Lecture 17: Capacitance, Resistance, and the Method of Images

ECE221: Electric and Magnetic Fields

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Outline

Resistance and Capacitance

$$R = \frac{V}{I} = \frac{-\int_C \boldsymbol{E} \cdot d\boldsymbol{l}}{\int_S \underbrace{\sigma \boldsymbol{E}}_{\vec{J}} \cdot d\boldsymbol{s}}$$

$$C = \frac{\oint_S \overbrace{\epsilon \boldsymbol{E}}^{\vec{D}} \cdot d\boldsymbol{s}}{-\int_C \boldsymbol{E} \cdot d\boldsymbol{\ell}}$$

$$RC = \frac{\cancel{-\int_C \vec{E} \cdot d\vec{l}}}{\int_S \sigma \vec{E} \cdot d\vec{s}} \times \frac{\oint_S \varepsilon \vec{E} \cdot d\vec{s}}{\cancel{-\int_C \vec{E} \cdot d\vec{l}}} = \frac{\varepsilon \cancel{\oint_S \vec{E} \cdot d\vec{s}}}{\sigma \cancel{\int_S \vec{E} \cdot d\vec{s}}} = \boxed{\frac{\varepsilon}{\sigma} = RC}$$

if ε & σ are uniform with position.

Procedure for Solving for Resistance

1. Choose a suitable coordinate system.
2. Solve for the electric field in the region of interest (resistive material region)
   - Find $\boldsymbol{E}$ from Q or $\rho_{v|s|l}$ using Gauss' Law or Coulomb's Law; or
   - Find V from solving Laplace's equation, and find $\boldsymbol{E} = -\boldsymbol{\nabla} V$.

   $\vec{J} = \sigma \vec{E}$
3. Find I:
$$I = \iint \boldsymbol{J} \cdot d\boldsymbol{s}$$
4. Determine
$$R = \frac{V_0}{I}$$

Parallel Plate Capacitor/Resistor

$\vec{E} = -V_0/d\,\hat{z} \rightarrow \vec{J} = \sigma\vec{E} = -\frac{\sigma V_0}{d}\hat{z}$

C remains unchanged if $\sigma \neq 0$ → now there's resistance btw two plates.

$$I = \iint_S \vec{J}\cdot d\vec{s} = \iint -\frac{\sigma V_0}{d}\hat{z}\cdot(-\hat{z})\,dy\,dx$$

direction of $d\vec{s}$

$$= \frac{\sigma V_0 A}{d} = I$$

$$R = V_0/I = V_0 \Big/ \frac{\sigma V_0 A}{d}$$

$$\boxed{R = \frac{d}{\sigma A}}$$

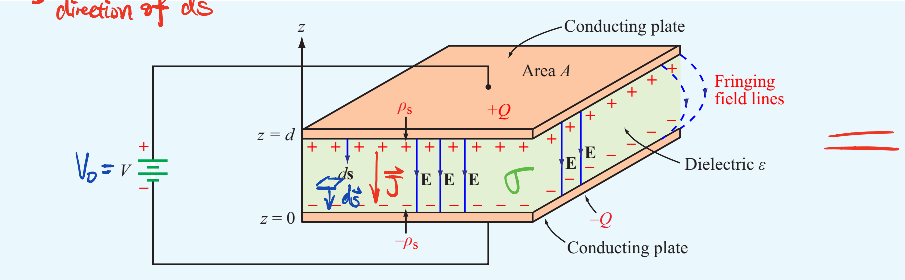

Image Credit: Ulaby and Ravaioli

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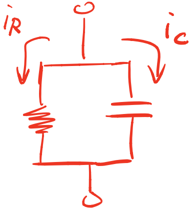


In parallel.

match!

$RC = \frac{\varepsilon}{\sigma} \rightarrow R = \frac{\varepsilon}{\sigma C}$ Recall $C = \frac{\varepsilon A}{d} \Rightarrow R = \frac{d}{\sigma A}$

matches

$R = \frac{\ell}{\sigma A}$ only applies when $\vec{E}/\vec{J}$ is uniform in resistive region.

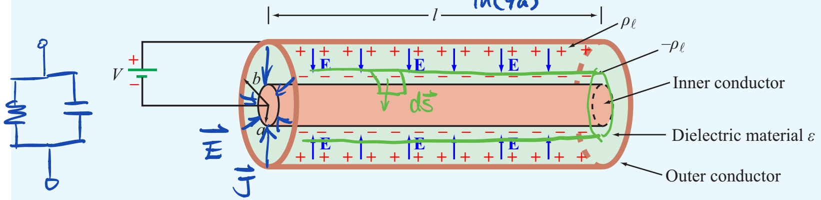
Coaxial Capacitor/Resistor

Recall $\vec{E} = -\frac{\hat{\rho} V_0}{\rho \ln(b/a)}$

$$I = \iint \vec{J}\cdot d\vec{s} = \iint \frac{-\hat{\rho}\,\sigma V_0}{\rho \ln(b/a)}\cdot d\vec{s} = \frac{\cancel{-\hat{\rho}}\,\sigma V_0}{\cancel{\rho}\ln(b/a)}\cdot \cancel{\rho}\, d\phi\, dz\, (\cancel{-\hat{\rho}})$$

$$= \frac{\sigma V_0}{\ln(b/a)}\, 2\pi l$$

$$R = V_0/I = \frac{\ln(b/a)}{\sigma 2\pi l} = \frac{1}{G} \qquad G = \frac{2\pi \sigma l}{\ln(b/a)}$$



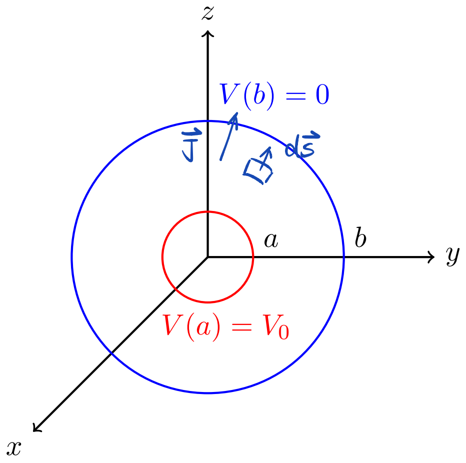
$G' = G$ per unit length $:= G/l$

$$G' = \frac{2\pi\sigma}{\ln(b/a)} \left[\frac{1}{\Omega m}\right] \text{ or } [S/m]$$

Figure 4-25 Coaxial capacitor filled with insulating material of permittivity ε (Example 4-12).

Image Credit: Ulaby and Ravaioli

Spherical Capacitor/Resistor



Recall: $\vec{E} = \hat{r}\frac{V_0}{r^2}\frac{1}{\frac{1}{a}-\frac{1}{b}}$

$$I = \int_0^{2\pi}\int_0^{\pi} \cancel{\hat{r}}\, \frac{\sigma V_0}{\cancel{r^2}} \frac{1}{\frac{1}{a}-\frac{1}{b}} \cdot (\cancel{\hat{r}}\, \cancel{r^2} \sin\theta \, d\theta \, d\phi)$$

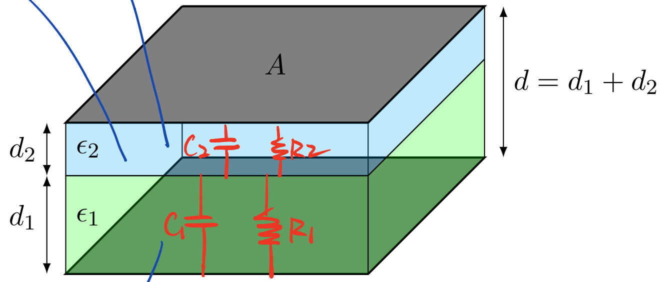
$$= \frac{\sigma V_0}{\frac{1}{a}-\frac{1}{b}}(4\pi)$$

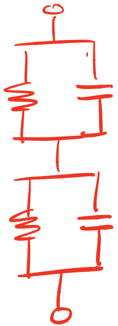
$$R = \frac{V_0}{I} = \frac{\frac{1}{a}-\frac{1}{b}}{4\pi\sigma}$$

Parallel Plate Capacitor/Resistor: Case 1

$C_2 = \frac{\varepsilon_2 A}{d}$

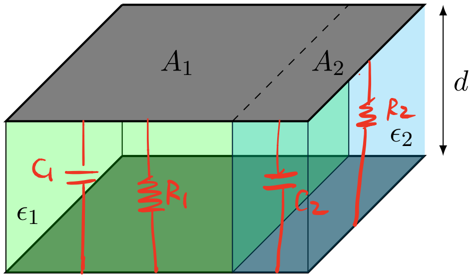
$R_2 = \frac{d_2}{\sigma_2 A}$

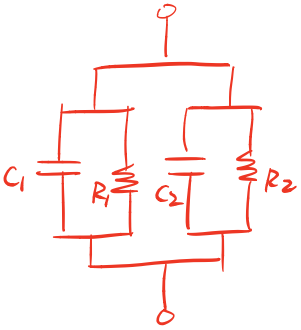




C_1, R_1

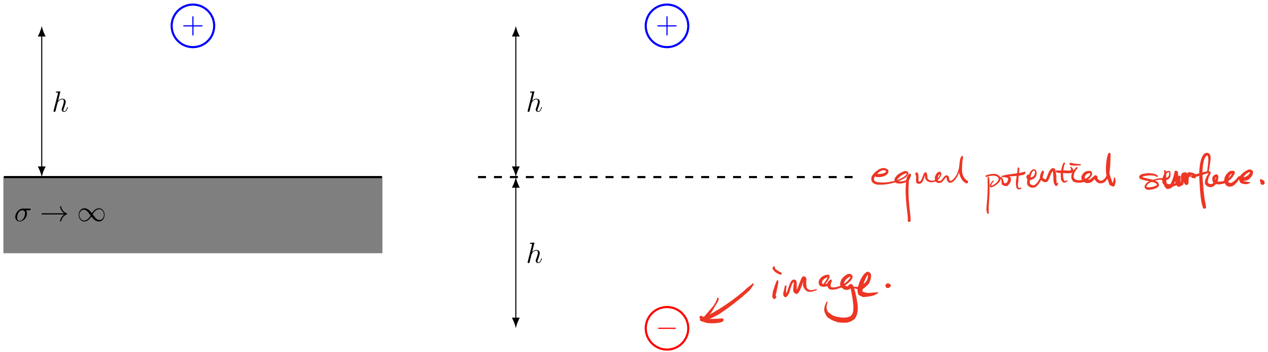
Parallel Plate Capacitor/Resistor: Case 2





Method of Images

Given a charge configuration above an infinite grounded PEC plane may be replaced by the charge configuration itself, its image, and an equipotential surface in place of the conducting plane.



Method of Images