Position, length and angle in C^N

Position: Determined by an N dimensional vector of ~~complex~~ of complex numbers

$$x = [x[0], x[1], x[2], \cdots\cdots x[n-1]]$$

$$x \in C^N \qquad x[k] \in C \qquad 0 < k < N-1$$

length:

Norm-squared: $$\|x\|^2 = \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} x[n](x[n])^*$$

Inner product for complex vectors $x \& y \in C^N$

$$\langle x, y \rangle = \sum_{n=0}^{N-1} x[n]\, y^*[n] \qquad \langle x, y \rangle \in C$$

properties of inner product. :

1. $\langle x, y \rangle^* = \left(\sum_{n=0}^{N-1} x[n]\, y^*[n] \right)^* = \sum_{n=0}^{N-1} x^*[n]\, y[n] = \langle y, x \rangle$

2. $\|x\|^2 = \langle x, x \rangle \in \mathbb{R}_+$ (positive real number)

3. $\langle \gamma x, y \rangle = \gamma \langle x, y \rangle \qquad x \& y \in C^N$

   $\langle x, \gamma y \rangle = \gamma^* \langle x, y \rangle \qquad \gamma \in C$

4. $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

Angle: $\cos(\theta) = \frac{\|\alpha_{opt.} y\|}{\|x\|}$, θ is the angle btw x & $y \in \mathbb{C}^N$

$$\alpha_{opt.} = \frac{\langle x, y\rangle}{\|y\|^2} \Rightarrow \cos(\theta) = \frac{\langle x, y\rangle}{\|x\|\|y\|} \quad //.$$

(*) Find the angle between y and $x - \alpha_{opt} y$, where $\alpha_{opt} = \frac{\langle x, y\rangle}{\|y\|^2}$
$x, y \in \mathbb{C}^N$

$$z = x - \alpha_{opt}\, y \qquad \cos(\theta) = \frac{\langle z, y\rangle}{\|z\|\,\|y\|}$$

$$\langle y, z\rangle = \langle y, x - \alpha_{opt.} y\rangle = \langle y, x\rangle - \langle y, \alpha_{opt.} y\rangle$$

$$= \langle y, x\rangle - \alpha_{opt.}^* \langle y, y\rangle$$

$$= \langle y, x\rangle - \alpha_{opt.}^* \|y\|^2$$

$$= \langle y, x\rangle - \left(\frac{\langle x, y\rangle}{\|y\|^2}\right)^* \|y\|^2$$

$$= \langle y, x\rangle - \frac{(\langle x, y\rangle)^*}{\cancel{(\|y\|^2)^*}} \cancel{\|y\|^2}$$

$$= \langle y, x\rangle - \langle y, x\rangle = 0$$

$$\theta = \cos^{-1}(0) = 90^\circ \quad //$$

Projection onto subspaces:

To get a better approx. of x, we can find its projection onto a subspace obtained by spanning multiple vectors $\{y_0, y_1, y_2, \dots, y_{N-1}\}$

Call that subspace S:

$$S = \left\{ z \in \mathbb{C}^N,\ z = \sum_{k=0}^{N-1} \alpha_k y_k,\ \alpha_k \in \mathbb{C} \right\}$$

→ A set of linearly independent vectors $\{y_0, y_1, \cdots y_{N-1}\}$ that defines S is said to be the basis for S.

→ If each pair of vectors in the $\{y_0, y_1, y_2, \cdots y_{N-1}\}$ is an orthogonal pair. then $\{y_0, y_1, \cdots y_{N-1}\}$ is a set of linearly independent vectors.

$$\cos(\theta_{ij}) = \frac{\langle y_i, y_j \rangle}{\|y_i\| \|y_j\|} = 0 \qquad \langle y_i, y_j \rangle = \begin{cases} \|y_i\|^2 & i=j \\ 0 & i \neq j \end{cases}$$

→ In this case, the best projection of x (arbitrary N dimensional complex vector) on the k-th vector, y_k, is given by:

$$\alpha_{opt,k} = \frac{\langle x, y_k \rangle}{\|y_k\|^2}$$

→ DT Fourier Series of the signal x is the projection of x onto the subspace spanned by complex exponentials:

$$\phi_k[n] = e^{-j 2\pi/N kn}, \quad n = 0, 1, \cdots, N-1$$

$$\langle \phi_m[n], \phi_h[n] \rangle = \begin{cases} N & m = h \\ 0 & m \neq h \end{cases}$$

$$\alpha_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j 2\pi/N kn} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \left(e^{j(2\pi/N)kn} \right)^*$$

$$= \frac{1}{N} \langle x[n], \phi_k[n] \rangle = \frac{\langle x[n], \phi_k[n] \rangle}{\|\phi_k\|^2}$$