Lecture 30: Mutual Inductance, Magnetic Energy

ECE221: Electric and Magnetic Fields

Prof. Sean V. Hum

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Outline

Mutual Inductance

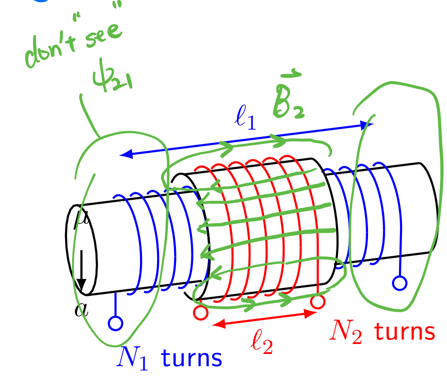
Mutual flux linkage

$$\Lambda_{12} = N_2\Psi_{12} \quad \Lambda_{21} = N_1\Psi_{21}$$

Mutual inductance

$$L_{12} = \frac{\Lambda_{12}}{I_1} \quad L_{21} = \frac{\Lambda_{21}}{I_2}$$

Reciprocity



$$n_1 = \frac{N_1}{\ell_1} \text{ turns/m}$$
$$n_2 = \frac{N_2}{\ell_2} \text{ turns/m}$$

When coil 1 is the source,

$$\Lambda_{12} = N_2\Psi_{12} = N_2\frac{\mu N_1 \pi a^2 I_1}{\ell_1}$$

$$L_{12} = \frac{\Lambda_{12}}{I_1} = \mu n_1 n_2 \ell_2 \pi a^2$$

When coil 2 is the source, coil 1 only sees a flux linkage of

$$\Lambda_{21} = N_1\Psi_{21} = \boxed{n_1\ell_2}\underbrace{\frac{\mu N_2 \pi a^2 I_2}{\ell_2}}_{\psi_{21}}$$

part of coil 1 that sees ψ_{21}

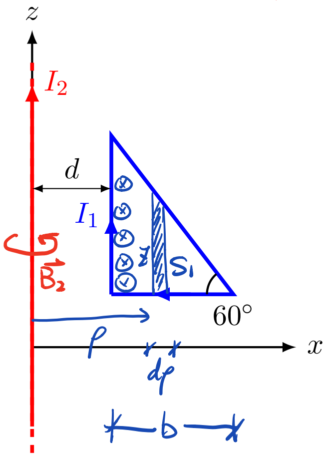
$$L_{21} = \frac{\Lambda_{21}}{I_2} = \mu n_1 n_2 \ell_2 \pi a^2 = L_{12}!$$

$L_{21} = L_{12} = M$

Example: Mutual Inductance Between Wire and Loop

Determine the mutual inductance between an infinitely long wire along the z-axis and the triangular loop as shown.

Find the flux Ψ_{21} produced by I_2 over coil 1



$$\vec{B}_2 = \frac{\mu_0 I_2}{2\pi\rho}\hat{\phi} \qquad \Psi_{21} = \int_{S_1} \vec{B}_2 \cdot d\vec{s}_1$$

$$d\vec{s}_1 = z\,d\rho\,\hat{\phi} \qquad z = [\rho - (d+b)]\overbrace{\tan 60^\circ}^{\sqrt{3}}$$

$$\Psi_{21} = \Lambda_{21} = -\frac{\sqrt{3}\,\mu_0 I_2}{2\pi}\int_d^{d+b} \frac{1}{\rho}[\rho - (d+b)]\,d\rho$$

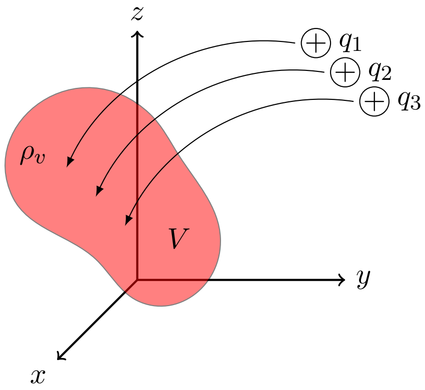
$$= -\frac{\sqrt{3}\,\mu_0 I_2}{2\pi}\left[(d+b)\ln(1+b/d) - b\right]$$

$$L_{21} = \frac{\Lambda_{21}}{I_2}$$

Reminder: Electric Energy

Recall what we did to find electric energy stored in a charge distribution:

1. Starting in empty space we brought in a charge from infinity;
2. Then we brought in another charge and calculated the work needed to position the charge;
3. Then another...
4. Repeat until we have a volume charge density ρ_v

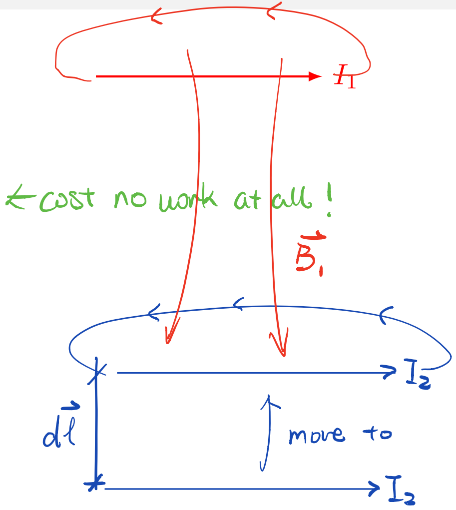


$$W_E = \frac{1}{2}\sum_{m=1}^{N} q_m V_m = \frac{1}{2}\iiint_V \rho_v V \, d\nu$$

Magnetic Energy

Can we apply the same process to compute magnetic energy? Let's try with some line currents.

1. Starting in empty space we bring in a line current I_1 from infinity;
2. Then we bring in another line current I_2 from infinity:
   - There is a force between the two conductors, so work is involved in moving the current I_2
   - How would we compute the incremental work done?



$dW_m = ?$

Magnetic Energy

For now, let's leverage the dual relationship between electric and magnetic fields:

Electrostatics

$$W_e = \frac{1}{2} \iiint_V \boldsymbol{D} \cdot \boldsymbol{E} d\nu \text{ [J]}$$

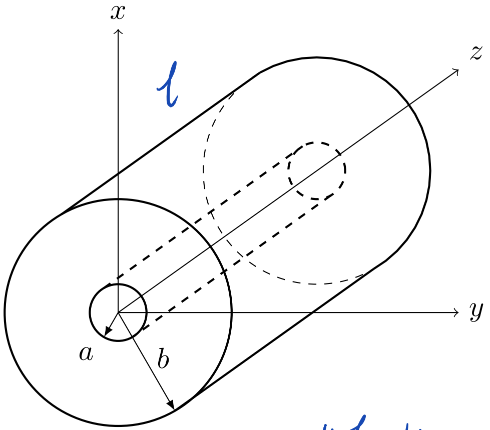
$$\frac{\partial W_e}{\partial \nu} = \frac{1}{2} \boldsymbol{D} \cdot \boldsymbol{E} \text{ [J/m}^3\text{]}$$

Magnetostatics

$$W_m = \frac{1}{2} \iiint_V \boldsymbol{B} \cdot \boldsymbol{H} d\nu \text{ [J]}$$

$$\frac{\partial W_m}{\partial \nu} = \frac{1}{2} \boldsymbol{B} \cdot \boldsymbol{H} \text{ [J/m}^3\text{]}$$

Example: Magnetic Energy Stored in a Coaxial Cable



$$\vec{H} = \frac{I}{2\pi\rho}\hat{\phi}$$

$$\vec{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}$$

$$\frac{1}{2}\vec{B}\cdot\vec{H} = \frac{\mu_0 I^2}{(2\pi)^2\rho^2} \times \frac{1}{2}$$

$$W_m = \int_0^{l}\int_0^{2\pi}\int_a^{b} \frac{1}{2}\,\frac{\mu_0 I^2}{(2\pi)^2\rho^2}\cdot\rho\, d\rho\, d\phi\, dz$$

$$= 2\pi l \times \frac{\mu_0 I^2}{2(2\pi)^2}\int_a^b \frac{1}{\rho}\,d\rho$$

$$= \frac{\mu_0 I^2 l}{4\pi}\ln(b/a)$$

but $L = \frac{\mu_0 l}{2\pi}\ln b/a$

$$\boxed{W_m = \frac{1}{2}LI^2}$$

Summary of Electrostatics and Magnetostatics

Maxwell's equations so far are:

Law	Point Form	Differential Form
Gauss' Law	$\nabla \cdot \boldsymbol{D} = \rho_v$	$\oint_S \boldsymbol{D} \cdot d\boldsymbol{s} = Q$
Conservative E-field	$\nabla \times \boldsymbol{E} = 0$	$\oint_C \boldsymbol{E} \cdot d\boldsymbol{\ell} = 0$
No magnetic monopoles	$\nabla \cdot \boldsymbol{B} = 0$	$\oint_S \boldsymbol{B} \cdot d\boldsymbol{s} = 0$
Ampère's Law	$\nabla \times \boldsymbol{H} = \boldsymbol{J}$	$\oint_C \boldsymbol{H} \cdot d\boldsymbol{\ell} = I$

The top two equations are **completely decoupled** from the bottom two.

Faraday's Law

- We have seen that a current can produce a magnetic field
- Michael Faraday wondered: Is the reverse possible?