# Lecture 9: Potential and Energy

ECE221: Electric and Magnetic Fields

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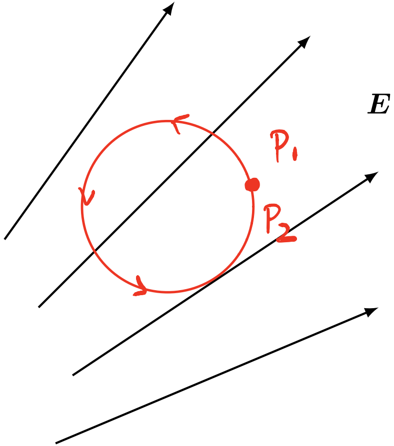
# Outline

# Conservative Property of the Electrostatic Field




total work done moving a charge along a closed countour C is zero

$$\boxed{V = -\oint_C \vec{E} \cdot d\vec{\ell} = 0}$$

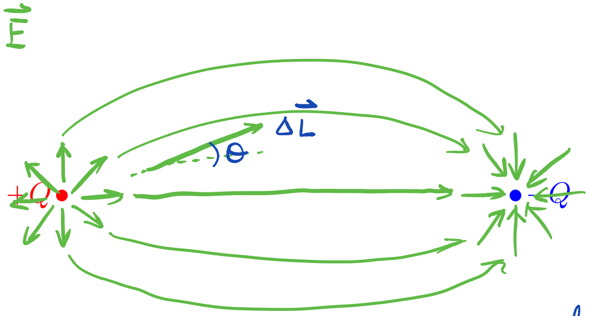
this is KVL

$$\sum_k V_k = 0$$

What is the circuit equivalent of this property?

# Potential Gradient

- We have seen how to go from $\boldsymbol{E}$ to $V$; we now wish to do the reverse.
- Consider the E-field between two opposite point charges:




Let's find potential difference $\Delta V$ along $\Delta\vec{L}$

$$\Delta V \simeq -\vec{E}\cdot\Delta\vec{L}$$
$$= -E\,\Delta L\cos\theta$$

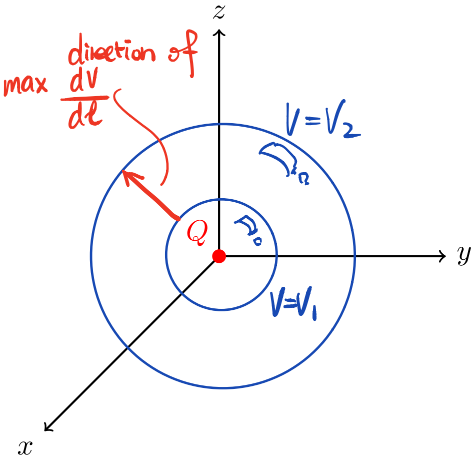
As $\Delta L \to 0$

$$\lim_{\Delta L\to 0}\frac{\Delta V}{\Delta L} = -E\cos\theta = \frac{dV}{dL}$$

max value $\dfrac{\Delta V}{\Delta L} = E$ when $\theta = 180^\circ$

direction of max. voltage change is anti-parallel to $\vec{E}$ (opposite direct.)

# Equipotential Surfaces



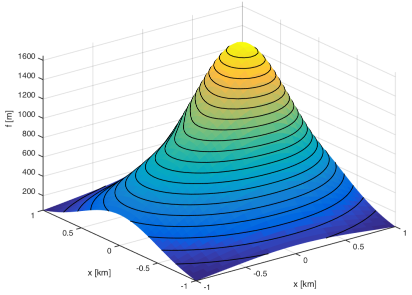

Draw the *equipotential surfaces* associated with a point charge.

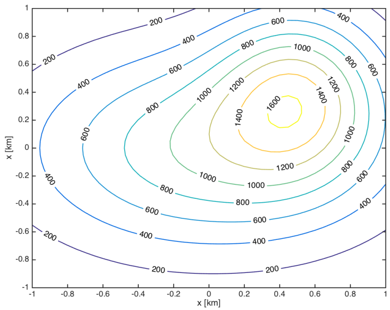
spheres in this case.

$$V = \frac{Q}{4\pi\epsilon_0 R}$$

# Contour Plots

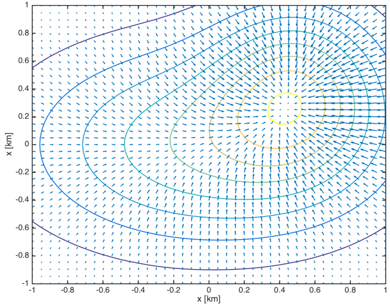
- Recall our plot of $f(x, y)$ in 2D, using *contours* to denote points where $f$ is constant
- If $f$ represents elevation, this is equivalent to a *topographic (topo) map*, where contours show lines of *constant elevation*



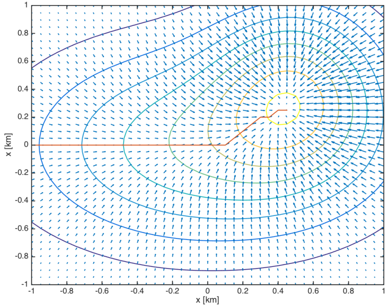
# Gradient Example

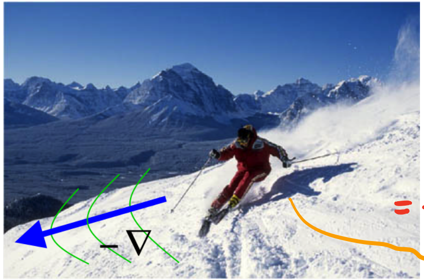
- $\nabla f$ evaluated along a grid of points is superimposed on the contour plot of $f$
- Vectors illustrate direction and magnitude of the gradient vector (by arrow length)



# Steepest Descent

- Follow the longest arrows down $-\nabla f$
- Notice $\nabla f$ always points perpendicular to contour lines!
- The steepest ways up/down is to follow a line perpendicular to the contours when they are closest together




$\vec{E} = \frac{dV}{dn}\hat{n}$

$V_1 = \text{const.}$

$\hat{n}$

$\vec{E}$

$V_2 = \text{const.}$

$\vec{E} = -\left(\frac{dV}{dx}\hat{x} + \frac{dV}{dy}\hat{y} + \frac{dV}{dz}\hat{z}\right)$

$= -\underbrace{\left(\frac{d}{dx}\hat{x} + \frac{d}{dy}\hat{y} + \frac{d}{dz}\hat{z}\right)}_{\text{gradient.}} V$

$\rho$

$r^2$

# Electric Dipole

$\vec{E} = -\nabla V$

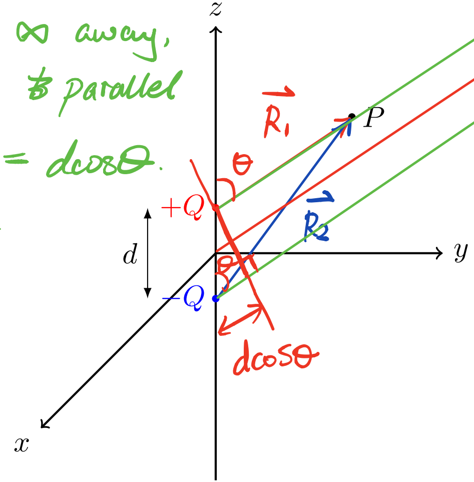
Let's revisit the example of two opposite charges separated by a distance.
What is $\boldsymbol{E}$ and $V$ at $P$?

$V = \frac{Q}{4\pi\varepsilon_0 R}$ for point charges

when P is $\infty$ away, $R_1$ & $R_2$ are ~~#~~ parallel

and $R_2 - R_1 = d\cos\theta$.

$R_1 R_2 \simeq r^2$




$$V_1 = \frac{Q}{4\pi\varepsilon_0 R_1} \qquad V_2 = \frac{-Q}{4\pi\varepsilon_0 R_2}$$

$$V_T = V_1 + V_2 = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{R_2 - R_1}{R_1 R_2}\right)$$

$$\boxed{V = \frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2}}$$

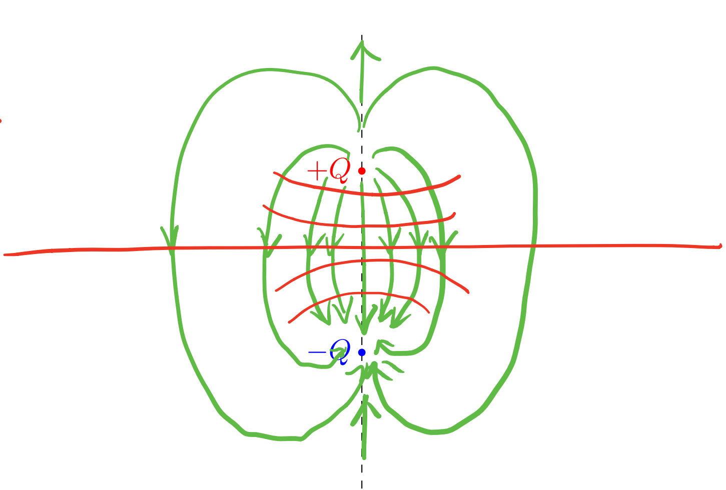
$$\vec{E} = -\nabla \cdot V = -\underbrace{\left(\frac{dV}{dr}\hat{r} + \frac{1}{r}\frac{dV}{d\theta}\hat{\theta} + \cancel{\frac{1}{r\sin\theta}\frac{dV}{d\phi}\hat{\phi}}\right)}_{\text{(sphere. coordinate } \nabla\text{)}} \rightarrow 0$$

$$\frac{dV}{dr} = \frac{-2Qd\cos\theta}{4\pi\varepsilon_0 r^3} \qquad \frac{dV}{d\theta} = \frac{-Qd\sin\theta}{4\pi\varepsilon_0 r^2}$$

# Electric Dipole E-field and Equipotential Surfaces

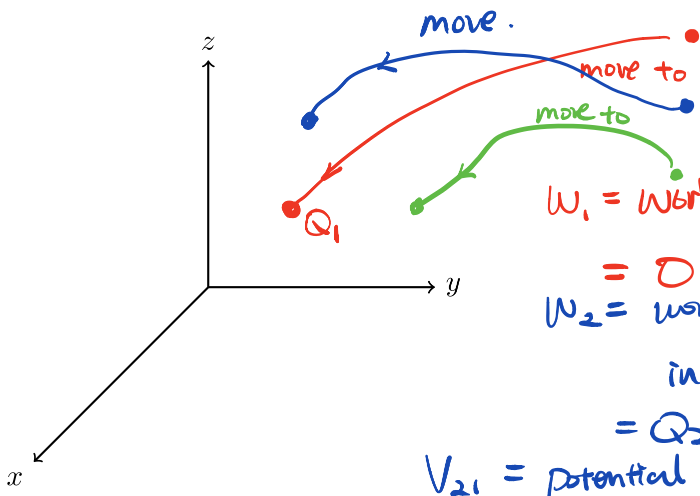
$$E = \frac{-Qd}{4\pi\varepsilon_0 r^2}\left(2r\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

- $\vec{E}$
- Equipotential surfaces.



# Total Work in Positioning Discrete Charges

真好.




$Q_1$ at $\infty$

$Q_2$ at $\infty$

$Q_3$ at $\infty$

$W_1$ = work done to position $Q_1$

$= 0$

$W_2$ = work done to position $Q_2$

in presense of $Q_1$

$= Q_2 V_{21}$

$V_{21}$ = potential @ location of $Q_2$ due to $Q_1$

$W_3 = Q_3 V_{31} + Q_3 V_{32}$

# Total Energy Associated with Continuous Charge Densities

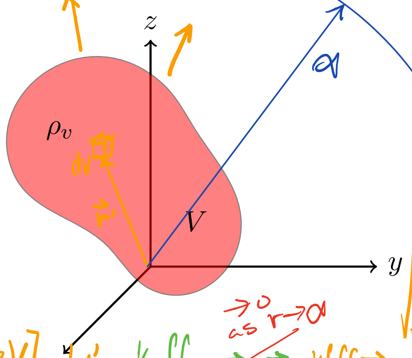
$W_T = W_1 + W_2 + W_3 + \dots + W_N = W_E$

$W_E = \frac{1}{2}\iiint_V \rho_v(\vec{r}\,')\,(\vec{V}\,')\,dv'$

Gauss's Law: (point form)

$\rho_v = \nabla \cdot \vec{D}$

$W_E = \frac{1}{2}\iiint \nabla \cdot \vec{D}\,\vec{V}\,dv'$




$W_E = \frac{1}{2}\underbrace{\iiint_V [\nabla\cdot(V\vec{D}) - \vec{D}\cdot\nabla V]\,dv'}_{\text{use divergence theorem.}} = \frac{1}{2}\cancel{\oint\!\!\oint_S V\vec{D}\cdot d\vec{s}} - \frac{1}{2}\iiint_V \vec{D}\cdot\nabla V\,dv' \rightarrow = \frac{1}{2}\iiint_V \varepsilon_0|\vec{E}|^2 dv'$

$\rightarrow 0$ as $r \rightarrow \infty$

sphere w/ $\infty$ radius.

$1/r^2$ $\quad r^2$

$$\boxed{W_E = \frac{1}{2}\iiint_V \vec{D}\cdot\vec{E}\,dv'}$$

What if we reverse the order in which we brought in the charges?

$W_E' = W_3' + W_2' + W_1' \equiv W_E$

$= 0 + Q_2 V_{23} + Q_1 V_{12} + Q_1 V_{13}$

$W_E + W_E' = 2W_E$

$= Q_1(\underbrace{V_{12} + V_{13}}_{V_1}) + Q_2(\underbrace{V_{21} + V_{23}}_{V_2}) + Q_3(\underbrace{V_{31} + V_{32}}_{V_3})$

$W_E = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3)$

Energy density per unit volume.

$$\frac{dW_E}{dV} = \frac{1}{2}\vec{D}\cdot\vec{E}$$



# Properties of Materials

In general: $V_1 = V_{12} + V_{13} + V_{14} + \cdots V_{1N}$

$V_2 = V_{21} + V_{23} + V_{24} + \cdots V_{2N}$

$\vdots$

For N charges

$$\boxed{W_E = \frac{1}{2}\sum_{m=1}^{N} Q_m V_m}$$

We can broadly characterize materials using their *conductivity* $\sigma$ [S/m or ℧/m or $\Omega^{-1}\mathrm{m}^{-1}$].

- Materials with *high conductivity* $(\sigma \gg 1)$ are *conductors*.
- Materials with *low conductivity* $(\sigma \ll 1)$ are *dielectrics*.
- Materials between these extremes are *semiconductors*.

We will be mainly concerned with conductors and dielectrics in this course.

A perfect electrical conductor (PEC) has $\sigma \to \infty$.
A perfect dielectric has $\sigma = 0$.

# Current $I$ and Current Density $\boldsymbol{J}$

- **Voltage** (or potential difference) and **current** are fundamental quantities in EE.
- Consider the current flowing in a wire as an example.