Lectures 3-4: Superposition and Charge Distributions

ECE221: Electric and Magnetic Fields

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Outline

Overview of Electrostatic Equations in Free Space

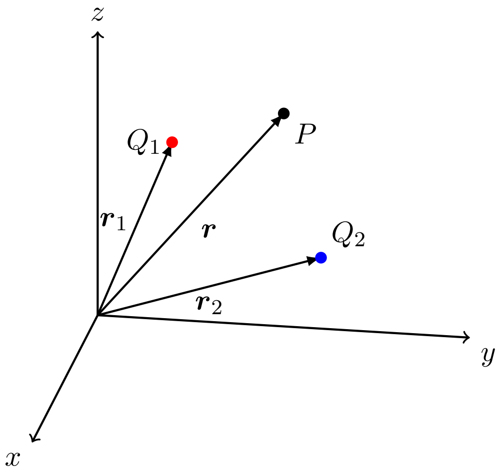
$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \rho_v(\boldsymbol{r}, t)/\epsilon$$
$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0$$

Coulomb's Law:

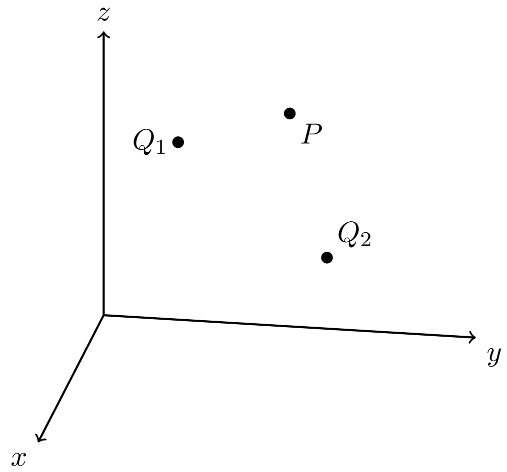
$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\boldsymbol{R}$$

Superposition of Two Charges

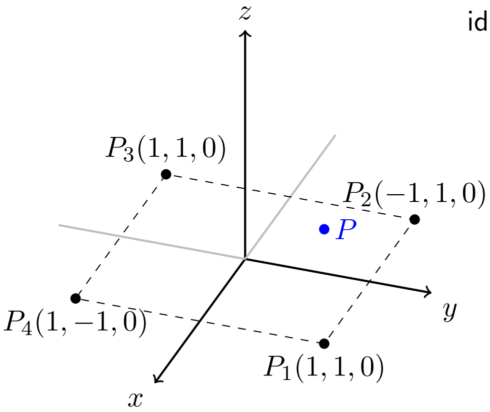
- Consider two charges placed in space.
- What is the E-field produced by Q_1 alone? Q_2 alone?



Extension to N Charges



Example



Find $\boldsymbol{E}$ at $P(1, 1, 1)$ caused by four identical 3 nC charges P_1–P_4.

Volume Distributions

- Imagine we have a discrete number of water molecules (masses).
- Add more water, molecule-by-molecule, to our mass of molecules
- Pack them extremely close together.

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- Pack them extremely close together.

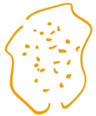
- Eventually we end up with a volume of water, which despite being composed of a **very large** collection of discrete molecules, we ascribe a **mass density** $(1\ \mathrm{g/cm^3})$.
- I could continue to add more molecules but what I really care about is what is happening **macroscopically** rather than microscopically.

Volumetric Charge Density ρ_v – charge per unit volume.

- Imagine we have a discrete number of point charges.
- Add more charges, charge-by-charge, to our volume of charges, packing them extremely close together

microscopic.
微观.

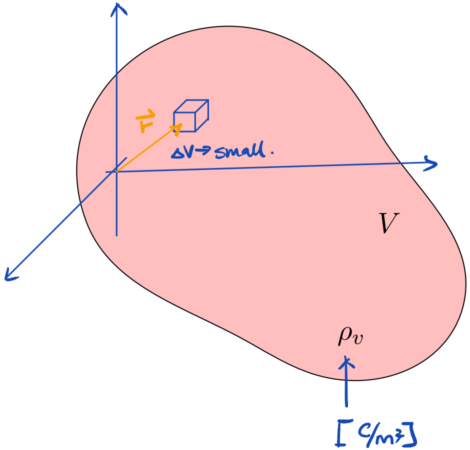
macroscopic.
宏观.



- Eventually we end up with a volume of charge, which despite being composed of a **very large** collection of discrete charges, we ascribe a **volumetric charge density** ρ_v $[\mathrm{C}/\mathrm{m}^3]$.
- This is fine so long as we are uninterested in the small irregularities in the field as we move from charge to charge – the **macroscopic** perspective

Volumetric Charge Density ρ_v [C/m³]

$$Q_{encl} = \iiint_V \rho_v(\vec{r})\, dV$$ ← What's the total charge in the volume V?



What's the charge ΔQ stored in a small volume ΔV?

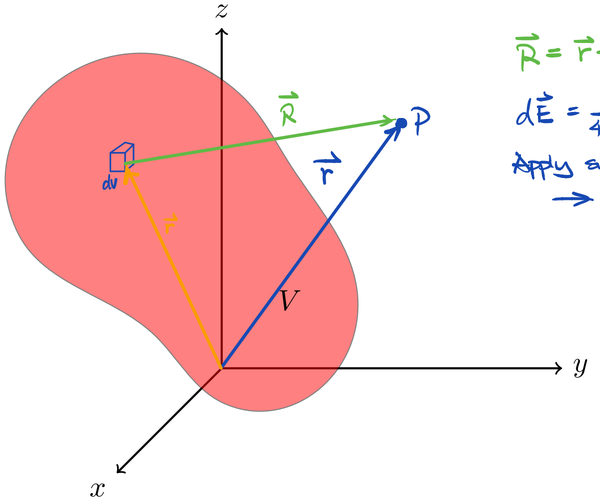
$$\Delta Q = \rho_v \Delta V \Rightarrow \rho_v = \lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V} \left[\frac{C}{m^3}\right]$$

$$= \rho_v(\vec{r})$$

"density function as a function of position."

Field from a Volumetric Charge Density

Reminder: for a point charge, $\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\boldsymbol{R}$



Find $\vec{E}$ @ P ?

$\vec{R} = \vec{r} - \vec{r}'$

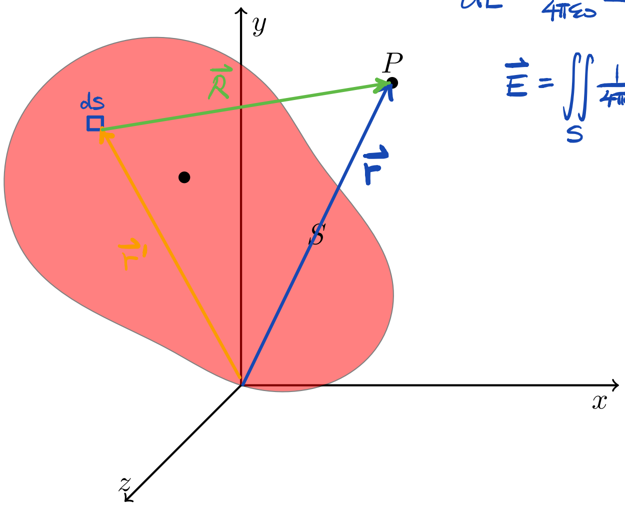
$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\vec{a}_R = \frac{1}{4\pi\epsilon_0}\frac{\rho_v(\vec{r}')\,dv}{R^2}\vec{a}_R$$

Apply superposition principle.

→ sum contribution over entire volume V

$$\vec{E} = \iiint_V d\vec{E} = \iiint_V \frac{1}{4\pi\epsilon_0}\frac{\rho_v(\vec{r}')\,dv}{R^2}\vec{a}_R$$

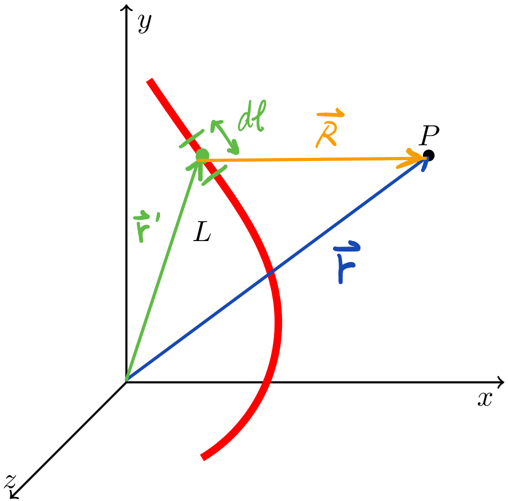
Surface Charge Density ρ_s [C/m²]



$$d\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{\rho_s(\vec{r}')\,ds}{R^2} \vec{a}_R$$

$$\vec{E} = \iint_S \frac{1}{4\pi\varepsilon_0} \frac{\rho_s(\vec{r}')\,ds}{R^2} \vec{a}_R$$

Line Charge Density ρ_l [C/m]



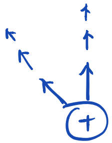
$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\,\frac{\rho_l(\vec{r}\,')\,dl}{R^2}\,\vec{a}_R$$

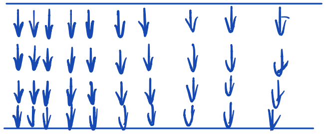
$$\vec{E} = \int_L \frac{1}{4\pi\varepsilon_0}\,\frac{\rho_l(\vec{r}\,')\,dl}{R^2}\,\vec{a}_R$$

Sketching Field Lines

The *strength* of the field can be indicated by either:

- the **length** of the vector arrows;
- their **spacing**.





Solving Field Problems

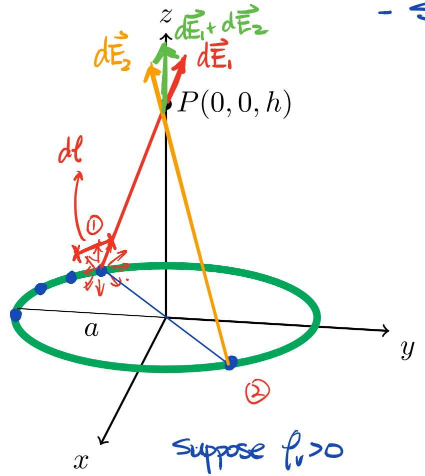
rectangular / cylindrical / sphereical.

Before solving problems, try use symmetry to deduce:

1. With which coordinates the field does *not* vary;
2. Which components of the field are *not* present.

Example: Ring Charge

Consider a uniformly charged ring charge of radius a with charge density ρ_l. Find $\boldsymbol{E}$ at a point $P(0,0,h)$.



- Solve in cylindrical coordinate: ρ, ϕ, z

→ $\vec{E}$ does not have a ϕ component.

→ $\vec{E}$ does not have a ρ_v component due to symmetry / cancellation of diametrically-opposed charge pairs.

$$d\vec{E} = d\vec{E}_1 = \frac{\rho_l\, dl}{4\pi\varepsilon_0 R^2}\, \vec{a}_R \quad , \quad \vec{R} = \vec{r} - \vec{r}'$$

$$\vec{r} = h\hat{z} \qquad \vec{r}' = \rho'\hat{\rho} \qquad \vec{R} = h\hat{z} - \rho'\hat{\rho}$$

$$dl = \rho\, d\phi = a\, d\phi \qquad R = |\vec{R}| = \sqrt{h^2 + \rho^2}$$

$$d\vec{E}_1 = \frac{\rho_l\, a\, d\phi}{4\pi\varepsilon_0 (\sqrt{h^2+a^2})^2} = \frac{\rho_l\, a\, d\phi\, (h\hat{z} - a\hat{\rho})}{4\pi\varepsilon_0 (\sqrt{h^2+a^2})^3}$$

Example: Finite Line Charge

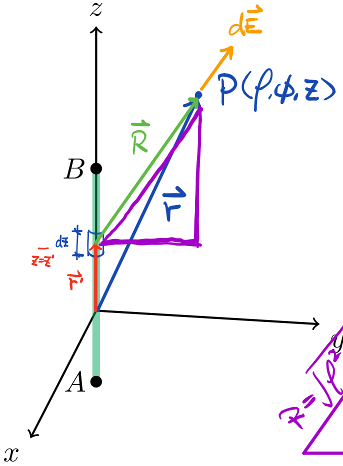
$$d\vec{E}_2 = \frac{\rho_l\, a\, d\phi\,(h\hat{z} + a\hat{\rho})}{4\pi\varepsilon_0 \left(\sqrt{h^2+a^2}\right)^3}$$

$$E_{Total} = \int_0^{2\pi} d\vec{E} = \int_0^{\pi} d\vec{E}_1 + d\vec{E}_2$$

$$= \int_0^{\pi} \frac{\rho_l\, a\, 2h\hat{z}\, d\phi}{4\pi\varepsilon_0 \sqrt{h^2+a^2}^{\,3}}$$

$$\vec{E}_{Total} = \frac{\rho_l\, a\, h\, \hat{z}\, \pi}{2\pi\varepsilon_0 (h^2+a^2)^{3/2}}$$

Determine the electric field of a uniform line charge along the z-axis extending between points A and B. The line charge density is ρ_l C/m.



Expect $\vec{E}$ to not vary w/ ϕ (azimuthal symmetry)

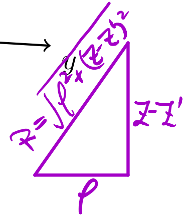
azimuth angle.

There's no E_ϕ component.

$$\vec{r} = \rho\hat{\rho} + z\hat{z} \qquad \vec{r}' = z'\hat{z} \qquad \vec{R} = \vec{r} - \vec{r}' = \rho\hat{\rho} + (z - z')\hat{z}$$

$$R = |\vec{R}| = \sqrt{\rho^2 + (z-z')^2}$$

$$d\vec{E} = \frac{\rho_l\, dz'\,(\rho\hat{\rho} + (z-z')\hat{z})}{4\pi\varepsilon_0 (\rho^2 + (z-z')^2)^{3/2}}$$

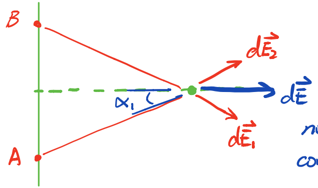


$$\rho = R\cos\alpha \rightarrow R = \rho/\cos\alpha = \rho\sec\alpha$$

$$\tan\alpha = \frac{z-z'}{\rho} \rightarrow z - z' = \rho\tan\alpha \leftarrow$$ differentiate both side.

Table of Integrals

Symmetric case:



no $\hat{z}$ component.

$\alpha_1 = -\alpha_2$

$\cos(\alpha_2) = \cos(-\alpha_1) = \cos\alpha_1$

$\vec{E} = \frac{2\rho_\ell}{4\pi\varepsilon_0\rho}\sin(\alpha_1)$

Infinite line case:

$\alpha_1 = \frac{\pi}{2} \quad \alpha_2 = -\frac{\pi}{2}$

$$\boxed{\vec{E} = \frac{\rho_\ell}{2\pi\varepsilon_0\rho}}$$

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Table of Integrals

Source: *M. Sadiku, Elements of Electromagnetics, 5[th] ed., Oxford University Press, New York, 2010.*

If $U = U(x)$, $V = V(x)$, and $a =$ constant,

$$\int a\,dx = ax + C$$

$$\int U\,dV = UV - \int V\,dU \quad \text{(integration by parts)}$$

$$\int U^n\,dU = \frac{U^{n+1}}{n+1} + C,\quad n \neq -1$$

$$\int \frac{dU}{U} = \ln U + C$$

$$\int a^U\,dU = \frac{a^U}{\ln a} + C,\quad a > 0, a \neq 1$$

$$\int e^U\,dU = e^U + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax - 1) + C$$

$$\int x^2e^{ax}dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2) + C$$

$$\int \ln x\,dx = x\ln x - x + C$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln\sec ax + C = -\frac{1}{a}\ln\cos ax + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ln(\sec ax + \tan ax) + C$$

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$$

$$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C$$

$$\int \frac{xdx}{\sqrt{x^2 + a^2}} = \sqrt{x^2 + a^2} + C$$

$$\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x/a^2}{\sqrt{x^2 + a^2}} + C$$

$$\int \frac{xdx}{(x^2 + a^2)^{3/2}} = -\frac{1}{\sqrt{x^2 + a^2}} + C$$

$$\int \frac{x^2dx}{\ldots}$$

$$\int x\sin ax\,dx = \frac{1}{a^2}(\sin ax - ax\cos ax) + C$$

$$\int x\cos ax\,dx = \frac{1}{a^2}(\cos ax + ax\sin ax) + C$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + C$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + C$$

$$\int \sin ax\sin bx\,dx = \frac{\sin(a - b)x}{2(a - b)} - \frac{\sin(a + b)x}{2(a + b)} + C,\quad a^2 \neq b^2$$

$$\int \sin ax\cos bx\,dx = -\frac{\cos(a - b)x}{2(a - b)} - \frac{\cos(a + b)x}{2(a + b)} + C,\quad a^2 \neq b^2$$

$$\int \cos ax\cos bx\,dx = \frac{\sin(a - b)x}{2(a - b)} + \frac{\sin(a + b)x}{2(a + b)} + C,\quad a^2 \neq b^2$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \tanh ax\,dx = \frac{1}{a}\ln\cosh ax + C$$

$$\int \frac{dx}{ax + b} = \frac{1}{a}\ln|ax + b|$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\int \frac{xdx}{x^2 + a^2} = \frac{1}{2}\ln(x^2 + a^2) + C$$

$$\int \frac{x^2\,dx}{x^2 + a^2} = x - a\tan^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{x^2 - a^2} = \begin{cases} \frac{1}{2a}\ln\frac{x - a}{x + a} + C, & x^2 > a^2 \\ \frac{1}{2a}\ln\frac{a - x}{a + x} + C, & x^2 < a^2 \end{cases}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln(x + \sqrt{x^2 \pm a^2}) + C$$

$-dz' = \rho\sec^2\alpha\,d\alpha$

$$d\vec{E}_\rho = \frac{\rho_\ell(-\rho\sec^2\alpha\,d\alpha)\cdot\rho}{4\pi\varepsilon_0(\rho^2 + \rho^2\tan^2\alpha)^{3/2}} = \frac{-\rho_\ell\rho^2\sec^2\alpha\,d\alpha}{4\pi\varepsilon_0\rho^3\sec^3\alpha} = \frac{-\rho_\ell\,d\alpha}{4\pi\varepsilon_0\rho\sec\alpha}$$

$$= \frac{-\rho_\ell\cos\alpha\,d\alpha}{4\pi\varepsilon_0\rho}$$

$$\vec{E}_\rho = -\int_{\alpha=\alpha_1}^{\alpha_2}\frac{\rho_\ell}{4\pi\varepsilon_0\rho}\cos\alpha\,d\alpha$$

(α_1 & α_2 are the start angle & end angle.)
pt. A pt. B

$$\vec{E}_\rho = -\frac{\rho_\ell\sin\alpha}{4\pi\varepsilon_0\rho}\Big|_{\alpha_1}^{\alpha_2} = \frac{\rho_\ell}{4\pi\varepsilon_0\rho}(\sin\alpha_1 - \sin\alpha_2)$$

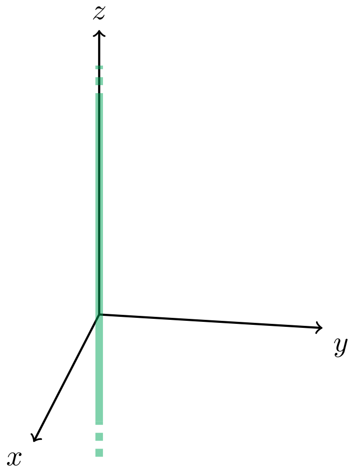
$$d\vec{E}_z = \frac{\rho_\ell(-\rho\sec^2\alpha\,d\alpha)\cdot\rho\tan\alpha}{4\pi\varepsilon_0(\rho^2 + \rho^2\tan^2\alpha)^{3/2}} = \frac{-\rho_\ell\tan\alpha\,d\alpha}{4\pi\varepsilon_0\rho\sec\alpha} = \frac{-\rho_\ell\sin\alpha\,d\alpha}{4\pi\varepsilon_0\rho}$$

$$\vec{E}_z = \frac{-\rho_\ell}{4\pi\varepsilon_0\rho}(\cos\alpha_1 - \cos\alpha_2)$$

$$\vec{E} = E_\rho\hat{\rho} + E_z\hat{z} = \frac{\rho_\ell}{4\pi\varepsilon_0\rho}\left[\hat{\rho}(\sin\alpha_1 - \sin\alpha_2) - \hat{z}(\cos\alpha_2 - \cos\alpha_1)\right] //$$

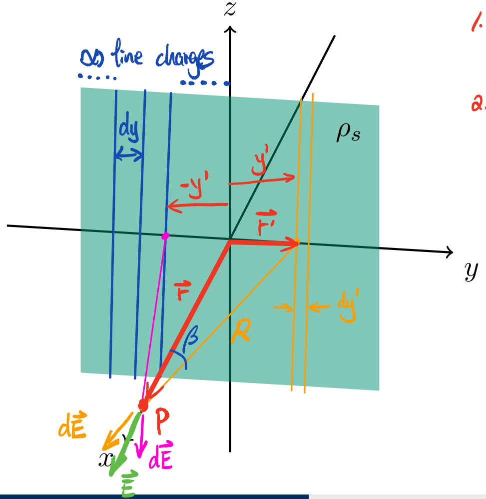
Example: Infinite Line Charge

Determine the electric field of an infinitely long and uniform line charge along the z-axis. The line charge density is ρ_l C/m.



Example: Infinite Surface Charge

Determine the electric field of an infinite sheet of charge with a uniform charge density ρ_s [C/m^2] placed on the yz plane $(x = 0)$.



1. What variables should $\vec{E}$ depend on?

→ Does not vary w/ y or z

2. What components should $\vec{E}$ have?

→ $\hat{x}$ (only) other component cancel out, see diagram.

$$d\vec{E} = \hat{\rho}\,\frac{\rho_s\, dy'}{2\pi\varepsilon_0 R} \qquad R = |\vec{r} - \vec{r}'| = |x\hat{x} - y'\hat{y}| = \sqrt{x^2 + y'^2}$$

$$dE_x = dE_\rho \cos\beta = \frac{\rho_s\, dy' \cos\beta}{2\pi\varepsilon_0 \sqrt{x^2+y'^2}}$$

$$= \frac{\rho_s\, dy' \cos\beta}{2\pi\varepsilon_0 \sqrt{x^2+y'^2}} \cdot \frac{\sqrt{x^2+y'^2}}{\sqrt{x^2+y'^2}} = \frac{\rho_s\, dy'\, x}{2\pi\varepsilon_0 (x^2+y'^2)}$$

Example: Two Parallel and Oppositely-Charged Infinite Surface Charge Densities

$$\vec{E}_x = \frac{\rho_s x}{2\pi\varepsilon_0}\int_{y'=-\infty} \frac{dy'}{x^2+y'^2}$$

$$\vec{E}_x = \frac{\rho_s}{2\pi\varepsilon_0}\tan^{-1}\left(\frac{y'}{x}\right)\Big|_{-\infty}^{\infty}$$

$$= \frac{\rho_s}{2\pi\varepsilon_0}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{\rho_s}{2\varepsilon_0}\hat{x} \quad x>0$$