Properties of Fourier Series.

$x(t)$, period T

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\omega_0)t} \Leftarrow \text{Synthesis}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk(2\pi/T)t} dt \Leftarrow \text{analysis.}$$

{① Linearity:} If $x(t) \xleftrightarrow{FS} a_k$, and $y(t) \xleftrightarrow{FS} b_k$,

then $\underbrace{A x(t) + B y(t)}_{c_k} \xleftrightarrow{FS} A a_k + B b_k$

A & B constant coefficient.

Proof:

$$C_k = \frac{1}{T}\int (A x(t) + B y(t)) e^{-jk(2\pi/T)t} dt$$

$$C_k = A \underbrace{\frac{1}{T}\int_T x(t) e^{-jk(2\pi/T)t}}_{a_k} + B \underbrace{\frac{1}{T}\int_T y(t) e^{-jk(2\pi/T)t}}_{b_k}$$

$$\therefore \quad A a_k + B b_k = C_k$$

{② Time-shift.}

If $x(t) \xleftrightarrow{FS} a_k$, then $x(t-t_0) \xleftrightarrow{FS} e^{-jk(2\pi/T)t_0} a_k$

Proof:

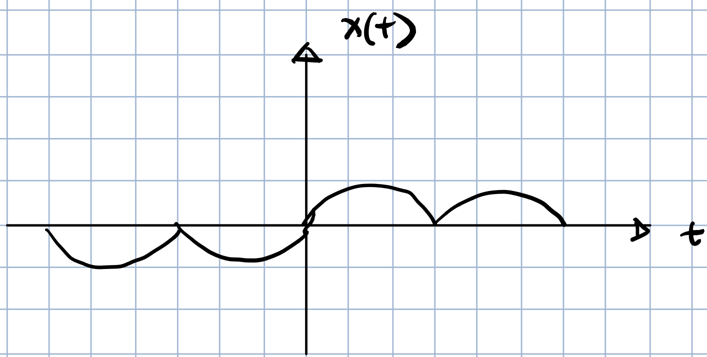
$$x(t-t_0) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)(t-t_0)}$$

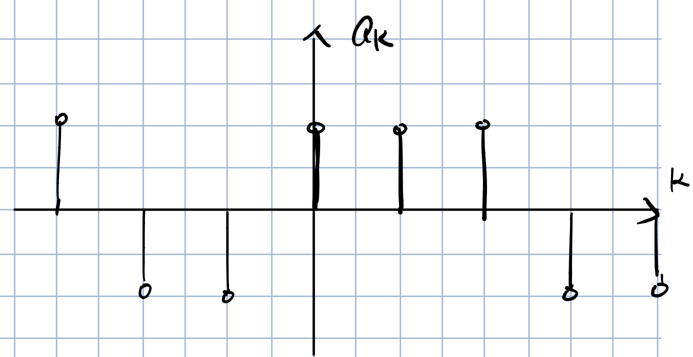
$$= \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} e^{-jk(2\pi/T)t_0}$$

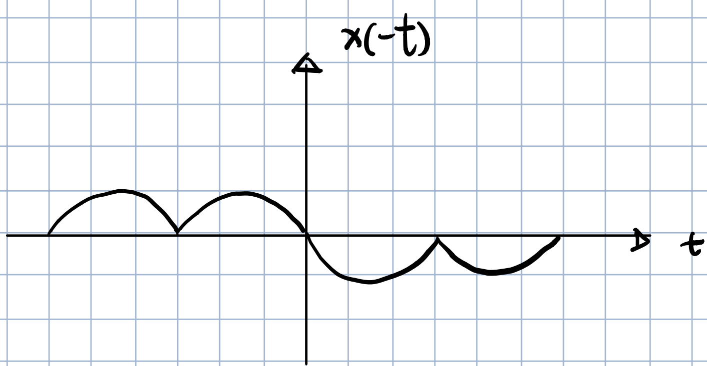
The magnitude of the FS coefficient does not change. only the angle (phase) changes.

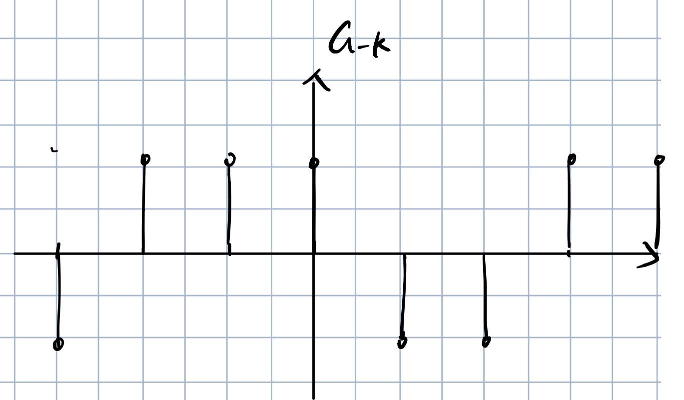
③ Time reversal

If $x(t) \stackrel{FS}{\longleftrightarrow} a_k$, then $x(-t) \stackrel{FS}{\longleftrightarrow} a_{-k}$









④ Time Scaling

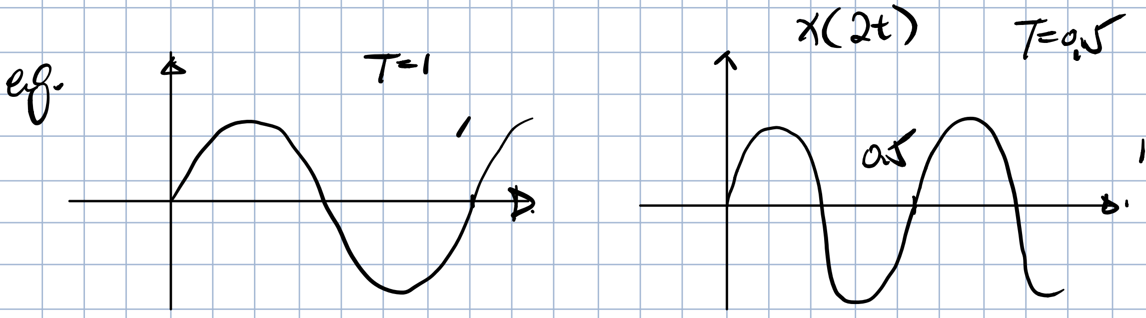
If $x(t) \stackrel{FS}{\longleftrightarrow} a_k$, $x(\alpha t) \stackrel{FS}{\longleftrightarrow} a_k$

Proof:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)(\alpha t)}$$

e.g.



$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)(\alpha t)} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\underbrace{(\omega_0 \alpha)}_{\frac{2\pi}{T/\alpha} = \frac{2\pi}{T}\alpha}t}$$

ω

ω_0

The Fourier series coefficient does not change, but the synthesis and analysis relations change.

⑤ Multiplication: if $x(t) \xleftrightarrow{FS} a_k$ & $y(t) \xleftrightarrow{FS} b_k$

then $x(t)y(t) \xleftrightarrow{FS} h_k = \underbrace{\sum_{l=-\infty}^{\infty} a_l b_{k-l}}_{\text{convolution of } \{a_k\} \,\&\, \{b_k\}}$

⑥ Conjugation.

if $x(t) \xleftrightarrow{FS} a_k$, then $x^*(t) = a^*_{-k}$

if $x(t)$ is real ($\mathbb{R}$), $x(t) = x^*(t)$.

then $a_k = a^*_{-k}$

⑦ Parseval's Relation for CT periodic signals.

$$\underbrace{\frac{1}{T}\int_T |x(t)|^2\,dt}_{\text{Total power of } x(t)} = \underbrace{\sum_{k=-\infty}^{\infty} |a_k|^2}_{\text{sum of powers of each harmonic.}}$$

$$\underbrace{\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt}_{\text{power of one harmonic.}} = \frac{1}{T}\int_T |a_k|^2\,dt = |a_k|^2$$