbol{\nabla} \times \boldsymbol{H}(x, y, z) = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

- Cylindrical coordinates

$$\boldsymbol{\nabla} \times \boldsymbol{H}(\rho, \phi, z) = \frac{1}{\rho} \begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_\rho & \rho H_\phi & H_z \end{vmatrix}$$

- Spherical coordinates

$$\boldsymbol{\nabla} \times \boldsymbol{H}(r, \theta, \phi) = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_r & rH_\theta & (r\sin\theta)H_\phi \end{vmatrix}$$

# Calculation of Curl

- Cylindrical coordinates

$$\boldsymbol{\nabla} \times \boldsymbol{H}(\rho, \phi, z) = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\hat{\boldsymbol{\phi}}$$
$$+ \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho H_\phi) - \frac{\partial H_\phi}{\partial \rho}\right]\hat{\boldsymbol{z}}$$

- Spherical coordinates

$$\boldsymbol{\nabla} \times \boldsymbol{H}(r, \theta, \phi) = \left[\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(H_\phi \sin\theta) - \frac{\partial H_\theta}{\partial \phi}\right]\hat{\boldsymbol{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \right.$$
$$\left.\frac{\partial}{\partial r}(rH_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

# Curl and Stokes' Theorem

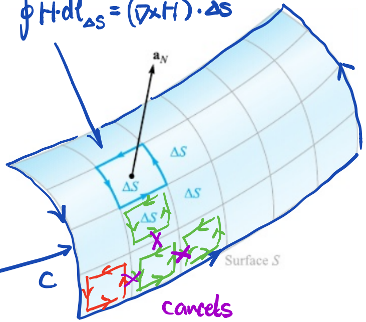
$\oint \vec{H} \cdot d\vec{\ell}_{\Delta S} = (\nabla \times \vec{H}) \cdot \Delta \vec{S}$

*Curl* is circulation per unit area:

$$\frac{\oint \boldsymbol{H} \cdot d\ell_{\Delta S}}{\Delta S} \equiv (\boldsymbol{\nabla} \times \boldsymbol{H}) \cdot \boldsymbol{a}_N$$

$\Delta \vec{S} = ds\, a_N$

What is the value of $\oint_C \boldsymbol{H} \cdot d\ell$?




$\oint_C \vec{H} \cdot d\vec{\ell} = \Sigma(\text{circulation of small loops})$

large loop

$$\boxed{\oint_C \vec{H} \cdot d\vec{\ell} = \iint_S (\nabla \times \vec{H})\, d\vec{s}}$$

Stokes theorem.

# Magnetix Flux Density

- *Magnetic field* $\boldsymbol{H}$ and *magnetic flux density* $\boldsymbol{B}$ are related through the **constitutive relation**

$$\boxed{\boldsymbol{B} = \mu \boldsymbol{H}}$$

$\vec{D} = \varepsilon \vec{E} \rightarrow$ electric flux $= \iint_S \vec{D} \cdot d\vec{s} = \phi$

$[C/m^2]$ $[C]$

where $\mu$ is called *magnetic permeability [H/m]* $\vec{J} = \sigma \vec{E}$

- $\mu_0 = 4\pi \times 10^{-7}$ H/m
- *Magnetic flux* $\Psi$ [Webers, Wb] and *magnetic flux density* are related through a usual **flux integral**

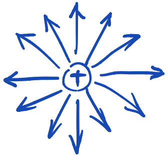
$[\frac{Wb}{m^2}]$

$$\Psi = \iint_S \boldsymbol{B} \cdot d\boldsymbol{s} \text{ [Wb]}$$

$[Wb/m^2] = [\text{Tesla}]$ $[T]$

# Fundamental Postulates of the Magnetic Field

E.S.



$\nabla \cdot \vec{D} \neq 0$

As we have seen before:

1. Magnetic field / magnetic flux density has no **divergence**

$$\boxed{\nabla \cdot \boldsymbol{B} = \nabla \cdot \boldsymbol{H} = 0}$$

   This is the same thing as saying that **there are no magnetic charges / monopoles**

2. Magnetic field / magnetic flux density is **solenoidal** – it has a **non-zero curl** and forms closed loops

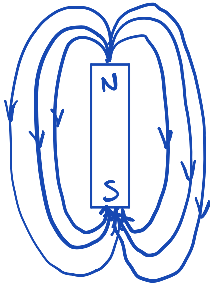
$$\nabla \times \boldsymbol{H} = \boldsymbol{J}$$

$\nabla \cdot \vec{D} = \rho_v$ $\quad \nabla \times \vec{B} = 0$

$\nabla \times \vec{E} = 0$ $\quad \nabla \times \vec{H} = \vec{J}$

MS

$\vec{B}$ $\vec{H}$




$\vec{H}$ $\quad I d\vec{l}$

$\vec{B} = \mu \vec{H}$

form a close loop

$\vec{B}$

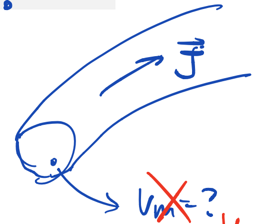
B has no divergence.

# Magnetic Potentials

- *Magnetic potentials* are useful mathematical aids to help us analyze magnetic structures called *magnetic circuits*.
- Unlike electric (scalar) potential, we cannot measure these potential easily; they are simply tools.
- We will need the following important vector identifies:
  1. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \mathbf{0}$
  2. $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{F} = 0$
  3. $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{F}) - \boldsymbol{\nabla}^2 \boldsymbol{F}$

# Scalar Potential $V_m$

$V_m = ?$




- Let's begin with *scalar magnetic potential* $V_m$.
- Similar to $\boldsymbol{E} = -\boldsymbol{\nabla} V$, define

$$\boldsymbol{H} = -\boldsymbol{\nabla} V_m \quad \boxed{\text{if } \boldsymbol{J} = 0}$$

- Then

$$\overbrace{\boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{H}}^{\text{Ampere's law}} = \boldsymbol{\nabla} \times (-\boldsymbol{\nabla} V_m) = \mathbf{0}$$

identity #1 from last page.

- Therefore, $V_m$ is only defined where $\boldsymbol{J} = 0$, in the source-free region.
- We can find $V_m$ using

$$\boldsymbol{\nabla} \cdot \mu_0 \boldsymbol{H} = 0 \Rightarrow \mu_0 \boldsymbol{\nabla} \cdot (-\boldsymbol{\nabla} V_m) = 0$$

$$\boldsymbol{\nabla}^2 V_m = 0$$

- This is Laplace's equation (in the source-free region)!

$V_m$ is not defined in source region.

# Vector Magnetic Potential $\boldsymbol{A}$

- This form of potential is more useful.
- **To uniquely define a vector we must define:**
  1. Its curl
  2. Its divergence
- We know:

$$\nabla \cdot \boldsymbol{B} = 0$$

- Define *vector magnetic potential* $\boldsymbol{A}$ such that:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad \nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0$$

- We have now defined the *curl* of $\boldsymbol{A}$ that is consistent with Maxwell's equations for magnetostatics.

# Vector Magnetic Potential $\boldsymbol{A}$

$\vec{B} = \mu_0 \vec{H}$

- Ampère's Law also tells us:

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} \quad \Rightarrow \quad \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

- Using identity 3,

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A}$$

Define $\nabla \cdot \vec{A} = 0$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \overset{\boldsymbol{A}}{\cancel{\boldsymbol{B}}}) - \boldsymbol{\nabla}^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}$$

$$\boxed{\boldsymbol{\nabla}^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}}$$

- Poisson's equation!

# Poisson's Equation

scalar

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J} \quad \Rightarrow \quad \begin{cases} \nabla^2 A_x = -\mu_0 J_x \\ \nabla^2 A_y = -\mu_0 J_y \\ \nabla^2 A_z = -\mu_0 J_z \end{cases}$$

We know how to solve equations like these!

# Poisson's Equation

**Electrostatics**

$$\underbrace{\nabla^2 V = -\rho_v/\epsilon_0}$$ Scalar.

$$V = \iiint_V \frac{\rho_v dv}{4\pi\epsilon_0 R}$$

$$= \iint_S \frac{\rho_s ds}{4\pi\epsilon_0 R}$$

$$= \int_c \frac{\rho_l dl}{4\pi\epsilon_0 R}$$

**Magnetostatics**

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$$

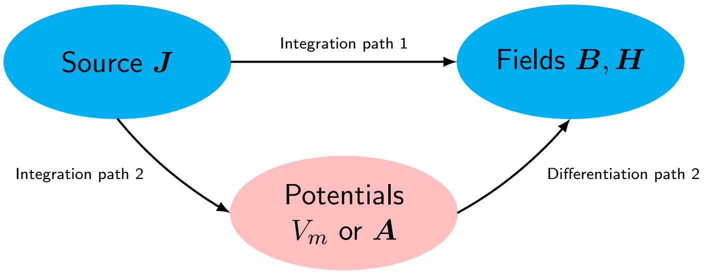
$$A_x = \iiint \frac{\mu_0 J_x}{4\pi R} dV$$

$$\vec{A} = \iiint_V \frac{\mu_0 \vec{J}}{4\pi R} dV.$$

$$= \iint_S \frac{\mu_0 \vec{K}\, ds}{4\pi R} \qquad = \int_c \frac{\mu_0 I\, d\vec{l}}{4\pi R}$$

# Poisson's Equation

| Magnetic Circuits | Ampère's Law |
|---|---|
| $\boldsymbol{\nabla}^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$ | $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$ |
| Easy to solve for $\boldsymbol{A}$<br>Then find $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ | Hard to solve for $\boldsymbol{B}$<br>from point form |



# Vector Potential and Magnetic Flux

$$\Psi = \iint_S \boldsymbol{B} \cdot d\boldsymbol{S}$$

Apply Stokes theorem.