Lecture 21: Ampère's Law

ECE221: Electric and Magnetic Fields

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Outline

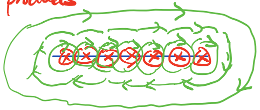
1. Examples of Applying Biot-Savart's Law
2. Ampère's Law
3. Examples of Applying Ampère's Law
4. Point Form of Ampère's Law

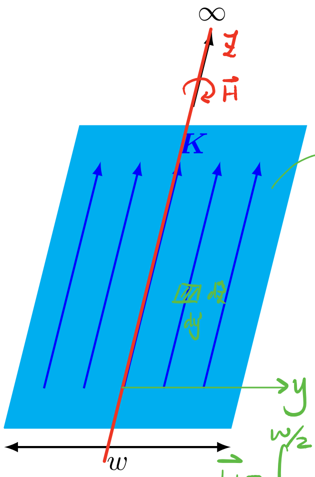
Example: Infinitely Long Current Strip or Sheet

↳ superposition of ∞-long wire filaments.

A single filament along z-axis produces

$$\vec{H} = \frac{I}{2\pi\rho}\hat{\phi}$$





$$d\vec{H} = \frac{I d\vec{l} \times \vec{R}}{4\pi R^3} = \frac{\vec{K}\times\vec{R}\, ds}{4\pi R^3}$$

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ $\quad \vec{r}\,' = y'\hat{y}$ $\quad \vec{R} = x\hat{x} + (y-y')\hat{y} + z\hat{z}$

→ don't expect any change w.r.t. z

$$d\vec{H} = K(dy'dz')\frac{\hat{z}\times\vec{R}}{4\pi R^3}$$

$$\hat{z}\times\vec{R} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ x & y-y' & z \end{vmatrix} = -(y-y')\hat{x} + x\hat{y}$$

$$\vec{H} = \int_{-w/2}^{w/2}\int_{-\infty}^{\infty} K\left[\frac{-\hat{x}(y-y') + x\hat{y}}{4\pi\left[(x^2)+(y-y')^2\right]^{3/2}}\right] \int dz'dy'$$

Ampère's Law

- We saw in electrostatics that applying Coulomb's law to problems was very tedious and was simplified by the use of Gauss' Law.
- Is there the same kind of thing for magnetic fields?
- Yes $\rightarrow$ Ampère's Law, which can be derived from Biot-Savart Law (advanced topic involving *magnetic potential* [later])

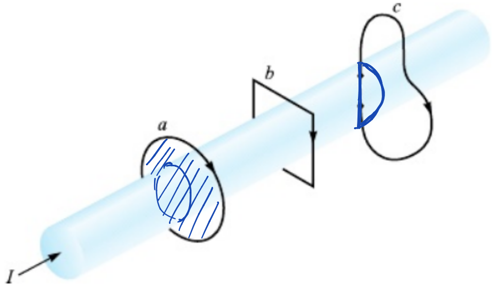
Ampère's Law

The line integral of $\boldsymbol{H}$ about any closed path is equal to the current enclosed by that path,

$$\oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = I$$

Ampèrian Contours

Examples of Ampèrian Contours

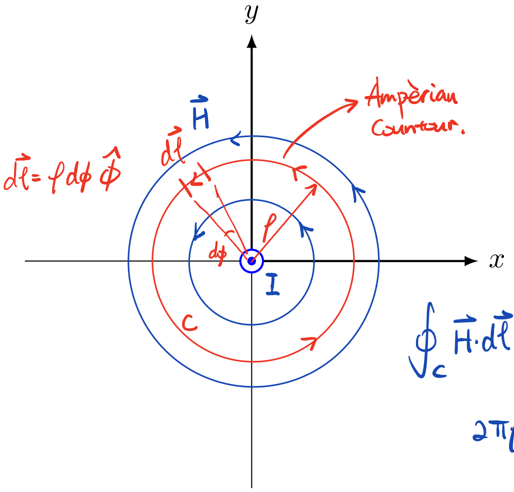


Source: Hayt and Buck, *Engineering Electromagnetics*, 8/e

$$I = \oint_a \vec{H} \cdot d\vec{l} = \oint_b \vec{H} \cdot d\vec{l} > \oint_c \vec{H} \cdot d\vec{l}$$

Filamentary Wire Along z-axis

Find the magnetic field $\boldsymbol{H}$ everywhere.



$$\oint_{\boxed{C}} \boldsymbol{H} \cdot d\boldsymbol{l} = I$$

1. choose C so that:
   a. H & dl are parallel so that $\vec{H} \cdot \vec{dl} = H\,dl$ (scalar product)
   b. H is constant over C so that it can be pulled outside integral.

$$\oint_C \vec{H} \cdot \vec{dl} = \int_0^{2\pi} H_\phi \hat{\phi} \cdot \rho\, d\phi\, \hat{\phi} = H_\phi \rho\, 2\pi = 2\pi \rho H_\phi$$

$$2\pi\rho H_\phi = I$$

$$H_\phi = \frac{I}{2\pi\rho}$$

$$\boxed{\vec{H} = \frac{I}{2\pi\rho}\hat{\phi}}$$

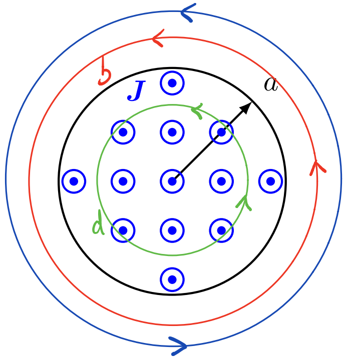
matches w/ Biot-Savart's Law.

Thick Wire

Find the magnetic field $\boldsymbol{H}$ everywhere.

The total current carried by the wire is I_T

$$\oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = I$$



$$\oint_b \vec{H} \cdot d\vec{l} > \oint_d \vec{H} \cdot d\vec{l}$$

For $0 \le \rho \le a$

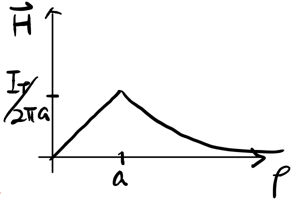
$$\oint_d \vec{H} \cdot d\vec{l} = 2\pi\rho H_\phi = I = \frac{\pi\rho^2}{\pi a^2} I_T$$

$$\boxed{\vec{H} = \frac{\rho}{2\pi a^2} I_T \hat{\phi}}$$

For $\rho > a$.

$$\oint_b \vec{H} \cdot d\vec{l} = 2\pi\rho H_\phi = I_T$$

$$\boxed{\vec{H} = \frac{I_T}{2\pi\rho} \hat{\phi}}$$



Infinite Current Sheet

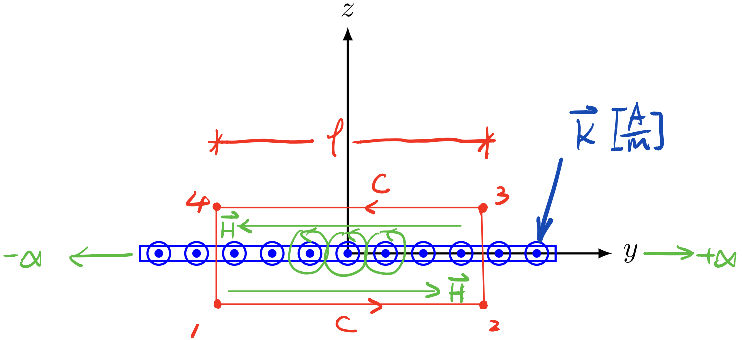
$$\oint \vec{H}\cdot d\vec{l} = \int_1^2 + \cancel{\int_2^3} + \int_3^4 + \cancel{\int_4^1} = I_{encl.}$$

$\vec{H}\cdot d\vec{l} = 0$ b/c $\vec{H} \perp d\vec{l}$

$$= \int_1^2 H\cdot dl + \int_3^4 H\cdot dl = H_y\int_1^2 dl + H_y\int_3^4 dl$$

$$= H_y l + H_y l = 2H_y l = Kl$$

Find the magnetic field $\boldsymbol{H}$ everywhere.

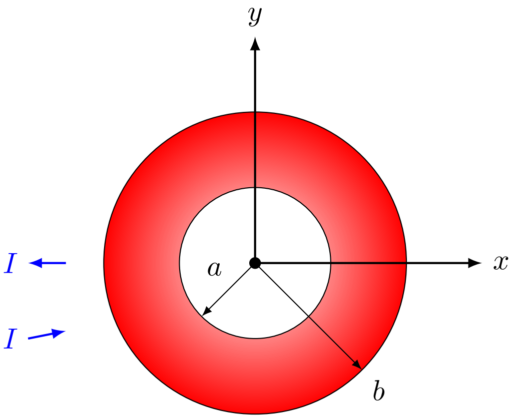


$$\boxed{\vec{H} = \begin{cases} -\hat{y}\,\frac{K}{2} & z>0 \\ \hat{y}\,\frac{K}{2} & z<0 \end{cases}}$$

* we will see this again when we study B.C.S

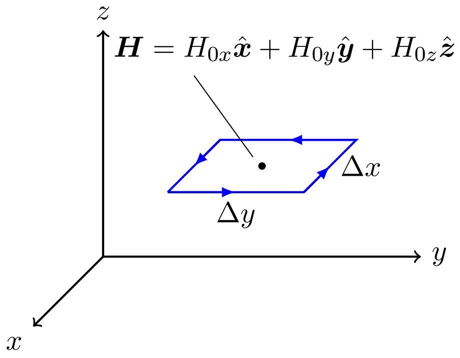
Toroidal Coil in xy Plane

Find the magnetic field $\boldsymbol{H}$ everywhere.



Point form of Ampère's Law

Let's try applying Ampère's Law to an *infinitesimal* small loop deep inside a current distribution $\boldsymbol{J}$.



Summary of Loop Analysis

1. Loop in xy plane:

$$[\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r})]_z = \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = J_z$$

This process can be repeated with the loop rotated to be placed in the xz and yz planes:

1. Loop in xz plane:

$$[\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r})]_y = \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) = J_y$$

2. Loop in yz plane:

$$[\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r})]_x = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) = J_x$$

Point Form of Ampère's Law

Point (Differential) Form of Ampère's Law

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{\boldsymbol{z}} = \boldsymbol{J}$$