Discrete time Fourier Series.

Recall: CTFS

$x(t)$ with period T, $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega k t}$

$$a_k = \frac{1}{T}\int_0^T x(t) e^{-j\omega k t} dt$$

suppose $x[n]$ has period N,

try $x[n] = \sum_{k=-\infty}^{\infty} a_k e^{j\omega k n}$

Let $\omega = 2\pi/N$, Family of basis signals $e^{j\underbrace{2\pi/N\, k}_{\omega_k} n}$, $k \in \mathbb{Z}$

Observe: $\frac{\omega_k}{\pi} = \frac{2k}{N}$ is rational

$e^{j 2\pi/N\, kn}$ is periodic

$$e^{j\omega n} = e^{j\omega n} e^{j2\pi n} = e^{j(\omega + 2\pi)n}$$

In DT, only consider fequencies $\omega \in [0, 2\pi)$

Consider $e^{j3.9\pi n} = e^{j1.9\pi n} = e^{j(-0.1)\pi n}$

Consider

$$e^{j 2\pi/N\, kn} = e^{j 2\pi/N\, kn} e^{j2\pi n} = e^{j(2\pi/N\, k + \frac{2\pi N}{N})n}$$

$$= e^{j(\frac{2\pi}{N}(k+N))n}$$

For DTFS, instead assume $x[n] = \sum_{k=0}^{N-1} a_k e^{j 2\pi/N\, kn}$

Detour: geometric series.

$$\sum_{k=0}^{\infty} \alpha^k = \alpha^0 + \sum_{k=1}^{\infty} \alpha^k = 1 + \sum_{k=0}^{\infty} \alpha^{k+1} = 1 + \alpha \sum_{k=0}^{\infty} \alpha^k$$

$$\Rightarrow (1-\alpha) \sum_{k=0}^{\infty} a_k = 1$$

$$= \sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha} \quad \text{, valid for } |\alpha| < 1$$

Finite geometric series

$$\sum_{k=0}^{N-1} \alpha^k = \sum_{k=0}^{\infty} \alpha^k - \sum_{k=N}^{\infty} \alpha^k = \frac{1}{1-\alpha} - \sum_{k=0}^{\infty} \alpha^{k+N}$$

$$= \frac{1}{1-\alpha} - \alpha^N \cancel{\sum_{k=0}^{\infty} \alpha^k} \frac{1}{1-\alpha}$$

$$= \frac{1-\alpha^N}{1-\alpha}$$

$$\sum_{k=0}^{N-1} \alpha^k = \begin{cases} \dfrac{1-\alpha^N}{1-\alpha} & \text{for } \alpha \neq 1 \\ N & \text{for } \alpha = 1 \end{cases}$$

Derive analysis eq'n.

recall

$$x[n] = \sum_{k=0}^{N-1} a_k e^{j \frac{2\pi}{N} k n}$$

Consider. :

$$\frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} l n} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} a_k e^{j \frac{2\pi}{N} k n} e^{-j \frac{2\pi}{N} l n}$$

$$= \frac{1}{N} \sum^{N-1} \sum^{N-1} a_k e^{j \frac{2\pi}{N} (k-l) n}$$

$$\frac{\ }{N} \sum_{k=0} \sum_{n=0}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} a_k \underbrace{\sum_{n=0}^{N-1} e^{j 2\pi/N (k-l) n}}$$

<u>case ①:</u> $k = l$

Then $\displaystyle\sum_{n=0}^{N-1} e^{j 2\pi/N \, 0 n} = \sum_{n=0}^{N-1} 1 = N$

<u>case ②</u> $k \neq l$

set $\alpha = e^{j 2\pi/N (k-l)}$

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

$$= \frac{1 - e^{j 2\pi/\cancel{N} (l-k) \cancel{N}}}{1 - e^{j 2\pi/N (l-k)}} = \frac{1 - e^{j 2\pi (l-k)}}{1 - e^{j 2\pi/N (l-k)}} = 0 _{//}$$

Then,

$$\frac{1}{N} \sum_{k=0}^{N-1} a_k \sum_{n=0}^{N-1} e^{j 2\pi/N (k-l)} = \frac{1}{N} \sum_{k=0}^{N-1} a_k \begin{cases} N & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

$$= \frac{1}{N} \left(a_l N + \sum_{k=0}^{N-1} a_k \, 0 \right)$$

$$= \frac{1}{N} a_l N = a_l$$

$$x[n] = \sum_{k=0}^{N-1} a_k e^{j 2\pi/N kn}$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j 2\pi/N kn}$$

$$N-1 \qquad 1 \; N-1$$

$$a_0 = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{0} = \frac{1}{N}\sum_{n=0} x[n]$$