Lecture 25: Magnetic Forces and Torques

ECE221: Electric and Magnetic Fields

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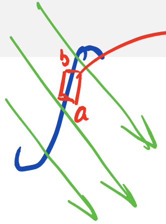
Outline

1. Magnetic Force on a Current-Carrying Conductor
2. Force and Torque on a Current-Carrying Loop
3. Magnetic Dipole

Force on a Current Element

Review:

$$\boldsymbol{F} = \int_C I d\boldsymbol{\ell} \times \boldsymbol{B}$$



Zoom: $\vec{B}$ is uniform over a very short length "ℓ" of conductor.

$$\vec{F} = I\int_a^b \vec{\ell} \times B$$

$$= I \vec{\ell}_{ab} \times B$$

If the current path is closed:

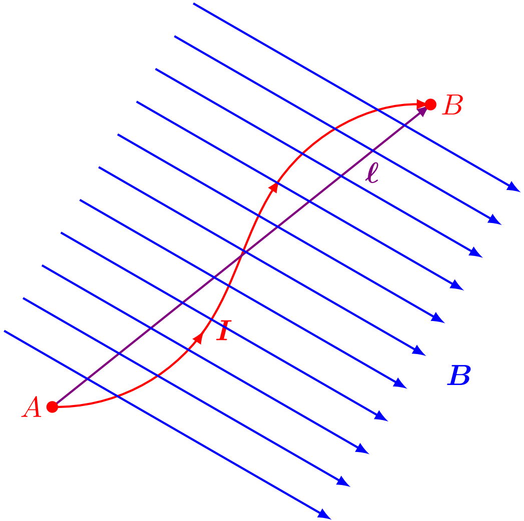
$$\boldsymbol{F} = \oint_C I d\boldsymbol{\ell} \times \boldsymbol{B}$$

$\vec{F} = q\vec{E}$ $\quad \vec{E} = \dfrac{\vec{F}}{q}$

Magnetic flux density B can be thought of as the force per unit current element, much as electric field is force per unit charge.

$$\boldsymbol{F} = \iint_s \boldsymbol{K} \times \boldsymbol{B} dS = \iiint_V \boldsymbol{J} \times \boldsymbol{B} dv$$

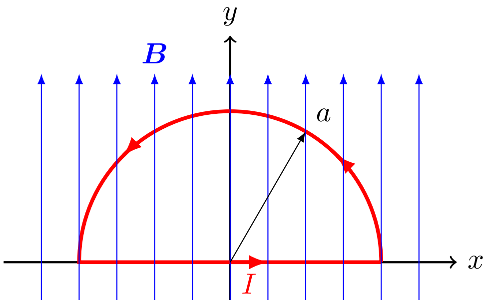
Contour Integrals for Force Calculations



If $\vec{B}$ is uniform, it does not matter which path we take from a to b when calculating $\vec{F}$

Example: Force on a Semicircular Conductor

A semicircular loop lies in a uniform field $\boldsymbol{B} = \hat{\boldsymbol{y}} B_0$. Calculate the force on the loop if it carries a current I.



Determine force $\vec{F}_1$ on straight section.

$\ell = 2a$; $\vec{\ell} = \hat{x}\, 2a$

$\vec{F}_1 = I\vec{\ell} \times \vec{B} = \hat{x}\, 2aI \times \hat{y} B_0 = \hat{z}\, 2IaB_0$ [N]

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Force $\vec{F}_2$ on curved section

$d\vec{\ell} = a\,d\phi\,\hat{\phi}$ $\qquad \hat{\phi} \times \hat{y} = -\hat{z}\sin\phi$

$\vec{F}_2 = I\int d\vec{\ell} \times \vec{B} = -\hat{z}\, I \int_0^{\pi} a B_0 \sin\phi\, d\phi = -2\hat{z}\, IaB_0$

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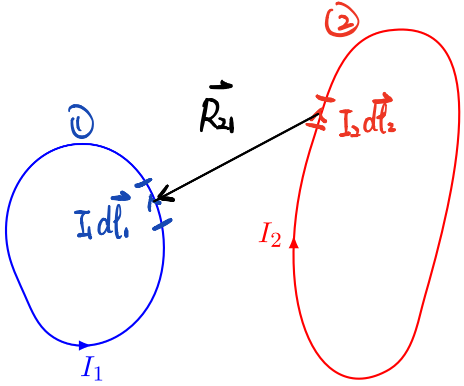
$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 = 0$

... but there will be some torque on the loop

Forces Between Two Current Elements



Find force on loop ①

Find the (differential) force on element $I_1 d\vec{l}_1$ produced by $I_2 dl_2$

$$d(d\vec{F}_1) = I_1 d\vec{l}_1 \times d\vec{B}_2$$

$$d\vec{B}_2 = \mu_0 \frac{I_2 d\vec{l}_2 \times \vec{R}_{21}}{R_{21}^3}$$

$$d(d\vec{F}_1) = \frac{\mu_0 I_1 d\vec{l}_1 \times (I_2 d\vec{l}_2 \times \vec{R}_{21})}{4\pi R_{21}^3}$$

$$d\vec{F}_1 = \frac{\mu_0 I_1 I_2}{4\pi} d\vec{l}_1 \times \oint_{②} \frac{d\vec{l}_2 \times \vec{R}_{21}}{R_{21}^3}$$

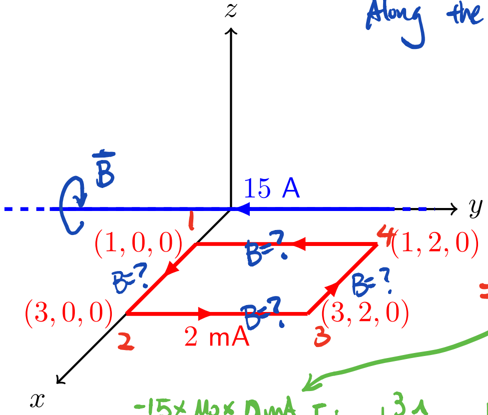
$$\vec{F} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{①} \oint_{②} \frac{d\vec{l}_1 \times (d\vec{l}_2 \times \vec{R}_{21})}{R_{21}^3}$$

Example: Force on a Loop

Calculate the force on the loop if it carries a current as shown.

magnetic analog of Coulomb's Law

$$F_1 = \frac{Q_1 Q_2 \vec{R_{21}}}{4\pi\epsilon_0 R_{21}^3}$$



Along the loop: $\vec{H} = \hat{z}\frac{I}{2\pi x} = \frac{15}{2\pi x}\hat{z}$ [A/m]

$$\vec{B} = \mu_0\vec{H} = \frac{15\mu_0}{2\pi x}\hat{z} \text{ [T]}$$

$$\vec{F} = -I\oint B\times d\vec{l} = \int_1^2 + \int_2^3 + \int_3^4 + \int_4^1$$

$$= \frac{-15\mu_0 I}{2\pi}\left[\int_{x=1}^{x=3}\frac{\hat{z}}{x}\cdot dx\,\hat{x} + \int_{y=0}^{2}\frac{\hat{z}}{3}\times dy\,\hat{y} + \int_{x=3}^{1}\frac{\hat{z}}{x}\times dx\,\hat{x} + \int_{y=2}^{0}\frac{\hat{z}}{1}\times dy\,\hat{y}\right]$$

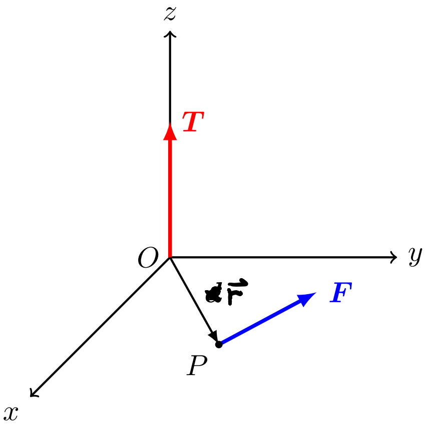
$$= \frac{-15\times\mu_0\times 2\text{mA}}{2\pi}\left[\ln x\Big|_1^3\hat{y} + \frac{1}{3}y\Big|_0^2(-\hat{x}) + \ln x\Big|_3^1\hat{y} + y\Big|_2^0(-\hat{x})\right]$$

$= -8\hat{x}\ \mu N$ (non-zero)!

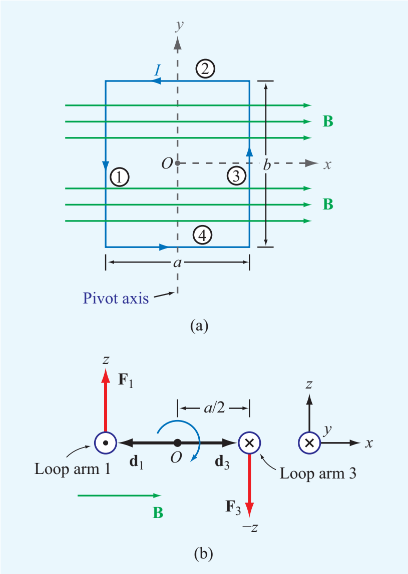
Force and Torque on a Current-Carrying Loop

Torque, or **moment**, is defined about a point O such that if $\boldsymbol{r}$ (a **moment arm** joins O and P,

$$\boldsymbol{T} = \boldsymbol{r} \times \boldsymbol{F} \quad [\mathrm{N \cdot m}]$$



Torque Analysis: Magnetic Field in Plane of a Loop



Remainder: there's no net force on the loop.

① $\vec{F}_1 = I(-\hat{y}b) \times (\hat{x}B_0) = \hat{z}\, I b B_0$

② $\vec{F}_2 = 0 \quad (B \parallel I d\vec{l})$

③ $\vec{F}_3 = I(\hat{y}b) \times (\hat{x}B_0) = -\hat{z}\, I b B_0$

④ $\vec{F}_4 = 0 \quad (B \parallel I d\vec{l})$

Torque about O:

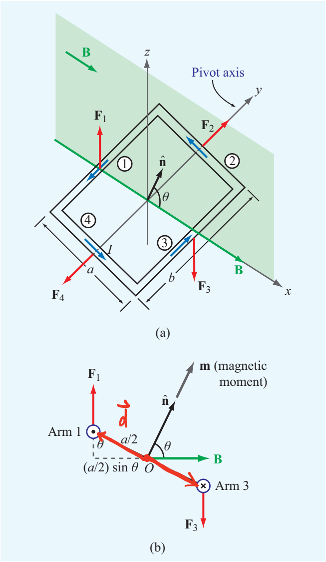
$$\vec{T} = \vec{d}_1 \times \vec{F}_1 + \vec{d}_3 \times \vec{F}_3$$

$$= \left(\frac{-a}{2}\hat{x}\right) \times \left(\hat{z}\, I b B_0\right) + \left(\frac{a}{2}\hat{x}\right) \times \left(-\hat{z}\, I b B_0\right)$$

$$= \hat{y}\, I \underbrace{ab}_{} B_0$$

$$\vec{T} = \hat{y}\, I A_{loop} B_0$$

Torque Analysis: B Perpendicular to Axis of a Loop



$\vec{F}_2$ & $\vec{F}_4$ are non-zero (current is at an angle) but they cancell out $I\vec{l}_2 \times \vec{B} \neq 0$

$$\vec{T} = (-\hat{d}\,\tfrac{a}{2}) \times (\hat{z}\, I b B_0) + (\hat{d}\,\tfrac{a}{2}) \times (-\hat{z}\, I b B_0)$$

$$= \hat{y}\, I \tfrac{a}{2} b B_0 \sin\theta + \hat{y}\, \tfrac{Iab}{2} B_0 \sin\theta$$

$$= \hat{y}\, I ab B_0 \sin\theta = \hat{y}\, I A_{loop} B_0 \sin\theta$$

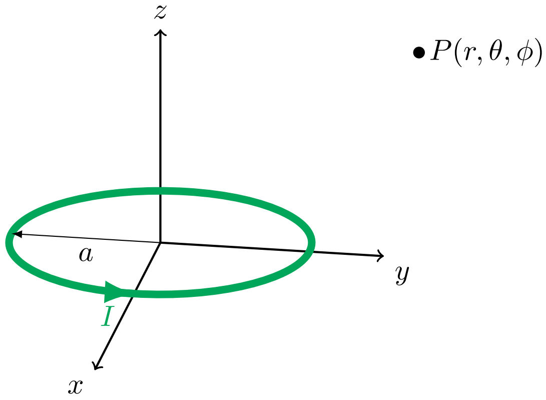
$$\vec{T} \propto \sin\theta$$

$\therefore$ when loop is horizontal, T is maximum.
loop is perpendicular, T is 0

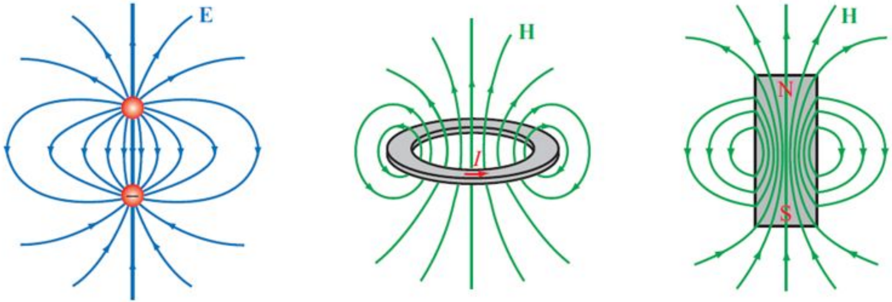
The Magnetic Dipole

Recall the circular loop we analyzed before. If the loop is **small** such that $r \gg a$, then

$$\boldsymbol{A} \approx \frac{\mu_0 I \pi a^2 \sin\theta \hat{\boldsymbol{\phi}}}{4\pi r^2}$$



Dipoles



(a) Electric dipole (b) Magnetic dipole (c) Bar magnet

Source: Ulaby, Ravaioli: *Fundamentals of Applied Electromagnetics*, 7th ed.