Lecture 22: Ampère's Law in Point Form, Magnetic Flux Density

ECE221: Electric and Magnetic Fields

Prof. Sean V. Hum

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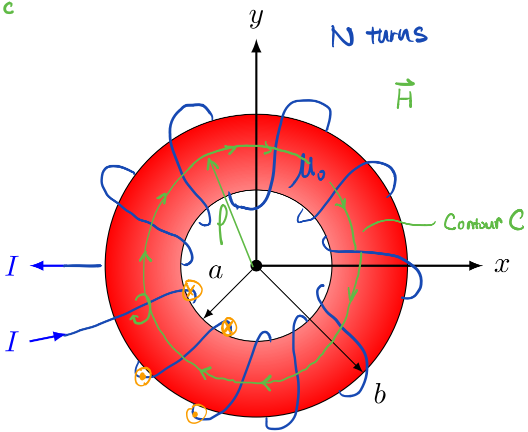
Outline

Toroidal Coil in xy Plane

Find the magnetic field $\boldsymbol{H}$ everywhere.

$$\oint_C \vec{H}\cdot d\vec{l} = I_{encl.}$$



$\vec{H}$ does not vary with ϕ

If $\rho < a$, $\vec{H} = 0$ $(I_{encl} = 0)$

If $\rho > b$, $\vec{H} = 0$ $(I_{encl} = 0)$

current goes out ⊙ cancels with current goes in ⊗

If $a < \rho < b$, $I_{encl.} = -NI$ (因为 into the page, $-\hat{z}$ direction)

$\vec{H}$ points in $-\hat{\phi}$ direction.

$$\oint \vec{H}\cdot d\vec{l} = \int_0^{2\pi} H_\phi(-\hat{\phi})\,\rho\, d\phi\, \hat{\phi} = -2\pi\rho H_\phi$$

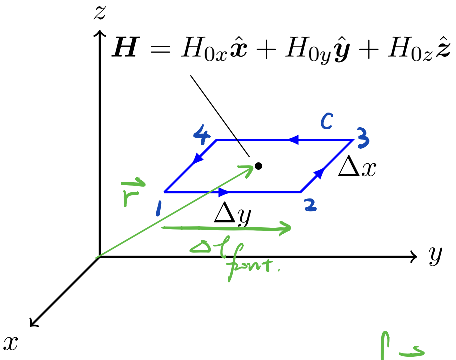
$$-2\pi\rho H_\phi = I_{encl} = -NI$$

$$H_\phi = \frac{NI}{2\pi\rho} \qquad a < \rho < b$$

$$\boxed{\vec{H} = -\hat{\phi}\,\frac{NI}{2\pi\rho} \qquad a < \rho < b}$$

Point form of Ampère's Law

Let's try applying Ampère's Law to an *infinitesimal* small loop deep inside a current distribution $\boldsymbol{J}$.



$$\oint_C \vec{H}\cdot d\vec{l} = \int_1^2 + \int_2^3 + \int_3^4 + \int_4^1$$

$$= \vec{H}(\vec{r}_{front})\cdot\Delta\vec{l}_{front} + \vec{H}(\vec{r}_{right})\cdot\Delta\vec{l}_{right}$$

$$+ \vec{H}(\vec{r}_{back})\cdot\Delta\vec{l}_{back} + \vec{H}(\vec{r}_{left})\cdot\Delta\vec{l}_{left}$$

$$= \left(H_y(\vec{r}) + \frac{\partial H_y}{\partial x}\cdot\frac{\Delta x}{2}\right)\cdot\Delta y\,\hat{y} + \int_2^3$$

$$- \left(H_y(\vec{r}) + \frac{\partial H_y}{\partial x}\cdot\frac{-\Delta x}{2}\right) + \int_4^1$$

$$\int_C \vec{H}\cdot d\vec{l} = \underbrace{\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)}\Delta x\,\Delta y = I_{encl} = J_z\,\Delta x\,\Delta y$$

$[\nabla \times \vec{H}]_z = (\nabla \times \vec{H}) \cdot z$

Summary of Loop Analysis

1. Loop in xy plane:

$$[\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r})]_z = \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = J_z$$

This process can be repeated with the loop rotated to be placed in the xz and yz planes:

2. Loop in xz plane:

$$[\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r})]_y = \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) = J_y$$

3. Loop in yz plane:

$$[\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r})]_x = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) = J_x$$

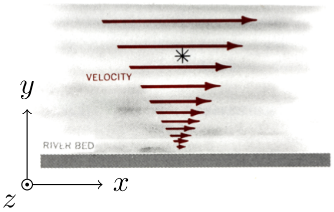
Point Form of Ampère's Law

Point (Differential) Form of Ampère's Law

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{\boldsymbol{z}} = \boldsymbol{J}$$

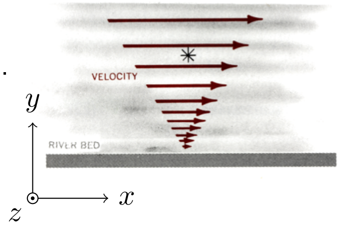
Curl: Physical Interpretation (1)

- Consider the velocity (a vector field) of water in a river
- The velocity of the water must be zero at the river bed (boundary condition), producing the velocity field shown
- A paddle-wheel whose axis (axle) is normal to the page is inserted
- The wheel axis is along the z-direction
- Does the wheel turn?



Curl: Physical Interpretation (2)

- Yes: it turns clockwise indicating a circulating velocity field
- This is one component of the curl: the z-component, which is the same as the paddle wheel axis
- Only the x- and y-components of the velocity field affect this curl component.
- To get the other two components of curl, we would need to rotate the paddle-wheel so its axle was along:
  2. The y-axis, where the wheel will be affected only by the x and z components;
  3. The x-axis, where the wheel will be affected only by the y and z components.



Calculation of Curl

- Remember that curl is an *operator formed from gradient* $\boldsymbol{\nabla}$!
- Employ definition cross product definition to find curl rather than resorting to aid sheets:

$$\boldsymbol{\nabla} \times \boldsymbol{H}(x, y, z) = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

- Cylindrical coordinates

$$\boldsymbol{\nabla} \times \boldsymbol{H}(\rho, \phi, z) = \frac{1}{\rho} \begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_\rho & \rho H_\phi & H_z \end{vmatrix}$$

- Spherical coordinates

$$\boldsymbol{\nabla} \times \boldsymbol{H}(r, \theta, \phi) = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_r & rH_\theta & (r\sin\theta)H_\phi \end{vmatrix}$$

Calculation of Curl

- Cylindrical coordinates

$$\boldsymbol{\nabla} \times \boldsymbol{H}(\rho, \phi, z) = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho H_\phi) - \frac{\partial H_\phi}{\partial \rho}\right]\hat{\boldsymbol{z}}$$

- Spherical coordinates

$$\boldsymbol{\nabla} \times \boldsymbol{H}(r, \theta, \phi) = \left[\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(H_\phi \sin\theta) - \frac{\partial H_\theta}{\partial \phi}\right]\hat{\boldsymbol{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial}{\partial r}(rH_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Magnetix Flux Density

- *Magnetic field* $\boldsymbol{H}$ and *magnetic flux density* $\boldsymbol{B}$ are related through the **constitutive relation**

$$\boldsymbol{B} = \mu \boldsymbol{H}$$

  where μ is called *magnetic permeability [H/m]*
- $\mu_0 = 4\pi \times 10^{-7}$ H/m
- *Magnetic flux* Φ [Webers, Wb] and *magnetic flux density* are related through a usual **flux integral**

$$\Psi = \iint_S \boldsymbol{B} \cdot d\boldsymbol{s} \;[\mathrm{Wb}]$$

Fundamental Postulates of the Magnetic Field

As we have seen before:

1. Magnetic field / magnetic flux density has no **divergence**

$$\nabla \cdot \boldsymbol{B} = \nabla \cdot \boldsymbol{H} = 0$$

   This is the same thing as saying that **there are no magnetic charges / monopoles**

2. Magnetic field / magnetic flux density is **solenoidal** – it has a **non-zero curl** and forms closed loops

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}$$