Lecture 13: Dielectrics and the Equation of Continuity

ECE221: Electric and Magnetic Fields

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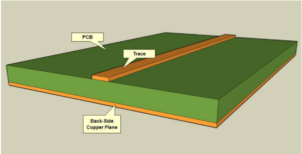
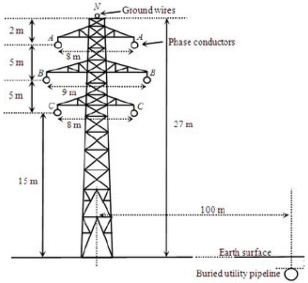
Outline

Examples of Boundaries

Source: imgur





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Boundary Conditions: Dielectric-Dielectric Interface

$$\oint_C \vec{E}\cdot d\vec{\ell} = \int_a^b + \int_b^c + \int_c^d + \int_d^a = 0$$

if Δw is small

$$= \underbrace{\int \vec{E}_{t_1}\cdot d\vec{\ell}_{ab}}$$

$$= E_{t_1}\cdot\Delta w - E_{n_1}(\text{at } b)\Delta h - E_{t_2}\cdot\Delta w + E_{n_2}(\text{at } a)\Delta h$$

Take the limit as $\Delta h \to 0$

$$= E_{t_1}\cdot\Delta w - \cancel{E_{n_1}(\text{at } b)\Delta h} - E_{t_2}\cdot\Delta w + \cancel{E_{n_2}(\text{at } a)\Delta h} = 0$$

$$\Rightarrow E_{t_1}\cdot\Delta w = E_{t_2}\cdot\Delta w$$

$$\boxed{E_{t_1} = E_{t_2}}$$

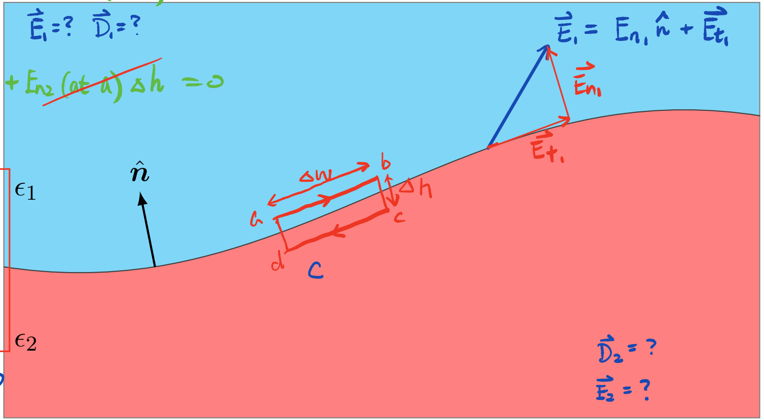
(The tangential component of $\vec{E}$ must remain continuous as we cross the interface)

$$\boxed{\vec{n}\times\vec{E}_1 = \vec{n}\times\vec{E}_2}$$

$$\vec{n}\times(\vec{n}_2-\vec{n}_1) = 0$$

Consider an interface between two dielectric materials, and apply

$$\oint_C \boldsymbol{E}\cdot d\boldsymbol{\ell} = 0$$



Boundary Conditions: Dielectric-Dielectric Interface

$$\oint_S \vec{D}\cdot d\vec{s} = Q_{encl.}$$

Consider an interface between two dielectric materials, and apply **Gauss' Law**.

$$\hat{n}\times\frac{D_1}{\varepsilon_1} = \hat{n}\times\frac{D_2}{\varepsilon_2}$$

$$\oint_S \vec{D}\cdot d\vec{s} = \iint_{top} + \iint_{side} + \iint_{bottom} = Q_{encl.}$$

$$= \oint D_{top}\cdot ds_{top}$$

$$= D_{n_1}\Delta S + \cancel{\iint_{side}\vec{D}\cdot d\vec{s}} - D_{n_2}\Delta S = \iint_{middle}\rho_s ds \approx \rho_s \Delta S$$

≈ 0 if $\Delta H \to 0$

$$\boxed{D_{n_1} - D_{n_2} = \rho_s}$$

Normal component of $\vec{D}$ is discontinuous by an amount ρ_s as we cross the surface.

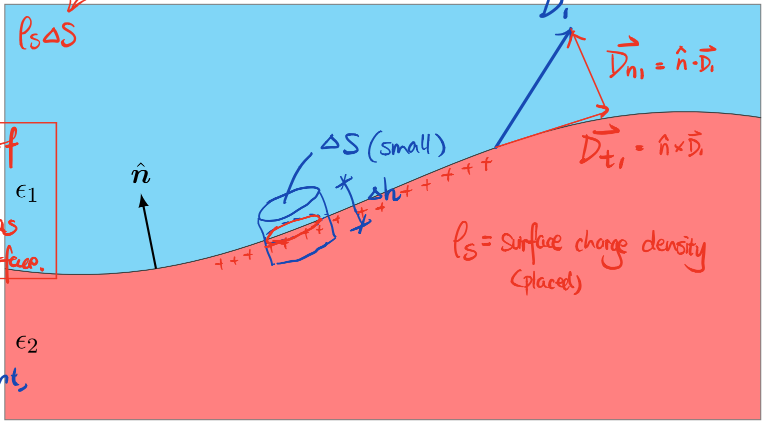
$$\Downarrow$$

$$\varepsilon_0 E_{n_1} - \varepsilon E_{n_2} = \rho_s$$

$$\boxed{\hat{n}\cdot\vec{D}_1 - \hat{n}\times\vec{D}_2 = \rho_s}$$

If there's no charge density (ρ_s) present,

$$\vec{n}\cdot\vec{D}_1 = \vec{n}\times\vec{D}_2$$



Boundary Conditions: Dielectric-Conductor Interface

$E_{t1} = \cancel{E_{t2}}^{0} \Rightarrow \vec{E}_t = 0$

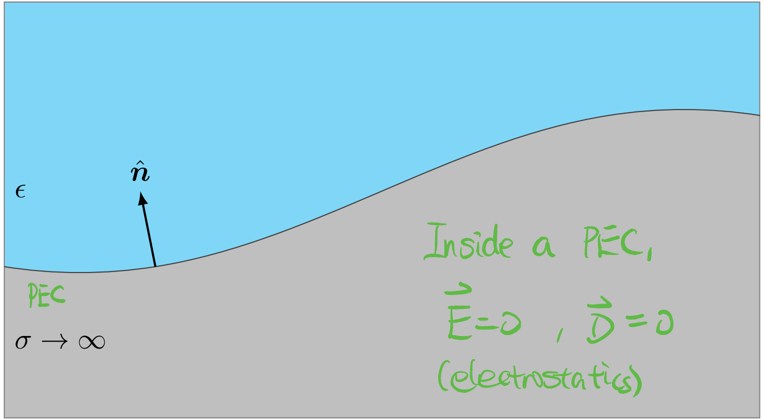
At the surface of a PEC,

$\hat{n} \times \vec{E} = 0 \quad \vec{E}_t = 0$

$\hat{n} \cdot \vec{D}_1 - \cancel{\hat{n} \cdot \vec{D}_2}^{0} = \rho_s$

$D_n = \rho_s$

$\varepsilon E = \rho_s$



$\vec{E} = \frac{\vec{J}}{\sigma}$

as $\sigma \to \infty$

$E \to 0$

Poisson's and Laplace's Equation

Gauss's Law $\nabla \cdot \boldsymbol{D} = \rho_v$ $\vec{D} = \varepsilon \vec{E}$

$\nabla \cdot \varepsilon \vec{E} = -\nabla(\varepsilon \nabla V) = \rho_v$

$$\boldsymbol{E} = -\nabla V$$

if medium is homogeneous. (i.e. ε is not a function of space)

$\underbrace{\nabla \cdot \nabla}V = -\rho_v/\varepsilon$ $\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \cdot \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)$

source.

$\nabla \cdot \nabla V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \equiv \boxed{\nabla^2 V = -\rho_v/\varepsilon}$ Poisson's eqn.

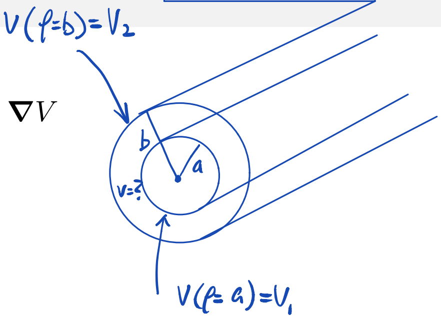
laplceian operator.

If we consider regions where there's no charges. ($\rho_v = 0$)

Laplacian Operator

$\boxed{\nabla^2 V = 0}$ Laplace's Eqn. (2nd order homogeneous PDE)

$$\nabla^2 V = \nabla \cdot \nabla V$$



Laplacian Operator

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$