Lecture 2: Fields, Coulomb's Law, Electric Field

ECE221: Electric and Magnetic Fields

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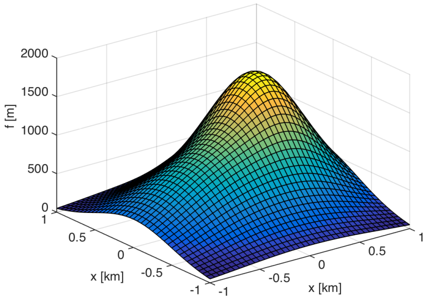
Outline

Derivatives of Fields

- Consider a 2D scalar function of space $f(x, y)$
- How do we take the derivative of a field that depends on more than one spatial variable?



How do we represent the **slope** of the mountain as a function of both x and y?

Gradient Operator ∇

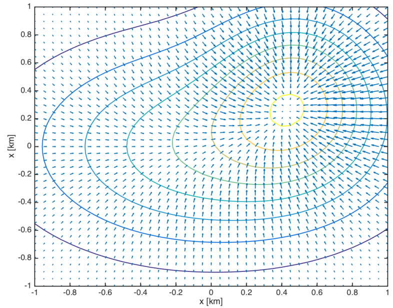
Gradient Operator ∇ in Three Dimensions

The gradient of a three-dimensional function $f(x, y, z)$ is defined as:

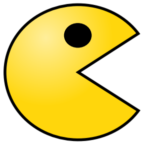
$$\nabla f(x, y, z) = \text{grad } f(x, y, z) = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$$

Gradient Example: Plot of ∇f

- Vectors illustrate direction and magnitude of the gradient vector (by arrow length)



"Hungriness" of ∇



∇ is a hungry operator!

$$\nabla F = \frac{\partial F}{\partial x}\hat{x} + \frac{\partial F}{\partial y}\hat{y} + \frac{\partial F}{\partial z}\hat{z}$$

$$\neq$$

$$F\nabla = F\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\square$$

Operations with ∇: Divergence

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \right) \cdot \left(F_x \hat{x} + F_y \hat{y} + F_z \hat{z} \right)$$

$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \text{div } F$$

$$= \text{scalar} \quad \text{(cartesian coordinates)}$$

Operations with ∇: Curl

$$\nabla \times \vec{F} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \times \vec{F}$$

$$= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{x} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{z}$$

$$= \text{curl } \vec{F}$$

Second Order Derivatives

- It is possible for form second-order derivatives from the ∇ operator, too
- Try evaluating the following:
  1. $\nabla \cdot (\nabla T)$
  2. $\nabla \times (\nabla T)$
  3. $\nabla (\nabla \cdot \boldsymbol{F})$
  4. $\nabla \cdot (\nabla \times \boldsymbol{F})$
  5. $\nabla \times (\nabla \times \boldsymbol{F})$
- The first in the list is called *Laplacian*:

$$\nabla \cdot (\nabla T) = \nabla^2 T$$

- Input: scalar, output: scalar (a vector/vector version is also possible)

Overview of Maxwell's Equations

$\nabla \cdot \varepsilon_0 \vec{E} = \rho_v \rightarrow \nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon_0}.$

$$\boldsymbol{\nabla} \cdot \boldsymbol{D}(\boldsymbol{r}, t) = \rho_v(\boldsymbol{r}, t)$$

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}, t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r}, \boldsymbol{t})}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B}(\boldsymbol{r}, t) = 0 \quad \nabla \cdot \vec{H} = 0$$

$$\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{J}(\boldsymbol{r}, t) + \frac{\partial \boldsymbol{D}(\boldsymbol{r}, t)}{\partial t}$$

$$= \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Coupling between $\vec{E}$ and $\vec{H}$

Constitutive Relations (in a vacuum / free space)

$$\vec{D}(\vec{r},t) = \varepsilon_0 \vec{E}(\vec{r},t)$$

ε_0 = permittivity of free space/vacuum

$= 8.854 \times 10^{-12}$ F/m

Farads.

$$\vec{B}(\vec{r},t) = \mu_0 \vec{H}(\vec{r},t)$$

μ_0 = permeability of free space

$= 4\pi \times 10^{-7}$ H/m

Henry.

What if There is No Time Variation?

$$\nabla \cdot \vec{E}(\vec{r}) = \rho_v(\vec{r}) / \varepsilon_0$$

$$\nabla \times \vec{E}(\vec{r}) = 0$$

electrostatics. no curl!

$$\nabla \cdot \vec{H}(\vec{r}) = 0$$

$$\nabla \times \vec{H}(\vec{r}) = \vec{J}(\vec{r})$$

magnetostatics. no div!

$\vec{E}$ and $\vec{H}$ are decoupled when there's no time variation.

没关系了

Coulomb's Law

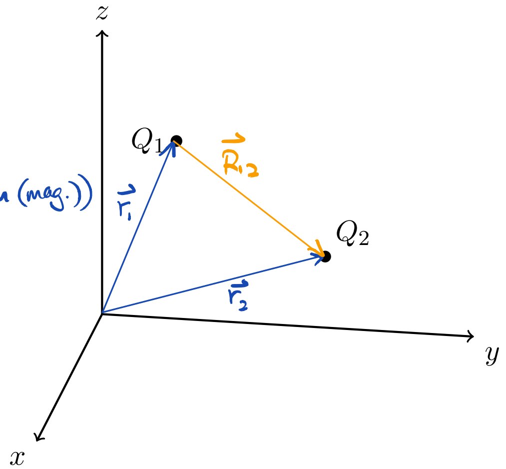
Consider two charges Q_1 and Q_2.

What is the force on Q_2 due to Q_1?

$$\vec{F}_2 = \frac{1}{4\pi\varepsilon_0} \frac{Q_1 Q_2}{R_{12}^2} \hat{r}_{12}$$

$$\vec{R}_{12} = \vec{r}_2 - \vec{r}_1$$ (R_{12} is the length (mag.))

$$\hat{r}_{12} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$$

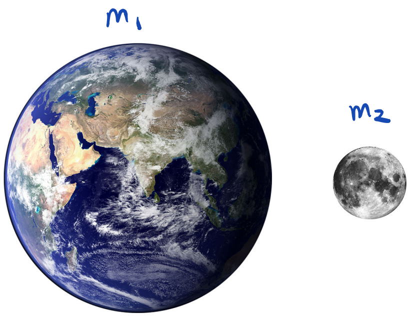


A Familiar Field: Gravity

$$\vec{F}_1 = -G\frac{m_1 m_2}{R_{12}^2}\hat{r}_{12}$$

G vs. $\frac{1}{4\pi\varepsilon_0}$.

20 order of mag. diff.



Source: NASA

$$G = 6.674 \times 10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}$$

Electric Field

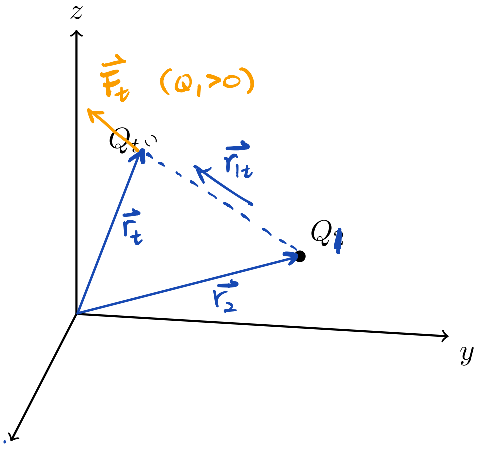
Consider a situation where we replace Q_2 with a positive test charge Q_t and see what force it feels:

$$\vec{F}_t = \frac{1}{4\pi\varepsilon_0}\frac{Q_t Q_1}{R_{1t}^2}\hat{r}_{1t}$$

$$\frac{\vec{F}_t}{Q_t} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{R_{1t}^2}\hat{r}_{1t} \left[\frac{N}{C}\right]$$

$$= \vec{E} = \text{electric field.}$$

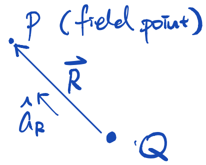
$$= \text{force per unit charge.}$$



Electric Field

General Expression

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{R^2} \hat{a}_R$$

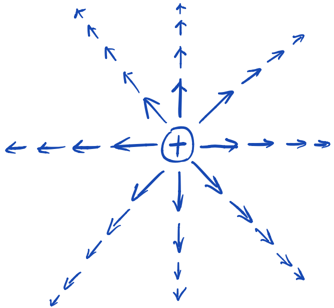


$$\vec{E} \rightarrow \left[\frac{N}{C}\right] \equiv \left[V/m\right]$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{R^3} \vec{R}$$ (subst. $\hat{a}_R$ with $\frac{\vec{R}}{R}$, where R is $|\vec{R}|$)

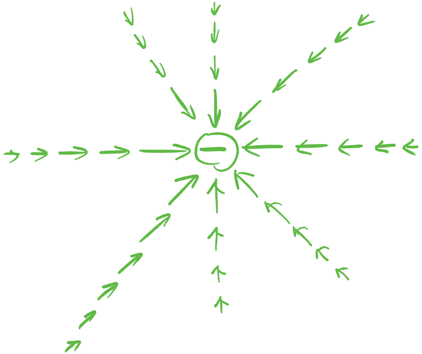
Field Plots for Point Charges

Positive charge.



diverging field.

Negative Charge.



negative diverging field.

(converging)

Fundamental Postulates of the Static Electric Field

Electrostatic eqns: 1. $\nabla \cdot \vec{E} = \rho_v / \varepsilon_0$

$$\nabla \times \vec{E} = 0$$

- Electrostatic field always pt away or towards the pt. charge.
  → diverging field. "$\nabla \cdot E \neq 0$"

- Electrostatic field never have curl / circulation.