Lecture 14: The Uniqueness Theorem and Solving Poisson's/Laplace's Equation

ECE221: Electric and Magnetic Fields

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Outline

Poisson's Equation

$$\boldsymbol{\nabla}^2 V = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} V = -\frac{\rho_v}{\epsilon}$$

Laplacian:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$ Cylindrical

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$ Spherical.

Laplace's Equation

- If ρ_v is zero, this means:
  - This means that the volume charge density is zero, but we could still have point, line, and surface charges at singular locations and the formula applies every except those locations;
  - Or that we have a volumetric charge density present, but so long as we are not in the volume V where the density is present, $\rho_v = 0$ (source free region)
- Under these conditions,

$$\nabla^2 V = 0$$

- Fundamental question: **Is the solution to Laplace's Equation unique?**

The Uniqueness Theorem

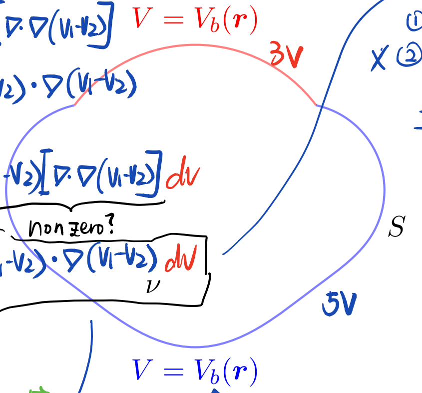
vector identity:

$\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f)$

Let $f = V_1 - V_2$, $\vec{A} = \nabla(V_1 - V_2)$

$\nabla\cdot[(V_1-V_2)\nabla(V_1-V_2)] = (V_1-V_2)[\nabla\cdot\nabla(V_1-V_2)] + \nabla(V_1-V_2)\cdot\nabla(V_1-V_2)$

Consider a volume ν bounded by an equipotential surface S at voltage $V = V_b$.



$\underbrace{\iiint_V \nabla\cdot[(V_1-V_2)\nabla(V_1-V_2)]\,dv}_{\text{div. theorm}} = \iiint_V \underbrace{(V_1-V_2)[\nabla\cdot\nabla(V_1-V_2)]}_{\text{non zero?}}\,dv + \iiint_V \nabla(V_1-V_2)\cdot\nabla(V_1-V_2)\,dv$

$\oint_S [(V_1-V_2)\nabla(V_1-V_2)]\cdot d\vec{s}$

$= \oint_S [(V_{1b}-V_{2b})\cancel{\nabla(V_{1b}-V_{2b})}]\cdot d\vec{s}$ 0

↑ "boundary"

zero if:

① integrand is zero over V, or

✗ ② integrand is +ve in some region & -ve in others such that they cancel out

∵ 它括$(\ )^2 > 0$ 不可能是这个

- Let $V = V_1$ be one solution to $\nabla^2 V = 0$ so that $\nabla^2 V_1 = 0$.
- Suppose another solution $V = V_2$ is also possible, $\nabla^2 V_2 = 0$, meaning that the solution is not unique. Let's try to prove this is possible.

$\nabla^2(V_1 - V_2) = 0$

V_1 must satisfy boundary conditions $V_{1b} = V_b$

V_2 " " " " $V_{2b} = V_b$

Parallel Plate Setup

∴ 除 $\nabla(V_1-V_2)=0$

$\Rightarrow V_1-V_2$ is constant.

But V_1 & V_2 must meet Boundary cond.

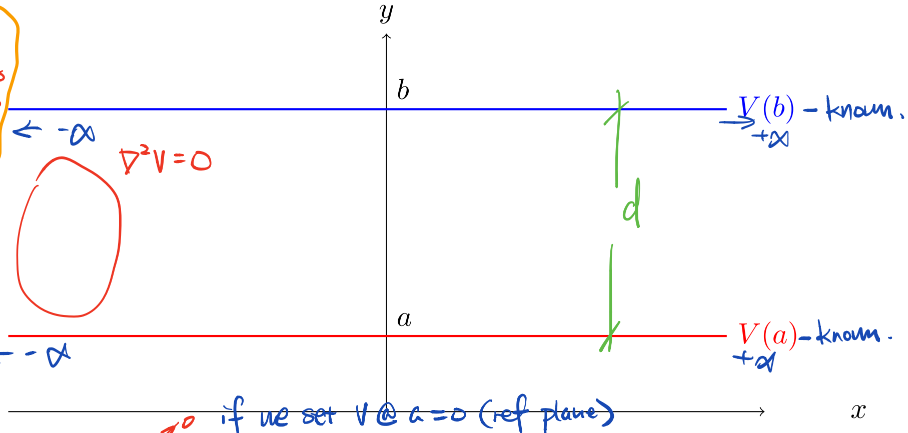
$V_1 = V_{1b} = V_b$ on S

$V_2 = V_{2b} = V_b$ on S

$\boxed{V_1-V_2=0}$

tells us there's only one unique solution.

Find the voltage and electric field in the region $a \leq y \leq b$.



$$\cancel{\frac{\partial^2 V}{\partial x^2}} + \frac{\partial^2 V}{\partial y^2} + \cancel{\frac{\partial^2 V}{\partial z^2}} = 0$$

$\int$ once $\frac{\partial V}{\partial y} = A \leftarrow$ constant

$\int$ again $V(y) = Ay + B$

Apply boundary condition to find

if we set V @ $a = 0$ (ref plane)

$V(a) = \cancel{V_a}\ 0 = Aa + B \quad B = -Aa \quad V(b) = V_0 = Ab - A(a)$

$\Rightarrow A = \frac{V_0}{\ }$

A & B.

Coaxial Cable (infinite)

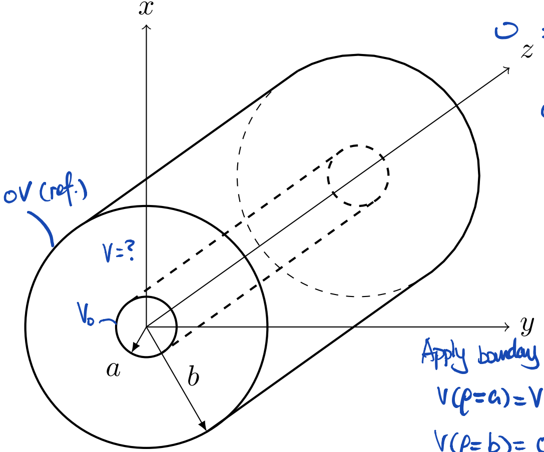
$\Rightarrow A = \frac{V_0}{b-a} \quad B = -\frac{V_0}{b-a}a$

$\therefore V(y) = \frac{V_0}{b-a}y - \frac{V_0 a}{b-a}$ //

$E = -\nabla V = \frac{-V_0}{b-a}\hat{y}$

$= \frac{-V_0}{d}\hat{y}$

Find the voltage and electric field in the region $a \leq \rho \leq b$.



$0 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \cancel{\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2}} + \cancel{\frac{\partial^2 V}{\partial z^2}}$

$0 = \frac{\partial}{\partial \rho}\left(\rho\frac{dV}{\partial \rho}\right)$

$A = \rho\frac{dV}{\partial \rho}$

$\frac{\partial V}{\partial \rho} = \frac{A}{\rho}$

$V(\rho) = A\ln\rho + B$

Apply boundary conditions.

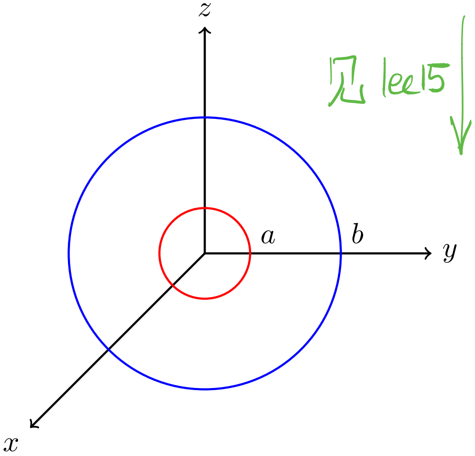
$V(\rho = a) = V_0 = A\ln a + B$

$V(\rho = b) = 0 = A\ln a + B \Rightarrow B = -A\ln b$

$\Rightarrow A\ln a - A\ln b = A\ln\left(\frac{a}{b}\right) = V_0$

Voltage Between Two Spheres

Find the voltage and electric field in the region $a \leq r \leq b$.



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$$A = \frac{V_0}{\ln\left(\frac{a}{b}\right)}$$

$$V(\rho) = \frac{V_0}{\ln(a/b)} \ln\rho - \frac{V_0}{\ln(a/b)} b$$

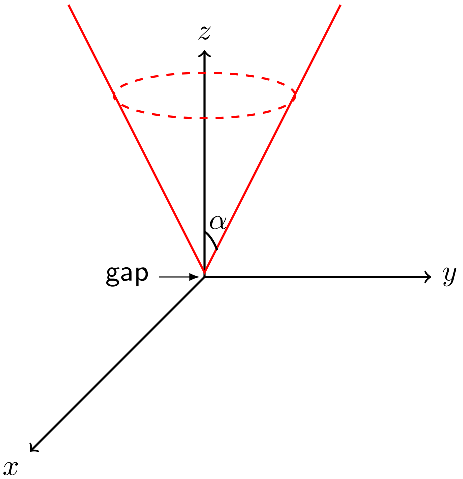
$$\boxed{V(\rho) = \frac{V_0 \ln(\rho/b)}{\ln(a/b)}}$$

$$\nabla V = \hat{\rho}\frac{\partial V}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial V}{\partial \phi} + \hat{z}\frac{\partial V}{\partial z}$$

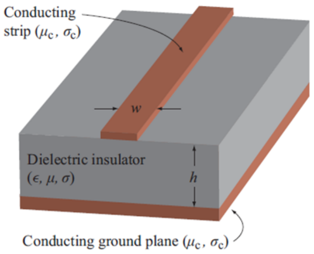
$$\boxed{\vec{E} = -\nabla V = \frac{\hat{\rho} V_0}{\ln(a/b)} \frac{1}{\rho/b} \cdot \frac{1}{b}}$$

Voltage Between Infinite Cone and Infinite Ground Plane

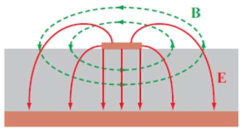
Find the voltage and electric field in the region $\alpha \leq \theta \leq \pi/2$, if there is an infinite ground plane located in the plane $\theta = \pi/2$.



Application of Laplace's Equation: Planar Transmission Lines



(a) Longitudinal view



(b) Cross-sectional view with **E** and **B** field lines

- Laplace's equation can be used to find the voltage and electric fields between conductors forming *planar transmission lines* on printed circuit boards.
- The high-frequency characteristics of such structures are of paramount importance in:
    - RF circuit design
    - High speed electronics (e.g. signal integrity of IC interconnects)

Example: IC Interposer

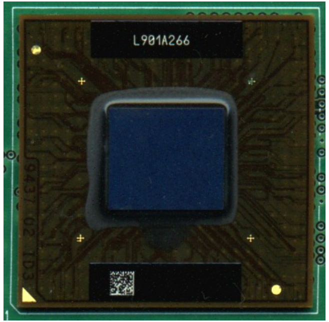
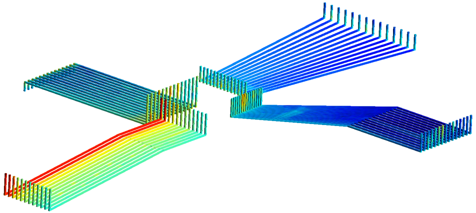


Image courtesy Wikipedia



- Knowing the voltage and current distribution produce by the conductors allows prediction of:
  - The speed of light along the transmission line
  - The *characteristic impedance* of the transmission line
- We will learn more about transmission lines later (and in ECE320).