Properties of DT FT:

⑧ Frequency differentiation.

$$x[n] \xleftrightarrow{ft} X(e^{j\omega})$$

$$\rightarrow \quad -jn\,x[n] \xleftrightarrow{ft} \frac{dX(e^{j\omega})}{d\omega}$$

⑨ Convolution property:

$$x[n] \xleftrightarrow{ft} X(e^{j\omega}) \quad \& \quad h[n] \xleftrightarrow{ft} H(e^{j\omega})$$

$$y[n] = x[n] * h[n] \xleftrightarrow{ft} Y(e^{j\omega}) = X(e^{j\omega})\,H(e^{j\omega})$$

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h[n]\end{matrix}} \longrightarrow y[n]$$

$$X(e^{j\omega}) \cdot H(e^{j\omega}) = Y(e^{j\omega})$$

$h[n]$: Impulse Response.

$H(e^{j\omega}) = \mathcal{F}\{h[n]\}$ Frequency Response.

(*) For an LTI system:

$$h[n] = \alpha^n u[n], \quad |\alpha| < 1$$

$$x[n] = \beta^n u[n], \quad |\beta| < 1$$

What's the output $y[n]$?

$$X(e^{j\omega}) = \mathcal{F}\{\beta^n u[n]\} = \frac{1}{1-\beta e^{-j\omega}}$$

$$H(e^{j\omega}) = \mathcal{F}\{\alpha^n u[n]\} = \frac{1}{1-\alpha e^{-j\omega}}$$

$$Y(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}$$

$$y[n] = \mathcal{F}^{-1}\{Y(e^{j\omega})\}$$

1) if $\alpha \neq \beta$

$$\frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})} = \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}$$

$$A(1-\beta e^{-j\omega}) + B(1-\alpha e^{-j\omega}) = 1$$

if $e^{-j\omega} = \frac{1}{\alpha}$ $\quad 1 = A(1-\beta\cdot\frac{1}{\alpha}) \rightarrow A = 1/(1-\beta/\alpha) = \frac{\alpha}{\alpha-\beta}$ //

if $e^{-j\omega} = \frac{1}{\beta}$ $\quad 1 = B(1-\alpha\cdot\frac{1}{\beta}) \rightarrow B = 1/(1-\alpha/\beta) = \frac{\beta}{\beta-\alpha}$ //

$$y[n] = \left(\frac{\alpha}{\alpha-\beta}\right)\cdot\alpha^n u[n] + \frac{\beta}{\beta-\alpha}\beta^n u[n]$$

$$= \frac{u[n]}{\alpha-\beta}\left[\alpha^{n+1} - \beta^{n+1}\right]$$

2) If $\alpha = \beta \rightarrow Y(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})^2}$

$$\alpha^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-\alpha e^{-j\omega}}$$

$|\alpha| < 1$

$$\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) = \frac{0-(-\alpha)(-j)e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} = \frac{-\alpha j e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} \text{ //}$$

$$Y(e^{j\omega}) = \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) \times \frac{-e^{+j\omega}}{j\alpha}$$

$$= j\frac{e^{+j\omega}}{\alpha}\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

∵ Freq. Differentiation

$$-jn\alpha^n u[n] \xleftrightarrow{f} \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$\times \frac{j}{\alpha}$

$$\frac{j}{\alpha}(-jn)\,\alpha^n u[n] \xleftrightarrow{f} \frac{j}{\alpha}\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$n\alpha^{n-1} u[n] \xleftrightarrow{f} \frac{j}{\alpha}\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

∵ time shift. $(n+1)\,\alpha^n u[n+1] \xleftrightarrow{f} \frac{j}{\alpha}e^{j\omega}\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$

$$y[n] = (n+1)\,\alpha^n u[n]$$

⊛ The input-output relation of an LTI system is:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

What's the output if the input is $x[n] = (\frac{1}{3})^n u[n]$?

$$Y(e^{j\omega}) - \frac{3}{4}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{8}e^{-2j\omega}Y(e^{j\omega}) = 2X(e^{j\omega})$$

$$Y(e^{j\omega})\left[1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega}\right] = 2X(e^{j\omega})$$

$$Y(e^{j\omega}) = H(e^{j\omega})\,X(e^{j\omega}) \qquad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}}$$

$$= \frac{2}{\cdots}$$

$$= \overline{(1-\tfrac{1}{2}e^{-j\omega})(1-\tfrac{1}{4}e^{-j\omega})}$$

$$Y(e^{j\omega}) = H \cdot X = \frac{2}{(1-\tfrac{1}{2}e^{-j\omega})(1-\tfrac{1}{4}e^{-j\omega})(1-\tfrac{1}{3}e^{-j\omega})}$$

$$Y(e^{j\omega}) = \frac{A}{1-\tfrac{1}{2}e^{-j\omega}} + \frac{B}{1-\tfrac{1}{3}e^{-j\omega}} + \frac{C}{1-\tfrac{1}{4}e^{-j\omega}}$$

$$A = 12, \quad B = -16 \quad C = 6$$

$$y[n] = \left(12\left(\tfrac{1}{2}\right)^n - 16\left(\tfrac{1}{3}\right)^n + 6\left(\tfrac{1}{4}\right)^n\right) u[n]$$