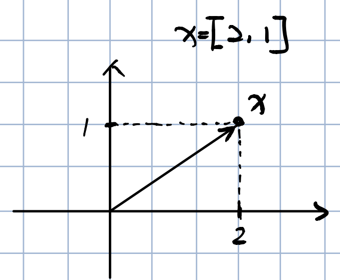
Geometric Perspective :

Objective = To show that FS coefficients of x are the projections of x onto the N dimensional space created by orthogonal complex exponential.

⊛ Position, Length, and angle in $\mathbb{R}^N$:

① **Position** : is determined by an N dimensional vector of real numbers :



$$x = [x[0], x[1], \ldots\ldots, x[n-1]] \in \mathbb{R}^N$$

$$x[n] \in \mathbb{R}$$

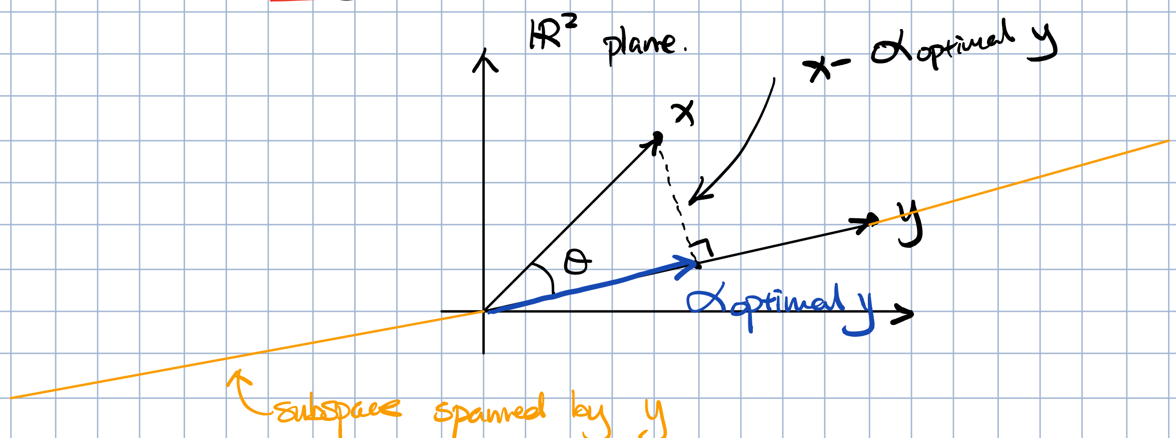
② **Length** : Defined by $\|x\| = \sqrt{\sum_{n=0}^{N-1} (x[n])^2}$ ← norm of vector x

More commonly :

$$\|x\|^2 = \sum_{n=0}^{N-1} (x[n])^2$$

For the previous example: $\|x\|^2 = 2^2 + 1^2 = 5$

③ **Angle** :



(obtained by multiplying y by $\alpha \in \mathbb{R}$.)

$$\cos\theta = \frac{\|\alpha_{optimal}\, y\|}{\|x\|}$$

$$\|x - \alpha y\|^2 = \sum_{n=0}^{N-1} (x[n] - \alpha\, y[n])^2$$

$$\frac{d\|x-\alpha y\|^2}{d\alpha} = \sum_{n=0}^{N-1} 2(x[n] - \alpha\, y[n])\, y[n] = 0$$

$$\sum_{n=0}^{N-1} x[n]\,y[n] - \alpha_{opt.} \sum_{n=0}^{N-1} y[n]\,y[n] = 0$$

$$\alpha_{opt} = \frac{\sum_{n=0}^{N-1} x[n]\,y[n]}{\sum_{n=0}^{N-1} y[n]\,y[n]}$$

generalization of Dot product.

Inner product of x and y : $\langle x, y\rangle = \sum_{n=0}^{N-1} x[n]\,y[n]$

$$\alpha_{opt} = \frac{\langle x, y\rangle}{\langle y, y\rangle}$$

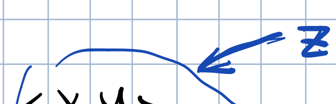
properties of inner product.

1) $\langle x, y\rangle = \langle y, x\rangle$

2) $\langle \gamma x, y\rangle = \langle x, \gamma y\rangle = \gamma\langle x, y\rangle \quad \gamma \in \mathbb{R}.$

3) $\|z\|^2 = \langle z, z\rangle \geq 0$

4) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$



$$\cos\theta = \frac{\|\alpha_{opt}\, y\|}{\|x\|} = \frac{\left\| \boxed{\frac{\langle x, y\rangle}{\langle y, y\rangle}}\, y \right\|}{\|x\|}$$

$$\|z\| = \sqrt{\|z\|^2}$$

$$= \sqrt{\langle z, z\rangle}$$

$$= \sqrt{\left\langle \frac{\langle x, y\rangle}{\langle y, y\rangle} y, \frac{\langle x, y\rangle}{\langle y, y\rangle} y \right\rangle}$$

$$= \sqrt{\frac{(\langle x, y\rangle)^2}{\|y\|^2\|y\|^2} \langle y, y\rangle}$$

$$= \sqrt{\frac{(\langle x, y\rangle)^2}{\|y\|^2}} = \frac{\langle x, y\rangle}{\|y\|}$$

$$\cos(\theta) = \frac{|\langle x, y\rangle|}{\|x\|\,\|y\|} \quad //$$

Example: Find the angle btw. vectors y and $x = \gamma y$,
x & $y \in \mathbb{R}^N$ $\quad \gamma \in \mathbb{R}$

$$\cos(\theta) = \frac{\langle x, y\rangle}{\|x\|\,\|y\|} = \frac{\langle \gamma y, y\rangle}{\|x\|\,\|y\|} = \frac{\gamma\langle y, y\rangle}{\|x\|\,\|y\|} = \frac{\gamma\|y\|^2}{\|x\|\,\|y\|}$$

$$= \frac{\gamma\|y\|^2}{\|\gamma y\|\,\|y\|} = \frac{\gamma\|y\|^2}{\gamma\|y\|^2}$$

$$= 1 \qquad \theta = \cos^{-1}(1)$$

$$= 0 \quad //$$

Example Find the angle between y and $x - \alpha_{opt.}\, y = z$

$$\cos\theta = \frac{|\langle z, y\rangle|}{\ldots}, \qquad \alpha_{opt} = \frac{\langle x, y\rangle}{\ldots}$$

$\|z\| \|y\|$ $\langle y, y\rangle$

$\langle z, y\rangle = 0$?

$$\Rightarrow \langle x - \alpha_{opt} \cdot y, y\rangle = \langle x, y\rangle - \langle \alpha_{opt} \cdot y, y\rangle$$

$$= \langle x, y\rangle - \frac{\langle x, y\rangle}{\cancel{\langle y, y\rangle}} \cancel{\langle y, y\rangle}$$

$$= 0$$

$$\theta = \cos^{-1}(0) = \pi/2 = 90^\circ \; //$$