Lecture 33: Displacement Current, Maxwell's Equations

ECE221: Electric and Magnetic Fields

Prof. Sean V. Hum

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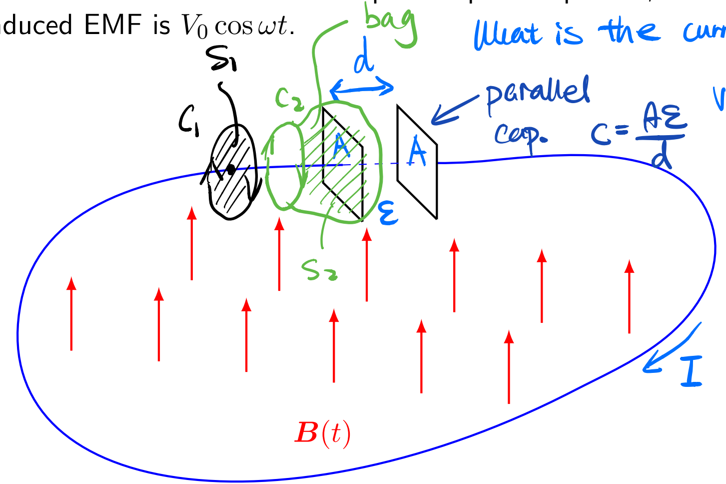
Outline

Displacement Current

Consider a closed circuit with a parallel-plate capacitor, in which the induced EMF is $V_0 \cos \omega t$.

What is the current flowing in this circuit?



$C = \frac{A\varepsilon}{d}$

Voltage across capacitor is

$V_c = V_0 \cos \omega t$

$$i_c = C \frac{dV_c}{dt}$$

$$= -CV_0 \omega \sin \omega t$$

$$= -\frac{\varepsilon A}{d} V_0 \omega \sin \omega t$$

Ampere's Maxwell Law:

$$\oint_{C_1} \vec{H} \cdot d\vec{l} = I \leftarrow \text{conduction current}$$

$$\oint_{C_2} \vec{H} \cdot d\vec{l} = \ ? \leftarrow \underline{\underline{\text{no}}} \text{ conduction current passing through } S_2!$$

= displacement current.

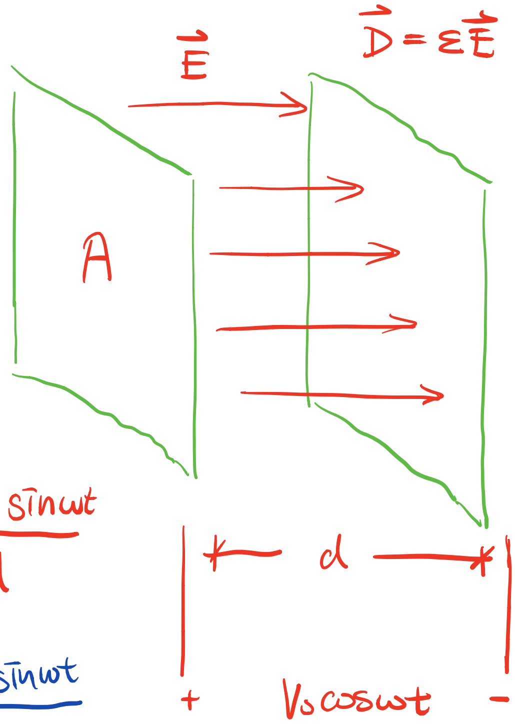
$$E = \frac{V}{d} = \frac{V_0 \cos\omega t}{d}$$

$$D = \frac{\varepsilon V_0 \cos\omega t}{d}.$$

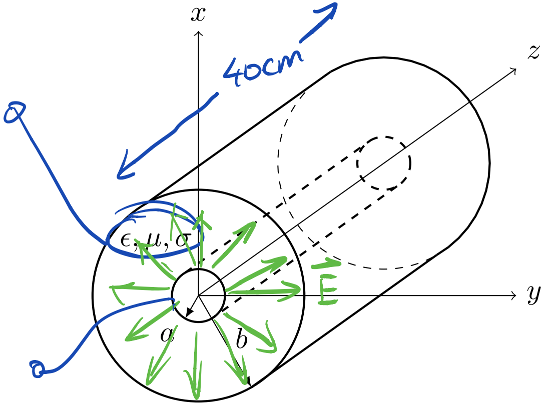
$$\frac{\partial D}{\partial t} = -\frac{\varepsilon V_0 \omega \sin\omega t}{d}$$

$$\int_S \frac{\partial \vec{D}}{\partial s} \cdot d\vec{s} = \frac{\partial D}{\partial s} \cdot A = -\frac{\varepsilon V_0 \omega A \sin\omega t}{d}$$

$$I_d = -\frac{C V_0 \omega \sin\omega t}{d}$$



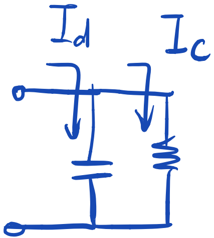
Example: Coaxial Cable



Let the internal dimensions of a coaxial capacitor be $a = 1.2$ cm, $b = 4$ cm, and $l = 40$ cm. The material within the capacitor has the parameters $\epsilon = 10^{-11}$ F/m, $\mu = 10^{-5}$ H/m, and $\sigma = 10^{-5}$ S/m. The electric field intensity between the cylinders is

$$\boldsymbol{E} = \frac{10^6}{\rho} \cos 10^5 t \hat{\boldsymbol{\rho}}$$

Find (a) $\boldsymbol{J}$; (b) the total conduction current I through the capacitor; (c) the total displacement current through the capacitor I_d.



a) $\vec{J} = \sigma \vec{E} = \frac{10}{\rho} \cos 10^5 t \, \hat{\rho}$

b) $I_{cond} = \int_s \vec{J} \cdot d\vec{s} = \int_s \frac{10}{\rho} \cos 10^5 t \, \hat{\rho} \cdot \hat{\rho} \rho \, d\phi \, dz = \int_0^l \int_0^{2\pi} 10 \cos 10^5 t \, d\phi \, dz$

$= 2\pi (40cm)(10 \cos 10^5 t)$

$= 8\pi \cos 10^5 t$

c) $I_d \rightarrow \vec{G} = \frac{\partial \vec{D}}{\partial t}$ $\quad \vec{D} = \varepsilon \vec{E} = \frac{10^{-5}}{\rho} \cos 10^5 t \, \hat{\rho}$

$G = \frac{-\sin 10^5 t}{\rho} \hat{\rho}$

$I_d = \int_0^{2\pi} \int_0^{40} \frac{-\sin 10^5 t}{\rho} \hat{\rho} \cdot \hat{\rho} \, \rho \, dz \, d\phi = -0.8\pi \sin(10^5 t)$

Maxwell's Equations

Integral form	Point form	
$\oint_S \boldsymbol{D}(\boldsymbol{r},t)\cdot d\boldsymbol{S} = Q(t)$	$\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{r},t) = \rho_v(\boldsymbol{r},t)$	Gauss's Law.
$\oint_S \boldsymbol{B}(\boldsymbol{r},t)\cdot d\boldsymbol{S} = 0$	$\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r},t) = 0$	No magnetic charges
$\oint_C \boldsymbol{E}(\boldsymbol{r},t)\cdot d\boldsymbol{l} = -\frac{d}{dt}\int_S \boldsymbol{B}(\boldsymbol{r},t)\cdot d\boldsymbol{S}$	$\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t}$	Faraday's Law.
$\oint_C \boldsymbol{H}(\boldsymbol{r},t)\cdot d\boldsymbol{l} = I(t) + \int_S \frac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t}\cdot d\boldsymbol{S}$	$\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r},t) + \frac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t}$	Ampere Maxwell.
$\oint_S \boldsymbol{J}(\boldsymbol{r},t)\cdot d\boldsymbol{S} = -\frac{dQ(t)}{dt}$	$\boldsymbol{\nabla}\times\boldsymbol{J}(\boldsymbol{r},t) = \frac{\partial \rho_v(\boldsymbol{r},t)}{\partial t}$	Equation of continuity.

Auxiliary Equations

- We also have auxiliary equations, the **constitutive relations**, linking field quantities:

$$\boldsymbol{D}(\boldsymbol{r},t) = \epsilon \boldsymbol{E}(\boldsymbol{r},t)$$

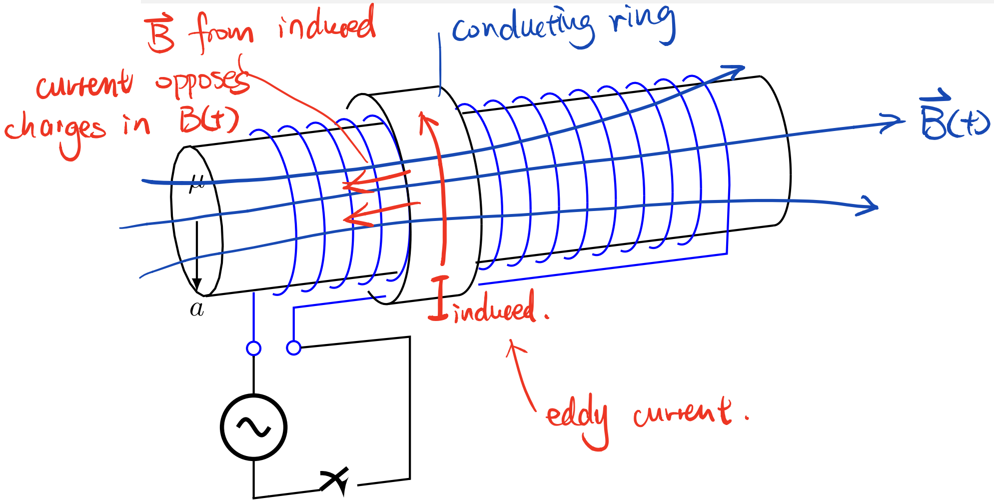
$$\boldsymbol{B}(\boldsymbol{r},t) = \mu \boldsymbol{H}(\boldsymbol{r},t)$$

$$\boldsymbol{J}(\boldsymbol{r},t) = \sigma \boldsymbol{E}(\boldsymbol{r},t)$$

- We can also relate conduction (convection) current density to ρ_v through

$$\boldsymbol{J}(\boldsymbol{r},t) = \rho_v(\boldsymbol{r},t)\boldsymbol{u}(\boldsymbol{r},t)$$

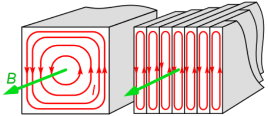
Application 1: Electromagnetic Induction



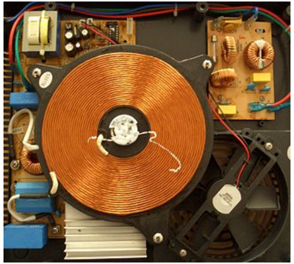
Application 2: Electromagnetic Induction

Electromagnetic induction induces **eddy currents** into conductors in the vicinity of a time-varying current. This can be . . .

. . . undesired, since it reduces efficiency of devices like transformers;



. . . useful as a way for efficiently producing heat.



Application 2: Electromagnetic Waves

- The **sources** in Maxwell's equations are fundamentally **charges** and **currents**.
- In time-varying problems, the charges tend to be in motion: only **currents** matter.
- In a **time-varying system**, Maxwell's equations force $\boldsymbol{E}$ and $\boldsymbol{H}$ to be related.
- Consider what happens away from the sources.

Application 2: Electromagnetic Waves

Example: Determine $\boldsymbol{H}$ if

$$\boldsymbol{E} = \hat{\boldsymbol{x}} A \cos(\omega t - kz)$$

The Electromagnetic Spectrum

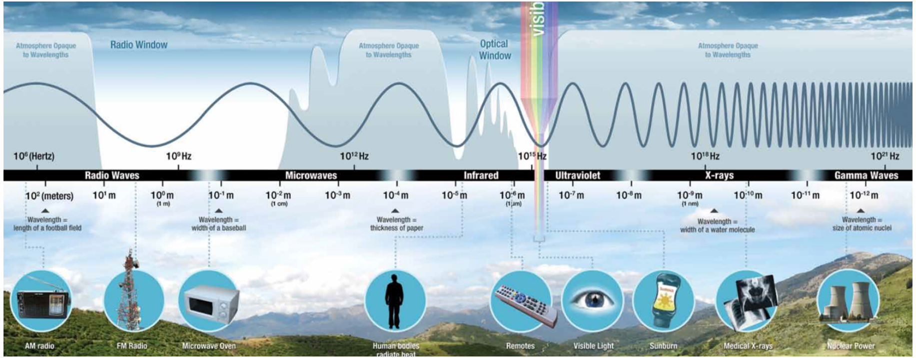


Image courtesy NASA

The Electromagnetic Spectrum