Week 9

Tuesday, March 12, 2019 2:12 PM

从下面部分开始看.

Properties of CTFT

analysis: $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$

synthesis: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$

$x(t) \xleftrightarrow{FT} X(j\omega)$

① Linearity

$x_1(t) \xleftrightarrow{FT} X_1(j\omega)$

$x_2(t) \xleftrightarrow{FT} X_2(j\omega)$

and a_1 & $a_2 \in C$,

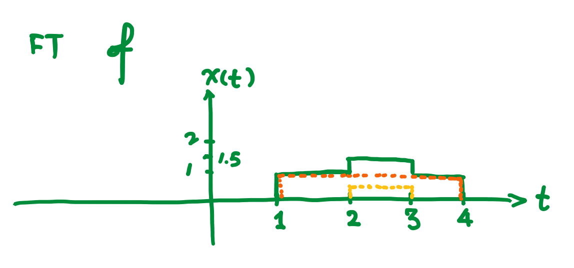
$a_1x_1 + a_2x_2 \xleftrightarrow{FT} a_1X_1 + a_2X_2$

② Time shift

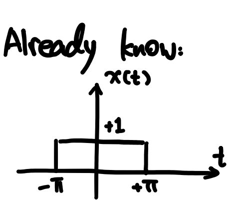
$x(t) \xleftrightarrow{FT} X(j\omega)$

$x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(j\omega)$

Find FT of



Already know:



$X_0(j\omega) = \frac{2\sin(\omega T_1)}{\omega}$

$x(t) = x_1(t-2.5) + x_2(t-2.5)$

$\therefore X(j\omega) = e^{-j2.5\omega}\,\frac{1}{2}\cdot\frac{2\sin(0.5\omega)}{\omega} + e^{-j2.5\omega}\cdot\frac{2\sin(1.5\omega)}{\omega}$

③ Conjugation

$x(t) \xleftrightarrow{FT} X(j\omega) \Rightarrow x^*(t) \xleftrightarrow{FT} X^*(-j\omega)$

If $x(t)$ is real ($x(t) = x^*(t)$), then

$X(j\omega) = X^*(-j\omega)$

$\Rightarrow X(-j\omega) = X^*(j\omega)$

$Re\{X(-j\omega)\} + jIm\{X(-j\omega)\} = Re\{X(j\omega)\} - jIm\{X(j\omega)\}$

$Re\{X(-j\omega)\} = Re\{X(j\omega)\}$, Even signal

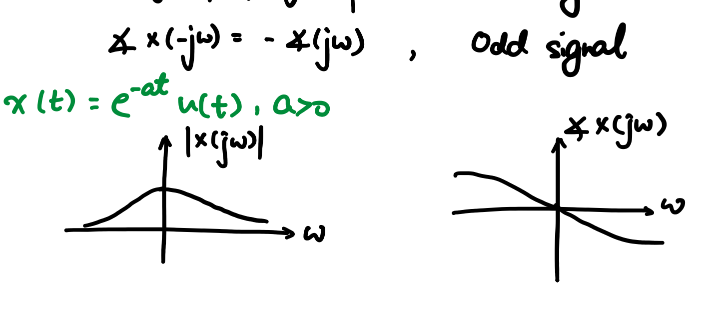
$Im\{X(-j\omega)\} = -Im\{X(j\omega)\}$, odd signal

$\Longrightarrow |X(-j\omega)|\, e^{j\measuredangle X(-j\omega)} = |X(j\omega)|\, e^{-j\measuredangle(j\omega)}$

$|X(-j\omega)| = |X(j\omega)|$, Even signal

$\measuredangle X(-j\omega) = -\measuredangle(j\omega)$, Odd signal

$x(t) = e^{-at}u(t)$, $a>0$



④ Time differentiation and integration

$x(t) \xleftrightarrow{FT} X(j\omega) \Longrightarrow \frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(j\omega)$

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$

$\Rightarrow \frac{dx}{dt} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{(j\omega\cdot X(j\omega))}_{\text{new FT}} e^{j\omega t} d\omega$

$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$

Examples:

(1) Find the FT of $u(t)$

method 1. $X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$

$= \int_{0}^{+\infty} e^{-j\omega t}dt$

$= -\frac{1}{j\omega} e^{-j\omega t}\Big|_0^{+\infty}$

$= \frac{1}{j\omega}$

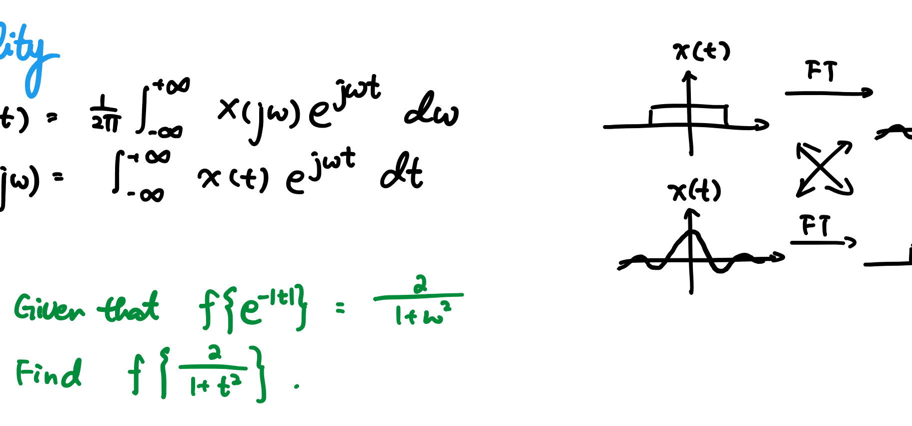
method 2.

$FT\{\delta(t)\} = \int_{-\infty}^{+\infty}\delta(t)e^{-j\omega t}dt = 1$

$u(t) = \int_{-\infty}^{t}\delta(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega}\cdot 1 + \pi\cdot 1\cdot\delta(\omega)$

$u(t) \xleftrightarrow{f.T} \frac{1}{j\omega} + \pi\delta(\omega)$

Find fT of



$fT\{g(t)\} = \frac{2\sin(\omega)}{\omega} - e^{-j\omega}\cdot 1 - e^{j\omega}\cdot 1 = G(j\omega)$

$\lim_{\omega\to 0} G(j\omega) = 0 \longrightarrow fT\{x(t)\} = \frac{2\sin(\omega)}{(j\omega)\cdot\omega} - \frac{e^{-j\omega}}{j\omega} - \frac{e^{+j\omega}}{j\omega}$

$= \frac{2\sin\omega}{j\omega^2} - \frac{1}{j\omega}2\cos(\omega)$

⑤ Duality

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$

$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{j\omega t}dt$

Example: Given that $f\{e^{-|t|}\} = \frac{2}{1+\omega^2}$

Find $f\{\frac{2}{1+t^2}\}$.

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$

$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{2}{1+\omega^2}e^{j\omega t}d\omega$

$\xrightarrow[t\to -t]{\times 2\pi}$ $2\pi\cdot e^{-|t|} = \int_{-\infty}^{+\infty}\frac{2}{1+\omega^2}e^{-j\omega t}d\omega$

$\xrightarrow[\omega\to t]{t\to\omega}$ $2\pi\cdot e^{-|\omega|} = \int_{-\infty}^{+\infty}\frac{2}{1+t^2}e^{-j\omega t}dt$

$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$

$X(j\omega) = 2\pi e^{-|\omega|}$

⑥ frequency differentiation

(Time differentiation

$x(t) \xleftrightarrow{fT} X(j\omega)$

$\frac{dx(t)}{dt} \xleftrightarrow{fT} j\omega X(j\omega)$)

$x(t) \xleftrightarrow{fT} X(j\omega)$

$-jt\,x(t) \xleftrightarrow{fT} \frac{dX(j\omega)}{d\omega}$

proof $X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$

$\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{+\infty}[-j\omega\cdot x(t)]e^{-j\omega t}dt$

Example: Find FT of $x(t) = te^{-at}u(t)$ $a>0$

$e^{-at} \xleftrightarrow{fT} \frac{1}{a+j\omega}$

$-jt\cdot e^{-at} \xleftrightarrow{fT} -\frac{1}{(a+j\omega)^2}\cdot j$

$\xrightarrow{\times -j}$ $te^{-at} \xleftrightarrow{fT} \frac{1}{(a+j\omega)^2}$

$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \xleftrightarrow{fT} \int_{-\infty}^{\omega} X(\eta)d\eta$

⑦ Parseval's Thm:

$\int_{-\infty}^{+\infty}|x(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega$, Energy.

⑧ Convolution property

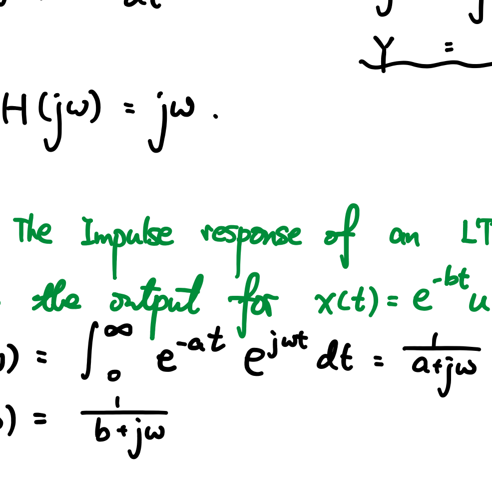
$x(t) \xleftrightarrow{f.T.} X(j\omega)$

$h(t) \xleftrightarrow{f.T.} H(j\omega)$

$x(t) * h(t) \xleftrightarrow{f.T} X(j\omega)\cdot H(j\omega)$

$h(t)$: Impulse response

$H(j\omega)$: Frequency response



Example: Find frequency response of a CT differentiator.

$y(t) = \frac{dx(t)}{dt} \longrightarrow Y(j\omega) = j\omega\cdot X(j\omega)$

$Y = H\cdot X$

$H(j\omega) = j\omega$.

Example. The Impulse response of an LTI system is $h(t) = e^{-at}u(t)$, $a>0$. What is the output for $x(t) = e^{-bt}u(t)$, $b>0$

$H(j\omega) = \int_0^{\infty} e^{-at}e^{j\omega t}dt = \frac{1}{a+j\omega}$

$X(j\omega) = \frac{1}{b+j\omega}$

$H(j\omega)\cdot X(j\omega) = \frac{1}{a+j\omega}\cdot\frac{1}{b+j\omega} = Y(j\omega)$

If $a \neq b$:

$Y(j\omega) = \frac{\frac{1}{b-a}}{a+j\omega} + \frac{\frac{1}{a-b}}{b+j\omega}$

$\therefore y(t) = \frac{1}{b-a}e^{-at}u(t) + \frac{1}{a-b}e^{-bt}u(t)$

If $a = b$.

$Y(j\omega) = \frac{1}{(a+j\omega)^2}$

$\therefore y(t) = te^{-at}u(t)$

⑨ Frequency shifting

(Reminder: Time shifting

$x(t-t_0) \xleftrightarrow{f.s.} e^{-j\omega t_0}X(j\omega)$)

$x(t) \xleftrightarrow{f.s.} X(j\omega)$

$e^{j\omega_0 t}x(t) \xleftrightarrow{f.s.} X(j(\omega-\omega_0))$

⑩ frequency and time scaling

$x(t) \xleftrightarrow{f.} X(j\omega)$

$x(at) \xleftrightarrow{f.} \frac{1}{|a|}X(j\frac{\omega}{a})$

$\xrightarrow{a=-1}$ $x(-t) \xleftrightarrow{f.} X(-j\omega)$

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}d\omega$

$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

If $\omega_1 < \omega_2$,

Discrete Time Fourier Transform

(Reminder for CTFT

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}d\omega$

$X(j\omega) = \int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt$)

$x[n] = \frac{1}{2\pi}\int_{2\pi}X(e^{j\omega})\cdot e^{j\omega n}d\omega$ Inverse FT/synthesis

$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty}x[n]\cdot e^{-j\omega n}$

→ $X(e^{j\omega})$ is periodic with period 2π.

→ $X(e^{j\omega})$ is larger when ω is close to the odd multiples of π for a rapidly varying $x[n]$.

→ $X(e^{j\omega})$ is larger when ω is close to the even multiples of π for a slowly varying $x[n]$.

Example: Find the DTFT of

$x[n] = a^n u[n]$ $|a|<1$

$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty}a^n u[n]\cdot e^{-j\omega n}$

$= \sum_{n=0}^{+\infty}a^n\cdot e^{-j\omega n}$

$= \sum_{n=0}^{+\infty}(a\cdot e^{-j\omega})^n$

$= \frac{1}{1-a\cdot e^{-j\omega}}$

$|X(e^{j\omega})| = \frac{1}{\sqrt{(1-a\cos(\omega))^2 + (1-a\sin(\omega))^2}}$

$0<a<1$, $-1<a<0$