

A Spectrum of Type Soundness and Performance

Supplementary Material

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	Benchmark	Untyped LOC	Annotation LOC	# Modules	
111	sieve	35	17 (49%)	2	166
112	fsm	182	56 (31%)	4	167
113	morsecode	159	38 (24%)	4	168
114	zombie	302	27 (9%)	4	169
115	jpeg	1432	165 (12%)	5	170
116	suffixtree	537	129 (24%)	6	171
117	kcfa	229	53 (23%)	7	172
118	snake	160	51 (32%)	8	173
119	tetris	246	107 (43%)	9	174
120	synth	835	139 (17%)	10	175
121					176
122					177
123					178

Fig. 1. Benchmark Size

A Benchmark Descriptions

128 **sieve** from Ben Greenman

129 Computes prime numbers using the sieve of Eratosthenes.

130 **fsm** from Linh Chi Nguyen

131 Simulates the interactions of economic agents modeled as
132 finite-state automata.

134 **morsecode** from John B. Clements & Neil Van Dyke

135 Computes Levenshtein distances and morse code translations
136 for a fixed sequence of pairs of words.

137 **zombie** from David Van Horn

138 Implements a game where players avoid enemies. The benchmark
139 runs a fixed sequence of moves (representing user
140 input).

142 **jpeg** from Andy Wingo

143 Parses a bytestream of JPEG data to an internal representa-
144 tion, then serializes the result.

145 **suffixtree** from Danny Yoo

146 Computes longest common subsequences between strings.

148 **kcfa** from Matt Might

149 Performs 1-CFA on a lambda calculus equation built from
150 Church numerals.

151 **snake** from David Van Horn

152 Implements the Snake game; the benchmark replays a fixed
153 sequence of moves.

155 **tetris** from David Van Horn

156 Replays a pre-recorded game of Tetris.

158 **synth** from Vincent St. Amour & Neil Toronto

159 Converts a description of notes and drum beats to WAV format.

160 For additional details about the benchmarks, their source
161 code, and links to more (object-oriented) benchmarks, see:
162 docs.racket-lang.org/gtp-benchmarks/index.html

221	B Performance vs. Number of Typed	276
222	Modules	277
223	Figures 2, 3, 4, 5, and 6 plot every running time in the dataset.	278
224	Each point represents one measurement; specifically, a point	279
225	at position (X, Y) reports one running time of Y milliseconds	280
226	for one configuration with X “typed units” — in this paper,	281
227	one typed unit is one typed module. There are eight such	282
228	points for each configuration. To make these points easier to	283
229	see, the eight points for one configuration are evenly spaced	284
230	along the x -axis within a bucket (delimited by solid vertical	285
231	lines). The left-most point plots the running time of the first	286
232	trial, the second-from-left point corresponds to the second	287
233	trial, and so on.	288
234	The figures support three broad conclusions. First, the	289
235	orange points for TR-1 are often lower (i.e., show better per-	290
236	formance) than the points for TR-H. Second, the performance	291
237	of TR-1 tends to degrade linearly as the number of typed	292
238	modules increases. This slowdown tapers off at the right-	293
239	most end because the implementation skips the codomain-	294
240	check for calls to statically-typed identifiers. Third, the per-	295
241	formance of TR-H has a steep umbrella shape. The worst	296
242	performance is in the middle, but improves significantly as	297
243	the number of typed modules approaches the maximum.	298
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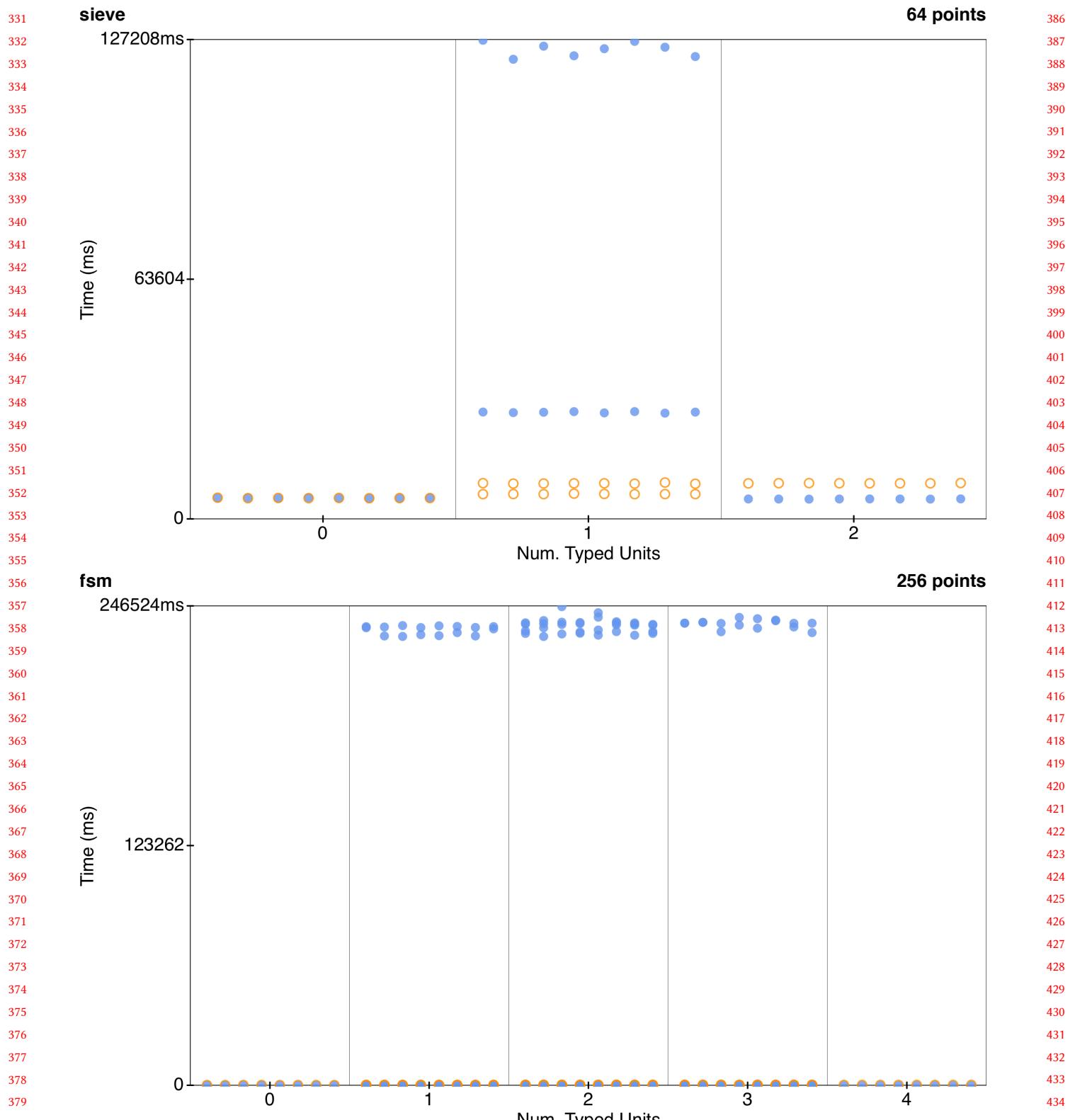


Fig. 2. Running time of TR-H configurations (blue) and TR-1 configurations (orange), part 1/5

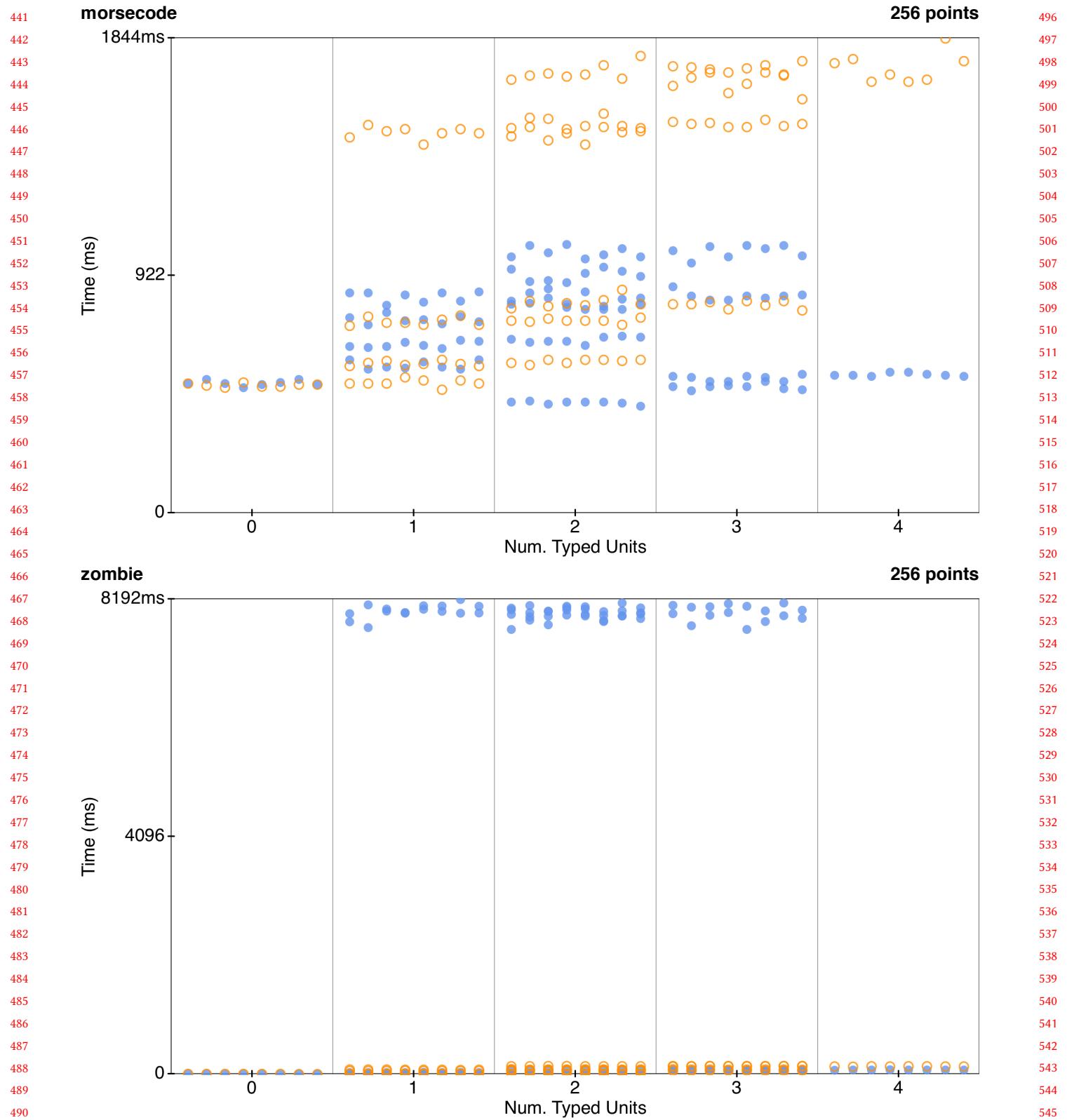
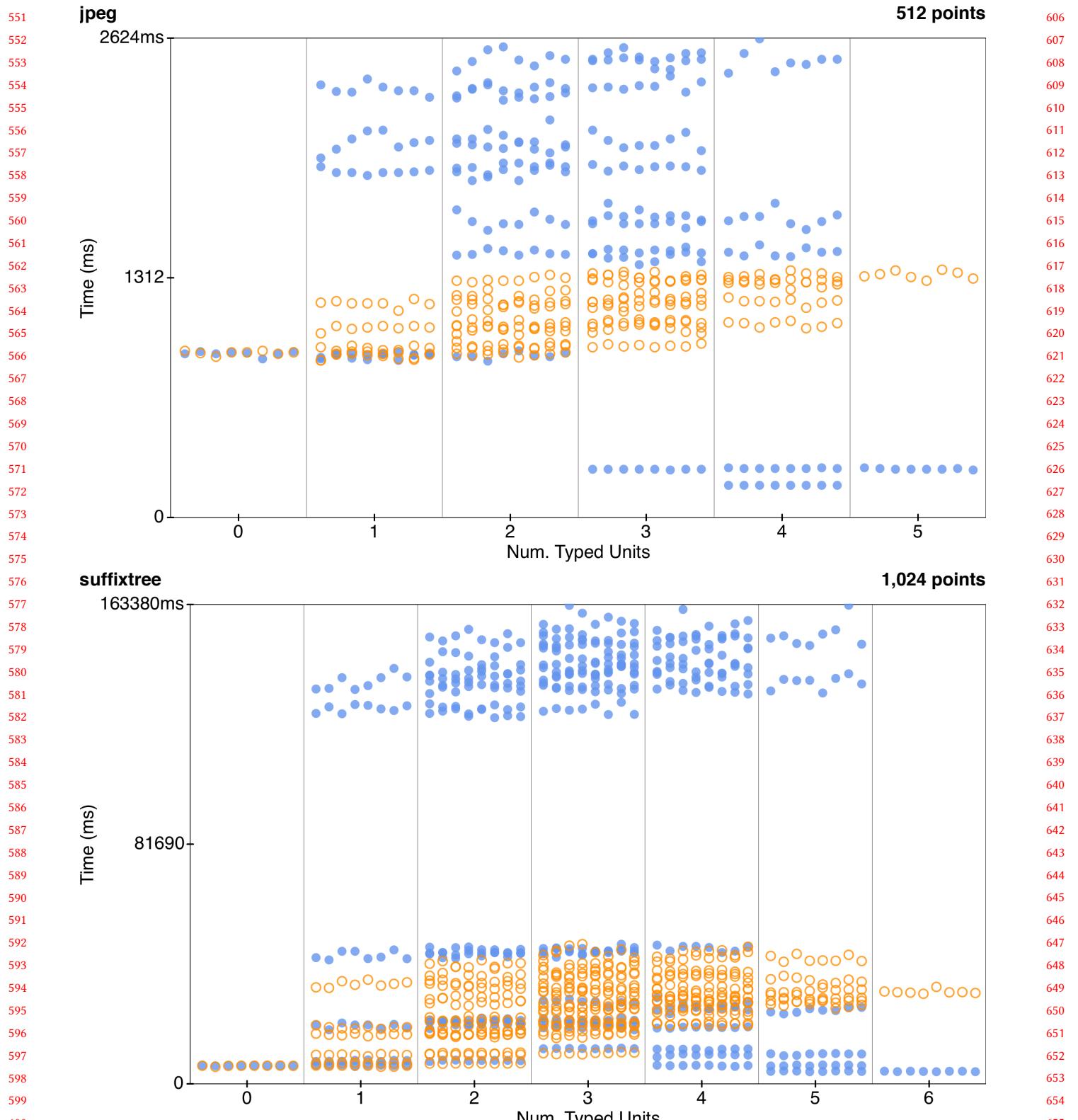


Fig. 3. Running time of TR-H configurations (blue) and TR-1 configurations (orange), part 2/5

Fig. 4. Running time of TR-H configurations (blue ■) and TR-1 configurations (orange ○), part 3/5

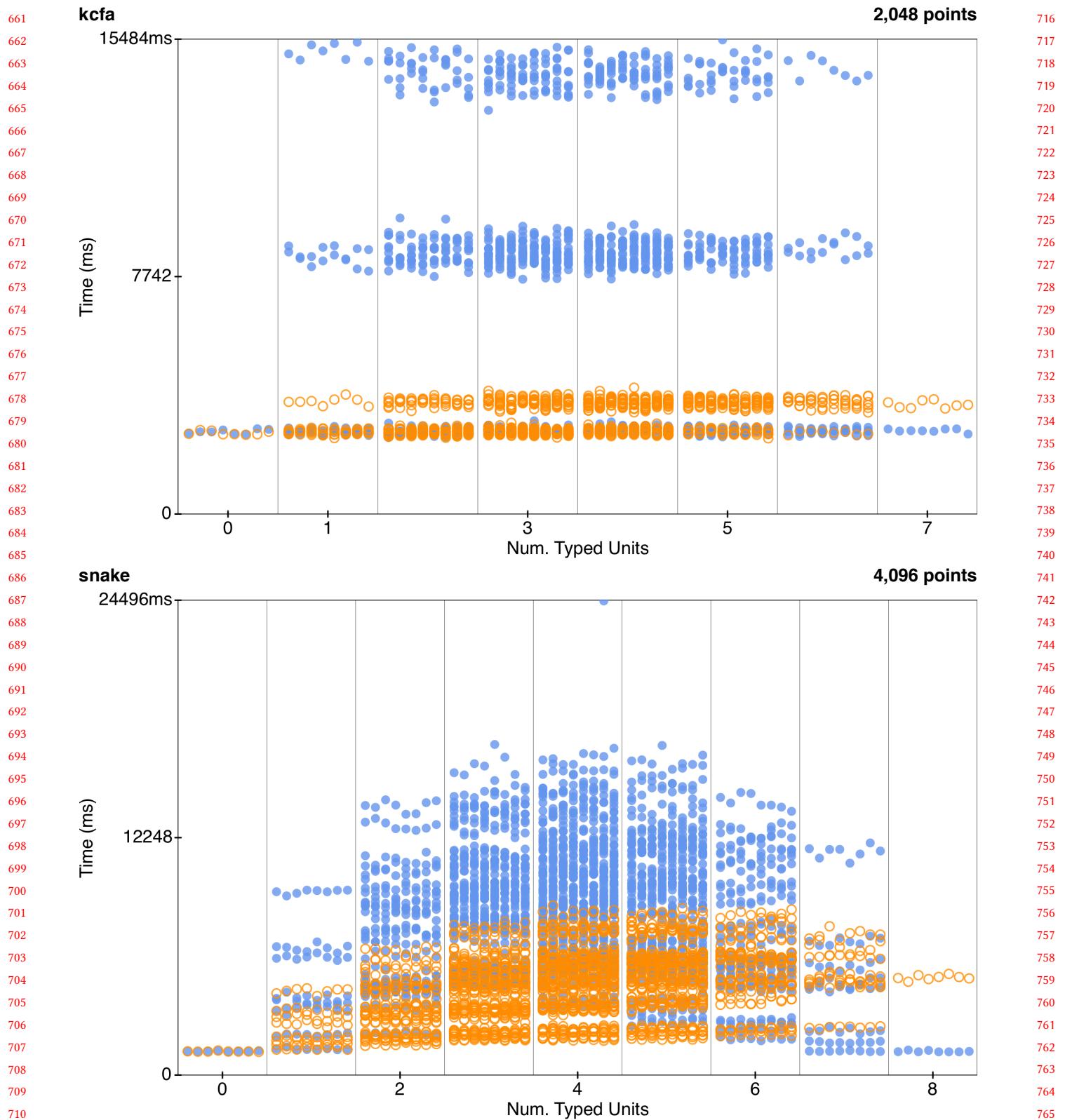
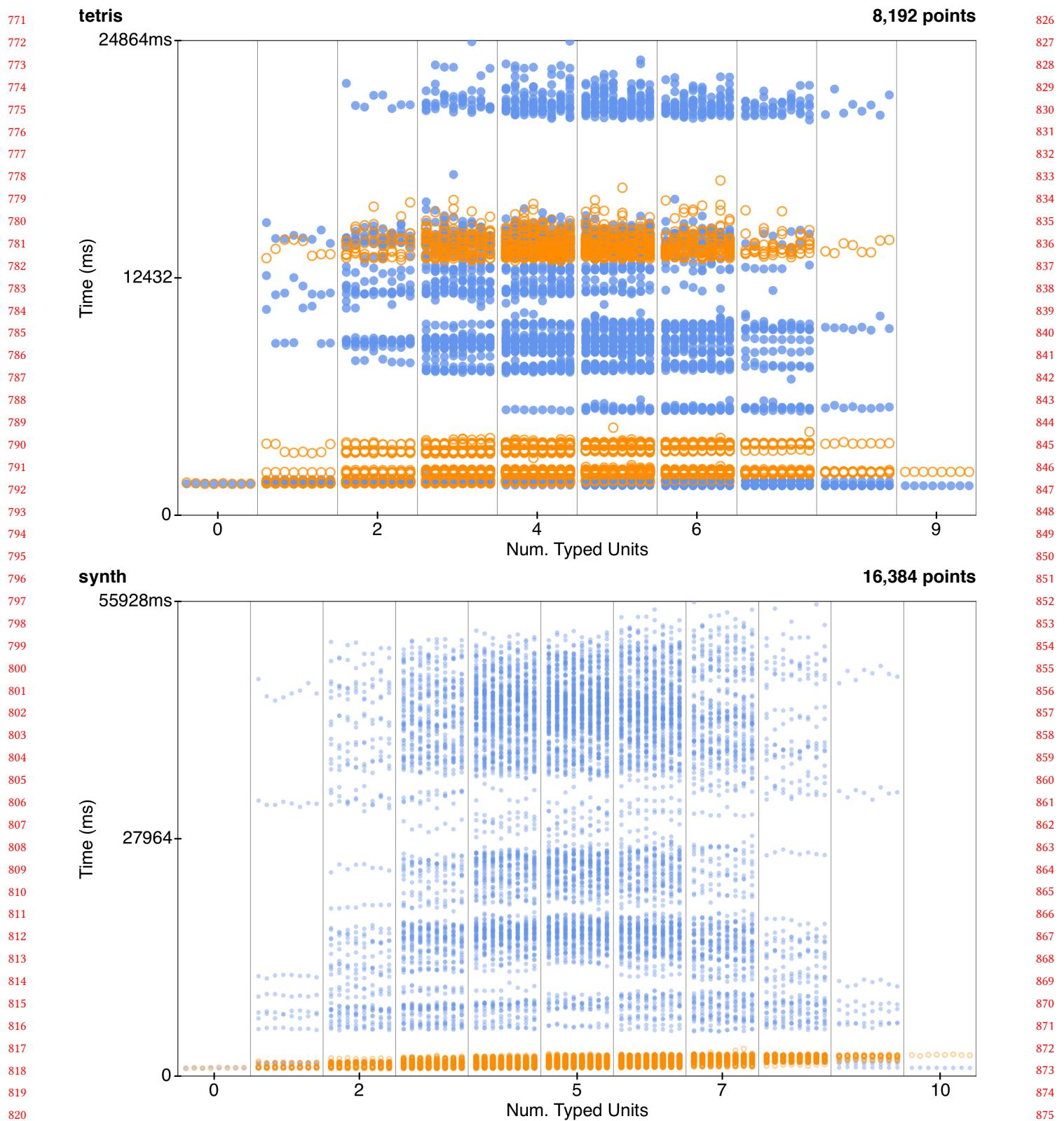


Fig. 5. Running time of TR-H configurations (blue and TR-1 configurations (orange)

Fig. 6. Running time of TR-H configurations (blue █) and TR-1 configurations (orange █), part 5/5

881	C Implementing Tagged Racket	936
882	The high-level architecture of TR-H is to:	937
883	1. type-check a module,	938
884	2. use the type environment to generate contracts,	939
885	3. optimize the contracts for the module,	940
886	4. output Racket bytecode.	941
887	For TR-1, we modified step 2 and replaced step 3.	942
888		943
889	C.1 Generating Type-Constructor Contracts	944
890	Typed Racket defines a function <code>type->contract</code> that (1)	945
891	expects a type, (2) compiles the type to a so-called <i>static contract</i> ,	946
892	(3) optimizes the representation of the static contract,	947
893	and (4) compiles the static contract to Racket code that will	948
894	generate an appropriate contract.	949
895	We modified the <code>type->contract</code> function to generate	950
896	type-constructor checks by adding a new method to the	951
897	internal API for static contracts. For example, the method	952
898	converts the contract for a list of elements into a contract	953
899	that checks the <code>list?</code> predicate.	954
900		955
901	C.2 Defending Typed Code	956
902	The TR-1 prototype replaces the Typed Racket optimizer	957
903	with a completion function that adds type-constructor checks	958
904	to typed code. The function implements a fold over the syn-	959
905	tax of a type-annotated program, and performs two kinds of	960
906	rewrites.	961
907	First, the completion function rewrites <i>most</i> applications	962
908	(<code>f x</code>) to (<code>check K (f x)</code>), where <code>K</code> is the static type of	963
909	the application. If <code>f</code> is an identifier, however, there are two	964
910	exceptional cases:	965
911		966
912	• <code>f</code> may be a built-in function that is certain to return a	967
913	value of the correct type constructor (e.g., <code>map</code> always	968
914	returns a list); and	969
915	• <code>f</code> may be statically typed, in which case soundness	970
916	guarantees that <code>f</code> returns a value that matches its static	971
917	type constructor. (there is one exception: accessor func-	972
918	tions for user-defined <code>structs</code> are unsafe like any other	973
919	accessor, e.g., <code>car</code>),	974
920	For these exceptional cases, the completion function does	975
921	not insert a type-constructor check.	976
922	Second, the completion function defends typed functions	977
923	from dynamically-typed arguments by translating a function	978
924	like (<code>λ (x) e</code>) to (<code>λ (x) (check x) e</code>). The structure of	979
925	the check is based on the domain type of the function.	980
926		981
927	C.3 Diff vs. Racket v6.10.1	982
928	The repository for this paper contains the TR-1 prototype	983
929	and a diff between the prototype and Typed Racket v6.10.1.	984
930	github.com/nuprl/tag-sound?path=src/locally-defensive.patch	985
931		986
932		987
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935		990

991	D Existing Systems	1046
992	This section illustrates prior work on gradual typing using	1047
993	the semantic framework of the paper. The goal is to demon-	1048
994	strate that the framework is able to express the <i>type bound-</i>	1049
995	<i>aries</i> and <i>boundary checks</i> of existing systems, and to outline	1050
996	a formal comparison.	1051
997		1052
998	This section does not attempt to summarize the novelties	1053
999	and subtleties of each system. The interested reader must	1054
1000	seek out the primary sources.	1055
1001	The subsections also give canonical forms lemmas for each	1056
1002	system as a taste of their logical implications.	1057
1003	URLs All URLs accessed on 2018-06-28.	1058
1004	• Gradualtalk : https://pleiad.cl/research/software/gradualtalk	1059
1005	• Typed Racket : https://github.com/racket/typed-racket	1060
1006	• TPD : https://github.com/jack-williams/tpd	1061
1007	• StrongScript : https://plg.uwaterloo.ca/~dynjs/strongscript/	1062
1008	• ActionScript : https://www.adobe.com/devnet/actionscript.html	1063
1009	• mypy : http://mypy-lang.org/	1064
1010	• Flow : https://flow.org/	1065
1011	• Hack : http://hacklang.org/	1066
1012	• Pyre : https://pyre-check.org/	1067
1013	• Pytype : https://opensource.google.com/projects/pytype	1068
1014	• rtc : https://github.com/plum-umd/rtc	1069
1015	• Strongtalk : http://strongtalk.org/	1070
1016	• TypeScript : https://www.typescriptlang.org/	1071
1017	• Typed Clojure : http://typedclojure.org/	1072
1018	• Typed Lua : https://github.com/andremm/typedlua	1073
1019	• Pyret : https://www.pyret.org/	1074
1020	• Thorn : http://janvitek.org/yearly.htm	1075
1021	• Dart 2 : https://www.dartlang.org/dart-2	1076
1022	• Nom : https://www.cs.cornell.edu/~ross/publications/nomalive/	1077
1023	• Reticulated : https://github.com/mvitousek/reticulated	1078
1024	• SafeTS : https://www.microsoft.com/en-us/research/publication/safe-efficient-gradual-typing-for-typescript-3/	1079
1025	• TR-1 : https://github.com/bennn/typed-racket/releases/tag/ld1.0	1080
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1101	$\boxed{\text{Thorn}}$ (sketch)	1156
1102	$\tau = C \mid \text{like } C \mid \text{dyn}$	1157
1103	$v = p \mid (\text{dyn})p \mid (\text{like } C)p$	1158
1104	$p = C(f = v, \dots)$	1159
1105	$\boxed{\mathcal{D} : \tau \times v \rightarrow e}$ (undefined, all values have a static type)	1160
1106		1161
1107	$\boxed{\mathcal{S} : \tau \times v \rightarrow e}$ (via explicit type-cast)	1162
1108	$\mathcal{S}(C, v) = p$	1163
1109	if $v = p$ or $v = (\text{dyn})p$ or $v = (\text{like } C')p$	1164
1110	and $p = C''(f = v, \dots)$	1165
1111	and $C'' \leqslant C$	1166
1112	$\mathcal{S}(\text{like } C, v) = (\text{like } C)p$	1167
1113	if $v = p$ or $v = (\text{dyn})p$ or $v = (\text{like } C')p$	1168
1114	$\mathcal{S}(\text{dyn}, v) = (\text{dyn})p$	1169
1115	if $v = p$ or $v = (\text{dyn})p$ or $v = (\text{like } C')p$	1170
1116	$\mathcal{S}(\tau, v) = \text{BndryErr}$	1171
1117	otherwise	1172
1118		1173

Fig. 7. Thorn types, values, and boundary functions. The \mathcal{D} function is undefined for all inputs.

1119		1174
1120		1175
1121		1176
1122	D.1 Thorn	1177
1123		1178
1124	Thorn (figure 7) is a nominally-typed object-oriented lan-	1179
1125	guage. The idea is that a program may: declare typed classes,	1180
1126	use the classes to create typed objects, and manipulate the	1181
1127	objects in gradually-typed methods. If a method expects a	1182
1128	dynamically-typed object, the type checker lets the method	1183
1129	perform any operation on the object and the run-time sys-	1184
1130	tem dynamically checks whether the operations are actually	1185
1131	valid.	1186

The types τ include *concrete* class names C , *like* class names ($\text{like } C$), and a dynamic type (dyn). The values are possibly-wrapped pointers to instances of classes; a value is either a direct pointer p , a dynamically-typed *view* to a pointer ($\text{dyn})p$, or a like-typed view to a pointer ($\text{like } C)p$. Informally, a view is a method-local pointer to an object.

One main invariant of Thorn is that every value comes with a type. In the figure, every value is an instance of a class and has the class name as its type. Because of this invariant, Thorn can efficiently check whether a value is compatible with some other type annotation at runtime. The \mathcal{S} function demonstrates this compatibility check.

The \mathcal{D} function is undefined for all inputs because there is no such thing as a dynamically-typed value. Put another way, the Thorn surface language is a single statically-typed language as opposed to a pair of languages. (The statically-typed language includes a dynamic to make it easy to experiment with statically-typed values, but nevertheless all values are statically typed to ensure safety and efficiency.)

Lemma 0.0 : Thorn canonical forms

- If $\vdash v : C$ then $v = C'(f = v_f, \dots)$ and $C' \leqslant C$.
- If $\vdash v : \text{like } C$ then $v = (\text{like } C')p$ and $C' \leqslant C$
- If $\vdash v : \text{dyn}$ then $v = (\text{dyn})p$

1211 **D.2 StrongScript**

1212 StrongScript (figure 8) adapts the ideas from Thorn to a type
 1213 system for JavaScript. The types τ include concrete class
 1214 names ($!C$), like class names (C), a dynamic type (any), and
 1215 function types ($\tau \Rightarrow \tau$). The values are objects and functions.
 1216

Every object and function comes with an intrinsic type.
 For an object imported from JavaScript, this type is any. (For
 a function imported from JavaScript, this type is presumably
 any \Rightarrow any.) A typed object cannot inherit from a JavaScript
 object, and vice-versa.

The S function checks the intrinsic type of a value against
 a type annotation. The idea is, if the check succeeds then
 a context may assume that the type annotation accurately
 describes the value.

The \mathcal{D} function is undefined for all inputs because the
 StrongScript paper does not directly model interactions with
 JavaScript. Instead, a JavaScript object is modeled as an object
 with type any, as mentioned above.

Lemma 0.1 : StrongScript canonical forms

- If $\vdash v : !C$ then $v = \{(s:v) \dots m \dots \| C'\}$ and $C' \leqslant C$
- If $\vdash v : C$ then either:
 - $v = \{(s:v) \dots m \dots \| C'\}$
 - $v = \{(s:v) \dots m \dots \| \text{any}\}$
- If $\vdash v : \tau_d \Rightarrow \tau_c$ then $v = \text{func}(x:\tau_d)\{\text{return } e:\tau_c\}$
- If $\vdash v : \text{any}$ then either:
 - $v = \{(s:v) \dots m \dots \| C'\}$
 - $v = \{(s:v) \dots m \dots \| \text{any}\}$
 - $v = \text{func}(x:\text{any})\{\text{return } e:\text{any}\}$

StrongScript (sketch)

$\tau = !C \mid C \mid \text{any} \mid \tau \Rightarrow \tau$	1266
$v = \{(s:v) \dots m \dots \ \tau\} \mid \text{func}(x:\tau)\{\text{return } e:\tau\}$	1267
$s \in \text{Strings}$	1268
$m = x(x:\tau)\{\text{return } e:\tau\}$	1269
$\mathcal{D} : \tau \times v \rightarrow e$	1270
	(undefined, all values have a static type)
$S : \tau \times v \rightarrow e$	1271
	(via explicit type-cast)
$S(\tau_d \Rightarrow \tau_c, v) = v'$	1272
if $v = \text{func}(x:\tau_x)\{\text{return } e:\tau_v\}$	1273
where $v' = \text{func}(x:\tau_d)\{\text{return } (\langle \tau_c \rangle(v((\langle \tau_x \rangle x))): \tau_c\}$	1274
$S(C, v) = v$	1275
if $v = \{(s:v) \dots m \dots \ C'\}$	1276
and $C' \leqslant C$	1277
$S(\text{any}, v) = v$	1278
if $v = \{(s:v) \dots m \dots \ \text{any}\}$	1279
or $v = \{(s:v) \dots m \dots \ C\}$	1280
$S(\text{any}, v) = v'$	1281
if $v = \text{func}(x:\tau_d)\{\text{return } e:\tau_c\}$	1282
where $v' = \text{func}(x:\text{any})\{\text{return } (\langle \text{any} \rangle(v((\langle \tau_d \rangle x))):\text{any}\}$	1283
$S(\tau, v) = \text{BndryErr}$	1284
otherwise	1285

Fig. 8. StrongScript boundary functions. The \mathcal{D} function is undefined for all inputs.

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1321 **D.3 Dart 2**

1322 Dart 2 is a new language with some support for dynamic
 1323 typing. For details, see: dartlang.org/dart-2

1324 Figure 9 summarizes the key aspects of dynamic typing in
 1325 Dart for a few types. The types represent integers (int), in-
 1326 tegers and decimal numbers (num), lists (List(τ)), functions
 1327 ($\tau \Rightarrow \tau$), and a dynamic type (dynamic). The base values b
 1328 match these types.

1329 Dart programs do not directly interact with base values.
 1330 Instead, base values are stored on a typed heap. The values v
 1331 in figure 9 model this indirection by associating a base value
 1332 with a compatible type.

1333 Just like in Thorn, a typed value may be used in a context
 1334 that expects a less precise type. Also like Thorn, a value of
 1335 type dynamic is an object that the type checker assumes can
 1336 receive any method call. The run-time system checks that
 1337 such method calls are actually safe for the given value.

1338 The S boundary function checks a value against a type
 1339 annotation by checking the value's associated type. The D
 1340 function is undefined for all inputs because it is not possible
 1341 to define (or interact with) an untyped value.

1342 **Lemma 0.2 : Dart canonical forms**

- 1343 • If $\vdash v : \tau$ then $v = b :: \tau'$ and $\tau' \leqslant \tau$

1344 *Remark:* a Dart function type must be declared explicitly.
 1345 For example, to use the type $\text{int} \Rightarrow \text{int}$ one must first define
 1346 an alias:

1347

```
typedef int IntFun(int _);
```

1348 Then the name `IntFun` may appear in other type annotations,
 1349 e.g., in a method signature.

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Dart (sketch)

1376 $\tau = \text{int} \mid \text{num} \mid \text{List}(\tau) \mid \tau \Rightarrow \tau \mid \text{dynamic}$

1377 $v = b :: \tau$

1378 $b = i \mid d \mid [v \dots] \mid (\tau x) \Rightarrow e$

1379 $\mathcal{D} : \tau \times v \rightarrow e$ (undefined, all values have a static type)

 $S : \tau \times v \rightarrow e$

1380 $S(\tau, b :: \tau') = b :: \tau'$

1381 if $\tau' \leqslant \tau$

1382 $S(\tau, v) = \text{BndryErr}$

1383 otherwise

Fig. 9. Dart boundary functions for a restricted grammar of types. The \mathcal{D} function is undefined for all inputs.

1431 **D.4 Pyret**

1432 Pyret is a dynamically-typed language with optional type
 1433 annotations and an optional static type checker. For details,
 1434 see: pyret.org.

1435 A type annotation in a Pyret program acts as a type-
 1436 constructor check at run-time. Figure 10 illustrates this as-
 1437 pect of Pyret in the \triangleright_S and \triangleright_D notions of reduction. Both
 1438 check the argument and result of a typed function against
 1439 the function's type. The X boundary function performs the
 1440 check by matching a type constructor K against a value.
 1441

1442 One aspect of Pyret that is missing from figure 10 is the
 1443 translation that maps type annotations in the source code to
 1444 run-time constructor checks. This translation could be mod-
 1445 eled with a completion function (\rightsquigarrow), similar to the model of
 1446 the first-order embedding in the paper.

1447 **Lemma 0.3 : Pyret assert-canonical forms**

1448 If v is a value with the static type τ then v may be any
 1449 kind of value; however, if v is assigned to a variable x with
 1450 the programmer-assigned type τ , then one of the following
 1451 holds:

- 1452 • If $\vdash x : \tau_0 \times \tau_1$ then $v = \langle v_0, v_1 \rangle$
- 1453 • If $\vdash x : \tau_d \Rightarrow \tau_c$ then either:
 - 1454 – $v = \lambda y. e$
 - 1455 – $v = \lambda((y : \tau'_d) : \tau'_c). e$
- 1456 • If $\vdash x : \text{Int}$ then $v \in \mathbb{Z}$
- 1457 • If $\vdash x : \text{Nat}$ then $v \in \mathbb{N}$

1486 **Pyret (sketch)**

1487 $\tau = \text{Int} \mid \text{Nat} \mid \tau \times \tau \mid \tau \Rightarrow \tau$
 1488 $K = \text{Int} \mid \text{Nat} \mid \text{Pair} \mid \text{Fun}$
 1489 $v = i \mid \langle v, v \rangle \mid \lambda x. e \mid \lambda((x : \tau) : \tau). e$
 1490 $i \in \mathbb{Z}$
 1491 $e = \dots \mid \text{chk } K \ e$
 1492

$$\boxed{\mathcal{D} : \tau \times v \longrightarrow e} \quad \mathcal{D}(\tau, v) = X(\lfloor \tau \rfloor, v)$$

$$\boxed{\mathcal{S} : \tau \times v \longrightarrow e} \quad \mathcal{S}(\tau, v) = X(\lfloor \tau \rfloor, v)$$

1493 **$X : K \times v \longrightarrow e$**

1494 $X(\text{Fun}, v) = v$
 1495 if $v = \lambda x. e$ or $v = \lambda((x : \tau'_d) : \tau'_c). e$
 $X(\text{Pair}, v) = v$
 1496 if $v = \langle v_0, v_1 \rangle$
 $X(\text{Int}, i) = i$
 $X(\text{Nat}, i) = i$
 1497 if $i \in \mathbb{N}$
 $X(K, v) = \text{BndryErr}$
 otherwise

1498 **$e \triangleright_S e$**

1499 $(\lambda x. e) v \quad \triangleright_S e[x \leftarrow v]$
 $(\lambda((x : \tau_d) : \tau_c). e) v \triangleright_S \text{BndryErr}$
 1500 if $X(\lfloor \tau_d \rfloor, v) = \text{BndryErr}$
 $(\lambda((x : \tau_d) : \tau_c). e) v \triangleright_S \text{chk } \lfloor \tau_c \rfloor (e[x \leftarrow X(\lfloor \tau_d \rfloor, v)])$
 1501 if $X(\lfloor \tau_d \rfloor, v) \neq \text{BndryErr}$
 $\text{chk } K \ v \quad \triangleright_S X(K, v)$

1502 **$e \triangleright_D e$**

1503 $(\lambda x. e) v \quad \triangleright_D e[x \leftarrow v]$
 $(\lambda((x : \tau_d) : \tau_c). e) v \triangleright_D \text{BndryErr}$
 1504 if $X(\lfloor \tau_d \rfloor, v) = \text{BndryErr}$
 $(\lambda((x : \tau_d) : \tau_c). e) v \triangleright_D \text{chk } \lfloor \tau_c \rfloor (e[x \leftarrow X(\lfloor \tau_d \rfloor, v)])$
 1505 if $X(\lfloor \tau_d \rfloor, v) \neq \text{BndryErr}$
 $\text{chk } K \ v \quad \triangleright_D X(K, v)$

Fig. 10. Pyret boundary functions and semantics for a restricted grammar of types.

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1541 **D.5 SafeTS**

1542 SafeTS is a core model of Safe TypeScript, which is a sound
 1543 type system for JavaScript (as opposed to TypeScript).

1544 Figure 11 demonstrates SafeTS on three types: a type
 1545 for numbers (number), a type for an object with two fields
 1546 ($\{\text{fst} : \tau, \text{snd} : \tau\}$), and a type for an object with one method
 1547 ($\{\text{call}(\tau) : \tau\}$). The latter types are intended to represent
 1548 tuples and anonymous functions.

1549 Every value in a SafeTS program has an intrinsic type;
 1550 there is no notion of a value that is defined in dynamically-
 1551 typed code and imported to statically typed code. A typed
 1552 value may, however, be used in a context that expects values
 1553 with a different type by means of a type cast. The differ-
 1554 ent type may contain new fields, but otherwise must be a
 1555 supertype of the value's intrinsic type.

1556 The \mathcal{S} boundary function illustrates the run-time checks
 1557 that SafeTS performs for our number, pair, and function types.
 1558 For type number, SafeTS checks that the value is a number.
 1559 For a pair type, SafeTS checks that the value is a pair of
 1560 compatible type and recursively transports the components.
 1561 For a function type, SafeTS checks that the value is a function
 1562 of compatible type.

1563 The \mathcal{D} function is undefined because the SafeTS model
 1564 does not define interactions with a model of JavaScript.

1565 **Lemma 0.4 : SafeTS canonical forms**

- 1566 • If $\vdash v : \{\text{fst} : \tau_0, \text{snd} : \tau_1\}$ then $v = \{\text{fst} : \tau'_0 v_0, \text{snd} : \tau'_1 v_1\}$ and
 $\tau'_0 \leqslant \tau_0$ and $\tau'_1 \leqslant \tau_1$
- 1567 • If $\vdash v : \{\text{call}(\tau_d) : \tau_c\}$ then $v = \{\text{call}(x : \tau'_d) : \tau'_c\{\text{return } e\}$
 1568 and $\{\text{call}(\tau'_d) : \tau'_c\} \leqslant \{\text{call}(\tau_d) : \tau_c\}$
- 1569 • If $\vdash v : \text{number}$ then $v = i$

1570 *Remark:* if a SafeTS cast adds new fields to a value, the
 1571 fields are recorded in an external “tag heap” of run-time type
 1572 information. Figure 11 does not model the tag heap because
 1573 it is not relevant to the types in the figure.

1596 **SafeTS (sketch)**

1597 $\tau = \text{number} \mid \{\text{fst} : \tau, \text{snd} : \tau\} \mid \{\text{call}(\tau) : \tau\} \mid \text{any}$
 1598 $v = i \mid \{\text{fst} : \tau v, \text{snd} : \tau v\} \mid \{\text{call}(x : \tau) : \tau\{\text{return } e\}\}$

1599 $\mathcal{D} : \tau \times v \rightarrow e$ (undefined, all values have a static type)

1600 **$\mathcal{S} : \tau \times v \rightarrow e$**

1601 $\mathcal{S}(\{\text{call}(\tau_d) : \tau_c\}, v) = v$
 1602 if $v = \{\text{call}(x : \tau'_d) : \tau'_c\{\text{return } e\}$
 1603 and $\{\text{call}(\tau'_d) : \tau'_c\} \leqslant \{\text{call}(\tau_d) : \tau_c\}$

1604 $\mathcal{S}(\{\text{fst} : \tau_0, \text{snd} : \tau_1\}, v) = \{\text{fst} : \tau'_0 \mathcal{S}(\tau_0, v_0), \text{snd} : \tau'_1 \mathcal{S}(\tau_1, v_1)\}$
 1605 if $v = \{\text{fst} : \tau'_0 v_0, \text{snd} : \tau'_1 v_1\}$
 1606 and $\tau'_0 \leqslant \tau_0$ and $\tau'_1 \leqslant \tau_1$

1607 $\mathcal{S}(\text{number}, i) = i$
 1608 $\mathcal{S}(\tau, v) = \text{BndryErr}$
 1609 otherwise

Fig. 11. SafeTS. The \mathcal{D} function is undefined for all inputs.

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1651 **D.6 Nom**

1652 Nom is a nominal object oriented language. Types include a
 1653 top type (\top), class names (C), and a dynamic type (**dynamic**).
 1654 Values are instances of classes. Each value has an intrinsic
 1655 type; namely, the name of its class.

1656 The \mathcal{S} function checks that the intrinsic type of a value
 1657 is compatible with a given type annotation. The \mathcal{D} function
 1658 is undefined for all inputs because there way to define or
 1659 import an untyped value.

1660 **Lemma 0.5 : Nom canonical forms**

- 1661 • If $\vdash v : C$ then $v = C'(v', \dots)$ and $C' \leqslant C$

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1706 **Nom** (sketch)1707 $\tau = \top \mid C \mid \mathbf{dynamic}$ 1708 $v = C(v, \dots)$ 1709 $\mathcal{D} : \tau \times v \rightarrow e$ (undefined, all values have a static type)1710 $\mathcal{S} : \tau \times v \rightarrow e$ 1711 $\mathcal{S}(\top, v) = v$ 1712 $\mathcal{S}(C, C'(v, \dots)) = C'(v, \dots)$
if $C' \leqslant C$ 1713 $\mathcal{S}(\mathbf{dynamic}, v) = v$ 1714 $\mathcal{S}(\tau, v) = \text{BndryErr}$
otherwise

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1761	E Models	1816
1762	This section contains full definitions of the languages and	1817
1763	full proofs of our claims about each language.	1818
1764	Aside from the common notions in section 5.1, the defi-	1819
1765	nition and proofs of each model are independent and self-	1820
1766	contained.	1821
1767		1822
1768	E.1 Preliminaries	1823
1769	Definition 1.0 : \rightarrow^* divergence	1824
1770	Given a reduction relation \rightarrow^* , an expression e diverges if for	1825
1771	all e' such that $e \rightarrow^* e'$ there exists an e'' such that $e' \rightarrow e''$.	1826
1772		1827
1773	Convention 1.1 : variable convention	1828
1774	All λ -bound variables in an expression are distinct from one	1829
1775	another, and from any free variables in the expression.	1830
1776		1831
1777	Assumption 1.2 : \vdash permutation	1832
1778	For all typing judgments and properties \vdash :	1833
1779	• If $x, x', \Gamma \vdash e$ then $x', x, \Gamma \vdash e$	1834
1780	• If $(x:\tau), (x':\tau'), \Gamma \vdash e$ then $(x':\tau'), (x:\tau), \Gamma \vdash e$	1835
1781	Definition 1.3 : \vdash boundary-free	1836
1782	An expression e is <i>boundary free</i> if e does not contain a	1837
1783	subterm of the form $(\text{dyn } \tau' e')$, nor a subterm of the form	1838
1784	$(\text{stat } \tau' e')$.	1839
1785		1840
1786	Notes:	1841
1787	• The upcoming models use a common surface syn-	1842
1788	tax and typing system, but to keep each model self-	1843
1789	contained we reprint this system in each definition.	1844
1790	• The proofs are written in a structured style, typically	1845
1791	as a list of basic steps where each step is justified by an	1846
1792	assumption, a lemma, or a previous step. Lemma names	1847
1793	are <i>italicized</i> and hyperlinked to the actual lemma.	1848
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1871 **E.2 (H) Higher-Order Embedding**

1872 **E.2.1 Higher-Order Definitions**

1873 **Language H**

1875 $e = x \mid v \mid \langle e, e \rangle \mid e \cdot e \mid op^1 e \mid op^2 e \cdot e \mid$
 dyn τ $e \mid$ stat τ $e \mid$ Err
 1876 $v = i \mid \langle v, v \rangle \mid \lambda x. e \mid \lambda(x: \tau). e \mid$
 mon ($\tau \Rightarrow \tau$) v
 1877 $\tau = \text{Nat} \mid \text{Int} \mid \tau \times \tau \mid \tau \Rightarrow \tau$
 1878 $\Gamma = \cdot \mid x, \Gamma \mid (x: \tau), \Gamma$
 1879 Err = BndryErr | TagErr
 1880 $r = v \mid \text{Err}$
 1881 $E^\bullet = [] \mid E^\bullet e \mid v E^\bullet \mid \langle E^\bullet, e \rangle \mid \langle v, E^\bullet \rangle \mid$
 1882 $op^1 E^\bullet \mid op^2 E^\bullet e \mid op^2 v E^\bullet$
 1883 $E = E^\bullet \mid E e \mid v E \mid \langle E, e \rangle \mid \langle v, E \rangle \mid op^1 E \mid$
 1884 $op^2 E e \mid op^2 v E \mid \text{dyn } \tau E \mid \text{stat } \tau E$

1885 $\Delta : op^1 \times \tau \rightarrow \tau$

1886 $\Delta(\text{fst}, \tau_0 \times \tau_1) = \tau_0$

1887 $\Delta(\text{snd}, \tau_0 \times \tau_1) = \tau_1$

1888 $\Delta : op^2 \times \tau \times \tau \rightarrow \tau$

1889 $\Delta(op^2, \text{Nat}, \text{Nat}) = \text{Nat}$

1890 $\Delta(op^2, \text{Int}, \text{Int}) = \text{Int}$

1891 $\tau \leqslant: \tau$

1892 $\frac{\text{Nat} \leqslant: \text{Int}}{\tau \leqslant: \tau} \quad \frac{\tau'_d \leqslant: \tau_d \quad \tau_c \leqslant: \tau'_c \quad \tau_0 \leqslant: \tau'_0 \quad \tau_1 \leqslant: \tau'_1}{\tau_d \Rightarrow \tau_c \leqslant: \tau'_d \Rightarrow \tau'_c \quad \tau_0 \times \tau_1 \leqslant: \tau'_0 \times \tau'_1}$

1893 $\frac{}{\tau \leqslant: \tau} \quad \frac{\tau \leqslant: \tau' \quad \tau' \leqslant: \tau''}{\tau \leqslant: \tau''}$

1894 $\boxed{\Gamma \vdash e}$

1895 $\frac{x \in \Gamma}{\Gamma \vdash x} \quad \frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x. e} \quad \frac{}{\Gamma \vdash i} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}$

1896 $\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0 e_1} \quad \frac{\Gamma \vdash e}{\Gamma \vdash op^1 e} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2 e_0 e_1} \quad \frac{}{\Gamma \vdash \text{Err}}$

1897 $\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{stat } \tau e}$

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1926 $\boxed{\Gamma \vdash e : \tau}$

1927 $\frac{(x: \tau) \in \Gamma}{\Gamma \vdash x : \tau} \quad \frac{(x: \tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x: \tau_d). e : \tau_d \Rightarrow \tau_c} \quad \frac{i \in \mathbb{N}}{\Gamma \vdash i : \text{Nat}}$

1928 $\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1} \quad \frac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c}{\Gamma \vdash e_0 e_1 : \tau_c}$

1929 $\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e : \tau'}{\Gamma \vdash op^1 e_0 e_1 : \tau} \quad \frac{\Delta(op^1, \tau_0) = \tau}{\Gamma \vdash op^1 e_0 : \tau} \quad \frac{\Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash op^2 e_0 e_1 : \tau} \quad \frac{\tau' \leqslant: \tau}{\Gamma \vdash e : \tau} \quad \frac{}{\Gamma \vdash \text{Err} : \tau}$

1930 $\frac{\Gamma \vdash e}{\Gamma \vdash \text{dyn } \tau e : \tau}$

1931 $\boxed{\Gamma \vdash_H e}$

1932 $\frac{x \in \Gamma}{\Gamma \vdash_H x} \quad \frac{x, \Gamma \vdash_H e}{\Gamma \vdash_H \lambda x. e} \quad \frac{}{\Gamma \vdash_H i} \quad \frac{\Gamma \vdash_H e_0 \quad \Gamma \vdash_H e_1}{\Gamma \vdash_H \langle e_0, e_1 \rangle}$

1933 $\frac{\Gamma \vdash_H e_0 \quad \Gamma \vdash_H e_1}{\Gamma \vdash_H e_0 e_1} \quad \frac{\Gamma \vdash_H e}{\Gamma \vdash_H op^1 e} \quad \frac{\Gamma \vdash_H e_0 \quad \Gamma \vdash_H e_1}{\Gamma \vdash_H op^2 e_0 e_1} \quad \frac{}{\Gamma \vdash_H \text{Err}}$

1934 $\frac{\Gamma \vdash_H e : \tau}{\Gamma \vdash_H \text{stat } \tau e} \quad \frac{\Gamma \vdash_H v : \tau_d \Rightarrow \tau_c}{\Gamma \vdash_H \text{mon } (\tau_d \Rightarrow \tau_c) v}$

1935 $\boxed{\Gamma \vdash_H e : \tau}$

1936 $\frac{(x: \tau) \in \Gamma}{\Gamma \vdash_H x : \tau} \quad \frac{(x: \tau_d), \Gamma \vdash_H e : \tau_c}{\Gamma \vdash_H \lambda(x: \tau_d). e : \tau_d \Rightarrow \tau_c} \quad \frac{i \in \mathbb{N}}{\Gamma \vdash_H i : \text{Nat}}$

1937 $\frac{\Gamma \vdash_H e_0 : \tau_0 \quad \Gamma \vdash_H e_1 : \tau_1}{\Gamma \vdash_H \langle e_0, e_1 \rangle : \tau_0 \times \tau_1} \quad \frac{\Gamma \vdash_H e_0 : \tau_d \Rightarrow \tau_c}{\Gamma \vdash_H e_0 e_1 : \tau_c}$

1938 $\frac{\Gamma \vdash_H e_0 : \tau_0 \quad \Gamma \vdash_H e_1 : \tau_1 \quad \Gamma \vdash_H e : \tau'}{\Gamma \vdash_H op^1 e_0 e_1 : \tau} \quad \frac{\Delta(op^1, \tau_0) = \tau}{\Gamma \vdash_H op^1 e_0 : \tau} \quad \frac{\Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash_H op^2 e_0 e_1 : \tau} \quad \frac{\tau' \leqslant: \tau}{\Gamma \vdash_H e : \tau}$

1939 $\frac{\Gamma \vdash_H e}{\Gamma \vdash_H \text{Err} : \tau} \quad \frac{\Gamma \vdash_H e}{\Gamma \vdash_H \text{dyn } \tau e : \tau}$

1940 $\frac{\Gamma \vdash_H v}{\Gamma \vdash_H \text{mon } (\tau_d \Rightarrow \tau_c) v : (\tau_d \Rightarrow \tau_c)}$

1941 $\boxed{\delta(op^1, v) = e}$

1942 $\delta(\text{fst}, \langle v_0, v_1 \rangle) = v_0$
 $\delta(\text{snd}, \langle v_0, v_1 \rangle) = v_1$

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1981	$\delta(op^2, v, v) = e$	2036
1982	$\delta(\text{sum}, i_0, i_1) = i_0 + i_1$	2037
1983	$\delta(\text{quotient}, i_0, 0) = \text{BndryErr}$	2038
1984	$\delta(\text{quotient}, i_0, i_1) = \lfloor i_0/i_1 \rfloor$	2039
1985	if $i_1 \neq 0$	2040
1986	$\mathcal{D}_H : \tau \times v \rightarrow e$	2041
1987	$\mathcal{D}_H(\tau_d \Rightarrow \tau_c, v) = \text{mon}(\tau_d \Rightarrow \tau_c) v$	2042
1988	if $v = \lambda x. e$ or $v = \text{mon } \tau' v'$	2043
1989	$\mathcal{D}_H(\tau_0 \times \tau_1, \langle v_0, v_1 \rangle) = \langle \text{dyn } \tau_0 v_0, \text{dyn } \tau_1 v_1 \rangle$	2044
1990	$\mathcal{D}_H(\text{Int}, i) = i$	2045
1991	$\mathcal{D}_H(\text{Nat}, i) = i$	2046
1992	if $i \in \mathbb{N}$	2047
1993	$\mathcal{D}_H(\tau, v) = \text{BndryErr}$	2048
1994	otherwise	2049
1995		2050
1996	$\mathcal{S}_H : \tau \times v \rightarrow e$	2051
1997	$\mathcal{S}_H(\tau_d \Rightarrow \tau_c, v) = \text{mon}(\tau_d \Rightarrow \tau_c) v$	2052
1998	$\mathcal{S}_H(\tau_0 \times \tau_1, \langle v_0, v_1 \rangle) = \langle \text{stat } \tau_0 v_0, \text{stat } \tau_1 v_1 \rangle$	2053
1999	$\mathcal{S}_H(\text{Int}, v) = v$	2054
2000	$\mathcal{S}_H(\text{Nat}, v) = v$	2055
2001	$e \triangleright_{H-S} e$	2056
2002	$\text{dyn } \tau v \triangleright_{H-S} \mathcal{D}_H(\tau, v)$	2057
2003	$(\text{mon}(\tau_d \Rightarrow \tau_c) v_f) v \triangleright_{H-S} \text{dyn } \tau_c (v_f (\text{stat } \tau_d v))$	2058
2004	$(\lambda(x:\tau). e) v \triangleright_{H-S} e[x \leftarrow v]$	2059
2005	$op^1 v \triangleright_{H-S} \delta(op^1, v)$	2060
2006	$op^2 v_0 v_1 \triangleright_{H-S} \delta(op^2, v_0, v_1)$	2061
2007		2062
2008	$e \triangleright_{H-D} e$	2063
2009	$\text{stat } \tau v \triangleright_{H-D} \mathcal{S}_H(\tau, v)$	2064
2010	$v_0 v_1 \triangleright_{H-D} \text{TagErr}$	2065
2011	if $v_0 \in \mathbb{Z}$ or $v_0 = \langle v, v' \rangle$	2066
2012	$(\text{mon } \tau_d \Rightarrow \tau_c v_f) v \triangleright_{H-D} \text{stat } \tau_c (v_f (\text{dyn } \tau_d v))$	2067
2013	$(\lambda x. e) v \triangleright_{H-D} e[x \leftarrow v]$	2068
2014	$op^1 v \triangleright_{H-D} \text{TagErr}$	2069
2015	if $\delta(op^1, v)$ is undefined	2070
2016	$op^1 v \triangleright_{H-D} \delta(op^1, v)$	2071
2017	$op^2 v_0 v_1 \triangleright_{H-D} \text{TagErr}$	2072
2018	if $\delta(op^2, v_0, v_1)$ is undefined	2073
2019	$op^2 v_0 v_1 \triangleright_{H-D} \delta(op^2, v_0, v_1)$	2074
2020		2075
2021	$E^*[e] \rightarrow_{H-S} E^*[e']$	2076
2022	if $e \triangleright_{H-S} e'$	2077
2023	$E[\text{stat } \tau E^*[e]] \rightarrow_{H-S} E[\text{stat } \tau E^*[e']]$	2078
2024	if $e \triangleright_{H-S} e'$	2079
2025	$E[\text{dyn } \tau E^*[e]] \rightarrow_{H-S} E[\text{dyn } \tau E^*[e']]$	2080
2026	if $e \triangleright_{H-D} e'$	2081
2027	$E[\text{Err}] \rightarrow_{H-S} \text{Err}$	2082
2028		2083
2029		2084
2030		2085
2031		2086
2032		2087
2033		2088
2034		2089
2035		2090

2091 **E.2.2 Higher-Order Theorems**

2092 **Theorem 2.0 : static H-soundness**

2093 If $\vdash e : \tau$ then $\vdash_H e : \tau$ and one of the following holds:

- 2095 • $e \rightarrow_{H-S}^* v$ and $\vdash_H v : \tau$
- 2096 • $e \rightarrow_{H-S}^* E[\text{dyn } \tau' E'[e']]$ and $e' \triangleright_{H-D} \text{TagErr}$
- 2097 • $e \rightarrow_{H-S}^* \text{BndryErr}$
- 2098 • e diverges

2099 *Proof:*

- 2100 1. $\vdash_H e : \tau$
by *static subset*
- 2101 2. QED by *H static progress* and *H static preservation*.

2102 □

2103 **Theorem 2.1 : dynamic H-soundness**

2104 If $\vdash e$ then $\vdash_H e$ and one of the following holds:

- 2105 • $e \rightarrow_{H-D}^* v$ and $\vdash_H v$
- 2106 • $e \rightarrow_{H-D}^* E[e']$ and $e' \triangleright_{H-D} \text{TagErr}$
- 2107 • $e \rightarrow_{H-D}^* \text{BndryErr}$
- 2108 • e diverges

2109 *Proof:*

- 2110 1. $\vdash_H e$
by *dynamic subset*
- 2111 2. QED by *H dynamic progress* and *H dynamic preservation*.

2112 □

2113 **Corollary 2.2 : H static soundness**

2114 If $\vdash e : \tau$ and e is boundary-free, then one of the following holds:

- 2115 • $e \rightarrow_H^* v$ and $\vdash_H v : \tau$
- 2116 • $e \rightarrow_H^* \text{BndryErr}$
- 2117 • e diverges

2118 *Proof:*

2119 Consequence of the proof for *static H-soundness*

2120 □

2121 **Corollary 2.3 : H compilation**

2122 If $\vdash e : \tau$

2123 and \mathcal{D}'_H extends \mathcal{D}_H with a rule to monitor a typed function:

$$2124 \quad \mathcal{D}'_H(\tau_d \Rightarrow \tau_c, \lambda(x:\tau). e) = \text{mon}(\tau_d \Rightarrow \tau_c)(\lambda(x:\tau). e)$$

2125 and \triangleright_{H-D}' extends \triangleright_{H-D} with a rule to apply a typed function:

$$2126 \quad (\lambda(x:\tau). e) v \triangleright_{H-D'} e[x \leftarrow v]$$

2127 and $e \rightarrow_{H-D}' e$ is defined as:

$$2128 \quad \begin{aligned} E[e] &\rightarrow_{H-D'} E[e'] \\ 2129 \quad \text{if } e &\triangleright_{H-D'} e' \end{aligned}$$

$$2130 \quad E[\text{stat } \tau v] \rightarrow_{H-D'} E[\mathcal{D}'_H(\tau, v)]$$

$$2131 \quad E[\text{dyn } \tau v] \rightarrow_{H-D'} E[\mathcal{D}'_H(\tau, v)]$$

$$2132 \quad E[\text{Err}] \rightarrow_{H-D'} \text{Err}$$

2133 and $\rightarrow_{H-D}'^*$ is the reflexive transitive closure of \rightarrow_{H-D}'

2134 then one of the following holds:

- 2135 • $e \rightarrow_{H-D}'^* v$ and $\vdash_H v : \tau$
- 2136 • $e \rightarrow_{H-D}'^* \text{TagErr}$
- 2137 • $e \rightarrow_{H-D}'^* \text{BndryErr}$
- 2138 • e diverges

2139 *Proof:*

2140 By *static H-soundness* and the fact that S_H and \triangleright_{H-S} are
2141 subsets of \mathcal{D}'_H and \triangleright_{H-D}' , respectively.

2142 □

2143 2144 2145 2146 2147 2148 2149 2150 2151 2152 2153 2154 2155 2156 2157 2158 2159 2160 2161 2162 2163 2164 2165 2166 2167 2168 2169 2170 2171 2172 2173 2174 2175 2176 2177 2178 2179 2180 2181 2182 2183 2184 2185 2186 2187 2188 2189 2190 2191 2192 2193 2194 2195 2196 2197 2198 2199 2200

2201 **E.2.3 Higher-Order Lemmas**2202 **Lemma 2.4 : \mathcal{D}_H soundness**2203 If $\vdash_H v$ then $\vdash_H \mathcal{D}_H(\tau, v) : \tau$ 2204 *Proof:*2205 **CASE** $\mathcal{D}_H(\tau_d \Rightarrow \tau_c, v) = \text{mon}(\tau_d \Rightarrow \tau_c) v :$ 2206 1. $\vdash_H \text{mon}(\tau_d \Rightarrow \tau_c) v : \tau_d \Rightarrow \tau_c$ 2207 by $\vdash_H v$

2208 2. QED

2209 **CASE** $v = \langle v_0, v_1 \rangle$ 2210 $\wedge \mathcal{D}_H(\tau_0 \times \tau_1, v) = \langle \text{dyn } \tau_0 v_0, \text{dyn } \tau_1 v_1 \rangle :$ 2211 1. $\vdash_H v_0$ 2212 $\wedge \vdash_H v_1$ 2213 by *H inversion*2214 2. $\vdash_H \text{dyn } \tau_0 v_0 : \tau_0$ 2215 $\wedge \vdash_H \text{dyn } \tau_1 v_1 : \tau_1$

2216 by (1)

2217 3. QED (2)

2218 **CASE** $v = i$ 2219 $\wedge \mathcal{D}_H(\text{Int}, v) = v :$

2220 1. QED

2221 **CASE** $v \in \mathbb{N}$ 2222 $\wedge \mathcal{D}_H(\text{Nat}, v) = v :$

2223 1. QED

2224 **CASE** $\mathcal{D}_H(\tau, v) = \text{BndryErr} :$

2225 1. QED

2226 \square 2227 **Lemma 2.5 : \mathcal{S}_H soundness**2228 If $\vdash_H v : \tau$ then $\vdash_H \mathcal{S}_H(\tau, v)$ 2229 *Proof:*2230 **CASE** $\vdash_H v : \tau_d \Rightarrow \tau_c$ 2231 $\wedge \mathcal{S}_H(\tau_d \Rightarrow \tau_c, v) = \text{mon}(\tau_d \Rightarrow \tau_c) v :$

2232 1. QED

2233 **CASE** $\vdash_H v : \tau_0 \times \tau_1$ 2234 $\wedge \mathcal{S}_H(\tau_0 \times \tau_1, v) = \langle \text{stat } \tau_0 v_0, \text{stat } \tau_1 v_1 \rangle :$ 2235 1. $v = \langle v_0, v_1 \rangle$ 2236 by *canonical forms*2237 2. $\vdash_H v_0 : \tau_0$ 2238 $\wedge \vdash_H v_1 : \tau_1$ 2239 by *H inversion* (1)2240 3. $\vdash_H \text{stat } \tau_0 v_0 : \tau_0$

2241 by the induction hypothesis (2)

2242 4. $\vdash_H \text{stat } \tau_1 v_1 : \tau_1$

2243 by the induction hypothesis (2)

2244 5. QED

2245 **CASE** $\vdash_H v : \text{Int}$ 2246 $\wedge \mathcal{S}_H(\text{Int}, v) = v :$

2247 1. QED

2248 **CASE** $\vdash_H v : \text{Nat}$ 2249 $\wedge \mathcal{S}_H(\text{Nat}, v) = v :$

2250 1. QED

2251 \square 2252 **Lemma 2.6 : H static subset**2253 If $\Gamma \vdash e : \tau$ then $\Gamma \vdash_H e : \tau$.

2254

2255 *Proof:*2256 By structural induction on the derivation of $\Gamma \vdash e : \tau$.

2257

2258 **CASE**
$$\frac{(x:\tau) \in \Gamma}{\Gamma \vdash x : \tau} :$$
2259 1. $\Gamma \vdash_H x : \tau$ 2260 by $(x:\tau) \in \Gamma$

2261 2. QED

2262 **CASE**
$$\frac{(x:\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x:\tau_d). e : \tau_d \Rightarrow \tau_c} :$$
2263 1. $(x:\tau_d), \Gamma \vdash_H e : \tau_c$

2264 by the induction hypothesis

2265 2. $\Gamma \vdash_H \lambda(x:\tau_d). e : \tau_d \Rightarrow \tau_c$

2266 3. QED

2267 **CASE**
$$\frac{i \in \mathbb{N}}{\Gamma \vdash i : \text{Nat}} :$$

2268 1. QED

2269 **CASE**
$$\frac{}{\Gamma \vdash i : \text{Int}} :$$

2270 1. QED

2271 **CASE**
$$\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1} :$$
2272 1. $\Gamma \vdash_H e_0 : \tau_0$ 2273 $\wedge \Gamma \vdash_H e_1 : \tau_1$

2274 by the induction hypothesis

2275 2. $\Gamma \vdash_H \langle e_0, e_1 \rangle : \tau_0 \times \tau_1$

2276 3. QED

2277 **CASE**
$$\frac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash e_1 : \tau_d}{\Gamma \vdash e_0 e_1 : \tau_c} :$$
2278 1. $\Gamma \vdash_H e_0 : \tau_d \Rightarrow \tau_c$ 2279 $\wedge \Gamma \vdash_H e_1 : \tau_d$

2280 by the induction hypothesis

2281 2. $\Gamma \vdash_H e_0 e_1 : \tau_c$

2282 3. QED

2283 **CASE**
$$\frac{\Gamma \vdash e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash op^1 e_0 : \tau} :$$
2284 1. $\Gamma \vdash_H e_0 : \tau_0$

2285 by the induction hypothesis

2286 2. $\Gamma \vdash_H op^1 e_0 : \tau$

2287 3. QED

2288 **CASE**
$$\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash op^2 e_0 e_1 : \tau} :$$
2289 1. $\Gamma \vdash_H e_0 : \tau_0$

2290 by the induction hypothesis

2291 2. $\Gamma \vdash_H op^2 e_0 e_1 : \tau$

2292 3. QED

2293 **CASE**
$$\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^3, \tau_0, \tau_1) = \tau}{\Gamma \vdash op^3 e_0 e_1 : \tau} :$$
2294 1. $\Gamma \vdash_H e_0 : \tau_0$

2295 by the induction hypothesis

2296 2. $\Gamma \vdash_H op^3 e_0 e_1 : \tau$

2297 3. QED

2298

2299 \square

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2311	CASE	$\frac{\Gamma \vdash e : \tau' \quad \tau' <: \tau}{\Gamma \vdash e : \tau}$:
2312			
2313	1.	$\Gamma \vdash_H e : \tau'$	
2314		by the induction hypothesis	
2315	2.	$\Gamma \vdash_H e : \tau$	
2316	3.	QED	
2317	CASE	$\frac{}{\Gamma \vdash \text{Err} : \tau}$:
2318			
2319	1.	QED	
2320	CASE	$\frac{\Gamma \vdash e}{\Gamma \vdash \text{dyn } \tau \ e : \tau}$:
2321			
2322	1.	$\Gamma \vdash_H e$	
2323		by <i>dynamic subset</i>	
2324	2.	$\Gamma \vdash_H \text{dyn } \tau \ e : \tau$	
2325		by (1)	
2326	3.	QED	
2327			□
2328			

Lemma 2.7 : H dynamic subsetIf $\Gamma \vdash e$ then $\Gamma \vdash_H e$.*Proof:*By structural induction on the derivation of $\Gamma \vdash e$.

2333	CASE	$\frac{x \in \Gamma}{\Gamma \vdash x}$:
2334			
2335	1.	$\Gamma \vdash_H x$	
2336		by $x \in \Gamma$	
2337	2.	QED	
2338	CASE	$\frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x. e}$:
2339			
2340	1.	$x, \Gamma \vdash_H e$	
2341		by the induction hypothesis	
2342	2.	$\Gamma \vdash_H \lambda x. e$	
2343		by (1)	
2344	3.	QED	
2345	CASE	$\frac{}{\Gamma \vdash i}$:
2346			
2347	1.	QED	
2348	CASE	$\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}$:
2349			
2350	1.	$\Gamma \vdash_H e_0$	
2351		$\wedge \Gamma \vdash_H e_1$	
2352		by the induction hypothesis	
2353	2.	$\Gamma \vdash_H \langle e_0, e_1 \rangle$	
2354		by (1)	
2355	3.	QED	
2356	CASE	$\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0 \ e_1}$:
2357			
2358	1.	$\Gamma \vdash_H e_0$	
2359		$\wedge \Gamma \vdash_H e_1$	
2360		by the induction hypothesis	
2361	2.	$\Gamma \vdash_H e_0 \ e_1$	
2362		by (1)	
2363	3.	QED	
2364	CASE	$\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0 \ e_1}$:
2365			

2366	1.	$\Gamma \vdash_H e_0$	
2367		$\wedge \Gamma \vdash_H e_1$	
2368		by the induction hypothesis	
2369	2.	$\Gamma \vdash_H e_0 \ e_1$	
2370		by (1)	
2371	3.	QED	
2372	CASE	$\frac{\Gamma \vdash e}{\Gamma \vdash op^1 e}$:
2373			
2374	1.	$\Gamma \vdash_H e$	
2375		by the induction hypothesis	
2376	2.	$\Gamma \vdash_H op^1 e$	
2377		by (1)	
2378	3.	QED	
2379	CASE	$\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2 e_0 \ e_1}$:
2380			
2381	1.	$\Gamma \vdash_H e_0$	
2382		$\wedge \Gamma \vdash_H e_1$	
2383		by the induction hypothesis	
2384	2.	$\Gamma \vdash_H op^2 e_0 \ e_1$	
2385		by (1)	
2386	3.	QED	
2388	CASE	$\frac{}{\Gamma \vdash \text{Err}}$:
2389			
2390	1.	QED	
2392	CASE	$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{stat } \tau \ e}$:
2393			
2394	1.	$\Gamma \vdash_H e : \tau$	
2395		by <i>static subset</i>	
2396	2.	$\Gamma \vdash_H \text{stat } \tau \ e$	
2397		by (1)	
2398	3.	QED	
2399			□
2400			
2401	Lemma 2.8 : H static progress		
2402	If $\vdash_H e : \tau$ then one of the following holds:		
2403	• e is a value		
2404	• $e \in \text{Err}$		
2405	• $e \rightarrow_{H-S} e'$		
2406	• $e \rightarrow_{H-S} \text{BndryErr}$		
2407	• $e = E[\text{dyn } \tau' \ E'[e']]$ and $e' \triangleright_{H-D} \text{TagErr}$		
2408	<i>Proof:</i>		
2409	By the <i>boundary factoring</i> lemma, there are seven possible cases.		
2410	CASE e is a value :		
2411	1.	QED	
2412	CASE $e = E^\bullet[v_0 \ v_1]$:		
2413	1.	$\vdash_H v_0 \ v_1 : \tau'$	
2414		by <i>static hole typing</i>	
2415	2.	$\vdash_H v_0 : \tau_d \Rightarrow \tau_c$	
2416		$\wedge \vdash_H v_1 : \tau_d$	
2417		by <i>H inversion</i>	
2418			
2419			
2420			

2421	3. $v_0 = \lambda(x:\tau'_d). e'$	2476
2422	$\vee v_0 = \text{mon}(\tau'_d \Rightarrow \tau'_c) v_f$	2477
2423	by <i>canonical forms</i>	2478
2424	4. IF $v_0 = \lambda(x:\tau'_d). e' :$	2479
2425	a. $e \rightarrow_{H-S} E^\bullet[e'[x \leftarrow v_1]]$	2480
2426	by $v_0 v_1 \triangleright_{H-S} e'[x \leftarrow v_1]$	2481
2427	b. QED	
2428	ELSE $v_0 = \text{mon}(\tau'_d \Rightarrow \tau'_c) v_f :$	2482
2429	a. $e \rightarrow_{H-S} E^\bullet[\text{dyn } \tau'_c(v_f \text{ (stat } \tau'_d v_1))]$	2483
2430	by $v_0 v_1 \triangleright_{H-S} \text{dyn } \tau'_c(v_f \text{ (stat } \tau'_d v_1))$	2484
2431	b. QED	2485
2432	CASE $e = E^\bullet[op^1 v] :$	2486
2433	1. $\vdash_H op^1 v : \tau'$	2487
2434	by <i>static hole typing</i>	2488
2435	2. $\vdash_H v : \tau_0 \times \tau_1$	2489
2436	by <i>H inversion</i>	2490
2437	3. $v = \langle v_0, v_1 \rangle$	2491
2438	by <i>canonical forms</i>	2492
2439	4. IF $op^1 = \text{fst} :$	2493
2440	a. $\delta(\text{fst}, \langle v_0, v_1 \rangle) = v_0$	2494
2441	b. $e \rightarrow_{H-S} E^\bullet[v_0]$	2495
2442	by $op^1 v \triangleright_{H-S} v_0$	2496
2443	c. QED	2497
2444	ELSE $op^1 = \text{snd} :$	2498
2445	a. $\delta(\text{snd}, \langle v_0, v_1 \rangle) = v_1$	2499
2446	b. $e \rightarrow_{H-S} E^\bullet[v_1]$	2500
2447	by $op^1 v \triangleright_{H-S} v_1$	2501
2448	c. QED	2502
2449	CASE $e = E^\bullet[op^2 v_0 v_1] :$	2503
2450	1. $\vdash_H op^2 v_0 v_1 : \tau'$	2504
2451	by <i>static hole typing</i>	2505
2452	2. $\vdash_H v_0 : \tau_0$	2506
2453	$\wedge \vdash_H v_1 : \tau_1$	2507
2454	$\wedge \Delta(op^2, \tau_0, \tau_1) = \tau''$	2508
2455	by <i>H inversion</i>	2509
2456	3. $\delta(op^2, v_0, v_1) = e'$	2510
2457	by <i>Δ type soundness</i> (2)	2511
2458	4. $op^2 v_0 v_1 \triangleright_{H-S} e'$	2512
2459	by (3)	2513
2460	5. QED by $e \rightarrow_{H-S} E^\bullet[e']$	2514
2461	CASE $e = E[\text{dyn } \tau' e']$ and e' is boundary-free :	2515
2462	1. e' is a value	2516
2463	$\vee e' \in \text{Err}$	2517
2464	$\vee e' \rightarrow_{H-D} e''$	2518
2465	$\vee e' \rightarrow_{H-D} \text{BndryErr}$	2519
2466	$\vee e' = E'[e'']$ and $e'' \triangleright_{H-D} \text{TagErr}$	2520
2467	by <i>H dynamic progress</i>	2521
2468	2. IF e' is a value :	2522
2469	a. QED $e \rightarrow_{H-S} E[\mathcal{D}_H(\tau', e')]$	2523
2470	IF $e' \in \text{Err} :$	2524
2471	a. QED $e \rightarrow_{H-S} e'$	2525
2472	IF $e' \rightarrow_{H-D} e'' :$	2526
2473	a. QED $e \rightarrow_{H-S} E[\text{dyn } \tau' e'']$	2527
2474		2528
2475		2529
		2530

```

2531      1.  $(op^1 v) \triangleright_{H-D} e'$ 
2532      2. QED
2533  ELSE  $\delta(op^1, v)$  is undefined :
2534    1.  $e \rightarrow_{H-D} \text{TagErr}$ 
2535      by  $(op^1 v) \triangleright_{H-D} \text{TagErr}$ 
2536    2. QED
2537 CASE  $e = E^\bullet[op^2 v_0 v_1]$  :
2538  IF  $\delta(op^2, v_0, v_1) = e''$  :
2539    1.  $op^2 v_0 v_1 \triangleright_{H-D} e''$ 
2540    2. QED
2541 ELSE  $\delta(op^2, v_0, v_1)$  is undefined :
2542  1.  $e \rightarrow_{H-D} \text{TagErr}$ 
2543      by  $op^2 v_0 v_1 \triangleright_{H-D} \text{TagErr}$ 
2544  2. QED
2545 CASE  $e = E[\text{dyn } \tau' e']$  and  $e'$  is boundary-free :
2546  1.  $e'$  is a value
2547     $\vee e' \in \text{Err}$ 
2548     $\vee e' \rightarrow_{H-D} e''$ 
2549     $\vee e' \rightarrow_{H-D} \text{BndryErr}$ 
2550     $\vee e' = E[e'']$  and  $e'' \triangleright_{H-D} \text{TagErr}$ 
2551    by H dynamic progress
2552  2. IF  $e'$  is a value :
2553    a. QED  $e \rightarrow_{H-D} E[\mathcal{D}_H(\tau', e')]$ 
2554  IF  $e' \in \text{Err}$  :
2555    a. QED  $e \rightarrow_{H-D} e'$ 
2556  IF  $e' \rightarrow_{H-D} e''$  :
2557    a. QED  $e \rightarrow_{H-S} E[\text{dyn } \tau' e'']$ 
2558  IF  $e' \rightarrow_{H-D} \text{BndryErr}$  :
2559    a. QED  $e \rightarrow_{H-D} E[\text{dyn } \tau' \text{BndryErr}]$ 
2560 ELSE  $e' = E[e'']$  and  $e'' \triangleright_{H-D} \text{TagErr}$  :
2561  a.  $E \in E^\bullet$ 
2562    by  $e'$  is boundary-free
2563  b. QED
2564 CASE  $e = E[\text{stat } \tau' e']$  and  $e'$  is boundary-free :
2565  1.  $e'$  is a value
2566     $\vee e' \in \text{Err}$ 
2567     $\vee e' \rightarrow_{H-S} e''$ 
2568     $\vee e' \rightarrow_{H-S} \text{BndryErr}$ 
2569     $\vee e' = E''[\text{dyn } \tau'' E^\bullet[e'']]$  and  $e'' \triangleright_{H-D} \text{TagErr}$ 
2570    by H static progress
2571  2. IF  $e'$  is a value :
2572    a. QED  $e \rightarrow_{H-S} E[S_H(\tau', e')]$ 
2573  IF  $e' \in \text{Err}$  :
2574    a. QED  $e \rightarrow_{H-S} e'$ 
2575  IF  $e' \rightarrow_{H-S} e''$  :
2576    a. QED  $e \rightarrow_{H-S} E[\text{stat } \tau' e'']$ 
2577  IF  $e' \rightarrow_{H-S} \text{BndryErr}$  :
2578    a. QED  $e \rightarrow_{H-S} E[\text{stat } \tau' \text{BndryErr}]$ 
2579 ELSE  $e' = E''[\text{dyn } \tau'' E^\bullet[e'']]$  and  $e'' \triangleright_{H-D} \text{TagErr}$ 
2580 :
2581  a. Contradiction by  $e'$  is boundary-free
2582 CASE  $e = E[\text{Err}]$  :
2583  1. QED  $e \rightarrow_{H-D} \text{Err}$ 
2584 □

```

Lemma 2.10 : H static preservation

If $\vdash_H e : \tau$ and $e \rightarrow_{H-S} e'$ then $\vdash_H e' : \tau$

Proof:

By the *boundary factoring* lemma there are seven cases.

CASE e is a value :

- Contradiction by $e \rightarrow_{H-S} e'$

CASE $e = E^\bullet[v_0 v_1]$:

- $v_0 = \lambda(x:\tau_x). e'$
 $\wedge e \rightarrow_{H-S} E^\bullet[e'[x \leftarrow v_1]]$
- $\vdash_H v_0 v_1 : \tau'$
 \wedge *static hole typing*
- $\vdash_H v_0 : \tau_d \Rightarrow \tau_c$
 $\wedge \vdash_H v_1 : \tau_d$
 $\wedge \tau_c \leqslant \tau'$
 \wedge *H inversion*
- $\tau_d \leqslant \tau_x$
 \wedge *canonical forms* (2)
- $(x:\tau_x) \vdash_H e' : \tau_c$
 \wedge *H inversion* (2)
- $\vdash_H v_1 : \tau_x$
 \wedge (2, 3)
- $\vdash_H e'[x \leftarrow v_1] : \tau_c$
 \wedge *substitution* (4, 5)
- $\vdash_H e'[x \leftarrow v_1] : \tau'$
 \wedge (2, 6)
- QED by *hole substitution* (7)

ELSE $v_0 = \text{mon}(\tau_d \Rightarrow \tau_c) v_f$
 $\wedge e \rightarrow_{H-S} E^\bullet[\text{dyn } \tau_c (v_f (\text{stat } \tau_d v_1))]$:

- $\vdash_H v_0 v_1 : \tau'$
 \wedge *static hole typing*
- $\vdash_H v_0 : \tau'_d \Rightarrow \tau'_c$
 $\wedge \vdash_H v_1 : \tau'_d$
 $\wedge \tau'_c \leqslant \tau'$
 \wedge *H inversion*
- $\vdash_H v_f$
 \wedge *H inversion* (2)
- $\tau_d \Rightarrow \tau_c \leqslant \tau'_d \Rightarrow \tau'_c$
 \wedge *canonical forms* (2)
- $\tau'_d \leqslant \tau_d$
 $\wedge \tau_c \leqslant \tau'_c$
 \wedge (4)
- $\vdash_H v_1 : \tau_d$
 \wedge (2, 5)
- $\vdash_H \text{stat } \tau_d v_1$
 \wedge (6)
- $\vdash_H v_f (\text{stat } \tau_d v_1)$
 \wedge (3, 7)
- $\vdash_H \text{dyn } \tau_c v_f (\text{stat } \tau_d v_1) : \tau_c$
 \wedge (8)
- $\vdash_H \text{dyn } \tau_c v_f (\text{stat } \tau_d v_1) : \tau'$
 \wedge (2, 5, 9)
- QED by *hole substitution* (10)

CASE $e = E^\bullet[op^1 v]$:

2641	IF $v = \langle v_0, v_1 \rangle$	2696
2642	$\wedge op^1 = fst$	2697
2643	$\wedge e \rightarrow_{H-S} E^\bullet[v_0] :$	2698
2644	1. $\vdash_H fst \langle v_0, v_1 \rangle : \tau'$	2699
2645	by <i>static hole typing</i>	2700
2646	2. $\vdash_H \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$	2701
2647	$\wedge \tau_0 \leqslant: \tau'$	2702
2648	by <i>H inversion</i> (1)	2703
2649	3. $\vdash_H v_0 : \tau_0$	2704
2650	by <i>H inversion</i> (2)	2705
2651	4. $\vdash_H v_0 : \tau'$	2706
2652	by (2, 3)	2707
2653	5. QED by <i>hole substitution</i> (4)	2708
2654	ELSE $v = \langle v_0, v_1 \rangle$	2709
2655	$\wedge op^1 = snd$	2710
2656	$\wedge e \rightarrow_{H-S} E^\bullet[v_1] :$	2711
2657	1. $\vdash_H snd \langle v_0, v_1 \rangle : \tau'$	2712
2658	by <i>static hole typing</i>	2713
2659	2. $\vdash_H \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$	2714
2660	$\wedge \tau_1 \leqslant: \tau'$	2715
2661	by <i>H inversion</i> (1)	2716
2662	3. $\vdash_H v_1 : \tau_1$	2717
2663	by <i>H inversion</i> (2)	2718
2664	4. $\vdash_H v_1 : \tau'$	2719
2665	by (2, 3)	2720
2666	5. QED by <i>hole substitution</i> (4)	2721
2667	CASE $e = E^\bullet[op^2 v_0 v_1]$:	2722
2668	1. $e \rightarrow_{H-S} E^\bullet[\delta(op^2, v_0, v_1)]$	2723
2669	by $e \rightarrow_{H-S} e'$	2724
2670	2. $\vdash_H op^2 v_0 v_1 : \tau'$	2725
2671	by <i>static hole typing</i>	2726
2672	3. $\vdash_H v_0 : \tau_0$	2727
2673	$\wedge \vdash_H v_1 : \tau_1$	2728
2674	$\wedge \Delta(op^2, \tau_0, \tau_1) = \tau''$	2729
2675	$\wedge \tau'' \leqslant: \tau'$	2730
2676	by <i>H inversion</i> (1)	2731
2677	4. $\vdash_H \delta(op^2, v_0, v_1) : \tau''$	2732
2678	by <i>Δ type soundness</i> (2)	2733
2679	5. $\vdash_H \delta(op^2, v_0, v_1) : \tau'$	2734
2680	by (2, 3)	2735
2681	6. QED by <i>hole substitution</i> (4)	2736
2682	CASE $e = E[dyn \tau' e']$ and e' is boundary-free :	2737
2683	IF e' is a value :	2738
2684	1. $e \rightarrow_{H-S} E[\mathcal{D}_H(\tau', e')]$	2739
2685	2. $\vdash_H dyn \tau' e' : \tau'$	2740
2686	by <i>boundary hole typing</i>	2741
2687	3. $\vdash_H e'$	2742
2688	by <i>H inversion</i> (2)	2743
2689	4. $\vdash_H \mathcal{D}_H(\tau', e') : \tau'$	2744
2690	by <i>D_H soundness</i> (3)	2745
2691	5. QED by <i>hole substitution</i> (4)	2746
2692	ELSE $e' \rightarrow_{H-D} e''$:	2747
2693	1. $e \rightarrow_{H-S} E[dyn \tau' e'']$	2748
2694		2749
2695		2750

2751	ELSE $v_0 = \text{mon}(\tau_d \Rightarrow \tau_c) v_f$	2806
2752	$\wedge e \rightarrow_{\text{H-D}} E^\bullet[\text{stat } \tau_c (v_f (\text{dyn } \tau_d v_1))] :$	2807
2753	1. $\vdash_H v_0 v_1$	2808
2754	by <i>dynamic hole typing</i>	2809
2755	2. $\vdash_H v_0$	2810
2756	$\wedge \vdash_H v_1$	2811
2757	by <i>H inversion</i> (1)	2812
2758	3. $\vdash_H v_f : \tau_d \Rightarrow \tau_c$	2813
2759	by <i>H inversion</i> (2)	2814
2760	4. $\vdash_H \text{dyn } \tau_d v_1 : \tau_d$	2815
2761	by (2)	2816
2762	5. $\vdash_H v_f (\text{dyn } \tau_d v_1) : \tau_c$	2817
2763	by (3, 4)	2818
2764	6. $\vdash_H \text{stat } \tau_c v_f (\text{dyn } \tau_d v_1)$	2819
2765	by (5)	2820
2766	7. QED by <i>hole substitution</i>	2821
2767	CASE $e = E^\bullet[\text{op}^1 v] :$	2822
2768	IF $v = \langle v_0, v_1 \rangle$	2823
2769	$\wedge \text{op}^1 = \text{fst}$	2824
2770	$\wedge e \rightarrow_{\text{H-D}} E^\bullet[v_0] :$	2825
2771	1. $\vdash_H \text{op}^1 v$	2826
2772	by <i>dynamic hole typing</i>	2827
2773	2. $\vdash_H v$	2828
2774	by <i>H inversion</i> (1)	2829
2775	3. $\vdash_H v_0$	2830
2776	by <i>H inversion</i> (2)	2831
2777	4. QED by <i>hole substitution</i>	2832
2778	ELSE $v = \langle v_0, v_1 \rangle$	2833
2779	$\wedge \text{op}^1 = \text{snd}$	2834
2780	$\wedge e \rightarrow_{\text{H-D}} E^\bullet[v_1] :$	2835
2781	1. $\vdash_H \text{op}^1 v$	2836
2782	by <i>dynamic hole typing</i>	2837
2783	2. $\vdash_H v$	2838
2784	by <i>H inversion</i> (1)	2839
2785	3. $\vdash_H v_1$	2840
2786	by <i>H inversion</i> (2)	2841
2787	4. QED by <i>hole substitution</i>	2842
2788	CASE $e = E^\bullet[\text{op}^2 v_0 v_1] :$	2843
2789	1. $e \rightarrow_{\text{H-D}} E^\bullet[\delta(\text{op}^2, v_0, v_1)]$	2844
2790	2. $\vdash_H \text{op}^2 v_0 v_1$	2845
2791	by <i>dynamic hole typing</i>	2846
2792	3. $\vdash_H v_0$	2847
2793	$\wedge \vdash_H v_1$	2848
2794	by <i>H inversion</i> (1)	2849
2795	4. $\vdash_H \delta(\text{op}^2, v_0, v_1)$	2850
2796	by <i>δ preservation</i> (2)	2851
2797	5. QED by <i>hole substitution</i> (3)	2852
2798	CASE $e = E[\text{dyn } \tau' e']$ and e' is boundary-free :	2853
2799	IF e' is a value :	2854
2800	1. $e \rightarrow_{\text{H-D}} E[\mathcal{D}_H(\tau', e')]$	2855
2801	2. $\vdash_H \text{dyn } \tau' e' : \tau'$	2856
2802	by <i>boundary hole typing</i>	2857
2803	3. $\vdash_H e'$	2858
2804	by <i>H inversion</i> (2)	2859
2805		2860

2861	CASE $e = E[v_0 v_1]$:		2916
2862	1. $E = E^\bullet$		2917
2863	$\vee E = E'[\text{dyn } \tau E^\bullet]$		2918
2864	$\vee E = E'[\text{stat } \tau E^\bullet]$		2919
2865	by <i>inner boundary</i>		2920
2866	2. IF $E = E^\bullet$:		2921
2867	a. QED $e = E^\bullet[v_0 v_1]$		2922
2868	IF $E = E'[\text{dyn } \tau E^\bullet]$:		2923
2869	a. QED $e = E'[\text{dyn } \tau E^\bullet[v_0 v_1]]$		2924
2870	ELSE $E = E'[\text{stat } \tau E^\bullet]$:		2925
2871	a. QED $e = E'[\text{stat } \tau E^\bullet[v_0 v_1]]$		2926
2872	CASE $e = E[\text{op}^1 v]$:		2927
2873	1. $E = E^\bullet$		2928
2874	$\vee E = E'[\text{dyn } \tau E^\bullet]$		2929
2875	$\vee E = E'[\text{stat } \tau E^\bullet]$		2930
2876	by <i>inner boundary</i>		2931
2877	2. IF $E = E^\bullet$:		2932
2878	a. QED $e = E^\bullet[\text{op}^1 v]$		2933
2879	IF $E = E'[\text{dyn } \tau E^\bullet]$:		2934
2880	a. QED $e = E'[\text{dyn } \tau E^\bullet[\text{op}^1 v]]$		2935
2881	ELSE $E = E'[\text{stat } \tau E^\bullet]$:		2936
2882	a. QED $e = E'[\text{stat } \tau E^\bullet[\text{op}^1 v]]$		2937
2883	CASE $e = E[\text{op}^2 v_0 v_1]$:		2938
2884	1. $E = E^\bullet$		2939
2885	$\vee E = E'[\text{dyn } \tau E^\bullet]$		2940
2886	$\vee E = E'[\text{stat } \tau E^\bullet]$		2941
2887	by <i>inner boundary</i>		2942
2888	2. IF $E = E^\bullet$:		2943
2889	a. QED $e = E^\bullet[\text{op}^2 v_0 v_1]$		2944
2890	IF $E = E'[\text{dyn } \tau E^\bullet]$:		2945
2891	a. QED $e = E'[\text{dyn } \tau E^\bullet[\text{op}^2 v_0 v_1]]$		2946
2892	ELSE $E = E'[\text{stat } \tau E^\bullet]$:		2947
2893	a. QED $e = E'[\text{stat } \tau E^\bullet[\text{op}^2 v_0 v_1]]$		2948
2894	CASE $e = E[\text{dyn } \tau v]$:		2949
2895	1. QED v is boundary-free		2950
2896	CASE $e = E[\text{stat } \tau v]$:		2951
2897	1. QED v is boundary-free		2952
2898	CASE $e = E[\text{Err}]$:		2953
2899	1. QED		2954
2900	□		2955
2901	Lemma 2.13 : \mathbb{H} unique static evaluation contexts		2956
2902	If $\vdash_{\mathbb{H}} e : \tau$ then one of the following holds:		2957
2903	• e is a value		2958
2904	• $e = E[v_0 v_1]$		2959
2905	• $e = E[\text{op}^1 v]$		2960
2906	• $e = E[\text{op}^2 v_0 v_1]$		2961
2907	• $e = E[\text{dyn } \tau v]$		2962
2908	• $e = E[\text{stat } \tau v]$		2963
2909	• $e = E[\text{Err}]$		2964
2910	<i>Proof:</i>		2965
2911	By induction on the structure of e .		2966
2912	CASE $e = x$		2967
2913	$\vee e = \lambda x. e'$		2968
2914	$\vee e = \text{stat } \tau e'$:		2969

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2971      1.  $E = []$                                      3026
2972      2. QED  $e = E[op^1 e_0]$                    3027
2973 CASE  $e = op^2 e_0 e_1$  :                      3028
2974   IF  $e_0 \notin v$  :                           3029
2975     1.  $\vdash_H e_0 : \tau_0$   
       by H inversion 3030
2976     2.  $e_0 = E_0[e'_0]$   
       by the induction hypothesis (1) 3031
2977     3.  $E = op^2 E_0 e_1$                      3032
2978     4. QED  $e = E[e'_0]$                       3033
2979   IF  $e_0 \in v$                             3034
2980      $\wedge e_1 \notin v$  :                      3035
2981     1.  $\vdash_H e_1 : \tau_1$   
       by H inversion 3036
2982     2.  $e_1 = E_1[e'_1]$   
       by the induction hypothesis (1) 3037
2983     3.  $E = op^2 e_0 E_1$                      3038
2984     4. QED  $e = E[e'_1]$                       3039
2985 ELSE  $e_0 \in v$                            3040
2986    $\wedge e_1 \in v$  :                         3041
2987   1.  $E = []$                                 3042
2988   2. QED  $e = E[op^2 e_0 e_1]$              3043
2989 CASE  $e = \text{dyn } \tau e_0$  :          3044
2990   IF  $e_0 \notin v$  :                           3045
2991     1.  $\vdash_H e_0$   
       by H inversion 3046
2992     2.  $e_0 = E_0[e'_0]$   
       by unique dynamic evaluation contexts (1) 3047
2993     3.  $E = \text{dyn } \tau E_0$                3048
2994     4. QED  $e = E[e'_0]$                       3049
2995 ELSE  $e_0 \in v$                            3050
2996   1.  $E = []$                                 3051
2997   2. QED  $e = E[\text{dyn } \tau e_0]$          3052
2998 CASE  $e = \text{stat } \tau e_0$  :          3053
2999   Contradiction by  $\vdash_H e : \tau$            3054
3000 CASE  $e = \text{Err}$  :                      3055
3001   1.  $E = []$                                 3056
3002   2. QED  $e = E[\text{Err}]$                   3057
3003 □
3004 Lemma 2.14 : H inner boundary
3005 For all contexts  $E$ , one of the following holds:
3006 •  $E = E^\bullet$                                3058
3007 •  $E = E'[\text{dyn } \tau E^\bullet]$            3059
3008 •  $E = E'[\text{stat } \tau E^\bullet]$           3060
3009 Proof:
3010 By induction on the structure of  $E$ .
3011 CASE  $E = E^\bullet$  :                      3061
3012   1. QED
3013 CASE  $E = E_0 e_1$  :                      3062
3014   1.  $E_0 = E^\bullet$   
      $\vee E_0 = E'_0[\text{dyn } \tau E^\bullet]$  3063
3015      $\vee E_0 = E'_0[\text{stat } \tau E^\bullet]$   
     by the induction hypothesis
3016   2. IF  $E_0 = E^\bullet$  :                  3064
3017     1. QED
3018     CASE  $E = E_0 e_1$  :                  3065
3019       1.  $E_0 = E^\bullet$   
          $\vee E_0 = E'_0[\text{dyn } \tau E^\bullet]$  3066
3020          $\vee E_0 = E'_0[\text{stat } \tau E^\bullet]$   
         by the induction hypothesis
3021       2. IF  $E_0 = E^\bullet$  :              3067
3022         1. QED E is boundary-free
3023           IF  $E_0 = E'_0[\text{dyn } \tau E^\bullet]$  : 3068
3024             1.  $E' = \langle v_0, E'_1 \rangle$   
               QED  $E = E'[\text{dyn } \tau E^\bullet]$  3069
3025             2.  $E' = \langle v_0, E'_1 \rangle$   
               QED  $E = E'[\text{stat } \tau E^\bullet]$  3070
3026           ELSE  $E_0 = E'_1[\text{stat } \tau E^\bullet]$  : 3071
3027             1.  $E' = \langle v_0, E'_1 \rangle$   
               QED  $E = E'[\text{stat } \tau E^\bullet]$  3072
3028           CASE  $E = op^1 E_0$  :              3073
3029             1.  $E_0 = E^\bullet$   
                $\vee E_0 = E'_0[\text{dyn } \tau E^\bullet]$  3074
3030                $\vee E_0 = E'_0[\text{stat } \tau E^\bullet]$   
               by the induction hypothesis
3031             2. IF  $E_0 = E^\bullet$  :          3075
3032               1. QED E is boundary-free
3033                 IF  $E_0 = E'_0[\text{dyn } \tau E^\bullet]$  : 3076
3034                   1.  $E' = E'_0 e_1$   
                     QED  $E = E'[\text{dyn } \tau E^\bullet]$  3077
3035                     2. QED  $E = E'[\text{stat } \tau E^\bullet]$  3078
3036               ELSE  $E_0 = E'_1[\text{stat } \tau E^\bullet]$  : 3079
3037                 1.  $E' = E'_1 e_1$   
                   QED  $E = E'[\text{stat } \tau E^\bullet]$  3080
3038

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3081      a.  $E' = op^1 E'_0$ 
3082      b. QED  $E = E'[\text{dyn } \tau E^\bullet]$ 
3083  ELSE  $E_0 = E'_0[\text{stat } \tau E^\bullet] :$ 
3084    a.  $E' = op^1 E'_0$ 
3085    b. QED  $E = E'[\text{stat } \tau E^\bullet]$ 
3086 CASE  $E = op^2 E_0 e_1 :$ 
3087  1.  $E_0 = E^\bullet$ 
3088     $\vee E_0 = E'_0[\text{dyn } \tau E^\bullet]$ 
3089     $\vee E_0 = E'_0[\text{stat } \tau E^\bullet]$ 
3090    by the induction hypothesis
3091  2. IF  $E_0 = E^\bullet :$ 
3092    a. QED  $E$  is boundary-free
3093    IF  $E_0 = E'_0[\text{dyn } \tau E^\bullet] :$ 
3094      a.  $E' = op^2 E'_0 e_1$ 
3095      b. QED  $E = E'[\text{dyn } \tau E^\bullet]$ 
3096    ELSE  $E_0 = E'_0[\text{stat } \tau E^\bullet] :$ 
3097      a.  $E' = op^2 E'_0 e_1$ 
3098      b. QED  $E = E'[\text{stat } \tau E^\bullet]$ 
3099 CASE  $E = op^2 v_0 E_1 :$ 
3100  1.  $E_1 = E^\bullet$ 
3101     $\vee E_1 = E'_1[\text{dyn } \tau E^\bullet]$ 
3102     $\vee E_1 = E'_1[\text{stat } \tau E^\bullet]$ 
3103    by the induction hypothesis
3104  2. IF  $E_1 = E^\bullet :$ 
3105    a. QED  $E$  is boundary-free
3106    IF  $E_1 = E'_1[\text{dyn } \tau E^\bullet] :$ 
3107      a.  $E' = op^2 v_0 E'_1$ 
3108      b. QED  $E = E'[\text{dyn } \tau E^\bullet]$ 
3109    ELSE  $E_1 = E'_1[\text{stat } \tau E^\bullet] :$ 
3110      a.  $E' = op^2 v_0 E'_1$ 
3111      b. QED  $E = E'[\text{stat } \tau E^\bullet]$ 
3112 CASE  $E = \text{dyn } \tau E_0 :$ 
3113  1.  $E_0 = E^\bullet$ 
3114     $\vee E_0 = E'_0[\text{dyn } \tau' E^\bullet]$ 
3115     $\vee E_0 = E'_0[\text{stat } \tau' E^\bullet]$ 
3116    by the induction hypothesis
3117  2. IF  $E_0 = E^\bullet :$ 
3118    a. QED
3119    IF  $E_0 = E'_0[\text{dyn } \tau' E^\bullet] :$ 
3120      a.  $E' = \text{dyn } \tau E'_0$ 
3121      b. QED  $E = E'[\text{dyn } \tau' E^\bullet]$ 
3122    ELSE  $E_0 = E'_0[\text{stat } \tau' E^\bullet] :$ 
3123      a.  $E' = \text{dyn } \tau E'_0$ 
3124      b. QED  $E = E'[\text{stat } \tau' E^\bullet]$ 
3125 CASE  $E = \text{stat } \tau E_0 :$ 
3126  1.  $E_0 = E^\bullet$ 
3127     $\vee E_0 = E'_0[\text{dyn } \tau' E^\bullet]$ 
3128     $\vee E_0 = E'_0[\text{stat } \tau' E^\bullet]$ 
3129    by the induction hypothesis
3130  2. IF  $E_0 = E^\bullet :$ 
3131    a. QED
3132    IF  $E_0 = E'_0[\text{dyn } \tau' E^\bullet] :$ 
3133      a.  $E' = \text{stat } \tau E'_0$ 

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b. QED $E = E'[\text{dyn } \tau' E^\bullet]$	3136
ELSE $E_0 = E'_0[\text{stat } \tau' E^\bullet] :$	3137
a. $E' = \text{stat } \tau E'_0$	3138
b. QED $E = E'[\text{stat } \tau' E^\bullet]$	3139

 \square **Lemma 2.15 : H dynamic boundary factoring**If $\vdash_H e$ then one of the following holds:

- e is a value
- $e = E^\bullet[v_0 v_1]$
- $e = E^\bullet[op^1 v]$
- $e = E^\bullet[op^2 v_0 v_1]$
- $e = E[\text{dyn } \tau e']$ where e' is boundary-free
- $e = E[\text{stat } \tau e']$ where e' is boundary-free
- $e = E[\text{Err}]$

*Proof:*By the *unique dynamic evaluation contexts* lemma, there are seven cases.CASE e is a value :

1. QED

CASE $e = E[v_0 v_1] :$ 1. $E = E^\bullet$ $\vee E = E'[\text{dyn } \tau E^\bullet]$ $\vee E = E'[\text{stat } \tau E^\bullet]$ by *inner boundary*2. IF $E = E^\bullet :$ a. QED $e = E^\bullet[v_0 v_1]$ IF $E = E'[\text{dyn } \tau E^\bullet] :$ a. QED $e = E'[\text{dyn } \tau E^\bullet[v_0 v_1]]$ ELSE $E = E'[\text{stat } \tau E^\bullet] :$ a. QED $e = E'[\text{stat } \tau E^\bullet[v_0 v_1]]$ CASE $e = E[op^1 v] :$ 1. $E = E^\bullet$ $\vee E = E'[\text{dyn } \tau E^\bullet]$ $\vee E = E'[\text{stat } \tau E^\bullet]$ by *inner boundary*2. IF $E = E^\bullet :$ a. QED $e = E^\bullet[op^1 v]$ IF $E = E'[\text{dyn } \tau E^\bullet] :$ a. QED $e = E'[\text{dyn } \tau E^\bullet[op^1 v]]$ ELSE $E = E'[\text{stat } \tau E^\bullet] :$ a. QED $e = E'[\text{stat } \tau E^\bullet[op^1 v]]$ CASE $e = E[op^2 v_0 v_1] :$ 1. $E = E^\bullet$ $\vee E = E'[\text{dyn } \tau E^\bullet]$ $\vee E = E'[\text{stat } \tau E^\bullet]$ by *inner boundary*2. IF $E = E^\bullet :$ a. QED $e = E^\bullet[op^2 v_0 v_1]$ IF $E = E'[\text{dyn } \tau E^\bullet] :$ a. QED $e = E'[\text{dyn } \tau E^\bullet[op^2 v_0 v_1]]$ ELSE $E = E'[\text{stat } \tau E^\bullet] :$ a. QED $e = E'[\text{stat } \tau E^\bullet[op^2 v_0 v_1]]$ CASE $e = E[\text{dyn } \tau v] :$ 1. QED v is boundary-free

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3191 CASE e = E[stat τ v] :
3192   1. QED v is boundary-free
3193 CASE e = E[Err] :
3194   1. QED
3195 □
3196 Lemma 2.16 : H unique dynamic evaluation contexts
3197 If  $\vdash_H e$  then one of the following holds:
3198 • e is a value
3199 • e = E[v0 v1]
3200 • e = E[op1 v]
3201 • e = E[op2 v0 v1]
3202 • e = E[dyn τ v]
3203 • e = E[stat τ v]
3204 • e = E[Err]
3205 Proof:
3206 By induction on the structure of e.
3207 CASE e = x
3208   ∨ e = λ(x:τ). e'
3209   ∨ e = dyn τ e' :
3210   1. Contradiction by  $\vdash_H e$ 
3211 CASE e = i
3212   ∨ e = λx. e'
3213   ∨ e = mon(τd ⇒ τc)v :
3214   1. QED e is a value
3215 CASE e = Err :
3216   1. E = []
3217   2. QED e = E[Err]
3218 CASE e = ⟨e0, e1⟩ :
3219   IF e0 ∉ v :
3220     1.  $\vdash_H e_0$ 
3221       by H inversion
3222     2. e0 = E0[e'0]
3223       by the induction hypothesis (1)
3224     3. E = ⟨E0, e1⟩
3225     4. QED e = E[e'0]
3226   IF e0 ∈ v
3227     ∧ e1 ∉ v :
3228     1.  $\vdash_H e_1$ 
3229       by H inversion
3230     2. e1 = E1[e'1]
3231       by the induction hypothesis (1)
3232     3. E = ⟨e0, E1⟩
3233     4. QED e = E[e'1]
3234 ELSE e0 ∈ v
3235   ∧ e1 ∈ v :
3236   1. E = []
3237   2. QED e is a value
3238 CASE e = e0 e1 :
3239   IF e0 ∉ v :
3240     1.  $\vdash_H e_0$ 
3241       by H inversion
3242     2. e0 = E0[e'0]
3243       by the induction hypothesis (1)
3244     3. E = E0 e1
3245
3246 4. QED e = E[e'0]
3247 IF e0 ∈ v
3248   ∧ e1 ∉ v :
3249   1.  $\vdash_H e_1$ 
3250     by H inversion
3251   2. e1 = E1[e'1]
3252     by the induction hypothesis (1)
3253   3. E = e0 E1
3254   4. QED e = E[e'1]
3255 ELSE e0 ∈ v
3256   ∧ e1 ∈ v :
3257   1. E = []
3258   2. QED e = E[e0 e1]
3259 CASE e = op1 e0 :
3260   IF e0 ∉ v :
3261     1.  $\vdash_H e_0$ 
3262       by H inversion
3263     2. e0 = E0[e'0]
3264       by the induction hypothesis (1)
3265     3. E = op1 E0
3266     4. QED e = E[e'0]
3267 ELSE e0 ∈ v :
3268   1. E = []
3269   2. QED e = E[op1 e0]
3270 CASE e = op2 e0 e1 :
3271   IF e0 ∉ v :
3272     1.  $\vdash_H e_0$ 
3273       by H inversion
3274     2. e0 = E0[e'0]
3275       by the induction hypothesis (1)
3276     3. E = op2 E0 e1
3277     4. QED e = E[e'0]
3278 IF e0 ∈ v
3279   ∧ e1 ∉ v :
3280   1.  $\vdash_H e_1$ 
3281     by H inversion
3282   2. e1 = E1[e'1]
3283     by the induction hypothesis (1)
3284   3. E = op2 e0 E1
3285   4. QED e = E[e'1]
3286 ELSE e0 ∈ v
3287   ∧ e1 ∈ v :
3288   1. E = []
3289   2. QED e = E[op2 e0 e1]
3290 CASE e = stat τ e0 :
3291   IF e0 ∉ v :
3292     1.  $\vdash_H e_0$ 
3293       by H inversion
3294     2. e0 = E0[e'0]
3295       by unique static evaluation contexts (1)
3296     3. E = stat τ E0
3297     4. QED e = E[e'0]
3298 ELSE e0 ∈ v :
3299   1. E = []
3300

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3301	2. QED $e = E[\text{stat } \tau e_0]$	
3302	□	
3303	Lemma 2.17 : H static hole typing	
3304	If $\vdash_H E^\bullet[e] : \tau$ then the derivation contains a sub-term $\vdash_H e : \tau'$	
3305	<i>Proof:</i>	
3306	By induction on the structure of E^\bullet :	
3307	CASE $E^\bullet = []$:	
3308	1. QED $E^\bullet[e] = e$	
3309	CASE $E^\bullet = E^\bullet_0 e_1$:	
3310	1. $E^\bullet[e] = E^\bullet_0[e] e_1$	
3311	2. $\vdash_H E^\bullet_0[e] : \tau_d \Rightarrow \tau_c$	
3312	by H inversion	
3313	3. QED by the induction hypothesis (2)	
3314	CASE $E^\bullet = v_0 E^\bullet_1$:	
3315	1. $E^\bullet[e] = v_0 E^\bullet_1[e]$	
3316	2. $\vdash_H E^\bullet_1[e] : \tau_d$	
3317	by H inversion	
3318	3. QED by the induction hypothesis (2)	
3319	CASE $E^\bullet = \langle E^\bullet_0, e_1 \rangle$:	
3320	1. $E^\bullet[e] = \langle E^\bullet_0[e], e_1 \rangle$	
3321	2. $\vdash_H E^\bullet_0[e] : \tau_0$	
3322	by H inversion	
3323	3. QED by the induction hypothesis (2)	
3324	CASE $E^\bullet = \langle v_0, E^\bullet_1 \rangle$:	
3325	1. $E^\bullet[e] = \langle v_0, E^\bullet_1[e] \rangle$	
3326	2. $\vdash_H E^\bullet_1[e] : \tau_1$	
3327	by H inversion	
3328	3. QED by the induction hypothesis (2)	
3329	CASE $E^\bullet = op^1 E^\bullet_0$:	
3330	1. $E^\bullet[e] = op^1 E^\bullet_0[e]$	
3331	2. $\vdash_H E^\bullet_0[e] : \tau_0$	
3332	by H inversion	
3333	3. QED by the induction hypothesis (2)	
3334	CASE $E^\bullet = op^2 E^\bullet_0 e_1$:	
3335	1. $E^\bullet[e] = op^2 E^\bullet_0[e] e_1$	
3336	2. $\vdash_H E^\bullet_0[e] : \tau_0$	
3337	by H inversion	
3338	3. QED by the induction hypothesis (2)	
3339	CASE $E^\bullet = op^2 v_0 E^\bullet_1$:	
3340	1. $E^\bullet[e] = op^2 v_0 E^\bullet_1[e]$	
3341	2. $\vdash_H E^\bullet_1[e] : \tau_1$	
3342	by H inversion	
3343	3. QED by the induction hypothesis (2)	
3344	□	
3345	Lemma 2.18 : H dynamic hole typing	
3346	If $\vdash_H E^\bullet[e]$ then the derivation contains a sub-term $\vdash_H e$	
3347	<i>Proof:</i>	
3348	By induction on the structure of E^\bullet .	
3349	CASE $E^\bullet = []$:	
3350	1. QED $E^\bullet[e] = e$	
3351	CASE $E^\bullet = E^\bullet_0 e_1$:	
3352	1. $E^\bullet[e] = E^\bullet_0[e] e_1$	
3353		
3354		
3355		

3356	2. $\vdash_H E^\bullet_0[e]$	
3357	by H inversion	
3358	3. QED by the induction hypothesis (2)	
3359	CASE $E^\bullet = v_0 E^\bullet_1$:	
3360	1. $E^\bullet[e] = v_0 E^\bullet_1[e]$	
3361	2. $\vdash_H E^\bullet_1[e]$	
3362	by H inversion	
3363	3. QED by the induction hypothesis (2)	
3364	CASE $E^\bullet = \langle E^\bullet_0, e_1 \rangle$:	
3365	1. $E^\bullet[e] = \langle E^\bullet_0[e], e_1 \rangle$	
3366	2. $\vdash_H E^\bullet_0[e]$	
3367	by H inversion	
3368	3. QED by the induction hypothesis (2)	
3369	CASE $E^\bullet = \langle v_0, E^\bullet_1 \rangle$:	
3370	1. $E^\bullet[e] = \langle v_0, E^\bullet_1[e] \rangle$	
3371	2. $\vdash_H E^\bullet_1[e]$	
3372	by H inversion	
3373	3. QED by the induction hypothesis (2)	
3374	CASE $E^\bullet = op^1 E^\bullet_0$:	
3375	1. $E^\bullet[e] = op^1 E^\bullet_0[e]$	
3376	2. $\vdash_H E^\bullet_0[e]$	
3377	by H inversion	
3378	3. QED by the induction hypothesis (2)	
3379	CASE $E^\bullet = op^2 E^\bullet_0 e_1$:	
3380	1. $E^\bullet[e] = op^2 E^\bullet_0[e] e_1$	
3381	2. $\vdash_H E^\bullet_0[e]$	
3382	by H inversion	
3383	3. QED by the induction hypothesis (2)	
3384	CASE $E^\bullet = op^2 v_0 E^\bullet_1$:	
3385	1. $E^\bullet[e] = op^2 v_0 E^\bullet_1[e]$	
3386	2. $\vdash_H E^\bullet_1[e]$	
3387	by H inversion	
3388	3. QED by the induction hypothesis (2)	
3389	□	
3390	Lemma 2.19 : H boundary hole typing	
3391	• If $\vdash_H E[\text{dyn } \tau e] : \tau'$ then the derivation contains a sub-term $\vdash_H \text{dyn } \tau e : \tau$	
3392		
3393	• If $\vdash_H E[\text{dyn } \tau e]$ then the derivation contains a sub-term $\vdash_H \text{dyn } \tau e : \tau$	
3394		
3395	• If $\vdash_H E[\text{stat } \tau e] : \tau'$ then the derivation contains a sub-term $\vdash_H \text{stat } \tau e$	
3396		
3397	• If $\vdash_H E[\text{stat } \tau e]$ then the derivation contains a sub-term $\vdash_H \text{stat } \tau e$	
3398		
3399		
3400	<i>Proof:</i>	
3401	By the following four lemmas: static dyn hole typing , dynamic dyn hole typing , static stat hole typing , and dynamic stat hole typing .	
3402		
3403	□	
3404	Lemma 2.20 : H static dyn hole typing	
3405	If $\vdash_H E[\text{dyn } \tau e] : \tau'$ then the derivation contains a sub-term $\vdash_H \text{dyn } \tau e : \tau$	
3406		
3407	<i>Proof:</i>	
3408	By induction on the structure of E .	
3409		
3410		

3411	CASE $E \in E^\bullet :$	By induction on the structure of E .	3466
3412	1. $\vdash_H \text{dyn } \tau e : \tau''$ by <i>static hole typing</i>	1. Contradiction by $\vdash_H E[\text{dyn } \tau e]$	3467
3413	2. $\vdash_H \text{dyn } \tau e : \tau$ by <i>H inversion</i> (1)	2. $\vdash_H E_0[\text{dyn } \tau e]$ by <i>H inversion</i>	3468
3414	3. QED	3. QED by the induction hypothesis (2)	3469
3415	CASE $E = E_0 e_1 :$	CASE $E = E_0 e_1 :$	3470
3416	1. $E[\text{dyn } \tau e] = E_0[\text{dyn } \tau e] e_1$	1. $E[\text{dyn } \tau e] = E_0[\text{dyn } \tau e] e_1$	3471
3417	2. $\vdash_H E_0[\text{dyn } \tau e] : \tau_0$ by <i>H inversion</i>	2. $\vdash_H E_0[\text{dyn } \tau e]$ by <i>H inversion</i>	3472
3418	3. QED by the induction hypothesis (2)	3. QED by the induction hypothesis (2)	3473
3419	CASE $E = v_0 E_1 :$	CASE $E = v_0 E_1 :$	3474
3420	1. $E[\text{dyn } \tau e] = v_0 E_1[\text{dyn } \tau e]$	1. $E[\text{dyn } \tau e] = v_0 E_1[\text{dyn } \tau e]$	3475
3421	2. $\vdash_H E_1[\text{dyn } \tau e] : \tau_1$ by <i>H inversion</i>	2. $\vdash_H E_1[\text{dyn } \tau e]$ by <i>H inversion</i>	3476
3422	3. QED by the induction hypothesis (2)	3. QED by the induction hypothesis (2)	3477
3423	CASE $E = \langle E_0, e_1 \rangle :$	CASE $E = \langle E_0, e_1 \rangle :$	3478
3424	1. $E[\text{dyn } \tau e] = \langle E_0[\text{dyn } \tau e], e_1 \rangle$	1. $E[\text{dyn } \tau e] = \langle E_0[\text{dyn } \tau e], e_1 \rangle$	3479
3425	2. $\vdash_H E_0[\text{dyn } \tau e] : \tau_0$ by <i>H inversion</i>	2. $\vdash_H E_0[\text{dyn } \tau e]$ by <i>H inversion</i>	3480
3426	3. QED by the induction hypothesis (2)	3. QED by the induction hypothesis (2)	3481
3427	CASE $E = \langle v_0, E_1 \rangle :$	CASE $E = \langle v_0, E_1 \rangle :$	3482
3428	1. $E[\text{dyn } \tau e] = \langle v_0, E_1[\text{dyn } \tau e] \rangle$	1. $E[\text{dyn } \tau e] = \langle v_0, E_1[\text{dyn } \tau e] \rangle$	3483
3429	2. $\vdash_H E_1[\text{dyn } \tau e] : \tau_1$ by <i>H inversion</i>	2. $\vdash_H E_1[\text{dyn } \tau e]$ by <i>H inversion</i>	3484
3430	3. QED by the induction hypothesis (2)	3. QED by the induction hypothesis (2)	3485
3431	CASE $E = op^1 E_0 :$	CASE $E = op^1 E_0 :$	3486
3432	1. $E[\text{dyn } \tau e] = op^1 E_0[\text{dyn } \tau e]$	1. $E[\text{dyn } \tau e] = op^1 E_0[\text{dyn } \tau e]$	3487
3433	2. $\vdash_H E_0[\text{dyn } \tau e] : \tau_0$ by <i>H inversion</i>	2. $\vdash_H E_0[\text{dyn } \tau e]$ by <i>H inversion</i>	3488
3434	3. QED by the induction hypothesis (2)	3. QED by the induction hypothesis (2)	3489
3435	CASE $E = op^1 E_1 :$	CASE $E = op^1 E_1 :$	3490
3436	1. $E[\text{dyn } \tau e] = \langle v_0, E_1[\text{dyn } \tau e] \rangle$	1. $E[\text{dyn } \tau e] = op^1 E_1[\text{dyn } \tau e]$	3491
3437	2. $\vdash_H E_1[\text{dyn } \tau e] : \tau_1$ by <i>H inversion</i>	2. $\vdash_H E_1[\text{dyn } \tau e]$ by <i>H inversion</i>	3492
3438	3. QED by the induction hypothesis (2)	3. QED by the induction hypothesis (2)	3493
3439	CASE $E = op^2 E_0 e_1 :$	CASE $E = op^2 E_0 e_1 :$	3494
3440	1. $E[\text{dyn } \tau e] = op^2 E_0[\text{dyn } \tau e] e_1$	1. $E[\text{dyn } \tau e] = op^2 E_0[\text{dyn } \tau e] e_1$	3495
3441	2. $\vdash_H E_0[\text{dyn } \tau e] : \tau_0$ by <i>H inversion</i>	2. $\vdash_H E_0[\text{dyn } \tau e]$ by <i>H inversion</i>	3496
3442	3. QED by the induction hypothesis (2)	3. QED by the induction hypothesis (2)	3497
3443	CASE $E = op^2 E_0 e_1 :$	CASE $E = op^2 v_0 E_1 :$	3498
3444	1. $E[\text{dyn } \tau e] = op^2 v_0 E_1[\text{dyn } \tau e]$	1. $E[\text{dyn } \tau e] = op^2 v_0 E_1[\text{dyn } \tau e]$	3499
3445	2. $\vdash_H E_1[\text{dyn } \tau e] : \tau_1$ by <i>H inversion</i>	2. $\vdash_H E_1[\text{dyn } \tau e]$ by <i>H inversion</i>	3500
3446	3. QED by the induction hypothesis (2)	3. QED by the induction hypothesis (2)	3501
3447	CASE $E = op^2 v_0 E_1 :$	CASE $E = dyn \tau E_0 :$	3502
3448	1. $E[\text{dyn } \tau e] = op^2 v_0 E_1[\text{dyn } \tau e]$	1. Contradiction by $\vdash_H E[\text{dyn } \tau e]$	3503
3449	2. $\vdash_H E_1[\text{dyn } \tau e] : \tau_1$ by <i>H inversion</i>	2. $\vdash_H E_0[\text{dyn } \tau e]$ by <i>H inversion</i>	3504
3450	3. QED by the induction hypothesis (2)	3. QED by <i>dynamic dyn hole typing</i> (2)	3505
3451	CASE $E = dyn \tau_0 E_0 :$	CASE $E = stat \tau_0 E_0 :$	3506
3452	1. $E[\text{dyn } \tau e] = dyn \tau_0 E_0[\text{dyn } \tau e]$	1. $E[\text{dyn } \tau e] = stat \tau_0 E_0[\text{dyn } \tau e]$	3507
3453	2. $\vdash_H E_0[\text{dyn } \tau e]$ by <i>H inversion</i>	2. $\vdash_H E_0[\text{dyn } \tau e] : \tau_0$ by <i>H inversion</i>	3508
3454	3. QED by <i>dynamic dyn hole typing</i> (2)	3. QED by <i>static dyn hole typing</i> (2)	3509
3455	CASE $E = stat \tau_0 E_0 :$	Proof:	3510
3456	1. Contradiction by $\vdash_H E[\text{dyn } \tau e] : \tau'$	By induction on the structure of E .	3511
3457	Lemma 2.21 : $\vdash_H \text{dynamic dyn hole typing}$	CASE $E \in E^\bullet :$	3512
3458	If $\vdash_H E[\text{dyn } \tau e]$ then the derivation contains a sub-term	1. Contradiction by $\vdash_H E[\text{stat } \tau e] : \tau'$	3513
3459	$\vdash_H \text{dyn } \tau e : \tau$.	2. $\vdash_H \text{stat } \tau e$.	3514
3460	<i>Proof:</i>	3. Proof:	3515
3461		By induction on the structure of E .	3516
3462		CASE $E \in E^\bullet :$	3517
3463		1. Contradiction by $\vdash_H E[\text{stat } \tau e] : \tau'$	3518
3464			3519
3465			3520

3521	CASE $E = E_0 e_1 :$		
3522	1. $E[\text{stat } \tau e] = E_0[\text{stat } \tau e] e_1$		3576
3523	2. $\vdash_H E_0[\text{stat } \tau e] : \tau_0$	by H inversion	3577
3524	by H inversion		3578
3525	3. QED by the induction hypothesis (2)		3579
3526	CASE $E = v_0 E_1 :$		3580
3527	1. $E[\text{stat } \tau e] = v_0 E_1[\text{stat } \tau e]$		3581
3528	2. $\vdash_H E_1[\text{stat } \tau e] : \tau_1$	by H inversion	3582
3529	by H inversion		3583
3530	3. QED by the induction hypothesis (2)		3584
3531	CASE $E = \langle E_0, e_1 \rangle :$		3585
3532	1. $E[\text{stat } \tau e] = \langle E_0[\text{stat } \tau e], e_1 \rangle$		3586
3533	2. $\vdash_H E_0[\text{stat } \tau e] : \tau_0$	by H inversion	3587
3534	by H inversion		3588
3535	3. QED by the induction hypothesis (2)		3589
3536	CASE $E = \langle v_0, E_1 \rangle :$		3590
3537	1. $E[\text{stat } \tau e] = \langle v_0, E_1[\text{stat } \tau e] \rangle$		3591
3538	2. $\vdash_H E_1[\text{stat } \tau e] : \tau_1$	by H inversion	3592
3539	by H inversion		3593
3540	3. QED by the induction hypothesis (2)		3594
3541	CASE $E = op^1 E_0 :$		3595
3542	1. $E[\text{stat } \tau e] = op^1 E_0[\text{stat } \tau e]$		3596
3543	2. $\vdash_H E_0[\text{stat } \tau e] : \tau_0$	by H inversion	3597
3544	by H inversion		3598
3545	3. QED by the induction hypothesis (2)		3599
3546	CASE $E = op^2 E_0 e_1 :$		3600
3547	1. $E[\text{stat } \tau e] = op^2 E_0[\text{stat } \tau e] e_1$		3601
3548	2. $\vdash_H E_0[\text{stat } \tau e] : \tau_0$	by H inversion	3602
3549	by H inversion		3603
3550	3. QED by the induction hypothesis (2)		3604
3551	CASE $E = op^2 v_0 E_1 :$		3605
3552	1. $E[\text{stat } \tau e] = op^2 v_0 E_1[\text{stat } \tau e]$		3606
3553	2. $\vdash_H E_1[\text{stat } \tau e] : \tau_1$	by H inversion	3607
3554	by H inversion		3608
3555	3. QED by the induction hypothesis (2)		3609
3556	CASE $E = \text{dyn } \tau_0 E_0 :$		3610
3557	1. $E[\text{stat } \tau e] = \text{dyn } \tau_0 E_0[\text{stat } \tau e]$		3611
3558	2. $\vdash_H E_0[\text{stat } \tau e]$	by H inversion	3612
3559	by H inversion		3613
3560	3. QED by dynamic stat hole typing (2)		3614
3561	CASE $E = \text{stat } \tau_0 E_0 :$		3615
3562	1. Contradiction by $\vdash_H E[\text{stat } \tau e] : \tau'$		3616
3563	□		
3564	Lemma 2.23 : H dynamic stat hole typing		3617
3565	If $\vdash_H E[\text{stat } \tau e]$ then the derivation contains a sub-term		3618
3566	$\vdash_H \text{stat } \tau e$.		3619
3567	<i>Proof:</i>		3620
3568	By induction on the structure of E .		3621
3569	CASE $E \in E^\bullet :$		3622
3570	1. QED by dynamic hole typing		3623
3571	CASE $E = E_0 e_1 :$		3624
3572	1. $E[\text{stat } \tau e] = E_0[\text{stat } \tau e] e_1$		3625
3573			3626
3574			3627
3575			3628
			3629
			3630

3631	5. QED by (1, 4)	3686
3632	CASE $E^\bullet = E^\bullet_0 e_1 :$	
3633	1. $E^\bullet[e] = E^\bullet_0[e] e_1$	3687
3634	$\wedge E^\bullet[e'] = E^\bullet_0[e'] e_1$	3688
3635	2. $\vdash_H E^\bullet_0[e] e_1 : \tau$	3689
3636	3. $\vdash_H E^\bullet_0[e] : \tau_0$	3690
3637	$\wedge \vdash_H e_1 : \tau_1$	3691
3638	by H inversion	3692
3639	4. $\vdash_H E^\bullet_0[e'] : \tau_0$	3693
3640	by the induction hypothesis (3)	
3641	5. $\vdash_H E^\bullet_0[e'] e_1 : \tau$	3694
3642	by (2, 3, 4)	3695
3643	6. QED by (1, 5)	3696
3644	CASE $E^\bullet = v_0 E^\bullet_1 :$	
3645	1. $E^\bullet[e] = v_0 E^\bullet_1[e]$	3697
3646	$\wedge E^\bullet[e'] = v_0 E^\bullet_1[e']$	3698
3647	2. $\vdash_H v_0 E^\bullet_1[e] : \tau$	3699
3648	3. $\vdash_H v_0 : \tau_0$	3700
3649	$\wedge \vdash_H E^\bullet_1[e] : \tau_1$	3701
3650	by H inversion	3702
3651	4. $\vdash_H E^\bullet_1[e'] : \tau_1$	3703
3652	by the induction hypothesis (3)	
3653	5. $\vdash_H v_0 E^\bullet_1[e'] : \tau$	3704
3654	by (2, 3, 4)	3705
3655	6. QED by (1, 5)	3706
3656	CASE $E^\bullet = op^1 E^\bullet_0 :$	
3657	1. $E^\bullet[e] = op^1 E^\bullet_0[e]$	3707
3658	$\wedge E^\bullet[e'] = op^1 E^\bullet_0[e']$	3708
3659	2. $\vdash_H op^1 E^\bullet_0[e] : \tau$	3709
3660	3. $\vdash_H E^\bullet_0[e] : \tau_0$	3710
3661	by H inversion	3711
3662	4. $\vdash_H E^\bullet_0[e'] : \tau_0$	3712
3663	by the induction hypothesis (3)	
3664	5. $\vdash_H op^1 E^\bullet_0[e'] : \tau$	3713
3665	by (2, 3, 4)	3714
3666	6. QED by (1, 5)	3715
3667	CASE $E^\bullet = \langle E^\bullet_0, e_1 \rangle :$	
3668	1. $E^\bullet[e] = \langle E^\bullet_0[e], e_1 \rangle$	3716
3669	$\wedge E^\bullet[e'] = \langle E^\bullet_0[e'], e_1 \rangle$	3717
3670	2. $\vdash_H \langle E^\bullet_0[e], e_1 \rangle : \tau$	3718
3671	3. $\vdash_H E^\bullet_0[e] : \tau_0$	
3672	$\wedge \vdash_H e_1 : \tau_1$	
3673	by H inversion	
3674	4. $\vdash_H E^\bullet_0[e'] : \tau_0$	
3675	by the induction hypothesis (3)	
3676	5. $\vdash_H \langle E^\bullet_0[e'], e_1 \rangle : \tau$	
3677	by (2, 3, 4)	
3678	6. QED by (1, 5)	
3679	CASE $E^\bullet = \langle v_0, E^\bullet_1 \rangle :$	
3680	1. $E^\bullet[e] = \langle v_0, E^\bullet_1[e] \rangle$	3722
3681	$\wedge E^\bullet[e'] = \langle v_0, E^\bullet_1[e'] \rangle$	3723
3682	2. $\vdash_H \langle v_0, E^\bullet_1[e] \rangle : \tau$	3724
3683		
3684		
3685		
	3. $\vdash_H v_0 : \tau_0$	3725
	$\wedge \vdash_H E^\bullet_1[e] : \tau_1$	3726
	by H inversion	3727
	4. $\vdash_H E^\bullet_1[e'] : \tau_1$	3728
	by the induction hypothesis (3)	
	5. $\vdash_H op^2 E^\bullet_0[e'] e_1 : \tau$	3729
	by (2, 3, 4)	3730
	6. QED by (1, 5)	3731
	□	3732
	Lemma 2.25 : H dynamic hole substitution	3733
	■ If $\vdash_H E^\bullet[e]$ and $\vdash_H e'$ then $\vdash_H E^\bullet[e']$	3734
	<i>Proof:</i>	3735
	By induction on the structure of E^\bullet	
	CASE $E^\bullet = [] :$	
	1. QED $E^\bullet[e'] = e'$	3736
	CASE $E^\bullet = \langle E^\bullet_0, e_1 \rangle :$	
	1. $E^\bullet[e] = \langle E^\bullet_0[e], e_1 \rangle$	3737
	$\wedge E^\bullet[e'] = \langle E^\bullet_0[e'], e_1 \rangle$	3738
	2. $\vdash_H \langle E^\bullet_0[e], e_1 \rangle : \tau$	3739
	3. $\vdash_H E^\bullet_0[e] : \tau_0$	
	$\wedge \vdash_H e_1 : \tau_1$	
	by H inversion	
	4. $\vdash_H E^\bullet_0[e'] : \tau_0$	
	by the induction hypothesis (3)	
	5. $\vdash_H \langle E^\bullet_0[e'], e_1 \rangle$	3740
	by (3, 4)	
	6. QED by (1, 5)	
	CASE $E^\bullet = \langle v_0, E^\bullet_1 \rangle :$	

3741	1. $E^\bullet[e] = \langle v_0, E^\bullet_1[e] \rangle$	3796
3742	$\wedge E^\bullet[e'] = \langle v_0, E^\bullet_1[e'] \rangle$	3797
3743	2. $\vdash_H \langle v_0, E^\bullet_1[e] \rangle$	3798
3744	3. $\vdash_H v_0$	3799
3745	$\wedge \vdash_H E^\bullet_1[e]$	3800
3746	by H inversion	3801
3747	4. $\vdash_H E^\bullet_1[e']$	3802
3748	by the induction hypothesis (3)	3803
3749	5. $\vdash_H \langle v_0, E^\bullet_1[e'] \rangle$	3804
3750	by (3, 4)	3805
3751	6. QED by (1, 5)	3806
3752	CASE $E^\bullet = E^\bullet_0 e_1 :$	3807
3753	1. $E^\bullet[e] = E^\bullet_0[e] e_1$	3808
3754	$\wedge E^\bullet[e'] = E^\bullet_0[e'] e_1$	3809
3755	2. $\vdash_H E^\bullet_0[e] e_1$	3810
3756	3. $\vdash_H E^\bullet_0[e]$	3811
3757	$\wedge \vdash_H e_1$	3812
3758	by H inversion	3813
3759	4. $\vdash_H E^\bullet_0[e']$	3814
3760	by the induction hypothesis (3)	3815
3761	5. $\vdash_H E^\bullet_0[e'] e_1$	3816
3762	by (3, 4)	3817
3763	6. QED by (1, 5)	3818
3764	CASE $E^\bullet = v_0 E^\bullet_1 :$	3819
3765	1. $E^\bullet[e] = v_0 E^\bullet_1[e]$	3820
3766	$\wedge E^\bullet[e'] = v_0 E^\bullet_1[e']$	3821
3767	2. $\vdash_H v_0 E^\bullet_1[e]$	3822
3768	3. $\vdash_H v_0$	3823
3769	$\wedge \vdash_H E^\bullet_1[e]$	3824
3770	by H inversion	3825
3771	4. $\vdash_H E^\bullet_1[e']$	3826
3772	by the induction hypothesis (3)	3827
3773	5. $\vdash_H v_0 E^\bullet_1[e']$	3828
3774	by (3, 4)	3829
3775	6. QED by (1, 5)	3830
3776	CASE $E^\bullet = op^1 E^\bullet_0 :$	3831
3777	1. $E^\bullet[e] = op^1 E^\bullet_0[e]$	3832
3778	$\wedge E^\bullet[e'] = op^1 E^\bullet_0[e']$	3833
3779	2. $\vdash_H op^1 E^\bullet_0[e]$	3834
3780	3. $\vdash_H E^\bullet_0[e]$	3835
3781	by H inversion	3836
3782	4. $\vdash_H E^\bullet_0[e']$	3837
3783	by the induction hypothesis (3)	3838
3784	5. $\vdash_H op^1 E^\bullet_0[e']$	3839
3785	by (4)	3840
3786	6. QED by (1, 5)	3841
3787	CASE $E^\bullet = op^2 E^\bullet_0 e_1 :$	3842
3788	1. $E^\bullet[e] = op^2 E^\bullet_0[e] e_1$	3843
3789	$\wedge E^\bullet[e'] = op^2 E^\bullet_0[e'] e_1$	3844
3790	2. $\vdash_H op^2 E^\bullet_0[e] e_1$	3845
3791	3. $\vdash_H E^\bullet_0[e]$	3846
3792	$\wedge \vdash_H e_1$	3847
3793	by H inversion	3848
3794		3849
3795		3850

3851	2. $\vdash_{\mathcal{H}} E_0[e]$	
3852	by \mathcal{H} inversion	
3853	3. QED by the induction hypothesis (2)	
3854	CASE $E = \langle v_0, E_1 \rangle :$	
3855	1. $E[e] = \langle v_0, E_1[e] \rangle$	
3856	2. $\vdash_{\mathcal{H}} E_1[e]$	
3857	by \mathcal{H} inversion	
3858	3. QED by the induction hypothesis (2)	
3859	CASE $E = op^1 E_0 :$	
3860	1. $E[e] = op^1 E_0[e]$	
3861	2. $\vdash_{\mathcal{H}} E_0[e]$	
3862	by \mathcal{H} inversion	
3863	3. QED by the induction hypothesis (2)	
3864	CASE $E = op^2 E_0 e_1 :$	
3865	1. $E[e] = op^2 E_0[e] e_1$	
3866	2. $\vdash_{\mathcal{H}} E_0[e]$	
3867	by \mathcal{H} inversion	
3868	3. QED by the induction hypothesis (2)	
3869	CASE $E = op^2 v_0 E_1 :$	
3870	1. $E[e] = op^2 v_0 E_1[e]$	
3871	2. $\vdash_{\mathcal{H}} E_1[e]$	
3872	by \mathcal{H} inversion	
3873	3. QED by the induction hypothesis (2)	
3874	CASE $E = dyn \tau'' E_0 :$	
3875	1. Contradiction by $\vdash_{\mathcal{H}} E[e]$	
3876	CASE $E = stat \tau_0 E_0 :$	
3877	1. $E[e] = stat \tau_0 E_0[e]$	
3878	2. $\vdash_{\mathcal{H}} E_0[e] : \tau_0$	
3879	by \mathcal{H} inversion	
3880	3. QED by <i>static context static hole substitution</i> (2)	
3881	□	
3882	Lemma 2.28 : \mathcal{H} dynamic context dynamic hole substitution	
3883	If $\vdash_{\mathcal{H}} E[e]$ and contains $\vdash_{\mathcal{H}} e$, and furthermore $\vdash_{\mathcal{H}} e'$, then	
3884	$\vdash_{\mathcal{H}} E[e']$	
3885	<i>Proof:</i>	
3886	By induction on the structure of E .	
3887	CASE $E \in E^\bullet :$	
3888	1. QED by <i>dynamic boundary-free hole substitution</i>	
3889	CASE $E = E_0 e_1 :$	
3890	1. $E[e] = E_0[e] e_1$	
3891	2. $\vdash_{\mathcal{H}} E_0[e]$	
3892	by \mathcal{H} inversion	
3893	3. QED by the induction hypothesis (2)	
3894	CASE $E = v_0 E_1 :$	
3895	1. $E[e] = v_0 E_1[e]$	
3896	2. $\vdash_{\mathcal{H}} E_1[e]$	
3897	by \mathcal{H} inversion	
3898	3. QED by the induction hypothesis (2)	
3899	CASE $E = \langle E_0, e_1 \rangle :$	
3900	1. $E[e] = \langle E_0[e], e_1 \rangle$	
3901	2. $\vdash_{\mathcal{H}} E_0[e]$	
3902	by \mathcal{H} inversion	
3903	3. QED by the induction hypothesis (2)	
3904		
3905		

3906	CASE $E = \langle v_0, E_1 \rangle :$	
3907	1. $E[e] = \langle v_0, E_1[e] \rangle$	
3908	2. $\vdash_{\mathcal{H}} E_1[e]$	
3909	by \mathcal{H} inversion	
3910	3. QED by the induction hypothesis (2)	
3911	CASE $E = op^1 E_0 :$	
3912	1. $E[e] = op^1 E_0[e]$	
3913	2. $\vdash_{\mathcal{H}} E_0[e]$	
3914	by \mathcal{H} inversion	
3915	3. QED by the induction hypothesis (2)	
3916	CASE $E = op^2 E_0 e_1 :$	
3917	1. $E[e] = op^2 E_0[e] e_1$	
3918	2. $\vdash_{\mathcal{H}} E_0[e]$	
3919	by \mathcal{H} inversion	
3920	3. QED by the induction hypothesis (2)	
3921	CASE $E = op^2 v_0 E_1 :$	
3922	1. $E[e] = op^2 v_0 E_1[e]$	
3923	2. $\vdash_{\mathcal{H}} E_1[e]$	
3924	by \mathcal{H} inversion	
3925	3. QED by the induction hypothesis (2)	
3926	CASE $E = dyn \tau'' E_0 :$	
3927	1. Contradiction by $\vdash_{\mathcal{H}} E[e]$	
3928	CASE $E = stat \tau_0 E_0 :$	
3929	1. $E[e] = stat \tau_0 E_0[e]$	
3930	2. $\vdash_{\mathcal{H}} E_0[e] : \tau_0$	
3931	by \mathcal{H} inversion	
3932	3. QED by <i>static context dynamic hole substitution</i> (2)	
3933	□	
3934	Lemma 2.29 : \mathcal{H} static context static hole substitution	
3935	If $\vdash_{\mathcal{H}} E[e] : \tau$ and contains $\vdash_{\mathcal{H}} e : \tau'$, and furthermore $\vdash_{\mathcal{H}} e' : \tau'$,	
3936	then $\vdash_{\mathcal{H}} E[e'] : \tau$	
3937	<i>Proof:</i>	
3938	By induction on the structure of E .	
3939	CASE $E \in E^\bullet :$	
3940	1. QED by <i>static boundary-free hole substitution</i>	
3941	CASE $E = E_0 e_1 :$	
3942	1. $E[e] = E_0[e] e_1$	
3943	2. $\vdash_{\mathcal{H}} E_0[e] : \tau_0$	
3944	by \mathcal{H} inversion	
3945	3. QED by the induction hypothesis (2)	
3946	CASE $E = v_0 E_1 :$	
3947	1. $E[e] = v_0 E_1[e]$	
3948	2. $\vdash_{\mathcal{H}} E_1[e] : \tau_1$	
3949	by \mathcal{H} inversion	
3950	3. QED by the induction hypothesis (2)	
3951	CASE $E = \langle E_0, e_1 \rangle :$	
3952	1. $E[e] = \langle E_0[e], e_1 \rangle$	
3953	2. $\vdash_{\mathcal{H}} E_0[e] : \tau_0$	
3954	by \mathcal{H} inversion	
3955	3. QED by the induction hypothesis (2)	
3956	CASE $E = \langle v_0, E_1 \rangle :$	
3957	1. $E[e] = \langle v_0, E_1[e] \rangle$	
3958		
3959		

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3961   2.  $\vdash_H E_1[e] : \tau_1$                                      4016
3962     by H inversion                                         4017
3963   3. QED by the induction hypothesis (2)                      4018
3964 CASE  $E = op^1 E_0 :$                                          4019
3965   1.  $E[e] = op^1 E_0[e]$                                          4020
3966   2.  $\vdash_H E_0[e] : \tau_0$                                          4021
3967     by H inversion                                         4022
3968   3. QED by the induction hypothesis (2)                      4023
3969 CASE  $E = op^2 E_0 e_1 :$                                          4024
3970   1.  $E[e] = op^2 E_0[e] e_1$                                          4025
3971   2.  $\vdash_H E_0[e] : \tau_0$                                          4026
3972     by H inversion                                         4027
3973   3. QED by the induction hypothesis (2)                      4028
3974 CASE  $E = op^2 v_0 E_1 :$                                          4029
3975   1.  $E[e] = op^2 v_0 E_1[e]$                                          4030
3976   2.  $\vdash_H E_1[e] : \tau_1$                                          4031
3977     by H inversion                                         4032
3978   3. QED by the induction hypothesis (2)                      4033
3979 CASE  $E = dyn \tau_0 E_0 :$                                          4034
3980   1.  $E[e] = dyn \tau_0 E_0[e]$                                          4035
3981   2.  $\vdash_H E_0[e]$                                          4036
3982     by H inversion                                         4037
3983   3. QED by static dyn hole typing (2)                      4038
3984 CASE  $E = stat \tau_0 E_0 :$                                          4039
3985   1. Contradiction by  $\vdash_H E[e] : \tau$                       4040
3986  $\square$                                                        4041
3987 Lemma 2.30 : H static context dynamic hole substitution 4042
3988 If  $\vdash_H E[e] : \tau$  and contains  $\vdash_H e$ , and furthermore  $\vdash_H e'$ , then 4043
3989  $\vdash_H E[e'] : \tau$                                          4044
3990 Proof:                                                 4045
3991 By induction on the structure of  $E$ .                           4046
3992 CASE  $E \in E^\bullet :$                                          4047
3993   1. Contradiction by  $\vdash_H E[e] : \tau$                       4048
3994 CASE  $E = E_0 e_1 :$                                          4049
3995   1.  $E[e] = E_0[e] e_1$                                          4050
3996   2.  $\vdash_H E_0[e] : \tau_0$                                          4051
3997     by H inversion                                         4052
3998   3. QED by the induction hypothesis (2)                      4053
3999 CASE  $E = v_0 E_1 :$                                          4054
4000   1.  $E[e] = v_0 E_1[e]$                                          4055
4001   2.  $\vdash_H E_1[e] : \tau_1$                                          4056
4002     by H inversion                                         4057
4003   3. QED by the induction hypothesis (2)                      4058
4004 CASE  $E = \langle E_0, e_1 \rangle :$                                          4059
4005   1.  $E[e] = \langle E_0[e], e_1 \rangle$                                          4060
4006   2.  $\vdash_H E_0[e] : \tau_0$                                          4061
4007     by H inversion                                         4062
4008   3. QED by the induction hypothesis (2)                      4063
4009 CASE  $E = \langle v_0, E_1 \rangle :$                                          4064
4010   1.  $E[e] = \langle v_0, E_1[e] \rangle$                                          4065
4011   2.  $\vdash_H E_1[e] : \tau_1$                                          4066
4012     by H inversion                                         4067
4013   3. QED by the induction hypothesis (2)                      4068
4014
4015

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CASE  $E = op^1 E_0 :$                                          4016
  1.  $E[e] = op^1 E_0[e]$                                          4017
  2.  $\vdash_H E_0[e] : \tau_0$                                          4018
    by H inversion                                         4019
  3. QED by the induction hypothesis (2)                      4020
CASE  $E = op^2 E_0 e_1 :$                                          4021
  1.  $E[e] = op^2 E_0[e] e_1$                                          4022
  2.  $\vdash_H E_0[e] : \tau_0$                                          4023
    by H inversion                                         4024
  3. QED by the induction hypothesis (2)                      4025
CASE  $E = op^2 v_0 E_1 :$                                          4026
  1.  $E[e] = op^2 v_0 E_1[e]$                                          4027
  2.  $\vdash_H E_1[e] : \tau_1$                                          4028
    by H inversion                                         4029
  3. QED by the induction hypothesis (2)                      4030
CASE  $E = dyn \tau_0 E_0 :$                                          4031
  1.  $E[e] = dyn \tau_0 E_0[e]$                                          4032
  2.  $\vdash_H E_0[e]$                                          4033
    by H inversion                                         4034
  3. QED by dynamic stat hole typing (2)                      4035
CASE  $E = stat \tau_0 E_0 :$                                          4036
  1. Contradiction by  $\vdash_H E[e] : \tau$                       4037
 $\square$                                                        4038
Lemma 2.31 : H static inversion                                         4039


- If  $\Gamma \vdash_H x : \tau$  then  $(x : \tau') \in \Gamma$  and  $\tau' \leqslant \tau$           4040
- If  $\Gamma \vdash_H \lambda(x : \tau_d). e' : \tau$  then  $(x : \tau'_d), \Gamma \vdash_H e' : \tau'_c$  and  $\tau'_d \Rightarrow \tau'_c \leqslant \tau$           4041
- If  $\Gamma \vdash_H \langle e_0, e_1 \rangle : \tau$  then  $\Gamma \vdash_H e_0 : \tau_0$  and  $\Gamma \vdash_H e_1 : \tau_1$  and  $\tau_0 \times \tau_1 \leqslant \tau$           4043
- If  $\Gamma \vdash_H e_0 e_1 : \tau_c$  then  $\Gamma \vdash_H e_0 : \tau'_d \Rightarrow \tau'_c$  and  $\Gamma \vdash_H e_1 : \tau'_d$  and  $\tau'_c \leqslant \tau_c$           4045
- If  $\Gamma \vdash_H \text{fst } e : \tau$  then  $\Gamma \vdash_H e : \tau_0 \times \tau_1$  and  $\Delta(\text{fst}, \tau_0 \times \tau_1) = \tau_0$  and  $\tau_0 \leqslant \tau$           4047
- If  $\Gamma \vdash_H \text{snd } e : \tau$  then  $\Gamma \vdash_H e : \tau_0 \times \tau_1$  and  $\Delta(\text{snd}, \tau_0 \times \tau_1) = \tau_1$  and  $\tau_1 \leqslant \tau$           4049
- If  $\Gamma \vdash_H op^2 e_0 e_1 : \tau$  then  $\Gamma \vdash_H e_0 : \tau_0$  and  $\Gamma \vdash_H e_1 : \tau_1$  and  $\Delta(op^2, \tau_0, \tau_1) = \tau'$  and  $\tau' \leqslant \tau$           4051
- If  $\Gamma \vdash_H \text{mon } \tau_d \Rightarrow \tau_c v' : \tau$  then  $\Gamma \vdash_H v'$  and  $\tau_d \Rightarrow \tau_c \leqslant \tau$           4053
- If  $\Gamma \vdash_H \text{dyn } \tau' e' : \tau$  then  $\Gamma \vdash_H e'$  and  $\tau' \leqslant \tau$           4054

Proof:                                                 4055
  QED by the definition of  $\Gamma \vdash_H e : \tau$                       4056
 $\square$                                                        4057
Lemma 2.32 : H dynamic inversion                                         4058


- If  $\Gamma \vdash_H x$  then  $x \in \Gamma$                                          4059
- If  $\Gamma \vdash_H \lambda x. e'$  then  $x, \Gamma \vdash_H e'$                                          4060
- If  $\Gamma \vdash_H \langle e_0, e_1 \rangle$  then  $\Gamma \vdash_H e_0$  and  $\Gamma \vdash_H e_1$                                          4061
- If  $\Gamma \vdash_H e_0 e_1$  then  $\Gamma \vdash_H e_0$  and  $\Gamma \vdash_H e_1$                                          4062
- If  $\Gamma \vdash_H op^1 e_0$  then  $\Gamma \vdash_H e_0$                                          4063
- If  $\Gamma \vdash_H op^2 e_0 e_1$  then  $\Gamma \vdash_H e_0$  and  $\Gamma \vdash_H e_1$                                          4064
- If  $\Gamma \vdash_H \text{mon } \tau_d \Rightarrow \tau_c v' : \tau$  then  $\Gamma \vdash_H v' : \tau_d \Rightarrow \tau_c$           4065
- If  $\Gamma \vdash_H \text{stat } \tau' e' : \tau$  then  $\Gamma \vdash_H e'$  and  $\tau' \leqslant \tau$           4066

Proof:                                                 4067
  QED by the definition of  $\Gamma \vdash_H e$                       4068
 $\square$                                                        4069

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4071 □
 4072 **Lemma 2.33 : H canonical forms**
 4073 • If $\vdash_H v : \tau_0 \times \tau_1$ then $v = \langle v_0, v_1 \rangle$
 4074 • If $\vdash_H v : \tau_d \Rightarrow \tau_c$ then either:
 4075 – $v = \lambda(x:\tau_x). e'$
 4076 $\wedge \tau_d \leqslant \tau_x$
 4077 – or $v = \text{mon}(\tau'_d \Rightarrow \tau'_c) v'$
 4078 $\wedge \tau'_d \Rightarrow \tau'_c \leqslant \tau_d \Rightarrow \tau_c$
 4079 • If $\vdash_H v : \text{Int}$ then $v \in i$
 4080 • If $\vdash_H v : \text{Nat}$ then $v \in \mathbb{N}$
 4081 *Proof:*
 4082 QED by definition of $\vdash_H e : \tau$
 4083 □
 4084 **Lemma 2.34 : Δ type soundness**
 4085 If $\vdash_H v_0 : \tau_0$ and $\vdash_H v_1 : \tau_1$ and $\Delta(\text{op}^2, \tau_0, \tau_1) = \tau$ then $\vdash_H \delta(\text{op}^2, v_0, v_1) : \tau$.
 4086
 4087 *Proof:*
 4088 By case analysis on the definition of Δ .
 4089 **CASE** $\Delta(\text{sum}, \text{Nat}, \text{Nat}) = \text{Nat}$:
 4090 1. $v_0 = i_0, i_0 \in \mathbb{N}$
 4091 $\wedge v_1 = i_1, i_1 \in \mathbb{N}$
 4092 by *canonical forms*
 4093 2. $\delta(\text{sum}, i_0, i_1) = i_0 + i_1 \in \mathbb{N}$
 4094 3. QED
 4095 **CASE** $\Delta(\text{sum}, \text{Int}, \text{Int}) = \text{Int}$:
 4096 1. $v_0 = i_0$
 4097 $\wedge v_1 = i_1$
 4098 by *canonical forms*
 4099 2. $\delta(\text{sum}, i_0, i_1) = i_0 + i_1 \in i$
 4100 3. QED
 4101 **CASE** $\Delta(\text{quotient}, \text{Nat}, \text{Nat}) = \text{Nat}$:
 4102 1. $v_0 = i_0, i_0 \in \mathbb{N}$
 4103 $\wedge v_1 = i_1, i_1 \in \mathbb{N}$
 4104 by *canonical forms*
 4105 2. **IF** $i_1 = 0$:
 4106 a. $\delta(\text{quotient}, i_0, i_1) = \text{BndryErr}$
 4107 b. QED by $\vdash_H \text{BndryErr} : \tau$
 4108 **ELSE** $i_1 \neq 0$:
 4109 a. $\delta(\text{quotient}, i_0, i_1) = \lfloor i_0/i_1 \rfloor \in \mathbb{N}$
 4110 b. QED
 4111 **CASE** $\Delta(\text{quotient}, \text{Int}, \text{Int}) = \text{Int}$:
 4112 1. $v_0 = i_0$
 4113 $\wedge v_1 = i_1$
 4114 by *canonical forms*
 4115 2. **IF** $i_1 = 0$:
 4116 a. $\delta(\text{quotient}, i_0, i_1) = \text{BndryErr}$
 4117 b. QED by $\vdash_H \text{BndryErr} : \tau$
 4118 **ELSE** $i_1 \neq 0$:
 4119 a. $\delta(\text{quotient}, i_0, i_1) = \lfloor i_0/i_1 \rfloor \in i$
 4120 b. QED
 4121 □
 4122 **Lemma 2.35 : δ preservation**
 4123
 4124

4126 • If $\vdash_H v$ and $\delta(\text{op}^1, v) = e$ then $\vdash_H e$
 4127 • If $\vdash_H v_0$ and $\vdash_H v_1$ and $\delta(\text{op}^2, v_0, v_1) = e$ then $\vdash_H e$
 4128
Proof:
 4129 **CASE** $\delta(\text{fst}, \langle v_0, v_1 \rangle) = v_0$:
 4130 1. $\vdash_H v_0$
 4131 by *H inversion*
 4132 2. QED
 4133 **CASE** $\delta(\text{snd}, \langle v_0, v_1 \rangle) = v_1$:
 4134 1. $\vdash_H v_1$
 4135 by *H inversion*
 4136 2. QED
 4137 **CASE** $\delta(\text{sum}, v_0, v_1) = v_0 + v_1$:
 4138 1. QED
 4139 **CASE** $\delta(\text{quotient}, v_0, v_1) = \lfloor v_0/v_1 \rfloor$:
 4140 1. QED
 4141 **CASE** $\delta(\text{op}^2, v_0, v_1) = \text{BndryErr}$:
 4142 1. QED
 4143
 4144 **Lemma 2.36 : H substitution**
 4145 • If $(x:\tau_x), \Gamma \vdash_H e$ and $\vdash_H v : \tau_x$ then $\Gamma \vdash_H e[x \leftarrow v]$
 4146 • If $x, \Gamma \vdash_H e$ and $\vdash_H v$ then $\Gamma \vdash_H e[x \leftarrow v]$
 4147 • If $(x:\tau_x), \Gamma \vdash_H e : \tau$ and $\vdash_H v : \tau_x$ then $\Gamma \vdash_H e[x \leftarrow v] : \tau$
 4148 • If $x, \Gamma \vdash_H e : \tau$ and $\vdash_H v$ then $\Gamma \vdash_H e[x \leftarrow v] : \tau$
 4149
Proof:
 4150 By the following four lemmas: *dynamic context static value substitution*, *dynamic context dynamic value substitution*, *static context static value substitution*, and *static context dynamic value substitution*.
 4151
 4152
 4153
 4154
 4155 **Lemma 2.37 : H dynamic-static substitution**
 4156 If $(x:\tau_x), \Gamma \vdash_H e$ and $\vdash_H v : \tau_x$ then $\Gamma \vdash_H e[x \leftarrow v]$
 4157
Proof:
 4158 By induction on the structure of e .
 4159 **CASE** $e = x$:
 4160 1. Contradiction by $(x:\tau_x), \Gamma \vdash_H e$
 4161 **CASE** $e = x'$:
 4162 1. QED by $(x'[x \leftarrow v]) = x'$
 4163 **CASE** $e = i$:
 4164 1. QED by $i[x \leftarrow v] = i$
 4165 **CASE** $e = \lambda x'. e'$:
 4166 1. $e[x \leftarrow v] = \lambda x'. (e'[x \leftarrow v])$
 4167 2. $x', (x:\tau_x), \Gamma \vdash_H e'$
 4168 by *H inversion*
 4169 3. $x', \Gamma \vdash_H e'[x \leftarrow v]$
 4170 by the induction hypothesis (2)
 4171 4. $\Gamma \vdash_H \lambda x'. (e'[x \leftarrow v])$
 4172 by (3)
 4173 5. QED
 4174 **CASE** $e = \lambda(x':\tau'). e'$:
 4175 1. Contradiction by $(x:\tau_x), \Gamma \vdash_H e$
 4176 **CASE** $e = \text{mon}(\tau_d \Rightarrow \tau_c) v'$:
 4177 1. $e[x \leftarrow v] = \text{mon}(\tau_d \Rightarrow \tau_c) v'[x \leftarrow v]$
 4178
 4179
 4180

4181	2. $(x:\tau_x), \Gamma \vdash_H v' : \tau_d \Rightarrow \tau_c$	4236
4182	by H inversion	4237
4183	3. $\Gamma \vdash_H v'[x \leftarrow v] : \tau_d \Rightarrow \tau_c$	4238
4184	by static context static value substitution (2)	4239
4185	4. $\Gamma \vdash_H \text{mon}(\tau_d \Rightarrow \tau_c) v'[x \leftarrow v]$	4240
4186	by (3)	4241
4187	5. QED	4242
4188	CASE $e = \langle e_0, e_1 \rangle :$	4243
4189	1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$	4244
4190	2. $(x:\tau_x), \Gamma \vdash_H e_0$	4245
4191	$\wedge (x:\tau_x), \Gamma \vdash_H e_1$	4246
4192	by H inversion	4247
4193	3. $\Gamma \vdash_H e_0[x \leftarrow v]$	4248
4194	$\wedge \Gamma \vdash_H e_1[x \leftarrow v]$	4249
4195	by the induction hypothesis (2)	4250
4196	4. $\Gamma \vdash_H \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$	4251
4197	by (3)	4252
4198	5. QED	4253
4199	CASE $e = e_0 e_1 :$	4254
4200	1. $e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]$	4255
4201	2. $(x:\tau_x), \Gamma \vdash_H e_0$	4256
4202	$\wedge (x:\tau_x), \Gamma \vdash_H e_1$	4257
4203	by H inversion	4258
4204	3. $\Gamma \vdash_H e_0[x \leftarrow v]$	4259
4205	$\wedge \Gamma \vdash_H e_1[x \leftarrow v]$	4260
4206	by the induction hypothesis (2)	4261
4207	4. $\Gamma \vdash_H e_0[x \leftarrow v] e_1[x \leftarrow v]$	4262
4208	by (3)	4263
4209	5. QED	4264
4210	CASE $e = op^1 e_0 :$	4265
4211	1. $e[x \leftarrow v] = op^1 e_0[x \leftarrow v]$	4266
4212	2. $(x:\tau_x), \Gamma \vdash_H e_0$	4267
4213	by H inversion	4268
4214	3. $\Gamma \vdash_H e_0[x \leftarrow v]$	4269
4215	by the induction hypothesis (2)	4270
4216	4. $\Gamma \vdash_H op^1 e_0[x \leftarrow v]$	4271
4217	by (3)	4272
4218	5. QED	4273
4219	CASE $e = op^2 e_0 e_1 :$	4274
4220	1. $e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$	4275
4221	2. $(x:\tau_x), \Gamma \vdash_H e_0$	4276
4222	$\wedge (x:\tau_x), \Gamma \vdash_H e_1$	4277
4223	by H inversion	4278
4224	3. $\Gamma \vdash_H e_0[x \leftarrow v]$	4279
4225	$\wedge \Gamma \vdash_H e_1[x \leftarrow v]$	4280
4226	by the induction hypothesis (2)	4281
4227	4. $\Gamma \vdash_H op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$	4282
4228	by (3)	4283
4229	5. QED	4284
4230	CASE $e = \text{stat } \tau' e' :$	4285
4231	1. $e[x \leftarrow v] = \text{stat } \tau' e'[x \leftarrow v]$	4286
4232	2. $(x:\tau_x), \Gamma \vdash_H e' : \tau'$	4287
4233	by H inversion	4288
4234	3. $\Gamma \vdash_H e'[x \leftarrow v]$	4289
4235	4. $\Gamma \vdash_H \text{stat } \tau' e'[x \leftarrow v]$	4290
	by (3)	
	5. QED	
	CASE $e = e_0 e_1 :$	
	1. $QED \quad \text{Err} = \text{Err}[x \leftarrow v]$	
	□	
	Lemma 2.38 : H dynamic-dynamic substitution	
	If $x, \Gamma \vdash_H e$ and $\vdash_H v$ then $\Gamma \vdash_H e[x \leftarrow v]$	
	<i>Proof:</i>	
	By induction on the structure of e	
	CASE $e = x :$	
	1. $e[x \leftarrow v] = v$	
	2. $\Gamma \vdash_H v$	
	by weakening	
	3. QED	
	CASE $e = x' :$	
	1. QED by $(x'[x \leftarrow v]) = x'$	
	CASE $e = i :$	
	1. QED by $i[x \leftarrow v] = i$	
	CASE $e = \lambda x'. e' :$	
	1. $e[x \leftarrow v] = \lambda x'. (e'[x \leftarrow v])$	
	2. $x', x, \Gamma \vdash_H e'$	
	by H inversion	
	3. $x', \Gamma \vdash_H e'[x \leftarrow v]$	
	by the induction hypothesis (2)	
	4. $\Gamma \vdash_H \lambda x'. (e'[x \leftarrow v])$	
	by (3)	
	5. QED	
	CASE $e = \lambda(x':\tau'). e' :$	
	1. Contradiction by $x, \Gamma \vdash_H e$	
	CASE $e = \text{mon } (\tau_d \Rightarrow \tau_c) v' :$	
	1. $e[x \leftarrow v] = \text{mon } (\tau_d \Rightarrow \tau_c) v'[x \leftarrow v]$	
	2. $x, \Gamma \vdash_H v' : \tau_d \Rightarrow \tau_c$	
	by H inversion	
	3. $\Gamma \vdash_H v'[x \leftarrow v] : \tau_d \Rightarrow \tau_c$	
	by static context dynamic value substitution (2)	
	4. $\Gamma \vdash_H \text{mon } (\tau_d \Rightarrow \tau_c) v'[x \leftarrow v]$	
	by (3)	
	5. QED	
	CASE $e = \langle e_0, e_1 \rangle :$	
	1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$	
	2. $x, \Gamma \vdash_H e_0$	
	$\wedge x, \Gamma \vdash_H e_1$	
	by H inversion	
	3. $\Gamma \vdash_H e_0[x \leftarrow v]$	
	$\wedge \Gamma \vdash_H e_1[x \leftarrow v]$	
	by the induction hypothesis (2)	
	4. $\Gamma \vdash_H \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$	
	by (3)	
	5. QED	
	CASE $e = e_0 e_1 :$	
	1. $QED \quad \text{Err} = \text{Err}[x \leftarrow v]$	

4291	1. $e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]$	4346
4292	2. $x, \Gamma \vdash_H e_0$	4347
4293	$\wedge x, \Gamma \vdash_H e_1$	4348
4294	by H inversion	4349
4295	3. $\Gamma \vdash_H e_0[x \leftarrow v]$	4350
4296	$\wedge \Gamma \vdash_H e_1[x \leftarrow v]$	4351
4297	by the induction hypothesis (2)	4352
4298	4. $\Gamma \vdash_H e_0[x \leftarrow v] e_1[x \leftarrow v]$	4353
4299	by (3)	4354
4300	5. QED	4355
4301	CASE $e = op^1 e_0$:	4356
4302	1. $e[x \leftarrow v] = op^1 e_0[x \leftarrow v]$	4357
4303	2. $x, \Gamma \vdash_H e_0$	4358
4304	by H inversion	4359
4305	3. $\Gamma \vdash_H e_0[x \leftarrow v]$	4360
4306	by the induction hypothesis (2)	4361
4307	4. $\Gamma \vdash_H op^1 e_0[x \leftarrow v]$	4362
4308	by (3)	4363
4309	5. QED	4364
4310	CASE $e = op^2 e_0 e_1$:	4365
4311	1. $e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$	4366
4312	2. $x, \Gamma \vdash_H e_0$	4367
4313	$\wedge x, \Gamma \vdash_H e_1$	4368
4314	by H inversion	4369
4315	3. $\Gamma \vdash_H e_0[x \leftarrow v]$	4370
4316	$\wedge \Gamma \vdash_H e_1[x \leftarrow v]$	4371
4317	by the induction hypothesis (2)	4372
4318	4. $\Gamma \vdash_H op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$	4373
4319	by (3)	4374
4320	5. QED	4375
4321	CASE $e = stat \tau' e'$:	4376
4322	1. $e[x \leftarrow v] = stat \tau' e'[x \leftarrow v]$	4377
4323	2. $x, \Gamma \vdash_H e' : \tau'$	4378
4324	by H inversion	4379
4325	3. $\Gamma \vdash_H e'[x \leftarrow v] : \tau'$	4380
4326	by static context static value substitution (2)	4381
4327	4. $\Gamma \vdash_H stat \tau' e'[x \leftarrow v]$	4382
4328	by (3)	4383
4329	5. QED	4384
4330	CASE $e = Err$:	4385
4331	1. QED $Err = Err[x \leftarrow v]$	4386
4332	□	4387
4333	Lemma 2.39 : H static-static substitution	4388
4334	If $(x:\tau_x), \Gamma \vdash_H e : \tau$ and $\vdash_H v : \tau_x$ then $\Gamma \vdash_H e[x \leftarrow v] : \tau$	4389
4335	<i>Proof:</i>	4390
4336	By induction on the structure of e .	4391
4337	CASE $e = x$:	4392
4338	1. $e[x \leftarrow v] = v$	4393
4339	2. $(x:\tau_x), \Gamma \vdash_H x : \tau$	4394
4340	3. $\tau_x \leqslant \tau$	4395
4341	by H inversion	4396
4342	4. $\vdash_H v : \tau$	4397
4343	by (3)	4398
4344		4399
4345		4400

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4401   4.  $\Gamma \vdash_{\mathcal{H}} op^1 e_0[x \leftarrow v] : \tau$  4456
4402     by (3) 4457
4403   5. QED 4458
4404 CASE  $e = op^2 e_0 e_1 :$  4459
4405   1.  $e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$  4460
4406   2.  $(x:\tau_x), \Gamma \vdash_{\mathcal{H}} e_0 : \tau_0$  4461
4407      $\wedge (x:\tau_x), \Gamma \vdash_{\mathcal{H}} e_1 : \tau_1$  4462
4408     by H inversion 4463
4409   3.  $\Gamma \vdash_{\mathcal{H}} e_0[x \leftarrow v] : \tau_0$  4464
4410      $\wedge \Gamma \vdash_{\mathcal{H}} e_1[x \leftarrow v] : \tau_1$  4465
4411     by the induction hypothesis (2) 4466
4412   4.  $\Gamma \vdash_{\mathcal{H}} op^2 e_0[x \leftarrow v] e_1[x \leftarrow v] : \tau$  4467
4413     by (3) 4468
4414   5. QED 4469
4415 CASE  $e = \text{dyn } \tau' e' :$  4470
4416   1.  $e[x \leftarrow v] = \text{dyn } \tau' e'[x \leftarrow v]$  4471
4417   2.  $(x:\tau_x), \Gamma \vdash_{\mathcal{H}} e'$  4472
4418     by H inversion 4473
4419   3.  $\Gamma \vdash_{\mathcal{H}} e'[x \leftarrow v]$  4474
4420     by dynamic context static value substitution (2) 4475
4421   4.  $\Gamma \vdash_{\mathcal{H}} \text{dyn } \tau' e'[x \leftarrow v] : \tau$  4476
4422     by (3) 4477
4423   5. QED 4478
4424 CASE  $e = \text{Err} :$  4479
4425   1. QED by  $\text{Err} = \text{Err}[x \leftarrow v]$  4480
4426  $\square$ 
4427 Lemma 2.40 :  $\mathcal{H}$  static-dynamic substitution 4481
4428 If  $x, \Gamma \vdash_{\mathcal{H}} e : \tau$  and  $\vdash_{\mathcal{H}} v$  then  $\Gamma \vdash_{\mathcal{H}} e[x \leftarrow v] : \tau$  4482
4429 Proof: 4483
4430 By induction on the structure of  $e$ . 4484
4431 CASE  $e = x :$  4485
4432   1. Contradiction by  $x, \Gamma \vdash_{\mathcal{H}} x : \tau$  4486
4433 CASE  $e = x' :$  4487
4434   1. QED by  $(x'[x \leftarrow v]) = x'$  4488
4435 CASE  $e = i :$  4489
4436   1. QED by  $i[x \leftarrow v] = i$  4490
4437 CASE  $e = \lambda x'. e' :$  4491
4438   1. Contradiction by  $(x:\tau_x), \Gamma \vdash_{\mathcal{H}} e : \tau$  4492
4439 CASE  $e = \lambda(x':\tau'). e' :$  4493
4440   1.  $e[x \leftarrow v] = \lambda(x':\tau'). (e'[x \leftarrow v])$  4494
4441   2.  $(x':\tau'), x, \Gamma \vdash_{\mathcal{H}} e' : \tau'_c$  4495
4442      $\wedge \tau' \Rightarrow \tau'_c \leqslant \tau$  4496
4443   3.  $(x':\tau'), \Gamma \vdash_{\mathcal{H}} e'[x \leftarrow v] : \tau'_c$  4497
4444     by the induction hypothesis (2) 4498
4445   4.  $\Gamma \vdash_{\mathcal{H}} \lambda(x':\tau'). e' : \tau$  4499
4446   5. QED 4500
4447 CASE  $e = \text{mon } (\tau_d \Rightarrow \tau_c) v' :$  4501
4448   1.  $e[x \leftarrow v] = \text{mon } (\tau_d \Rightarrow \tau_c) v'[x \leftarrow v]$  4502
4449   2.  $x, \Gamma \vdash_{\mathcal{H}} v'$  4503
4450     by H inversion 4504
4451   3.  $\Gamma \vdash_{\mathcal{H}} v'[x \leftarrow v]$  4505
4452     by dynamic context dynamic value substitution (2) 4506
4453   4.  $\Gamma \vdash_{\mathcal{H}} \text{dyn } \tau' e'[x \leftarrow v] : \tau$  4507
4454     by (3) 4508
4455   5. QED 4509

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4456   4.  $\Gamma \vdash_{\mathcal{H}} \text{mon } (\tau_d \Rightarrow \tau_c) v'[x \leftarrow v] : \tau$  4456
4457     by (3) 4457
4458   5. QED 4458
4459 CASE  $e = \langle e_0, e_1 \rangle :$  4459
4460   1.  $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$  4460
4461   2.  $x, \Gamma \vdash_{\mathcal{H}} e_0 : \tau_0$  4461
4462      $\wedge x, \Gamma \vdash_{\mathcal{H}} e_1 : \tau_1$  4462
4463     by H inversion 4463
4464   3.  $\Gamma \vdash_{\mathcal{H}} e_0[x \leftarrow v] : \tau_0$  4464
4465      $\wedge \Gamma \vdash_{\mathcal{H}} e_1[x \leftarrow v] : \tau_1$  4465
4466     by the induction hypothesis (2) 4466
4467   4.  $\Gamma \vdash_{\mathcal{H}} \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle : \tau$  4467
4468     by (3) 4468
4469   5. QED 4469
4470 CASE  $e = e_0 e_1 :$  4470
4471   1.  $e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]$  4471
4472   2.  $x, \Gamma \vdash_{\mathcal{H}} e_0 : \tau_0$  4472
4473      $\wedge x, \Gamma \vdash_{\mathcal{H}} e_1 : \tau_1$  4473
4474     by H inversion 4474
4475   3.  $\Gamma \vdash_{\mathcal{H}} e_0[x \leftarrow v] : \tau_0$  4475
4476      $\wedge \Gamma \vdash_{\mathcal{H}} e_1[x \leftarrow v] : \tau_1$  4476
4477     by the induction hypothesis (2) 4477
4478   4.  $\Gamma \vdash_{\mathcal{H}} e_0[x \leftarrow v] e_1[x \leftarrow v] : \tau$  4478
4479     by (3) 4479
4480   5. QED 4480
4481 CASE  $e = op^1 e_0 :$  4481
4482   1.  $e[x \leftarrow v] = op^1 e_0[x \leftarrow v]$  4482
4483   2.  $x, \Gamma \vdash_{\mathcal{H}} e_0 : \tau_0$  4483
4484     by H inversion 4484
4485   3.  $\Gamma \vdash_{\mathcal{H}} e_0[x \leftarrow v] : \tau_0$  4485
4486     by the induction hypothesis (2) 4486
4487   4.  $\Gamma \vdash_{\mathcal{H}} op^1 e_0[x \leftarrow v] : \tau$  4487
4488     by (3) 4488
4489   5. QED 4489
4490 CASE  $e = op^2 e_0 e_1 :$  4490
4491   1.  $e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$  4491
4492   2.  $x, \Gamma \vdash_{\mathcal{H}} e_0 : \tau_0$  4492
4493      $\wedge x, \Gamma \vdash_{\mathcal{H}} e_1 : \tau_1$  4493
4494     by H inversion 4494
4495   3.  $\Gamma \vdash_{\mathcal{H}} e_0[x \leftarrow v] : \tau_0$  4495
4496      $\wedge \Gamma \vdash_{\mathcal{H}} e_1[x \leftarrow v] : \tau_1$  4496
4497     by the induction hypothesis (2) 4497
4498   4.  $\Gamma \vdash_{\mathcal{H}} op^2 e_0[x \leftarrow v] e_1[x \leftarrow v] : \tau$  4498
4499     by (3) 4499
4500   5. QED 4500
4501 CASE  $e = \text{dyn } \tau' e' :$  4501
4502   1.  $e[x \leftarrow v] = \text{dyn } \tau' e'[x \leftarrow v]$  4502
4503   2.  $x, \Gamma \vdash_{\mathcal{H}} e'$  4503
4504     by H inversion 4504
4505   3.  $\Gamma \vdash_{\mathcal{H}} e'[x \leftarrow v]$  4505
4506     by dynamic context dynamic value substitution (2) 4506
4507   4.  $\Gamma \vdash_{\mathcal{H}} \text{dyn } \tau' e'[x \leftarrow v] : \tau$  4507
4508     by (3) 4508
4509   5. QED 4509

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4511	CASE $e = \text{Err} :$	4566
4512	1. QED by $\text{Err} = \text{Err}[x \leftarrow v]$	4567
4513	□	4568
4514	Lemma 2.41 : <i>weakening</i>	4569
4515	• If $\Gamma \vdash_H e$ then $x, \Gamma \vdash_H e$	4570
4516	• If $\Gamma \vdash_H e : \tau$ then $(x:\tau'), \Gamma \vdash_H e : \tau$	4571
4517	<i>Proof:</i>	4572
4518	• e is closed under Γ	4573
4519	by $\Gamma \vdash_H e$	4574
4520	$\vee \Gamma \vdash_H e : \tau$	4575
4521	• QED x is unused in the derivation	4576
4522	□	4577
4523		4578
4524		4579
4525		4580
4526		4581
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4546		4601
4547		4602
4548		4603
4549		4604
4550		4605
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4554		4609
4555		4610
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4621 **E.3 (E) Erasure Embedding**

4622 **E.3.1 Erasure Definitions**

4623 **Language E**

4625 $e = x \mid v \mid \langle e, e \rangle \mid e \cdot e \mid op^1 e \mid op^2 e \cdot e \mid$
 4626 dyn $\tau e \mid stat \tau e \mid Err$
 4627 $v = i \mid \langle v, v \rangle \mid \lambda x. e \mid \lambda(x:\tau). e$
 4628 $\tau = Nat \mid Int \mid \tau \times \tau \mid \tau \Rightarrow \tau$
 4629 $\Gamma = \cdot \mid x, \Gamma \mid (x:\tau), \Gamma$
 4630 $Err = BndryErr \mid TagErr$
 4631 $r = v \mid Err$
 4632 $E^\bullet = [] \mid E^\bullet e \mid v E^\bullet \mid \langle E^\bullet, e \rangle \mid \langle v, E^\bullet \rangle \mid$
 4633 $op^1 E^\bullet \mid op^2 E^\bullet e \mid op^2 v E^\bullet$
 4634 $E = E^\bullet \mid E e \mid v E \mid \langle E, e \rangle \mid \langle v, E \rangle \mid op^1 E \mid$
 4635 $op^2 E e \mid op^2 v E \mid dyn \tau E \mid stat \tau E$

4636 $\Delta : op^1 \times \tau \rightarrow \tau$

4637 $\Delta(fst, \tau_0 \times \tau_1) = \tau_0$
 4638 $\Delta(snd, \tau_0 \times \tau_1) = \tau_1$

4640 $\Delta : op^2 \times \tau \times \tau \rightarrow \tau$

4641 $\Delta(op^2, Nat, Nat) = Nat$
 4642 $\Delta(op^2, Int, Int) = Int$

4643 $\tau \leqslant: \tau$

4644 $\frac{}{Nat \leqslant: Int}$ $\frac{\tau'_d \leqslant: \tau_d \quad \tau_c \leqslant: \tau'_c \quad \tau_0 \leqslant: \tau'_0 \quad \tau_1 \leqslant: \tau'_1}{\tau_d \Rightarrow \tau_c \leqslant: \tau'_d \Rightarrow \tau'_c} \quad \frac{}{\tau_0 \times \tau_1 \leqslant: \tau'_0 \times \tau'_1}$
 4645 $\frac{}{\tau \leqslant: \tau} \quad \frac{\tau \leqslant: \tau' \quad \tau' \leqslant: \tau''}{\tau \leqslant: \tau''}$

4646 $\boxed{\Gamma \vdash e}$

4647 $\frac{x \in \Gamma}{\Gamma \vdash x} \quad \frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x. e} \quad \frac{}{\Gamma \vdash i} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}$

4648 $\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0 \cdot e_1} \quad \frac{\Gamma \vdash e}{\Gamma \vdash op^1 e} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2 e_0 \cdot e_1} \quad \frac{}{\Gamma \vdash Err}$

4649 $\frac{\Gamma \vdash e : \tau}{\Gamma \vdash stat \tau e}$

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4676 $\boxed{\Gamma \vdash e : \tau}$

4677 $\frac{(x:\tau) \in \Gamma}{\Gamma \vdash x : \tau} \quad \frac{(x:\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x:\tau_d). e : \tau_d \Rightarrow \tau_c} \quad \frac{i \in \mathbb{N}}{\Gamma \vdash i : Nat}$

4678 $\frac{}{\Gamma \vdash i : Int} \quad \frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1} \quad \frac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c}{\Gamma \vdash e_0 \cdot e_1 : \tau_c}$

4679 $\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e : \tau'}{\Gamma \vdash op^1 e_0 \cdot e_1 : \tau} \quad \frac{\Delta(op^1, \tau_0) = \tau \quad \Delta(op^2, \tau_0, \tau_1) = \tau \quad \tau' \leqslant: \tau}{\Gamma \vdash op^2 e_0 \cdot e_1 : \tau} \quad \frac{}{\Gamma \vdash e : \tau} \quad \frac{}{\Gamma \vdash Err : \tau}$

4680 $\frac{\Gamma \vdash e}{\Gamma \vdash dyn \tau e : \tau}$

4681 $\boxed{\Gamma \vdash_E e}$

4682 $\frac{x \in \Gamma}{\Gamma \vdash_E x} \quad \frac{(x:\tau) \in \Gamma}{\Gamma \vdash_E x} \quad \frac{x, \Gamma \vdash_E e}{\Gamma \vdash_E \lambda x. e} \quad \frac{(x:\tau), \Gamma \vdash_E e}{\Gamma \vdash_E \lambda(x:\tau). e} \quad \frac{}{\Gamma \vdash_E i}$

4683 $\frac{\Gamma \vdash_E e_0 \quad \Gamma \vdash_E e_1}{\Gamma \vdash_E \langle e_0, e_1 \rangle} \quad \frac{\Gamma \vdash_E e_0 \quad \Gamma \vdash_E e_1}{\Gamma \vdash_E e_0 \cdot e_1} \quad \frac{\Gamma \vdash_E e}{\Gamma \vdash_E op^1 e}$

4684 $\frac{\Gamma \vdash_E e_0 \quad \Gamma \vdash_E e_1}{\Gamma \vdash_E op^2 e_0 \cdot e_1} \quad \frac{\Gamma \vdash_E e}{\Gamma \vdash_E Err} \quad \frac{\Gamma \vdash_E e}{\Gamma \vdash_E dyn \tau e} \quad \frac{\Gamma \vdash_E e}{\Gamma \vdash_E stat \tau e}$

4685 $\boxed{\delta(op^1, v) = e}$

4686 $\delta(fst, \langle v_0, v_1 \rangle) = v_0$

4687 $\delta(snd, \langle v_0, v_1 \rangle) = v_1$

4688 $\boxed{\delta(op^2, v, v) = e}$

4689 $\delta(sum, i_0, i_1) = i_0 + i_1$

4690 $\delta(quotient, i_0, 0) = BndryErr$

4691 $\delta(quotient, i_0, i_1) = \lfloor i_0 / i_1 \rfloor$
 4692 if $i_1 \neq 0$

4693 $\boxed{\mathcal{D}_E : \tau \times v \rightarrow e}$

4694 $\mathcal{D}_E(\tau, v) = v$

4695 $\boxed{\mathcal{S}_E : \tau \times v \rightarrow e}$

4696 $\mathcal{S}_E(\tau, v) = v$

4731	$e \triangleright_{E-S} e$	4786
4732	$\text{dyn } \tau v \quad \triangleright_{E-S} \mathcal{D}_E(\tau, v)$	4787
4733	$\text{stat } \tau v \quad \triangleright_{E-S} \mathcal{S}_E(\tau, v)$	4788
4734	$v_0 v_1 \quad \triangleright_{E-S} \text{TagErr}$	4789
4735	if $v_0 \in \mathbb{Z}$ or $v_0 = \langle v, v' \rangle$	4790
4736	$(\lambda(x:\tau).e) v \triangleright_{E-S} e[x \leftarrow v]$	4791
4737	$(\lambda x. e) v \quad \triangleright_{E-S} e[x \leftarrow v]$	4792
4738	$op^1 v \quad \triangleright_{E-S} \text{TagErr}$	4793
4739	if $\delta(op^1, v)$ is undefined	4794
4740	$op^1 v \quad \triangleright_{E-S} \delta(op^1, v)$	4795
4741	$op^2 v_0 v_1 \quad \triangleright_{E-S} \text{TagErr}$	4796
4742	if $\delta(op^2, v_0, v_1)$ is undefined	4797
4743	$op^2 v_0 v_1 \quad \triangleright_{E-S} \delta(op^2, v_0, v_1)$	4798
4744	$e \triangleright_{E-D} e$	4799
4745	$\text{stat } \tau v \quad \triangleright_{E-D} \mathcal{S}_E(\tau, v)$	4800
4746	$\text{dyn } \tau v \quad \triangleright_{E-D} \mathcal{D}_E(\tau, v)$	4801
4747	$v_0 v_1 \quad \triangleright_{E-D} \text{TagErr}$	4802
4748	if $v_0 \in \mathbb{Z}$ or $v_0 = \langle v, v' \rangle$	4803
4749	$(\lambda(x:\tau).e) v \triangleright_{E-D} e[x \leftarrow v]$	4804
4750	$(\lambda x. e) v \quad \triangleright_{E-D} e[x \leftarrow v]$	4805
4751	$op^1 v \quad \triangleright_{E-D} \text{TagErr}$	4806
4752	if $\delta(op^1, v)$ is undefined	4807
4753	$op^1 v \quad \triangleright_{E-D} \delta(op^1, v)$	4808
4754	$op^2 v_0 v_1 \quad \triangleright_{E-D} \text{TagErr}$	4809
4755	if $\delta(op^2, v_0, v_1)$ is undefined	4810
4756	$op^2 v_0 v_1 \quad \triangleright_{E-D} \delta(op^2, v_0, v_1)$	4811
4757	$e \rightarrow_{E-S} e$	4812
4758	$E[e] \rightarrow_{E-S} E[e']$	4813
4759	if $e \triangleright_{E-S} e'$	4814
4760	$E[\text{Err}] \rightarrow_{E-S} \text{Err}$	4815
4761	$e \rightarrow_{E-D} e$	4816
4762	$E[e] \rightarrow_{E-D} E[e']$	4817
4763	if $e \triangleright_{E-D} e'$	4818
4764	$E[\text{Err}] \rightarrow_{E-D} \text{Err}$	4819
4765	$e \rightarrow_{E-S}^* e$ reflexive, transitive closure of \rightarrow_{E-S}	4820
4766		4821
4767		4822
4768	$e \rightarrow_{E-D}^* e$ reflexive, transitive closure of \rightarrow_{E-D}	4823
4769		4824
4770		4825
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4841	E.3.2 Erasure Theorems	4896
4842	Theorem 3.0 : static E-soundness	4897
4843	If $\vdash e : \tau$ then $\vdash_E e$ and one of the following holds:	4898
4844	• $e \rightarrow_{E-S}^* v$ and $\vdash_E v$	4899
4845	• $e \rightarrow_{E-S}^* \text{TagErr}$	4900
4846	• $e \rightarrow_{E-S}^* \text{BndryErr}$	4901
4847	• e diverges	4902
4848	<i>Proof:</i>	4903
4849	1. $\vdash_E e$	4904
4850	by <i>static subset</i>	4905
4851	2. QED by <i>E progress</i> and <i>E preservation</i>	4906
4852	□	4907
4853	Theorem 3.1 : dynamic E-soundness	4908
4854	If $\vdash e$ then $\vdash_E e$ and one of the following holds:	4909
4855	• $e \rightarrow_{E-D}^* v$ and $\vdash_E v$	4910
4856	• $e \rightarrow_{E-D}^* \text{TagErr}$	4911
4857	• $e \rightarrow_{E-D}^* \text{BndryErr}$	4912
4858	• e diverges	4913
4859	<i>Proof:</i>	4914
4860	1. $\rightarrow_{E-D}^* = \rightarrow_{E-S}^*$	4915
4861	by definition	4916
4862	2. QED by <i>static E soundness</i>	4917
4863	□	4918
4864	Remark 3.2 : E-compilation	4919
4865	The \rightarrow_{E-S}^* and \rightarrow_{E-D}^* relations are identical. In practice, uses	4920
4866	of \rightarrow_{E-S}^* may be replaced with \rightarrow_{E-D}^* .	4921
4867		4922
4868		4923
4869	Theorem 3.3 : boundary-free E-soundness	4924
4870	If $\vdash e : \tau$ and e is boundary-free then one of the following	4925
4871	holds:	4926
4872	• $e \rightarrow_{E-S}^* v$ and $\vdash v : \tau$	4927
4873	• $e \rightarrow_{E-S}^* \text{BndryErr}$	4928
4874	• e diverges	4929
4875	<i>Proof:</i>	4930
4876	QED by <i>boundary-free progress</i> and <i>boundary-free preservation</i> .	4931
4877	□	4932
4878		4933
4879		4934
4880		4935
4881		4936
4882		4937
4883		4938
4884		4939
4885		4940
4886		4941
4887		4942
4888		4943
4889		4944
4890		4945
4891		4946
4892		4947
4893		4948
4894		4949
4895		4950

4951 **E.3.3 Erasure Lemmas**4952 **Lemma 3.4 : \mathcal{D}_E soundness**4953 **| If $\vdash_E v$ then $\vdash_E \mathcal{D}_E(\tau, v)$.**4954 *Proof:*4955 **CASE** $\mathcal{D}_E(\tau, v) = v :$

4956 1. QED

4957 \square 4958 **Lemma 3.5 : \mathcal{S}_E soundness**4959 **| If $\vdash_E v$ then $\vdash_E \mathcal{S}_E(\tau, v)$.**4960 *Proof:*4961 **CASE** $\mathcal{S}_E(\tau, v) = v :$

4962 1. QED

4963 \square 4964 **Lemma 3.6 : static subset**4965 **| If $\Gamma \vdash e : \tau$ then $\Gamma \vdash_E e$.**4966 *Proof:*

4967 By structural induction on the typing relation.

4968 **CASE**
$$\boxed{(x:\tau) \in \Gamma} : \quad$$
4969 1. $(x:\tau) \in \Gamma$ 4970 2. $\Gamma \vdash_E x$
 by (1)

4971 3. QED

4972 **CASE**
$$\boxed{(x:\tau_d), \Gamma \vdash e : \tau_c} : \quad$$
4973 1. $(x:\tau_d), \Gamma \vdash_E e$
 by the induction hypothesis4974 2. $\Gamma \vdash_E \lambda(x:\tau_d). e : \tau_d \Rightarrow \tau_c$
 by (1)

4975 3. QED

4976 **CASE**
$$\boxed{i \in \mathbb{N}} : \quad$$

4977 1. QED

4978 **CASE**
$$\boxed{\Gamma \vdash_E i : \text{Int}} : \quad$$

4979 1. QED

4980 **CASE**
$$\boxed{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1} : \quad$$
4981 1. $\Gamma \vdash_E e_0$
 $\wedge \Gamma \vdash_E e_1$
 by the induction hypothesis4982 2. $\Gamma \vdash_E \langle e_0, e_1 \rangle : \tau_0 \times \tau_1$
 by (1)

4983 3. QED

4984 **CASE**
$$\boxed{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash e_1 : \tau_d} : \quad$$
4985 1. $\Gamma \vdash_E e_0$
 $\wedge \Gamma \vdash_E e_1$
 by the induction hypothesis4986 \square 5006 2. $\Gamma \vdash_E e_0 e_1$

5007 by (1)

5008 3. QED

5009 **CASE**
$$\boxed{\Gamma \vdash e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau} : \quad$$
5010 $\Gamma \vdash op^1 e_0 : \tau$ 5011 1. $\Gamma \vdash_E e_0$

5012 by the induction hypothesis

5013 2. $\Gamma \vdash_E op^1 e_0$

5014 by (1)

5015 3. QED

5016 **CASE**
$$\boxed{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau} : \quad$$
5017 $\Gamma \vdash op^2 e_0 e_1 : \tau$ 5018 1. $\Gamma \vdash_E e_0$ 5019 $\wedge \Gamma \vdash_E e_1$

5020 by the induction hypothesis

5021 2. $\Gamma \vdash_E op^2 e_0 e_1$

5022 by (1)

5023 3. QED

5024 **CASE**
$$\boxed{\Gamma \vdash e : \tau' \quad \tau' <: \tau} : \quad$$
5025 $\Gamma \vdash e : \tau$ 5026 1. $\Gamma \vdash_E e$

5027 by the induction hypothesis

5028 2. QED

5029 **CASE**
$$\boxed{} : \quad$$
5030 $\Gamma \vdash \text{Err} : \tau$

5031 1. QED

5032 **CASE**
$$\boxed{\Gamma \vdash e} : \quad$$
5033 $\Gamma \vdash \text{dyn } \tau e : \tau$ 5034 1. $\Gamma \vdash_E e$ 5035 by *dynamic subset*5036 2. $\Gamma \vdash_E \text{dyn } \tau e$

5037 by (1)

5038 3. QED

5039 \square 5040 **Lemma 3.7 : dynamic subset**5041 **| If $\Gamma \vdash e$ then $\Gamma \vdash_E e$.**5042 *Proof:*5043 By structural induction on the \vdash relation.5044 **CASE**
$$\boxed{x \in \Gamma} : \quad$$
5045 1. $x \in \Gamma$ 5046 2. $\Gamma \vdash_E x$

5047 by (1)

5048 3. QED

5049 **CASE**
$$\boxed{x, \Gamma \vdash e} : \quad$$
5050 $\Gamma \vdash \lambda x. e$ 5051 1. $x, \Gamma \vdash_E e$

5052 by the induction hypothesis

5053 2. QED

5054 3. QED

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5061   2.  $\Gamma \vdash_E \lambda x. e$ 
5062     by (1)
5063   3. QED
5064 CASE  $\boxed{\Gamma \vdash i}$  :
5065   1. QED
5066
5067 CASE  $\boxed{\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}}$  :
5068   1.  $\Gamma \vdash_E e_0$ 
5069      $\wedge \Gamma \vdash_E e_1$ 
5070     by the induction hypothesis
5071   2.  $\Gamma \vdash_E \langle e_0, e_1 \rangle$ 
5072     by (1)
5073   3. QED
5074
5075 CASE  $\boxed{\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0 \ e_1}}$  :
5076   1.  $\Gamma \vdash_E e_0$ 
5077      $\wedge \Gamma \vdash_E e_1$ 
5078     by the induction hypothesis
5079   2.  $\Gamma \vdash_E e_0 \ e_1$ 
5080     by (1)
5081   3. QED
5082
5083 CASE  $\boxed{\frac{\Gamma \vdash e}{\Gamma \vdash op^1 \ e}}$  :
5084   1.  $\Gamma \vdash_E e_0$ 
5085     by the induction hypothesis
5086   2.  $\Gamma \vdash_E op^1 \ e_0$ 
5087     by (1)
5088   3. QED
5089
5090 CASE  $\boxed{\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2 \ e_0 \ e_1}}$  :
5091   1.  $\Gamma \vdash_E e_0$ 
5092      $\wedge \Gamma \vdash_E e_1$ 
5093     by the induction hypothesis
5094   2.  $\Gamma \vdash_E op^2 \ e_0 \ e_1$ 
5095     by (1)
5096   3. QED
5097
5098 CASE  $\boxed{\Gamma \vdash Err}$  :
5099   1. QED
5100
5101 CASE  $\boxed{\frac{\Gamma \vdash e : \tau}{\Gamma \vdash stat \tau \ e}}$  :
5102   1.  $\Gamma \vdash_E e$ 
5103     by static subset
5104   2.  $\Gamma \vdash_E stat \tau \ e$ 
5105     by (1)
5106   3. QED
5107
5108  $\square$ 
5109
5110
5111
5112
5113 Lemma 3.8 : E progress
5114

```

If $\vdash_E e$ then one of the following holds:

- e is a value
- $e \in \text{Err}$
- $e \rightarrow_{E-S} e'$
- $e \rightarrow_{E-S} \text{TagErr}$
- $e \rightarrow_{E-S} \text{BndryErr}$

Proof: By the *unique evaluation contexts* lemma, there are seven possible cases.

CASE e is a value :

1. QED

CASE $e = E[v_0 \ v_1]$:

- IF** $v_0 = \lambda x. e'$:
- 1. $e \rightarrow_{E-S} E[e'[x \leftarrow v_1]]$
by $v_0 \ v_1 \triangleright_{E-S} e'[x \leftarrow v_1]$
- 2. QED

- IF** $v_0 = \lambda(x:\tau). e'$:
- 1. $e \rightarrow_{E-S} E[e'[x \leftarrow v_1]]$
by $v_0 \ v_1 \triangleright_{E-S} e'[x \leftarrow v_1]$
- 2. QED

- ELSE** $v_0 = i$
 $\vee v_0 = \langle v, v' \rangle$:
- 1. $e \rightarrow_{E-S} \text{TagErr}$
by $v_0 \ v_1 \triangleright_{E-S} \text{TagErr}$
- 2. QED

CASE $e = E[op^1 \ v]$:

- IF** $\delta(op^1, v) = e''$:
- 1. $e \rightarrow_{E-S} E[e'']$
by $(op^1 \ v) \triangleright_{E-S} e''$
- 2. QED

- ELSE** $\delta(op^1, v)$ is undefined :
- 1. $e \rightarrow_{E-S} \text{TagErr}$
by $(op^1 \ v) \triangleright_{E-S} \text{TagErr}$
- 2. QED

CASE $e = E[op^2 \ v_0 \ v_1]$:

- IF** $\delta(op^2, v_0, v_1) = e''$:
- 1. $e \rightarrow_{E-S} E[e'']$
by $(op^2 \ v_0 \ v_1) \triangleright_{E-S} e''$
- 2. QED

- IF** $\delta(op^2, v_0, v_1) = \text{BndryErr}$:
- 1. $e \rightarrow_{E-S} \text{BndryErr}$
by $(op^2 \ v_0 \ v_1) \triangleright_{E-S} \text{BndryErr}$
- 2. QED

- ELSE** $\delta(op^2, v_0, v_1)$ is undefined :
- 1. $e \rightarrow_{E-S} \text{TagErr}$
by $(op^2 \ v_0 \ v_1) \triangleright_{E-S} \text{TagErr}$
- 2. QED

CASE $e = E[dyn \ \tau \ v]$:

- 1. $e \rightarrow_{E-S} E[\mathcal{D}_E(\tau, v)]$
- 2. QED

CASE $e = E[stat \ \tau \ v]$:

- 1. $e \rightarrow_{E-S} E[\mathcal{S}_E(\tau, v)]$
- 2. QED

CASE $eE[\text{Err}]$:

5171	1. $e \rightarrow_{E-S} \text{Err}$	
5172	2. QED	
5173	□	
5174	Lemma 3.9 : E preservation	
5175	If $\vdash_E e$ and $e \rightarrow_{E-S} e'$ then $\vdash_E e'$.	
5176	<i>Proof:</i>	
5177	By <i>unique evaluation contexts</i> there are seven cases.	
5178	CASE e is a value :	
5179	1. Contradiction by $e \rightarrow_{E-S} e'$	
5180	CASE $e = E[v_0 v_1]$:	
5181	1. $v_0 = \lambda x. e'$ or $v_0 = \lambda(x:\tau). e'$	
5182	$\wedge E[v_0 v_1] \rightarrow_{E-S} E[e'[x \leftarrow v_1]]$	
5183	2. $\vdash_E v_0 v_1$	
5184	by <i>hole typing</i>	
5185	3. $\vdash_E v_0$	
5186	$\wedge \vdash_E v_1$	
5187	by <i>E inversion</i> (2)	
5188	4. $x \vdash_E e'$	
5189	by <i>E inversion</i> (3)	
5190	5. $\vdash_E e'[x \leftarrow v_1]$	
5191	by <i>substitution</i> (3, 4)	
5192	6. QED by <i>hole substitution</i> (5)	
5193	CASE $e = E[op^1 v]$:	
5194	1. $E[op^1 v] \rightarrow_{E-S} E[v']$	
5195	$\wedge \delta(op^1, v) = e''$	
5196	2. $\vdash_E op^1 v$	
5197	by <i>hole typing</i>	
5198	3. $\vdash_E v$	
5199	by <i>E inversion</i> (2)	
5200	4. $\vdash_E e''$	
5201	by <i>δ preservation</i> (1,3)	
5202	5. QED by <i>hole substitution</i> (4)	
5203	CASE $e = E[op^2 v_0 v_1]$:	
5204	1. $E[op^2 v_0 v_1] \rightarrow_{E-S} E[v']$	
5205	$\wedge \delta(op^2, v_0, v_1) = e''$	
5206	2. $\vdash_E op^2 v_0 v_1$	
5207	by <i>hole typing</i>	
5208	3. $\vdash_E v_0$	
5209	$\wedge \vdash_E v_1$	
5210	by <i>E inversion</i> (2)	
5211	4. $\vdash_E e''$	
5212	by <i>δ preservation</i> (3)	
5213	5. QED by <i>hole substitution</i> (4)	
5214	CASE $e = E[\text{dyn } \tau v]$:	
5215	1. $E[\text{dyn } \tau v] \rightarrow_{E-S} E[\mathcal{D}_E(\tau, v)]$	
5216	2. $\vdash_E \text{dyn } \tau v$	
5217	by <i>hole typing</i>	
5218	3. $\vdash_E v$	
5219	by <i>E inversion</i> (2)	
5220	4. $\vdash_E \mathcal{D}_E(\tau, v)$	
5221	by <i>D_E soundness</i> (3)	
5222	5. QED by <i>hole substitution</i> (4)	
5223	CASE $e = E[\text{stat } \tau v]$:	
5224		

5225	1. $E[\text{stat } \tau v] \rightarrow_{E-S} \mathcal{S}_E(\tau, v)$	5226
5226	2. $\vdash_E \text{stat } \tau v$	5227
5227	by <i>hole typing</i>	5228
5228	3. $\vdash_E v$	5229
5229	by <i>E inversion</i> (2)	5230
5230	4. $\vdash_E \mathcal{S}_E(\tau, v)$	5231
5231	by <i>S_E soundness</i> (3)	5232
5232	5. QED by <i>hole substitution</i> (4)	5233
5233	CASE $e = E[\text{Err}]$:	5234
5234	1. $E[\text{Err}] \rightarrow_{E-S} \text{Err}$	5235
5235	2. QED	5236
5236	□	5237
5237	Lemma 3.10 : E boundary-free progress	5238
5238	If $\vdash e : \tau$ and e is boundary-free, then one of the following holds:	5239
5239	• e is a value	5240
5240	• $e \in \text{Err}$	5241
5241	• $e \rightarrow_{E-S} e'$	5242
5242	• $e \rightarrow_{E-S} \text{BndryErr}$	5243
5243	<i>Proof:</i>	5244
5244	By the <i>unique static evaluation contexts</i> lemma, there are five cases:	5245
5245	CASE $e = v$:	5246
5246	1. QED	5247
5247	CASE $e = E[v_0 v_1]$:	5248
5248	IF $v_0 = \lambda(x:\tau'). e' :$	5249
5249	1. $e \rightarrow_{E-S} E[e'[x \leftarrow v_1]]$	5250
5250	by $v_0 v_1 \triangleright_{E-S} e'[x \leftarrow v_1]$	5251
5251	2. QED	5252
5252	ELSE $v_0 = \lambda x. e'$	5253
5253	$\vee v_0 = i$	5254
5254	$\vee v_0 = \langle v, v' \rangle :$	5255
5255	1. Contradiction by $\vdash e : \tau$	5256
5256	CASE $e = E[op^1 v]$:	5257
5257	IF $\delta(op^1, v) = e'' :$	5258
5258	1. $e \rightarrow_{E-S} E[e'']$	5259
5259	by $(op^1 v) \triangleright_{E-S} e''$	5260
5260	2. QED	5261
5261	ELSE $\delta(op^1, v)$ is undefined :	5262
5262	1. Contradiction by $\vdash e : \tau$	5263
5263	CASE $e = E[op^2 v_0 v_1]$:	5264
5264	IF $\delta(op^2, v_0, v_1) = e'' :$	5265
5265	1. $e \rightarrow_{E-S} E[e'']$	5266
5266	by $(op^2 v_0 v_1) \triangleright_{E-S} e''$	5267
5267	2. QED	5268
5268	IF $\delta(op^2, v_0, v_1) = \text{BndryErr} :$	5269
5269	1. $e \rightarrow_{E-S} \text{BndryErr}$	5270
5270	by $(op^2 v_0 v_1) \triangleright_{E-S} \text{BndryErr}$	5271
5271	2. QED	5272
5272	ELSE $\delta(op^2, v_0, v_1)$ is undefined :	5273
5273	1. Contradiction by $\vdash e : \tau$	5274
5274	CASE $e = E[\text{Err}]$:	5275
5275	1. $E[\text{Err}] \rightarrow_{E-S} \text{Err}$	5276
5276	2. QED	5277
5277		5278
5278		5279
5279		5280

5281 \square 5282 **Lemma 3.11 : E boundary-free preservation**5283 If $\vdash e : \tau$ and e is boundary-free and $e \rightarrow_{E-S} e'$ then $\vdash e' : \tau$
5284 and e' is boundary-free.5285 *Proof:*5286 By the *unique static evaluation contexts* lemma, there are
5287 five cases.5288 **CASE** e is a value :5289 1. Contradiction by $e \rightarrow_{E-S} e'$ 5290 **CASE** $e = E[v_0 v_1]$:5291 **IF** $v_0 = \lambda(x:\tau_d). e'$:5292 1. $E[v_0 v_1] \rightarrow_{E-S} E[e'[x \leftarrow v_1]]$ 5293 2. $\vdash v_0 v_1 : \tau_c$ 5294 3. $\vdash v_0 : \tau_d \Rightarrow \tau_c$ 5295 $\wedge \vdash v_1 : \tau_d$

5296 by (2)

5297 4. $(x:\tau_d) \vdash e' : \tau_c$

5298 by (3)

5299 5. $\vdash e'[x \leftarrow v_1] : \tau_c$ 5300 by *substitution* (3, 4)5301 6. $e'[x \leftarrow v_1]$ is boundary-free5302 by e' and v_1 are boundary-free

5303 7. QED

5304 **ELSE** :5305 1. Contradiction by $\vdash e : \tau$ 5306 **CASE** $e = E[op^1 v]$:5307 1. $E[op^1 v] \rightarrow_{E-S} E[v']$ 5308 $\wedge \delta(op^1, v) = e''$ 5309 2. $\vdash op^1 v : \tau'$ 5310 3. $\vdash v : \tau_0$ 5311 4. $\vdash e'' : \tau'$ 5312 by *δ preservation* (3)

5313 5. QED

5314 **CASE** $e = E[op^2 v_0 v_1]$:5315 1. $E[op^2 v_0 v_1] \rightarrow_{E-S} E[v']$ 5316 $\wedge \delta(op^2, v_0, v_1) = e''$ 5317 2. $\vdash op^2 v_0 v_1 : \tau'$ 5318 3. $\vdash v_0 : \tau_0$ 5319 $\wedge \vdash v_1 : \tau_1$ 5320 4. $\vdash e'' : \tau'$ 5321 by *δ preservation* (3)

5322 5. QED

5323 **CASE** $e = E[Err]$:5324 1. $E[Err] \rightarrow_{E-S} Err$ 5325 2. QED by $\vdash Err : \tau$ 5326 \square 5327 **Lemma 3.12 : E unique evaluation contexts**

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5336 If $\vdash_E e$ then one of the following holds:

- e is a value
- $e = E[v_0 v_1]$
- $e = E[op^1 v]$
- $e = E[op^2 v_0 v_1]$
- $e = E[dyn \tau v]$
- $e = E[stat \tau v]$
- $e = E[Err]$

5337 *Proof:*5338 By induction on the structure of e .5339 **CASE** $e = x$:5340 1. Contradiction by $\vdash_E e$ 5341 **CASE** $e = i$ 5342 $\vee e = \lambda x. e'$ 5343 $\vee e = \lambda(x:\tau_d). e' :$

5344 1. QED

5345 **CASE** $e = \langle e_0, e_1 \rangle$:5346 **IF** $e_0 \notin v$:5347 1. $\vdash_E e_0$
by *E inversion*5348 2. $e_0 = E_0[e'_0]$

5349 by the induction hypothesis

5350 3. $E = \langle E_0, e_1 \rangle$ 5351 4. QED $e = E[e'_0]$ 5352 **IF** $e_0 \in v$:5353 $\wedge e_1 \notin v$:5354 1. $\vdash_E e_1$
by *E inversion*5355 2. $e_1 = E_1[e'_1]$

5356 by the induction hypothesis

5357 3. $E = \langle e_0, E_1 \rangle$ 5358 4. QED $e = E[e'_1]$ 5359 **ELSE** $e_0 \in v$ 5360 $\wedge e_1 \in v$:

5361 1. QED

5362 **CASE** $e = e_0 e_1$:5363 **IF** $e_0 \notin v$:5364 1. $\vdash_E e_0$
by *E inversion*5365 2. $e_0 = E_0[e'_0]$

5366 by the induction hypothesis

5367 3. $E = E_0 e_1$ 5368 4. QED $e = E[e'_0]$ 5369 **IF** $e_0 \in v$:5370 $\wedge e_1 \notin v$:5371 1. $\vdash_E e_1$
by *E inversion*5372 2. $e_1 = E_1[e'_1]$

5373 by the induction hypothesis

5374 3. $E = e_0 E_1$ 5375 4. QED $e = E[e'_1]$ 5376 **ELSE** $e_0 \in v$ 5377 $\wedge e_1 \in v$:5378 1. $E = []$

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5391      2. QED
5392  CASE  $e = op^1 e_0 :$ 
5393    IF  $e_0 \notin v :$ 
5394      1.  $\vdash_E e_0$ 
5395        by E inversion
5396      2.  $e_0 = E_0[e'_0]$ 
5397        by the induction hypothesis
5398      3.  $E = op^1 E_0$ 
5399      4. QED  $e = E[e'_0]$ 
5400  ELSE  $e_0 \in v :$ 
5401    1.  $E = []$ 
5402    2. QED
5403  CASE  $e = op^2 e_0 e_1 :$ 
5404    IF  $e_0 \notin v :$ 
5405      1.  $\vdash_E e_0$ 
5406        by E inversion
5407      2.  $e_0 = E_0[e'_0]$ 
5408        by the induction hypothesis
5409      3.  $E = op^2 E_0 e_1$ 
5410      4. QED  $e = E[e'_0]$ 
5411    IF  $e_0 \in v$ 
5412       $\wedge e_1 \notin v :$ 
5413      1.  $\vdash_E e_1$ 
5414        by E inversion
5415      2.  $e_1 = E_1[e'_1]$ 
5416        by the induction hypothesis
5417      3.  $E = op^2 e_0 E_1$ 
5418      4. QED  $e = E[e'_1]$ 
5419  ELSE  $e_0 \in v$ 
5420     $\wedge e_1 \in v :$ 
5421    1.  $E = []$ 
5422    2. QED
5423  CASE  $e = \text{dyn } \tau e_0 :$ 
5424    IF  $e_0 \notin v :$ 
5425      1.  $\vdash_E e_0$ 
5426        by E inversion
5427      2.  $e_0 = E_0[e'_0]$ 
5428        by the induction hypothesis
5429      3.  $E = \text{dyn } \tau E_0$ 
5430      4. QED  $e = E[e'_0]$ 
5431    ELSE  $e_0 \in v :$ 
5432    1.  $E = []$ 
5433    2. QED
5434  CASE  $e = \text{stat } \tau e_0 :$ 
5435    IF  $e_0 \notin v :$ 
5436      1.  $\vdash_E e_0$ 
5437        by E inversion
5438      2.  $e_0 = E_0[e'_0]$ 
5439        by the induction hypothesis
5440      3.  $E = \text{stat } \tau E_0$ 
5441      4. QED  $e = E[e'_0]$ 
5442    ELSE  $e_0 \in v :$ 
5443    1.  $E = []$ 
5444    2. QED

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5446  CASE  $e = \text{Err} :$ 
5447    1.  $E = []$ 
5448    2. QED
5449  $\square$ 
5450  Lemma 3.13 : E hole typing
5451  If  $\vdash_E E[e]$  then the derivation contains a sub-term  $\vdash_E e$ 
5452  Proof:
5453  By induction on the structure of  $E$ .
5454  CASE  $E = [] :$ 
5455    1. QED  $E[e] = e$ 
5456  CASE  $E = E_0 e_1 :$ 
5457    1.  $E[e] = E_0[e] e_1$ 
5458    2.  $\vdash_E E_0[e]$ 
5459      by E inversion
5460    3. QED by the induction hypothesis (2)
5461  CASE  $E = v_0 E_1 :$ 
5462    1.  $E[e] = v_0 E_1[e]$ 
5463    2.  $\vdash_E E_1[e]$ 
5464      by E inversion
5465    3. QED by the induction hypothesis (2)
5466  CASE  $E = \langle E_0, e_1 \rangle :$ 
5467    1.  $E[e] = \langle E_0[e], e_1 \rangle$ 
5468    2.  $\vdash_E E_0[e]$ 
5469      by E inversion
5470    3. QED by the induction hypothesis (2)
5471  CASE  $E = \langle v_0, E_1 \rangle :$ 
5472    1.  $E[e] = \langle v_0, E_1[e] \rangle$ 
5473    2.  $\vdash_E E_1[e]$ 
5474      by E inversion
5475    3. QED by the induction hypothesis (2)
5476  CASE  $E = op^1 E_0 :$ 
5477    1.  $E[e] = op^1 E_0[e]$ 
5478    2.  $\vdash_E E_0[e]$ 
5479      by E inversion
5480    3. QED by the induction hypothesis (2)
5481  CASE  $E = op^2 E_0 e_1 :$ 
5482    1.  $E[e] = op^2 E_0[e] e_1$ 
5483    2.  $\vdash_E E_0[e]$ 
5484      by E inversion
5485    3. QED by the induction hypothesis (2)
5486  CASE  $E = op^2 v_0 E_1 :$ 
5487    1.  $E[e] = op^2 v_0 E_1[e]$ 
5488    2.  $\vdash_E E_1[e]$ 
5489      by E inversion
5490    3. QED by the induction hypothesis (2)
5491  CASE  $E = \text{dyn } \tau E_0 :$ 
5492    1.  $E[e] = \text{dyn } \tau E_0[e]$ 
5493    2.  $\vdash_E E_0[e]$ 
5494      by E inversion
5495    3. QED by the induction hypothesis (2)
5496  CASE  $E = \text{stat } \tau E_0 :$ 
5497    1.  $E[e] = \text{stat } \tau E_0[e]$ 
5498    2.  $\vdash_E E_0[e]$ 
5499      by E inversion
5500    3. QED by the induction hypothesis (2)

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5501 2. $\vdash_E E_0[e]$
 5502 by *E inversion*
 5503 3. QED by the induction hypothesis (2)
 5504 □
 5505 **Lemma 3.14 : E hole substitution**
 5506 | If $\vdash_E E[e]$ and $\vdash_E e'$ then $\vdash_E E[e']$
 5507 | *Proof:*
 5508 | By induction on the structure of E .
 5509 | **CASE** $E = []$:
 5510 | 1. QED $E[e'] = e'$
 5511 | **CASE** $E = \langle E_0, e_1 \rangle$:
 5512 | 1. $E[e] = \langle E_0[e], e_1 \rangle$
 5513 | $\wedge E[e'] = \langle E_0[e'], e_1 \rangle$
 5514 | 2. $\vdash_E \langle E_0[e], e_1 \rangle$
 5515 | 3. $\vdash_E E_0[e]$
 5516 | $\wedge \vdash_E e_1$
 5517 | by *E inversion*
 5518 | 4. $\vdash_E E_0[e']$
 5519 | by the induction hypothesis (3)
 5520 | 5. $\vdash_E \langle E_0[e'], e_1 \rangle$
 5521 | by (3, 4)
 5522 | 6. QED by (1, 5)
 5523 | **CASE** $E = \langle v_0, E_1 \rangle$:
 5524 | 1. $E[e] = \langle v_0, E_1[e] \rangle$
 5525 | $\wedge E[e'] = \langle v_0, E_1[e'] \rangle$
 5526 | 2. $\vdash_E \langle v_0, E_1[e] \rangle$
 5527 | 3. $\vdash_E v_0$
 5528 | $\wedge \vdash_E E_1[e]$
 5529 | by *E inversion*
 5530 | 4. $\vdash_E E_1[e']$
 5531 | by the induction hypothesis (3)
 5532 | 5. $\vdash_E \langle v_0, E_1[e'] \rangle$
 5533 | by (3, 4)
 5534 | 6. QED by (1, 5)
 5535 | **CASE** $E = E_0 e_1$:
 5536 | 1. $E[e] = E_0[e] e_1$
 5537 | $\wedge E[e'] = E_0[e'] e_1$
 5538 | 2. $\vdash_E E_0[e] e_1$
 5539 | 3. $\vdash_E E_0[e]$
 5540 | $\wedge \vdash_E e_1$
 5541 | by *E inversion*
 5542 | 4. $\vdash_E E_0[e']$
 5543 | by the induction hypothesis (3)
 5544 | 5. $\vdash_E E_0[e'] e_1$
 5545 | by (3, 4)
 5546 | 6. QED by (1, 5)
 5547 | **CASE** $E = v_0 E_1$:
 5548 | 1. $E[e] = v_0 E_1[e]$
 5549 | $\wedge E[e'] = v_0 E_1[e']$
 5550 | 2. $\vdash_E v_0 E_1[e]$
 5551 | 3. $\vdash_E v_0$
 5552 | $\wedge \vdash_E E_1[e]$
 5553 | by *E inversion*
 5554

5555 | 4. $\vdash_E E_1[e']$
 5556 | by the induction hypothesis (3)
 5557 | 5. $\vdash_E v_0 E_1[e']$
 5558 | by (3, 4)
 5559 | 6. QED by (1, 5)
 5560 | **CASE** $E = op^1 E_0$:
 5561 | 1. $E[e] = op^1 E_0[e]$
 5562 | $\wedge E[e'] = op^1 E_0[e']$
 5563 | 2. $\vdash_E op^1 E_0[e]$
 5564 | 3. $\vdash_E E_0[e]$
 5565 | by *E inversion*
 5566 | 4. $\vdash_E E_0[e']$
 5567 | by the induction hypothesis (3)
 5568 | 5. $\vdash_E op^1 E_0[e']$
 5569 | by (3, 4)
 5570 | 6. QED by (1, 5)
 5571 | **CASE** $E = op^2 E_0 e_1$:
 5572 | 1. $E[e] = op^2 E_0[e] e_1$
 5573 | $\wedge E[e'] = op^2 E_0[e'] e_1$
 5574 | 2. $\vdash_E op^2 E_0[e] e_1$
 5575 | 3. $\vdash_E E_0[e]$
 5576 | $\wedge \vdash_E e_1$
 5577 | by *E inversion*
 5578 | 4. $\vdash_E E_0[e']$
 5579 | by the induction hypothesis (3)
 5580 | 5. $\vdash_E op^2 E_0[e'] e_1$
 5581 | by (3, 4)
 5582 | 6. QED by (1, 5)
 5583 | **CASE** $E = op^2 v_0 E_1$:
 5584 | 1. $E[e] = op^2 v_0 E_1[e]$
 5585 | $\wedge E[e'] = op^2 v_0 E_1[e']$
 5586 | 2. $\vdash_E op^2 v_0 E_1[e]$
 5587 | 3. $\vdash_E v_0$
 5588 | $\wedge \vdash_E E_1[e]$
 5589 | by *E inversion*
 5590 | 4. $\vdash_E E_1[e']$
 5591 | by the induction hypothesis (3)
 5592 | 5. $\vdash_E op^2 v_0 E_1[e']$
 5593 | by (3, 4)
 5594 | 6. QED by (1, 5)
 5595 | **CASE** $E = dyn \tau E_0$:
 5596 | 1. $E[e] = dyn \tau E_0[e]$
 5597 | $\wedge E[e'] = dyn \tau E_0[e']$
 5598 | 2. $\vdash_E dyn \tau E_0[e]$
 5599 | 3. $\vdash_E E_0[e]$
 5600 | by *E inversion*
 5601 | 4. $\vdash_E E_0[e']$
 5602 | by the induction hypothesis (3)
 5603 | 5. $\vdash_E dyn \tau E_0[e']$
 5604 | by (3, 4)
 5605 | 6. QED by (1, 5)
 5606 | **CASE** $E = stat \tau E_0$:
 5607 | 1. $E[e] = stat \tau E_0[e]$
 5608 | $\wedge E[e'] = stat \tau E_0[e']$
 5609 | 6. QED by (1, 5)

- 5611 2. $\vdash_E \text{stat } \tau E_0[e]$
 5612 3. $\vdash_E E_0[e]$
 5613 by *E inversion*
 5614 4. $\vdash_E E_0[e']$
 5615 by the induction hypothesis (3)
 5616 5. $\vdash_E \text{stat } \tau E_0[e']$
 5617 by (3, 4)
 5618 6. QED by (1, 5)

 \square **Lemma 3.15 : \vdash_E inversion**

- If $\Gamma \vdash_E e_0 e_1$ then $\Gamma \vdash_E e_0$ and $\Gamma \vdash_E e_1$
- If $\Gamma \vdash_E \lambda x. e$ then $x, \Gamma \vdash_E e$
- If $\Gamma \vdash_E \lambda(x:\tau). e$ then $(x:\tau), \Gamma \vdash_E e$
- If $\Gamma \vdash_E op^1 e$ then $\Gamma \vdash_E e$
- If $\Gamma \vdash_E op^2 e_0 e_1$ then $\Gamma \vdash_E e_0$ and $\Gamma \vdash_E e_1$
- If $\Gamma \vdash_E \text{dyn } \tau e$ then $\Gamma \vdash_E e$
- If $\Gamma \vdash_E \text{stat } \tau e$ then $\Gamma \vdash_E e$

*Proof:*QED by the definition of $\vdash_E e$. \square **Lemma 3.16 : \vdash_E substitution**

- If $x, \Gamma \vdash_E e$ or $(x:\tau), \Gamma \vdash_E e$, and $\vdash_E v$ then $\Gamma \vdash_E e[x \leftarrow v]$

*Proof:*By induction on the structure of e .**CASE $e = x$:**

1. $e[x \leftarrow v] = v$
2. $\Gamma \vdash_E v$
by *weakening*
3. QED

CASE $e = x'$:

1. QED by $x'[x \leftarrow v] = x'$

CASE $e = i$:

1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x. e'$:

1. QED by $(\lambda x. e')[x \leftarrow v] = \lambda x. e'$

CASE $e = \lambda(x:\tau'). e'$:

1. QED by $(\lambda(x:\tau'). e')[x \leftarrow v] = \lambda(x:\tau'). e'$

CASE $e = \lambda x'. e'$:

1. $e[x \leftarrow v] = \lambda x'. (e'[x \leftarrow v])$

CASE $e = \lambda x'. e'$:

2. $x', x, \Gamma \vdash_E e'$
by *E inversion*
3. $x', \Gamma \vdash_E e'[x \leftarrow v]$
by the induction hypothesis (2)

CASE $e = \lambda x'. e'[x \leftarrow v]$:

1. by (3)

CASE $e = \lambda(x':\tau'). e'$:

1. $e[x \leftarrow v] = \lambda(x':\tau'). (e'[x \leftarrow v])$
2. $(x':\tau'), x, \Gamma \vdash_E e'$
by *E inversion*

CASE $e = \lambda(x':\tau'). e'[x \leftarrow v]$:

1. by the induction hypothesis (2)

 \square

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4. $\Gamma \vdash_E \lambda(x':\tau'). (e'[x \leftarrow v])$

5666

by (3)

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5. QED

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CASE $e = \langle e_0, e_1 \rangle$:

5669

1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$

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2. $x, \Gamma \vdash_E e_0$

5671

 $\wedge x, \Gamma \vdash_E e_1$

5672

by *E inversion*

5673

3. $\Gamma \vdash_E e_0[x \leftarrow v]$

5674

 $\wedge \Gamma \vdash_E e_1[x \leftarrow v]$

5675

by the induction hypothesis (2)

5676

4. $\Gamma \vdash_E \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$

5677

by (3)

5678

5. QED

5679

CASE $e = e_0 e_1$:

5680

1. $e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]$

5681

2. $x, \Gamma \vdash_E e_0$

5682

 $\wedge x, \Gamma \vdash_E e_1$

5683

by *E inversion*

5684

3. $\Gamma \vdash_E e_0[x \leftarrow v]$

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 $\wedge \Gamma \vdash_E e_1[x \leftarrow v]$

5686

by the induction hypothesis (2)

5687

4. $\Gamma \vdash_E e_0[x \leftarrow v] e_1[x \leftarrow v]$

5688

by (3)

5689

5. QED

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CASE $e = op^1 e_0$:

5691

1. $e[x \leftarrow v] = op^1 e_0[x \leftarrow v]$

5692

2. $x, \Gamma \vdash_E e_0$

5693

by *E inversion*

5694

3. $\Gamma \vdash_E e_0[x \leftarrow v]$

5695

by the induction hypothesis (2)

5696

4. $\Gamma \vdash_E op^1 e_0[x \leftarrow v]$

5697

by (3)

5698

5. QED

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CASE $e = op^2 e_0 e_1$:

5700

1. $e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$

5701

2. $x, \Gamma \vdash_E e_0$

5702

 $\wedge x, \Gamma \vdash_E e_1$

5703

by *E inversion*

5704

3. $\Gamma \vdash_E e_0[x \leftarrow v]$

5705

 $\wedge \Gamma \vdash_E e_1[x \leftarrow v]$

5706

by the induction hypothesis (2)

5707

4. $\Gamma \vdash_E op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$

5708

by (3)

5709

5. QED

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CASE $e = \text{dyn } \tau' e'$:

5711

1. $e[x \leftarrow v] = \text{dyn } \tau' e'[x \leftarrow v]$

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2. $x, \Gamma \vdash_E e'$

5713

by *E inversion*

5714

3. $\Gamma \vdash_E e'[x \leftarrow v]$

5715

by the induction hypothesis (2)

5716

4. $\Gamma \vdash_E \text{dyn } \tau' (e'[x \leftarrow v])$

5717

by (3)

5718

5. QED

5719

5720

5721 **CASE** $e = \text{stat } \tau' e'$:
 5722 1. $e[x \leftarrow v] = \text{stat } \tau' e'[x \leftarrow v]$
 5723 2. $x, \Gamma \vdash_E e'$
 by *E inversion*
 5724 3. $\Gamma \vdash_E e'[x \leftarrow v]$
 by the induction hypothesis (2)
 5725 4. $\Gamma \vdash_E \text{stat } \tau' (e'[x \leftarrow v])$
 by (3)
 5726 5. QED
 5727 **CASE** $e = \text{Err}$:
 5728 1. QED by $\text{Err}[x \leftarrow v] = \text{Err}$
 5729 □
 5730 **Lemma 3.17 : δ preservation**
 5731 • If $\vdash_E v$ and $\delta(op^1, v) = e'$ then $\vdash_E e'$
 5732 • If $\vdash_E v_0$ and $\vdash_E v_1$ and $\delta(op^2, v_0, v_1) = e'$ then $\vdash_E v'$
 5733 *Proof:*
 5734 **CASE** $\delta(\text{fst}, \langle v_0, v_1 \rangle) = v_0$:
 5735 1. $\vdash_E v_0$
 by *E inversion*
 5736 2. QED
 5737 **CASE** $\delta(\text{snd}, \langle v_0, v_1 \rangle) = v_1$:
 5738 1. $\vdash_E v_1$
 by *E inversion*
 5739 2. QED
 5740 **CASE** $\delta(\text{sum}, v_0, v_1) = v_0 + v_1$:
 5741 1. QED
 5742 **CASE** $\delta(\text{quotient}, v_0, v_1) = \lfloor v_0/v_1 \rfloor$:
 5743 1. QED
 5744 **CASE** $\delta(\text{quotient}, v_0, v_1) = \text{BndryErr}$:
 5745 1. QED
 5746 □
 5747 **Lemma 3.18 : weakening**
 5748 • If $\Gamma \vdash_E e$ then $x, \Gamma \vdash_E e$
 5749 • If $\Gamma \vdash_E e$ then $(x:\tau), \Gamma \vdash_E e$
 5750 *Proof:*
 5751 QED because e is closed under Γ
 5752 □
 5753 **Lemma 3.19 : unique static evaluation contexts**
 5754 If $\vdash e : \tau$ then one of the following holds:
 5755 • e is a value
 5756 • $e = E[v_0 v_1]$
 5757 • $e = E[op^1 v]$
 5758 • $e = E[op^2 v_0 v_1]$
 5759 • $e = E[\text{Err}]$
 5760 *Proof:*
 5761 By induction on the structure of e .
 5762 **CASE** $e = x$:
 5763 1. Contradiction by $\vdash e : \tau$
 5764 **CASE** $e = i$:
 5765 $\vee e = \lambda(x:\tau_d). e'$:
 5766 1. QED e is a value
 5767 **CASE** $e = \text{stat } \tau e'$:
 5768 1. Contradiction by $\vdash_1 e : K$

5776 **CASE** $e = \langle e_0, e_1 \rangle$:
 5777 **IF** $e_0 \notin v$:
 5778 1. $e_0 = E_0[e'_0]$
 by the induction hypothesis
 5779 2. $E = \langle E_0, e_1 \rangle$
 5780 3. QED by $e = E[e'_0]$
 5781 **IF** $e_0 \in v$
 $\wedge e_1 \notin v$:
 5782 1. $e_1 = E_1[e'_1]$
 by the induction hypothesis
 5783 2. $E = \langle e_0, E_1 \rangle$
 5784 3. QED by $e = E[e'_1]$
 5785 **ELSE** $e_0 \in v$
 $\wedge e_1 \in v$:
 5786 1. $E = []$
 5787 2. QED $e = E[\langle e_0, e_1 \rangle]$
 5788 **CASE** $e = e_0 e_1$:
 5789 **IF** $e_0 \notin v$:
 5790 1. $e_0 = E_0[e'_0]$
 by the induction hypothesis
 5791 2. $E = E_0 e_1$
 5792 3. QED by $e = E[e'_0]$
 5793 **IF** $e_0 \in v$
 $\wedge e_1 \notin v$:
 5794 1. $e_1 = E_1[e'_1]$
 by the induction hypothesis
 5795 2. $E = e_0 E_1$
 5796 3. QED by $e = E[e'_1]$
 5797 **ELSE** $e_0 \in v$
 $\wedge e_1 \in v$:
 5798 1. $E = []$
 5799 2. QED $e = E[e_0 e_1]$
 5800 **CASE** $e = op^1 e_0$:
 5801 **IF** $e_0 \notin v$:
 5802 1. $e_0 = E_0[e'_0]$
 by the induction hypothesis
 5803 2. $E = op^1 E_0$
 5804 3. QED by $e = E[e'_0]$
 5805 **ELSE** $e_0 \in v$
 $\wedge e_1 \in v$:
 5806 1. $E = []$
 5807 2. QED $e = E[op^1 e_0]$
 5808 **CASE** $e = op^2 e_0 e_1$:
 5809 **IF** $e_0 \notin v$:
 5810 1. $e_0 = E_0[e'_0]$
 by the induction hypothesis
 5811 2. $E = op^2 E_0 e_1$
 5812 3. QED $e = E[e'_0]$
 5813 **ELSE** $e_0 \in v$:
 5814 1. $E = []$
 5815 2. QED $e = E[op^2 e_0]$
 5816 **CASE** $e = \text{stat } \tau e'$:
 5817 **IF** $e_0 \notin v$:
 5818 1. $e_0 = E_0[e'_0]$
 by the induction hypothesis
 5819 2. $E = \text{stat } \tau e'$
 5820 3. QED $e = E[e'_0]$
 5821 **IF** $e_0 \in v$
 $\wedge e_1 \notin v$:
 5822 1. $e_1 = E_1[e'_1]$
 by the induction hypothesis
 5823 2. $E = \text{stat } \tau e'$
 5824 3. QED $e = E[e'_1]$
 5825 **ELSE** $e_0 \in v$
 $\wedge e_1 \in v$:
 5826 1. $E = \langle E_0, E_1 \rangle$
 5827 2. QED $e = E[e'_1]$
 5828 **IF** $e_0 \in v$
 $\wedge e_1 \notin v$:
 5829 1. $e_1 = E_1[e'_1]$
 by the induction hypothesis
 5830 2. QED $e = E[e'_1]$

```

5831   ELSE  $e_0 \in v$ 
5832      $\wedge e_1 \in v :$ 
5833     1.  $E = []$ 
5834     2. QED  $e = E[op^2 e_0 e_1]$ 
5835   CASE  $e = \text{dyn } \tau e_0 :$ 
5836     1. Contradiction by  $e$  is boundary-free
5837   CASE  $e = \text{stat } \tau e_0 :$ 
5838     1. Contradiction by  $\vdash e : \tau$ 
5839   CASE  $e = \text{Err} :$ 
5840     1.  $E = []$ 
5841     2. QED  $e = E[\text{Err}]$ 
5842    $\square$ 

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Lemma 3.20 : $\vdash \text{static inversion}$

- If $\Gamma \vdash x : \tau$ then $(x:\tau') \in \Gamma$ and $\tau' \leqslant \tau$
- If $\Gamma \vdash \lambda(x:\tau'_d).e' : \tau$ then $(x:\tau'_d), \Gamma \vdash e' : \tau'_c$ and $\tau'_d \Rightarrow \tau'_c \leqslant \tau$
- If $\Gamma \vdash \langle e_0, e_1 \rangle : \tau$ then $\Gamma \vdash e_0 : \tau_0$ and $\Gamma \vdash e_1 : \tau_1$ and $\tau_0 \times \tau_1 \leqslant \tau$
- If $\Gamma \vdash e_0 e_1 : \tau_c$ then $\Gamma \vdash e_0 : \tau'_d \Rightarrow \tau'_c$ and $\Gamma \vdash e_1 : \tau'_d$ and $\tau'_c \leqslant \tau_c$
- If $\Gamma \vdash \text{fst } e : \tau$ then $\Gamma \vdash e : \tau_0 \times \tau_1$ and $\Delta(\text{fst}, \tau_0 \times \tau_1) = \tau_0$ and $\tau_0 \leqslant \tau$
- If $\Gamma \vdash \text{snd } e : \tau$ then $\Gamma \vdash e : \tau_0 \times \tau_1$ and $\Delta(\text{snd}, \tau_0 \times \tau_1) = \tau_1$ and $\tau_1 \leqslant \tau$
- If $\Gamma \vdash op^2 e_0 e_1 : \tau$ then $\Gamma \vdash e_0 : \tau_0$ and $\Gamma \vdash e_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau'$ and $\tau' \leqslant \tau$
- If $\Gamma \vdash \text{dyn } \tau' e' : \tau$ then $\Gamma \vdash e'$ and $\tau' \leqslant \tau$

Proof:

QED by the definition of $\vdash e : \tau$

 \square **Lemma 3.21** : canonical forms

- If $\vdash v : \tau_0 \times \tau_1$ then $v = \langle v_0, v_1 \rangle$
- If $\vdash v : \tau_d \Rightarrow \tau_c$ then $v = \lambda(x:\tau_x).e'$
 $\wedge \tau_d \leqslant \tau_x$
- If $\vdash v : \text{Int}$ then $v = i$
- If $\vdash v : \text{Nat}$ then $v = i$ and $v \in \mathbb{N}$

Proof:

QED by definition of $\vdash e : \tau$

 \square **Lemma 3.22** : substitution

If $(x:\tau_x), \Gamma \vdash e : \tau$, and e is boundary-free and $\vdash v : \tau_x$ then
 $\Gamma \vdash e[x \leftarrow v] : \tau$

Proof:

By induction on the structure of e .

- CASE** $e = x :$
 - $e[x \leftarrow v] = v$
 - $\tau_x = \tau$
 - $\Gamma \vdash v : \tau$
by *weakening*
 - QED
- CASE** $e = x' :$
 - QED by $x'[x \leftarrow v] = x'$
- CASE** $e = i :$
 - QED by $i[x \leftarrow v] = i$

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- | | |
|---|------|
| CASE $e = \lambda x. e' :$ | 5886 |
| 1. Contradiction by $(x:\tau_x), \Gamma \vdash e : \tau$ | 5887 |
| CASE $e = \lambda(x:\tau'). e' :$ | 5888 |
| 1. QED by $(\lambda(x:\tau'). e')[x \leftarrow v] = \lambda(x:\tau'). e'$ | 5889 |
| CASE $e = \lambda(x':\tau'). e' :$ | 5890 |
| 1. $e[x \leftarrow v] = \lambda(x':\tau'). (e'[x \leftarrow v])$ | 5891 |
| 2. $(x':\tau'), x, \Gamma \vdash e'$
by <i>static inversion forms</i> | 5892 |
| 3. $(x':\tau'), \Gamma \vdash e'[x \leftarrow v]$
by the induction hypothesis (2) | 5893 |
| 4. $\Gamma \vdash \lambda(x':\tau'). (e'[x \leftarrow v])$
by (3) | 5894 |
| 5. QED | 5895 |
| CASE $e = \langle e_0, e_1 \rangle :$ | 5896 |
| 1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$ | 5897 |
| 2. $x, \Gamma \vdash e_0$
$\wedge x, \Gamma \vdash e_1$
by <i>static inversion forms</i> | 5898 |
| 3. $\Gamma \vdash e_0[x \leftarrow v]$
$\wedge \Gamma \vdash e_1[x \leftarrow v]$
by the induction hypothesis (2) | 5899 |
| 4. $\Gamma \vdash \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
by (3) | 5900 |
| 5. QED | 5901 |
| CASE $e = e_0 e_1 :$ | 5902 |
| 1. $e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]$ | 5903 |
| 2. $x, \Gamma \vdash e_0$
$\wedge x, \Gamma \vdash e_1$
by <i>static inversion forms</i> | 5904 |
| 3. $\Gamma \vdash e_0[x \leftarrow v]$
$\wedge \Gamma \vdash e_1[x \leftarrow v]$
by the induction hypothesis (2) | 5905 |
| 4. $\Gamma \vdash e_0[x \leftarrow v] e_1[x \leftarrow v]$
by (3) | 5906 |
| 5. QED | 5907 |
| CASE $e = op^1 e_0 :$ | 5908 |
| 1. $e[x \leftarrow v] = op^1 e_0[x \leftarrow v]$ | 5909 |
| 2. $x, \Gamma \vdash e_0$
by <i>static inversion forms</i> | 5910 |
| 3. $\Gamma \vdash e_0[x \leftarrow v]$
$\wedge \Gamma \vdash e_1[x \leftarrow v]$
by the induction hypothesis (2) | 5911 |
| 4. $\Gamma \vdash op^1 e_0[x \leftarrow v]$
by (3) | 5912 |
| 5. QED | 5913 |
| CASE $e = op^2 e_0 e_1 :$ | 5914 |
| 1. $e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$ | 5915 |
| 2. $x, \Gamma \vdash e_0$
$\wedge x, \Gamma \vdash e_1$
by <i>static inversion forms</i> | 5916 |
| 3. $\Gamma \vdash e_0[x \leftarrow v]$
$\wedge \Gamma \vdash e_1[x \leftarrow v]$
by the induction hypothesis (2) | 5917 |
| 4. $\Gamma \vdash op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$
by (3) | 5918 |
| 5. QED | 5919 |
| CASE $e = \text{dyn } \tau' e' :$ | 5920 |
| 1. $e[x \leftarrow v] = \text{dyn } \tau' e'[x \leftarrow v]$ | 5921 |
| 2. $x, \Gamma \vdash e'$
by <i>static inversion forms</i> | 5922 |
| 3. $\Gamma \vdash e'[x \leftarrow v]$
$\wedge \Gamma \vdash e_1[x \leftarrow v]$
by the induction hypothesis (2) | 5923 |
| 4. $\Gamma \vdash \text{dyn } \tau' e_1[x \leftarrow v]$
by (3) | 5924 |
| 5. QED | 5925 |
| CASE $e = \text{fst } e :$ | 5926 |
| 1. $e[x \leftarrow v] = \text{fst } e[x \leftarrow v]$ | 5927 |
| 2. $x, \Gamma \vdash e$
$\wedge x, \Gamma \vdash e_0$
by <i>static inversion forms</i> | 5928 |
| 3. $\Gamma \vdash e_0[x \leftarrow v]$
$\wedge \Gamma \vdash e_1[x \leftarrow v]$
by the induction hypothesis (2) | 5929 |
| 4. $\Gamma \vdash \text{fst } e_0[x \leftarrow v]$
by (3) | 5930 |
| 5. QED | 5931 |
| CASE $e = \text{snd } e :$ | 5932 |
| 1. $e[x \leftarrow v] = \text{snd } e[x \leftarrow v]$ | 5933 |
| 2. $x, \Gamma \vdash e$
$\wedge x, \Gamma \vdash e_0$
by <i>static inversion forms</i> | 5934 |
| 3. $\Gamma \vdash e_0[x \leftarrow v]$
$\wedge \Gamma \vdash e_1[x \leftarrow v]$
by the induction hypothesis (2) | 5935 |
| 4. $\Gamma \vdash \text{snd } e_1[x \leftarrow v]$
by (3) | 5936 |
| 5. QED | 5937 |
| CASE $e = \text{Err} :$ | 5938 |
| 1. $e[x \leftarrow v] = \text{Err}$ | 5939 |

5941	5. QED	5996
5942	CASE $e = \text{dyn } \tau' e'$:	5997
5943	1. Contradiction by e is boundary-free	5998
5944	CASE $e = \text{stat } \tau' e'$:	5999
5945	1. Contradiction by e is boundary-free	6000
5946	CASE $e = \text{Err}$:	6001
5947	1. QED $\text{Err}[x \leftarrow v] = \text{Err}$	6002
5948	□	6003
5949	Lemma 3.23 : δ preservation	6004
5950	• If $\vdash v$ and $\delta(\text{op}^1, v) = v'$ then $\vdash e'$	6005
5951	• If $\vdash v_0$ and $\vdash v_1$ and $\delta(\text{op}^2, v_0, v_1) = e'$ then $\vdash v'$	6006
5952	<i>Proof:</i>	6007
5953	CASE $\delta(\text{fst}, \langle v_0, v_1 \rangle) = v_0$:	6008
5954	1. $\vdash v_0$	6009
5955	by <i>static inversion forms</i>	6010
5956	2. QED	6011
5957	CASE $\delta(\text{snd}, \langle v_0, v_1 \rangle) = v_1$:	6012
5958	1. $\vdash v_1$	6013
5959	by <i>static inversion forms</i>	6014
5960	2. QED	6015
5961	CASE $\delta(\text{sum}, v_0, v_1) = v_0 + v_1$:	6016
5962	1. QED	6017
5963	CASE $\delta(\text{quotient}, v_0, v_1) = \lfloor v_0/v_1 \rfloor$:	6018
5964	1. QED	6019
5965	CASE $\delta(\text{quotient}, v_0, v_1) = \text{BndryErr}$:	6020
5966	1. QED	6021
5967	□	6022
5968	Lemma 3.24 : weakening	6023
5969	• If $\Gamma \vdash e$ then $x, \Gamma \vdash e$	6024
5970	• If $\Gamma \vdash e$ then $(x : \tau), \Gamma \vdash e$	6025
5971	<i>Proof:</i>	6026
5972	QED because e is closed under Γ	6027
5973	□	6028
5974		6029
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6051 **E.4 (1) First-Order Embedding**

6052 **E.4.1 First-Order Definitions**

6053 **Language 1**

6055 $e = x \mid v \mid e \cdot e \mid \langle e, e \rangle \mid op^1 e \mid op^2 e \cdot e \mid$
 dyn τ $e \mid$ stat τ $e \mid$ Err \mid chk K $e \mid$ dyn $e \mid$ stat e

6056 $v = i \mid \langle v, v \rangle \mid \lambda x. e \mid \lambda(x: \tau). e$

6057 $\tau = \text{Nat} \mid \text{Int} \mid \tau \times \tau \mid \tau \Rightarrow \tau$

6058 $K = \text{Nat} \mid \text{Int} \mid \text{Pair} \mid \text{Fun} \mid \text{Any}$

6059 $\Gamma = \cdot \mid x, \Gamma \mid (x: \tau), \Gamma$

6060 $\text{Err} = \text{BndryErr} \mid \text{TagErr}$

6061 $r = v \mid \text{Err}$

6062 $E^\bullet = [] \mid E^\bullet e \mid v E^\bullet \mid \langle E^\bullet, e \rangle \mid \langle v, E^\bullet \rangle \mid$
 $op^1 E^\bullet \mid op^2 E^\bullet e \mid op^2 v E^\bullet \mid \text{chk } K E^\bullet$

6063 $E = E^\bullet \mid E e \mid v E \mid \langle E, e \rangle \mid \langle v, E \rangle \mid op^1 E \mid$
 $op^2 E e \mid op^2 v E \mid \text{dyn } \tau E \mid \text{stat } \tau E \mid$
 $\text{chk } K E \mid \text{dyn } E \mid \text{stat } E$

6068 **$\Delta : op^1 \times \tau \rightarrow \tau$**

6069 $\Delta(\text{fst}, \tau_0 \times \tau_1) = \tau_0$

6070 $\Delta(\text{snd}, \tau_0 \times \tau_1) = \tau_1$

6071 **$\Delta : op^2 \times \tau \times \tau \rightarrow \tau$**

6072 $\Delta(op^2, \text{Nat}, \text{Nat}) = \text{Nat}$

6073 $\Delta(op^2, \text{Int}, \text{Int}) = \text{Int}$

6074 **$\tau \leqslant: \tau$**

6075 $\frac{\text{Nat} \leqslant: \text{Int}}{\tau_d \Rightarrow \tau_c \leqslant: \tau'_d \Rightarrow \tau'_c} \quad \frac{\tau'_d \leqslant: \tau_d \quad \tau_c \leqslant: \tau'_c}{\tau \leqslant: \tau'}$

6076 $\frac{\text{Nat} \leqslant: \text{Int}}{\tau \leqslant: \tau} \quad \frac{\tau \leqslant: \tau' \quad \tau' \leqslant: \tau''}{\tau \leqslant: \tau''}$

6077 **$\Gamma \vdash e$**

6078 $\frac{x \in \Gamma}{\Gamma \vdash x} \quad \frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x. e} \quad \frac{}{\Gamma \vdash i} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}$

6079 $\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0 e_1} \quad \frac{\Gamma \vdash e}{\Gamma \vdash op^1 e} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2 e_0 e_1} \quad \frac{}{\Gamma \vdash \text{Err}}$

6080 $\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{stat } \tau e}$

6081 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6082 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6083 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6084 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6085 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6086 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6087 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6088 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6089 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6090 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6091 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6092 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6093 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6094 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6095 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6096 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6097 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6098 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6099 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6100 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6101 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6102 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6103 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6104 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6105 $\frac{}{\Gamma \vdash \text{stat } \tau e}$

6106 $\boxed{\Gamma \vdash e : \tau}$

6107 $\frac{(x: \tau) \in \Gamma}{\Gamma \vdash x : \tau}$

6108 $\frac{(x: \tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x: \tau_d). e : \tau_d \Rightarrow \tau_c}$

6109 $\frac{i \in \mathbb{N}}{\Gamma \vdash i : \text{Nat}}$

6110 $\boxed{\Gamma \vdash i : \text{Int}}$

6111 $\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1}$

6112 $\frac{\Gamma \vdash e_1 : \tau_d}{\Gamma \vdash e_0 e_1 : \tau_c}$

6113 $\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash op^1 e_0 e_1 : \tau}$

6114 $\frac{\Delta(op^2, \tau_0, \tau_1) = \tau \quad \tau' \leqslant: \tau}{\Gamma \vdash op^2 e_0 e_1 : \tau}$

6115 $\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{dyn } \tau e : \tau}$

6123 $\boxed{K \leqslant: K}$

6124 $\boxed{K \leqslant: \text{Any}}$

6125 $\boxed{\text{Nat} \leqslant: \text{Int}}$

6126 $\boxed{K \leqslant: K}$

6127 $\frac{K \leqslant: K' \quad K' \leqslant: K''}{K \leqslant: K''}$

6128 $\boxed{\lfloor \tau \rfloor = K}$

6129 $\lfloor \text{Nat} \rfloor = \text{Nat}$

6130 $\lfloor \text{Int} \rfloor = \text{Int}$

6131 $\lfloor \tau_0 \times \tau_1 \rfloor = \text{Pair}$

6132 $\lfloor \tau_d \Rightarrow \tau_c \rfloor = \text{Fun}$

6133 $\boxed{\Gamma \vdash e \rightsquigarrow e}$

6134 $\frac{\Gamma \vdash e_0 \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 \rightsquigarrow e'_1}{\Gamma \vdash i \rightsquigarrow i}$

6135 $\frac{x, \Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \langle e_0, e_1 \rangle \rightsquigarrow \langle e'_0, e'_1 \rangle}$

6136 $\frac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \lambda x. e \rightsquigarrow \lambda x. e'}$

6137 $\frac{\Gamma \vdash e_0 \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 \rightsquigarrow e'_1}{\Gamma \vdash e_0 e_1 \rightsquigarrow e'_0 e'_1}$

6138 $\frac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash op^1 e \rightsquigarrow op^1 e'}$

6139 $\frac{\Gamma \vdash e_0 \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 \rightsquigarrow e'_1}{\Gamma \vdash op^2 e_0 e_1 \rightsquigarrow op^2 e'_0 e'_1}$

6140 $\frac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \text{Err} \rightsquigarrow \text{Err}}$

6141 $\frac{\Gamma \vdash e_0 \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 \rightsquigarrow e'_1}{\Gamma \vdash \text{Err} \rightsquigarrow \text{Err}}$

6142 $\frac{\Gamma \vdash e_0 \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 \rightsquigarrow e'_1}{\Gamma \vdash op^2 e_0 e_1 \rightsquigarrow op^2 e'_0 e'_1}$

6143 $\frac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \text{Err} \rightsquigarrow \text{Err}}$

6144 $\frac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \text{Err} \rightsquigarrow \text{Err}}$

6145 $\frac{\Gamma \vdash e : \tau \rightsquigarrow e'}{\Gamma \vdash \text{stat } \tau e \rightsquigarrow \text{stat } \tau e'}$

6146 $\frac{\Gamma \vdash e : \tau \rightsquigarrow e'}{\Gamma \vdash \text{stat } \tau e \rightsquigarrow \text{stat } \tau e'}$

6161	$\boxed{\Gamma \vdash e : \tau \rightsquigarrow e}$
6162	$\frac{}{\Gamma \vdash i : \text{Nat} \rightsquigarrow i} \quad \frac{}{\Gamma \vdash i : \text{Int} \rightsquigarrow i}$
6163	
6164	
6165	$\frac{\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 : \tau_1 \rightsquigarrow e'_1}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1 \rightsquigarrow \langle e'_0, e'_1 \rangle}$
6166	
6167	
6168	$(x:\tau_d), \Gamma \vdash e : \tau_c \rightsquigarrow e'$
6169	$\frac{\Gamma \vdash \lambda(x:\tau_d). e : \tau_d \Rightarrow \tau_c \rightsquigarrow \lambda(x:\tau_d). e' \quad \Gamma \vdash x : \tau \rightsquigarrow x}{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 : \tau_d \rightsquigarrow e'_1 \quad \lfloor \tau_c \rfloor = K}$
6170	
6171	$\frac{}{\Gamma \vdash e_0 e_1 : \tau_c \rightsquigarrow \text{chk } K(e'_0 e'_1)}$
6172	
6173	
6174	
6175	$\frac{\Gamma \vdash e : \tau_0 \times \tau_1 \rightsquigarrow e' \quad \lfloor \tau_0 \rfloor = K}{\Gamma \vdash \text{fst } e : \tau_0 \rightsquigarrow \text{chk } K(\text{fst } e')}$
6176	
6177	
6178	$\frac{\Gamma \vdash e : \tau_0 \times \tau_1 \rightsquigarrow e' \quad \lfloor \tau_1 \rfloor = K}{\Gamma \vdash \text{snd } e : \tau_1 \rightsquigarrow \text{chk } K(\text{snd } e')}$
6179	
6180	
6181	
6182	$\frac{\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 : \tau_1 \rightsquigarrow e'_1}{\Gamma \vdash op^2 e_0 e_1 : \tau \rightsquigarrow op^2 e'_0 e'_1} \quad \frac{\Gamma \vdash e : \tau' \rightsquigarrow e' \quad \tau' \leqslant \tau}{\Gamma \vdash e : \tau \rightsquigarrow e'}$
6183	
6184	
6185	
6186	
6187	$\frac{}{\Gamma \vdash \text{Err} : \tau \rightsquigarrow \text{Err}} \quad \frac{}{\Gamma \vdash \text{dyn } \tau e : \tau \rightsquigarrow \text{dyn } \tau e'}$
6188	
6189	$\boxed{\Gamma \vdash_1 e}$
6190	$\frac{}{\Gamma \vdash_1 i} \quad \frac{\Gamma \vdash_1 e_0 \quad \Gamma \vdash_1 e_1}{\Gamma \vdash_1 \langle e_0, e_1 \rangle} \quad \frac{x, \Gamma \vdash_1 e}{\Gamma \vdash_1 \lambda x. e} \quad \frac{(x:\tau), \Gamma \vdash_1 e : \text{Any}}{\Gamma \vdash_1 \lambda(x:\tau). e}$
6191	
6192	
6193	
6194	$\frac{x \in \Gamma}{\Gamma \vdash_1 x} \quad \frac{(x:\tau) \in \Gamma}{\Gamma \vdash_1 x} \quad \frac{\Gamma \vdash_1 e_0 \quad \Gamma \vdash_1 e_1}{\Gamma \vdash_1 e_0 e_1} \quad \frac{}{\Gamma \vdash_1 op^1 e}$
6195	
6196	
6197	$\frac{\Gamma \vdash_1 e_0 \quad \Gamma \vdash_1 e_1}{\Gamma \vdash_1 op^2 e_0 e_1} \quad \frac{}{\Gamma \vdash_1 \text{Err}} \quad \frac{\Gamma \vdash_1 e : \lfloor \tau \rfloor}{\Gamma \vdash_1 \text{stat } \tau e} \quad \frac{\Gamma \vdash_1 e : \text{Any}}{\Gamma \vdash_1 \text{stat } e}$
6198	
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6216	$\boxed{\Gamma \vdash_1 e : K}$
6217	
6218	$i \in \mathbb{N} \quad \frac{}{\Gamma \vdash_1 i : \text{Nat}} \quad \frac{}{\Gamma \vdash_1 i : \text{Int}} \quad \frac{\Gamma \vdash_1 e_0 : \text{Any} \quad \Gamma \vdash_1 e_1 : \text{Any}}{\Gamma \vdash_1 \langle e_0, e_1 \rangle : \text{Pair}}$
6219	
6220	$x, \Gamma \vdash_1 e \quad \frac{(x:\tau), \Gamma \vdash_1 e : \text{Any}}{\Gamma \vdash_1 \lambda(x:\tau). e : \text{Fun}} \quad \frac{x \in \Gamma}{\Gamma \vdash_1 x : \text{Any}}$
6221	
6222	$\frac{}{\Gamma \vdash_1 \text{fst } e : \text{Any}} \quad \frac{(x:\tau) \in \Gamma}{\Gamma \vdash_1 x : K} \quad \frac{\lfloor \tau \rfloor = K}{\Gamma \vdash_1 e_0 e_1 : \text{Any}} \quad \frac{\Gamma \vdash_1 e : \text{Pair}}{\Gamma \vdash_1 \text{fst } e : \text{Any}}$
6223	
6224	
6225	$\frac{\Gamma \vdash_1 e_0 : K_0 \quad \Gamma \vdash_1 e_1 : K_1}{\Delta(op^2, K_0, K_1) = K} \quad \frac{\Gamma \vdash_1 \text{snd } e : \text{Any}}{\Gamma \vdash_1 op^2 e_0 e_1 : K}$
6226	
6227	
6228	
6229	$\frac{\Gamma \vdash_1 e : \text{Pair}}{\Gamma \vdash_1 \text{snd } e : \text{Any}} \quad \frac{\Gamma \vdash_1 e_0 : K_0 \quad \Gamma \vdash_1 e_1 : K_1}{\Delta(op^2, K_0, K_1) = K} \quad \frac{\Gamma \vdash_1 e : \text{Any}}{\Gamma \vdash_1 \text{dyn } \tau e : K}$
6230	
6231	
6232	
6233	$\frac{\Gamma \vdash_1 e : K' \quad K' \leqslant K}{\Gamma \vdash_1 e : K} \quad \frac{\Gamma \vdash_1 \text{Err} : K}{\Gamma \vdash_1 \text{dyn } \tau e : K} \quad \frac{\Gamma \vdash_1 e \quad \lfloor \tau \rfloor = K}{\Gamma \vdash_1 \text{chk } K e : K}$
6234	
6235	
6236	
6237	$\frac{\Gamma \vdash_1 e}{\Gamma \vdash_1 \text{dyn } e : \text{Any}} \quad \frac{\Gamma \vdash_1 e : \text{Any}}{\Gamma \vdash_1 \text{chk } K e : K}$
6238	
6239	
6240	$\boxed{\delta(op^1, v) = e}$
6241	$\delta(\text{fst}, \langle v_0, v_1 \rangle) = v_0$
6242	$\delta(\text{snd}, \langle v_0, v_1 \rangle) = v_1$
6243	
6244	$\boxed{\delta(op^2, v, v) = e}$
6245	$\delta(\text{sum}, i_0, i_1) = i_0 + i_1$
6246	$\delta(\text{quotient}, i_0, 0) = \text{BndryErr}$
6247	$\delta(\text{quotient}, i_0, i_1) = \lfloor i_0 / i_1 \rfloor$
6248	$\text{if } i_1 \neq 0$
6249	$\boxed{\mathcal{D}_1 : \tau \times v \longrightarrow v}$
6250	$\mathcal{D}_1(\tau, v) = \mathcal{X}(\lfloor \tau \rfloor, v)$
6251	
6252	$\boxed{\mathcal{S}_1 : \tau \times v \longrightarrow v}$
6253	$\mathcal{S}_1(\tau, v) = v$
6254	$\boxed{\mathcal{X} : K \times v \longrightarrow v}$
6255	$\mathcal{X}(\text{Fun}, \lambda x. e) = \lambda x. e$
6256	$\mathcal{X}(\text{Fun}, \lambda(x:\tau). e) = \lambda(x:\tau). e$
6257	$\mathcal{X}(\text{Pair}, \langle v_0, v_1 \rangle) = \langle v_0, v_1 \rangle$
6258	$\mathcal{X}(\text{Int}, i) = i$
6259	$\mathcal{X}(\text{Nat}, i) = i$
6260	$\text{if } i \in \mathbb{N}$
6261	$\mathcal{X}(K, v) = \text{BndryErr}$
6262	otherwise
6263	
6264	
6265	
6266	
6267	
6268	
6269	
6270	

6271	$e \triangleright_{1-S} e$	6326
6272	dyn $v \triangleright_{1-S} v$	6327
6273	dyn $\tau v \triangleright_{1-S} \mathcal{D}(\tau, v)$	6328
6274	chk $K v \triangleright_{1-S} X(K, v)$	6329
6275	$(\lambda(x:\tau).e) v \triangleright_{1-S} \text{BndryErr}$	6330
6276	if $X([\tau], v) = \text{BndryErr}$	6331
6277	$(\lambda(x:\tau).e) v \triangleright_{1-S} e[x \leftarrow X([\tau], v)]$	6332
6278	$(\lambda x. e) v \triangleright_{1-S} \text{dyn}(e[x \leftarrow v])$	6333
6279	$op^1 v \triangleright_{1-S} \delta(op^1, v)$	6334
6280	$op^2 v_0 v_1 \triangleright_{1-S} \delta(op^2, v_0, v_1)$	6335
6281	$e \triangleright_{1-D} e$	6336
6282	stat $v \triangleright_{1-D} v$	6337
6283	stat $\tau v \triangleright_{1-D} \mathcal{S}(\tau, v)$	6338
6284	$v_0 v_1 \triangleright_{1-D} \text{TagErr}$	6339
6285	if $v_0 \in \mathbb{Z}$ or $v_0 = \langle v, v' \rangle$	6340
6286	$(\lambda(x:\tau).e) v \triangleright_{1-D} \text{BndryErr}$	6341
6287	if $X([\tau], v) = \text{BndryErr}$	6342
6288	$(\lambda(x:\tau).e) v \triangleright_{1-D} \text{stat}(e[x \leftarrow X([\tau], v)])$	6343
6289	$(\lambda x. e) v \triangleright_{1-D} e[x \leftarrow v]$	6344
6290	$op^1 v \triangleright_{1-D} \text{TagErr}$	6345
6291	if $\delta(op^1, v)$ is undefined	6346
6292	$op^1 v \triangleright_{1-D} \delta(op^1, v)$	6347
6293	$op^2 v_0 v_1 \triangleright_{1-D} \text{TagErr}$	6348
6294	if $\delta(op^2, v_0, v_1)$ is undefined	6349
6295	$op^2 v_0 v_1 \triangleright_{1-D} \delta(op^2, v_0, v_1)$	6350
6296	$e \rightarrow_{1-S} e$	6351
6297	$E^\bullet[e] \rightarrow_{1-S} E^\bullet[e']$	6352
6298	if $e \triangleright_{1-S} e'$	6353
6299	$E[\text{stat } \tau E^\bullet[e]] \rightarrow_{1-S} E[\text{stat } \tau E^\bullet[e']]$	6354
6300	if $e \triangleright_{1-S} e'$	6355
6301	$E[\text{dyn } \tau E^\bullet[e]] \rightarrow_{1-S} E[\text{dyn } \tau E^\bullet[e']]$	6356
6302	if $e \triangleright_{1-D} e'$	6357
6303	$E[\text{Err}] \rightarrow_{1-S} \text{Err}$	6358
6304	$e \rightarrow_{1-D} e$	6359
6305	$E^\bullet[e] \rightarrow_{1-D} E^\bullet[e']$	6360
6306	if $e \triangleright_{1-D} e'$	6361
6307	$E[\text{stat } \tau E^\bullet[e]] \rightarrow_{1-D} E[\text{stat } \tau E^\bullet[e']]$	6362
6308	if $e \triangleright_{1-S} e'$	6363
6309	$E[\text{dyn } \tau E^\bullet[e]] \rightarrow_{1-D} E[\text{dyn } \tau E^\bullet[e']]$	6364
6310	if $e \triangleright_{1-D} e'$	6365
6311	$E[\text{Err}] \rightarrow_{1-D} \text{Err}$	6366
6312	$e \rightarrow_{1-S}^* e$ reflexive, transitive closure of \rightarrow_{1-S}	6367
6313	$e \rightarrow_{1-D}^* e$ reflexive, transitive closure of \rightarrow_{1-D}	6368
6314		6369
6315		6370
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6381 E.4.2 First-Order Theorems

6382 Theorem 4.1 : static 1-soundness

6383 If $\vdash e : \tau$ then $\vdash e : \tau \rightsquigarrow e''$ and $\vdash e'' : [\tau]$ and one of the
 6384 following holds:
 6385 • $e'' \rightarrow_{1-S}^* v$ and $\vdash_1 v : [\tau]$
 6386 • $e'' \rightarrow_{1-S}^* E[\text{dyn } \tau' E^*[e']]$ and $e' \triangleright_{1-D} \text{TagErr}$
 6387 • $e'' \rightarrow_{1-S}^* E[\text{dyn } E^*[e']]$ and $e' \triangleright_{1-D} \text{TagErr}$
 6388 • $e'' \rightarrow_{1-S}^* \text{BndryErr}$
 6389 • e'' diverges

6390 *Proof:*

- 6391 1. $\vdash_1 e : \tau \rightsquigarrow e''$
 6392 $\wedge \vdash_1 e'' : [\tau]$
 6393 by \rightsquigarrow static soundness
- 6394 2. QED by 1 static progress and 1 static preservation

6395 \square

6396 Theorem 4.2 : dynamic 1-soundness

6397 If $\vdash e$ then $\vdash e \rightsquigarrow e''$ and $\vdash_1 e''$ and one of the following holds:
 6398 • $e'' \rightarrow_{1-D}^* v$ and $\vdash_1 v$
 6399 • $e'' \rightarrow_{1-D}^* E[e']$ and $e' \triangleright_{1-D} \text{TagErr}$
 6400 • $e'' \rightarrow_{1-D}^* \text{BndryErr}$
 6401 • e'' diverges

6402 *Proof:*

- 6403 1. $\vdash_1 e \rightsquigarrow e''$
 6404 $\wedge \vdash_1 e''$
 6405 by \rightsquigarrow dynamic soundness
- 6406 2. QED by 1 dynamic progress and 1 dynamic preservation

6407 \square

6408 Theorem 4.3 : boundary-free 1-soundness

6409 If $\vdash e : \tau$ and e is boundary-free then one of the following
 6410 holds:
 6411 • $e \rightarrow_{1-S}^* v$ and $\vdash v : \tau$
 6412 • $e \rightarrow_{1-S}^* \text{BndryErr}$
 6413 • e diverges

6414 *Proof:*

- 6415 1. QED by progress and preservation

6416 \square

6417 Theorem 4.4 : H/1 base type equivalence

6418 If $\vdash e : \tau$ and all boundary terms in e are of the following
 6419 four forms:
 6420 • dyn Int e'
 6421 • stat Int e'
 6422 • stat Nat e'
 6423 • dyn Nat e'

6424 and $\vdash e : \tau \rightsquigarrow e''$, then $e \rightarrow_{H-S}^* v$ if and only if $e'' \rightarrow_{1-S}^* v$.
 6425 *Proof:*

- 6426 1. $\mathcal{D}_H(\text{Int}, v) = \mathcal{D}_1(\text{Int}, v)$
 6427 by definition
- 6428 2. $\mathcal{D}_H(\text{Nat}, v) = \mathcal{D}_1(\text{Nat}, v)$
 6429 by definition
- 6430 3. $\mathcal{S}_H(\text{Int}, v) = \mathcal{S}_1(\text{Int}, v)$
 6431 by definition
- 6432 4. $\mathcal{S}_H(\text{Nat}, v) = \mathcal{S}_1(\text{Nat}, v)$
 6433 by definition

6434

6435 5. QED

6436 \square

6437 Corollary 4.5 : 1 compilation

6438 If $\vdash e : \tau$
 6439 and $\vdash e : \tau \rightsquigarrow e''$
 6440 and $\vdash_1 e'' : [\tau]$
 6441 and \triangleright_{1-D}' is similar to \triangleright_{1-D} but without the no-op boundaries, as follows:
 6442 $\text{chk } K v \triangleright_{1-D}' X(K, v)$
 6443 $v_0 v_1 \triangleright_{1-D}' \text{TagErr}$
 6444 if $v_0 \in \mathbb{Z}$ or $v_0 = \langle v, v' \rangle$
 6445 $(\lambda(x:\tau). e) v \triangleright_{1-D}' \text{BndryErr}$
 6446 if $X([\tau], v) = \text{BndryErr}$
 6447 $(\lambda(x:\tau). e) v \triangleright_{1-D}' e[x \leftarrow X([\tau], v)]$
 6448 $(\lambda x. e) v \triangleright_{1-D}' e[x \leftarrow v]$
 6449 $op^1 v \triangleright_{1-D}' \text{TagErr}$
 6450 if $\delta(op^1, v)$ is undefined
 6451 $op^1 v \triangleright_{1-D}' \delta(op^1, v)$
 6452 $op^2 v_0 v_1 \triangleright_{1-D}' \text{TagErr}$
 6453 if $\delta(op^2, v_0, v_1)$ is undefined
 6454 $op^2 v_0 v_1 \triangleright_{1-D}' \delta(op^2, v_0, v_1)$
 6455 and $e \rightarrow_{1-D}' e$ is defined as:
 6456 $E[e] \rightarrow_{1-D}' E[e']$
 6457 if $e \triangleright_{1-D}' e'$
 6458 $E[\text{stat } \tau v] \rightarrow_{1-D}' E[\mathcal{D}_1(\tau, v)]$
 6459 $E[\text{dyn } \tau v] \rightarrow_{1-D}' E[\mathcal{D}_1(\tau, v)]$
 6460 $E[\text{Err}] \rightarrow_{1-D}' \text{Err}$
 6461 and $\rightarrow_{1-D}'^*$ is the reflexive transitive closure of \rightarrow_{1-D}'

6462 then one of the following holds:
 6463 • $e'' \rightarrow_{1-D}'^* v$ and $\vdash_1 v : [\tau]$
 6464 • $e'' \rightarrow_{1-D}'^* \text{TagErr}$
 6465 • $e'' \rightarrow_{1-D}'^* \text{BndryErr}$
 6466 • e'' diverges

6467 *Proof(sketch):* By static 1-soundness and the fact that \triangleright_{1-S} is a
 6468 subset of \triangleright_{1-D}' (modulo the dyn e and stat e boundaries). \square

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6488	6488
6489	6489
6490	6490

6491 **E.4.3 First-Order Lemmas**6492 **Lemma 4.6 : \mathcal{D}_1 soundness**6493 If $\vdash_1 v$ then $\vdash_1 \mathcal{D}_1(\tau, v) : \lfloor \tau \rfloor$.6494 *Proof:*

- $\mathcal{D}_1(\tau, v) = X(\lfloor \tau \rfloor, v)$
- QED by *check soundness*

6497 \square 6498 **Lemma 4.7 : \mathcal{S}_1 soundness**6499 If $\vdash_1 v : \tau$ then $\vdash_1 \mathcal{S}_1(\tau, v)$.6500 *Proof:*

- $\mathcal{S}_1(\tau, v) = X(\lfloor \tau \rfloor, v)$
- QED *check soundness*

6504 \square 6505 **Lemma 4.8 : \rightsquigarrow static soundness**6506 If $\Gamma \vdash e : \tau$ then $\Gamma \vdash e : \tau \rightsquigarrow e'$ and $\Gamma \vdash_1 e' : \lfloor \tau \rfloor$.6507 *Proof:*6508 By induction on the structure of $\Gamma \vdash e : \tau$.6509 **CASE** $\boxed{(x:\tau) \in \Gamma} :$

$$\boxed{\Gamma \vdash x : \tau}$$

1. $\Gamma \vdash x \rightsquigarrow x$
2. $\Gamma \vdash_1 x : \lfloor \tau \rfloor$
by $(x:\tau) \in \Gamma$
3. QED

6510 **CASE** $\boxed{(x:\tau_d), \Gamma \vdash e : \tau_c} :$

$$\boxed{\Gamma \vdash \lambda(x:\tau_d). e : \tau_d \Rightarrow \tau_c}$$

1. $\Gamma \vdash e : \tau_c \rightsquigarrow e'$
 $\wedge (x:\tau_d), \Gamma \vdash e' : \lfloor \tau_c \rfloor$
by the induction hypothesis
2. $(x:\tau_d), \Gamma \vdash e' : \text{Any}$
by $\lfloor \tau_c \rfloor <: \text{Any}$
3. $\lambda(x:\tau_d). e : \tau_d \Rightarrow \tau_c \rightsquigarrow \lambda(x:\tau_d). e'$
4. $\Gamma \vdash_1 \lambda(x:\tau_d). e' : \text{Fun}$
by (2)
5. QED (3, 4)

6511 **CASE** $\boxed{i \in \mathbb{N}} :$

$$\boxed{\Gamma \vdash i : \text{Nat}}$$

1. $\Gamma \vdash i : \text{Nat} \rightsquigarrow i$
2. QED by $\Gamma \vdash_1 i : \text{Nat}$

6512 **CASE** $\boxed{\Gamma \vdash i : \text{Int}} :$

$$\boxed{\Gamma \vdash i : \text{Int}}$$

1. $\Gamma \vdash i : \text{Int} \rightsquigarrow i$
2. QED by $\Gamma \vdash_1 i : \text{Int}$

6513 **CASE** $\boxed{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1} :$

$$\boxed{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1}$$

1. $\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e'_0$
 $\wedge \Gamma \vdash e'_0 : \lfloor \tau_0 \rfloor$
by the induction hypothesis

6514 \square 6515 \square 6516 \square 6517 \square 6518 \square 6519 \square 6520 \square 6521 \square 6522 \square 6523 \square 6524 \square 6525 \square 6526 \square 6527 \square 6528 \square 6529 \square 6530 \square 6531 \square 6532 \square 6533 \square 6534 \square 6535 \square 6536 \square 6537 \square 6538 \square 6539 \square 6540 \square 6541 \square 6542 \square 6543 \square 6544 \square 6545 \square 6546 **2.** $\Gamma \vdash e_1 : \tau_1 \rightsquigarrow e'_1$ 6547 $\wedge \Gamma \vdash e'_1 : \lfloor \tau_1 \rfloor$

6548 by the induction hypothesis

6549 **3.** $\Gamma \vdash_1 e_0 : \text{Any}$ 6550 by $\lfloor \tau_0 \rfloor <: \text{Any}$ 6551 **4.** $\Gamma \vdash_1 e_1 : \text{Any}$ 6552 by $\lfloor \tau_1 \rfloor <: \text{Any}$ 6553 **5.** $\Gamma \vdash \langle e_0, e_1 \rangle : \tau \rightsquigarrow \langle e'_0, e'_1 \rangle$

6554 by (1, 2)

6555 **6.** $\Gamma \vdash_1 \langle e'_0, e'_1 \rangle : \text{Pair}$

6556 by (3, 4)

6557 **7.** QED by (5, 6)6558 **CASE**
$$\boxed{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash e_1 : \tau_d} :$$

$$\boxed{\Gamma \vdash e_0 e_1 : \tau_c}$$
6559 **1.** $\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \rightsquigarrow e'_0$ 6560 $\wedge \Gamma \vdash_1 e'_0 : \lfloor \tau_d \Rightarrow \tau_c \rfloor$

6561 by the induction hypothesis

6562 **2.** $\Gamma \vdash e_1 : \tau_d \rightsquigarrow e'_1$ 6563 $\wedge \Gamma \vdash_1 e'_1 : \lfloor \tau_d \rfloor$

6564 by the induction hypothesis

6565 **3.** $\Gamma \vdash_1 e'_0 : \text{Fun}$ 6566 by $\lfloor \tau_d \Rightarrow \tau_c \rfloor = \text{Fun}$ 6567 **4.** $\Gamma \vdash_1 e'_1 : \text{Any}$ 6568 by $\lfloor \tau_c \rfloor <: \text{Any}$ 6569 **5.** $\Gamma \vdash e_0 e_1 : \tau_c \rightsquigarrow \text{chk } \lfloor \tau_c \rfloor (e'_0 e'_1)$

6570 by (1, 2)

6571 **6.** $\Gamma \vdash_1 \text{chk } \lfloor \tau_c \rfloor (e'_0 e'_1) : \lfloor \tau_c \rfloor$

6572 by (3, 4)

6573 **7.** QED by (5, 6)6574 **CASE**
$$\boxed{\Gamma \vdash e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau} :$$

$$\boxed{\Gamma \vdash op^1 e_0 : \tau}$$
6575 **IF** $op^1 = \text{fst} :$ 6576 **1.** $\Delta(\text{fst}, \tau_0) = \tau$ 6577 **2.** $\tau_0 = \tau \times \tau'$ 6578 by Δ *inversion*6579 **3.** $\Gamma \vdash e_0 : \tau \times \tau' \rightsquigarrow e'_0$ 6580 $\wedge \Gamma \vdash_1 e'_0 : \lfloor \tau \times \tau' \rfloor$

6581 by the induction hypothesis

6582 **4.** $\Gamma \vdash_1 e'_0 : \text{Pair}$ 6583 by $\lfloor \tau \times \tau' \rfloor = \text{Pair}$ 6584 **5.** $\Gamma \vdash \text{fst } e_0 : \tau \rightsquigarrow \text{chk } \lfloor \tau \rfloor (\text{fst } e'_0)$

6585 by (2)

6586 **6.** $\Gamma \vdash_1 \text{chk } \lfloor \tau \rfloor (\text{fst } e'_0) : \lfloor \tau \rfloor$

6587 by (3)

6588 **7.** QED by 4,56589 **ELSE** $op^1 = \text{snd} :$ 6590 **1.** $\Delta(\text{snd}, \tau_0) = \tau$ 6591 **2.** $\tau_0 = \tau' \times \tau$ 6592 by Δ *inversion*6593 **3.** $\Gamma \vdash e_0 : \tau' \times \tau \rightsquigarrow e'_0$ 6594 $\wedge \Gamma \vdash_1 e'_0 : \lfloor \tau' \times \tau \rfloor$

6595 by the induction hypothesis

6596 **4.** $\Gamma \vdash_1 e'_0 : \text{Pair}$ 6597 by $\lfloor \tau' \times \tau \rfloor = \text{Pair}$ 6598 **5.** $\Gamma \vdash \text{fst } e_0 : \tau' \rightsquigarrow \text{chk } \lfloor \tau' \rfloor (\text{fst } e'_0)$

6599 by (2)

6600 by (3)

6601	4. $\Gamma \vdash_1 e'_0 : \text{Pair}$	
6602	by $\lfloor \tau' \times \tau \rfloor = \text{Pair}$	
6603	5. $\Gamma \vdash \text{snd } e_0 : \tau \rightsquigarrow \text{chk } \lfloor \tau \rfloor (\text{snd } e'_0)$	
6604	by (2)	
6605	6. $\Gamma \vdash_1 \text{chk } \lfloor \tau \rfloor (\text{snd } e'_0) : \lfloor \tau \rfloor$	
6606	by (3)	
6607	7. QED by 4,5	
6608	CASE $\boxed{\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash op^2 e_0 e_1 : \tau}}$	
6609		
6610		
6611	1. $\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e'_0$	
6612	$\wedge \Gamma \vdash e'_0 : \lfloor \tau_0 \rfloor$	
6613	by the induction hypothesis	
6614	2. $\Gamma \vdash e_1 : \tau_1 \rightsquigarrow e'_1$	
6615	$\wedge \Gamma \vdash e'_1 : \lfloor \tau_1 \rfloor$	
6616	by the induction hypothesis	
6617	3. $\Delta(op^2, \lfloor \tau_0 \rfloor, \lfloor \tau_1 \rfloor) = \lfloor \tau \rfloor$	
6618	by <i>Δ tag preservation</i>	
6619	4. $\Gamma \vdash op^2 e_0 e_1 : \tau \rightsquigarrow op^2 e'_0 e'_1$	
6620	by (1, 2)	
6621	5. $\Gamma \vdash_1 op^2 e'_0 e'_1 : \lfloor \tau \rfloor$	
6622	by (1, 2, 3)	
6623	6. QED by (5, 6)	
6624	CASE $\boxed{\frac{\Gamma \vdash e : \tau' \quad \tau' \leqslant \tau}{\Gamma \vdash e : \tau}}$	
6625		
6626		
6627	1. $\Gamma \vdash e : \tau' \rightsquigarrow e'$	
6628	$\wedge \Gamma \vdash_1 e' : \lfloor \tau' \rfloor$	
6629	by the induction hypothesis	
6630	2. $\lfloor \tau' \rfloor \leqslant \lfloor \tau \rfloor$	
6631	by <i>subtyping preservation</i>	
6632	3. $\Gamma \vdash_1 e' : \lfloor \tau \rfloor$	
6633	by (2)	
6634	4. QED by (1, 3)	
6635	CASE $\boxed{\frac{}{\Gamma \vdash \text{Err} : \tau}}$	
6636		
6637	1. $\Gamma \vdash \text{Err} : \tau \rightsquigarrow \text{Err}$	
6638	2. $\Gamma \vdash_1 \text{Err} : \tau$	
6639	3. QED	
6640	CASE $\boxed{\frac{\Gamma \vdash e}{\Gamma \vdash \text{dyn } \tau e : \tau}}$	
6641		
6642		
6643	1. $\Gamma \vdash e \rightsquigarrow e'$	
6644	$\wedge \Gamma \vdash_1 e'$	
6645	by \rightsquigarrow <i>dynamic soundness</i>	
6646	2. $\Gamma \vdash \text{dyn } \tau e : \tau \rightsquigarrow \text{dyn } \tau e'$	
6647	by (1)	
6648	3. $\Gamma \vdash_1 \text{dyn } \tau e' : \lfloor \tau \rfloor$	
6649	by (1)	
6650	4. QED by (2, 3)	
6651	□	
6652	Lemma 4.9 : \rightsquigarrow <i>dynamic soundness</i>	
6653	If $\Gamma \vdash e$ then $\Gamma \vdash e \rightsquigarrow e'$ and $\Gamma \vdash_1 e'$	
6654	Proof:	
6655		

By induction on the structure of $\Gamma \vdash e$.	6656
CASE $\boxed{\frac{x \in \Gamma}{\Gamma \vdash x}} :$	6657
1. $\Gamma \vdash x \rightsquigarrow x$	6658
2. $\Gamma \vdash_1 x$	6659
by $x \in \Gamma$	6660
3. QED	6661
CASE $\boxed{\frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x. e}} :$	6662
1. $x, \Gamma \vdash e \rightsquigarrow e'$	6663
$\wedge x, \Gamma \vdash_1 e'$	6664
by the induction hypothesis	6665
2. $\Gamma \vdash \lambda x. e \rightsquigarrow \lambda x. e'$	6666
by (1)	6667
3. $\Gamma \vdash_1 \lambda x. e'$	6668
by (1)	6669
4. QED by (2, 3)	6670
CASE $\boxed{\frac{}{\Gamma \vdash i}} :$	6671
1. $\Gamma \vdash i \rightsquigarrow i$	6672
2. $\Gamma \vdash_1 i$	6673
3. QED	6674
CASE $\boxed{\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}} :$	6675
1. $\Gamma \vdash e_0 \rightsquigarrow e'_0$	6676
$\wedge \Gamma \vdash_1 e'_0$	6677
by the induction hypothesis	6678
2. $\Gamma \vdash e_1 \rightsquigarrow e'_1$	6679
$\wedge \Gamma \vdash_1 e'_1$	6680
by the induction hypothesis	6681
3. $\Gamma \vdash \langle e_0, e_1 \rangle \rightsquigarrow \langle e'_0, e'_1 \rangle$	6682
by (1, 2)	6683
4. $\Gamma \vdash_1 \langle e'_0, e'_1 \rangle$	6684
by (1, 2)	6685
5. QED by (3, 4)	6686
CASE $\boxed{\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0 e_1}} :$	6687
1. $\Gamma \vdash e_0 \rightsquigarrow e'_0$	6688
$\wedge \Gamma \vdash_1 e'_0$	6689
by the induction hypothesis	6690
2. $\Gamma \vdash e_1 \rightsquigarrow e'_1$	6691
$\wedge \Gamma \vdash_1 e'_1$	6692
by the induction hypothesis	6693
3. $\Gamma \vdash e_0 e_1 \rightsquigarrow e'_0 e'_1$	6694
by (1, 2)	6695
4. $\Gamma \vdash_1 e'_0 e'_1$	6696
by (1, 2)	6697
5. QED by (3, 4)	6698

6711	CASE	$\boxed{\Gamma \vdash e} : \Gamma \vdash op^1 e$	
6712		1. $\Gamma \vdash e \rightsquigarrow e'$	
6713		$\wedge \Gamma \vdash_1 e'$	
6714		by the induction hypothesis	
6715		2. $\Gamma \vdash op^1 e \rightsquigarrow op^1 e'$	
6716		by (1)	
6717		3. $\vdash_1 op^1 e'$	
6718		by (1)	
6719		4. QED by (2, 3)	
6720	CASE	$\boxed{\Gamma \vdash e_0 \quad \Gamma \vdash e_1} : \Gamma \vdash op^2 e_0 e_1$	
6721		1. $\Gamma \vdash e_0 \rightsquigarrow e'_0$	
6722		$\wedge \Gamma \vdash_1 e'_0$	
6723		by the induction hypothesis	
6724		2. $\Gamma \vdash e_1 \rightsquigarrow e'_1$	
6725		$\wedge \Gamma \vdash_1 e'_1$	
6726		by the induction hypothesis	
6727		3. $\Gamma \vdash op^2 e_0 e_1 \rightsquigarrow op^2 e'_0 e'_1$	
6728		by (1, 2)	
6729		4. $\vdash_1 op^2 e'_0 e'_1$	
6730		by (1, 2)	
6731		5. QED by 3,4	
6732	CASE	$\boxed{} : \Gamma \vdash Err$	
6733		1. $\Gamma \vdash Err \rightsquigarrow Err$	
6734		2. $\vdash_1 Err$	
6735		3. QED	
6736	CASE	$\boxed{\Gamma \vdash e : \tau} : \Gamma \vdash stat \tau e$	
6737		1. $\Gamma \vdash e : \tau \rightsquigarrow e'$	
6738		$\wedge \Gamma \vdash_1 e' : [\tau]$	
6739		by \rightsquigarrow static soundness	
6740		2. $\Gamma \vdash stat \tau e \rightsquigarrow stat \tau e'$	
6741		by (1)	
6742		3. $\vdash_1 stat \tau e$	
6743		by (1)	
6744		4. QED by (2,3)	
6752	□		

Lemma 4.10 : 1 static progressIf $\vdash_1 e : K$ then one of the following holds:

- e is a value
- $e \in Err$
- $e \rightarrow_{1-S} e'$
- $e \rightarrow_{1-S} BndryErr$
- $e = E[\text{dyn } \tau' E^\bullet[e']]$ and $e' \rightarrow_{1-D} TagErr$
- $e = E[\text{dyn } E^\bullet[e']]$ and $e' \rightarrow_{1-D} TagErr$

Proof:

By the *boundary factoring* lemma, there are ten cases.**CASE** e is a value :

1. QED

6765	CASE	$e = E^\bullet[v_0 v_1] :$	6766
6766		1. $\vdash_1 v_0 v_1 : K'$	6767
6767		by static hole typing	6768
6768		2. $\vdash_1 v_0 : \text{Fun}$	6769
6769		by 1 inversion (1)	6770
6770		3. $v_0 = \lambda x. e'$	6771
6771		$\vee v_0 = \lambda(x:\tau_d). e'$	6772
6772		by canonical forms (2)	6773
6773		4. IF $v_0 = \lambda x. e' :$	6774
6774		a. $e \rightarrow_{1-S} E^\bullet[\text{dyn } (e'[x \leftarrow v_1])]$	6775
6775		by $(\lambda x. e') v_1 \triangleright_{1-S} (\text{dyn } (e'[x \leftarrow v_1]))$	6776
6776		b. QED	6777
6777		IF $v_0 = \lambda(x:\tau_d). e'$	6778
6778		$\wedge X([\tau_d], v_1) = v_1 :$	6779
6779		a. $e \rightarrow_{1-S} E^\bullet[e'[x \leftarrow v_1]]$	6780
6780		by $(\lambda(x:\tau_d). e') v_1 \triangleright_{1-S} e'[x \leftarrow v_1]$	6781
6781		b. QED	6782
6782		ELSE $v_0 = \lambda(x:\tau_d). e'$	6783
6783		$\wedge X([\tau_d], v_1) = \text{BndryErr} :$	6784
6784		a. $e \rightarrow_{1-S} E^\bullet[\text{BndryErr}]$	6785
6785		by $(\lambda(x:\tau_d). e') v_1 \triangleright_{1-S} \text{BndryErr}$	6786
6786		b. QED	6787
6787	CASE	$e = E^\bullet[op^1 v] :$	6788
6788		1. $op^1 = \text{fst}$	6789
6789		$\vee op^1 = \text{snd}$	6790
6790		2. $\vdash_1 op^1 v : K'$	6791
6791		by static hole typing	6792
6792		3. $\vdash_1 v : \text{Pair}$	6793
6793		by 1 inversion (2)	6794
6794		4. $v = \langle v_0, v_1 \rangle$	6795
6795		by canonical forms (3)	6796
6796		5. $\delta(op^1, v) = v_i$ where $i \in \{0, 1\}$	6797
6797		by (1, 3)	6798
6798		6. $e \rightarrow_{1-S} E^\bullet[v_i]$	6799
6799		by $(op^1 v) \triangleright_{1-S} v_i$	6800
6800		7. QED	6801
6801	CASE	$e = E^\bullet[op^2 v_0 v_1] :$	6802
6802		1. $\vdash_1 op^2 v_0 v_1 : K'$	6803
6803		by static hole typing	6804
6804		2. $\vdash_1 v_0 : K_0$	6805
6805		$\wedge \vdash_1 v_1 : K_1$	6806
6806		$\wedge \Delta(op^2, K_0, K_1) = K_2$	6807
6807		by 1 inversion (1)	6808
6808		3. $\delta(op^2, v_0, v_1) = e''$	6809
6809		by Δ tag soundness	6810
6810		4. QED by $e \rightarrow_{1-S} E^\bullet[e'']$	6811
6811	CASE	$e = E^\bullet[\text{chk } K v_0] :$	6812
6812		1. $e \rightarrow_{1-S} E^\bullet[X(K, v)]$	6813
6813		2. QED	6814
6814	CASE	$e = E[\text{dyn } e']$ where e' is boundary-free :	6815
6815		1. e' is a value	6816
6816		$\vee e' \in Err$	6817
6817		$\vee e' \rightarrow_{1-D} e''$	6818
6818			6819
6819			6820

6821 $\vee e' \rightarrow_{1-D} \text{BndryErr}$
 6822 $\vee e' = E'[e''] \text{ and } e'' \triangleright_{1-D} \text{TagErr}$
 6823 by 1 *dynamic progress*
 6824 2. **IF** e' is a value :
 6825 a. QED $e \rightarrow_{1-S} E[v]$
 6826 **IF** $e' \in \text{Err}$:
 6827 a. QED $e \rightarrow_{1-S} e'$
 6828 **IF** $e' \rightarrow_{1-D} e''$:
 6829 a. QED $e \rightarrow_{1-S} E[\text{dyn } e'']$
 6830 **IF** $e' \rightarrow_{1-D} \text{BndryErr}$:
 6831 a. QED $e \rightarrow_{1-S} E[\text{dyn BndryErr}]$
 6832 **ELSE** $e' = E'[e''] \text{ and } e'' \triangleright_{1-D} \text{TagErr}$:
 6833 a. $E' \in E^\bullet$
 6834 by e' is boundary-free
 6835 b. QED
 6836 **CASE** $e = E[\text{stat } e']$ where e' is boundary-free :
 6837 1. e' is a value
 6838 $\vee e' \in \text{Err}$
 6839 $\vee e' \rightarrow_{1-S} e''$
 6840 $\vee e' \rightarrow_{1-S} \text{BndryErr}$
 6841 $\vee e' = E''[\text{dyn } \tau'' E^\bullet[e'']] \text{ and } e'' \triangleright_{1-D} \text{TagErr}$
 6842 $\vee e' = E''[\text{dyn } E^\bullet[e'']] \text{ and } e'' \triangleright_{1-D} \text{TagErr}$
 6843 by 1 *static progress*
 6844 2. **IF** e' is a value :
 6845 a. QED $e \rightarrow_{1-S} E[e']$
 6846 **IF** $e' \in \text{Err}$:
 6847 a. QED $e \rightarrow_{1-S} e'$
 6848 **IF** $e' \rightarrow_{1-S} e''$:
 6849 a. QED $e \rightarrow_{1-S} E[\text{stat } e'']$
 6850 **IF** $e' \rightarrow_{1-S} \text{BndryErr}$:
 6851 a. QED $e \rightarrow_{1-S} E[\text{stat BndryErr}]$
 6852 **IF** $e' = E''[\text{dyn } \tau'' E^\bullet[e'']] \text{ and } e'' \triangleright_{1-D} \text{TagErr}$
 6853 :
 6854 a. Contradiction by e' is boundary-free
 6855 **ELSE** $e' = E''[\text{dyn } E^\bullet[e'']] \text{ and } e'' \triangleright_{1-D} \text{TagErr}$
 6856 :
 6857 a. Contradiction by e' is boundary-free
 6858 **CASE** $e = E[\text{dyn } \tau' e']$ where e' is boundary-free :
 6859 1. e' is a value
 6860 $\vee e' \in \text{Err}$
 6861 $\vee e' \rightarrow_{1-D} e''$
 6862 $\vee e' \rightarrow_{1-D} \text{BndryErr}$
 6863 $\vee e' = E'[e''] \text{ and } e'' \triangleright_{1-D} \text{TagErr}$
 6864 by 1 *dynamic progress*
 6865 2. **IF** e' is a value :
 6866 a. QED $e \rightarrow_{1-S} E[\mathcal{D}_1(\tau', e')]$
 6867 **IF** $e' \in \text{Err}$:
 6868 a. QED $e \rightarrow_{1-S} e'$
 6869 **IF** $e' \rightarrow_{1-D} e''$:
 6870 a. QED $e \rightarrow_{1-S} E[\text{dyn } \tau' e'']$
 6871 **IF** $e' \rightarrow_{1-D} \text{BndryErr}$:
 6872 a. QED $e \rightarrow_{1-S} E[\text{dyn } \tau' \text{ BndryErr}]$
 6873 **ELSE** $e' = E'[e''] \text{ and } e'' \triangleright_{1-D} \text{TagErr}$:
 6874

a. $E' \in E^\bullet$ 6876
 by e' is boundary-free 6877
 b. QED 6878
CASE $e = E[\text{stat } \tau' e']$ where e' is boundary-free : 6879
 1. e' is a value 6880
 $\vee e' \in \text{Err}$ 6881
 $\vee e' \rightarrow_{1-S} e''$ 6882
 $\vee e' \rightarrow_{1-S} \text{BndryErr}$ 6883
 $\vee e' = E''[\text{dyn } \tau'' E^\bullet[e'']] \text{ and } e'' \triangleright_{1-D} \text{TagErr}$ 6884
 $\vee e' = E''[\text{dyn } E^\bullet[e'']] \text{ and } e'' \triangleright_{1-D} \text{TagErr}$ 6885
by 1 *static progress* 6886
2. **IF** e' is a value : 6887
a. QED $e \rightarrow_{1-S} E[\mathcal{S}_1(\tau', e')]$ 6888
IF $e' \in \text{Err}$: 6889
a. QED $e \rightarrow_{1-S} e'$ 6890
IF $e' \rightarrow_{1-S} e''$: 6891
a. QED $e \rightarrow_{1-S} E[\text{stat } \tau' e'']$ 6892
IF $e' \rightarrow_{1-S} \text{BndryErr}$: 6893
a. QED $e \rightarrow_{1-S} E[\text{stat } \tau' \text{ BndryErr}]$ 6894
IF $e' = E''[\text{dyn } \tau'' E^\bullet[e'']] \text{ and } e'' \triangleright_{1-D} \text{TagErr}$ 6895
:
a. Contradiction by e' is boundary-free 6897
ELSE $e' = E''[\text{dyn } E^\bullet[e'']] \text{ and } e'' \triangleright_{1-D} \text{TagErr}$ 6898
:
a. Contradiction by e' is boundary-free 6900
CASE $e = E[\text{Err}]$: 6901
1. QED $e \rightarrow_{1-S} \text{Err}$ 6902
□ 6903
Lemma 4.11 : 1 *dynamic progress* 6904
If $\vdash_1 e : K$ then one of the following holds: 6905
• e is a value 6906
• $e \in \text{Err}$ 6907
• $e \rightarrow_{1-D} e'$ 6908
• $e \rightarrow_{1-D} \text{BndryErr}$ 6909
• $e = E[e'] \text{ and } e' \rightarrow_{1-D} \text{TagErr}$ 6910
Proof: 6911
By the *boundary factoring* lemma, there are nine cases. 6912
CASE $e = v$: 6913
1. QED 6914
CASE $e = E^\bullet[v_0 v_1]$: 6915
IF $v_0 = \lambda x. e_0$: 6916
1. $e \rightarrow_{1-D} E^\bullet[e_0[x \leftarrow v_1]]$ 6917
by $(\lambda x. e_0) v_1 \triangleright_{1-D} e_0[x \leftarrow v_1]$ 6918
2. QED 6919
IF $v_0 = \lambda(x:\tau_d). e_0$: 6920
 $\wedge X(\lfloor \tau_d \rfloor, v_1) = v_1$: 6921
1. $e \rightarrow_{1-D} E^\bullet[\text{stat } (e_0[x \leftarrow v_1])]$ 6922
by $(\lambda(x:\tau_d). e_0) v_1 \triangleright_{1-D} (\text{stat } e_0[x \leftarrow v_1])$ 6923
2. QED 6924
IF $v_0 = \lambda(x:\tau_d). e_0$: 6925
 $\wedge X(\lfloor \tau_d \rfloor, v_1) = \text{BndryErr}$: 6926
1. $e \rightarrow_{1-D} E^\bullet[\text{BndryErr}]$ 6927
by $(\lambda(x:\tau_d). e_0) v_1 \triangleright_{1-D} \text{BndryErr}$ 6928
2. QED 6929

6931	ELSE $v_0 = i$	6986
6932	$\vee v_0 = \langle v, v' \rangle :$	6987
6933	1. $e \rightarrow_{1-D} E^\bullet[\text{TagErr}]$	6988
6934	by $v_0 v_1 \triangleright_{1-D} \text{TagErr}$	6989
6935	2. QED	6990
6936	CASE $e = E^\bullet[op^1 v] :$	6991
6937	IF $\delta(op^1, v) = v' :$	6992
6938	1. $e \rightarrow_{1-D} E^\bullet[v']$	6993
6939	by $(op^1 v) \triangleright_{1-D} v'$	6994
6940	2. QED	6995
6941	ELSE $\delta(op^1, v)$ is undefined :	6996
6942	1. $e \rightarrow_{1-D} \text{TagErr}$	6997
6943	by $(op^1 v) \triangleright_{1-D} \text{TagErr}$	6998
6944	2. QED	6999
6945	CASE $e = E^\bullet[op^2 v_0 v_1] :$	7000
6946	IF $\delta(op^2, v_0, v_1) = e'' :$	7001
6947	1. QED by $e \rightarrow_{1-D} E[e'']$	7002
6948	ELSE $\delta(op^2, v_0, v_1)$ is undefined :	7003
6949	1. $e \rightarrow_{1-D} E^\bullet[\text{TagErr}]$	7004
6950	by $(op^2 v_0 v_1) \triangleright_{1-D} \text{TagErr}$	7005
6951	2. QED	7006
6952	CASE $e = E^\bullet[\text{chk } K v_0] :$	7007
6953	1. Contradiction by $\vdash_1 e$	7008
6954	CASE $e = E[\text{dyn } v]$ where e' is boundary-free :	7009
6955	1. e' is a value	7010
6956	$\vee e' \in \text{Err}$	7011
6957	$\vee e' \rightarrow_{1-D} e''$	7012
6958	$\vee e' \rightarrow_{1-D} \text{BndryErr}$	7013
6959	$\vee e' = E'[e'']$ and $e'' \triangleright_{1-D} \text{TagErr}$	7014
6960	by 1 <i>dynamic progress</i>	7015
6961	2. IF e' is a value :	7016
6962	a. QED $e \rightarrow_{1-S} E[v]$	7017
6963	IF $e' \in \text{Err} :$	7018
6964	a. QED $e \rightarrow_{1-S} e'$	7019
6965	IF $e' \rightarrow_{1-D} e'' :$	7020
6966	a. QED $e \rightarrow_{1-S} E[\text{dyn } e'']$	7021
6967	IF $e' \rightarrow_{1-D} \text{BndryErr} :$	7022
6968	a. QED $e \rightarrow_{1-S} E[\text{dyn } \text{BndryErr}]$	7023
6969	ELSE $e' = E'[e'']$ and $e'' \triangleright_{1-D} \text{TagErr} :$	7024
6970	a. $E' \in E^\bullet$	7025
6971	by e' is boundary-free	7026
6972	b. QED	7027
6973	CASE $e = E[\text{stat } e']$ where e' is boundary-free :	7028
6974	1. e' is a value	7029
6975	$\vee e' \in \text{Err}$	7030
6976	$\vee e' \rightarrow_{1-S} e''$	7031
6977	$\vee e' \rightarrow_{1-S} \text{BndryErr}$	7032
6978	$\vee e' = E''[\text{dyn } \tau'' E^\bullet[e'']]$ and $e'' \triangleright_{1-D} \text{TagErr}$	7033
6979	$\vee e' = E''[\text{dyn } E^\bullet[e'']]$ and $e'' \triangleright_{1-D} \text{TagErr}$	7034
6980	by 1 <i>static progress</i>	7035
6981	2. IF e' is a value :	7036
6982	a. QED $e \rightarrow_{1-S} E[e']$	7037
6983	IF $e' \in \text{Err} :$	7038
6984	a. QED $e \rightarrow_{1-S} e'$	7039

7041 \square

7042 **Lemma 4.12 : 1 static preservation**

7043 If $\vdash_1 e : K$ and $e \rightarrow_{1-S} e'$ then $\vdash_1 e' : K$

7044 *Proof:*

7045 By the *boundary factoring* lemma, there are ten cases to consider.

7046 **CASE** e is a value :

- 7048 1. Contradiction by $e \rightarrow_{1-S} e'$

7049 **CASE** $e = E^\bullet[v_0 v_1]$:

7050 **IF** $v_0 = \lambda x. e'$

- 7051 $\wedge e \rightarrow_{1-S} E^\bullet[\text{dyn } e'[x \leftarrow v_1]]$:
- 7052 1. $\vdash_1 v_0 v_1 : \text{Any}$
by *static hole typing*
- 7053 2. $\vdash_1 v_0 : \text{Fun}$
 $\wedge \vdash_1 v_1 : \text{Any}$
by *1 inversion*
- 7054 3. $x \vdash_1 e'$
by *1 inversion* (2)
- 7055 4. $\vdash_1 v_1$
by *static value inversion* (2)
- 7056 5. $\vdash_1 e'[x \leftarrow v_1]$
by *substitution* (3, 4)
- 7057 6. $\vdash_1 \text{dyn}(e'[x \leftarrow v_1]) : \text{Any}$
by (5)
- 7058 7. QED by *hole substitution*

7059 **IF** $v_0 = \lambda(x:\tau). e'$

- 7060 $\wedge X(\lfloor \tau \rfloor, v_1) = \text{BndryErr}$
- 7061 $\wedge e \rightarrow_{1-D} E^\bullet[\text{BndryErr}]$:
- 7062 1. $\vdash_1 v_0 v_1 : \text{Any}$
by *static hole typing*
- 7063 2. $\vdash_1 \text{BndryErr} : \text{Any}$
- 7064 3. QED by *hole substitution* (2)

7065 **ELSE** $v_0 = \lambda(x:\tau). e'$

- 7066 $\wedge e \rightarrow_{1-S} E^\bullet[e'[x \leftarrow X(\lfloor \tau \rfloor, v_1)]]$:
- 7067 1. $\vdash_1 v_0 v_1 : \text{Any}$
by *static hole typing*
- 7068 2. $\vdash_1 v_0 : \text{Fun}$
 $\wedge \vdash_1 v_1 : \text{Any}$
by *1 inversion* (1)
- 7069 3. $(x:\tau) \vdash_1 e' : \text{Any}$
by *1 inversion* (2)
- 7070 4. $\vdash_1 X(\lfloor \tau \rfloor, v_1) : \lfloor \tau \rfloor$
by *check soundness* (2)
- 7071 5. $\vdash_1 e[x \leftarrow X(\lfloor \tau \rfloor, v_1)] : \text{Any}$
by *substitution* (3, 4)
- 7072 6. QED by *hole substitution*

7073 **CASE** $e = E^\bullet[op^1 v]$

- 7074 $\wedge \delta(op^1, v) = v'$
- 7075 $\wedge e \rightarrow_{1-S} E^\bullet[v']$:
- 7076 1. $\vdash_1 op^1 v : \text{Any}$
by *static hole typing*
- 7077 2. $\vdash_1 v : \text{Pair}$
by *1 inversion*

7078 3. $v = \langle v_0, v_1 \rangle$
by *canonical forms*

7079 4. $\vdash_1 v_0 : \text{Any}$
 $\wedge \vdash_1 v_1 : \text{Any}$
by *1 inversion* (2, 3)

7080 5. $v' = v_0$
 $\vee v' = v_1$
by $\delta(\text{fst}, v) = v_0$
 $\wedge \delta(\text{snd}, v) = v_1$

7081 6. QED by *hole substitution* (5)

7082 **CASE** $e = E^\bullet[op^2 v_0 v_1]$

- 7083 $\wedge \delta(op^2, v_0, v_1) = e''$
- 7084 $\wedge e \rightarrow_{1-S} E^\bullet[e'']$:
- 7085 1. $\vdash_1 op^2 v_0 v_1 : K'$
by *static hole typing*
- 7086 2. $\vdash_1 v_0 : K_0$
 $\wedge \vdash_1 v_1 : K_1$
 $\wedge \Delta(op^2, K_0, K_1) = K''$
 $\wedge K'' <: K'$
by *1 inversion* (1)
- 7087 3. $\vdash_1 e'' : K''$
by *Δ tag soundness* (3)
- 7088 4. $\vdash_1 e'' : K'$
by (2, 3)
- 7089 5. QED by *hole substitution* (4)

7090 **CASE** $e = E^\bullet[\text{chk } K_0 v_0]$:

- 7091 1. $E^\bullet[\text{chk } K_0 v_0] \rightarrow_{1-S} E^\bullet[X(K_0, v_0)]$
- 7092 2. $\vdash_1 \text{chk } K_0 v : K''$
by *static hole typing*
- 7093 3. $K_0 \leqslant K''$
by *1 inversion*
- 7094 4. $\vdash_1 X(K_0, v_0) : K_0$
by *check soundness*
- 7095 5. QED by (3, 4, *hole substitution*)

7096 **CASE** $e = E[\text{dyn } e']$ where e' is boundary-free :

7097 **IF** e' is a value :

- 7098 1. $e \rightarrow_{1-S} E[e']$
- 7099 2. $\vdash_1 \text{dyn } e' : \text{Any}$
by *boundary hole typing*
- 7100 3. $\vdash_1 e'$
by *1 inversion* (2)
- 7101 4. $\vdash_1 e' : \text{Any}$
by *dynamic value inversion* (3)
- 7102 5. QED by *hole substitution* (4)

7103 **ELSE** $e' \rightarrow_{1-D} e''$:

- 7104 1. $e \rightarrow_{1-S} E[\text{dyn } e'']$
- 7105 2. $\vdash_1 \text{dyn } e' : \text{Any}$
by *boundary hole typing*
- 7106 3. $\vdash_1 e'$
by *1 inversion* (2)
- 7107 4. $\vdash_1 e''$
by *1 dynamic preservation* (3)

7151 5. $\vdash_1 \text{dyn } e'' : \text{Any}$
 7152 by (4)
 7153 6. QED by *hole substitution* (5)
CASE $e = E[\text{stat } e']$ where e' is boundary-free :
 7155 **IF** e' is a value :
 7156 1. $e \rightarrow_{1-S} E[e']$
 7157 2. $\vdash_1 \text{stat } e'$
 7158 by *boundary hole typing*
 7159 3. $\vdash_1 e' : \text{Any}$
 7160 by *1 inversion* (2)
 7161 4. $\vdash_1 e'$
 7162 by *static value inversion* (3)
 7163 5. QED by *hole substitution* (4)
ELSE $e' \rightarrow_{1-S} e'' :$
 7165 1. $e \rightarrow_{1-S} E[\text{stat } e'']$
 7166 2. $\vdash_1 \text{stat } e'$
 7167 by *boundary hole typing*
 7168 3. $\vdash_1 e' : \text{Any}$
 7169 by *1 inversion* (2)
 7170 4. $\vdash_1 e'' : \text{Any}$
 7171 by *1 static preservation* (3)
 7172 5. $\vdash_1 \text{stat } e''$
 7173 by (4)
 7174 6. QED by *hole substitution* (5)
CASE $e = E[\text{dyn } \tau \ e']$ where e' is boundary-free :
 7176 **IF** e' is a value :
 7177 1. $e \rightarrow_{1-S} E[\mathcal{D}_1(\tau', e')]$
 7178 2. $\vdash_1 \text{dyn } \tau' \ e' : \lfloor \tau' \rfloor$
 7179 by *boundary hole typing*
 7180 3. $\vdash_1 e'$
 7181 by *1 inversion* (2)
 7182 4. $\vdash_1 \mathcal{D}_1(\tau', e') : \lfloor \tau' \rfloor$
 7183 by *D₁ soundness* (3)
 7184 5. QED by *hole substitution* (4)
ELSE $e' \rightarrow_{1-D} e'' :$
 7186 1. $e \rightarrow_{1-S} E[\text{dyn } \tau' \ e'']$
 7187 2. $\vdash_1 \text{dyn } \tau' \ e' : \lfloor \tau' \rfloor$
 7188 by *boundary hole typing*
 7189 3. $\vdash_1 e'$
 7190 by *1 inversion* (2)
 7191 4. $\vdash_1 e''$
 7192 by *1 dynamic preservation* (3)
 7193 5. $\vdash_1 \text{dyn } \tau' \ e'' : \lfloor \tau' \rfloor$
 7194 by (4)
 7195 6. QED by *hole substitution* (5)
CASE $e = E[\text{stat } \tau \ e']$ where e' is boundary-free :
 7197 **IF** e' is a value :
 7198 1. $e \rightarrow_{1-S} E[\mathcal{S}_1(\tau', e')]$
 7199 2. $\vdash_1 \text{stat } \tau' \ e'$
 7200 by *boundary hole typing*
 7201 3. $\vdash_1 e' : \lfloor \tau' \rfloor$
 7202 by *1 inversion* (2)
 7203 4. $\vdash_1 \mathcal{S}_1(\tau', e')$
 7204 by *S₁ soundness* (3)

7205 5. QED by *hole substitution* (4)
ELSE $e' \rightarrow_{1-S} e'' :$
 7206 1. $e \rightarrow_{1-S} E[\text{stat } \tau' \ e'']$
 7207 2. $\vdash_1 \text{stat } \tau' \ e'$
 7208 by *boundary hole typing*
 7209 3. $\vdash_1 e' : \lfloor \tau' \rfloor$
 7210 by *1 inversion* (2)
 7211 4. $\vdash_1 e'' : \lfloor \tau' \rfloor$
 7212 by *1 static preservation* (3)
 7213 5. $\vdash_1 \text{stat } \tau' \ e''$
 7214 by (4)
 7215 6. QED by *hole substitution* (5)
CASE $e = E[\text{Err}] :$
 7216 1. $e \rightarrow_{1-S} \text{Err}$
 7217 2. QED $\vdash_1 \text{Err} : K$
 □
Lemma 4.13 : 1 dynamic preservation
 If $\vdash_1 e$ and $e \rightarrow_{1-D} e'$ then $\vdash_1 e'$
Proof:
 By *boundary factoring* there are nine cases.
CASE e is a value :
 1. Contradiction by $e \rightarrow_{1-D} e'$
CASE $e = E^\bullet[v_0 \ v_1] :$
IF $v_0 = \lambda x. e'$
 $\wedge e \rightarrow_{1-D} E^\bullet[e'[x \leftarrow v_1]] :$
 1. $\vdash_1 v_0 \ v_1$
 by *dynamic hole typing*
 2. $\vdash_1 v_0$
 $\wedge \vdash_1 v_1$
by *1 inversion* (1)
3. $x \vdash_1 e'$
by *1 inversion* (2)
4. $\vdash_1 e'[x \leftarrow v_1]$
by *substitution* (2, 3)
5. QED by *hole substitution*
IF $v_0 = \lambda(x:\tau). e'$
 $\wedge X(\lfloor \tau \rfloor, v_1) = \text{BndryErr}$
 $\wedge e \rightarrow_{1-D} E^\bullet[\text{BndryErr}] :$
1. $\vdash_1 v_0 \ v_1$
by *dynamic hole typing*
2. $\vdash_1 \text{BndryErr}$
3. QED by *hole substitution* (2)
ELSE $v_0 = \lambda(x:\tau). e'$
 $\wedge e \rightarrow_{1-D} E^\bullet[\text{stat}(e'[x \leftarrow X(\lfloor \tau \rfloor, v_1)])] :$
1. $\vdash_1 v_0 \ v_1$
by *dynamic hole typing*
2. $\vdash_1 v_0$
 $\wedge \vdash_1 v_1$
by *1 inversion* (1)
3. $(x:\tau) \vdash_1 e : \text{Any}$
by *1 inversion* (2)
4. $\vdash_1 X(\lfloor \tau \rfloor, v_1) : \lfloor \tau \rfloor$
by *check soundness* (2)

7261	$\vdash_1 e[x \leftarrow X(\lfloor \tau \rfloor, v_1)] : \text{Any}$	7316
7262	by <i>substitution</i> (3, 4)	7317
7263	6. $\vdash_1 \text{stat}(e[x \leftarrow X(\lfloor \tau \rfloor, v_1)])$	7318
7264	by (5)	7319
7265	7. QED by <i>hole substitution</i> (6)	7320
7266	CASE $e = E^\bullet[op^1 v]$	7321
7267	$\wedge \delta(op^1, v) = v'$	7322
7268	$\wedge e \rightarrow_{1\text{-D}} E^\bullet[v'] :$	7323
7269	1. $\vdash_1 op^1 v$	7324
7270	by <i>dynamic hole typing</i>	7325
7271	2. $\vdash_1 v$	7326
7272	by <i>1 inversion</i> (1)	7327
7273	3. $\vdash_1 v'$	7328
7274	by <i>δ preservation</i> (2)	7329
7275	4. QED by <i>hole substitution</i> (3)	7330
7276	CASE $e = E^\bullet[op^2 v_0 v_1]$	7331
7277	$\wedge \delta(op^2, v_0, v_1) = e''$	7332
7278	$\wedge e \rightarrow_{1\text{-D}} E^\bullet[e''] :$	7333
7279	1. $\vdash_1 op^2 v_0 v_1$	7334
7280	by <i>dynamic hole typing</i>	7335
7281	2. $\vdash_1 v_0$	7336
7282	$\wedge \vdash_1 v_1$	7337
7283	by <i>1 inversion</i> (1)	7338
7284	3. $\vdash_1 e''$	7339
7285	by <i>δ preservation</i> (2)	7340
7286	4. QED by <i>hole substitution</i> (3)	7341
7287	CASE $e = E[\text{dyn } e']$ where e' is boundary-free :	7342
7288	IF e' is a value :	7343
7289	1. $e \rightarrow_{1\text{-D}} E[e']$	7344
7290	2. $\vdash_1 \text{dyn } e' : \text{Any}$	7345
7291	by <i>boundary hole typing</i>	7346
7292	3. $\vdash_1 e'$	7347
7293	by <i>1 inversion</i> (2)	7348
7294	4. $\vdash_1 e' : \text{Any}$	7349
7295	by <i>\mathcal{D}_1 soundness</i> (3)	7350
7296	5. QED by <i>hole substitution</i> (4)	7351
7297	ELSE $e' \rightarrow_{1\text{-D}} e'' :$	7352
7298	1. $e \rightarrow_{1\text{-D}} E[\text{dyn } e'']$	7353
7299	2. $\vdash_1 \text{dyn } e' : \text{Any}$	7354
7300	by <i>boundary hole typing</i>	7355
7301	3. $\vdash_1 e'$	7356
7302	by <i>1 inversion</i> (2)	7357
7303	4. $\vdash_1 e''$	7358
7304	by <i>1 dynamic preservation</i> (3)	7359
7305	5. $\vdash_1 \text{dyn } e'' : \text{Any}$	7360
7306	by (4)	7361
7307	6. QED by <i>hole substitution</i> (5)	7362
7308	CASE $e = E[\text{stat } e']$ where e' is boundary-free :	7363
7309	IF $e' \in v :$	7364
7310	1. $e \rightarrow_{1\text{-D}} E[e']$	7365
7311	2. $\vdash_1 \text{stat } e'$	7366
7312	by <i>boundary hole typing</i>	7367
7313	3. $\vdash_1 e' : \text{Any}$	7368
7314	by <i>1 inversion</i> (2)	7369
7315		7370

7371 5. $\vdash_1 \text{stat } \tau' e''$
 7372 by (4)
 7373 6. QED by *hole substitution* (5)
CASE $e = E[\text{Err}] :$
 7375 1. $e \rightarrow_{1-\text{D}} \text{Err}$
 7376 2. QED $\vdash_1 \text{Err}$
 7377 \square
Lemma 4.14 : boundary-free progress
 7379 If $\vdash e : \tau$ and e is boundary-free, then one of the following
 7380 holds:
 7381 • e is a value
 7382 • $e \rightarrow_{1-S} e'$
 7383 • $e \rightarrow_{1-S} \text{BndryErr}$
Proof:
 7385 By the *L unique static evaluation contexts* lemma, there
 7386 are five cases:
CASE $e = v :$
 7388 1. QED
CASE $e = E^\bullet[v_0 v_1] :$
 7390 **IF** $v_0 = \lambda(x:\tau'). e' :$
 7391 1. $e \rightarrow_{1-S} E^\bullet[e'[x \leftarrow v_1]]$
 7392 by $v_0 v_1 \triangleright_{1-S} e'[x \leftarrow v_1]$
 7393 2. QED
ELSE $v_0 = \lambda x. e'$
 7394 $\vee v_0 = i$
 7395 $\vee v_0 = \langle v, v' \rangle :$
 7396 1. Contradiction by $\vdash e : \tau$
CASE $e = E^\bullet[op^1 v] :$
 7399 **IF** $\delta(op^1, v) = e'' :$
 7400 1. $e \rightarrow_{1-S} E^\bullet[e'']$
 7401 by $(op^1 v) \triangleright_{1-S} e''$
 7402 2. QED
ELSE $\delta(op^1, v)$ is undefined :
 7403 1. Contradiction by $\vdash e : \tau$
CASE $e = E^\bullet[op^2 v_0 v_1] :$
 7405 **IF** $\delta(op^2, v_0, v_1) = e'' :$
 7406 1. $e \rightarrow_{1-S} E^\bullet[e'']$
 7407 by $(op^2 v_0 v_1) \triangleright_{1-S} e''$
 7408 2. QED
IF $\delta(op^2, v_0, v_1) = \text{BndryErr} :$
 7409 1. $e \rightarrow_{1-S} \text{BndryErr}$
 7410 by $(op^2 v_0 v_1) \triangleright_{1-S} \text{BndryErr}$
 7411 2. QED
ELSE $\delta(op^2, v_0, v_1)$ is undefined :
 7412 1. Contradiction by $\vdash e : \tau$
CASE $e = E^\bullet[\text{Err}] :$
 7413 1. $E^\bullet[\text{Err}] \rightarrow_{1-S} \text{Err}$
 7414 2. QED
 7415 \square
Lemma 4.15 : 1 boundary-free preservation
 7416 If $\vdash e : \tau$ and e is boundary-free and $e \rightarrow_{1-S} e'$ then $\vdash e' : \tau$
 7417 and e' is boundary-free.
Proof:

By the *L unique static evaluation contexts* lemma, there
 7426 are five cases.
CASE e is a value :
 7428 1. Contradiction by $e \rightarrow_{1-S} e'$
CASE $e = E^\bullet[v_0 v_1] :$
 7430 **IF** $v_0 = \lambda(x:\tau_d). e' :$
 7431 1. $E^\bullet[v_0 v_1] \rightarrow_{1-S} E^\bullet[e'[x \leftarrow v_1]]$
 7432 2. $\vdash v_0 v_1 : \tau_c$
 7433 3. $\vdash v_0 : \tau_d \Rightarrow \tau_c$
 7434 by (2)
 7435 4. $(x:\tau_d) \vdash e' : \tau_c$
 7436 by (3)
 7437 5. $\vdash e'[x \leftarrow v_1] : \tau_c$
 7438 by *substitution* (3, 4)
 7439 6. $e'[x \leftarrow v_1]$ is boundary-free
 7440 by e' and v_1 are boundary-free
 7441 7. QED
ELSE :
 7442 1. Contradiction by $\vdash e : \tau$
CASE $e = E^\bullet[op^1 v] :$
 7444 1. $E^\bullet[op^1 v] \rightarrow_{1-S} E^\bullet[v']$
 7445 by $\delta(op^1, v) = e'$
 7446 2. $\vdash op^1 v : \tau'$
 7447 3. $\vdash v : \tau_0$
 7448 4. $\vdash e' : \tau'$
 7449 by *δ preservation* (3)
 7450 5. QED
CASE $e = E^\bullet[op^2 v_0 v_1] :$
 7451 1. $E^\bullet[op^2 v_0 v_1] \rightarrow_{1-S} E^\bullet[v']$
 7452 by $\delta(op^2, v_0, v_1) = e'$
 7453 2. $\vdash op^2 v_0 v_1 : \tau'$
 7454 3. $\vdash v_0 : \tau_0$
 7455 by $\delta(op^2, v_0, v_1) = e'$
 7456 4. $\vdash e' : \tau'$
 7457 by *δ preservation* (3)
 7458 5. QED
CASE $e = E^\bullet[\text{Err}] :$
 7459 1. $E^\bullet[\text{Err}] \rightarrow_{1-S} \text{Err}$
 7460 2. QED by $\vdash \text{Err} : \tau$
 7461 \square
Lemma 4.16 : X soundness
 7462 For all K and v , $\vdash_1 X(K, v) : K$.
Proof:
 7463 **CASE** $\vdash_1 v : K :$
 7464 1. $X(K, v) = v$
 7465 2. QED
CASE $\nvdash_1 v : K :$
 7466 1. $X(K, v) = \text{BndryErr}$
 7467 2. QED
 7468 \square
Lemma 4.17 : 1 static boundary factoring
 7469

7481 If $\vdash_1 e : K$ then one of the following holds:
 7482 • e is a value
 7483 • $e = E^\bullet[v_0 v_1]$
 7484 • $e = E^\bullet[op^1 v]$
 7485 • $e = E^\bullet[op^2 v_0 v_1]$
 7486 • $e = E^\bullet[\text{chk } K v]$
 7487 • $e = E[\text{dyn } e']$ where e' is boundary-free
 7488 • $e = E[\text{stat } e']$ where e' is boundary-free
 7489 • $e = E[\text{dyn } \tau e']$ where e' is boundary-free
 7490 • $e = E[\text{stat } \tau e']$ where e' is boundary-free
 7491 • $e = E[\text{Err}]$

Proof:

By the *unique evaluation contexts* lemma, there are ten cases.

CASE e is a value :

1. QED

CASE $e = E[v_0 v_1]$:

1. $E = E^\bullet$

$\vee E = E'[\text{dyn } E^\bullet]$

$\vee E = E'[\text{stat } E^\bullet]$

$\vee E = E'[\text{dyn } \tau E^\bullet]$

$\vee E = E'[\text{stat } \tau E^\bullet]$

by *inner boundary*

2. IF $E = E^\bullet$:

a. QED $e = E^\bullet[v_0 v_1]$

IF $E = E'[\text{dyn } E^\bullet]$:

a. QED $e = E'[\text{dyn } E^\bullet[v_0 v_1]]$

IF $E = E'[\text{stat } E^\bullet]$:

a. QED $e = E'[\text{stat } E^\bullet[v_0 v_1]]$

IF $E = E'[\text{dyn } \tau E^\bullet]$:

a. QED $e = E'[\text{dyn } \tau E^\bullet[v_0 v_1]]$

ELSE $E = E'[\text{stat } \tau E^\bullet]$:

a. QED $e = E'[\text{stat } \tau E^\bullet[v_0 v_1]]$

CASE $e = E[op^1 v]$:

1. $E = E^\bullet$

$\vee E = E'[\text{dyn } E^\bullet]$

$\vee E = E'[\text{stat } E^\bullet]$

$\vee E = E'[\text{dyn } \tau E^\bullet]$

$\vee E = E'[\text{stat } \tau E^\bullet]$

by *inner boundary*

2. IF $E = E^\bullet$:

a. QED $e = E^\bullet[op^1 v]$

IF $E = E'[\text{dyn } E^\bullet]$:

a. QED $e = E'[\text{dyn } E^\bullet[op^1 v]]$

IF $E = E'[\text{stat } E^\bullet]$:

a. QED $e = E'[\text{stat } E^\bullet[op^1 v]]$

IF $E = E'[\text{dyn } \tau E^\bullet]$:

a. QED $e = E'[\text{dyn } \tau E^\bullet[op^1 v]]$

ELSE $E = E'[\text{stat } \tau E^\bullet]$:

a. QED $e = E'[\text{stat } \tau E^\bullet[op^1 v]]$

CASE $e = E[op^2 v_0 v_1]$:

1. $E = E^\bullet$

$\vee E = E'[\text{dyn } E^\bullet]$

$\vee E = E'[\text{stat } E^\bullet]$

$\vee E = E'[\text{dyn } \tau E^\bullet]$ 7536
 $\vee E = E'[\text{stat } \tau E^\bullet]$ 7537
 by *inner boundary* 7538
 2. IF $E = E^\bullet$: 7539
 a. QED $e = E^\bullet[op^2 v_0 v_1]$ 7540
 IF $E = E'[\text{dyn } E^\bullet]$: 7541
 a. QED $e = E'[\text{dyn } E^\bullet[op^2 v_0 v_1]]$ 7542
 IF $E = E'[\text{stat } E^\bullet]$: 7543
 a. QED $e = E'[\text{stat } E^\bullet[op^2 v_0 v_1]]$ 7544
 IF $E = E'[\text{dyn } \tau E^\bullet]$: 7545
 a. QED $e = E'[\text{dyn } \tau E^\bullet[op^2 v_0 v_1]]$ 7546
 ELSE $E = E'[\text{stat } \tau E^\bullet]$: 7547
 a. QED $e = E'[\text{stat } \tau E^\bullet[op^2 v_0 v_1]]$ 7548

CASE $e = E[\text{dyn } v]$: 7549

1. QED v is boundary-free 7550

CASE $e = E[\text{stat } v]$: 7551

1. QED v is boundary-free 7552

CASE $e = E[\text{dyn } \tau v]$: 7553

1. QED v is boundary-free 7554

CASE $e = E[\text{stat } \tau v]$: 7555

1. QED v is boundary-free 7556

CASE $e = E[\text{Err}]$: 7557

1. QED 7558

□ 7559

Lemma 4.18 : 1 unique static evaluation contexts 7560

If $\vdash_1 e : K$ then one of the following holds: 7561

- e is a value 7562
- $e = E[v_0 v_1]$ 7563
- $e = E[op^1 v]$ 7564
- $e = E[op^2 v_0 v_1]$ 7565
- $e = E[\text{chk } K v]$ 7566
- $e = E[\text{dyn } v]$ 7567
- $e = E[\text{stat } v]$ 7568
- $e = E[\text{dyn } \tau v]$ 7569
- $e = E[\text{stat } \tau v]$ 7570
- $e = E[\text{Err}]$ 7571

Proof:

By induction on the structure of e . 7573

CASE $e = x$: 7574

1. Contradiction by $\vdash_1 e : K$ 7575

CASE $e = i$: 7576

$\vee e = \lambda x. e'$ 7577

$\vee e = \lambda(x:\tau_d). e'$: 7578

1. QED e is a value 7579

CASE $e = \langle e_0, e_1 \rangle$: 7580

IF $e_0 \notin v$: 7581

1. $e_0 = E_0[e'_0]$ 7582

by the induction hypothesis 7583

2. $E = \langle E_0, e_1 \rangle$ 7584

3. QED by $e = E[e'_0]$ 7585

IF $e_0 \in v$ 7586

$\wedge e_1 \notin v$: 7587

1. $e_1 = E_1[e'_1]$ 7588

by the induction hypothesis 7589

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7591      2.  $E = \langle e_0, E_1 \rangle$                                      7646
7592      3. QED by  $e = E[e'_1]$                                  7647
7593      ELSE  $e_0 \in v$  :                                         7648
7594           $\wedge e_1 \in v$  :                                         7649
7595          1.  $E = []$                                          7650
7596          2. QED  $e = E[\langle e_0, e_1 \rangle]$                   7651
7597 CASE  $e = e_0 e_1$  :                                         7652
7598     IF  $e_0 \notin v$  :                                         7653
7599         1.  $e_0 = E_0[e'_0]$                                 7654
7600             by the induction hypothesis                      7655
7601         2.  $E = E_0 e_1$                                          7656
7602         3. QED by  $e = E[e'_0]$                                7657
7603     IF  $e_0 \in v$  :                                         7658
7604          $\wedge e_1 \notin v$  :                                         7659
7605         1.  $e_1 = E_1[e'_1]$                                 7660
7606             by the induction hypothesis                      7661
7607         2.  $E = e_0 E_1$                                          7662
7608         3. QED by  $e = E[e'_1]$                                7663
7609     ELSE  $e_0 \in v$  :                                         7664
7610          $\wedge e_1 \in v$  :                                         7665
7611         1.  $E = []$                                          7666
7612         2. QED  $e = E[e_0 e_1]$                              7667
7613 CASE  $e = op^1 e_0$  :                                         7668
7614     1. IF  $e_0 \notin v$  :                                         7669
7615         a.  $e_0 = E_0[e'_0]$                                 7670
7616             by the induction hypothesis                      7671
7617         b.  $E = op^1 E_0$                                          7672
7618         c. QED  $e = E[e'_0]$                                7673
7619     2. ELSE  $e_0 \in v$  :                                         7674
7620         a.  $E = []$                                          7675
7621         b. QED  $e = E[op^1 e_0]$                             7676
7622 CASE  $e = op^2 e_0 e_1$  :                                         7677
7623     IF  $e_0 \notin v$  :                                         7678
7624         1.  $e_0 = E_0[e'_0]$                                 7679
7625             by the induction hypothesis                      7680
7626         2.  $E = op^2 E_0 e_1$                                7681
7627         3. QED  $e = E[e'_0]$                                7682
7628     IF  $e_0 \in v$  :                                         7683
7629          $\wedge e_1 \notin v$  :                                         7684
7630         1.  $e_1 = E_1[e'_1]$                                 7685
7631             by the induction hypothesis                      7686
7632         2.  $E = op^2 e_0 E_1$                                7687
7633         3. QED  $e = E[e'_1]$                                7688
7634     ELSE  $e_0 \in v$  :                                         7689
7635          $\wedge e_1 \in v$  :                                         7690
7636         1.  $E = []$                                          7691
7637         2. QED  $e = E[op^2 e_0 e_1]$                          7692
7638 CASE  $e = \text{chk } K e_0$  :                                         7693
7639     IF  $e_0 \notin v$  :                                         7694
7640         1.  $e_0 = E_0[e'_0]$                                 7695
7641             by the induction hypothesis                      7696
7642         2.  $E = \text{chk } K E_0$                                7697
7643         3. QED  $e = E[e'_0]$                                7698
7644     ELSE  $e_0 \in v$  :                                         7699
7645

```

□

Lemma 4.19 : 1 inner boundary

For all contexts E , one of the following holds:

- $E = E^\bullet$
- $E = E'[\text{dyn } v]$
- $E = E'[\text{stat } v]$
- $E = E'[\text{dyn } \tau E^\bullet]$
- $E = E'[\text{stat } \tau E^\bullet]$

Proof:

By induction on the structure of E .

CASE $E = E^\bullet$:

1. QED

CASE $E = E_0 e_1$:

1. $E_0 = E^\bullet$
 - $\vee E_0 = E'_0[\text{dyn } E^\bullet]$
 - $\vee E_0 = E'_0[\text{stat } E^\bullet]$
 - $\vee E_0 = E'_0[\text{dyn } \tau E^\bullet]$
 - $\vee E_0 = E'_0[\text{stat } \tau E^\bullet]$

by the induction hypothesis

2. IF $E_0 = E^\bullet$:

- a. QED E is boundary-free

IF $E_0 = E'_0[\text{dyn } E^\bullet]$:

- a. $E' = E'_0 e_1$

7701	b. QED $E = E'[\text{dyn } E^\bullet]$	7756
7702	IF $E_0 = E'_0[\text{stat } E^\bullet]$:	7757
7703	a. $E' = E'_0 e_1$	7758
7704	b. QED $E = E'[\text{stat } E^\bullet]$	7759
7705	IF $E_0 = E'_0[\text{dyn } \tau E^\bullet]$:	7760
7706	a. $E' = E'_0 e_1$	7761
7707	b. QED $E = E'[\text{dyn } \tau E^\bullet]$	7762
7708	ELSE $E_0 = E'_0[\text{stat } \tau E^\bullet]$:	7763
7709	a. $E' = E'_0 e_1$	7764
7710	b. QED $E = E'[\text{stat } \tau E^\bullet]$	7765
7711	CASE $E = v_0 E_1$:	7766
7712	1. $E_1 = E^\bullet$	7767
7713	$\vee E_1 = E'_1[\text{dyn } E^\bullet]$	7768
7714	$\vee E_1 = E'_1[\text{stat } E^\bullet]$	7769
7715	$\vee E_1 = E'_1[\text{dyn } \tau E^\bullet]$	7770
7716	$\vee E_1 = E'_1[\text{stat } \tau E^\bullet]$	7771
7717	by the induction hypothesis	7772
7718	2. IF $E_1 = E^\bullet$:	7773
7719	a. QED E is boundary-free	7774
7720	IF $E_1 = E'_1[\text{dyn } E^\bullet]$:	7775
7721	a. $E' = v_0 E'_1$	7776
7722	b. QED $E = E'[\text{dyn } E^\bullet]$	7777
7723	IF $E_1 = E'_1[\text{stat } E^\bullet]$:	7778
7724	a. $E' = v_0 E'_1$	7779
7725	b. QED $E = E'[\text{stat } E^\bullet]$	7780
7726	IF $E_1 = E'_1[\text{dyn } \tau E^\bullet]$:	7781
7727	a. $E' = v_0 E'_1$	7782
7728	b. QED $E = E'[\text{dyn } \tau E^\bullet]$	7783
7729	ELSE $E_1 = E'_1[\text{stat } \tau E^\bullet]$:	7784
7730	a. $E' = v_0 E'_1$	7785
7731	b. QED $E = E'[\text{stat } \tau E^\bullet]$	7786
7732	CASE $E = \langle E_0, e_1 \rangle$:	7787
7733	1. $E_0 = E^\bullet$	7788
7734	$\vee E_0 = E'_0[\text{dyn } E^\bullet]$	7789
7735	$\vee E_0 = E'_0[\text{stat } E^\bullet]$	7790
7736	$\vee E_0 = E'_0[\text{dyn } \tau E^\bullet]$	7791
7737	$\vee E_0 = E'_0[\text{stat } \tau E^\bullet]$	7792
7738	by the induction hypothesis	7793
7739	2. IF $E_0 = E^\bullet$:	7794
7740	a. QED E is boundary-free	7795
7741	IF $E_0 = E'_0[\text{dyn } E^\bullet]$:	7796
7742	a. $E' = \langle E'_0, e_1 \rangle$	7797
7743	b. QED $E = E'[\text{dyn } E^\bullet]$	7798
7744	IF $E_0 = E'_0[\text{stat } E^\bullet]$:	7799
7745	a. $E' = \langle E'_0, e_1 \rangle$	7800
7746	b. QED $E = E'[\text{stat } E^\bullet]$	7801
7747	IF $E_0 = E'_0[\text{dyn } \tau E^\bullet]$:	7802
7748	a. $E' = \langle E'_0, e_1 \rangle$	7803
7749	b. QED $E = E'[\text{dyn } \tau E^\bullet]$	7804
7750	ELSE $E_0 = E'_0[\text{stat } \tau E^\bullet]$:	7805
7751	a. $E' = \langle E'_0, e_1 \rangle$	7806
7752	b. QED $E = E'[\text{stat } \tau E^\bullet]$	7807
7753	CASE $E = \langle v_0, E_1 \rangle$:	7808
7754		7809
7755		7810

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7811   IF  $E_0 = E'_0[\text{stat } E^\bullet]$  :
7812     a.  $E' = op^2 E'_0 e_1$ 
7813     b. QED  $E = E'[\text{stat } E^\bullet]$ 
7814   IF  $E_0 = E'_0[\text{dyn } \tau E^\bullet]$  :
7815     a.  $E' = op^2 E'_0 e_1$ 
7816     b. QED  $E = E'[\text{dyn } \tau E^\bullet]$ 
7817   ELSE  $E_0 = E'_0[\text{stat } \tau E^\bullet]$  :
7818     a.  $E' = op^2 E'_0 e_1$ 
7819     b. QED  $E = E'[\text{stat } \tau E^\bullet]$ 
7820 CASE  $E = op^2 v_0 E_1$  :
7821   1.  $E_1 = E^\bullet$ 
7822      $\vee E_1 = E'_1[\text{dyn } E^\bullet]$ 
7823      $\vee E_1 = E'_1[\text{stat } E^\bullet]$ 
7824      $\vee E_1 = E'_1[\text{dyn } \tau E^\bullet]$ 
7825      $\vee E_1 = E'_1[\text{stat } \tau E^\bullet]$ 
7826     by the induction hypothesis
7827   2. IF  $E_1 = E^\bullet$  :
7828     a. QED  $E$  is boundary-free
7829     IF  $E_1 = E'_1[\text{dyn } E^\bullet]$  :
7830       a.  $E' = op^2 v_0 E'_1$ 
7831       b. QED  $E = E'[\text{dyn } E^\bullet]$ 
7832     IF  $E_1 = E'_1[\text{stat } E^\bullet]$  :
7833       a.  $E' = op^2 v_0 E'_1$ 
7834       b. QED  $E = E'[\text{stat } E^\bullet]$ 
7835     IF  $E_1 = E'_1[\text{dyn } \tau E^\bullet]$  :
7836       a.  $E' = op^2 v_0 E'_1$ 
7837       b. QED  $E = E'[\text{dyn } \tau E^\bullet]$ 
7838     ELSE  $E_1 = E'_1[\text{stat } \tau E^\bullet]$  :
7839       a.  $E' = op^2 v_0 E'_1$ 
7840       b. QED  $E = E'[\text{stat } \tau E^\bullet]$ 
7841 CASE  $E = \text{dyn } E_0$  :
7842   1.  $E_0 = E^\bullet$ 
7843      $\vee E_0 = E'_0[\text{dyn } E^\bullet]$ 
7844      $\vee E_0 = E'_0[\text{stat } E^\bullet]$ 
7845      $\vee E_0 = E'_0[\text{dyn } \tau' E^\bullet]$ 
7846      $\vee E_0 = E'_0[\text{stat } \tau' E^\bullet]$ 
7847     by the induction hypothesis
7848   2. IF  $E_0 = E^\bullet$  :
7849     a. QED
7850     IF  $E_0 = E'_0[\text{dyn } E^\bullet]$  :
7851       a.  $E' = \text{dyn } E'_0$ 
7852       b. QED  $E = E'[\text{dyn } E^\bullet]$ 
7853     IF  $E_0 = E'_0[\text{stat } E^\bullet]$  :
7854       a.  $E' = \text{dyn } E'_0$ 
7855       b. QED  $E = E'[\text{stat } E^\bullet]$ 
7856     IF  $E_0 = E'_0[\text{dyn } \tau' E^\bullet]$  :
7857       a.  $E' = \text{dyn } E'_0$ 
7858       b. QED  $E = E'[\text{dyn } \tau' E^\bullet]$ 
7859     ELSE  $E_0 = E'_0[\text{stat } \tau' E^\bullet]$  :
7860       a.  $E' = \text{dyn } E'_0$ 
7861       b. QED  $E = E'[\text{stat } \tau' E^\bullet]$ 
7862 CASE  $E = \text{stat } E_0$  :
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7921	a. $E' = \text{stat } \tau E'_0$	7976
7922	b. $\text{QED } E = E'[\text{stat } E^\bullet]$	7977
7923	IF $E_0 = E'_0[\text{dyn } \tau' E^\bullet]$:	7978
7924	a. $E' = \text{stat } \tau E'_0$	7979
7925	b. $\text{QED } E = E'[\text{dyn } \tau' E^\bullet]$	7980
7926	ELSE $E_0 = E'_0[\text{stat } \tau' E^\bullet]$:	7981
7927	a. $E' = \text{stat } \tau E'_0$	7982
7928	b. $\text{QED } E = E'[\text{stat } \tau' E^\bullet]$	7983
7929	CASE $E = \text{chk } K_0 E_0$:	7984
7930	1. $E_0 = E^\bullet$	7985
7931	$\vee E_0 = E'_0[\text{dyn } E^\bullet]$	7986
7932	$\vee E_0 = E'_0[\text{stat } E^\bullet]$	7987
7933	$\vee E_0 = E'_0[\text{dyn } \tau E^\bullet]$	7988
7934	$\vee E_0 = E'_0[\text{stat } \tau E^\bullet]$	7989
7935	by the induction hypothesis	7990
7936	2. IF $E_0 = E^\bullet$:	7991
7937	a. $\text{QED } E$ is boundary-free	7992
7938	IF $E_0 = E'_0[\text{dyn } E^\bullet]$:	7993
7939	a. $E' = \text{chk } K_0 E'_0$	7994
7940	b. $\text{QED } E = E'[\text{dyn } E^\bullet]$	7995
7941	IF $E_0 = E'_0[\text{stat } E^\bullet]$:	7996
7942	a. $E' = \text{chk } K_0 E'_0$	7997
7943	b. $\text{QED } E = E'[\text{stat } E^\bullet]$	7998
7944	IF $E_0 = E'_0[\text{dyn } \tau E^\bullet]$:	7999
7945	a. $E' = \text{chk } K_0 E'_0$	8000
7946	b. $\text{QED } E = E'[\text{dyn } \tau E^\bullet]$	8001
7947	ELSE $E_0 = E'_0[\text{stat } \tau E^\bullet]$:	8002
7948	a. $E' = \text{chk } K_0 E'_0$	8003
7949	b. $\text{QED } E = E'[\text{stat } \tau E^\bullet]$	8004
7950	□	8005
7951	Lemma 4.20 : 1 dynamic boundary factoring	8006
7952	If $\vdash_1 e$ then one of the following holds:	8007
7953	• e is a value	8008
7954	• $e = E^\bullet[v_0 v_1]$	8009
7955	• $e = E^\bullet[op^1 v]$	8010
7956	• $e = E^\bullet[op^2 v_0 v_1]$	8011
7957	• $e = E[\text{dyn } e']$ where e' is boundary-free	8012
7958	• $e = E[\text{stat } e']$ where e' is boundary-free	8013
7959	• $e = E[\text{dyn } \tau e']$ where e' is boundary-free	8014
7960	• $e = E[\text{stat } \tau e']$ where e' is boundary-free	8015
7961	• $e = E[\text{Err}]$	8016
7962	Proof:	8017
7963	By the <i>unique evaluation contexts</i> lemma, there are ten cases.	8018
7964		8019
7965	CASE e is a value :	8020
7966	1. QED	8021
7967	CASE $e = E[v_0 v_1]$:	8022
7968	1. $E = E^\bullet$	8023
7969	$\vee E = E'[\text{dyn } E^\bullet]$	8024
7970	$\vee E = E'[\text{stat } E^\bullet]$	8025
7971	$\vee E = E'[\text{dyn } \tau E^\bullet]$	8026
7972	$\vee E = E'[\text{stat } \tau E^\bullet]$	8027
7973	by <i>inner boundary</i>	8028
7974	2. IF $E = E^\bullet$:	8029
7975		8030

```

8031   IF  $E = E'[\text{stat } E^\bullet]$  :
8032     a. QED  $e = E'[\text{stat } E^\bullet[\text{chk } K' v]]$ 
8033   IF  $E = E'[\text{dyn } \tau E^\bullet]$  :
8034     a. QED  $e = E'[\text{dyn } \tau E^\bullet[\text{chk } K' v]]$ 
8035   ELSE  $E = E'[\text{stat } \tau E^\bullet]$  :
8036     a. QED  $e = E'[\text{stat } \tau E^\bullet[\text{chk } K' v]]$ 
8037   CASE  $e = E[\text{dyn } v]$  :
8038     1. QED  $v$  is boundary-free
8039   CASE  $e = E[\text{stat } v]$  :
8040     1. QED  $v$  is boundary-free
8041   CASE  $e = E[\text{dyn } \tau v]$  :
8042     1. QED  $v$  is boundary-free
8043   CASE  $e = E[\text{stat } \tau v]$  :
8044     1. QED  $v$  is boundary-free
8045   CASE  $e = E[\text{Err}]$  :
8046     1. QED
8047  $\square$ 

```

Lemma 4.21 : 1 unique dynamic evaluation contexts

If $\vdash_1 e$ then one of the following holds:

- e is a value
- $e = E[v_0 v_1]$
- $e = E[op^1 v]$
- $e = E[op^2 v_0 v_1]$
- $e = E[\text{chk } K v]$
- $e = E[\text{dyn } v]$
- $e = E[\text{stat } v]$
- $e = E[\text{dyn } \tau v]$
- $e = E[\text{stat } \tau v]$
- $e = E[\text{Err}]$

Proof:

By induction on the structure of e .

CASE $e = x$:

1. Contradiction by $\vdash_1 e$

CASE $e = i$

$$\vee e = \lambda x. e'$$

$$\vee e = \lambda(x:\tau_d). e' :$$

1. QED e is a value

CASE $e = \langle e_0, e_1 \rangle$:

IF $e_0 \notin v$:

$$1. e_0 = E_0[e'_0]$$

by the induction hypothesis

$$2. E = \langle E_0, e_1 \rangle$$

$$3. \text{QED } e = E[e'_0]$$

IF $e_0 \in v$

$\wedge e_1 \notin v$:

$$1. e_1 = E_1[e'_1]$$

by the induction hypothesis

$$2. E = \langle e_0, E_1 \rangle$$

$$3. \text{QED } e = E[e'_1]$$

ELSE $e_0 \in v$

$\wedge e_1 \in v$:

$$1. E = []$$

$$2. \text{QED } e = E[\langle e_0, e_1 \rangle]$$

CASE $e = e_0 e_1$:

```

8031   IF  $e_0 \notin v$  :
8032     1.  $e_0 = E_0[e'_0]$ 
8033       by the induction hypothesis
8034     2.  $E = E_0 e_1$ 
8035     3. QED  $e = E[e'_0]$ 
8036   IF  $e_0 \in v$ 
8037      $\wedge e_1 \notin v$  :
8038       1.  $e_1 = E_1[e'_1]$ 
8039         by the induction hypothesis
8040       2.  $E = e_0 E_1$ 
8041       3. QED  $e = E[e'_1]$ 
8042   ELSE  $e_0 \in v$ 
8043      $\wedge e_1 \in v$  :
8044       1.  $E = []$ 
8045       2. QED  $e = E[e_0 e_1]$ 
8046   CASE  $e = op^1 e_0$  :
8047     IF  $e_0 \notin v$  :
8048       1.  $e_0 = E_0[e'_0]$ 
8049         by the induction hypothesis
8050       2.  $E = op^1 E_0$ 
8051       3. QED  $e = E[e'_0]$ 
8052   ELSE  $e_0 \in v$  :
8053     1.  $E = []$ 
8054     2. QED  $e = E[op^1 e_0]$ 
8055   CASE  $e = op^2 e_0 e_1$  :
8056     IF  $e_0 \notin v$  :
8057       1.  $e_0 = E_0[e'_0]$ 
8058         by the induction hypothesis
8059       2.  $E = op^2 E_0 e_1$ 
8060       3. QED  $e = E[e'_0]$ 
8061     IF  $e_0 \in v$ 
8062        $\wedge e_1 \notin v$  :
8063         1.  $e_1 = E_1[e'_1]$ 
8064           by the induction hypothesis
8065         2.  $E = op^2 e_0 E_1$ 
8066         3. QED  $e = E[e'_1]$ 
8067   ELSE  $e_0 \in v$ 
8068      $\wedge e_1 \in v$  :
8069       1.  $E = []$ 
8070       2. QED  $e = E[op^2 e_0 e_1]$ 
8071   CASE  $e = \text{chk } K e'$  :
8072     1. Contradiction by  $\vdash_1 e$ 
8073   CASE  $e = \text{dyn } e_0$  :
8074     1. Contradiction by  $\vdash_1 e$ 
8075   CASE  $e = \text{stat } e_0$  :
8076     IF  $e_0 \notin v$  :
8077       1.  $e_0 = E_0[e'_0]$ 
8078         by the induction hypothesis
8079       2.  $E = \langle e_0, E_1 \rangle$ 
8080       3. QED  $e = E[e'_0]$ 
8081     ELSE  $e_0 \in v$ 
8082        $\wedge e_1 \in v$  :
8083         1.  $E = []$ 
8084         2. QED  $e = E[\langle e_0, e_1 \rangle]$ 
8085   CASE  $e = e_0 e_1$  :

```

```

8141      2. QED  $e = E[\text{stat } e_0]$ 
8142 CASE  $e = \text{dyn } e_0 :$ 
8143   Contradiction by  $\vdash_1 e$ 
8144 CASE  $e = \text{stat } K_0 e_0 :$ 
8145   IF  $e_0 \notin v :$ 
8146     1.  $\vdash_1 e_0$ 
8147       by 1 inversion
8148     2.  $e_0 = E_0[e'_0]$ 
8149       by unique evaluation contexts (1)
8150     3.  $E = \text{stat } \tau E_0$ 
8151     4. QED  $e = E[e'_0]$ 
8152 ELSE  $e_0 \in v :$ 
8153   1.  $E = []$ 
8154   2. QED  $e = E[\text{stat } \tau e_0]$ 
8155  $\square$ 
8156 Lemma 4.22 : 1 static hole typing
8157 If  $\vdash_1 E^*[e] : K$  then the typing derivation contains a sub-term
8158  $\vdash_1 e : K'$  for some  $K'$ .
8159 Proof:
8160   By induction on the structure of  $E^*$ .
8161 CASE  $E^* = [] :$ 
8162   1. QED  $E^*[e] = e$ 
8163 CASE  $E^* = E^*_0 e_1 :$ 
8164   1.  $E^*[e] = E^*_0[e] e_1$ 
8165   2.  $\vdash_1 E^*_0[e] : \text{Fun}$ 
8166     by 1 inversion
8167   3. QED by the induction hypothesis (2)
8168 CASE  $E^* = v_0 E^*_1 :$ 
8169   1.  $E^*[e] = v_0 E^*_1[e]$ 
8170   2.  $\vdash_1 E^*_1[e] : \text{Any}$ 
8171     by 1 inversion
8172   3. QED by the induction hypothesis (2)
8173 CASE  $E^* = \langle E^*_0, e_1 \rangle :$ 
8174   1.  $E^*[e] = \langle E^*_0[e], e_1 \rangle$ 
8175   2.  $\vdash_1 E^*_0[e] : \text{Any}$ 
8176     by 1 inversion
8177   3. QED by the induction hypothesis (2)
8178 CASE  $E^* = \langle v_0, E^*_1 \rangle :$ 
8179   1.  $E^*[e] = \langle v_0, E^*_1[e] \rangle$ 
8180   2.  $\vdash_1 E^*_1[e] : \text{Any}$ 
8181     by 1 inversion
8182   3. QED by the induction hypothesis (2)
8183 CASE  $E^* = op^1 E^*_0 :$ 
8184   1.  $E^*[e] = op^1 E^*_0[e]$ 
8185   2.  $\vdash_1 E^*_0[e] : \text{Pair}$ 
8186     by 1 inversion
8187   3. QED by the induction hypothesis (2)
8188 CASE  $E^* = op^2 E^*_0 e_1 :$ 
8189   1.  $E^*[e] = op^2 E^*_0[e] e_1$ 
8190   2.  $\vdash_1 E^*_0[e] : K_0$ 
8191     by 1 inversion
8192   3. QED by the induction hypothesis (2)
8193 CASE  $E^* = op^2 v_0 E^*_1 :$ 

```

1. $E^*[e] = op^2 v_0 E^*_1[e]$	8196
2. $\vdash_1 E^*_1[e] : K_1$	8197
by 1 <i>inversion</i>	8198
3. QED the induction hypothesis (2)	8199
CASE $E^* = \text{chk } K E^*_0 :$	8200
1. $E^*[e] = \text{chk } K E^*_0[e]$	8201
2. $\vdash_1 E^*_0[e] : \text{Any}$	8202
by 1 <i>inversion</i>	8203
3. QED the induction hypothesis (2)	8204
\square	8205
Lemma 4.23 : 1 dynamic hole typing	8206
If $\vdash_1 E^*[e] : K$ then the derivation contains a sub-term $\vdash_1 e$	8207
<i>Proof:</i>	8208
By induction on the structure of E^* .	8209
CASE $E^* = [] :$	8210
1. QED $E^*[e] = e$	8211
CASE $E^* = E^*_0 e_1 :$	8212
1. $E^*[e] = E^*_0[e] e_1$	8213
2. $\vdash_1 E^*_0[e] : e_1$	8214
by 1 <i>inversion</i>	8215
3. QED the induction hypothesis (2)	8216
CASE $E^* = v_0 E^*_1 :$	8217
1. $E^*[e] = v_0 E^*_1[e]$	8218
2. $\vdash_1 E^*_1[e] : e_1$	8219
by 1 <i>inversion</i>	8220
3. QED the induction hypothesis (2)	8221
CASE $E^* = \langle E^*_0, e_1 \rangle :$	8222
1. $E^*[e] = \langle E^*_0[e], e_1 \rangle$	8223
2. $\vdash_1 E^*_0[e] : e_1$	8224
by 1 <i>inversion</i>	8225
3. QED the induction hypothesis (2)	8226
CASE $E^* = \langle v_0, E^*_1 \rangle :$	8227
1. $E^*[e] = \langle v_0, E^*_1[e] \rangle$	8228
2. $\vdash_1 E^*_1[e] : v_0$	8229
by 1 <i>inversion</i>	8230
3. QED the induction hypothesis (2)	8231
CASE $E^* = op^1 E^*_0 :$	8232
1. $E^*[e] = op^1 E^*_0[e]$	8233
2. $\vdash_1 E^*_0[e] : e_1$	8234
by 1 <i>inversion</i>	8235
3. QED the induction hypothesis (2)	8236
CASE $E^* = op^2 E^*_0 e_1 :$	8237
1. $E^*[e] = op^2 E^*_0[e] e_1$	8238
2. $\vdash_1 E^*_0[e] : e_1$	8239
by 1 <i>inversion</i>	8240
3. QED the induction hypothesis (2)	8241
CASE $E^* = op^2 v_0 E^*_1 :$	8242
1. $E^*[e] = op^2 v_0 E^*_1[e]$	8243
2. $\vdash_1 E^*_1[e] : v_0$	8244
by 1 <i>inversion</i>	8245
3. QED the induction hypothesis (2)	8246
CASE $E^* = \text{chk } K E^*_0 :$	8247
1. Contradiction by $\vdash_1 E^*[e]$	8248

8251 □

Lemma 4.24 : 1 boundary hole typing

- 8252 • If $\vdash_1 E[\text{dyn } e]$ then the derivation contains a sub-term $\vdash_1 \text{dyn } e : \text{Any}$
- 8253 • If $\vdash_1 E[\text{dyn } e] : K'$ then the derivation contains a sub-term
- 8254 $\vdash_1 \text{dyn } e : \text{Any}$
- 8255 • If $\vdash_1 E[\text{stat } e]$ then the derivation contains a sub-term $\vdash_1 \text{stat } e$
- 8256 • If $\vdash_1 E[\text{stat } e] : K'$ then the derivation contains a sub-term
- 8257 $\vdash_1 \text{stat } e$
- 8258 • If $\vdash_1 E[\text{dyn } \tau e]$ then the derivation contains a sub-term
- 8259 $\vdash_1 \text{dyn } \tau e : [\tau]$
- 8260 • If $\vdash_1 E[\text{dyn } \tau e] : K'$ then the derivation contains a sub-term
- 8261 $\vdash_1 \text{dyn } \tau e : [\tau]$
- 8262 • If $\vdash_1 E[\text{stat } \tau e]$ then the derivation contains a sub-term
- 8263 $\vdash_1 \text{stat } \tau e$
- 8264 • If $\vdash_1 E[\text{stat } \tau e] : K'$ then the derivation contains a sub-term
- 8265 $\vdash_1 \text{stat } \tau e$

8266 *Proof:*

8267 By the following four lemmas: *static dyn hole typing*,
 8268 *dynamic dyn hole typing*, *static stat hole typing*, and
 8269 *dynamic stat hole typing*.

8270 □

Lemma 4.25 : 1 static dyn hole typing

- 8271 If $\vdash_1 E[\text{dyn } \tau e] : K'$ then the derivation contains a sub-term
- 8272 $\vdash_1 \text{dyn } \tau e : [\tau]$.

8273 *Proof:*8274 By induction on the structure of E .CASE $E \in E^*$:

- 8275 1. $\vdash_1 \text{dyn } \tau e : K''$
by *static hole typing*
- 8276 2. $\vdash_1 \text{dyn } \tau e : [\tau]$
by *1 inversion* (1)
- 8277 3. QED

CASE $E = E_0 e_1$:

- 8278 1. $E[\text{dyn } \tau e] = E_0[\text{dyn } \tau e] e_1$
- 8279 2. $\vdash_1 E_0[\text{dyn } \tau e] : K_0$
by *1 inversion*
- 8280 3. QED by the induction hypothesis (2)

CASE $E = v_0 E_1$:

- 8281 1. $E[\text{dyn } \tau e] = v_0 E_1[\text{dyn } \tau e]$
- 8282 2. $\vdash_1 E_1[\text{dyn } \tau e] : K_1$
by *1 inversion*
- 8283 3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:

- 8284 1. $E[\text{dyn } \tau e] = \langle E_0[\text{dyn } \tau e], e_1 \rangle$
- 8285 2. $\vdash_1 E_0[\text{dyn } \tau e] : K_0$
by *1 inversion*
- 8286 3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:

- 8287 1. $E[\text{dyn } \tau e] = \langle v_0, E_1[\text{dyn } \tau e] \rangle$
- 8288 2. $\vdash_1 E_1[\text{dyn } \tau e] : K_1$
by *1 inversion*

8289 *Proof:*8290 By induction on the structure of E .CASE $E \in E^*$:

- 8291 1. Contradiction by $\vdash_1 E[\text{dyn } \tau e] : \tau'$

CASE $E = \text{dyn } E_0$:

- 8292 1. $E[\text{dyn } \tau e] = \text{dyn } E_0[\text{dyn } \tau e]$
- 8293 2. $\vdash_1 E_0[\text{dyn } \tau e]$
by *1 inversion*
- 8294 3. QED by *dynamic dyn hole typing* (2)

CASE $E = \text{stat } E_0$:

- 8295 1. Contradiction by $\vdash_1 E[\text{dyn } \tau e] : \tau'$

CASE $E = \text{chk } K_0 E_0$:

- 8296 1. $E[\text{dyn } \tau e] = \text{chk } K_0 E_0[\text{dyn } \tau e]$
- 8297 2. $\vdash_1 E_0[\text{dyn } \tau e] : \text{Any}$
by *1 inversion*
- 8298 3. QED by the induction hypothesis (2)

□

Lemma 4.26 : 1 dynamic dyn hole typing

- If $\vdash_1 E[\text{dyn } \tau e]$ then the derivation contains a sub-term $\vdash_1 \text{dyn } \tau e : [\tau]$.

*Proof:*By induction on the structure of E .CASE $E \in E^*$:

1. Contradiction by $\vdash_1 E[\text{dyn } \tau e]$

CASE $E = E_0 e_1$:

1. $E[\text{dyn } \tau e] = E_0[\text{dyn } \tau e] e_1$
2. $\vdash_1 E_0[\text{dyn } \tau e] : K_0$
by *1 inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0 E_1$:

1. $E[\text{dyn } \tau e] = v_0 E_1[\text{dyn } \tau e]$
2. $\vdash_1 E_1[\text{dyn } \tau e]$
by *1 inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:

1. $E[\text{dyn } \tau e] = \langle E_0[\text{dyn } \tau e], e_1 \rangle$
2. $\vdash_1 E_0[\text{dyn } \tau e] : K_0$
by *1 inversion*
3. QED by the induction hypothesis (2)

8300 *Proof:*8301 By induction on the structure of E .CASE $E \in E^*$:8302 1. Contradiction by $\vdash_1 E[\text{dyn } \tau e]$ 8303 2. $\vdash_1 E[\text{dyn } \tau e] : K_0$
by *1 inversion*8304 *Proof:*

8305

8306 QED by the induction hypothesis (2) 8306
CASE $E = op^1 E_0$: 8307
 1. $E[\text{dyn } \tau e] = op^1 E_0[\text{dyn } \tau e]$ 8308
 2. $\vdash_1 E_0[\text{dyn } \tau e] : K_0$ 8309
 by *1 inversion* 8310
 3. QED by the induction hypothesis (2) 8311
CASE $E = op^2 E_0 e_1$: 8312
 1. $E[\text{dyn } \tau e] = op^2 E_0[\text{dyn } \tau e] e_1$ 8313
 2. $\vdash_1 E_0[\text{dyn } \tau e] : K_0$ 8314
 by *1 inversion* 8315
 3. QED by the induction hypothesis (2) 8316
CASE $E = op^2 v_0 E_1$: 8317
 1. $E[\text{dyn } \tau e] = op^2 v_0 E_1[\text{dyn } \tau e]$ 8318
 2. $\vdash_1 E_1[\text{dyn } \tau e] : K_1$ 8319
 by *1 inversion* 8320
 3. QED by the induction hypothesis (2) 8321
CASE $E = \text{dyn } E_0$: 8322
 1. $E[\text{dyn } \tau e] = \text{dyn } E_0[\text{dyn } \tau e]$ 8323
 2. $\vdash_1 E_0[\text{dyn } \tau e]$ 8324
 by *1 inversion* 8325
 3. QED by *dynamic dyn hole typing* (2) 8326
CASE $E = \text{stat } E_0$: 8327
 1. Contradiction by $\vdash_1 E[\text{dyn } \tau e] : \tau'$ 8328
CASE $E = \text{dyn } \tau_0 E_0$: 8329
 1. $E[\text{dyn } \tau e] = \text{dyn } \tau_0 E_0[\text{dyn } \tau e]$ 8330
 2. $\vdash_1 E_0[\text{dyn } \tau e]$ 8331
 by *1 inversion* 8332
 3. QED by *dynamic dyn hole typing* (2) 8333
CASE $E = \text{stat } \tau_0 E_0$: 8334
 1. Contradiction by $\vdash_1 E[\text{dyn } \tau e] : \tau'$ 8335
CASE $E = \text{chk } K_0 E_0$: 8336
 1. $E[\text{dyn } \tau e] = \text{chk } K_0 E_0[\text{dyn } \tau e]$ 8337
 2. $\vdash_1 E_0[\text{dyn } \tau e] : \text{Any}$ 8338
 by *1 inversion* 8339
 3. QED by the induction hypothesis (2) 8340
CASE $E = \langle E_0, e_1 \rangle$: 8341
 1. Contradiction by $\vdash_1 E[\text{dyn } \tau e]$ 8342
CASE $E = \langle v_0, E_1 \rangle$: 8343
 1. $E[\text{dyn } \tau e] = \langle v_0, E_1[\text{dyn } \tau e] \rangle$ 8344
CASE $E = v_0 E_1$: 8345
 1. $E[\text{dyn } \tau e] = v_0 E_1[\text{dyn } \tau e]$ 8346
CASE $E = E_0 e_1$: 8347
 1. Contradiction by $\vdash_1 E[\text{dyn } \tau e]$ 8348
CASE $E = E_0 e_1$: 8349
 1. $E[\text{dyn } \tau e] = E_0[\text{dyn } \tau e] e_1$ 8350
 2. $\vdash_1 E_0[\text{dyn } \tau e]$ 8351
 by *1 inversion* 8352
 3. QED by the induction hypothesis (2) 8353
CASE $E = v_0 E_1$: 8354
 1. $E[\text{dyn } \tau e] = v_0 E_1[\text{dyn } \tau e]$ 8355
 2. $\vdash_1 E_1[\text{dyn } \tau e]$ 8356
 by *1 inversion* 8357
 3. QED by the induction hypothesis (2) 8358
CASE $E = \langle E_0, e_1 \rangle$: 8359
 1. $E[\text{dyn } \tau e] = \langle E_0[\text{dyn } \tau e], e_1 \rangle$ 8360

8361 1. $E[\text{dyn } \tau e] = \langle E_0[\text{dyn } \tau e], e_1 \rangle$
 8362 2. $\vdash_1 E_0[\text{dyn } \tau e]$
 8363 by 1 *inversion*
 8364 3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle :$

8365 1. $E[\text{dyn } \tau e] = \langle v_0, E_1[\text{dyn } \tau e] \rangle$
 8366 2. $\vdash_1 E_1[\text{dyn } \tau e]$
 8367 by 1 *inversion*
 8368 3. QED by the induction hypothesis (2)

CASE $E = op^1 E_0 :$

8369 1. $E[\text{dyn } \tau e] = op^1 E_0[\text{dyn } \tau e]$
 8370 2. $\vdash_1 E_0[\text{dyn } \tau e]$
 8371 by 1 *inversion*
 8372 3. QED by the induction hypothesis (2)

CASE $E = op^2 E_0 e_1 :$

8373 1. $E[\text{dyn } \tau e] = op^2 E_0[\text{dyn } \tau e] e_1$
 8374 2. $\vdash_1 E_0[\text{dyn } \tau e]$
 8375 by 1 *inversion*
 8376 3. QED by the induction hypothesis (2)

CASE $E = op^2 v_0 E_1 :$

8377 1. $E[\text{dyn } \tau e] = op^2 v_0 E_1[\text{dyn } \tau e]$
 8378 2. $\vdash_1 E_1[\text{dyn } \tau e]$
 8379 by 1 *inversion*
 8380 3. QED by the induction hypothesis (2)

CASE $E = \text{dyn } E_0 :$

8381 1. Contradiction by $\vdash_1 E[\text{dyn } \tau e]$
 8382 2. $\vdash_1 E_1[\text{dyn } \tau e]$
 8383 by 1 *inversion*
 8384 3. QED by the induction hypothesis (2)

CASE $E = stat E_0 :$

8385 1. Contradiction by $\vdash_1 E[\text{dyn } \tau e]$
 8386 2. $\vdash_1 E_1[\text{dyn } \tau e]$
 8387 by 1 *inversion*
 8388 3. QED by *static dyn hole typing* (2)

CASE $E = \text{dyn } \tau E_0 :$

8389 1. Contradiction by $\vdash_1 E[\text{dyn } \tau e]$
 8390 2. $\vdash_1 E_0[\text{dyn } \tau e]$
 8391 by 1 *inversion*
 8392 3. QED by *static dyn hole typing* (2)

CASE $E = stat \tau_0 E_0 :$

8393 1. Contradiction by $\vdash_1 E[\text{dyn } \tau e]$
 8394 2. $\vdash_1 E_0[\text{dyn } \tau e]$
 8395 by 1 *inversion*
 8396 3. QED by *static dyn hole typing* (2)

CASE $E = \text{chk } K_0 E_0 :$

8397 1. Contradiction by $\vdash_1 E[\text{dyn } \tau e]$
 8398 2. $\vdash_1 E_0[\text{dyn } \tau e]$
 8399 by 1 *inversion*
 8400 3. QED by *dynamic stat hole typing* (2)

8401 \square

Lemma 4.27 : 1 static stat hole typing
 If $\vdash_1 E[\text{stat } \tau e] : K'$ then the derivation contains a sub-term $\vdash_1 \text{stat } \tau e$.

Proof:
 By induction on the structure of E .

CASE $E \in E^\bullet :$

8401 1. Contradiction by $\vdash_1 E[\text{stat } \tau e] : \tau'$
 8402 **CASE** $E = E_0 e_1 :$

8403 1. $E[\text{stat } \tau e] = E_0[\text{stat } \tau e] e_1$
 8404 2. $\vdash_1 E_0[\text{stat } \tau e] : K_0$
 8405 by 1 *inversion*
 8406 3. QED by the induction hypothesis (2)

8416 **CASE** $E = v_0 E_1 :$
 8417 1. $E[\text{stat } \tau e] = v_0 E_1[\text{stat } \tau e]$
 8418 2. $\vdash_1 E_1[\text{stat } \tau e] : K_1$
 8419 by 1 *inversion*
 8420 3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle :$

8421 1. $E[\text{stat } \tau e] = \langle E_0[\text{stat } \tau e], e_1 \rangle$
 8422 2. $\vdash_1 E_0[\text{stat } \tau e] : K_0$
 8423 by 1 *inversion*
 8424 3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle :$

8425 1. $E[\text{stat } \tau e] = \langle v_0, E_1[\text{stat } \tau e] \rangle$
 8426 2. $\vdash_1 E_1[\text{stat } \tau e] : K_1$
 8427 by 1 *inversion*
 8428 3. QED by the induction hypothesis (2)

CASE $E = op^1 E_0 :$

8429 1. $E[\text{stat } \tau e] = op^1 E_0[\text{stat } \tau e]$
 8430 2. $\vdash_1 E_0[\text{stat } \tau e] : K_0$
 8431 by 1 *inversion*
 8432 3. QED by the induction hypothesis (2)

CASE $E = op^2 E_0 e_1 :$

8433 1. $E[\text{stat } \tau e] = op^2 E_0[\text{stat } \tau e] e_1$
 8434 2. $\vdash_1 E_0[\text{stat } \tau e] : K_0$
 8435 by 1 *inversion*
 8436 3. QED by the induction hypothesis (2)

CASE $E = op^2 v_0 E_1 :$

8437 1. $E[\text{stat } \tau e] = op^2 v_0 E_1[\text{stat } \tau e]$
 8438 2. $\vdash_1 E_1[\text{stat } \tau e] : K_1$
 8439 by 1 *inversion*
 8440 3. QED by the induction hypothesis (2)

CASE $E = op^2 v_0 E_1 :$

8441 1. $E[\text{stat } \tau e] = op^2 v_0 E_1[\text{stat } \tau e]$
 8442 2. $\vdash_1 E_1[\text{stat } \tau e] : K_1$
 8443 by 1 *inversion*
 8444 3. QED by the induction hypothesis (2)

CASE $E = \text{dyn } E_0 :$

8445 1. $E[\text{stat } \tau e] = \text{dyn } E_0[\text{stat } \tau e]$
 8446 2. $\vdash_1 E_0[\text{stat } \tau e]$
 8447 by 1 *inversion*
 8448 3. QED by *dynamic stat hole typing* (2)

CASE $E = stat E_0 :$

8449 1. Contradiction by $\vdash_1 E[\text{stat } \tau e] : \tau'$
 8450 2. $\vdash_1 E_0[\text{stat } \tau e]$
 8451 by 1 *inversion*
 8452 3. QED by *dynamic stat hole typing* (2)

CASE $E = \text{dyn } \tau_0 E_0 :$

8453 1. $E[\text{stat } \tau e] = \text{dyn } \tau_0 E_0[\text{stat } \tau e]$
 8454 2. $\vdash_1 E_0[\text{stat } \tau e]$
 8455 by 1 *inversion*
 8456 3. QED by *dynamic stat hole typing* (2)

CASE $E = stat \tau_0 E_0 :$

8457 1. Contradiction by $\vdash_1 E[\text{stat } \tau e] : \tau'$
 8458 2. $\vdash_1 E_0[\text{stat } \tau e]$
 8459 by 1 *inversion*
 8460 3. QED by *dynamic stat hole typing* (2)

CASE $E = \text{chk } K_0 E_0 :$

8461 1. $E[\text{stat } \tau e] = \text{chk } K_0 E_0[\text{stat } \tau e]$
 8462 2. $\vdash_1 E_0[\text{stat } \tau e] : \text{Any}$
 8463 by 1 *inversion*
 8464 3. QED by the induction hypothesis (2)

8465 \square

Lemma 4.28 : 1 dynamic stat hole typing
 If $\vdash_1 E[\text{stat } \tau e]$ then the derivation contains a sub-term $\vdash_1 \text{stat } \tau e$.

Proof:

8471 By induction on the structure of E .
CASE $E \in E^\bullet$:
 1. QED by *dynamic hole typing*
CASE $E = E_0 e_1$:
 1. $E[\text{stat } \tau e] = E_0[\text{stat } \tau e] e_1$
 2. $\vdash_1 E_0[\text{stat } \tau e]$
 by 1 *inversion*
 3. QED by the induction hypothesis (2)
CASE $E = v_0 E_1$:
 1. $E[\text{stat } \tau e] = v_0 E_1[\text{stat } \tau e]$
 2. $\vdash_1 E_1[\text{stat } \tau e]$
 by 1 *inversion*
 3. QED by the induction hypothesis (2)
CASE $E = \langle E_0, e_1 \rangle$:
 1. $E[\text{stat } \tau e] = \langle E_0[\text{stat } \tau e], e_1 \rangle$
 2. $\vdash_1 E_0[\text{stat } \tau e]$
 by 1 *inversion*
 3. QED by the induction hypothesis (2)
CASE $E = \langle v_0, E_1 \rangle$:
 1. $E[\text{stat } \tau e] = \langle v_0, E_1[\text{stat } \tau e] \rangle$
 2. $\vdash_1 E_1[\text{stat } \tau e]$
 by 1 *inversion*
 3. QED by the induction hypothesis (2)
CASE $E = op^1 E_0$:
 1. $E[\text{stat } \tau e] = op^1 E_0[\text{stat } \tau e]$
 2. $\vdash_1 E_0[\text{stat } \tau e]$
 by 1 *inversion*
 3. QED by the induction hypothesis (2)
CASE $E = op^2 E_0 e_1$:
 1. $E[\text{stat } \tau e] = op^2 E_0[\text{stat } \tau e] e_1$
 2. $\vdash_1 E_0[\text{stat } \tau e]$
 by 1 *inversion*
 3. QED by the induction hypothesis (2)
CASE $E = op^2 v_0 E_1$:
 1. $E[\text{stat } \tau e] = op^2 v_0 E_1[\text{stat } \tau e]$
 2. $\vdash_1 E_1[\text{stat } \tau e]$
 by 1 *inversion*
 3. QED by the induction hypothesis (2)
CASE $E = \text{dyn } E_0$:
 1. Contradiction by $\vdash_1 E[\text{stat } \tau e]$
CASE $E = \text{stat } E_0$:
 1. $E[\text{stat } \tau e] = \text{stat } E_0[\text{stat } \tau e]$
 2. $\vdash_1 E_0[\text{stat } \tau e] : \lfloor \tau_0 \rfloor$
 by 1 *inversion*
 3. QED by *static stat hole typing* (2)
CASE $E = \text{dyn } \tau E_0$:
 1. Contradiction by $\vdash_1 E[\text{stat } \tau e]$
CASE $E = \text{stat } \tau_0 E_0$:
 1. $E[\text{stat } \tau e] = \text{stat } \tau_0 E_0[\text{stat } \tau e]$
 2. $\vdash_1 E_0[\text{stat } \tau e] : \lfloor \tau_0 \rfloor$
 by 1 *inversion*
 3. QED by *static stat hole typing* (2)
CASE $E = \text{chk } K_0 E_0$:
 1. Contradiction by $\vdash_1 E[\text{stat } \tau e]$

□
Lemma 4.29 : 1 static hole substitution
 If $\vdash_1 E^\bullet[e] : K$ and the derivation contains a sub-term $\vdash_1 e : K'$ and $\vdash_1 e' : K'$, then $\vdash_1 E^\bullet[e'] : K$
Proof:
 By induction on the structure of E^\bullet .
CASE $E^\bullet = []$:
 1. $E^\bullet[e] = e$
 $\wedge E^\bullet[e'] = e'$
 2. $\vdash_1 e : K$
 by (1)
 3. $K' = K$
 4. QED
CASE $E^\bullet = \langle E^\bullet_0, e_1 \rangle$:
 1. $E^\bullet[e] = \langle E^\bullet_0[e], e_1 \rangle$
 $\wedge E^\bullet[e'] = \langle E^\bullet_0[e'], e_1 \rangle$
 2. $\vdash_1 \langle E^\bullet_0[e], e_1 \rangle : K$
 3. $\vdash_1 E^\bullet_0[e] : K_0$
 $\wedge \vdash_1 e_1 : K_1$
 by 1 *inversion*
 4. $\vdash_1 E^\bullet_0[e'] : K_0$
 by the induction hypothesis (3)
 5. $\vdash_1 \langle E^\bullet_0[e'], e_1 \rangle : K$
 by (2, 3, 4)
 6. QED by (1, 5)
CASE $E^\bullet = \langle v_0, E^\bullet_1 \rangle$:
 1. $E^\bullet[e] = \langle v_0, E^\bullet_1[e] \rangle$
 $\wedge E^\bullet[e'] = \langle v_0, E^\bullet_1[e'] \rangle$
 2. $\vdash_1 \langle v_0, E^\bullet_1[e] \rangle : K$
 3. $\vdash_1 v_0 : K_0$
 $\wedge \vdash_1 E^\bullet_1[e] : K_1$
 by 1 *inversion*
 4. $\vdash_1 E^\bullet_1[e'] : K_1$
 by the induction hypothesis (3)
 5. $\vdash_1 \langle v_0, E^\bullet_1[e'] \rangle : K$
 by (2, 3, 4)
 6. QED by (1, 5)
CASE $E^\bullet = E^\bullet_0 e_1$:
 1. $E^\bullet[e] = E^\bullet_0[e] e_1$
 $\wedge E^\bullet[e'] = E^\bullet_0[e'] e_1$
 2. $\vdash_1 E^\bullet_0[e] e_1 : K$
 3. $\vdash_1 E^\bullet_0[e] : K_0$
 $\wedge \vdash_1 e_1 : K_1$
 by 1 *inversion*
 4. $\vdash_1 E^\bullet_0[e'] : K_0$
 by the induction hypothesis (3)
 5. $\vdash_1 E^\bullet_0[e'] e_1 : K$
 by (2, 3, 4)
 6. QED by (1, 5)
CASE $E^\bullet = v_0 E^\bullet_1$:
 1. $E^\bullet[e] = v_0 E^\bullet_1[e]$
 $\wedge E^\bullet[e'] = v_0 E^\bullet_1[e']$
 2. $\vdash_1 v_0 E^\bullet_1[e] : K$

8581	3. $\vdash_1 v_0 : K_0$	8636
8582	$\wedge \vdash_1 E^{\bullet}_1[e] : K_1$	
8583	by 1 <i>inversion</i>	8637
8584	4. $\vdash_1 E^{\bullet}_1[e'] : K_1$	8638
8585	by the induction hypothesis (3)	8639
8586	5. $\vdash_1 v_0 E^{\bullet}_1[e'] : K$	8640
8587	by (2, 3, 4)	8641
8588	6. QED by (1, 5)	8642
8589	CASE $E^{\bullet} = op^1 E^{\bullet}_0 :$	8643
8590	1. $E^{\bullet}[e] = op^1 E^{\bullet}_0[e]$	8644
8591	$\wedge E^{\bullet}[e'] = op^1 E^{\bullet}_0[e']$	8645
8592	2. $\vdash_1 op^1 E^{\bullet}_0[e] : K$	8646
8593	3. $\vdash_1 E^{\bullet}_0[e] : K_0$	8647
8594	by 1 <i>inversion</i>	8648
8595	4. $\vdash_1 E^{\bullet}_0[e'] : K_0$	8649
8596	by the induction hypothesis (3)	8650
8597	5. $\vdash_1 op^1 E^{\bullet}_0[e'] : K$	8651
8598	by (2, 3, 4)	8652
8599	6. QED by (1, 5)	8653
8600	CASE $E^{\bullet} = op^2 E^{\bullet}_0 e_1 :$	8654
8601	1. $E^{\bullet}[e] = op^2 E^{\bullet}_0[e] e_1$	8655
8602	$\wedge E^{\bullet}[e'] = op^2 E^{\bullet}_0[e'] e_1$	8656
8603	2. $\vdash_1 op^2 E^{\bullet}_0[e] e_1 : K$	8657
8604	3. $\vdash_1 E^{\bullet}_0[e] : K_0$	8658
8605	$\wedge \vdash_1 e_1 : K_1$	8659
8606	by 1 <i>inversion</i>	8660
8607	4. $\vdash_1 E^{\bullet}_0[e'] : K_0$	8661
8608	by the induction hypothesis (3)	8662
8609	5. $\vdash_1 op^2 E^{\bullet}_0[e'] e_1 : K$	8663
8610	by (2, 3, 4)	8664
8611	6. QED by (1, 5)	8665
8612	CASE $E^{\bullet} = op^2 v_0 E^{\bullet}_1 :$	8666
8613	1. $E^{\bullet}[e] = op^2 v_0 E^{\bullet}_1[e]$	8667
8614	$\wedge E^{\bullet}[e'] = op^2 v_0 E^{\bullet}_1[e']$	8668
8615	2. $\vdash_1 op^2 v_0 E^{\bullet}_1[e] : K$	8669
8616	3. $\vdash_1 v_0 : K_0$	8670
8617	$\wedge \vdash_1 E^{\bullet}_1[e] : K_1$	8671
8618	by 1 <i>inversion</i>	8672
8619	4. $\vdash_1 E^{\bullet}_1[e'] : K_1$	8673
8620	by the induction hypothesis (3)	8674
8621	5. $\vdash_1 op^2 v_0 E^{\bullet}_1[e'] : K$	8675
8622	by (2, 3, 4)	8676
8623	6. QED by (1, 5)	8677
8624	CASE $E^{\bullet} = chk K_c E^{\bullet}_0 :$	8678
8625	1. $E^{\bullet}[e] = chk K_c E^{\bullet}_0[e]$	8679
8626	$\wedge E^{\bullet}[e'] = chk K_c E^{\bullet}_0[e']$	8680
8627	2. $\vdash_1 chk K_c E^{\bullet}_0[e] : K$	8681
8628	3. $\vdash_1 E^{\bullet}_0[e] : K_0$	8682
8629	by 1 <i>inversion</i>	8683
8630	4. $\vdash_1 E^{\bullet}_0[e'] : K_0$	8684
8631	by the induction hypothesis (3)	8685
8632	5. $\vdash_1 chk K_c E^{\bullet}_0[e'] : K$	8686
8633	by (2, 3, 4)	8687
8634	6. QED by (1, 5)	8688
8635		8689

8691 5. $\vdash_1 v_0 E^\bullet_1[e']$
 8692 by (3, 4)
 8693 6. QED by (1, 5)
CASE $E^\bullet = op^1 E^\bullet_0 :$
 8695 1. $E^\bullet[e] = op^1 E^\bullet_0[e]$
 8696 $\wedge E^\bullet[e'] = op^1 E^\bullet_0[e']$
 8697 2. $\vdash_1 op^1 E^\bullet_0[e]$
 8698 3. $\vdash_1 E^\bullet_0[e]$
 8699 by 1 *inversion*
 8700 4. $\vdash_1 E^\bullet_0[e']$
 8701 by the induction hypothesis (3)
 8702 5. $\vdash_1 op^1 E^\bullet_0[e']$
 8703 by (3, 4)
 8704 6. QED by (1, 5)
CASE $E^\bullet = op^2 E^\bullet_0 e_1 :$
 8705 1. $E^\bullet[e] = op^2 E^\bullet_0[e] e_1$
 8706 $\wedge E^\bullet[e'] = op^2 E^\bullet_0[e'] e_1$
 8707 2. $\vdash_1 op^2 E^\bullet_0[e] e_1$
 8708 3. $\vdash_1 E^\bullet_0[e]$
 8709 $\wedge \vdash_1 e_1$
 8710 by 1 *inversion*
 8711 4. $\vdash_1 E^\bullet_0[e']$
 8712 by the induction hypothesis (3)
 8713 5. $\vdash_1 op^2 E^\bullet_0[e'] e_1$
 8714 by (3, 4)
 8715 6. QED by (1, 5)
CASE $E^\bullet = op^2 v_0 E^\bullet_1 :$
 8716 1. $E^\bullet[e] = op^2 v_0 E^\bullet_1[e]$
 8717 $\wedge E^\bullet[e'] = op^2 v_0 E^\bullet_1[e']$
 8718 2. $\vdash_1 op^2 v_0 E^\bullet_1[e]$
 8719 3. $\vdash_1 v_0$
 8720 $\wedge \vdash_1 E^\bullet_1[e]$
 8721 by 1 *inversion*
 8722 4. $\vdash_1 E^\bullet_1[e']$
 8723 by the induction hypothesis (3)
 8724 5. $\vdash_1 op^2 v_0 E^\bullet_1[e']$
 8725 by (3, 4)
 8726 6. QED by (1, 5)
CASE $E^\bullet = \text{chk } K_c E^\bullet_0 :$
 8727 1. Contradiction by $\vdash_1 E^\bullet[e]$

□

Lemma 4.31 : 1 hole substitution

- If $\vdash_1 E[e] : K$ and the derivation contains a sub-term $\vdash_1 e : K'$ and $\vdash_1 e' : K'$ then $\vdash_1 E[e'] : K$.
- If $\vdash_1 E[e] : K$ and the derivation contains a sub-term $\vdash_1 e$ and $\vdash_1 e'$ then $\vdash_1 E[e'] : K$.
- If $\vdash_1 E[e]$ and the derivation contains a sub-term $\vdash_1 e : K'$ and $\vdash_1 e' : K'$ then $\vdash_1 E[e']$.
- If $\vdash_1 E[e]$ and the derivation contains a sub-term $\vdash_1 e$ and $\vdash_1 e'$ then $\vdash_1 E[e']$.

Proof:

By the following four lemmas: *dynamic context static hole substitution*, *dynamic context dynamic hole substitution*,

static context static hole substitution, and *static context dynamic hole substitution*. □

Lemma 4.32 : 1 dynamic context static hole substitution

If $\vdash_1 E[e]$ and contains $\vdash_1 e : K'$, and furthermore $\vdash_1 e' : K'$, then $\vdash_1 E[e']$

Proof:

By induction on the structure of E .

CASE $E \in E^\bullet :$

1. Contradiction by $\vdash_1 E[e]$

CASE $E = E_0 e_1 :$

1. $E[e] = E_0[e] e_1$
2. $\vdash_1 E_0[e]$
by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0 E_1 :$

1. $E[e] = v_0 E_1[e]$
2. $\vdash_1 E_1[e]$
by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle :$

1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_1 E_0[e]$
by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle :$

1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_1 E_1[e]$
by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1 E_0 :$

1. $E[e] = op^1 E_0[e]$
2. $\vdash_1 E_0[e]$
by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2 E_0 e_1 :$

1. $E[e] = op^2 E_0[e] e_1$
2. $\vdash_1 E_0[e]$
by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2 v_0 E_1 :$

1. $E[e] = op^2 v_0 E_1[e]$
2. $\vdash_1 E_1[e]$
by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \text{dyn } E_0 :$

1. Contradiction by $\vdash_1 E[e]$

CASE $E = \text{stat } E_0 :$

1. $E[e] = \text{stat } E_0[e]$
2. $\vdash_1 E_0[e] : \text{Any}$
by 1 *inversion*
3. QED by *static context static hole substitution* (2)

CASE $E = \text{dyn } \tau'' E_0 :$

8801 1. Contradiction by $\vdash_1 E[e]$
CASE $E = \text{stat } \tau_0 E_0 :$
8802 1. $E[e] = \text{stat } \tau_0 E_0[e]$
8803 2. $\vdash_1 E_0[e] : \lfloor \tau_0 \rfloor$
8804 by 1 *inversion*
8805 3. QED by *static context static hole substitution* (2)
8806 **CASE** $E = \text{chk } K_0 E_0 :$
8807 1. Contradiction by $\vdash_1 E[e]$
8808 □

8810 **Lemma 4.33 : 1 dynamic context dynamic hole substitution**
8811 If $\vdash_1 E[e]$ and contains $\vdash_1 e$, and furthermore $\vdash_1 e' : K'$, then $\vdash_1 E[e']$
8812 *Proof:*
8813 By induction on the structure of E .
CASE $E \in E^\bullet :$
8814 1. QED by *dynamic boundary-free hole substitution*
CASE $E = E_0 e_1 :$
8815 1. $E[e] = E_0[e] e_1$
8816 2. $\vdash_1 E_0[e]$
8817 by 1 *inversion*
8818 3. QED by the induction hypothesis (2)
CASE $E = v_0 E_1 :$
8819 1. $E[e] = v_0 E_1[e]$
8820 2. $\vdash_1 E_1[e]$
8821 by 1 *inversion*
8822 3. QED by the induction hypothesis (2)
CASE $E = \langle E_0, e_1 \rangle :$
8823 1. $E[e] = \langle E_0[e], e_1 \rangle$
8824 2. $\vdash_1 E_0[e]$
8825 by 1 *inversion*
8826 3. QED by the induction hypothesis (2)
CASE $E = \langle v_0, E_1 \rangle :$
8827 1. $E[e] = \langle v_0, E_1[e] \rangle$
8828 2. $\vdash_1 E_1[e]$
8829 by 1 *inversion*
8830 3. QED by the induction hypothesis (2)
CASE $E = op^1 E_0 :$
8831 1. $E[e] = op^1 E_0[e]$
8832 2. $\vdash_1 E_0[e]$
8833 by 1 *inversion*
8834 3. QED by the induction hypothesis (2)
CASE $E = op^2 E_0 e_1 :$
8835 1. $E[e] = op^2 E_0[e] e_1$
8836 2. $\vdash_1 E_0[e]$
8837 by 1 *inversion*
8838 3. QED by the induction hypothesis (2)
CASE $E = op^2 v_0 E_1 :$
8839 1. $E[e] = op^2 v_0 E_1[e]$
8840 2. $\vdash_1 E_1[e]$
8841 by 1 *inversion*
8842 3. QED by the induction hypothesis (2)
CASE $E = \text{dyn } E_0 :$
8843 1. Contradiction by $\vdash_1 E[e]$
CASE $E = \text{stat } E_0 :$

8856 1. $E[e] = \text{stat } E_0[e]$
8857 2. $\vdash_1 E_0[e] : \text{Any}$
8858 by 1 *inversion*
8859 3. QED by *static context dynamic hole substitution* (2)
CASE $E = \text{dyn } \tau'' E_0 :$
8860 1. Contradiction by $\vdash_1 E[e]$
CASE $E = \text{stat } \tau_0 E_0 :$
8861 1. $E[e] = \text{stat } \tau_0 E_0[e]$
8862 2. $\vdash_1 E_0[e] : \lfloor \tau_0 \rfloor$
8863 by 1 *inversion*
8864 3. QED by *static context dynamic hole substitution* (2)
CASE $E = \text{chk } K_0 E_0 :$
8865 1. Contradiction by $\vdash_1 E[e]$
8866 □

8870 **Lemma 4.34 : 1 static context static hole substitution**
8871 If $\vdash_1 E[e] : K$ and contains $\vdash_1 e : K'$, and furthermore $\vdash_1 e' : K'$, then $\vdash_1 E[e'] : K$
Proof:
8873 By induction on the structure of E .
CASE $E \in E^\bullet :$
8874 1. QED by *static boundary-free hole substitution*
CASE $E = E_0 e_1 :$
8875 1. $E[e] = E_0[e] e_1$
8876 2. $\vdash_1 E_0[e] : K_0$
8877 by 1 *inversion*
8878 3. QED by the induction hypothesis (2)
CASE $E = v_0 E_1 :$
8879 1. $E[e] = v_0 E_1[e]$
8880 2. $\vdash_1 E_1[e] : K_1$
8881 by 1 *inversion*
8882 3. QED by the induction hypothesis (2)
CASE $E = \langle E_0, e_1 \rangle :$
8883 1. $E[e] = \langle E_0[e], e_1 \rangle$
8884 2. $\vdash_1 E_0[e] : K_0$
8885 by 1 *inversion*
8886 3. QED by the induction hypothesis (2)
CASE $E = \langle v_0, E_1 \rangle :$
8887 1. $E[e] = \langle v_0, E_1[e] \rangle$
8888 2. $\vdash_1 E_1[e] : K_1$
8889 by 1 *inversion*
8890 3. QED by the induction hypothesis (2)
CASE $E = op^1 E_0 :$
8891 1. $E[e] = op^1 E_0[e]$
8892 2. $\vdash_1 E_0[e] : K_0$
8893 by 1 *inversion*
8894 3. QED by the induction hypothesis (2)
CASE $E = op^2 E_0 e_1 :$
8895 1. $E[e] = op^2 E_0[e] e_1$
8896 2. $\vdash_1 E_0[e] : K_0$
8897 by 1 *inversion*
8898 3. QED by the induction hypothesis (2)
CASE $E = op^2 E_0 e_1 :$
8899 1. $E[e] = op^2 E_0[e] e_1$
8900 2. $\vdash_1 E_0[e] : K_0$
8901 by 1 *inversion*
8902 3. QED by the induction hypothesis (2)
CASE $E = op^2 v_0 E_1 :$
8903 1. $E[e] = op^2 v_0 E_1[e]$
8904 2. $\vdash_1 E_0[e] : K_0$
8905 by 1 *inversion*
8906 3. QED by the induction hypothesis (2)
CASE $E = \text{op}^2 v_0 E_1 :$
8907 1. $E[e] = \text{op}^2 v_0 E_1[e]$
8908 □

8911 2. $\vdash_1 E_1[e] : K_1$
 8912 by 1 *inversion*
 8913 3. QED by the induction hypothesis (2)
CASE $E = \text{dyn } E_0 :$
 8915 1. $E[e] = \text{dyn } E_0[e]$
 8916 2. $\vdash_1 E_0[e]$
 8917 by 1 *inversion*
 8918 3. QED by *static dyn hole typing* (2)
CASE $E = \text{stat } E_0 :$
 8920 1. Contradiction by $\vdash_1 E[e] : K$
CASE $E = \text{dyn } \tau_0 E_0 :$
 8922 1. $E[e] = \text{dyn } \tau_0 E_0[e]$
 8923 2. $\vdash_1 E_0[e]$
 8924 by 1 *inversion*
 8925 3. QED by *static dyn hole typing* (2)
CASE $E = \text{stat } \tau_0 E_0 :$
 8927 1. Contradiction by $\vdash_1 E[e] : K$
CASE $E = \text{chk } K_0 E_0 :$
 8929 1. $E[e] = \text{chk } K_0 E_0[e]$
 8930 2. $\vdash_1 E_0[e] : \text{Any}$
 8931 by 1 *inversion*
 8932 3. QED by the induction hypothesis (2)
 8933 \square

Lemma 4.35 : 1 static context dynamic hole substitution
 If $\vdash_1 E[e] : K$ and contains $\vdash_1 e$, and furthermore $\vdash_1 e'$, then
 $\vdash_1 E[e'] : K$

Proof:
 By induction on the structure of E .
CASE $E \in E^\bullet :$
 1. Contradiction by $\vdash_1 E[e] : K$
CASE $E = E_0 e_1 :$
 1. $E[e] = E_0[e] e_1$
 2. $\vdash_1 E_0[e] : K_0$
 by 1 *inversion*
 3. QED by the induction hypothesis (2)
CASE $E = v_0 E_1 :$
 1. $E[e] = v_0 E_1[e]$
 2. $\vdash_1 E_1[e] : K_1$
 by 1 *inversion*
 3. QED by the induction hypothesis (2)
CASE $E = \langle E_0, e_1 \rangle :$
 1. $E[e] = \langle E_0[e], e_1 \rangle$
 2. $\vdash_1 E_0[e] : K_0$
 by 1 *inversion*
 3. QED by the induction hypothesis (2)
CASE $E = \langle v_0, E_1 \rangle :$
 1. $E[e] = \langle v_0, E_1[e] \rangle$
 2. $\vdash_1 E_1[e] : K_1$
 by 1 *inversion*
 3. QED by the induction hypothesis (2)
CASE $E = op^1 E_0 :$
 1. $E[e] = op^1 E_0[e]$

8966 2. $\vdash_1 E_0[e] : K_0$
 8967 by 1 *inversion*
 8968 3. QED by the induction hypothesis (2)
CASE $E = op^2 E_0 e_1 :$
 8969 1. $E[e] = op^2 E_0[e] e_1$
 8970 2. $\vdash_1 E_0[e] : K_0$
 8971 by 1 *inversion*
 8972 3. QED by the induction hypothesis (2)
CASE $E = op^2 v_0 E_1 :$
 8974 1. $E[e] = op^2 v_0 E_1[e]$
 8975 2. $\vdash_1 E_1[e] : K_1$
 8976 by 1 *inversion*
 8977 3. QED by the induction hypothesis (2)
CASE $E = \text{dyn } E_0 :$
 8979 1. $E[e] = \text{dyn } E_0[e]$
 8980 2. $\vdash_1 E_0[e]$
 8981 by 1 *inversion*
 8982 3. QED by *dynamic stat hole typing* (2)
CASE $E = \text{stat } E_0 :$
 8984 1. Contradiction by $\vdash_1 E[e] : K$
CASE $E = \text{dyn } \tau_0 E_0 :$
 8986 1. $E[e] = \text{dyn } \tau_0 E_0[e]$
 8987 2. $\vdash_1 E_0[e]$
 8988 by 1 *inversion*
 8989 3. QED by *dynamic stat hole typing* (2)
CASE $E = \text{stat } \tau_0 E_0 :$
 8991 1. Contradiction by $\vdash_1 E[e] : K$
CASE $E = \text{chk } K_0 E_0 :$
 8993 1. $E[e] = \text{chk } K_0 E_0[e]$
 8994 2. $\vdash_1 E_0[e] : \text{Any}$
 8995 by 1 *inversion*
 8996 3. QED by the induction hypothesis (2)
 8997 \square

Lemma 4.36 : 1 static inversion

- If $\vdash_1 \langle e_0, e_1 \rangle : K$ then $\vdash_1 e_0 : \text{Any}$ and $\vdash_1 e_1 : \text{Any}$
- If $\vdash_1 \lambda x. e : K$ then $x \vdash_1 e$
- If $\vdash_1 \lambda(x:\tau). e : K$ then $(x:\tau) \vdash_1 e : \text{Any}$
- If $\vdash_1 e_0 e_1 : K$ then $K = \text{Any}$ and $\vdash_1 e_0 : \text{Fun}$ and $\vdash_1 e_1 : \text{Any}$
- If $\vdash_1 op^1 e_0 : K$ then $K = \text{Any}$ and $\vdash_1 e_0 : \text{Pair}$
- If $\vdash_1 op^2 e_0 e_1 : K$ then $\vdash_1 e_0 : K_0$ and $\vdash_1 e_1 : K_1$ and $\Delta(op^2, K_0, K_1) = K'$ and $K' \leq K$
- If $\vdash_1 \text{dyn } \tau e : K$ then $\vdash_1 e$ and $\lfloor \tau \rfloor \leq K$
- If $\vdash_1 \text{chk } K' e_0 : K$ then $\vdash_1 e_0 : \text{Any}$ and $K' \leq K$

Proof:
 QED by the definition of $\Gamma \vdash_1 e : \tau$
 \square

Lemma 4.37 : 1 dynamic inversion
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- If $\vdash_1 \langle e_0, e_1 \rangle$ then $\vdash_1 e_0$ and $\vdash_1 e_1$
- If $\vdash_1 \lambda x. e$ then $x \vdash_1 e$
- If $\vdash_1 \lambda(x:\tau). e$ then $(x:\tau) \vdash_1 e : \text{Any}$
- If $\vdash_1 e_0 e_1$ then $\vdash_1 e_0$ and $\vdash_1 e_1$
- If $\vdash_1 op^1 e_0$ then $\vdash_1 e_0$
- If $\vdash_1 op^2 e_0 e_1$ then $\vdash_1 e_0$ and $\vdash_1 e_1$
- If $\vdash_1 \text{stat } \tau e$ then $\vdash_1 e : \lfloor \tau \rfloor$
- If $\vdash_1 \text{stat } e$ then $\vdash_1 e : \text{Any}$

*Proof:*QED by the definition of $\vdash_1 e$.

□

Lemma 4.38 : 1 canonical forms

- If $\vdash_1 v : \text{Pair}$ then $v = \langle v_0, v_1 \rangle$
- If $\vdash_1 v : \text{Fun}$ then $v = \lambda x. e'$ or $v = \lambda(x:\tau_d). e'$
- If $\vdash_1 v : \text{Int}$ then $v = i$
- If $\vdash_1 v : \text{Nat}$ then $v \in \mathbb{N}$

*Proof:*QED by definition of $\vdash_1 \cdot : K$

□

Lemma 4.39 : Δ tag soundness

- If $\vdash_1 v_0 : K_0$ and $\vdash_1 v_1 : K_1$ and $\Delta(op^2, K_0, K_1) = K$ then
 $\vdash_1 \delta(op^2, v_0, v_1) : K$.

*Proof:*By case analysis on Δ .**CASE** $\Delta(\text{sum}, \text{Nat}, \text{Nat}) = \text{Nat}$:

1. $v_0 = i_0, i_0 \in \mathbb{N}$
 $\wedge v_1 = i_1, i_1 \in \mathbb{N}$
by *canonical forms*
2. $\delta(\text{sum}, i_0, i_1) = i_0 + i_1 \in \mathbb{N}$
3. QED

CASE $\Delta(\text{sum}, \text{Int}, \text{Int}) = \text{Int}$:

1. $v_0 = i_0$
 $\wedge v_1 = i_1$
by *canonical forms*
2. $\delta(\text{sum}, i_0, i_1) = i_0 + i_1 \in i$
3. QED

CASE $\Delta(\text{quotient}, \text{Nat}, \text{Nat}) = \text{Nat}$:

1. $v_0 = i_0, i_0 \in \mathbb{N}$
 $\wedge v_1 = i_1, i_1 \in \mathbb{N}$
by *canonical forms*
2. **IF** $i_1 = 0$:
 - a. $\delta(\text{quotient}, i_0, i_1) = \text{BndryErr}$
 - b. QED by $\vdash_1 \text{BndryErr} : K$

ELSE $i_1 \neq 0$:

- a. $\delta(\text{quotient}, i_0, i_1) = \lfloor i_0/i_1 \rfloor \in \mathbb{N}$
- b. QED

CASE $\Delta(\text{quotient}, \text{Int}, \text{Int}) = \text{Int}$:

1. $v_0 = i_0$
 $\wedge v_1 = i_1$
by *canonical forms*
2. **IF** $i_1 = 0$:
 - a. $\delta(\text{quotient}, i_0, i_1) = \text{BndryErr}$
 - b. QED by $\vdash_1 \text{BndryErr} : K$

ELSE $i_1 \neq 0$:

- a. $\delta(\text{quotient}, i_0, i_1) = \lfloor i_0/i_1 \rfloor \in i$
- b. QED

□

Lemma 4.40 : δ preservation

- If $\vdash_1 v$ and $\delta(op^1, v) = e$ then $\vdash_1 e$: Any
- If $\vdash_1 v_0$ and $\vdash_1 v_1$ and $\delta(op^2, v_0, v_1) = e$ then $\vdash_1 e$

*Proof:***CASE** $\delta(\text{fst}, \langle v_0, v_1 \rangle) = v_0$:

1. $\vdash_1 v_0$
by *1 inversion*
2. QED

CASE $\delta(\text{snd}, \langle v_0, v_1 \rangle) = v_1$:

1. $\vdash_1 v_1$
by *1 inversion*
2. QED

CASE $\delta(\text{sum}, v_0, v_1) = v_0 + v_1$:

1. QED

CASE $\delta(\text{quotient}, v_0, v_1) = \lfloor v_0/v_1 \rfloor$:

1. QED

CASE $\delta(op^2, v_0, v_1) = \text{BndryErr}$:

1. QED

□

Lemma 4.41 : Δ preservation

- If $\Delta(op^2, \tau_0, \tau_1) = \tau$ then $\Delta(op^2, \lfloor \tau_0 \rfloor, \lfloor \tau_1 \rfloor) = \lfloor \tau \rfloor$.

*Proof:*By case analysis on the definition of Δ **CASE** $\Delta(op^2, \text{Nat}, \text{Nat}) = \text{Nat}$:

1. QED by $\lfloor \text{Nat} \rfloor = \text{Nat}$
2. **CASE** $\Delta(op^2, \text{Int}, \text{Int}) = \text{Int}$:

1. QED by $\lfloor \text{Int} \rfloor = \text{Int}$

□

Lemma 4.42 : Δ inversion

- If $\Delta(\text{fst}, \tau) = \tau'$ then $\tau = \tau_0 \times \tau_1$ and $\tau' = \tau_0$
- If $\Delta(\text{snd}, \tau) = \tau'$ then $\tau = \tau_0 \times \tau_1$ and $\tau' = \tau_1$

*Proof:*QED by the definition of Δ

□

Lemma 4.43 : $<$: preservation

- If $\tau <: \tau'$ then $\lfloor \tau \rfloor \leqslant \lfloor \tau' \rfloor$

*Proof:*By case analysis on the last rule used to show $\tau <: \tau'$.**CASE** $\text{Nat} <: \text{Int}$:

1. QED $\lfloor \text{Nat} \rfloor <: \lfloor \text{Int} \rfloor$

CASE $\tau_d \Rightarrow \tau_c <: \tau'_d \Rightarrow \tau'_c$:

1. $\lfloor \tau_d \Rightarrow \tau_c \rfloor = \text{Fun}$
 $\wedge \lfloor \tau'_d \Rightarrow \tau'_c \rfloor = \text{Fun}$
2. QED

CASE $\tau_0 \times \tau_1 <: \tau'_0 \times \tau'_1$:

1. $\lfloor \tau_0 \times \tau_1 \rfloor = \text{Pair}$
 $\wedge \lfloor \tau'_0 \times \tau'_1 \rfloor = \text{Pair}$
2. QED

□

Lemma 4.44 : 1 static value inversion

9021	• If $\vdash_1 \langle e_0, e_1 \rangle$ then $\vdash_1 e_0$ and $\vdash_1 e_1$	9076
9022	• If $\vdash_1 \lambda x. e$ then $x \vdash_1 e$	9077
9023	• If $\vdash_1 \lambda(x:\tau). e$ then $(x:\tau) \vdash_1 e : \text{Any}$	9078
9024	• If $\vdash_1 e_0 e_1$ then $\vdash_1 e_0$ and $\vdash_1 e_1$	9079
9025	• If $\vdash_1 op^1 e_0$ then $\vdash_1 e_0$	9080
9026	• If $\vdash_1 op^2 e_0 e_1$ then $\vdash_1 e_0$ and $\vdash_1 e_1$	9081
9027	• If $\vdash_1 \text{stat } \tau e$ then $\vdash_1 e : \lfloor \tau \rfloor$	9082
9028	• If $\vdash_1 \text{stat } e$ then $\vdash_1 e : \text{Any}$	9083
9029	<i>Proof:</i>	
9030	QED by the definition of $\vdash_1 e$.	
9031		
9032	Lemma 4.38 : 1 canonical forms	
9033	• If $\vdash_1 v : \text{Pair}$ then $v = \langle v_0, v_1 \rangle$	
9034	• If $\vdash_1 v : \text{Fun}$ then $v = \lambda x. e'$ or $v = \lambda(x:\tau_d). e'$	
9035	• If $\vdash_1 v : \text{Int}$ then $v = i$	
9036	• If $\vdash_1 v : \text{Nat}$ then $v \in \mathbb{N}$	
9037	<i>Proof:</i>	
9038	QED by definition of $\vdash_1 \cdot : K$	
9039		
9040	Lemma 4.39 : Δ tag soundness	
9041	If $\vdash_1 v_0 : K_0$ and $\vdash_1 v_1 : K_1$ and $\Delta(op^2, K_0, K_1) = K$ then $\vdash_1 \delta(op^2, v_0, v_1) : K$.	
9042		
9043	<i>Proof:</i>	
9044	By case analysis on Δ .	
9045	CASE $\Delta(\text{sum}, \text{Nat}, \text{Nat}) = \text{Nat}$:	
9046	1. $v_0 = i_0, i_0 \in \mathbb{N}$ $\wedge v_1 = i_1, i_1 \in \mathbb{N}$ by <i>canonical forms</i>	
9047	2. $\delta(\text{sum}, i_0, i_1) = i_0 + i_1 \in \mathbb{N}$	
9048	3. QED	
9049		
9050		
9051	CASE $\Delta(\text{sum}, \text{Int}, \text{Int}) = \text{Int}$:	
9052	1. $v_0 = i_0$ $\wedge v_1 = i_1$ by <i>canonical forms</i>	
9053	2. $\delta(\text{sum}, i_0, i_1) = i_0 + i_1 \in i$	
9054	3. QED	
9055		
9056		
9057	CASE $\Delta(\text{quotient}, \text{Nat}, \text{Nat}) = \text{Nat}$:	
9058	1. $v_0 = i_0, i_0 \in \mathbb{N}$ $\wedge v_1 = i_1, i_1 \in \mathbb{N}$ by <i>canonical forms</i>	
9059	2. IF $i_1 = 0$: <ol style="list-style-type: none"> a. $\delta(\text{quotient}, i_0, i_1) = \text{BndryErr}$ b. QED by $\vdash_1 \text{BndryErr} : K$ 	
9060		
9061		
9062		
9063		
9064		
9065		
9066		
9067		
9068	CASE $\Delta(\text{quotient}, \text{Int}, \text{Int}) = \text{Int}$:	
9069	1. $v_0 = i_0$ $\wedge v_1 = i_1$ by <i>canonical forms</i>	
9070	2. IF $i_1 = 0$: <ol style="list-style-type: none"> a. $\delta(\text{quotient}, i_0, i_1) = \text{BndryErr}$ b. QED by $\vdash_1 \text{BndryErr} : K$ 	
9071		
9072		
9073		
9074		
9075		
9076	a. $\delta(\text{quotient}, i_0, i_1) = \lfloor i_0/i_1 \rfloor \in i$	9076
9077	b. QED	9077
9078		9078
9079	Lemma 4.40 : δ preservation	9079
9080	• If $\vdash_1 v$ and $\delta(op^1, v) = e$ then $\vdash_1 e$: Any	9080
9081	• If $\vdash_1 v_0$ and $\vdash_1 v_1$ and $\delta(op^2, v_0, v_1) = e$ then $\vdash_1 e$	9081
9082		9082
9083	<i>Proof:</i>	
9084	CASE $\delta(\text{fst}, \langle v_0, v_1 \rangle) = v_0$:	
9085	1. $\vdash_1 v_0$ by <i>1 inversion</i>	
9086	2. QED	9086
9087	CASE $\delta(\text{snd}, \langle v_0, v_1 \rangle) = v_1$:	9087
9088	1. $\vdash_1 v_1$ by <i>1 inversion</i>	9088
9089	2. QED	9089
9090		9090
9091	CASE $\delta(\text{sum}, v_0, v_1) = v_0 + v_1$:	9091
9092	1. QED	9092
9093	CASE $\delta(\text{quotient}, v_0, v_1) = \lfloor v_0/v_1 \rfloor$:	9093
9094	1. QED	9094
9095	CASE $\delta(op^2, v_0, v_1) = \text{BndryErr}$:	9095
9096	1. QED	9096
9097		9097
9098		9098
9099	Lemma 4.41 : Δ preservation	9099
9100	If $\Delta(op^2, \tau_0, \tau_1) = \tau$ then $\Delta(op^2, \lfloor \tau_0 \rfloor, \lfloor \tau_1 \rfloor) = \lfloor \tau \rfloor$.	9100
9101	<i>Proof:</i>	
9102	By case analysis on the definition of Δ	
9103	CASE $\Delta(op^2, \text{Nat}, \text{Nat}) = \text{Nat}$:	
9104	1. QED by $\lfloor \text{Nat} \rfloor = \text{Nat}$	
9105	CASE $\Delta(op^2, \text{Int}, \text{Int}) = \text{Int}$:	
9106	1. QED by $\lfloor \text{Int} \rfloor = \text{Int}$	
9107		9107
9108	Lemma 4.42 : Δ inversion	9108
9109	• If $\Delta(\text{fst}, \tau) = \tau'$ then $\tau = \tau_0 \times \tau_1$ and $\tau' = \tau_0$	9109
9110	• If $\Delta(\text{snd}, \tau) = \tau'$ then $\tau = \tau_0 \times \tau_1$ and $\tau' = \tau_1$	9110
9111	<i>Proof:</i>	
9112	QED by the definition of Δ	
9113		9113
9114	Lemma 4.43 : $<$: preservation	9114
9115	If $\tau <: \tau'$ then $\lfloor \tau \rfloor \leqslant \lfloor \tau' \rfloor$	9115
9116	<i>Proof:</i>	
9117	By case analysis on the last rule used to show $\tau <: \tau'$.	
9118	CASE $\text{Nat} <: \text{Int}$:	
9119	1. QED $\lfloor \text{Nat} \rfloor <: \lfloor \text{Int} \rfloor$	
9120	CASE $\tau_d \Rightarrow \tau_c <: \tau'_d \Rightarrow \tau'_c$:	
9121	1. $\lfloor \tau_d \Rightarrow \tau_c \rfloor = \text{Fun}$ $\wedge \lfloor \tau'_d \Rightarrow \tau'_c \rfloor = \text{Fun}$	
9122	2. QED	9122
9123	CASE $\tau_0 \times \tau_1 <: \tau'_0 \times \tau'_1$:	9123
9124	1. $\lfloor \tau_0 \times \tau_1 \rfloor = \text{Pair}$ $\wedge \lfloor \tau'_0 \times \tau'_1 \rfloor = \text{Pair}$	9124
9125	2. QED	9125
9126		9126
9127		9127
9128		9128
9129	Lemma 4.44 : 1 static value inversion	9129
9130		9130

9131 If $\vdash_1 v : \text{Any}$ then $\vdash_1 v$

9132 *Proof:*

9133 By induction on the structure of v .

9134 **CASE** $v = i$:

9135 1. QED by $\vdash_1 v$

9136 **CASE** $v = \langle v_0, v_1 \rangle$:

9137 1. $\vdash_1 v_0 : \text{Any}$

9138 $\wedge \vdash_1 v_1 : \text{Any}$

9139 by 1 *inversion*

9140 2. $\vdash_1 v_0$

9141 $\wedge \vdash_1 v_1$

9142 by the induction hypothesis

9143 3. QED by (2)

9144 **CASE** $v = \lambda x. e$:

9145 1. $x \vdash_1 e$

9146 by 1 *inversion*

9147 2. QED

9148 **CASE** $v = \lambda(x:\tau). e$:

9149 1. $(x:\tau) \vdash_1 e : \text{Any}$

9150 by 1 *inversion*

9151 2. QED

9152 \square

9153 **Lemma 4.45 : 1 dynamic value inversion**

9154 If $\vdash_1 v$ then $\vdash_1 v : \text{Any}$

9155 *Proof:*

9156 By induction on the structure of v .

9157 **CASE** $v = i$:

9158 1. $\vdash_1 v : \text{Int}$

9159 2. QED by $\text{Int} <: \text{Any}$

9160 **CASE** $v = \langle v_0, v_1 \rangle$:

9161 1. $\vdash_1 v_0$

9162 $\wedge \vdash_1 v_1$

9163 by 1 *inversion*

9164 2. $\vdash_1 v_0 : \text{Any}$

9165 $\wedge \vdash_1 v_1 : \text{Any}$

9166 by the induction hypothesis

9167 3. $\vdash_1 \langle v_0, v_1 \rangle : \text{Pair}$

9168 by (2)

9169 4. QED by $\text{Pair} <: \text{Any}$

9170 **CASE** $v = \lambda x. e$:

9171 1. $x \vdash_1 e$

9172 by 1 *inversion*

9173 2. $\vdash_1 \lambda x. e : \text{Fun}$

9174 by (1)

9175 3. QED by $\text{Fun} <: \text{Any}$

9176 **CASE** $v = \lambda(x:\tau). e$:

9177 1. $(x:\tau) \vdash_1 e : \text{Any}$

9178 by 1 *inversion*

9179 2. $\vdash_1 \lambda(x:\tau). e : \text{Fun}$

9180 by (1)

9181 3. QED by $\text{Fun} <: \text{Any}$

9182 \square

9183 **Lemma 4.46 : 1 substitution**

9184

- If $(x:\tau), \Gamma \vdash_1 e$ and $\vdash_1 v : \lfloor \tau \rfloor$ then $\Gamma \vdash_1 e[x \leftarrow v]$ 9186
- If $x, \Gamma \vdash_1 e$ and $\vdash_1 v$ then $\Gamma \vdash_1 e[x \leftarrow v]$ 9187
- If $(x:\tau_x), \Gamma \vdash_1 e : K$ and $\vdash_1 v : \lfloor \tau_x \rfloor$ then $\Gamma \vdash_1 e[x \leftarrow v] : K$ 9188
- If $x, \Gamma \vdash_1 e : K$ and $\vdash_1 v$ then $\Gamma \vdash_1 e[x \leftarrow v] : K$ 9189

9190 *Proof:*

9191 By the following four lemmas: *dynamic context static value substitution*, *dynamic context dynamic value substitution*, *static context static value substitution*, and *static context dynamic value substitution*. 9192

9193 \square

9194 **Lemma 4.47 : 1 dynamic-static substitution**

9195 If $(x:\tau), \Gamma \vdash_1 e$ and $\vdash_1 v : \lfloor \tau \rfloor$ then $\Gamma \vdash_1 e[x \leftarrow v]$ 9196

9197 *Proof:*

9198 By induction on the structure of e . 9199

9200 **CASE** $e = x$:

9201 1. $e[x \leftarrow v] = v$

9202 2. $\vdash_1 v : \text{Any}$

9203 by $\lfloor \tau \rfloor <: \text{Any}$

9204 3. $\vdash_1 v$

9205 by *static value inversion* (2)

9206 4. $\Gamma \vdash_1 v$

9207 by *weakening* (3)

9208 5. QED

9209 **CASE** $e = x'$:

9210 1. QED by $x'[x \leftarrow v] = x'$

9211 **CASE** $e = i$:

9212 1. QED by $i[x \leftarrow v] = i$

9213 **CASE** $e = \lambda x. e'$:

9214 1. QED by $(\lambda x. e')[x \leftarrow v] = \lambda x. e'$

9215 **CASE** $e = \lambda(x:\tau'). e'$:

9216 1. QED by $(\lambda(x:\tau'). e')[x \leftarrow v] = \lambda(x:\tau'). e'$

9217 **CASE** $e = \lambda x'. e'$:

9218 1. $e[x \leftarrow v] = \lambda x'. (e'[x \leftarrow v])$

9219 2. $x', (x:\tau), \Gamma \vdash_1 e'$

9220 by 1 *inversion*

9221 3. $x', \Gamma \vdash_1 e'[x \leftarrow v]$

9222 by *dynamic context static value substitution*

9223 4. $\Gamma \vdash_1 \lambda x'. e'[x \leftarrow v]$

9224 by (3)

9225 5. QED

9226 **CASE** $e = \lambda(x':\tau'). e'$:

9227 1. $e[x \leftarrow v] = \lambda(x':\tau'). (e'[x \leftarrow v])$

9228 2. $(x':\tau'), (x:\tau), \Gamma \vdash_1 e' : \text{Any}$

9229 by 1 *inversion*

9230 3. $(x':\tau'), \Gamma \vdash_1 e'[x \leftarrow v] : \text{Any}$

9231 by *static context static value substitution*

9232 4. $\Gamma \vdash_1 \lambda(x':\tau'). (e'[x \leftarrow v])$

9233 5. QED

9234 **CASE** $e = \langle e_0, e_1 \rangle$:

9235 1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$

9236 2. $(x:\tau), \Gamma \vdash_1 e_0$

9237 $\wedge (x:\tau), \Gamma \vdash_1 e_1$

9238 by 1 *inversion*

9239

9240

9241	3. $\Gamma \vdash_1 e_0[x \leftarrow v]$	9296
9242	$\wedge \Gamma \vdash_1 e_1[x \leftarrow v]$	9297
9243	by the induction hypothesis (2)	9298
9244	4. $\Gamma \vdash_1 \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$	9299
9245	by (3)	9300
9246	5. QED	9301
9247	CASE $e = e_0 e_1 :$	9302
9248	1. $e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]$	9303
9249	2. $(x:\tau), \Gamma \vdash_1 e_0$	9304
9250	$\wedge (x:\tau), \Gamma \vdash_1 e_1$	9305
9251	by 1 <i>inversion</i>	9306
9252	3. $\Gamma \vdash_1 e_0[x \leftarrow v]$	9307
9253	$\wedge \Gamma \vdash_1 e_1[x \leftarrow v]$	9308
9254	by the induction hypothesis (2)	9309
9255	4. $\Gamma \vdash_1 e_0[x \leftarrow v] e_1[x \leftarrow v]$	9310
9256	by (3)	9311
9257	5. QED	9312
9258	CASE $e = op^1 e_0 :$	9313
9259	1. $e[x \leftarrow v] = op^1 e_0[x \leftarrow v]$	9314
9260	2. $(x:\tau), \Gamma \vdash_1 e_0$	9315
9261	by 1 <i>inversion</i>	9316
9262	3. $\Gamma \vdash_1 e_0[x \leftarrow v]$	9317
9263	by the induction hypothesis (2)	9318
9264	4. $\Gamma \vdash_1 op^1 e_0[x \leftarrow v]$	9319
9265	by (3)	9320
9266	5. QED	9321
9267	CASE $e = op^2 e_0 e_1 :$	9322
9268	1. $e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$	9323
9269	2. $(x:\tau), \Gamma \vdash_1 e_0$	9324
9270	$\wedge (x:\tau), \Gamma \vdash_1 e_1$	9325
9271	by 1 <i>inversion</i>	9326
9272	3. $\Gamma \vdash_1 e_0[x \leftarrow v]$	9327
9273	$\wedge \Gamma \vdash_1 e_1[x \leftarrow v]$	9328
9274	by the induction hypothesis (2)	9329
9275	4. $\Gamma \vdash_1 op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$	9330
9276	by (3)	9331
9277	5. QED	9332
9278	CASE $e = dyn \tau' e'$	9333
9279	$\vee e = dyn e'$	9334
9280	$\vee e = chk K' e' :$	9335
9281	1. Contradiction by $(x:\tau), \Gamma \vdash_1 e$	9336
9282	CASE $e = stat \tau' e' :$	9337
9283	1. $e[x \leftarrow v] = stat \tau' e'[x \leftarrow v]$	9338
9284	2. $(x:\tau), \Gamma \vdash_1 e' : \lfloor \tau' \rfloor$	9339
9285	by 1 <i>inversion</i>	9340
9286	3. $\Gamma \vdash_1 e'[x \leftarrow v] : \lfloor \tau' \rfloor$	9341
9287	by <i>static context static value substitution</i> (2)	9342
9288	4. $\Gamma \vdash_1 stat \tau' (e'[x \leftarrow v])$	9343
9289	by (3)	9344
9290	5. QED	9345
9291	CASE $e = stat e' :$	9346
9292	1. $e[x \leftarrow v] = stat e'[x \leftarrow v]$	9347
9293	2. $(x:\tau), \Gamma \vdash_1 e' : Any$	9348
9294	by 1 <i>inversion</i>	9349
9295		9350
	3. $\Gamma \vdash_1 e'[x \leftarrow v] : Any$	
	by <i>static context static value substitution</i> (2)	
	4. $\Gamma \vdash_1 stat (e'[x \leftarrow v])$	
	by (3)	
	5. QED	
	□	
	Lemma 4.48 : 1 <i>dynamic-dynamic substitution</i>	
	If $x, \Gamma \vdash_1 e$ and $\vdash_1 v$ then $\Gamma \vdash_1 e[x \leftarrow v]$	
	<i>Proof:</i>	
	By induction on the structure of e .	
	CASE $e = x :$	
	1. $e[x \leftarrow v] = v$	
	2. $\Gamma \vdash_1 v$	
	by <i>weakening</i> (3)	
	3. QED	
	CASE $e = x' :$	
	1. QED by $x'[x \leftarrow v] = x'$	
	CASE $e = i :$	
	1. QED by $i[x \leftarrow v] = i$	
	CASE $e = \lambda x. e' :$	
	1. QED by $(\lambda x. e')[x \leftarrow v] = \lambda x. e'$	
	CASE $e = \lambda(x:\tau'). e' :$	
	1. QED by $(\lambda(x:\tau'). e')[x \leftarrow v] = \lambda(x:\tau'). e'$	
	CASE $e = \lambda x'. e' :$	
	1. $e[x \leftarrow v] = \lambda x'. (e'[x \leftarrow v])$	
	2. $x', x, \Gamma \vdash_1 e'$	
	by 1 <i>inversion</i>	
	3. $x', \Gamma \vdash_1 e'[x \leftarrow v]$	
	by the induction hypothesis (2)	
	4. $\Gamma \vdash_1 \lambda x'. e'[x \leftarrow v]$	
	by (3)	
	5. QED	
	CASE $e = \lambda(x':\tau'). e' :$	
	1. $e[x \leftarrow v] = \lambda(x':\tau'). (e'[x \leftarrow v])$	
	2. $(x':\tau'), x, \Gamma \vdash_1 e' : Any$	
	by 1 <i>inversion</i>	
	3. $(x':\tau'), \Gamma \vdash_1 e'[x \leftarrow v] : Any$	
	by <i>static context dynamic value substitution</i>	
	4. $\Gamma \vdash_1 \lambda(x':\tau'). (e'[x \leftarrow v])$	
	5. QED	
	CASE $e = \langle e_0, e_1 \rangle :$	
	1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$	
	2. $x, \Gamma \vdash_1 e_0$	
	$\wedge x, \Gamma \vdash_1 e_1$	
	by 1 <i>inversion</i>	
	3. $\Gamma \vdash_1 e_0[x \leftarrow v]$	
	$\wedge \Gamma \vdash_1 e_1[x \leftarrow v]$	
	by the induction hypothesis (2)	
	4. $\Gamma \vdash_1 \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$	
	by (3)	
	5. QED	
	CASE $e = e_0 e_1 :$	
	1. $e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]$	

9351	2. $x, \Gamma \vdash_1 e_0$	
9352	$\wedge x, \Gamma \vdash_1 e_1$	
9353	by 1 <i>inversion</i>	
9354	3. $\Gamma \vdash_1 e_0[x \leftarrow v]$	
9355	$\wedge \Gamma \vdash_1 e_1[x \leftarrow v]$	
9356	by the induction hypothesis (2)	
9357	4. $\Gamma \vdash_1 e_0[x \leftarrow v] e_1[x \leftarrow v]$	
9358	by (3)	
9359	5. QED	
9360	CASE $e = op^1 e_0 :$	
9361	1. $e[x \leftarrow v] = op^1 e_0[x \leftarrow v]$	
9362	2. $x, \Gamma \vdash_1 e_0$	
9363	by 1 <i>inversion</i>	
9364	3. $\Gamma \vdash_1 e_0[x \leftarrow v]$	
9365	by the induction hypothesis (2)	
9366	4. $\Gamma \vdash_1 op^1 e_0[x \leftarrow v]$	
9367	by (3)	
9368	5. QED	
9369	CASE $e = op^2 e_0 e_1 :$	
9370	1. $e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$	
9371	2. $x, \Gamma \vdash_1 e_0$	
9372	$\wedge x, \Gamma \vdash_1 e_1$	
9373	by 1 <i>inversion</i>	
9374	3. $\Gamma \vdash_1 e_0[x \leftarrow v]$	
9375	$\wedge \Gamma \vdash_1 e_1[x \leftarrow v]$	
9376	by the induction hypothesis (2)	
9377	4. $\Gamma \vdash_1 op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$	
9378	by (3)	
9379	5. QED	
9380	CASE $e = \text{dyn } \tau' e'$	
9381	$\vee e = \text{dyn } e'$	
9382	$\vee e = \text{chk } K' e' :$	
9383	1. Contradiction by $\Gamma \vdash_1 e$	
9384	CASE $e = \text{stat } \tau' e' :$	
9385	1. $e[x \leftarrow v] = \text{stat } \tau' e'[x \leftarrow v]$	
9386	2. $x, \Gamma \vdash_1 e' : \lfloor \tau' \rfloor$	
9387	by 1 <i>inversion</i>	
9388	3. $\Gamma \vdash_1 e'[x \leftarrow v] : \lfloor \tau' \rfloor$	
9389	by <i>static context dynamic value substitution</i> (2)	
9390	4. $\Gamma \vdash_1 \text{stat } \tau' (e'[x \leftarrow v])$	
9391	by (3)	
9392	5. QED	
9393	CASE $e = \text{stat } e' :$	
9394	1. $e[x \leftarrow v] = \text{stat } e'[x \leftarrow v]$	
9395	2. $x, \Gamma \vdash_1 e' : \text{Any}$	
9396	by 1 <i>inversion</i>	
9397	3. $\Gamma \vdash_1 e'[x \leftarrow v] : \text{Any}$	
9398	by <i>static context dynamic value substitution</i> (2)	
9399	4. $\Gamma \vdash_1 \text{stat } (e'[x \leftarrow v])$	
9400	by (3)	
9401	5. QED	
9402	□	
9403	Lemma 4.49 : 1 static-static substitution	
9404		
9405		

	If $(x:\tau), \Gamma \vdash_1 e : K$ and $\vdash_1 v : \lfloor \tau \rfloor$ then $\Gamma \vdash_1 e[x \leftarrow v] : K$	9406
	<i>Proof:</i>	9407
	By induction on the structure of e .	9408
	CASE $e = x :$	9409
	1. $\lfloor \tau \rfloor \leqslant K$	9410
	by $(x:\tau), \Gamma \vdash_1 x : K$	9411
	2. $e[x \leftarrow v] = v$	9412
	3. $\vdash_1 v : K$	9413
	by (1)	9414
	4. $\Gamma \vdash_1 v : K$	9415
	by <i>weakening</i> (3)	9416
	5. QED	9417
	CASE $e = x' :$	9418
	1. QED by $(x'[x \leftarrow v]) = x'$	9419
	CASE $e = i :$	9420
	1. QED by $i[x \leftarrow v] = i$	9421
	CASE $e = \lambda x. e' :$	9422
	1. QED by $(\lambda x. e')[x \leftarrow v] = \lambda x. e'$	9423
	CASE $e = \lambda x'. e' :$	9424
	1. $e[x \leftarrow v] = \lambda x'. (e'[x \leftarrow v])$	9425
	2. $x', (x:\tau), \Gamma \vdash_1 e'$	9426
	by 1 <i>inversion</i>	9427
	3. $x', \Gamma \vdash_1 e'[x \leftarrow v]$	9428
	by <i>dynamic context static value substitution</i>	9429
	4. $\Gamma \vdash_1 \lambda x'. e'[x \leftarrow v] : K$	9430
	by (3)	9431
	5. QED	9432
	CASE $e = \lambda(x':\tau'). e' :$	9433
	1. $e[x \leftarrow v] = \lambda(x':\tau'). (e'[x \leftarrow v])$	9434
	2. $(x':\tau'), (x:\tau), \Gamma \vdash_1 e' : \text{Any}$	9435
	by 1 <i>inversion</i>	9436
	3. $(x':\tau'), \Gamma \vdash_1 e'[x \leftarrow v] : \text{Any}$	9437
	by the induction hypothesis (2)	9438
	4. $\Gamma \vdash_1 \lambda(x':\tau'). (e'[x \leftarrow v]) : K$	9439
	5. QED	9440
	CASE $e = \langle e_0, e_1 \rangle :$	9441
	1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$	9442
	2. $(x:\tau), \Gamma \vdash_1 e_0 : \text{Any}$	9443
	$\wedge (x:\tau), \Gamma \vdash_1 e_1 : \text{Any}$	9444
	by 1 <i>inversion</i>	9445
	3. $\Gamma \vdash_1 e_0[x \leftarrow v] : \text{Any}$	9446
	$\wedge \Gamma \vdash_1 e_1[x \leftarrow v] : \text{Any}$	9447
	by the induction hypothesis (2)	9448
	4. $\Gamma \vdash_1 \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle : K$	9449
	by (3)	9450
	5. QED	9451
	CASE $e = e_0 e_1 :$	9452
	1. $e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]$	9453
	2. $(x:\tau), \Gamma \vdash_1 e_0 : K_0$	9454
	$\wedge (x:\tau), \Gamma \vdash_1 e_1 : K_1$	9455
	by 1 <i>inversion</i>	9456
	3. $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$	9457
	$\wedge \Gamma \vdash_1 e_1[x \leftarrow v] : K_1$	9458
	by the induction hypothesis (2)	9459
		9460

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9461   4.  $\Gamma \vdash_1 e_0[x \leftarrow v] e_1[x \leftarrow v] : K$  9516
9462     by (3) 9517
9463   5. QED 9518
9464 CASE  $e = op^1 e_0 :$  9519
9465   1.  $e[x \leftarrow v] = op^1 e_0[x \leftarrow v]$  9520
9466   2.  $(x:\tau), \Gamma \vdash_1 e_0 : K_0$  9521
9467     by 1 inversion 9522
9468   3.  $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$  9523
9469     by the induction hypothesis (2) 9524
9470   4.  $\Gamma \vdash_1 op^1 e_0[x \leftarrow v] : K$  9525
9471     by (3) 9526
9472   5. QED 9527
9473 CASE  $e = op^2 e_0 e_1 :$  9528
9474   1.  $e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$  9529
9475   2.  $(x:\tau), \Gamma \vdash_1 e_0 : K_0$  9530
9476      $\wedge (x:\tau), \Gamma \vdash_1 e_1 : K_1$  9531
9477     by 1 inversion 9532
9478   3.  $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$  9533
9479      $\wedge \Gamma \vdash_1 e_1[x \leftarrow v] : K_1$  9534
9480     by the induction hypothesis (2) 9535
9481   4.  $\Gamma \vdash_1 op^2 e_0[x \leftarrow v] e_1[x \leftarrow v] : K$  9536
9482     by (3) 9537
9483   5. QED 9538
9484 CASE  $e = dyn \tau' e' :$  9539
9485   1.  $e[x \leftarrow v] = dyn \tau' e'[x \leftarrow v]$  9540
9486   2.  $(x:\tau), \Gamma \vdash_1 e'$  9541
9487     by 1 inversion 9542
9488   3.  $\Gamma \vdash_1 e'[x \leftarrow v]$  9543
9489     by dynamic context static value substitution (2) 9544
9490   4.  $\Gamma \vdash_1 dyn \tau' (e'[x \leftarrow v]) : K$  9545
9491     by (3) 9546
9492   5. QED 9547
9493 CASE  $e = dyn e' :$  9548
9494   1.  $e[x \leftarrow v] = dyn e'[x \leftarrow v]$  9549
9495   2.  $(x:\tau), \Gamma \vdash_1 e'$  9550
9496     by 1 inversion 9551
9497   3.  $\Gamma \vdash_1 e'[x \leftarrow v]$  9552
9498     by dynamic context static value substitution (2) 9553
9499   4.  $\Gamma \vdash_1 dyn (e'[x \leftarrow v]) : K$  9554
9500     by (3) 9555
9501   5. QED 9556
9502 CASE  $e = chk K' e' :$  9557
9503   1.  $e[x \leftarrow v] = chk K' (e'[x \leftarrow v])$  9558
9504   2.  $(x:\tau), \Gamma \vdash_1 e' : Any$  9559
9505     by 1 inversion 9560
9506   3.  $\Gamma \vdash_1 e'[x \leftarrow v] : Any$  9561
9507     by the induction hypothesis (2) 9562
9508   4.  $\Gamma \vdash_1 chk K' (e'[x \leftarrow v]) : K$  9563
9509     by (3) 9564
9510   5. QED 9565
9511 CASE  $e = stat \tau' e'$  9566
9512    $\vee e = stat e' :$  9567
9513   1. Contradiction by  $\Gamma \vdash_1 e : K$  9568
9514
9515

```

□ 9516

Lemma 4.50 : 1 *static-dynamic substitution* 9517

If $x, \Gamma \vdash_1 e : K$ and $\vdash_1 v$ then $\Gamma \vdash_1 e[x \leftarrow v] : K$ 9518

Proof: 9519

By induction on the structure of e . 9520

CASE $e = x :$ 9521

1. $K = Any$ 9522
- by $x, \Gamma \vdash_1 x : K$ 9523
2. $e[x \leftarrow v] = v$ 9524
3. $\vdash_1 v : K$ 9525
- by *dynamic value inversion* 9526
4. $\vdash_1 v : Any$ 9527
- by $K \leqslant: Any$ 9528
5. $\Gamma \vdash_1 v : Any$ 9529
- by *weakening* (3) 9530
6. QED 9531

CASE $e = x' :$ 9532

1. QED by $x'[x \leftarrow v] = x'$ 9533

CASE $e = i :$ 9534

1. QED by $i[x \leftarrow v] = i$ 9535

CASE $e = \lambda x. e' :$ 9536

1. QED by $(\lambda x. e')[x \leftarrow v] = \lambda x. e'$ 9537

CASE $e = \lambda(x:\tau'). e' :$ 9538

1. QED by $(\lambda(x:\tau'). e')[x \leftarrow v] = \lambda(x:\tau'). e'$ 9539

CASE $e = \lambda x'. e' :$ 9540

1. $e[x \leftarrow v] = \lambda x'. (e'[x \leftarrow v])$ 9541
2. $x', x, \Gamma \vdash_1 e'$ 9542
- by 1 *inversion* 9543
3. $x', \Gamma \vdash_1 e'[x \leftarrow v]$ 9544
- by *dynamic context dynamic value substitution* 9545
4. $\Gamma \vdash_1 \lambda x'. e'[x \leftarrow v] : K$ 9546
- by (3) 9547
5. QED 9548

CASE $e = \lambda(x':\tau'). e' :$ 9549

1. $e[x \leftarrow v] = \lambda(x':\tau'). (e'[x \leftarrow v])$ 9550
2. $(x':\tau'), x, \Gamma \vdash_1 e' : Any$ 9551
- by 1 *inversion* 9552
3. $(x':\tau'), \Gamma \vdash_1 e'[x \leftarrow v] : Any$ 9553
- by *static context dynamic value substitution* 9554
4. $\Gamma \vdash_1 \lambda(x':\tau'). (e'[x \leftarrow v]) : K$ 9555
5. QED 9556

CASE $e = \langle e_0, e_1 \rangle :$ 9557

1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$ 9558
2. $x, \Gamma \vdash_1 e_0 : Any$ 9559
- $\wedge x, \Gamma \vdash_1 e_1 : Any$ 9560
- by 1 *inversion* 9561
3. $\Gamma \vdash_1 e_0[x \leftarrow v] : Any$ 9562
- $\wedge \Gamma \vdash_1 e_1[x \leftarrow v] : Any$ 9563
- by the induction hypothesis (2) 9564
4. $\Gamma \vdash_1 \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle : K$ 9565
- by (3) 9566
5. QED 9567

CASE $e = e_0 e_1 :$ 9568

9571	1. $e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]$	9626
9572	2. $x, \Gamma \vdash_1 e_0 : K_0$	9627
9573	$\wedge x, \Gamma \vdash_1 e_1 : K_1$	9628
9574	by 1 <i>inversion</i>	9629
9575	3. $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$	9630
9576	$\wedge \Gamma \vdash_1 e_1[x \leftarrow v] : K_1$	9631
9577	by the induction hypothesis (2)	9632
9578	4. $\Gamma \vdash_1 e_0[x \leftarrow v] e_1[x \leftarrow v] : K$	9633
9579	by (3)	9634
9580	5. QED	9635
9581	CASE $e = op^1 e_0 :$	9636
9582	1. $e[x \leftarrow v] = op^1 e_0[x \leftarrow v]$	9637
9583	2. $x, \Gamma \vdash_1 e_0 : K_0$	9638
9584	by 1 <i>inversion</i>	9639
9585	3. $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$	9640
9586	by the induction hypothesis (2)	9641
9587	4. $\Gamma \vdash_1 op^1 e_0[x \leftarrow v] : K$	9642
9588	by (3)	9643
9589	5. QED	9644
9590	CASE $e = op^2 e_0 e_1 :$	9645
9591	1. $e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$	9646
9592	2. $x, \Gamma \vdash_1 e_0 : K_0$	9647
9593	$\wedge x, \Gamma \vdash_1 e_1 : K_1$	9648
9594	by 1 <i>inversion</i>	9649
9595	3. $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$	9650
9596	$\wedge \Gamma \vdash_1 e_1[x \leftarrow v] : K_1$	9651
9597	by the induction hypothesis (2)	9652
9598	4. $\Gamma \vdash_1 op^2 e_0[x \leftarrow v] e_1[x \leftarrow v] : K$	9653
9599	by (3)	9654
9600	5. QED	9655
9601	CASE $e = dyn \tau' e' :$	9656
9602	1. $e[x \leftarrow v] = dyn \tau' e'[x \leftarrow v]$	9657
9603	2. $x, \Gamma \vdash_1 e'$	9658
9604	by 1 <i>inversion</i>	9659
9605	3. $\Gamma \vdash_1 e'[x \leftarrow v]$	9660
9606	by <i>dynamic context dynamic value substitution</i> (2)	9661
9607	4. $\Gamma \vdash_1 dyn \tau' (e'[x \leftarrow v]) : K$	9662
9608	by (3)	9663
9609	5. QED	9664
9610	CASE $e = dyn e' :$	9665
9611	1. $e[x \leftarrow v] = dyn e'[x \leftarrow v]$	9666
9612	2. $x, \Gamma \vdash_1 e'$	9667
9613	by 1 <i>inversion</i>	9668
9614	3. $\Gamma \vdash_1 e'[x \leftarrow v]$	9669
9615	by <i>dynamic context dynamic value substitution</i> (2)	9670
9616	4. $\Gamma \vdash_1 dyn (e'[x \leftarrow v]) : K$	9671
9617	by (3)	9672
9618	5. QED	9673
9619	CASE $e = chk K' e' :$	9674
9620	1. $e[x \leftarrow v] = chk K' (e'[x \leftarrow v])$	9675
9621	2. $x, \Gamma \vdash_1 e' : \text{Any}$	9676
9622	by 1 <i>inversion</i>	9677
9623	3. $\Gamma \vdash_1 e'[x \leftarrow v] : \text{Any}$	9678
9624	by the induction hypothesis (2)	9679
9625		9680
	4. $\Gamma \vdash_1 \text{chk } K' (e'[x \leftarrow v]) : K$	
	by (3)	
	5. QED	
	CASE $e = \text{stat } \tau' e'$	
	$\vee e = \text{stat } e' :$	
	1. Contradiction by $\Gamma \vdash_1 e : K$	
	□	
	Lemma 4.51 : weakening	
	• If $\Gamma \vdash_1 e$ then $x, \Gamma \vdash_1 e$	
	• If $\Gamma \vdash_1 e : \tau$ then $(x:\tau'), \Gamma \vdash_1 e : \tau$	
	<i>Proof:</i>	
	• e is closed under Γ	
	by $\Gamma \vdash_1 e$	
	$\vee \Gamma \vdash_1 e : \tau$ QED	
	□	
	Lemma 4.52 : unique static evaluation contexts	
	If $\vdash e : \tau$ and e is boundary-free then one of the following holds:	
	• e is a value	
	• $e = E^\bullet[v_0 v_1]$	
	• $e = E^\bullet[op^1 v]$	
	• $e = E^\bullet[op^2 v_0 v_1]$	
	• $e = E^\bullet[\text{Err}]$	
	<i>Proof:</i>	
	By induction on the structure of e .	
	CASE $e = x :$	
	1. Contradiction by $\vdash e : \tau$	
	CASE $e = i$	
	$\vee e = \lambda(x:\tau_d). e'$:	
	1. QED e is a value	
	CASE $e = \langle e_0, e_1 \rangle :$	
	IF $e_0 \notin v :$	
	1. $e_0 = E^\bullet_0[e'_0]$	
	by the induction hypothesis	
	2. $E^\bullet = \langle E^\bullet_0, e_1 \rangle$	
	3. QED by $e = E^\bullet[e'_0]$	
	IF $e_0 \in v$	
	$\wedge e_1 \notin v :$	
	1. $e_1 = E^\bullet_1[e'_1]$	
	by the induction hypothesis	
	2. $E^\bullet = \langle e_0, E^\bullet_1 \rangle$	
	3. QED by $e = E^\bullet[e'_1]$	
	ELSE $e_0 \in v$	
	$\wedge e_1 \in v :$	
	1. $E^\bullet = []$	
	2. QED $e = E^\bullet[\langle e_0, e_1 \rangle]$	
	CASE $e = e_0 e_1 :$	
	IF $e_0 \notin v :$	
	1. $e_0 = E^\bullet_0[e'_0]$	
	by the induction hypothesis	
	2. $E^\bullet = E^\bullet_0 e_1$	
	3. QED by $e = E^\bullet[e'_0]$	

```

9681   IF  $e_0 \in v$                                      9736
9682      $\wedge e_1 \notin v :$ 
9683       1.  $e_1 = E^{\bullet}_1[e'_1]$                  9737
9684         by the induction hypothesis
9685       2.  $E^{\bullet} = e_0 E^{\bullet}_1$ 
9686       3. QED by  $e = E^{\bullet}[e'_1]$ 
9687   ELSE  $e_0 \in v$ 
9688      $\wedge e_1 \in v :$ 
9689       1.  $E^{\bullet} = []$ 
9690       2. QED  $e = E^{\bullet}[e_0 e_1]$ 
9691 CASE  $e = op^1 e_0 :$ 
9692   IF  $e_0 \notin v :$ 
9693     1.  $e_0 = E^{\bullet}_0[e'_0]$                  9738
9694       by the induction hypothesis
9695     2.  $E^{\bullet} = op^1 E^{\bullet}_0$ 
9696     3. QED  $e = E^{\bullet}[e'_0]$ 
9697   ELSE  $e_0 \in v :$ 
9698     1.  $E^{\bullet} = []$ 
9699     2. QED  $e = E^{\bullet}[op^1 e_0]$ 
9700 CASE  $e = op^2 e_0 e_1 :$ 
9701   IF  $e_0 \notin v :$ 
9702     1.  $e_0 = E^{\bullet}_0[e'_0]$                  9739
9703       by the induction hypothesis
9704     2.  $E^{\bullet} = op^2 E^{\bullet}_0 e_1$ 
9705     3. QED  $e = E^{\bullet}[e'_0]$ 
9706   IF  $e_0 \in v$ 
9707      $\wedge e_1 \notin v :$ 
9708       1.  $e_1 = E^{\bullet}_1[e'_1]$                  9740
9709         by the induction hypothesis
9710       2.  $E^{\bullet} = op^2 e_0 E^{\bullet}_1$ 
9711       3. QED  $e = E^{\bullet}[e'_1]$ 
9712   ELSE  $e_0 \in v$ 
9713      $\wedge e_1 \in v :$ 
9714       1.  $E^{\bullet} = []$ 
9715       2. QED  $e = E^{\bullet}[op^2 e_0 e_1]$ 
9716 CASE  $e = \text{chk } K' e' :$ 
9717   1. Contradiction by  $\vdash e : \tau$ 
9718 CASE  $e = \text{dyn } e_0 :$ 
9719   1. Contradiction by  $\vdash e : \tau$ 
9720 CASE  $e = \text{stat } e' :$ 
9721   1. Contradiction by  $\vdash e : \tau$ 
9722 CASE  $e = \text{dyn } \tau e_0 :$ 
9723   1. QED  $e$  is boundary-free
9724 CASE  $e = \text{stat } \tau e' :$ 
9725   1. Contradiction by  $\vdash e : \tau$ 
9726 CASE  $e = \text{Err} :$ 
9727   1.  $E^{\bullet} = []$ 
9728   2. QED
9729  $\square$ 

```

Lemma 4.53 : $\vdash \text{static inversion}$

- If $\Gamma \vdash x : \tau$ then $(x:\tau') \in \Gamma$ and $\tau' \leqslant \tau$ 9736
 - If $\Gamma \vdash \lambda(x:\tau'_d). e' : \tau$ then $(x:\tau'_d), \Gamma \vdash e' : \tau'_c$ and $\tau'_d \Rightarrow \tau'_c \leqslant \tau$ 9737
 - If $\Gamma \vdash \langle e_0, e_1 \rangle : \tau$ then $\Gamma \vdash e_0 : \tau_0$ and $\Gamma \vdash e_1 : \tau_1$ and $\tau_0 \times \tau_1 \leqslant \tau$ 9738
 - If $\Gamma \vdash e_0 e_1 : \tau_c$ then $\Gamma \vdash e_0 : \tau'_d \Rightarrow \tau'_c$ and $\Gamma \vdash e_1 : \tau'_d$ and $\tau'_c \leqslant \tau_c$ 9739
 - If $\Gamma \vdash \text{fst } e : \tau$ then $\Gamma \vdash e : \tau_0 \times \tau_1$ and $\Delta(\text{fst}, \tau_0 \times \tau_1) = \tau_0$ and $\tau_0 \leqslant \tau$ 9740
 - If $\Gamma \vdash \text{snd } e : \tau$ then $\Gamma \vdash e : \tau_0 \times \tau_1$ and $\Delta(\text{snd}, \tau_0 \times \tau_1) = \tau_1$ and $\tau_1 \leqslant \tau$ 9741
 - If $\Gamma \vdash op^2 e_0 e_1 : \tau$ then $\Gamma \vdash e_0 : \tau_0$ and $\Gamma \vdash e_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau'$ and $\tau' \leqslant \tau$ 9742
 - If $\Gamma \vdash \text{dyn } \tau' e' : \tau$ then $\Gamma \vdash e'$ and $\tau' \leqslant \tau$ 9743
- Proof:* 9744
- QED by the definition of $\vdash e : \tau$ 9745
- \square 9746
- Lemma 4.54 : canonical forms** 9747
- If $\vdash v : \tau_0 \times \tau_1$ then $v = \langle v_0, v_1 \rangle$ 9748
 - If $\vdash v : \tau_d \Rightarrow \tau_c$ then $v = \lambda(x:\tau_x). e'$ 9749
 - If $\vdash v : \text{Int}$ then $v = i$ 9750
 - If $\vdash v : \text{Nat}$ then $v = i$ and $v \in \mathbb{N}$ 9751
- Proof:* 9752
- QED by the definition of $\vdash e : \tau$ 9753
- \square 9754
- Lemma 4.55 : substitution** 9755
- If $(x:\tau_x), \Gamma \vdash e : \tau$, and e is boundary-free and $\vdash v : \tau_x$ then $\Gamma \vdash e[x \leftarrow v] : \tau$ 9756
- Proof:* 9757
- By induction on the structure of e . 9758
- CASE** $e = x :$ 9759
1. $e[x \leftarrow v] = v$ 9760
 2. $\tau_x = \tau$ 9761
 3. $\Gamma \vdash v : \tau$ 9762
 - by *weakening* 9763
 4. QED 9764
- CASE** $e = x' :$ 9765
1. QED by $x'[x \leftarrow v] = x'$ 9766
- CASE** $e = i :$ 9767
1. QED by $i[x \leftarrow v] = i$ 9768
- CASE** $e = \lambda x. e' :$ 9769
1. Contradiction by $(x:\tau_x), \Gamma \vdash e : \tau$ 9770
- CASE** $e = \lambda(x:\tau'). e' :$ 9771
1. QED by $(\lambda(x:\tau'). e')[x \leftarrow v] = \lambda(x:\tau'). e'$ 9772
- CASE** $e = \lambda(x':\tau'). e' :$ 9773
1. Contradiction by $(x':\tau'), \Gamma \vdash e' : \tau$ 9774
- CASE** $e = \lambda(x:\tau'). e' :$ 9775
1. QED by $i[x \leftarrow v] = i$ 9776
- CASE** $e = \lambda x. e' :$ 9777
1. Contradiction by $(x:\tau_x), \Gamma \vdash e : \tau$ 9778
- CASE** $e = \lambda(x:\tau'). e' :$ 9779
1. QED by $(\lambda(x:\tau'). e')[x \leftarrow v] = \lambda(x:\tau'). e'$ 9780
- CASE** $e = \lambda(x':\tau'). e' :$ 9781
1. $e[x \leftarrow v] = \lambda(x':\tau'). (e'[x \leftarrow v])$ 9782
 2. $(x':\tau'), x, \Gamma \vdash e'$ 9783
 - by *static inversion forms* 9784
 3. $(x':\tau'), \Gamma \vdash e'[x \leftarrow v]$ 9785
 - by the induction hypothesis (2) 9786
 4. $\Gamma \vdash \lambda(x':\tau'). (e'[x \leftarrow v])$ 9787
 - by (3) 9788
 5. QED 9789

```

9791 CASE  $e = \langle e_0, e_1 \rangle :$ 
9792   1.  $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$ 
9793   2.  $x, \Gamma \vdash e_0$ 
9794      $\wedge x, \Gamma \vdash e_1$ 
9795     by static inversion forms
9796   3.  $\Gamma \vdash e_0[x \leftarrow v]$ 
9797      $\wedge \Gamma \vdash e_1[x \leftarrow v]$ 
9798     by the induction hypothesis (2)
9799   4.  $\Gamma \vdash \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$ 
9800     by (3)
9801   5. QED
9802 CASE  $e = e_0 e_1 :$ 
9803   1.  $e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]$ 
9804   2.  $x, \Gamma \vdash e_0$ 
9805      $\wedge x, \Gamma \vdash e_1$ 
9806     by static inversion forms
9807   3.  $\Gamma \vdash e_0[x \leftarrow v]$ 
9808      $\wedge \Gamma \vdash e_1[x \leftarrow v]$ 
9809     by the induction hypothesis (2)
9810   4.  $\Gamma \vdash e_0[x \leftarrow v] e_1[x \leftarrow v]$ 
9811     by (3)
9812   5. QED
9813 CASE  $e = op^1 e_0 :$ 
9814   1.  $e[x \leftarrow v] = op^1 e_0[x \leftarrow v]$ 
9815   2.  $x, \Gamma \vdash e_0$ 
9816     by static inversion forms
9817   3.  $\Gamma \vdash e_0[x \leftarrow v]$ 
9818     by the induction hypothesis (2)
9819   4.  $\Gamma \vdash op^1 e_0[x \leftarrow v]$ 
9820     by (3)
9821   5. QED
9822 CASE  $e = op^2 e_0 e_1 :$ 
9823   1.  $e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$ 
9824   2.  $x, \Gamma \vdash e_0$ 
9825      $\wedge x, \Gamma \vdash e_1$ 
9826     by static inversion forms
9827   3.  $\Gamma \vdash e_0[x \leftarrow v]$ 
9828      $\wedge \Gamma \vdash e_1[x \leftarrow v]$ 
9829     by the induction hypothesis (2)
9830   4.  $\Gamma \vdash op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]$ 
9831     by (3)
9832   5. QED
9833 CASE  $e = \text{chk } K' e' :$ 
9834   1. Contradiction by  $\vdash e : \tau$ 
9835 CASE  $e = \text{dyn } e' :$ 
9836   1. Contradiction by  $\vdash e : \tau$ 
9837 CASE  $e = \text{stat } e' :$ 
9838   1. Contradiction by  $\vdash e : \tau$ 
9839 CASE  $e = \text{dyn } \tau' e' :$ 
9840   1. Contradiction by  $e$  is boundary-free
9841 CASE  $e = \text{stat } \tau' e' :$ 
9842   1. Contradiction by  $\vdash e : \tau$ 
9843 CASE  $e = \text{Err} :$ 
9844   1. QED  $\text{Err}[x \leftarrow v] = \text{Err}$ 
9845

```

<p style="margin: 0;">□</p>	<p style="margin: 0;">9846</p>
<p>Lemma 4.56 : δ preservation</p>	<p style="margin: 0;">9847</p>
• If $\vdash v$ and $\delta(op^1, v) = v'$ then $\vdash e'$	9848
• If $\vdash v_0$ and $\vdash v_1$ and $\delta(op^2, v_0, v_1) = e'$ then $\vdash v'$	9849
<i>Proof:</i>	9850
CASE $\delta(\text{fst}, \langle v_0, v_1 \rangle) = v_0 :$	9851
1. $\vdash v_0$	9852
by <i>static inversion forms</i>	9853
2. QED	9854
CASE $\delta(\text{snd}, \langle v_0, v_1 \rangle) = v_1 :$	9855
1. $\vdash v_1$	9856
by <i>static inversion forms</i>	9857
2. QED	9858
CASE $\delta(\text{sum}, v_0, v_1) = v_0 + v_1 :$	9859
1. QED	9860
CASE $\delta(\text{quotient}, v_0, v_1) = \lfloor v_0/v_1 \rfloor :$	9861
1. QED	9862
CASE $\delta(\text{quotient}, v_0, v_1) = \text{BndryErr} :$	9863
1. QED	9864
□	9865
Lemma 4.57 : weakening	9866
• If $\Gamma \vdash e$ then $x, \Gamma \vdash e$	9867
• If $\Gamma \vdash e$ then $(x : \tau), \Gamma \vdash e$	9868
<i>Proof:</i>	9869
QED because e is closed under Γ	9870
□	9871
9872	9872
9873	9873
9874	9874
9875	9875
9876	9876
9877	9877
9878	9878
9879	9879
9880	9880
9881	9881
9882	9882
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9899	9899
9900	9900

9901 **E.5 (HC) Co-Natural Embedding**

9902 **E.5.1 Co-Natural Definitions**

9903 **Language HC**

9905 $e = x \mid v \mid \langle e, e \rangle \mid e \ e \mid op^1 \ e \mid op^2 \ e \ e \mid$
 9906 $\text{dyn } \tau \ e \mid \text{stat } \tau \ e \mid \text{Err}$
 9907 $v = i \mid \langle v, v \rangle \mid \lambda x. \ e \mid \lambda(x: \tau). \ e$
 9908 $\text{mon}(\tau \Rightarrow \tau) \ v \mid \text{mon}(\tau \times \tau) \ v$
 9909 $\tau = \text{Nat} \mid \text{Int} \mid \tau \times \tau \mid \tau \Rightarrow \tau$
 9910 $\Gamma = \cdot \mid x, \Gamma \mid (x: \tau), \Gamma$
 9911 $\text{Err} = \text{BndryErr} \mid \text{TagErr}$
 9912 $r = v \mid \text{Err}$
 9913 $E^\bullet = [] \mid E^\bullet \ e \mid v \ E^\bullet \mid \langle E^\bullet, e \rangle \mid \langle v, E^\bullet \rangle \mid$
 9914 $op^1 \ E^\bullet \mid op^2 \ E^\bullet \ e \mid op^2 \ v \ E^\bullet$
 9915 $E = E^\bullet \mid E \ e \mid v \ E \mid \langle E, e \rangle \mid \langle v, E \rangle \mid op^1 \ E \mid$
 9916 $op^2 \ E \ e \mid op^2 \ v \ E \mid \text{dyn } \tau \ E \mid \text{stat } \tau \ E$

9917 $\Delta : op^1 \times \tau \rightarrow \tau$

$$\Delta(\text{fst}, \tau_0 \times \tau_1) = \tau_0$$

$$\Delta(\text{snd}, \tau_0 \times \tau_1) = \tau_1$$

9921 $\Delta : op^2 \times \tau \times \tau \rightarrow \tau$

$$\Delta(op^2, \text{Nat}, \text{Nat}) = \text{Nat}$$

$$\Delta(op^2, \text{Int}, \text{Int}) = \text{Int}$$

9924 $\tau \leqslant \tau$

$$\frac{\tau'_d \leqslant \tau_d \quad \tau_c \leqslant \tau'_c \quad \tau_0 \leqslant \tau'_0 \quad \tau_1 \leqslant \tau'_1}{\text{Nat} \leqslant \text{Int}} \quad \frac{\tau'_d \leqslant \tau_d \leqslant \tau'_c \Rightarrow \tau'_c}{\tau_d \Rightarrow \tau_c \leqslant \tau'_d \Rightarrow \tau'_c} \quad \frac{\tau_0 \times \tau_1 \leqslant \tau'_0 \times \tau'_1}{\tau_0 \leqslant \tau' \quad \tau' \leqslant \tau''}$$

9925 $\Gamma \vdash e$

$$\frac{x \in \Gamma}{\Gamma \vdash x} \quad \frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x. \ e} \quad \frac{}{\Gamma \vdash i} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}$$

$$\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0 \ e_1} \quad \frac{\Gamma \vdash e}{\Gamma \vdash op^1 \ e} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2 \ e_0 \ e_1} \quad \frac{}{\Gamma \vdash \text{Err}}$$

$$\frac{}{\Gamma \vdash e : \tau} \quad \frac{}{\Gamma \vdash \text{stat } \tau \ e}$$

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$\boxed{\Gamma \vdash e : \tau}$	$\frac{(x: \tau) \in \Gamma}{\Gamma \vdash x : \tau}$	$\frac{(x: \tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x: \tau_d). \ e : \tau_d \Rightarrow \tau_c} \quad i \in \mathbb{N}$	9956 9957 9958 9959 9960 9961 9962 9963 9964 9965 9966 9967 9968 9969 9970 9971 9972 9973 9974 9975 9976 9977 9978 9979 9980 9981 9982 9983 9984 9985 9986 9987 9988 9989 9990 9991 9992 9993 9994 9995 9996 9997 9998 9999 10000 10001 10002 10003 10004 10005 10006 10007 10008 10009 10010
$\boxed{\Gamma \vdash i : \text{Int}}$	$\frac{}{\Gamma \vdash i : \text{Int}}$	$\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1} \quad \frac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c}{\Gamma \vdash e_0 \ e_1 : \tau_c}$	9964 9965 9966 9967 9968 9969 9970 9971 9972 9973 9974 9975 9976 9977 9978 9979 9980 9981 9982 9983 9984 9985 9986 9987 9988 9989 9990 9991 9992 9993 9994 9995 9996 9997 9998 9999 10000 10001 10002 10003 10004 10005 10006 10007 10008 10009 10010
$\boxed{\Gamma \vdash e_0 : \tau_0}$	$\frac{\Gamma \vdash e_0 : \tau_0}{\Gamma \vdash op^1 \ e_0 : \tau}$	$\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash op^2 \ e_0 \ e_1 : \tau} \quad \frac{\Gamma \vdash e : \tau'}{\Gamma \vdash e : \tau} \quad \frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{Err} : \tau}$	9965 9966 9967 9968 9969 9970 9971 9972 9973 9974 9975 9976 9977 9978 9979 9980 9981 9982 9983 9984 9985 9986 9987 9988 9989 9990 9991 9992 9993 9994 9995 9996 9997 9998 9999 10000 10001 10002 10003 10004 10005 10006 10007 10008 10009 10010
$\boxed{\Gamma \vdash e : \tau}$	$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{dyn } \tau \ e : \tau}$		9971 9972 9973 9974 9975 9976 9977 9978 9979 9980 9981 9982 9983 9984 9985 9986 9987 9988 9989 9990 9991 9992 9993 9994 9995 9996 9997 9998 9999 10000 10001 10002 10003 10004 10005 10006 10007 10008 10009 10010
$\boxed{\Gamma \vdash e : \tau}$	$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{stat } \tau \ e}$		9971 9972 9973 9974 9975 9976 9977 9978 9979 9980 9981 9982 9983 9984 9985 9986 9987 9988 9989 9990 9991 9992 9993 9994 9995 9996 9997 9998 9999 10000 10001 10002 10003 10004 10005 10006 10007 10008 10009 10010
$\boxed{\Gamma \vdash e : \tau}$	$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{mon}(\tau_0 \times \tau_1) \ v : (\tau_0 \times \tau_1)}$	$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{mon}(\tau_d \Rightarrow \tau_c) \ v : (\tau_d \Rightarrow \tau_c)}$	9981 9982 9983 9984 9985 9986 9987 9988 9989 9990 9991 9992 9993 9994 9995 9996 9997 9998 9999 10000 10001 10002 10003 10004 10005 10006 10007 10008 10009 10010
$\boxed{\Gamma \vdash e : \tau}$	$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{mon}(\tau_0 \times \tau_1) \ v : (\tau_0 \times \tau_1)}$	$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{mon}(\tau_d \Rightarrow \tau_c) \ v : (\tau_d \Rightarrow \tau_c)}$	9981 9982 9983 9984 9985 9986 9987 9988 9989 9990 9991 9992 9993 9994 9995 9996 9997 9998 9999 10000 10001 10002 10003 10004 10005 10006 10007 10008 10009 10010

10011	$\delta(op^1, v) = e$
10012	$\delta(\text{fst}, \langle v_0, v_1 \rangle) = v_0$
10013	$\delta(\text{snd}, \langle v_0, v_1 \rangle) = v_1$
10014	$\delta(op^2, v, v) = e$
10015	$\delta(\text{sum}, i_0, i_1) = i_0 + i_1$
10016	$\delta(\text{quotient}, i_0, 0) = \text{BndryErr}$
10017	$\delta(\text{quotient}, i_0, i_1) = \lfloor i_0/i_1 \rfloor$
10018	if $i_1 \neq 0$
10019	$\mathcal{D}_C : \tau \times v \longrightarrow e$
10020	$\mathcal{D}_C(\tau_d \Rightarrow \tau_c, v) = \text{mon}(\tau_d \Rightarrow \tau_c)v$
10021	if $v = \lambda x. e$ or $v = \text{mon}(\tau'_d \Rightarrow \tau'_c)v'$
10022	$\mathcal{D}_C(\tau_0 \times \tau_1, v) = \text{mon}(\tau_0 \times \tau_1)v$
10023	if $v = \langle v_0, v_1 \rangle$ or $v = \text{mon}(\tau'_0 \times \tau'_1)v'$
10024	$\mathcal{D}_C(\text{Int}, i) = i$
10025	$\mathcal{D}_C(\text{Nat}, i) = i$
10026	if $i \in \mathbb{N}$
10027	$\mathcal{D}_C(\tau, v) = \text{BndryErr}$
10028	otherwise
10029	$\mathcal{S}_C : \tau \times v \longrightarrow e$
10030	$\mathcal{S}_C(\tau_d \Rightarrow \tau_c, v) = \text{mon}(\tau_d \Rightarrow \tau_c)v$
10031	$\mathcal{S}_C(\tau_0 \times \tau_1, v) = \text{mon}(\tau_0 \times \tau_1)v$
10032	$\mathcal{S}_C(\tau, v) = v$
10033	otherwise
10034	$e \triangleright_{S-C} e$
10035	$\text{dyn } \tau v \triangleright_{S-C} \mathcal{D}_C(\tau, v)$
10036	$(\text{mon}(\tau_d \Rightarrow \tau_c)v_f)v \triangleright_{S-C} \text{dyn } \tau_c(v_f e')$
10037	where $e' = \text{stat } \tau_d v$
10038	$(\lambda(x:\tau). e)v \triangleright_{S-C} e[x \leftarrow v]$
10039	$\text{fst}(\text{mon}(\tau_0 \times \tau_1)v) \triangleright_{S-C} \text{dyn } \tau_0(\text{fst } v)$
10040	$\text{snd}(\text{mon}(\tau_0 \times \tau_1)v) \triangleright_{S-C} \text{dyn } \tau_1(\text{snd } v)$
10041	$op^1 v \triangleright_{S-C} \delta(op^1, v)$
10042	$op^2 v_0 v_1 \triangleright_{S-C} \delta(op^2, v_0, v_1)$
10043	$e \triangleright_{D-C} e$
10044	$\text{stat } \tau v \triangleright_{D-C} \mathcal{S}_C(\tau, v)$
10045	$v_0 v_1 \triangleright_{D-C} \text{TagErr}$
10046	if $v_0 \in \mathbb{Z}$ or $v_0 = \langle v, v' \rangle$
10047	$(\text{mon}(\tau_d \Rightarrow \tau_c)v_f)v \triangleright_{D-C} \text{stat } \tau_c(v_f e')$
10048	where $e' = \text{dyn } \tau_d v$
10049	$(\lambda x. e)v \triangleright_{D-C} e[x \leftarrow v]$
10050	$\text{fst}(\text{mon}(\tau_0 \times \tau_1)v) \triangleright_{D-C} \text{stat } \tau_0(\text{fst } v)$
10051	$\text{snd}(\text{mon}(\tau_0 \times \tau_1)v) \triangleright_{D-C} \text{stat } \tau_1(\text{snd } v)$
10052	$op^1 v \triangleright_{D-C} \text{TagErr}$
10053	if $\delta(op^1, v)$ is undefined
10054	$op^2 v \triangleright_{D-C} \delta(op^1, v)$
10055	$op^2 v_0 v_1 \triangleright_{D-C} \text{TagErr}$
10056	if $\delta(op^2, v_0, v_1)$ is undefined
10057	$op^2 v_0 v_1 \triangleright_{D-C} \delta(op^2, v_0, v_1)$
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10066	$e \rightarrow_{C-S} e$
10067	$E^\bullet[e] \rightarrow_{C-S} E^\bullet[e']$
10068	if $e \triangleright_{S-C} e'$
10069	$E[\text{stat } \tau E^\bullet[e]] \rightarrow_{C-S} E[\text{stat } \tau E^\bullet[e']]$
10070	if $e \triangleright_{S-C} e'$
10071	$E[\text{dyn } \tau E^\bullet[e]] \rightarrow_{C-S} E[\text{dyn } \tau E^\bullet[e']]$
10072	if $e \triangleright_{D-C} e'$
10073	$E[\text{Err}] \rightarrow_{C-S} \text{Err}$
10074	$e \rightarrow_{C-D} e$
10075	$E^\bullet[e] \rightarrow_{C-D} E^\bullet[e']$
10076	if $e \triangleright_{D-C} e'$
10077	$E[\text{stat } \tau E^\bullet[e]] \rightarrow_{C-D} E[\text{stat } \tau E^\bullet[e']]$
10078	if $e \triangleright_{S-C} e'$
10079	$E[\text{dyn } \tau E^\bullet[e]] \rightarrow_{C-D} E[\text{dyn } \tau E^\bullet[e']]$
10080	if $e \triangleright_{D-C} e'$
10081	$E[\text{Err}] \rightarrow_{C-D} \text{Err}$
10082	$e \rightarrow_{C-S}^* e$ reflexive, transitive closure of \rightarrow_{C-S}
10083	$e \rightarrow_{C-D}^* e$ reflexive, transitive closure of \rightarrow_{C-D}
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10121	E.5.2 Co-Natural Theorems	10176
10122	Theorem 5.0 : static HC soundness	10177
10123	If $\vdash e : \tau$ then $\vdash_C e : \tau$ and one of the following holds:	10178
10124	• $e \rightarrow_{C-S}^* v$ and $\vdash_C v : \tau$	10179
10125	• $e \rightarrow_{C-S}^* E[\text{dyn } \tau' e']$ and $e' \triangleright_{D-C} \text{TagErr}$	10180
10126	• $e \rightarrow_{C-S}^* \text{BndryErr}$	10181
10127	• e diverges	10182
10128	<i>Proof:</i>	10183
10129	1. $\vdash_C e : \tau$	10184
10130	by <i>static subset</i>	10185
10131	2. QED by <i>static progress</i> and <i>static preservation</i> .	10186
10132	□	10187
10133	Theorem 5.1 : dynamic HC-soundness	10188
10134	If $\vdash e$ then $\vdash_C e$ and one of the following holds:	10189
10135	• $e \rightarrow_{C-D}^* v$ and $\vdash_C v$	10190
10136	• $e \rightarrow_{C-D}^* E[e']$ and $e' \triangleright_{D-C} \text{TagErr}$	10191
10137	• $e \rightarrow_{C-D}^* \text{BndryErr}$	10192
10138	• e diverges	10193
10139	<i>Proof:</i>	10194
10140	1. $\vdash_C e$	10195
10141	by <i>dynamic subset</i>	10196
10142	2. QED by <i>dynamic progress</i> and <i>dynamic preservation</i> .	10197
10143	□	10198
10144	Corollary 5.2 : HC static soundness	10199
10145	If $\vdash e : \tau$ and e is boundary-free, then one of the following holds:	10200
10146	holds:	10201
10147	• $e \rightarrow_{C-S}^* v$ and $\vdash_C v : \tau$	10202
10148	• $e \rightarrow_{C-S}^* \text{BndryErr}$	10203
10149	• e diverges	10204
10150	<i>Proof:</i>	10205
10151	Consequence of the proof for <i>static HC-soundness</i>	10206
10152	□	10207
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10231 **E.5.3 Co-Natural Lemmas**10232 **Lemma 5.3 : \mathcal{D}_C soundness**10233 **If $\vdash_C v$ then $\vdash_C \mathcal{D}_C(\tau, v) : \tau$** 10234 *Proof:*10235 **CASE** $\mathcal{D}_C(\tau_d \Rightarrow \tau_c, v) = \text{mon}(\tau_d \Rightarrow \tau_c) v :$ 10236 1. $\vdash_C \text{mon}(\tau_d \Rightarrow \tau_c) v : \tau_d \Rightarrow \tau_c$ 10237 by $\vdash_C v$

10238 2. QED

10239 **CASE** $\mathcal{D}_C(\tau_0 \times \tau_1, v) = \text{mon}(\tau_0 \times \tau_1) v :$ 10240 1. $\vdash_C \text{mon}(\tau_0 \times \tau_1) v : \tau_0 \times \tau_1$ 10241 by $\vdash_C v$

10242 2. QED

10243 **CASE** $v = i$ 10244 $\wedge \mathcal{D}_C(\text{Int}, v) = v :$

10245 1. QED

10246 **CASE** $v \in \mathbb{N}$ 10247 $\wedge \mathcal{D}_C(\text{Nat}, v) = v :$

10248 1. QED

10249 **CASE** $\mathcal{D}_C(\tau, v) = \text{BndryErr} :$

10250 1. QED

10251 \square 10252 **Lemma 5.4 : \mathcal{S}_C soundness**10253 **If $\vdash_C v : \tau$ then $\vdash_C \mathcal{S}_C(\tau, v)$** 10254 *Proof:*10255 **CASE** $\vdash_C v : \tau_d \Rightarrow \tau_c$ 10256 $\wedge \mathcal{S}_C(\tau_d \Rightarrow \tau_c, v) = \text{mon}(\tau_d \Rightarrow \tau_c) v :$

10257 1. QED

10258 **CASE** $\vdash_C v : \tau_0 \times \tau_1$ 10259 $\wedge \mathcal{S}_C(\tau_0 \times \tau_1, v) = \text{mon}(\tau_0 \times \tau_1) v :$

10260 1. QED

10261 **CASE** $\vdash_C v : \text{Int}$ 10262 $\wedge \mathcal{S}_C(\text{Int}, v) = v :$

10263 1. QED

10264 **CASE** $\vdash_C v : \text{Nat}$ 10265 $\wedge \mathcal{S}_C(\text{Nat}, v) = v :$

10266 1. QED

10267 \square 10268 **Corollary 5.5 : HC static subset**10269 **If $\Gamma \vdash e : \tau$ then $\Gamma \vdash_C e : \tau$.**10270 *Proof:*10271 Consequence of the proof for the higher-order *static subset* lemma; both \vdash_C and \vdash_H have the same typing rules for surface-language expressions.10272 \square 10273 **Corollary 5.6 : HC dynamic subset**10274 **If $\Gamma \vdash e$ then $\Gamma \vdash_C e$.**10275 *Proof:*10276 Consequence of the proof for the higher-order *dynamic subset* lemma.10277 \square 10278 **Lemma 5.7 : HC static progress**10279 \square 10280 \square 10281 \square 10282 \square 10283 \square 10284 \square 10285 \square 10286 If $\vdash_C e : \tau$ then one of the following holds:10287 • e is a value10288 • $e \in \text{Err}$ 10289 • $e \rightarrow_{C-S} e'$ 10290 • $e \rightarrow_{C-S} \text{BndryErr}$ 10291 • $e = E[\text{dyn } \tau' e']$ and $e' \rightarrow_{C-D} \text{TagErr}$ 10292 *Proof:*10293 By the *boundary factoring* lemma, there are seven possible cases.10294 **CASE** e is a value :

10295 1. QED

10296 **CASE** $e = E^\bullet[v_0 v_1] :$ 10297 1. $\vdash_C v_0 v_1 : \tau'$ 10298 by *static hole typing*10299 2. $\vdash_C v_0 : \tau_d \Rightarrow \tau_c$ 10300 $\wedge \vdash_C v_1 : \tau_d$ 10301 by *inversion*10302 3. $v_0 = \lambda(x : \tau'_d). e'$ 10303 $\vee v_0 = \text{mon}(\tau'_d \Rightarrow \tau'_c) v_f$ 10304 by *canonical forms*10305 4. **IF** $v_0 = \lambda(x : \tau'_d). e' :$ 10306 a. $e \rightarrow_{C-S} E^\bullet[e'[x \leftarrow v_1]]$ 10307 by $v_0 v_1 \triangleright_{S-C} e'[x \leftarrow v_1]$

10308 b. QED

10309 **ELSE** $v_0 = \text{mon}(\tau'_d \Rightarrow \tau'_c) v_f :$ 10310 a. $e \rightarrow_{C-S} E^\bullet[\text{dyn } \tau'_c (v_f (\text{stat } \tau'_d v_1))]$ 10311 by $v_0 v_1 \triangleright_{S-C} \text{dyn } \tau'_c (v_f (\text{stat } \tau'_d v_1))$

10312 b. QED

10313 **CASE** $e = E^\bullet[op^1 v] :$ 10314 1. $\vdash_C op^1 v : \tau'$ 10315 by *static hole typing*10316 2. $\vdash_C v : \tau_0 \times \tau_1$ 10317 by *inversion*10318 3. $v = \langle v_0, v_1 \rangle$ 10319 $\vee v = \text{mon}(\tau_0 \times \tau_1) v'$ 10320 by *canonical forms*10321 4. **IF** $v = \langle v_0, v_1 \rangle$ 10322 $\wedge op^1 = \text{fst} :$ 10323 a. $\delta(op^1, \langle v_0, v_1 \rangle) = v_0$

10324 by definition

10325 b. $e \rightarrow_{C-S} E^\bullet[v_0]$ 10326 by $op^1 v \triangleright_{S-C} v_0$

10327 c. QED

10328 **IF** $v = \langle v_0, v_1 \rangle$ 10329 $\wedge op^1 = \text{snd} :$ 10330 a. $\delta(op^1, \langle v_0, v_1 \rangle) = v_1$

10331 by definition

10332 b. $e \rightarrow_{C-S} E^\bullet[v_1]$ 10333 by $op^1 v \triangleright_{S-C} v_1$

10334 c. QED

10335 **IF** $v = \text{mon}(\tau_0 \times \tau_1) v'$ 10336 $\wedge op^1 = \text{fst} :$ 10337 a. $e \rightarrow_{C-S} E^\bullet[\text{dyn } \tau_0 (op^1 v')]$

10338 by definition

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10341	b. QED		10396
10342	ELSE $v = \text{mon}(\tau_0 \times \tau_1) v'$: a. Contradiction by e' is boundary-free	10397 10398
10343	$\wedge op^1 = \text{snd} :$	CASE $e = E[\text{Err}] :$	10399
10344	a. $e \rightarrow_{C-S} E^\bullet[\text{dyn } \tau_1 (op^1 v')]$	1. QED $e \rightarrow_{C-S} \text{Err}$	10400
10345	by definition	□	10401
10346	b. QED		
10347	CASE $e = E^\bullet[op^2 v_0 v_1] :$	Lemma 5.8 : HC dynamic progress	10402
10348	1. $\vdash_C op^2 v_0 v_1 : \tau'$	If $\vdash_C e$ then one of the following holds:	10403
10349	by <i>static hole typing</i>	• e is a value	10404
10350	2. $\vdash_C v_0 : \tau_0$	• $e \in \text{Err}$	10405
10351	$\wedge \vdash_C v_1 : \tau_1$	• $e \rightarrow_{C-D} e'$	10406
10352	$\wedge \Delta(op^2, \tau_0, \tau_1) = \tau''$	• $e \rightarrow_{C-D} \text{BndryErr}$	10407
10353	by <i>inversion</i>	• $e \rightarrow_{C-D} \text{TagErr}$	10408
10354	3. $\delta(op^2, v_0, v_1) = e'$	Proof:	10409
10355	by <i>Δ type soundness</i> (2)	By the <i>boundary factoring</i> lemma, there are seven cases.	10410
10356	4. $op^2 v_0 v_1 \triangleright_{S-C} e'$	CASE e is a value :	10411
10357	by (3)	1. QED	10412
10358	5. QED by $e \rightarrow_{C-S} E^\bullet[e']$	CASE $e = E^\bullet[v_0 v_1] :$	10413
10359	CASE $e = E[\text{dyn } \tau' e']$ and e' is boundary-free :	IF $v_0 = \lambda x. e' :$	10414
10360	1. e' is a value	1. $e \rightarrow_{C-D} E^\bullet[e'[x \leftarrow v_1]]$	10415
10361	$\vee e' \in \text{Err}$	by $v_0 v_1 \triangleright_{D-C} e'[x \leftarrow v_1]$	10416
10362	$\vee e' \rightarrow_{C-D} e''$	2. QED	10417
10363	$\vee e' \rightarrow_{C-D} \text{BndryErr}$	IF $v_0 = \text{mon}(\tau_d \Rightarrow \tau_c) v_f :$	10418
10364	$\vee e' = E'[e'']$ and $e'' \triangleright_{D-C} \text{TagErr}$	1. $e \rightarrow_{C-D} E^\bullet[\text{stat } \tau_c (v_f (\text{dyn } \tau_d v_1))]$	10419
10365	by <i>dynamic progress</i>	by $v_0 v_1 \triangleright_{D-C} \text{stat } \tau_c (v_f (\text{dyn } \tau_d v_1))$	10420
10366	2. IF e' is a value :	2. QED	10421
10367	a. QED $e \rightarrow_{C-S} E[\mathcal{D}_C(\tau', e')]$	ELSE $v_0 = i$	10422
10368	IF $e' \in \text{Err} :$	$\vee v_0 = \langle v, v' \rangle :$	10423
10369	a. QED $e \rightarrow_{C-S} e'$	1. $e \rightarrow_{C-D} \text{TagErr}$	10424
10370	IF $e' \rightarrow_{C-D} e'' :$	by $(v_0 v_1) \triangleright_{D-C} \text{TagErr}$	10425
10371	a. QED $e \rightarrow_{C-S} E[\text{dyn } \tau' e'']$	2. QED	10426
10372	IF $e' \rightarrow_{C-D} \text{BndryErr} :$	CASE $e = E^\bullet[op^1 v] :$	10427
10373	a. QED $e \rightarrow_{C-S} E[\text{dyn } \tau' \text{BndryErr}]$	IF $v = \text{mon}(\tau_0 \times \tau_1) v'$	10428
10374	ELSE $e' = E'[e'']$ and $e'' \triangleright_{D-C} \text{TagErr} :$	$\wedge op^1 = \text{fst} :$	10429
10375	a. $E' \in E^\bullet$	1. $e \rightarrow_{C-D} E^\bullet[\text{stat } \tau_0 op^1 v']$	10430
10376	by e' is boundary-free	by $op^1 v \triangleright_{D-C} \text{stat } \tau_0 op^1 v'$	10431
10377	b. QED	2. QED	10432
10378	CASE $e = E[\text{stat } \tau' e']$ and e' is boundary-free :	IF $v = \text{mon}(\tau_0 \times \tau_1) v'$	10433
10379	1. e' is a value	$\wedge op^1 = \text{snd} :$	10434
10380	$\vee e' \in \text{Err}$	1. $e \rightarrow_{C-D} E^\bullet[\text{stat } \tau_1 op^1 v']$	10435
10381	$\vee e' \rightarrow_{C-S} e''$	by $op^1 v \triangleright_{D-C} \text{stat } \tau_1 op^1 v'$	10436
10382	$\vee e' \rightarrow_{C-S} \text{BndryErr}$	2. QED	10437
10383	$\vee e' = E''[\text{dyn } \tau'' E''''[e'']]$ and $e'' \triangleright_{D-C} \text{TagErr}$	IF $\delta(op^1, v) = e' :$	10438
10384	by <i>static progress</i>	1. $(op^1 v) \triangleright_{D-C} e'$	10439
10385	2. IF e' is a value :	2. QED	10440
10386	a. QED $e \rightarrow_{C-S} E[\mathcal{S}_C(\tau', e')]$	ELSE $\delta(op^1, v)$ is undefined :	10441
10387	IF $e' \in \text{Err} :$	1. $e \rightarrow_{C-D} \text{TagErr}$	10442
10388	a. QED $e \rightarrow_{C-S} e'$	by $(op^1 v) \triangleright_{D-C} \text{TagErr}$	10443
10389	IF $e' \rightarrow_{C-S} e'' :$	2. QED	10444
10390	a. QED $e \rightarrow_{C-S} E[\text{stat } \tau' e'']$	CASE $e = E^\bullet[op^2 v_0 v_1] :$	10445
10391	IF $e' \rightarrow_{C-S} \text{BndryErr} :$	IF $\delta(op^2, v_0, v_1) = e'' :$	10446
10392	a. QED $e \rightarrow_{C-S} E[\text{stat } \tau' \text{BndryErr}]$	1. $op^2 v_0 v_1 \triangleright_{D-C} e''$	10447
10393		2. QED	10448
10394		ELSE $\delta(op^2, v_0, v_1)$ is undefined :	10449
10395			10450

10451 1. $e \rightarrow_{C-D} \text{TagErr}$
 10452 by $op^2 v_0 v_1 \triangleright_{D-C} \text{TagErr}$
 10453 2. QED
CASE $e = E[\text{dyn } \tau' e']$ and e' is boundary-free :
 10455 1. e' is a value
 10456 $\vee e' \in \text{Err}$
 10457 $\vee e' \rightarrow_{C-D} e''$
 10458 $\vee e' \rightarrow_{C-D} \text{BndryErr}$
 10459 $\vee e' = E[e'']$ and $e'' \triangleright_{D-C} \text{TagErr}$
 10460 by *dynamic progress*
 10461 2. **IF** e' is a value :
 10462 a. QED $e \rightarrow_{C-D} E[\mathcal{D}_C(\tau', e')]$
 10463 **IF** $e' \in \text{Err}$:
 10464 a. QED $e \rightarrow_{C-D} e'$
 10465 **IF** $e' \rightarrow_{C-D} e''$:
 10466 a. QED $e \rightarrow_{C-S} E[\text{dyn } \tau' e'']$
 10467 **IF** $e' \rightarrow_{C-D} \text{BndryErr}$:
 10468 a. QED $e \rightarrow_{C-D} E[\text{dyn } \tau' \text{BndryErr}]$
 10469 **ELSE** $e' = E[e'']$ and $e'' \triangleright_{D-C} \text{TagErr}$:
 10470 a. $E \in E^*$
 10471 by e' is boundary-free
 10472 b. QED
CASE $e = E[\text{stat } \tau' e']$ and e' is boundary-free :
 10473 1. e' is a value
 10474 $\vee e' \in \text{Err}$
 10475 $\vee e' \rightarrow_{C-S} e''$
 10476 $\vee e' \rightarrow_{C-S} \text{BndryErr}$
 10477 $\vee e' = E''[\text{dyn } \tau'' E^{*''}[e'']]$ and $e'' \triangleright_{D-C} \text{TagErr}$
 10478 by *static progress*
 10479 2. **IF** e' is a value :
 10480 a. QED $e \rightarrow_{C-S} E[\mathcal{S}_C(\tau', e')]$
 10481 **IF** $e' \in \text{Err}$:
 10482 a. QED $e \rightarrow_{C-S} e'$
 10483 **IF** $e' \rightarrow_{C-S} e''$:
 10484 a. QED $e \rightarrow_{C-S} E[\text{stat } \tau' e'']$
 10485 **IF** $e' \rightarrow_{C-S} \text{BndryErr}$:
 10486 a. QED $e \rightarrow_{C-S} E[\text{stat } \tau' \text{BndryErr}]$
 10487 **ELSE** $e' = E''[\text{dyn } \tau'' E^{*''}[e'']]$ and $e'' \triangleright_{D-C} \text{TagErr}$
 10488 :
 10489 a. Contradiction by e' is boundary-free
CASE $e = E[\text{Err}]$:
 10490 1. QED $e \rightarrow_{C-D} \text{Err}$
 10491 \square
Lemma 5.9 : HC static preservation
 10492 If $\vdash_C e : \tau$ and $e \rightarrow_{C-S} e'$ then $\vdash_C e' : \tau$
 10493 *Proof:*
 10494 By the *boundary factoring* lemma there are seven cases.
CASE e is a value :
 10495 1. Contradiction by $e \rightarrow_{C-S} e'$
CASE $e = E^\bullet[v_0 v_1]$:
 10501 **IF** $v_0 = \lambda(x:\tau_x). e'$
 10502 $\wedge e \rightarrow_{C-S} E^\bullet[e'[x \leftarrow v_1]]$:
 10503
 10504
 10505

1. $\vdash_C v_0 v_1 : \tau'$
 by *static hole typing*
 10506
 2. $\vdash_C v_0 : \tau_d \Rightarrow \tau_c$
 $\wedge \vdash_C v_1 : \tau_d$
 $\wedge \tau_c \leqslant \tau'$
 by *inversion*
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CASE $e = E^\bullet[op^1 v]$:
IF $v = \text{mon}(\tau_0 \times \tau_1) v'$
 $\wedge op^1 = \text{fst}$
 $\wedge e \rightarrow_{C-S} E^\bullet[\text{dyn } \tau_0 (\text{fst } v')]$:
 1. $\vdash_C \text{fst } v : \tau'$
 by *static hole typing*
 10553
 2. $\vdash_C v : \tau'_0 \times \tau'_1$
 $\wedge \tau'_0 <: \tau'$
 by *inversion*
 10554
 3. $\vdash_C v'$
 by *inversion* (2)
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10561	4. $\tau_0 \times \tau_1 \leqslant: \tau'_0 \times \tau'_1$	10616
10562	by <i>canonical forms</i> (2)	10617
10563	5. $\tau_0 \leqslant: \tau'_0$	10618
10564	6. $\vdash_C \text{fst } v'$	10619
10565	by (3)	10620
10566	7. $\vdash_C \text{dyn } \tau_0 (\text{fst } v') : \tau_0$	10621
10567	by (6)	10622
10568	8. $\vdash_C \text{dyn } \tau_0 (\text{fst } v') : \tau'$	10623
10569	by (2, 5, 7)	10624
10570	9. QED by <i>hole substitution</i>	10625
10571	IF $v = \text{mon}(\tau_0 \times \tau_1) \langle v_0, v_1 \rangle$	10626
10572	$\wedge op^1 = \text{snd}$	10627
10573	$\wedge e \rightarrow_{C-S} E^\bullet[\text{dyn } \tau_0 (\text{snd } v')] :$	10628
10574	1. $\vdash_C \text{snd } v : \tau'$	10629
10575	by <i>static hole typing</i>	10630
10576	2. $\vdash_C v : \tau'_0 \times \tau'_1$	10631
10577	$\wedge \tau'_1 <: \tau'$	10632
10578	by <i>inversion</i>	10633
10579	3. $\vdash_C v'$	10634
10580	by <i>inversion</i> (2)	10635
10581	4. $\tau_0 \times \tau_1 \leqslant: \tau'_0 \times \tau'_1$	10636
10582	by <i>canonical forms</i> (2)	10637
10583	5. $\tau_1 \leqslant: \tau'_1$	10638
10584	6. $\vdash_C \text{snd } v'$	10639
10585	by (3)	10640
10586	7. $\vdash_C \text{dyn } \tau_0 (\text{snd } v') : \tau_0$	10641
10587	by (5)	10642
10588	8. $\vdash_C \text{dyn } \tau_0 (\text{snd } v') : \tau'$	10643
10589	by (2, 5, 7)	10644
10590	9. QED by <i>hole substitution</i>	10645
10591	IF $v = \langle v_0, v_1 \rangle$	10646
10592	$\wedge op^1 = \text{fst}$	10647
10593	$\wedge e \rightarrow_{C-S} E^\bullet[v_0] :$	10648
10594	1. $\vdash_C \text{fst } \langle v_0, v_1 \rangle : \tau'$	10649
10595	by <i>static hole typing</i>	10650
10596	2. $\vdash_C \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$	10651
10597	$\wedge \tau_0 \leqslant: \tau'$	10652
10598	by <i>inversion</i> (1)	10653
10599	3. $\vdash_C v_0 : \tau_0$	10654
10600	by <i>inversion</i> (2)	10655
10601	4. $\vdash_C v_0 : \tau'$	10656
10602	by (2, 3)	10657
10603	5. QED by <i>hole substitution</i> (4)	10658
10604	ELSE $v = \langle v_0, v_1 \rangle$	10659
10605	$\wedge op^1 = \text{snd}$	10660
10606	$\wedge e \rightarrow_{C-S} E^\bullet[v_1] :$	10661
10607	1. $\vdash_C \text{snd } \langle v_0, v_1 \rangle : \tau'$	10662
10608	by <i>static hole typing</i>	10663
10609	2. $\vdash_C \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$	10664
10610	$\wedge \tau_1 \leqslant: \tau'$	10665
10611	by <i>inversion</i> (1)	10666
10612	3. $\vdash_C v_1 : \tau_1$	10667
10613	by <i>inversion</i> (2)	10668
10614		10669
10615		10670

10671 3. $\vdash_C e' : \tau'$
 10672 by *inversion* (2)
 10673 4. $\vdash_C e'' : \tau'$
 10674 by *static preservation* (3)
 10675 5. $\vdash_C \text{stat } \tau' e''$
 10676 by (4)
 10677 6. QED by *hole substitution* (5)
 10678 **CASE** $e = E[\text{Err}] :$
 10679 1. $e \rightarrow_{C-S} \text{Err}$
 10680 2. QED by $\vdash_C \text{Err} : \tau$
 10681 □

Lemma 5.10 : HC dynamic preservation

If $\vdash_C e$ and $e \rightarrow_{C-D} e'$ then $\vdash_C e'$

Proof:

By the *boundary factoring* lemma, there are seven cases.

CASE e is a value :

1. Contradiction by $e \rightarrow_{C-D} e'$

CASE $e = E^\bullet[v_0 v_1] :$

IF $v_0 = \lambda x. e'$

$\wedge e \rightarrow_{C-D} E^\bullet[e'[x \leftarrow v_1]] :$

1. $\vdash_C v_0 v_1$

by *dynamic hole typing*

2. $\vdash_C v_0$

$\wedge \vdash_C v_1$

by *inversion* (1)

3. $x \vdash_C e'$

by *inversion* (2)

4. $\vdash_C e'[x \leftarrow v_1]$

by *substitution* (2, 3)

5. QED *hole substitution* (4)

ELSE $v_0 = \text{mon}(\tau_d \Rightarrow \tau_c) v_f$

$\wedge e \rightarrow_{C-D} E^\bullet[\text{stat } \tau_c (v_f (\text{dyn } \tau_d v_1))] :$

1. $\vdash_C v_0 v_1$

by *dynamic hole typing*

2. $\vdash_C v_0$

$\wedge \vdash_C v_1$

by *inversion* (1)

3. $\vdash_C v_f : \tau_d \Rightarrow \tau_c$

by *inversion* (2)

4. $\vdash_C \text{dyn } \tau_d v_1 : \tau_d$

by (2)

5. $\vdash_C v_f (\text{dyn } \tau_d v_1) : \tau_c$

by (3, 4)

6. $\vdash_C \text{stat } \tau_c (v_f (\text{dyn } \tau_d v_1))$

by (5)

7. QED by *hole substitution*

CASE $e = E^\bullet[op^1 v] :$

IF $v = \text{mon}(\tau_0 \times \tau_1) v'$

$\wedge op^1 = \text{fst}$

$\wedge e \rightarrow_{C-D} E[\text{stat } \tau_0 (\text{fst } v')] :$

1. $\vdash_C op^1 v$

by *dynamic hole typing*

2. $\vdash_C \text{mon}(\tau_0 \times \tau_1) v'$
 10726 by *inversion* (1)
 10727
 3. $\vdash_C v' : \tau_0 \times \tau_1$
 10728 by *inversion* (2)
 10729
 4. $\vdash_C \text{fst } v' : \tau_0$
 10730 by (3)
 10731
 5. $\vdash_C \text{stat } \tau_0 (\text{fst } v')$
 10732 by (4)
 10733
 6. QED by *hole substitution*
 10734
IF $v = \text{mon}(\tau_0 \times \tau_1) v'$
 10735 $\wedge op^1 = \text{snd}$
 10736 $\wedge e \rightarrow_{C-D} E[\text{stat } \tau_0 (\text{snd } v')] :$
 10737
 1. $\vdash_C op^1 v$
 10738 by *dynamic hole typing*
 10739
 2. $\vdash_C \text{mon}(\tau_0 \times \tau_1) v'$
 10740 by *inversion* (1)
 10741
 3. $\vdash_C v' : \tau_0 \times \tau_1$
 10742 by *inversion* (2)
 10743
 4. $\vdash_C \text{snd } v' : \tau_1$
 10744 by (3)
 10745
 5. $\vdash_C \text{stat } \tau_1 (\text{snd } v')$
 10746 by (4)
 10747
 6. QED by *hole substitution*
 10748
IF $v = \langle v_0, v_1 \rangle$
 10749 $\wedge op^1 = \text{fst}$
 10750 $\wedge e \rightarrow_{C-D} E^\bullet[v_0] :$
 10751
 1. $\vdash_C op^1 v$
 10752 by *dynamic hole typing*
 10753
 2. $\vdash_C v$
 10754 by *inversion* (1)
 10755
 3. $\vdash_C v_0$
 10756 by *inversion* (2)
 10757
 4. QED by *hole substitution*
 10758
ELSE $v = \langle v_0, v_1 \rangle$
 10759 $\wedge op^1 = \text{snd}$
 10760 $\wedge e \rightarrow_{C-D} E^\bullet[v_1] :$
 10761
 1. $\vdash_C op^1 v$
 10762 by *dynamic hole typing*
 10763
 2. $\vdash_C v$
 10764 by *inversion* (1)
 10765
 3. $\vdash_C v_1$
 10766 by *inversion* (2)
 10767
 4. QED by *hole substitution*
 10768
CASE $e = E^\bullet[op^2 v_0 v_1] :$
 10769
 1. $e \rightarrow_{C-D} E^\bullet[\delta(op^2, v_0, v_1)]$
 10770
 2. $\vdash_C op^2 v_0 v_1$
 10771 by *dynamic hole typing*
 10772
 3. $\vdash_C v_0$
 10773 $\wedge \vdash_C v_1$
 10774 by *inversion* (1)
 10775
 4. $\vdash_C \delta(op^2, v_0, v_1)$
 10776 by *δ preservation* (2)
 10777
 5. QED by *hole substitution* (3)
 10778
CASE $e = E[\text{dyn } \tau' e']$ and e' is boundary-free :
 10779
 10780

```

10781 IF  $e'$  is a value :
10782   1.  $e \rightarrow_{C-D} E[\mathcal{D}_C(\tau', e')]$ 
10783   2.  $\vdash_C \text{dyn } \tau' e' : \tau'$ 
10784     by boundary hole typing
10785   3.  $\vdash_C e'$ 
10786     by inversion (2)
10787   4.  $\vdash_C \mathcal{D}_C(\tau', e') : \tau'$ 
10788     by  $\mathcal{D}_C$  soundness (3)
10789   5. QED by hole substitution (4)
10790 ELSE  $e' \rightarrow_{C-D} e'' :$ 
10791   1.  $e \rightarrow_{C-D} E[\text{dyn } \tau' e'']$ 
10792   2.  $\vdash_C \text{dyn } \tau' e' : \tau'$ 
10793     by boundary hole typing
10794   3.  $\vdash_C e'$ 
10795      $\wedge \tau' \leqslant \tau''$ 
10796     by inversion (2)
10797   4.  $\vdash_C e''$ 
10798     by dynamic preservation (3)
10799   5.  $\vdash_C \text{dyn } \tau' e'' : \tau'$ 
10800     by (4)
10801   6. QED by hole substitution (5)
10802 CASE  $e = E[\text{stat } \tau' e']$  and  $e'$  is boundary-free :
10803   IF  $e' \in v :$ 
10804     1.  $e \rightarrow_{C-D} E[\mathcal{S}_C(\tau', e')]$ 
10805     2.  $\vdash_C \text{stat } \tau' e'$ 
10806       by boundary hole typing
10807     3.  $\vdash_C e' : \tau'$ 
10808       by inversion (2)
10809     4.  $\vdash_C \mathcal{S}_C(\tau', e')$ 
10810       by  $\mathcal{S}_C$  soundness (3)
10811     5. QED by hole substitution (5)
10812 ELSE  $e' \rightarrow_{C-S} e'' :$ 
10813   1.  $e \rightarrow_{C-D} E[\text{stat } \tau' e'']$ 
10814   2.  $\vdash_C \text{stat } \tau' e'$ 
10815     by boundary hole typing
10816   3.  $\vdash_C e' : \tau'$ 
10817     by inversion (2)
10818   4.  $\vdash_C e'' : \tau'$ 
10819     by static preservation (3)
10820   5.  $\vdash_C \text{stat } \tau' e''$ 
10821     by (4)
10822   6. QED by hole substitution (5)
10823 CASE  $e = E[\text{Err}] :$ 
10824   1.  $e \rightarrow_{C-D} \text{Err}$ 
10825   2. QED  $\vdash_C \text{Err}$ 
10826  $\square$ 

```

Lemma 5.11 : HC static boundary factoring

- If $\vdash_C e : \tau$ then one of the following holds:
- e is a value
 - $e = E^\bullet[v_0 v_1]$
 - $e = E^\bullet[op^1 v]$
 - $e = E^\bullet[op^2 v_0 v_1]$
 - $e = E[\text{dyn } \tau e']$ where e' is boundary-free
 - $e = E[\text{stat } \tau e']$ where e' is boundary-free
 - $e = E[\text{Err}]$
- Proof:*
- By the *boundary factoring* lemma for the higher-order embedding. (The only difference is the meaning of e is a value.)
- \square
- Lemma 5.12 : HC dynamic boundary factoring**
- If $\vdash_C e$ then one of the following holds:
- e is a value
 - $e = E^\bullet[v_0 v_1]$
 - $e = E^\bullet[op^1 v]$
 - $e = E^\bullet[op^2 v_0 v_1]$
 - $e = E[\text{dyn } \tau e']$ where e' is boundary-free
 - $e = E[\text{stat } \tau e']$ where e' is boundary-free
 - $e = E[\text{Err}]$
- Proof:*
- By the *boundary factoring* lemma for the higher-order embedding.
- \square
- Lemma 5.13 : HC static hole typing**
- If $\vdash_C E^\bullet[e] : \tau$ then the derivation contains a sub-term $\vdash_C e : \tau'$
- Proof (sketch):* Similar to the *static hole typing* lemma for the higher-order embedding. \square
- Lemma 5.14 : HC dynamic hole typing**
- If $\vdash_C E^\bullet[e]$ then the derivation contains a sub-term $\vdash_C e$
- Proof (sketch):* Similar to the *static hole typing* lemma for the higher-order embedding. \square
- Lemma 5.15 : HC boundary hole typing**
- If $\vdash_C E[\text{dyn } \tau e] : \tau'$ then the derivation contains a sub-term $\vdash_C \text{dyn } \tau e : \tau$
 - If $\vdash_C E[\text{dyn } \tau e]$ then the derivation contains a sub-term $\vdash_C \text{dyn } \tau e : \tau$
 - If $\vdash_C E[\text{stat } \tau e] : \tau'$ then the derivation contains a sub-term $\vdash_C \text{stat } \tau e$
 - If $\vdash_C E[\text{stat } \tau e]$ then the derivation contains a sub-term $\vdash_C \text{stat } \tau e$
- Proof (sketch):* Similar to the proof for the higher-order *boundary hole typing* lemma. \square
- Lemma 5.16 : HC hole substitution**
- If $\vdash_C E[e]$ and the derivation contains a sub-term $\vdash_C e : \tau'$ and $\vdash_C e' : \tau'$ then $\vdash_C E[e']$.
 - If $\vdash_C E[e]$ and the derivation contains a sub-term $\vdash_C e$ and $\vdash_C e'$ then $\vdash_C E[e']$.
 - If $\vdash_C E[e] : \tau$ and the derivation contains a sub-term $\vdash_C e : \tau'$ and $\vdash_C e' : \tau'$ then $\vdash_C E[e'] : \tau$.
 - If $\vdash_C E[e] : \tau$ and the derivation contains a sub-term $\vdash_C e$ and $\vdash_C e'$ then $\vdash_C E[e'] : \tau$.

10891 *Proof (sketch):* Similar to the proof of the higher-order *hole*
 10892 *substitution* lemma, just replacing \vdash_H with \vdash_C . \square

10893 **Lemma 5.17 : \vdash_C static inversion**

- If $\Gamma \vdash_C x : \tau$ then $(x : \tau') \in \Gamma$ and $\tau' \leqslant \tau$
- If $\Gamma \vdash_C \lambda(x : \tau_d). e' : \tau$ then $(x : \tau'_d), \Gamma \vdash_C e' : \tau'_c$ and $\tau'_d \Rightarrow \tau'_c \leqslant \tau$
- If $\Gamma \vdash_C \langle e_0, e_1 \rangle : \tau_0 \times \tau_1$ then $\Gamma \vdash_C e_0 : \tau'_0$ and $\Gamma \vdash_C e_1 : \tau'_1$ and $\tau'_0 \leqslant \tau_0$ and $\tau'_1 \leqslant \tau_1$
- If $\Gamma \vdash_C e_0 e_1 : \tau_c$ then $\Gamma \vdash_C e_0 : \tau'_d \Rightarrow \tau'_c$ and $\Gamma \vdash_C e_1 : \tau'_d$ and $\tau'_c \leqslant \tau_c$
- If $\Gamma \vdash_C \text{fst } e : \tau$ then $\Gamma \vdash_C e : \tau_0 \times \tau_1$ and $\Delta(\text{fst}, \tau_0 \times \tau_1) = \tau_0$ and $\tau_0 \leqslant \tau$
- If $\Gamma \vdash_C \text{snd } e : \tau$ then $\Gamma \vdash_C e : \tau_0 \times \tau_1$ and $\Delta(\text{snd}, \tau_0 \times \tau_1) = \tau_1$ and $\tau_1 \leqslant \tau$
- If $\Gamma \vdash_C op^2 e_0 e_1 : \tau$ then $\Gamma \vdash_C e_0 : \tau_0$ and $\Gamma \vdash_C e_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau'$ and $\tau' \leqslant \tau$
- If $\Gamma \vdash_C \text{mon } \tau'_0 \times \tau'_1 v' : \tau_0 \times \tau_1$ then $\Gamma \vdash_C v'$ and $\tau'_0 \times \tau'_1 \leqslant \tau_0 \times \tau_1$
- If $\Gamma \vdash_C \text{mon } \tau'_d \Rightarrow \tau'_c v' : \tau_d \Rightarrow \tau_c$ then $\Gamma \vdash_C v'$ and $\tau'_d \Rightarrow \tau'_c \leqslant \tau_d \Rightarrow \tau_c$
- If $\Gamma \vdash_C \text{dyn } \tau' e' : \tau$ then $\Gamma \vdash_C e'$ and $\tau' \leqslant \tau$

10911 *Proof:*

10912 QED by the definition of $\Gamma \vdash_C e : \tau$ \square

10914 **Lemma 5.18 : \vdash_C dynamic inversion**

- If $\Gamma \vdash_C x$ then $x \in \Gamma$
- If $\Gamma \vdash_C \lambda x. e'$ then $x, \Gamma \vdash_C e'$
- If $\Gamma \vdash_C \langle e_0, e_1 \rangle$ then $\Gamma \vdash_C e_0$ and $\Gamma \vdash_C e_1$
- If $\Gamma \vdash_C e_0 e_1$ then $\Gamma \vdash_C e_0$ and $\Gamma \vdash_C e_1$
- If $\Gamma \vdash_C op^1 e_0$ then $\Gamma \vdash_C e_0$
- If $\Gamma \vdash_C op^2 e_0 e_1$ then $\Gamma \vdash_C e_0$ and $\Gamma \vdash_C e_1$
- If $\Gamma \vdash_C \text{mon } \tau_d \Rightarrow \tau_c v'$ then $\Gamma \vdash_C v' : \tau_d \Rightarrow \tau_c$
- If $\Gamma \vdash_C \text{mon } \tau_0 \times \tau_1 v'$ then $\Gamma \vdash_C v' : \tau_0 \times \tau_1$
- If $\Gamma \vdash_C \text{stat } \tau' e'$ then $\Gamma \vdash_C e' : \tau'$

10924 *Proof:*

10925 QED by the definition of $\Gamma \vdash_C e$ \square

10927 **Lemma 5.19 : HC canonical forms**

- If $\vdash_C v : \tau_0 \times \tau_1$ then either:
 - $v = \langle v_0, v_1 \rangle$
 - or $v = \text{mon}(\tau'_0 \times \tau'_1) v'$
 $\wedge \tau'_0 \times \tau'_1 \leqslant \tau_0 \times \tau_1$
- If $\vdash_C v : \tau_d \Rightarrow \tau_c$ then either:
 - $v = \lambda(x : \tau_x). e'$
 $\wedge \tau_d \leqslant \tau_x$
 - or $v = \text{mon}(\tau'_d \Rightarrow \tau'_c) v'$
 $\wedge \tau'_d \Rightarrow \tau'_c \leqslant \tau_d \Rightarrow \tau_c$
- If $\vdash_C v : \text{Int}$ then $v = i$
- If $\vdash_C v : \text{Nat}$ then $v = i$ and $v \in \mathbb{N}$

10939 *Proof:*

10940 QED by definition of $\vdash_C e : \tau$ \square

10942 **Lemma 5.20 : Δ type soundness**

If $\vdash_C v_0 : \tau_0$ and $\vdash_C v_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau$ then one of the following holds:

- $\delta(op^2, v_0, v_1) = v$ and $\vdash_C v : \tau$, or
- $\delta(op^2, v_0, v_1) = \text{BndryErr}$

Proof (sketch): Similar to the proof for the higher-order *type soundness* lemma. \square

10946 **Lemma 5.21 : δ preservation**

- If $\vdash_C v$ and $\delta(op^1, v) = v'$ then $\vdash_C v'$
- If $\vdash_C v_0$ and $\vdash_C v_1$ and $\delta(op^2, v_0, v_1) = v'$ then $\vdash_C v'$

Proof (sketch): Similar to the proof for the higher-order *preservation* lemma. \square

10947 **Lemma 5.22 : HC substitution**

- If $(x : \tau_x), \Gamma \vdash_C e$ and $\vdash_C v : \tau_x$ then $\Gamma \vdash_C e[x \leftarrow v]$
- If $x, \Gamma \vdash_C e$ and $\vdash_C v$ then $\Gamma \vdash_C e[x \leftarrow v]$
- If $(x : \tau_x), \Gamma \vdash_C e : \tau$ and $\vdash_C v : \tau_x$ then $\Gamma \vdash_C e[x \leftarrow v] : \tau$
- If $x, \Gamma \vdash_C e : \tau$ and $\vdash_C v$ then $\Gamma \vdash_C e[x \leftarrow v] : \tau$

Proof (sketch): Similar to the proof for the higher-order *substitution* lemma. \square

10948 **Lemma 5.23 : weakening**

- If $\Gamma \vdash_C e$ then $x, \Gamma \vdash_C e$
- If $\Gamma \vdash_C e : \tau$ then $(x : \tau'), \Gamma \vdash_C e : \tau$

Proof:

QED because e is closed under Γ \square

11001 **E.6 (HF) Forgetful Embedding**

11002 **E.6.1 Forgetful Definitions**

11003 **Language HF**

11005 $e = x \mid v \mid \langle e, e \rangle \mid e \cdot e \mid op^1 e \mid op^2 e \cdot e \mid$
 11006 $\text{dyn } \tau \cdot e \mid \text{stat } \tau \cdot e \mid \text{Err} \mid \text{chk } \tau \cdot e$
 11007 $v = i \mid \langle v, v \rangle \mid \lambda x. e \mid \lambda(x: \tau). e \mid$
 11008 $\text{mon } (\tau \Rightarrow \tau) (\lambda x. e) \mid \text{mon } (\tau \Rightarrow \tau) (\lambda(x: \tau). e) \mid$
 11009 $\text{mon } (\tau \times \tau) \langle v, v \rangle$
 11010 $\tau = \text{Nat} \mid \text{Int} \mid \tau \times \tau \mid \tau \Rightarrow \tau$
 11011 $\Gamma = \cdot \mid x, \Gamma \mid (x: \tau), \Gamma$
 11012 $\text{Err} = \text{BndryErr} \mid \text{TagErr}$
 11013 $r = v \mid \text{Err}$
 11014 $E^\bullet = [] \mid E^\bullet \cdot e \mid v \cdot E^\bullet \mid \langle E^\bullet, e \rangle \mid \langle v, E^\bullet \rangle \mid$
 11015 $op^1 E^\bullet \mid op^2 E^\bullet \cdot e \mid op^2 v \cdot E^\bullet$
 11016 $E = E^\bullet \mid E \cdot e \mid v \cdot E \mid \langle E, e \rangle \mid \langle v, E \rangle \mid op^1 E \mid$
 11017 $op^2 E \cdot e \mid op^2 v \cdot E \mid \text{dyn } \tau \cdot E \mid \text{stat } \tau \cdot E$

11018 **$\Delta : op^1 \times \tau \rightarrow \tau$**

11019 $\Delta(\text{fst}, \tau_0 \times \tau_1) = \tau_0$
 11020 $\Delta(\text{snd}, \tau_0 \times \tau_1) = \tau_1$

11022 **$\Delta : op^2 \times \tau \times \tau \rightarrow \tau$**

11023 $\Delta(op^2, \text{Nat}, \text{Nat}) = \text{Nat}$
 11024 $\Delta(op^2, \text{Int}, \text{Int}) = \text{Int}$

11025 $\tau \leqslant: \tau$

11027 $\frac{}{\text{Nat} \leqslant: \text{Int}} \quad \frac{\tau'_d \leqslant: \tau_d \quad \tau_c \leqslant: \tau'_c}{\tau_d \Rightarrow \tau_c \leqslant: \tau'_d \Rightarrow \tau'_c} \quad \frac{\tau_0 \leqslant: \tau'_0 \quad \tau_1 \leqslant: \tau'_1}{\tau_0 \times \tau_1 \leqslant: \tau'_0 \times \tau'_1}$

11030 $\frac{}{\tau \leqslant: \tau} \quad \frac{\tau \leqslant: \tau' \quad \tau' \leqslant: \tau''}{\tau \leqslant: \tau''}$

11033 **$\Gamma \vdash e$**

11034 $\frac{x \in \Gamma}{\Gamma \vdash x} \quad \frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x. e} \quad \frac{}{\Gamma \vdash i} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}$

11037 $\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0 \cdot e_1} \quad \frac{\Gamma \vdash e}{\Gamma \vdash op^1 e} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2 e_0 \cdot e_1} \quad \frac{}{\Gamma \vdash \text{Err}}$

11040 $\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{stat } \tau \cdot e}$

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$\boxed{\Gamma \vdash e : \tau}$

11056 $\frac{(x: \tau) \in \Gamma}{\Gamma \vdash x : \tau} \quad \frac{(x: \tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x: \tau_d). e : \tau_d \Rightarrow \tau_c} \quad \frac{i \in \mathbb{N}}{\Gamma \vdash i : \text{Nat}}$

11057 $\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1} \quad \frac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c}{\Gamma \vdash e_0 \cdot e_1 : \tau_c}$

11058 $\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash e_0 \cdot e_1 : \tau_d \Rightarrow \tau_c} \quad \frac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c}{\Gamma \vdash e_1 : \tau_d}$

11059 $\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash op^1 e_0 \cdot e_1 : \tau} \quad \frac{\Delta(op^2, \tau_0, \tau_1) = \tau \quad \tau' \leqslant: \tau}{\Gamma \vdash op^2 e_0 \cdot e_1 : \tau} \quad \frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{Err} : \tau}$

11060 $\frac{\Gamma \vdash e}{\Gamma \vdash \text{dyn } \tau \cdot e : \tau}$

11061 $\frac{x \in \Gamma}{\Gamma \vdash x} \quad \frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x. e} \quad \frac{}{\Gamma \vdash i} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}$

11062 $\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0 \cdot e_1} \quad \frac{\Gamma \vdash e}{\Gamma \vdash op^1 e} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2 e_0 \cdot e_1} \quad \frac{}{\Gamma \vdash \text{Err}}$

11063 $\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{stat } \tau \cdot e} \quad \frac{\Gamma \vdash v_0 : \tau'_0 \quad \Gamma \vdash v_1 : \tau'_1}{\Gamma \vdash \text{mon } (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle}$

11064 $\frac{\Gamma \vdash v_0 \quad \Gamma \vdash v_1}{\Gamma \vdash \text{mon } (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle} \quad \frac{\Gamma \vdash \lambda x. e}{\Gamma \vdash \text{mon } (\tau_d \Rightarrow \tau_c) \lambda x. e}$

11065 $\frac{\Gamma \vdash \lambda(x: \tau'_d). e : \tau'_d \Rightarrow \tau'_c}{\Gamma \vdash \text{mon } (\tau_d \Rightarrow \tau_c) \lambda(x: \tau'_d). e}$

11111	$\boxed{\Gamma \vdash_{\mathbb{F}} e : \tau}$
11112	$\frac{(x:\tau) \in \Gamma}{\Gamma \vdash_{\mathbb{F}} x : \tau}$
11113	$\frac{(x:\tau_d), \Gamma \vdash_{\mathbb{F}} e : \tau_c}{\Gamma \vdash_{\mathbb{F}} \lambda(x:\tau_d). e : \tau_d \Rightarrow \tau_c}$
11114	$\frac{i \in \mathbb{N}}{\Gamma \vdash_{\mathbb{F}} i : \text{Nat}}$
11115	
11116	$\frac{\Gamma \vdash_{\mathbb{F}} e_0 : \tau_0}{\Gamma \vdash_{\mathbb{F}} e_1 : \tau_1}$
11117	$\frac{\Gamma \vdash_{\mathbb{F}} e_0 : \tau_d \Rightarrow \tau_c}{\Gamma \vdash_{\mathbb{F}} e_1 : \tau_d}$
11118	$\frac{}{\Gamma \vdash_{\mathbb{F}} i : \text{Int}}$
11119	$\frac{}{\Gamma \vdash_{\mathbb{F}} \langle e_0, e_1 \rangle : \tau_0 \times \tau_1}$
11120	
11121	$\frac{\Gamma \vdash_{\mathbb{F}} e_0 : \tau_0}{\Delta(op^1, \tau_0) = \tau}$
11122	$\frac{\Gamma \vdash_{\mathbb{F}} e_1 : \tau_1}{\Delta(op^2, \tau_0, \tau_1) = \tau}$
11123	$\frac{\Gamma \vdash_{\mathbb{F}} e : \tau'}{\tau' <: \tau}$
11124	$\frac{\Gamma \vdash_{\mathbb{F}} op^1 e_0 : \tau}{\Gamma \vdash_{\mathbb{F}} op^2 e_0 e_1 : \tau}$
11125	
11126	$\frac{}{\Gamma \vdash_{\mathbb{F}} \text{Err} : \tau}$
11127	$\frac{\Gamma \vdash_{\mathbb{F}} e}{\Gamma \vdash_{\mathbb{F}} \text{dyn } \tau \ e : \tau}$
11128	
11129	$\frac{\Gamma \vdash_{\mathbb{F}} v_0 : \tau'_0 \quad \Gamma \vdash_{\mathbb{F}} v_1 : \tau'_1}{\Gamma \vdash_{\mathbb{F}} \text{mon}(\tau_0 \times \tau_1) \langle v_0, v_1 \rangle : (\tau_0 \times \tau_1)}$
11130	
11131	
11132	$\frac{\Gamma \vdash_{\mathbb{F}} v_0 \quad \Gamma \vdash_{\mathbb{F}} v_1}{\Gamma \vdash_{\mathbb{F}} \text{mon}(\tau_0 \times \tau_1) \langle v_0, v_1 \rangle : (\tau_0 \times \tau_1)}$
11133	
11134	
11135	
11136	$\frac{\Gamma \vdash_{\mathbb{F}} \lambda x. e}{\Gamma \vdash_{\mathbb{F}} \text{mon}(\tau_d \Rightarrow \tau_c) \lambda x. e : (\tau_d \Rightarrow \tau_c)}$
11137	
11138	
11139	$\frac{\Gamma \vdash_{\mathbb{F}} \lambda(x:\tau'_d). e : \tau'_d \Rightarrow \tau'_c}{\Gamma \vdash_{\mathbb{F}} \text{mon}(\tau_d \Rightarrow \tau_c) \lambda(x:\tau'_d). e : (\tau_d \Rightarrow \tau_c)}$
11140	$\frac{\Gamma \vdash_{\mathbb{F}} e : \tau'}{\Gamma \vdash_{\mathbb{F}} \text{chk } \tau \ e : \tau}$
11141	
11142	$\boxed{\delta(op^1, v) = e}$
11143	$\delta(\text{fst}, \langle v_0, v_1 \rangle) = v_0$
11144	$\delta(\text{snd}, \langle v_0, v_1 \rangle) = v_1$
11145	$\boxed{\delta(op^2, v, v) = e}$
11146	$\delta(\text{sum}, i_0, i_1) = i_0 + i_1$
11147	$\delta(\text{quotient}, i_0, 0) = \text{BndryErr}$
11148	$\delta(\text{quotient}, i_0, i_1) = \lfloor i_0 / i_1 \rfloor$
11149	if $i_1 \neq 0$
11150	
11151	$\boxed{\mathcal{D}_{\mathbb{F}} : \tau \times v \longrightarrow e}$
11152	$\mathcal{D}_{\mathbb{F}}(\tau, v) = \mathcal{X}(\tau, v)$
11153	$\boxed{\mathcal{S}_{\mathbb{F}} : \tau \times v \longrightarrow e}$
11154	$\mathcal{S}_{\mathbb{F}}(\tau, v) = \mathcal{X}(\tau, v)$
11155	
11156	
11157	
11158	
11159	
11160	
11161	
11162	
11163	
11164	
11165	

11166	$\boxed{X : \tau \times v \longrightarrow e}$
11167	$X(\tau_d \Rightarrow \tau_c, \lambda x. e) = \text{mon}(\tau_d \Rightarrow \tau_c)(\lambda x. e)$
11168	$X(\tau_d \Rightarrow \tau_c, \lambda(x:\tau). e) = \text{mon}(\tau_d \Rightarrow \tau_c)(\lambda(x:\tau). e)$
11169	$X(\tau_d \Rightarrow \tau_c, \text{mon}(\tau'_d \Rightarrow \tau'_c) v') = \text{mon}(\tau_d \Rightarrow \tau_c) v'$
11170	$X(\tau_0 \times \tau_1, \langle v_0, v_1 \rangle) = \text{mon}(\tau_0 \times \tau_1) \langle v_0, v_1 \rangle$
11171	$X(\tau_0 \times \tau_1, \text{mon}(\tau'_0 \times \tau'_1) v') = \text{mon}(\tau_0 \times \tau_1) v'$
11172	$X(\text{Int}, i) = i$
11173	$X(\text{Nat}, i) = i$
11174	if $i \in \mathbb{N}$
11175	$X(\tau, v) = \text{BndryErr}$
11176	otherwise
11177	$\boxed{e \triangleright_{S-1} e}$
11178	$\text{dyn } \tau \ v \triangleright_{S-1} \mathcal{D}_{\mathbb{F}}(\tau, v)$
11179	$\text{chk } \tau \ v \triangleright_{S-1} \mathcal{X}(\tau, v)$
11180	$(\text{mon}(\tau_d \Rightarrow \tau_c)(\lambda x. e)) v \triangleright_{S-1} \text{dyn } \tau_c \ e'$
11181	where $e' = (\lambda x. e)(X(\tau_d, v))$
11182	$(\text{mon}(\tau_d \Rightarrow \tau_c)(\lambda(x:\tau). e)) v \triangleright_{S-1} \text{chk } \tau_c \ e'$
11183	where $e' = (\lambda(x:\tau). e)(X(\tau, v))$
11184	$\text{fst}(\text{mon}(\tau_0 \times \tau_1) \langle v_0, v_1 \rangle) \triangleright_{S-1} X(\tau_0, v_0)$
11185	$\text{snd}(\text{mon}(\tau_0 \times \tau_1) \langle v_0, v_1 \rangle) \triangleright_{S-1} X(\tau_1, v_1)$
11186	$(\lambda(x:\tau). e) v \triangleright_{S-1} e[x \leftarrow v]$
11187	$op^1 v \triangleright_{S-1} \delta(op^1, v)$
11188	$op^2 v_0 v_1 \triangleright_{S-1} \delta(op^2, v_0, v_1)$
11189	$\boxed{e \triangleright_{D-1} e}$
11190	$\text{stat } \tau \ v \triangleright_{D-1} S_{\mathbb{F}}(\tau, v)$
11191	$v_0 \ v_1 \triangleright_{D-1} \text{TagErr}$
11192	if $v_0 \in \mathbb{Z}$ or $v_0 = \langle v, v' \rangle$
11193	$(\text{mon}(\tau_d \Rightarrow \tau_c)(\lambda x. e)) v \triangleright_{D-1} (\lambda x. e) v$
11194	$(\text{mon}(\tau_d \Rightarrow \tau_c)(\lambda(x:\tau). e)) v \triangleright_{D-1} \text{stat } \tau_c \ e'$
11195	where $e' = \text{chk } \tau_c ((\lambda(x:\tau). e)(X(\tau, v)))$
11196	$(\lambda x. e) v \triangleright_{H-D} e[x \leftarrow v]$
11197	$\text{fst}(\text{mon}(\tau_0 \times \tau_1) \langle v_0, v_1 \rangle) \triangleright_{D-1} X(\tau_0, v_0)$
11198	$\text{snd}(\text{mon}(\tau_0 \times \tau_1) \langle v_0, v_1 \rangle) \triangleright_{D-1} X(\tau_1, v_1)$
11199	$op^1 v \triangleright_{H-D} \text{TagErr}$
11200	if $\delta(op^1, v)$ is undefined
11201	$op^1 v \triangleright_{H-D} \delta(op^1, v)$
11202	$op^2 v_0 v_1 \triangleright_{H-D} \text{TagErr}$
11203	if $\delta(op^2, v_0, v_1)$ is undefined
11204	$op^2 v_0 v_1 \triangleright_{H-D} \delta(op^2, v_0, v_1)$
11205	$\boxed{e \rightarrow_{F-S} e}$
11206	$E^\bullet[e] \rightarrow_{F-S} E^\bullet[e']$
11207	if $e \triangleright_{S-1} e'$
11208	$E[\text{stat } \tau \ E^\bullet[e]] \rightarrow_{F-S} E[\text{stat } \tau \ E^\bullet[e']]$
11209	if $e \triangleright_{S-1} e'$
11210	$E[\text{dyn } \tau \ E^\bullet[e]] \rightarrow_{F-S} E[\text{dyn } \tau \ E^\bullet[e']]$
11211	if $e \triangleright_{D-1} e'$
11212	$E[\text{Err}] \rightarrow_{F-S} \text{Err}$

11221	$e \rightarrow_{F-D} e$	11276
11222	$E^*[e] \rightarrow_{F-D} E^*[e']$	11277
11223	if $e \triangleright_{D-1} e'$	11278
11224	$E[\text{stat } \tau E^*[e]] \rightarrow_{F-D} E[\text{stat } \tau E^*[e']]$	11279
11225	if $e \triangleright_{S-1} e'$	11280
11226	$E[\text{dyn } \tau E^*[e]] \rightarrow_{F-D} E[\text{dyn } \tau E^*[e']]$	11281
11227	if $e \triangleright_{D-1} e'$	11282
11228	$E[\text{Err}] \rightarrow_{F-D} \text{Err}$	11283
11229	$e \rightarrow_{F-S}^* e$ reflexive, transitive closure of \rightarrow_{F-S}	11284
11230		11285
11231	$e \rightarrow_{F-D}^* e$ reflexive, transitive closure of \rightarrow_{F-D}	11286
11232		11287
11233		11288
11234		11289
11235		11290
11236		11291
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11331	E.6.2 Forgetful Theorems	11386
11332	Theorem 6.0 : static HF-soundness	11387
11333	If $\vdash e : \tau$ then $\vdash_F e : \tau$ and one of the following holds:	11388
11334	• $e \rightarrow_{F-S}^* v$ and $\vdash_F v : \tau$	11389
11335	• $e \rightarrow_{F-S}^* E[\text{dyn } \tau' e']$ and $e' \triangleright_{D-1} \text{TagErr}$	11390
11336	• $e \rightarrow_{F-S}^* \text{BndryErr}$	11391
11337	• e diverges	11392
11338	<i>Proof:</i>	11393
11339	1. $\vdash_F e : \tau$	11394
11340	by <i>static subset</i>	11395
11341	2. QED by <i>static progress</i> and <i>static preservation</i> .	11396
11342	□	11397
11343	Theorem 6.1 : dynamic HF-soundness	11398
11344	If $\vdash e$ then $\vdash_F e$ and one of the following holds:	11399
11345	• $e \rightarrow_{F-D}^* v$ and $\vdash_F v$	11400
11346	• $e \rightarrow_{F-D}^* E[e']$ and $e' \triangleright_{D-1} \text{TagErr}$	11401
11347	• $e \rightarrow_{F-D}^* \text{BndryErr}$	11402
11348	• e diverges	11403
11349	<i>Proof:</i>	11404
11350	1. $\vdash_F e$	11405
11351	by <i>dynamic subset</i>	11406
11352	2. QED by <i>dynamic progress</i> and <i>dynamic preservation</i> .	11407
11353	□	11408
11354	Corollary 6.2 : HF static soundness	11409
11355	If $\vdash e : \tau$ and e is boundary-free, then one of the following holds:	11410
11356	holds:	11411
11357	• $e \rightarrow_{F-S}^* v$ and $\vdash_F v : \tau$	11412
11358	• $e \rightarrow_{F-S}^* \text{BndryErr}$	11413
11359	• e diverges	11414
11360	<i>Proof:</i>	11415
11361	Consequence of the proof for <i>static HF-soundness</i>	11416
11362	□	11417
11363		11418
11364		11419
11365		11420
11366		11421
11367		11422
11368		11423
11369		11424
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11382		11437
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11384		11439
11385		11440

11441 E.6.3 Forgetful Lemmas

11442 **Lemma 6.3 : $X(\cdot, \cdot)$ soundness**
 11443 If $\Gamma \vdash_F v$ or $\Gamma \vdash_F v : \tau$
 11444 and $X(\tau', v) = v'$,
 11445 then $\Gamma \vdash_F v'$ and $\Gamma \vdash_F v' : \tau'$
 11446
Proof:

11447 By case analysis of the definition of $X(\cdot, \cdot)$.

11448 **CASE** $X(\tau_d \Rightarrow \tau_c, v) = \text{mon}(\tau_d \Rightarrow \tau_c)v$:

11449 **IF** $v = \lambda x. e$
 11450 $\wedge \Gamma \vdash_F v$:
 11451 1. $\Gamma \vdash_F \text{mon}(\tau_d \Rightarrow \tau_c)v$
 11452 by $\Gamma \vdash_F v$
 11453 2. $\Gamma \vdash_F \text{mon}(\tau_d \Rightarrow \tau_c)v : \tau_d \Rightarrow \tau_c$
 11454 by $\Gamma \vdash_F v$
 11455 3. QED

11456 **ELSE** $v = \lambda(x:\tau_x). e$

11457 $\wedge \Gamma \vdash_F v : \tau'_d \Rightarrow \tau'_c$:
 11458 1. $\Gamma \vdash_F \text{mon}(\tau_d \Rightarrow \tau_c)v$
 11459 by $\Gamma \vdash_F v : \tau'_d \Rightarrow \tau'_c$
 11460 2. $\Gamma \vdash_F \text{mon}(\tau_d \Rightarrow \tau_c)v : \tau_d \Rightarrow \tau_c$
 11461 by $\Gamma \vdash_F v : \tau'_d \Rightarrow \tau'_c$
 11462 3. QED

11463 **CASE** $X(\tau_d \Rightarrow \tau_c, \text{mon}(\tau'_d \Rightarrow \tau'_c)v') = \text{mon}(\tau_d \Rightarrow \tau_c)v'$:

11464 **IF** $\Gamma \vdash_F \text{mon}(\tau'_d \Rightarrow \tau'_c)v'$:

11465 **IF** $v' = \lambda x. e'$
 11466 $\wedge \Gamma \vdash_F v'$:
 11467 1. $\Gamma \vdash_F \text{mon}(\tau_d \Rightarrow \tau_c)v'$
 11468 by $\Gamma \vdash_F v'$
 11469 2. $\Gamma \vdash_F \text{mon}(\tau_d \Rightarrow \tau_c)v' : \tau_d \Rightarrow \tau_c$
 11470 by $\Gamma \vdash_F v'$
 11471 3. QED

11472 **ELSE** $v' = \lambda(x:\tau_x). e'$

11473 $\wedge \Gamma \vdash_F v' : \tau''_d \Rightarrow \tau''_c$:
 11474 1. $\Gamma \vdash_F \text{mon}(\tau_d \Rightarrow \tau_c)v'$
 11475 by $\Gamma \vdash_F v' : \tau''_d \Rightarrow \tau''_c$
 11476 2. $\Gamma \vdash_F \text{mon}(\tau_d \Rightarrow \tau_c)v' : \tau_d \Rightarrow \tau_c$
 11477 by $\Gamma \vdash_F v' : \tau''_d \Rightarrow \tau''_c$
 11478 3. QED

11479 **ELSE** $\Gamma \vdash_F \text{mon}(\tau'_d \Rightarrow \tau'_c)v' : \tau'_d \Rightarrow \tau'_c$:

11480 **IF** $v' = \lambda x. e'$
 11481 $\wedge \Gamma \vdash_F v'$:
 11482 1. $\Gamma \vdash_F \text{mon}(\tau_d \Rightarrow \tau_c)v'$
 11483 by $\Gamma \vdash_F v'$
 11484 2. $\Gamma \vdash_F \text{mon}(\tau_d \Rightarrow \tau_c)v' : \tau_d \Rightarrow \tau_c$
 11485 by $\Gamma \vdash_F v'$
 11486 3. QED

11487 **ELSE** $v' = \lambda(x:\tau_x). e'$

11488 $\wedge \Gamma \vdash_F v' : \tau''_d \Rightarrow \tau''_c$:
 11489 1. $\Gamma \vdash_F \text{mon}(\tau_d \Rightarrow \tau_c)v'$
 11490 by $\Gamma \vdash_F v' : \tau''_d \Rightarrow \tau''_c$
 11491 2. $\Gamma \vdash_F \text{mon}(\tau_d \Rightarrow \tau_c)v' : \tau_d \Rightarrow \tau_c$
 11492 by $\Gamma \vdash_F v' : \tau''_d \Rightarrow \tau''_c$
 11493 3. QED

11494

11495

11496

CASE $X(\tau_0 \times \tau_1, \langle v_0, v_1 \rangle) = \text{mon}(\tau_0 \times \tau_1)\langle v_0, v_1 \rangle$:

11497 **IF** $\Gamma \vdash_F \langle v_0, v_1 \rangle$:
 11498 1. $\Gamma \vdash_F \text{mon}(\tau_0 \times \tau_1)\langle v_0, v_1 \rangle$
 11499 by $\Gamma \vdash_F \langle v_0, v_1 \rangle$
 11500 2. $\Gamma \vdash_F \text{mon}(\tau_0 \times \tau_1)\langle v_0, v_1 \rangle : \tau_0 \times \tau_1$
 11501 by $\Gamma \vdash_F \langle v_0, v_1 \rangle$
 11502 3. QED

11503 **ELSE** $\Gamma \vdash_F \langle v_0, v_1 \rangle : \tau'_0 \times \tau'_1$:
 11504 1. $\Gamma \vdash_F \text{mon}(\tau_0 \times \tau_1)\langle v_0, v_1 \rangle$
 11505 by $\Gamma \vdash_F \langle v_0, v_1 \rangle : \tau'_0 \times \tau'_1$
 11506 2. $\Gamma \vdash_F \text{mon}(\tau_0 \times \tau_1)\langle v_0, v_1 \rangle : \tau_0 \times \tau_1$
 11507 by $\Gamma \vdash_F \langle v_0, v_1 \rangle : \tau'_0 \times \tau'_1$
 11508 3. QED

11509 **CASE** $X(\tau_0 \times \tau_1, \text{mon}(\tau'_0 \times \tau'_1)\langle v_0, v_1 \rangle) = \text{mon}(\tau_0 \times \tau_1)\langle v_0, v_1 \rangle$:

11510 **IF** $\Gamma \vdash_F \langle v_0, v_1 \rangle$:
 11511 1. $\Gamma \vdash_F \text{mon}(\tau_0 \times \tau_1)\langle v_0, v_1 \rangle$
 11512 by $\Gamma \vdash_F \langle v_0, v_1 \rangle$
 11513 2. $\Gamma \vdash_F \text{mon}(\tau_0 \times \tau_1)\langle v_0, v_1 \rangle : \tau_0 \times \tau_1$
 11514 by $\Gamma \vdash_F \langle v_0, v_1 \rangle$
 11515 3. QED

11516 **ELSE** $\Gamma \vdash_F \langle v_0, v_1 \rangle : \tau''_0 \times \tau''_1$:

11517 1. $\Gamma \vdash_F \text{mon}(\tau_0 \times \tau_1)\langle v_0, v_1 \rangle$
 11518 by $\Gamma \vdash_F \langle v_0, v_1 \rangle : \tau''_0 \times \tau''_1$
 11519 2. $\Gamma \vdash_F \text{mon}(\tau_0 \times \tau_1)\langle v_0, v_1 \rangle : \tau_0 \times \tau_1$
 11520 by $\Gamma \vdash_F \langle v_0, v_1 \rangle : \tau''_0 \times \tau''_1$
 11521 3. QED

11522 **CASE** $X(\text{Int}, i) = i$:

11523 1. $\Gamma \vdash_F i$
 11524 2. $\Gamma \vdash_F i : \text{Int}$
 11525 3. QED

11526 **CASE** $X(\text{Nat}, i) = i$:

11527 1. $\Gamma \vdash_F i$
 11528 2. $\Gamma \vdash_F i : \text{Nat}$
 11529 by $i \in \mathbb{N}$
 11530 3. QED

11531 \square

11532 **Corollary 6.4 : \mathcal{D}_F soundness**

11533 **If** $\vdash_F v$ then $\vdash_F \mathcal{D}_F(\tau, v) : \tau$

11534 *Proof:*

11535 QED by X soundness

11536 \square

11537 **Corollary 6.5 : \mathcal{S}_F soundness**

11538 **If** $\vdash_F v : \tau$ then $\vdash_F \mathcal{S}_F(\tau, v)$

11539 *Proof:*

11540 QED by X soundness

11541 \square

11542 **Corollary 6.6 : HF static subset**

11543 **If** $\Gamma \vdash e : \tau$ then $\Gamma \vdash_F e : \tau$.

11544 *Proof:*

11545 Consequence of the proof for the higher-order *static subset* lemma; both \vdash_F and \vdash_H have the same typing rules for surface-language expressions.

□

Corollary 6.7 : HF dynamic subset

If $\Gamma \vdash e$ then $\Gamma \vdash_F e$.

Proof:

Consequence of the proof for the higher-order *dynamic subset* lemma.

□

Lemma 6.8 : HF static progress

If $\vdash_F e : \tau$ then one of the following holds:

- e is a value
- $e \in \text{Err}$
- $e \rightarrow_{F-S} e'$
- $e \rightarrow_{F-S} \text{BndryErr}$
- $e = E[\text{dyn } \tau' e']$ and $e' \rightarrow_{F-D} \text{TagErr}$

Proof:

By the *boundary factoring* lemma, there are eight possible cases.

CASE e is a value :

1. QED

CASE $e = E^\bullet[v_0 v_1]$:

1. $\vdash_F v_0 v_1 : \tau'$
by *static hole typing*
2. $\vdash_F v_0 : \tau_d \Rightarrow \tau_c$
 $\wedge \vdash_F v_1 : \tau_d$
by *inversion*
3. $v_0 = \lambda(x : \tau_d). e'$
 $\vee v_0 = \text{mon}(\tau_d' \Rightarrow \tau_c') \lambda x. e'$
 $\vee v_0 = \text{mon}(\tau_d' \Rightarrow \tau_c') \lambda(x : \tau_x). e'$
by *canonical forms*
4. **IF** $v_0 = \lambda(x : \tau_d'). e'$:
 - a. $e \rightarrow_{F-S} E^\bullet[e'[x \leftarrow v_1]]$
by $v_0 v_1 \triangleright_{S-1} e'[x \leftarrow v_1]$
 - b. QED**IF** $v_0 = \text{mon}(\tau_d' \Rightarrow \tau_c') \lambda x. e'$
 $\wedge X(\tau_d', v_1) = v'_1$:
 - a. $e \rightarrow_{F-S} E^\bullet[\text{dyn } \tau_c'(e'[x \leftarrow v'_1])]$
by $v_0 v_1 \triangleright_{S-1} \text{dyn } \tau_c'(e'[x \leftarrow v'_1])$
 - b. QED**IF** $v_0 = \text{mon}(\tau_d' \Rightarrow \tau_c') \lambda x. e'$
 $\wedge X(\tau_d', v_1) = \text{BndryErr}$:
 - a. $e \rightarrow_{F-S} \text{BndryErr}$
by $fst v \triangleright_{S-1} \text{BndryErr}$
 - b. QED**IF** $v_0 = \text{mon}(\tau_d' \Rightarrow \tau_c') \lambda x. e'$
 $\wedge X(\tau_d', v_1) = v'_1$:
 - a. $e \rightarrow_{F-S} E^\bullet[v'_1]$
by $snd v \triangleright_{S-1} v'_1$
 - b. QED**ELSE** $v_0 = \text{mon}(\tau_d' \Rightarrow \tau_c') \lambda x. e'$
 $\wedge X(\tau_d', v_1) = \text{BndryErr}$:
 - a. $e \rightarrow_{F-S} \text{BndryErr}$
by $snd v \triangleright_{S-1} \text{BndryErr}$
 - b. QED

CASE $e = E^\bullet[op^2 v_0 v_1]$:

1. $\vdash_F op^2 v_0 v_1 : \tau'$
by *static hole typing*
2. $\vdash_F v_0 : \tau_0$
 $\wedge \vdash_F v_1 : \tau_1$
 $\wedge \Delta(op^2, \tau_0, \tau_1) = \tau''$
by *inversion*
3. $\delta(op^2, v_0, v_1) = e'$
by *type soundness* (2)

CASE $e = E^\bullet[op^1 v]$:

1. $\vdash_F op^1 v : \tau'$
by *static hole typing*
2. $\vdash_F v : \tau_0 \times \tau_1$
by *inversion*
3. $v = \langle v_0, v_1 \rangle$
 $\vee v = \text{mon}(\tau_0' \times \tau_1') \langle v_0, v_1 \rangle$
by *canonical forms*
4. **IF** $v = \langle v_0, v_1 \rangle$
 $\wedge op^1 = \text{fst}$:
 - a. $\delta(\text{fst}, \langle v_0, v_1 \rangle) = v_0$
 - b. $e \rightarrow_{F-S} E^\bullet[v_0]$
by $op^1 v \triangleright_{S-1} v_0$
 - c. QED**IF** $v = \langle v_0, v_1 \rangle$
 $\wedge op^1 = \text{snd}$:
 - a. $\delta(\text{snd}, \langle v_0, v_1 \rangle) = v_1$
 - b. $e \rightarrow_{F-S} E^\bullet[v_1]$
by $op^1 v \triangleright_{S-1} v_1$
 - c. QED**IF** $v = \text{mon}(\tau_0' \times \tau_1') \langle v_0, v_1 \rangle$
 $\wedge op^1 = \text{fst}$
 $\wedge X(\tau_0', v_0) = v'_0$:
 - a. $e \rightarrow_{F-S} E^\bullet[v'_0]$
by $fst v \triangleright_{S-1} v'_0$
 - b. QED**IF** $v = \text{mon}(\tau_0' \times \tau_1') \langle v_0, v_1 \rangle$
 $\wedge op^1 = \text{fst}$
 $\wedge X(\tau_0', v_0) = \text{BndryErr}$:
 - a. $e \rightarrow_{F-S} \text{BndryErr}$
by $fst v \triangleright_{S-1} \text{BndryErr}$
 - b. QED**IF** $v = \text{mon}(\tau_0' \times \tau_1') \langle v_0, v_1 \rangle$
 $\wedge op^1 = \text{snd}$
 $\wedge X(\tau_1', v_1) = v'_1$:
 - a. $e \rightarrow_{F-S} E^\bullet[v'_1]$
by $snd v \triangleright_{S-1} v'_1$
 - b. QED**ELSE** $v = \text{mon}(\tau_0' \times \tau_1') \langle v_0, v_1 \rangle$
 $\wedge op^1 = \text{snd}$
 $\wedge X(\tau_1', v_1) = \text{BndryErr}$:
 - a. $e \rightarrow_{F-S} \text{BndryErr}$
by $snd v \triangleright_{S-1} \text{BndryErr}$
 - b. QED

11661 4. $op^2 v_0 v_1 \triangleright_{S-1} e'$
 11662 by (3)
 11663 5. QED by $e \rightarrow_{F-S} E^\bullet[e']$
CASE $e = E[\text{dyn } \tau' e']$ and e' is boundary-free :
 11665 1. e' is a value
 11666 $\vee e' \in \text{Err}$
 11667 $\vee e' \rightarrow_{F-D} e''$
 11668 $\vee e' \rightarrow_{F-D} \text{BndryErr}$
 11669 $\vee e' = E'[e'']$ and $e'' \triangleright_{D-1} \text{TagErr}$
 11670 by *dynamic progress*
 11671 2. **IF** e' is a value :
 11672 a. QED $e \rightarrow_{F-S} E[\mathcal{D}_F(\tau', e')]$
 11673 **IF** $e' \in \text{Err}$:
 11674 a. QED $e \rightarrow_{F-S} e'$
 11675 **IF** $e' \rightarrow_{F-D} e''$:
 11676 a. QED $e \rightarrow_{F-S} E[\text{dyn } \tau' e'']$
 11677 **IF** $e' \rightarrow_{F-D} \text{BndryErr}$:
 11678 a. QED $e \rightarrow_{F-S} E[\text{dyn } \tau' \text{BndryErr}]$
 11679 **ELSE** $e' = E'[e'']$ and $e'' \triangleright_{D-1} \text{TagErr}$:
 11680 a. $E' \in E^\bullet$
 11681 by e' is boundary-free
 11682 b. QED
CASE $e = E[\text{stat } \tau' e']$ and e' is boundary-free :
 11683 1. e' is a value
 11684 $\vee e' \in \text{Err}$
 11685 $\vee e' \rightarrow_{F-S} e''$
 11686 $\vee e' \rightarrow_{F-S} \text{BndryErr}$
 11687 $\vee e' = E''[\text{dyn } \tau'' E^\bullet[e'']]$ and $e'' \triangleright_{D-1} \text{TagErr}$
 11688 by *static progress*
 11689 2. **IF** e' is a value :
 11690 a. QED $e \rightarrow_{F-S} E[\mathcal{S}_F(\tau', e')]$
 11691 **IF** $e' \in \text{Err}$:
 11692 a. QED $e \rightarrow_{F-S} e'$
 11693 **IF** $e' \rightarrow_{F-S} e''$:
 11694 a. QED $e \rightarrow_{F-S} E[\text{stat } \tau' e'']$
 11695 **IF** $e' \rightarrow_{F-S} \text{BndryErr}$:
 11696 a. QED $e \rightarrow_{F-S} E[\text{stat } \tau' \text{BndryErr}]$
 11697 **ELSE** $e' = E''[\text{dyn } \tau'' E^\bullet[e'']]$ and $e'' \triangleright_{D-1} \text{TagErr}$
 11698 :
 11699 a. Contradiction by e' is boundary-free
CASE $e = E[\text{Err}]$:
 11700 1. QED $e \rightarrow_{F-S} \text{Err}$
CASE $e = E^\bullet[\text{chk } \tau' v]$:
 11701 **IF** $X(\tau, v) = v'$:
 11702 1. $e \rightarrow_{F-S} E^\bullet[v']$
 11703 by $(\text{chk } \tau v) \triangleright_{S-1} v'$
 11704 2. QED
 11705 **ELSE** $X(\tau, v) = \text{BndryErr}$:
 11706 1. $e \rightarrow_{F-S} \text{BndryErr}$
 11707 by $(\text{chk } \tau v) \triangleright_{S-1} \text{BndryErr}$
 11708 2. QED
 11709 \square
 11710 **Lemma 6.9** : HF *dynamic progress*

If $\vdash_F e$ then one of the following holds: 11716
 • e is a value 11717
 • $e \in \text{Err}$ 11718
 • $e \rightarrow_{F-D} e'$ 11719
 • $e \rightarrow_{F-D} \text{BndryErr}$ 11720
 • $e \rightarrow_{F-D} \text{TagErr}$ 11721
Proof: 11722
 By the *boundary factoring* lemma, there are seven cases. 11723
CASE e is a value : 11724
 1. QED 11725
CASE $e = E^\bullet[v_0 v_1]$: 11726
IF $v_0 = \lambda x. e'$: 11727
 1. $e \rightarrow_{F-D} E^\bullet[e'[x \leftarrow v_1]]$ 11728
 by $v_0 v_1 \triangleright_{D-1} e'[x \leftarrow v_1]$ 11729
 2. QED 11730
IF $v_0 = \text{mon } (\tau_d \Rightarrow \tau_c)(\lambda x. e')$: 11731
 1. $e \rightarrow_{F-D} E^\bullet[e'[x \leftarrow v_1]]$ 11732
 by $v_0 v_1 \triangleright_{D-1} e'[x \leftarrow v_1]$ 11733
 2. QED 11734
IF $v_0 = \text{mon } (\tau_d \Rightarrow \tau_c)(\lambda(x: \tau_x). e')$ 11735
 $\wedge X(\tau_x, v_1) = v'_1$: 11736
 1. $e \rightarrow_{F-D} E^\bullet[\text{stat } \tau_c (\text{chk } \tau_c e'[x \leftarrow v'_1])]$ 11737
 by $v_0 v_1 \triangleright_{D-1} \text{stat } \tau_c (\text{chk } \tau_c e'[x \leftarrow v'_1])$ 11738
 2. QED 11739
IF $v_0 = \text{mon } (\tau_d \Rightarrow \tau_c)(\lambda(x: \tau_x). e')$ 11740
 $\wedge X(\tau_x, v_1) = \text{BndryErr}$: 11741
 1. $e \rightarrow_{F-D} \text{BndryErr}$ 11742
 by $v_0 v_1 \triangleright_{D-1} \text{BndryErr}$ 11743
 2. QED 11744
IF $v_0 = \lambda(x: \tau_x). e'$: 11745
 1. Contradiction by $\vdash_F e$ 11746
ELSE $v_0 = i$ 11747
 $\vee v_0 = \langle v, v' \rangle$ 11748
 $\vee v_0 = \text{mon } \tau_0 \times \tau_1 v'$: 11749
 1. $e \rightarrow_{F-D} \text{TagErr}$ 11750
 by $(v_0 v_1) \triangleright_{D-1} \text{TagErr}$ 11751
 2. QED 11752
CASE $e = E^\bullet[op^1 v]$: 11753
IF $v = \text{mon } \tau_0 \times \tau_1 \langle v_0, v_1 \rangle$ 11754
 $\wedge op^1 = \text{fst}$ 11755
 $\wedge X(\tau_0, v_0) = v'_0$: 11756
 1. $e \rightarrow_{F-D} E^\bullet[v'_0]$ 11757
 by $op^1 v \triangleright_{D-1} v'_0$ 11758
 2. QED 11759
IF $v = \text{mon } \tau_0 \times \tau_1 \langle v_0, v_1 \rangle$ 11760
 $\wedge op^1 = \text{fst}$ 11761
 $\wedge X(\tau_0, v_0) = \text{BndryErr}$: 11762
 1. $e \rightarrow_{F-D} \text{BndryErr}$ 11763
 by $op^1 v \triangleright_{D-1} \text{BndryErr}$ 11764
 2. QED 11765
IF $v = \text{mon } \tau_0 \times \tau_1 \langle v_0, v_1 \rangle$ 11766
 $\wedge op^1 = \text{snd}$ 11767
 $\wedge X(\tau_1, v_1) = v'_1$: 11768

11771	1. $e \rightarrow_{F-D} E^\bullet[v'_1]$	11826
11772	by $op^1 v \triangleright_{D-1} v'_1$	11827
11773	2. QED	11828
11774	IF $v = \text{mon } \tau_0 \times \tau_1 \langle v_0, v_1 \rangle$	11829
11775	$\wedge op^1 = \text{snd}$	11830
11776	$\wedge X(\tau_1, v_1) = \text{BndryErr} :$	11831
11777	1. $e \rightarrow_{F-D} \text{BndryErr}$	11832
11778	by $op^1 v \triangleright_{D-1} \text{BndryErr}$	11833
11779	2. QED	11834
11780	IF $\delta(op^1, v) = e' :$	11835
11781	1. $(op^1 v) \triangleright_{D-1} e'$	11836
11782	2. QED by $e \rightarrow_{F-D} E^\bullet[e']$	11837
11783	ELSE $\delta(op^1, v)$ is undefined :	11838
11784	1. $e \rightarrow_{F-D} \text{TagErr}$	11839
11785	by $(op^1 v) \triangleright_{D-1} \text{TagErr}$	11840
11786	2. QED	11841
11787	CASE $e = E^\bullet[op^2 v_0 v_1] :$	11842
11788	IF $\delta(op^2, v_0, v_1) = e'' :$	11843
11789	1. $op^2 v_0 v_1 \triangleright_{D-1} e''$	11844
11790	2. QED	11845
11791	ELSE $\delta(op^2, v_0, v_1)$ is undefined :	11846
11792	1. $e \rightarrow_{F-D} \text{TagErr}$	11847
11793	by $op^2 v_0 v_1 \triangleright_{D-1} \text{TagErr}$	11848
11794	2. QED	11849
11795	CASE $e = E[\text{dyn } \tau' e']$ and e' is boundary-free :	11850
11796	1. e' is a value	11851
11797	$\vee e' \in \text{Err}$	11852
11798	$\vee e' \rightarrow_{F-D} e''$	11853
11799	$\vee e' \rightarrow_{F-D} \text{BndryErr}$	11854
11800	$\vee e' = E^\bullet[e'']$ and $e'' \triangleright_{D-1} \text{TagErr}$	11855
11801	by <i>dynamic progress</i>	11856
11802	2. IF e' is a value :	11857
11803	a. QED $e \rightarrow_{F-D} E[\mathcal{D}_F(\tau', e')]$	11858
11804	IF $e' \in \text{Err} :$	11859
11805	a. QED $e \rightarrow_{F-D} e'$	11860
11806	IF $e' \rightarrow_{F-D} e'' :$	11861
11807	a. QED $e \rightarrow_{F-S} E[\text{dyn } \tau' e']$	11862
11808	IF $e' \rightarrow_{F-D} \text{BndryErr} :$	11863
11809	a. QED $e \rightarrow_{F-D} E[\text{dyn } \tau' \text{ BndryErr}]$	11864
11810	ELSE $e' = E[e'']$ and $e'' \triangleright_{D-1} \text{TagErr} :$	11865
11811	a. $E \in E^\bullet$	11866
11812	by e' is boundary-free	11867
11813	b. QED	11868
11814	CASE $e = E[\text{stat } \tau' e']$ and e' is boundary-free :	11869
11815	1. e' is a value	11870
11816	$\vee e' \in \text{Err}$	11871
11817	$\vee e' \rightarrow_{F-S} e''$	11872
11818	$\vee e' \rightarrow_{F-S} \text{BndryErr}$	11873
11819	$\vee e' = E''[\text{dyn } \tau'' E^\bullet[e'']]$ and $e'' \triangleright_{D-1} \text{TagErr}$	11874
11820	by <i>static progress</i>	11875
11821	2. IF e' is a value :	11876
11822	a. QED $e \rightarrow_{F-S} E[\mathcal{S}_F(\tau', e')]$	11877
11823	IF $e' \in \text{Err} :$	11878
11824		11879
11825		11880

11881	7. $\vdash_{\mathbb{F}} (\lambda x. e') X(\tau_d, v_1)$	11936
11882	by (4, 6)	11937
11883	8. $\vdash_{\mathbb{F}} \text{dyn } \tau_c ((\lambda x. e') X(\tau_d, v_1)) : \tau_c$	11938
11884	by (7)	11939
11885	9. $\vdash_{\mathbb{F}} \text{dyn } \tau_c ((\lambda x. e') X(\tau_d, v_1)) : \tau'$	11940
11886	by (2, 5, 8)	11941
11887	10. QED by <i>hole substitution</i>	11942
11888	ELSE $v_0 = \text{mon } \tau_d \Rightarrow \tau_c (\lambda(x:\tau_x). e')$	11943
11889	$\wedge e \rightarrow_{\mathbb{F}-S} E^{\bullet}[\text{stat } \tau_c (\text{chk } \tau_c ((\lambda(x:\tau_x). e') X(\tau_x, v_1)))]$	11944
11890	:	11945
11891	1. $\vdash_{\mathbb{F}} v_0 v_1 : \tau'$	11946
11892	by <i>static hole typing</i>	11947
11893	2. $\vdash_{\mathbb{F}} v_0 : \tau'_d \Rightarrow \tau'_c$	11948
11894	$\wedge \vdash_{\mathbb{F}} v_1 : \tau'_d$	11949
11895	$\wedge \tau'_c \leqslant: \tau'$	11950
11896	by <i>inversion</i>	11951
11897	3. $\vdash_{\mathbb{F}} \lambda(x:\tau_x). e' : \tau_x \Rightarrow \tau'_x$	11952
11898	by <i>inversion</i>	11953
11899	4. $\tau_d \Rightarrow \tau_c \leqslant: \tau'_d \Rightarrow \tau'_c$	11954
11900	by <i>canonical forms</i> (2)	11955
11901	5. $\tau_c \leqslant: \tau'_c$	11956
11902	by (4)	11957
11903	6. $\vdash_{\mathbb{F}} X(\tau_x, v_1) : \tau_x$	11958
11904	by <i>X soundness</i>	11959
11905	7. $\vdash_{\mathbb{F}} (\lambda(x:\tau_x). e') X(\tau_x, v_1) : \tau'_x$	11960
11906	by (3, 6)	11961
11907	8. $\vdash_{\mathbb{F}} \text{chk } \tau_c ((\lambda(x:\tau_x). e') X(\tau_x, v_1)) : \tau_c$	11962
11908	by (7)	11963
11909	9. $\vdash_{\mathbb{F}} \text{stat } \tau_c (\text{chk } \tau_c ((\lambda(x:\tau_x). e') X(\tau_x, v_1))) : \tau_c$	11964
11910	by (8)	11965
11911	10. $\vdash_{\mathbb{F}} \text{stat } \tau_c (\text{chk } \tau_c ((\lambda(x:\tau_x). e') X(\tau_x, v_1))) : \tau'$	11966
11912	by (2, 5, 9)	11967
11913	11. QED by <i>hole substitution</i>	11968
11914	CASE $e = E^{\bullet}[op^1 v] :$	11969
11915	IF $v = \langle v_0, v_1 \rangle$	11970
11916	$\wedge op^1 = \text{fst}$	11971
11917	$\wedge e \rightarrow_{\mathbb{F}-S} E^{\bullet}[v_0] :$	11972
11918	1. $\vdash_{\mathbb{F}} \text{fst } \langle v_0, v_1 \rangle : \tau'$	11973
11919	by <i>static hole typing</i>	11974
11920	2. $\vdash_{\mathbb{F}} \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$	11975
11921	$\wedge \tau_0 \leqslant: \tau'$	11976
11922	by <i>inversion</i>	11977
11923	3. $\vdash_{\mathbb{F}} v_0 : \tau_0$	11978
11924	by <i>inversion</i>	11979
11925	4. $\vdash_{\mathbb{F}} v_0 : \tau'$	11980
11926	by (2)	11981
11927	5. QED by <i>hole substitution</i>	11982
11928	IF $v = \langle v_0, v_1 \rangle$	11983
11929	$\wedge op^1 = \text{snd}$	11984
11930	$\wedge e \rightarrow_{\mathbb{F}-S} E^{\bullet}[v_1] :$	11985
11931	1. $\vdash_{\mathbb{F}} \text{snd } \langle v_0, v_1 \rangle : \tau'$	11986
11932	by <i>static hole typing</i>	11987
11933		11988
11934		11989
11935		11990

11991	1. $\vdash_F op^2 v_0 v_1 : \tau'$	12046
11992	by <i>static hole typing</i>	12047
11993	2. $\vdash_F v_0 : \tau_0$	12048
11994	$\wedge \vdash_F v_1 : \tau_1$	12049
11995	$\wedge \Delta(op^2, \tau_0, \tau_1) = \tau''$	12050
11996	$\wedge \tau'' \leqslant \tau'$	12051
11997	by <i>inversion</i>	12052
11998	3. $\vdash_F v : \tau''$	12053
11999	by Δ <i>type soundness</i> (2)	12054
12000	4. $\vdash_F v : \tau'$	12055
12001	by (2, 3)	12056
12002	5. QED by <i>hole substitution</i> (4)	12057
12003	CASE $e = E[\text{dyn } \tau' e']$ and e' is boundary-free :	12058
12004	IF e' is a value :	12059
12005	1. $e \rightarrow_{F-S} E[\mathcal{D}_F(\tau', e')]$	12060
12006	2. $\vdash_F \text{dyn } \tau' e' : \tau'$	12061
12007	by <i>boundary hole typing</i>	12062
12008	3. $\vdash_F e'$	12063
12009	by <i>inversion</i> (2)	12064
12010	4. $\vdash_F \mathcal{D}_F(\tau', e') : \tau'$	12065
12011	by <i>D_F soundness</i> (3)	12066
12012	5. QED by <i>hole substitution</i> (4)	12067
12013	ELSE $e' \rightarrow_{F-D} e'' :$	12068
12014	1. $e \rightarrow_{F-S} E[\text{dyn } \tau' e'']$	12069
12015	2. $\vdash_F \text{dyn } \tau' e' : \tau'$	12070
12016	by <i>boundary hole typing</i>	12071
12017	3. $\vdash_F e'$	12072
12018	by <i>inversion</i> (2)	12073
12019	4. $\vdash_F e''$	12074
12020	by <i>dynamic preservation</i> (3)	12075
12021	5. $\vdash_F \text{dyn } \tau' e'' : \tau'$	12076
12022	by (4)	12077
12023	6. QED by <i>hole substitution</i> (5)	12078
12024	CASE $e = E[\text{stat } \tau' e']$ and e' is boundary-free :	12079
12025	IF e' is a value :	12080
12026	1. $e \rightarrow_{F-S} E[\mathcal{S}_F(\tau', e')]$	12081
12027	2. $\vdash_F \text{stat } \tau' e'$	12082
12028	by <i>boundary hole typing</i>	12083
12029	3. $\vdash_F e' : \tau'$	12084
12030	by <i>inversion</i> (2)	12085
12031	4. $\vdash_F \mathcal{S}_F(\tau', e')$	12086
12032	by <i>S_F soundness</i> (3)	12087
12033	5. QED by <i>hole substitution</i> (4)	12088
12034	ELSE $e' \rightarrow_{F-S} e'' :$	12089
12035	1. $e \rightarrow_{F-S} E[\text{stat } \tau' e'']$	12090
12036	2. $\vdash_F \text{stat } \tau' e'$	12091
12037	by <i>boundary hole typing</i>	12092
12038	3. $\vdash_F e' : \tau'$	12093
12039	by <i>inversion</i> (2)	12094
12040	4. $\vdash_F e'' : \tau'$	12095
12041	by <i>static preservation</i> (3)	12096
12042	5. $\vdash_F \text{stat } \tau' e''$	12097
12043	by (4)	12098
12044		12099
12045		12100

12101	4. $\vdash_{\mathcal{F}} X(\tau_x, v_1) : \tau_x$	12156
12102	by <i>X soundness</i> (2)	12157
12103	5. $\vdash_{\mathcal{F}} ((\lambda(x:\tau_x). e') X(\tau_x, v_1)) : \tau'_x$	12158
12104	by (3, 4)	12159
12105	6. $\vdash_{\mathcal{F}} \text{chk } \tau_c ((\lambda(x:\tau_x). e') X(\tau_x, v_1)) : \tau_c$	12160
12106	by (5)	12161
12107	7. $\vdash_{\mathcal{F}} \text{stat } \tau_c (\text{chk } \tau_c ((\lambda(x:\tau_x). e') X(\tau_x, v_1))) : \tau_c$	12162
12108	by (6)	12163
12109	8. QED <i>hole substitution</i>	12164
12110	CASE $e = E^{\bullet}[\text{op}^1 v] :$	12165
12111	IF $v = \langle v_0, v_1 \rangle$	12166
12112	$\wedge \text{op}^1 = \text{fst}$	12167
12113	$\wedge e \rightarrow_{\mathcal{F}\text{-D}} E^{\bullet}[v_0] :$	12168
12114	1. $\vdash_{\mathcal{F}} \text{op}^1 v$	12169
12115	by <i>dynamic hole typing</i>	12170
12116	2. $\vdash_{\mathcal{F}} v$	12171
12117	by <i>inversion</i> (1)	12172
12118	3. $\vdash_{\mathcal{F}} v_0$	12173
12119	by <i>inversion</i> (2)	12174
12120	4. QED by <i>hole substitution</i>	12175
12121	IF $v = \langle v_0, v_1 \rangle$	12176
12122	$\wedge \text{op}^1 = \text{snd}$	12177
12123	$\wedge e \rightarrow_{\mathcal{F}\text{-D}} E^{\bullet}[v_1] :$	12178
12124	1. $\vdash_{\mathcal{F}} \text{op}^1 v$	12179
12125	by <i>dynamic hole typing</i>	12180
12126	2. $\vdash_{\mathcal{F}} v$	12181
12127	by <i>inversion</i> (1)	12182
12128	3. $\vdash_{\mathcal{F}} v_1$	12183
12129	by <i>inversion</i> (2)	12184
12130	4. QED by <i>hole substitution</i>	12185
12131	IF $v = \text{mon } (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle$	12186
12132	$\wedge \text{op}^1 = \text{fst}$	12187
12133	$\wedge e \rightarrow_{\mathcal{F}\text{-D}} E^{\bullet}[X(\tau_0, v_0)] :$	12188
12134	1. $\vdash_{\mathcal{F}} \text{op}^1 v$	12189
12135	by <i>dynamic hole typing</i>	12190
12136	2. $\vdash_{\mathcal{F}} \text{mon } (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle$	12191
12137	by <i>inversion</i> (1)	12192
12138	3. $\vdash_{\mathcal{F}} v_0$	12193
12139	$\vee \vdash_{\mathcal{F}} v_0 : \tau'_0$	12194
12140	by <i>inversion</i> (2)	12195
12141	4. $\vdash_{\mathcal{F}} X(\tau_0, v_0)$	12196
12142	by <i>X soundness</i> (3)	12197
12143	5. QED by <i>hole substitution</i>	12198
12144	ELSE $v = \text{mon } (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle$	12199
12145	$\wedge \text{op}^1 = \text{snd}$	12200
12146	$\wedge e \rightarrow_{\mathcal{F}\text{-D}} E^{\bullet}[X(\tau_1, v_1)] :$	12201
12147	1. $\vdash_{\mathcal{F}} \text{op}^1 v$	12202
12148	by <i>dynamic hole typing</i>	12203
12149	2. $\vdash_{\mathcal{F}} \text{mon } (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle$	12204
12150	by <i>inversion</i> (1)	12205
12151	3. $\vdash_{\mathcal{F}} v_1$	12206
12152	$\vee \vdash_{\mathcal{F}} v_1 : \tau'_1$	12207
12153	by <i>inversion</i> (2)	12208
12154		12209
12155		12210

12211 5. $\vdash_F \text{stat } \tau' e''$
 12212 by (4)
 12213 6. QED by *hole substitution* (5)
CASE $e = E[\text{Err}]$:
 12215 1. $e \rightarrow_{F\text{-D}} \text{Err}$
 12216 2. QED $\vdash_F \text{Err}$
 12217 \square
Lemma 6.12 : HF static boundary factoring
 12219 If $\vdash_F e : \tau$ then one of the following holds:
 12220 • e is a value
 12221 • $e = E^\bullet[v_0 v_1]$
 12222 • $e = E^\bullet[op^1 v]$
 12223 • $e = E^\bullet[op^2 v_0 v_1]$
 12224 • $e = E^\bullet[\text{chk } \tau v]$
 12225 • $e = E[\text{dyn } \tau e']$ where e' is boundary-free
 12226 • $e = E[\text{stat } \tau e']$ where e' is boundary-free
 12227 • $e = E[\text{Err}]$
 12228 *Proof:*
 12229 By the *unique static evaluation contexts* lemma, there are
 1230 eight cases.
CASE e is a value :
 1232 1. QED
CASE $e = E[v_0 v_1]$:
 1234 1. $E = E^\bullet$
 1235 $\vee E = E'[\text{dyn } \tau E^\bullet]$
 1236 $\vee E = E'[\text{stat } \tau E^\bullet]$
 1237 by *inner boundary*
 1238 2. **IF** $E = E^\bullet$:
 1239 a. QED $e = E^\bullet[v_0 v_1]$
 1240 **IF** $E = E'[\text{dyn } \tau E^\bullet]$:
 1241 a. QED $e = E'[\text{dyn } \tau E^\bullet[v_0 v_1]]$
 1242 **ELSE** $E = E'[\text{stat } \tau E^\bullet]$:
 1243 a. QED $e = E'[\text{stat } \tau E^\bullet[v_0 v_1]]$
 1244 **CASE** $e = E[op^1 v]$:
 1245 1. $E = E^\bullet$
 1246 $\vee E = E'[\text{dyn } \tau E^\bullet]$
 1247 $\vee E = E'[\text{stat } \tau E^\bullet]$
 1248 by *inner boundary*
 1249 2. **IF** $E = E^\bullet$:
 1250 a. QED $e = E^\bullet[op^1 v]$
 1251 **IF** $E = E'[\text{dyn } \tau E^\bullet]$:
 1252 a. QED $e = E'[\text{dyn } \tau E^\bullet[op^1 v]]$
 1253 **ELSE** $E = E'[\text{stat } \tau E^\bullet]$:
 1254 a. QED $e = E'[\text{stat } \tau E^\bullet[op^1 v]]$
 1255 **CASE** $e = E[op^2 v_0 v_1]$:
 1256 1. $E = E^\bullet$
 1257 $\vee E = E'[\text{dyn } \tau E^\bullet]$
 1258 $\vee E = E'[\text{stat } \tau E^\bullet]$
 1259 by *inner boundary*
 1260 2. **IF** $E = E^\bullet$:
 1261 a. QED $e = E^\bullet[op^2 v_0 v_1]$
 1262 **IF** $E = E'[\text{dyn } \tau E^\bullet]$:
 1263 a. QED $e = E'[\text{dyn } \tau E^\bullet[op^2 v_0 v_1]]$
 1264 **ELSE** $E = E'[\text{stat } \tau E^\bullet]$:

12266 a. QED $e = E'[\text{stat } \tau E^\bullet[op^2 v_0 v_1]]$
CASE $e = E[\text{chk } \tau v]$:
 12267 1. $E = E^\bullet$
 12268 $\vee E = E'[\text{dyn } \tau E^\bullet]$
 12269 $\vee E = E'[\text{stat } \tau E^\bullet]$
 12270 by *inner boundary*
 12271 2. **IF** $E = E^\bullet$:
 12272 a. QED $e = E^\bullet[\text{chk } \tau v]$
 12273 **IF** $E = E'[\text{dyn } \tau E^\bullet]$:
 12274 a. Contradiction by $\vdash_F e : \tau$
 12275 **ELSE** $E = E'[\text{stat } \tau E^\bullet]$:
 12276 a. QED $e = E'[\text{stat } \tau E^\bullet[\text{chk } \tau v]]$
CASE $e = E[\text{dyn } \tau v]$:
 12278 1. QED v is boundary-free
CASE $e = E[\text{stat } \tau v]$:
 12280 1. QED v is boundary-free
CASE $e = E[\text{Err}]$:
 12282 1. QED
 12283 \square
Lemma 6.13 : HF unique static evaluation contexts
 12285 If $\vdash_F e : \tau$ then one of the following holds:
 12286 • e is a value
 12287 • $e = E[v_0 v_1]$
 12288 • $e = E[op^1 v]$
 12289 • $e = E[op^2 v_0 v_1]$
 12290 • $e = E[\text{chk } \tau v]$
 12291 • $e = E[\text{dyn } \tau v]$
 12292 • $e = E[\text{stat } \tau v]$
 12293 • $e = E[\text{Err}]$
 12294 *Proof:*
 12295 By induction on the structure of e .
CASE $e = x$
 12297 $\vee e = \lambda x. e'$
 12298 $\vee e = \text{stat } \tau e'$:
 12299 1. Contradiction by $\vdash_F e : \tau$
CASE $e = i$
 12301 $\vee e = \lambda(x:\tau_d). e'$
 12302 $\vee e = \text{mon } (\tau_d \Rightarrow \tau_c) v$:
 12303 1. QED e is a value
CASE $e = \langle e_0, e_1 \rangle$:
 12305 **IF** $e_0 \notin v$:
 12306 1. $\vdash_F e_0 : \tau_0$
 12307 by *inversion*
 12308 2. $e_0 = E_0[e'_0]$
 12309 by the induction hypothesis (1)
 12310 3. $E = \langle E_0, e_1 \rangle$
 12311 4. QED $e = E[e'_0]$
 12312 **IF** $e_0 \in v$
 12313 $\wedge e_1 \notin v$:
 12314 1. $\vdash_F e_1 : \tau_1$
 12315 by *inversion*
 12316 2. $e_1 = E_1[e'_1]$
 12317 by the induction hypothesis (1)
 12318 3. $E = \langle e_0, E_1 \rangle$
 12319 \square

```

12321   4. QED  $e = E[e'_1]$                                      12376
12322   ELSE  $e_0 \in v$                                          12377
12323      $\wedge e_1 \in v :$                                      12378
12324       1.  $E = []$                                          12379
12325       2. QED  $e$  is a value                                12380
12326 CASE  $e = e_0 e_1 :$                                      12381
12327   IF  $e_0 \notin v :$                                      12382
12328     1.  $\vdash_F e_0 : \tau_0$                                 12383
12329       by inversion                                    12384
12330     2.  $e_0 = E_0[e'_0]$                                  12385
12331       by the induction hypothesis (1)                  12386
12332     3.  $E = E_0 e_1$                                      12387
12333     4. QED  $e = E[e'_0]$                                 12388
12334 IF  $e_0 \in v$                                          12389
12335    $\wedge e_1 \notin v :$                                      12390
12336     1.  $\vdash_F e_1 : \tau_1$                                 12391
12337       by inversion                                    12392
12338     2.  $e_1 = E_1[e'_1]$                                  12393
12339       by the induction hypothesis (1)                  12394
12340     3.  $E = e_0 E_1$                                      12395
12341     4. QED  $e = E[e'_1]$                                 12396
12342 ELSE  $e_0 \in v$                                          12397
12343    $\wedge e_1 \in v :$                                      12398
12344     1.  $E = []$                                          12399
12345     2. QED  $e = E[e_0 e_1]$                                 12400
12346 CASE  $e = op^1 e_0 :$                                      12401
12347   IF  $e_0 \notin v :$                                      12402
12348     1.  $\vdash_F e_0 : \tau_0$                                 12403
12349       by inversion                                    12404
12350     2.  $e_0 = E_0[e'_0]$                                  12405
12351       by the induction hypothesis (1)                  12406
12352     3.  $E = op^1 E_0$                                      12407
12353     4. QED  $e = E[e'_0]$                                 12408
12354 ELSE  $e_0 \in v :$                                      12409
12355   1.  $E = []$                                          12410
12356   2. QED  $e = E[op^1 e_0]$                                 12411
12357 CASE  $e = op^2 e_0 e_1 :$                                      12412
12358   IF  $e_0 \notin v :$                                      12413
12359     1.  $\vdash_F e_0 : \tau_0$                                 12414
12360       by inversion                                    12415
12361     2.  $e_0 = E_0[e'_0]$                                  12416
12362       by the induction hypothesis (1)                  12417
12363     3.  $E = op^2 E_0 e_1$                                 12418
12364     4. QED  $e = E[e'_0]$                                 12419
12365 IF  $e_0 \in v$                                          12420
12366    $\wedge e_1 \notin v :$                                      12421
12367     1.  $\vdash_F e_1 : \tau_1$                                 12422
12368       by inversion                                    12423
12369     2.  $e_1 = E_1[e'_1]$                                  12424
12370       by the induction hypothesis (1)                  12425
12371     3.  $E = op^2 e_0 E_1$                                 12426
12372     4. QED  $e = E[e'_1]$                                 12427
12373 ELSE  $e_0 \in v$                                          12428
12374    $\wedge e_1 \in v :$                                      12429
12375
12321   1.  $E = []$                                          12376
12322   2. QED  $e = E[op^2 e_0 e_1]$                                 12377
12323 CASE  $e = \text{chk } \tau e_0 :$                                      12378
12324   IF  $e_0 \notin v :$                                      12379
12325     1.  $\vdash_F e_0 : \tau_0$                                 12380
12326       by inversion                                    12381
12327     2.  $e_0 = E_0[e'_0]$                                  12382
12328       by the induction hypothesis (1)                  12383
12329     3.  $E = \text{chk } \tau E_0$                                 12384
12330     4. QED  $e = E[e'_0]$                                 12385
12331 ELSE  $e_0 \in v :$                                      12386
12332   1.  $E = []$                                          12387
12333   2. QED  $e = E[\text{chk } \tau e_0]$                                 12388
12334 CASE  $e = \text{dyn } \tau e_0 :$                                      12389
12335   IF  $e_0 \notin v :$                                      12390
12336     1.  $\vdash_F e_0 : \tau_0$                                 12391
12337       by inversion                                    12392
12338     2.  $e_0 = E_0[e'_0]$                                  12393
12339       by unique dynamic evaluation contexts (1) 12394
12340     3.  $E = \text{dyn } \tau E_0$                                 12395
12341     4. QED  $e = E[e'_0]$                                 12396
12342 ELSE  $e_0 \in v :$                                      12397
12343   1.  $E = []$                                          12398
12344   2. QED  $e = E[\text{dyn } \tau e_0]$                                 12399
12345 CASE  $e = \text{Err} :$                                      12400
12346   1.  $E = []$                                          12401
12347   2. QED  $e = E[\text{Err}]$                                 12402
12348 □
12349 Lemma 6.14 : HF inner boundary
12350 For all contexts  $E$ , one of the following holds:
12351 •  $E = E^\bullet$ 
12352 •  $E = E'[\text{dyn } \tau E^\bullet]$ 
12353 •  $E = E'[\text{stat } \tau E^\bullet]$ 
12354 Proof:
12355 By induction on the structure of  $E$ .
12356 CASE  $E = E^\bullet :$ 
12357   1. QED
12358 CASE  $E = E_0 e_1 :$ 
12359   1.  $E_0 = E^\bullet$ 
12360      $\vee E_0 = E'_0[\text{dyn } \tau E^\bullet]$ 
12361      $\vee E_0 = E'_0[\text{stat } \tau E^\bullet]$ 
12362       by the induction hypothesis
12363   2. IF  $E_0 = E^\bullet :$ 
12364     a. QED  $E$  is boundary-free
12365     IF  $E_0 = E'_0[\text{dyn } \tau E^\bullet] :$ 
12366       a.  $E' = E'_0 e_1$ 
12367       b. QED  $E = E'[\text{dyn } \tau E^\bullet]$ 
12368     ELSE  $E_0 = E'_0[\text{stat } \tau E^\bullet] :$ 
12369       a.  $E' = E'_0 e_1$ 
12370       b. QED  $E = E'[\text{stat } \tau E^\bullet]$ 
12371 CASE  $E = v_0 E_1 :$ 
12372   1.  $E_1 = E^\bullet$ 
12373      $\vee E_1 = E'_1[\text{dyn } \tau E^\bullet]$ 
12374

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12431	$\vee E_1 = E'_1[\text{stat } \tau E^\bullet]$	12486
12432	by the induction hypothesis	12487
12433	2. IF $E_1 = E^\bullet$:	12488
12434	a. QED E is boundary-free	12489
12435	IF $E_1 = E'_1[\text{dyn } \tau E^\bullet]$:	12490
12436	a. $E' = v_0 E'_1$	12491
12437	b. QED $E = E'_1[\text{dyn } \tau E^\bullet]$	12492
12438	ELSE $E_1 = E'_1[\text{stat } \tau E^\bullet]$:	12493
12439	a. $E' = v_0 E'_1$	12494
12440	b. QED $E = E'_1[\text{stat } \tau E^\bullet]$	12495
12441	CASE $E = \langle E_0, e_1 \rangle$:	12496
12442	1. $E_0 = E^\bullet$	12497
12443	$\vee E_0 = E'_0[\text{dyn } \tau E^\bullet]$	12498
12444	$\vee E_0 = E'_0[\text{stat } \tau E^\bullet]$	12499
12445	by the induction hypothesis	12500
12446	2. IF $E_0 = E^\bullet$:	12501
12447	a. QED E is boundary-free	12502
12448	IF $E_0 = E'_0[\text{dyn } \tau E^\bullet]$:	12503
12449	a. $E' = \langle E'_0, e_1 \rangle$	12504
12450	b. QED $E = E'_0[\text{dyn } \tau E^\bullet]$	12505
12451	ELSE $E_0 = E'_0[\text{stat } \tau E^\bullet]$:	12506
12452	a. $E' = \langle E'_0, e_1 \rangle$	12507
12453	b. QED $E = E'_0[\text{stat } \tau E^\bullet]$	12508
12454	CASE $E = \langle v_0, E_1 \rangle$:	12509
12455	1. $E_1 = E^\bullet$	12510
12456	$\vee E_1 = E'_1[\text{dyn } \tau E^\bullet]$	12511
12457	$\vee E_1 = E'_1[\text{stat } \tau E^\bullet]$	12512
12458	by the induction hypothesis	12513
12459	2. IF $E_1 = E^\bullet$:	12514
12460	a. QED E is boundary-free	12515
12461	IF $E_1 = E'_1[\text{dyn } \tau E^\bullet]$:	12516
12462	a. $E' = \langle v_0, E'_1 \rangle$	12517
12463	b. QED $E = E'_1[\text{dyn } \tau E^\bullet]$	12518
12464	ELSE $E_1 = E'_1[\text{stat } \tau E^\bullet]$:	12519
12465	a. $E' = \langle v_0, E'_1 \rangle$	12520
12466	b. QED $E = E'_1[\text{stat } \tau E^\bullet]$	12521
12467	CASE $E = op^1 E_0$:	12522
12468	1. $E_0 = E^\bullet$	12523
12469	$\vee E_0 = E'_0[\text{dyn } \tau E^\bullet]$	12524
12470	$\vee E_0 = E'_0[\text{stat } \tau E^\bullet]$	12525
12471	by the induction hypothesis	12526
12472	2. IF $E_0 = E^\bullet$:	12527
12473	a. QED E is boundary-free	12528
12474	IF $E_0 = E'_0[\text{dyn } \tau E^\bullet]$:	12529
12475	a. $E' = op^1 E'_0$	12530
12476	b. QED $E = E'_0[\text{dyn } \tau E^\bullet]$	12531
12477	ELSE $E_0 = E'_0[\text{stat } \tau E^\bullet]$:	12532
12478	a. $E' = op^1 E'_0$	12533
12479	b. QED $E = E'_0[\text{stat } \tau E^\bullet]$	12534
12480	CASE $E = op^2 E_0 e_1$:	12535
12481	1. $E_0 = E^\bullet$	12536
12482	$\vee E_0 = E'_0[\text{dyn } \tau E^\bullet]$	12537
12483		12538
12484		12539
12485		12540

12541 $\vee E_0 = E'_0[\text{stat } \tau' E^\bullet]$
 12542 by the induction hypothesis
 12543 2. **IF** $E_0 = E^\bullet$:
 12544 a. QED
 12545 **IF** $E_0 = E'_0[\text{dyn } \tau' E^\bullet]$:
 12546 a. $E' = \text{stat } \tau E'_0$
 12547 b. QED $E = E'_0[\text{dyn } \tau' E^\bullet]$
 12548 **ELSE** $E_0 = E'_0[\text{stat } \tau' E^\bullet]$:
 12549 a. $E' = \text{stat } \tau E'_0$
 12550 b. QED $E = E'_0[\text{stat } \tau' E^\bullet]$
 12551 \square
 12552 **Lemma 6.15 : HF dynamic boundary factoring**
 12553 If $\vdash_F e$ then one of the following holds:
 12554 • e is a value
 12555 • $e = E^\bullet[v_0 v_1]$
 12556 • $e = E^\bullet[op^1 v]$
 12557 • $e = E^\bullet[op^2 v_0 v_1]$
 12558 • $e = E[\text{dyn } \tau e']$ where e' is boundary-free
 12559 • $e = E[\text{stat } \tau e']$ where e' is boundary-free
 12560 • $e = E[\text{Err}]$
 12561 *Proof:*
 12562 By the *unique dynamic evaluation contexts* lemma, there
 12563 are eight cases.
 12564 **CASE** e is a value :
 12565 1. QED
 12566 **CASE** $e = E[v_0 v_1]$:
 12567 1. $E = E^\bullet$
 12568 $\vee E = E'_0[\text{dyn } \tau E^\bullet]$
 12569 $\vee E = E'_0[\text{stat } \tau E^\bullet]$
 12570 by *inner boundary*
 12571 2. **IF** $E = E^\bullet$:
 12572 a. QED $e = E^\bullet[v_0 v_1]$
 12573 **IF** $E = E'_0[\text{dyn } \tau E^\bullet]$:
 12574 a. QED $e = E'_0[\text{dyn } \tau E^\bullet[v_0 v_1]]$
 12575 **ELSE** $E = E'_0[\text{stat } \tau E^\bullet]$:
 12576 a. QED $e = E'_0[\text{stat } \tau E^\bullet[v_0 v_1]]$
 12577 **CASE** $e = E[op^1 v]$:
 12578 1. $E = E^\bullet$
 12579 $\vee E = E'_0[\text{dyn } \tau E^\bullet]$
 12580 $\vee E = E'_0[\text{stat } \tau E^\bullet]$
 12581 by *inner boundary*
 12582 2. **IF** $E = E^\bullet$:
 12583 a. QED $e = E^\bullet[op^1 v]$
 12584 **IF** $E = E'_0[\text{dyn } \tau E^\bullet]$:
 12585 a. QED $e = E'_0[\text{dyn } \tau E^\bullet[op^1 v]]$
 12586 **ELSE** $E = E'_0[\text{stat } \tau E^\bullet]$:
 12587 a. QED $e = E'_0[\text{stat } \tau E^\bullet[op^1 v]]$
 12588 **CASE** $e = E[op^2 v_0 v_1]$:
 12589 1. $E = E^\bullet$
 12590 $\vee E = E'_0[\text{dyn } \tau E^\bullet]$
 12591 $\vee E = E'_0[\text{stat } \tau E^\bullet]$
 12592 by *inner boundary*
 12593 2. **IF** $E = E^\bullet$:
 12594 a. QED $e = E^\bullet[op^2 v_0 v_1]$
 12595

12596 **IF** $E = E'_0[\text{dyn } \tau E^\bullet]$:
 12597 a. QED $e = E'_0[\text{dyn } \tau E^\bullet[op^2 v_0 v_1]]$
 12598 **ELSE** $E = E'_0[\text{stat } \tau E^\bullet]$:
 12599 a. QED $e = E'_0[\text{stat } \tau E^\bullet[op^2 v_0 v_1]]$
 12600 **CASE** $e = E[\text{chk } \tau' v]$:
 12601 1. $E = E^\bullet$
 12602 $\vee E = E'_0[\text{dyn } \tau E^\bullet]$
 12603 $\vee E = E'_0[\text{stat } \tau E^\bullet]$
 12604 by *inner boundary*
 12605 2. **IF** $E = E^\bullet$:
 12606 a. Contradiction by $\vdash_F e$
 12607 **IF** $E = E'_0[\text{dyn } \tau E^\bullet]$:
 12608 a. Contradiction by $\vdash_F e$
 12609 **ELSE** $E = E'_0[\text{stat } \tau E^\bullet]$:
 12610 a. QED $e = E'_0[\text{stat } \tau E^\bullet[\text{chk } \tau' v]]$
 12611 **CASE** $e = E[\text{dyn } \tau v]$:
 12612 1. QED v is boundary-free
 12613 **CASE** $e = E[\text{stat } \tau v]$:
 12614 1. QED v is boundary-free
 12615 **CASE** $e = E[\text{Err}]$:
 12616 1. QED
 12617 \square
 12618 **Lemma 6.16 : HF unique dynamic evaluation contexts**
 12619 If $\vdash_F e$ then one of the following holds:
 12620 • e is a value
 12621 • $e = E[v_0 v_1]$
 12622 • $e = E[op^1 v]$
 12623 • $e = E[op^2 v_0 v_1]$
 12624 • $e = E[\text{chk } \tau v]$
 12625 • $e = E[\text{dyn } \tau v]$
 12626 • $e = E[\text{stat } \tau v]$
 12627 • $e = E[\text{Err}]$
 12628 *Proof:*
 12629 By induction on the structure of e .
 12630 **CASE** $e = x$
 12631 $\vee e = \lambda(x:\tau). e'$
 12632 $\vee e = \text{dyn } \tau e'$
 12633 1. Contradiction by $\vdash_F e$
 12634 **CASE** $e = i$
 12635 $\vee e = \lambda x. e'$
 12636 $\vee e = \text{mon}(\tau_d \Rightarrow \tau_c) v$:
 12637 1. QED e is a value
 12638 **CASE** $e = \text{Err}$:
 12639 1. $E = []$
 12640 2. QED $e = E[\text{Err}]$
 12641 **CASE** $e = \langle e_0, e_1 \rangle$:
 12642 **IF** $e_0 \notin v$:
 12643 1. $\vdash_F e_0$
 12644 by *inversion*
 12645 2. $e_0 = E_0[e'_0]$
 12646 by the induction hypothesis (1)
 12647 3. $E = \langle E_0, e_1 \rangle$
 12648 4. QED $e = E[e'_0]$
 12649
 12650

12651	IF $e_0 \in v$	1. $\vdash_F e_1$	12706
12652	$\wedge e_1 \notin v :$	by <i>inversion</i>	12707
12653	1. $\vdash_F e_1$	2. $e_1 = E_1[e'_1]$	12708
12654	by <i>inversion</i>	by the induction hypothesis (1)	12709
12655	2. $e_1 = E_1[e'_1]$	3. $E = op^2 e_0 E_1$	12710
12656	by the induction hypothesis (1)	4. QED $e = E[e'_1]$	12711
12657	3. $E = \langle e_0, E_1 \rangle$	ELSE $e_0 \in v$	12712
12658	4. QED $e = E[e'_1]$	$\wedge e_1 \in v :$	12713
12659	ELSE $e_0 \in v$	1. $E = []$	12714
12660	$\wedge e_1 \in v :$	2. QED $e = E[op^2 e_0 e_1]$	12715
12661	1. $E = []$	CASE $e = \text{chk } \tau e_0 :$	12716
12662	2. QED e is a value	Contradiction by $\vdash_F e$	12717
12663	CASE $e = e_0 e_1 :$	CASE $e = \text{stat } \tau e_0 :$	12718
12664	IF $e_0 \notin v :$	IF $e_0 \notin v :$	12719
12665	1. $\vdash_F e_0$	1. $\vdash_F e_0$	12720
12666	by <i>inversion</i>	by <i>inversion</i>	12721
12667	2. $e_0 = E_0[e'_0]$	2. $e_0 = E_0[e'_0]$	12722
12668	by the induction hypothesis (1)	by <i>unique static evaluation contexts</i> (1)	12723
12669	3. $E = E_0 e_1$	3. $E = \text{stat } \tau E_0$	12724
12670	4. QED $e = E[e'_0]$	4. QED $e = E[e'_0]$	12725
12671	IF $e_0 \in v$	ELSE $e_0 \in v :$	12726
12672	$\wedge e_1 \notin v :$	1. $E = []$	12727
12673	1. $\vdash_F e_1$	2. QED $e = E[\text{stat } \tau e_0]$	12728
12674	by <i>inversion</i>	□	12729
12675	2. $e_1 = E_1[e'_1]$		
12676	by the induction hypothesis (1)		
12677	3. $E = e_0 E_1$		
12678	4. QED $e = E[e'_1]$		
12679	ELSE $e_0 \in v$		
12680	$\wedge e_1 \in v :$		
12681	1. $E = []$		
12682	2. QED $e = E[e_0 e_1]$		
12683	CASE $e = op^1 e_0 :$		
12684	IF $e_0 \notin v :$		
12685	1. $\vdash_F e_0$		
12686	by <i>inversion</i>		
12687	2. $e_0 = E_0[e'_0]$		
12688	by the induction hypothesis (1)		
12689	3. $E = op^1 E_0$		
12690	4. QED $e = E[e'_0]$		
12691	ELSE $e_0 \in v :$		
12692	1. $E = []$		
12693	2. QED $e = E[op^1 e_0]$		
12694	CASE $e = op^2 e_0 e_1 :$		
12695	IF $e_0 \notin v :$		
12696	1. $\vdash_F e_0$		
12697	by <i>inversion</i>		
12698	2. $e_0 = E_0[e'_0]$		
12699	by the induction hypothesis (1)		
12700	3. $E = op^2 E_0 e_1$		
12701	4. QED $e = E[e'_0]$		
12702	IF $e_0 \in v$		
12703	$\wedge e_1 \notin v :$		
12704			12759
12705			12760

2. $\vdash_{\mathbb{F}} E^{\bullet}_0[e] : \tau_0$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = op^2 E^{\bullet}_0 e_1 :$

1. $E^{\bullet}[e] = op^2 E^{\bullet}_0[e] e_1$
2. $\vdash_{\mathbb{F}} E^{\bullet}_0[e] : \tau_0$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = op^2 v_0 E^{\bullet}_1 :$

1. $E^{\bullet}[e] = op^2 v_0 E^{\bullet}_1[e]$
2. $\vdash_{\mathbb{F}} E^{\bullet}_1[e] : \tau_1$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = \text{chk } \tau E^{\bullet}_0 :$

1. $E^{\bullet}[e] = \text{chk } \tau E^{\bullet}_0[e]$
2. $\vdash_{\mathbb{F}} E^{\bullet}_0[e] : \tau_0$
by *inversion*
3. QED by the induction hypothesis (2)

Lemma 6.18 : HF dynamic hole typing

If $\vdash_F E^\bullet[e]$ then the derivation contains a *Proof*:

By induction on the structure of E^\bullet .

CASE $E^\bullet = []$:

- QED $E^\bullet[e] = e$

CASE $E^\bullet = E^\bullet_0 e_1$:

- $E^\bullet[e] = E^\bullet_0[e] e_1$
- $\vdash_F E^\bullet_0[e]$
by *inversion*
- QED by the induction hypothesis (2)

CASE $E^\bullet = v_0 E^\bullet_1$:

- $E^\bullet[e] = v_0 E^\bullet_1[e]$
- $\vdash_F E^\bullet_1[e]$
by *inversion*
- QED by the induction hypothesis (2)

CASE $E^\bullet = \langle E^\bullet_0, e_1 \rangle$:

- $E^\bullet[e] = \langle E^\bullet_0[e], e_1 \rangle$
- $\vdash_F E^\bullet_0[e]$
by *inversion*
- QED by the induction hypothesis (2)

CASE $E^\bullet = \langle v_0, E^\bullet_1 \rangle$:

- $E^\bullet[e] = \langle v_0, E^\bullet_1[e] \rangle$
- $\vdash_F E^\bullet_1[e]$
by *inversion*
- QED by the induction hypothesis (2)

CASE $E^\bullet = op^1 E^\bullet_0$:

- $E^\bullet[e] = op^1 E^\bullet_0[e]$
- $\vdash_F E^\bullet_0[e]$
by *inversion*
- QED by the induction hypothesis (2)

CASE $E^\bullet = op^2 E^\bullet_0 e_1$:

- $E^\bullet[e] = op^2 E^\bullet_0[e] e_1$

2. $\vdash_F E^{\bullet}_0[e]$
by *inversion*

3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = op^2 v_0 E^{\bullet}_1 :$

1. $E^{\bullet}[e] = op^2 v_0 E^{\bullet}_1[e]$
2. $\vdash_F E^{\bullet}_1[e]$
by *inversion*
3. QED by the induction hypothesis (2)

□

Lemma 6.19 : HF boundary hole typing

- If $\vdash_F E[\text{dyn } \tau e] : \tau'$ then the derivation contains a sub-term
 $\vdash_F \text{dyn } \tau e : \tau$
- If $\vdash_F E[\text{dyn } \tau e]$ then the derivation contains a sub-term
 $\vdash_F \text{dyn } \tau e : \tau$
- If $\vdash_F E[\text{stat } \tau e] : \tau'$ then the derivation contains a sub-term
 $\vdash_F \text{stat } \tau e$
- If $\vdash_F E[\text{stat } \tau e]$ then the derivation contains a sub-term
 $\vdash_F \text{stat } \tau e$

Proof:

Proof: By the following four lemmas: *static dyn hole typing*, *dynamic dyn hole typing*, *static stat hole typing*, and *dynamic stat hole typing*. 12834 12835 12836 12837

Lemma 6.20 : HF static dyn hole typing	12839
If $\vdash_{\text{F}} E[\text{dyn } \tau e] : \tau'$ then the derivation contains a sub-term	12840
$\vdash_{\text{F}} \text{dyn } \tau e : \tau$.	12841
<i>Proof:</i>	12842
By induction on the structure of E .	12843
CASE $E \in E^*$:	12844
1. $\vdash_{\text{F}} \text{dyn } \tau e : \tau''$ by <i>static hole typing</i>	12845
2. $\vdash_{\text{F}} \text{dyn } \tau e : \tau$ by <i>inversion</i> (1)	12847
3. QED	12849
CASE $E = E_0 e_1$:	12850
1. $E[\text{dyn } \tau e] = E_0[\text{dyn } \tau e] e_1$	12851
2. $\vdash_{\text{F}} E_0[\text{dyn } \tau e] : \tau_0$ by <i>inversion</i>	12852
3. QED by the induction hypothesis (2)	12854
CASE $E = v_0 E_1$:	12855
1. $E[\text{dyn } \tau e] = v_0 E_1[\text{dyn } \tau e]$	12856
2. $\vdash_{\text{F}} E_1[\text{dyn } \tau e] : \tau_1$ by <i>inversion</i>	12857
3. QED by the induction hypothesis (2)	12859
CASE $E = \langle E_0, e_1 \rangle$:	12860
1. $E[\text{dyn } \tau e] = \langle E_0[\text{dyn } \tau e], e_1 \rangle$	12861
2. $\vdash_{\text{F}} E_0[\text{dyn } \tau e] : \tau_0$ by <i>inversion</i>	12862
3. QED by the induction hypothesis (2)	12864
CASE $E = \langle v_0, E_1 \rangle$:	12865
1. $E[\text{dyn } \tau e] = \langle v_0, E_1[\text{dyn } \tau e] \rangle$	12866
2. $\vdash_{\text{F}} E_1[\text{dyn } \tau e] : \tau_1$ by <i>inversion</i>	12867
	12868

12871 3. QED by the induction hypothesis (2)
CASE $E = op^1$:
12872 1. $E[\text{dyn } \tau e] = op^1 E_0[\text{dyn } \tau e]$
12873 2. $\vdash_F E_0[\text{dyn } \tau e] : \tau_0$
12874 by *inversion*
12875 3. QED by the induction hypothesis (2)
CASE $E = op^2 E_0 e_1$:
12876 1. $E[\text{dyn } \tau e] = op^2 E_0[\text{dyn } \tau e] e_1$
12877 2. $\vdash_F E_0[\text{dyn } \tau e] : \tau_0$
12878 by *inversion*
12879 3. QED by the induction hypothesis (2)
CASE $E = op^2 v_0 E_1$:
12880 1. $E[\text{dyn } \tau e] = op^2 v_0 E_1[\text{dyn } \tau e]$
12881 2. $\vdash_F E_1[\text{dyn } \tau e] : \tau_1$
12882 by *inversion*
12883 3. QED by the induction hypothesis (2)
CASE $E = \text{chk } \tau'' E_0$:
12884 1. $E[\text{dyn } \tau e] = \text{chk } \tau'' E_0[\text{dyn } \tau e]$
12885 2. $\vdash_F E_0[\text{dyn } \tau e] : \tau_0$
12886 by *inversion*
12887 3. QED by the induction hypothesis (2)
CASE $E = \text{dyn } \tau_0 E_0$:
12888 1. $E[\text{dyn } \tau e] = \text{dyn } \tau_0 E_0[\text{dyn } \tau e]$
12889 2. $\vdash_F E_0[\text{dyn } \tau e] : \tau_0$
12890 by *inversion*
12891 3. QED by the induction hypothesis (2)
CASE $E = \text{stat } \tau_0 E_0$:
12892 1. Contradiction by $\vdash_F E[\text{dyn } \tau e]$
12893 □

Lemma 6.21 : HF dynamic dyn hole typing

If $\vdash_F E[\text{dyn } \tau e]$ then the derivation contains a sub-term
 $\vdash_F \text{dyn } \tau e : \tau$.

Proof:

By induction on the structure of E .

CASE $E \in E^\bullet$:
1. Contradiction by $\vdash_F E[\text{dyn } \tau e]$

CASE $E = E_0 e_1$:
1. $E[\text{dyn } \tau e] = E_0[\text{dyn } \tau e] e_1$

2. $\vdash_F E_0[\text{dyn } \tau e]$
by *inversion*

3. QED by the induction hypothesis (2)

CASE $E = v_0 E_1$:
1. $E[\text{dyn } \tau e] = v_0 E_1[\text{dyn } \tau e]$

2. $\vdash_F E_1[\text{dyn } \tau e]$
by *inversion*

3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[\text{dyn } \tau e] = \langle E_0[\text{dyn } \tau e], e_1 \rangle$

2. $\vdash_F E_0[\text{dyn } \tau e]$
by *inversion*

3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[\text{dyn } \tau e] = \langle v_0, E_1[\text{dyn } \tau e] \rangle$

12925

2. $\vdash_F E_1[\text{dyn } \tau e]$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1 E_0$:
1. $E[\text{dyn } \tau e] = op^1 E_0[\text{dyn } \tau e]$
2. $\vdash_F E_0[\text{dyn } \tau e]$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2 E_0 e_1$:
1. $E[\text{dyn } \tau e] = op^2 E_0[\text{dyn } \tau e] e_1$
2. $\vdash_F E_0[\text{dyn } \tau e]$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2 v_0 E_1$:
1. $E[\text{dyn } \tau e] = op^2 v_0 E_1[\text{dyn } \tau e]$
2. $\vdash_F E_1[\text{dyn } \tau e]$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \text{chk } \tau'' E_0$:
1. Contradiction by $\vdash_F E[\text{dyn } \tau e]$

□

CASE $E = \text{dyn } \tau E_0$:
1. Contradiction by $\vdash_F E[\text{dyn } \tau e]$

CASE $E = \text{stat } \tau_0 E_0$:
1. $E[\text{dyn } \tau e] = \text{stat } \tau_0 E_0[\text{dyn } \tau e]$
2. $\vdash_F E_0[\text{dyn } \tau e] : \tau_0$
by *inversion*
3. QED by static dyn hole typing (2)

}

Lemma 6.22 : HF static stat hole typing

If $\vdash_F E[\text{stat } \tau e] : \tau'$ then the derivation contains a sub-term
 $\vdash_F \text{stat } \tau e$.

Proof:

By induction on the structure of E .

CASE $E \in E^\bullet$:
1. Contradiction by $\vdash_F E[\text{stat } \tau e] : \tau'$

CASE $E = E_0 e_1$:
1. $E[\text{stat } \tau e] = E_0[\text{stat } \tau e] e_1$
2. $\vdash_F E_0[\text{stat } \tau e] : \tau_0$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0 E_1$:
1. $E[\text{stat } \tau e] = v_0 E_1[\text{stat } \tau e]$
2. $\vdash_F E_1[\text{stat } \tau e] : \tau_1$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[\text{stat } \tau e] = \langle E_0[\text{stat } \tau e], e_1 \rangle$
2. $\vdash_F E_0[\text{stat } \tau e] : \tau_0$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[\text{stat } \tau e] = \langle v_0, E_1[\text{stat } \tau e] \rangle$

12981	2. $\vdash_F E_1[\text{stat } \tau e] : \tau_1$	13036
12982	by <i>inversion</i>	13037
12983	3. QED by the induction hypothesis (2)	13038
12984	CASE $E = op^1 E_0 :$	13039
12985	1. $E[\text{stat } \tau e] = op^1 E_0[\text{stat } \tau e]$	13040
12986	2. $\vdash_F E_0[\text{stat } \tau e] : \tau_0$	13041
12987	by <i>inversion</i>	13042
12988	3. QED by the induction hypothesis (2)	13043
12989	CASE $E = op^2 E_0 e_1 :$	13044
12990	1. $E[\text{stat } \tau e] = op^2 E_0[\text{stat } \tau e] e_1$	13045
12991	2. $\vdash_F E_0[\text{stat } \tau e] : \tau_0$	13046
12992	by <i>inversion</i>	13047
12993	3. QED by the induction hypothesis (2)	13048
12994	CASE $E = op^2 v_0 E_1 :$	13049
12995	1. $E[\text{stat } \tau e] = op^2 v_0 E_1[\text{stat } \tau e]$	13050
12996	2. $\vdash_F E_1[\text{stat } \tau e] : \tau_1$	13051
12997	by <i>inversion</i>	13052
12998	3. QED by the induction hypothesis (2)	13053
12999	CASE $E = \text{chk } \tau'' E_0 :$	13054
13000	1. $E[\text{stat } \tau e] = \text{chk } \tau'' E_0[\text{stat } \tau e]$	13055
13001	2. $\vdash_F E_0[\text{stat } \tau e] : \tau_0$	13056
13002	by <i>inversion</i>	13057
13003	3. QED by the induction hypothesis (2)	13058
13004	CASE $E = \text{dyn } \tau_0 E_0 :$	13059
13005	1. $E[\text{stat } \tau e] = \text{dyn } \tau_0 E_0[\text{stat } \tau e]$	13060
13006	2. $\vdash_F E_0[\text{stat } \tau e]$	13061
13007	by <i>inversion</i>	13062
13008	3. QED by <i>dynamic stat hole typing</i> (2)	13063
13009	CASE $E = \text{stat } \tau_0 E_0 :$	13064
13010	1. Contradiction by $\vdash_F E[\text{stat } \tau e] : \tau'$	13065
13011	□	
13012	Lemma 6.23 : HF dynamic stat hole typing	13065
13013	If $\vdash_F E[\text{stat } \tau e]$ then the derivation contains a sub-term $\vdash_F \text{stat } \tau e$.	13066
13014		13067
13015	<i>Proof:</i>	13068
13016	By induction on the structure of E .	13069
13017	CASE $E \in E^\bullet :$	13070
13018	1. QED by <i>dynamic hole typing</i>	13071
13019	CASE $E = E_0 e_1 :$	13072
13020	1. $E[\text{stat } \tau e] = E_0[\text{stat } \tau e] e_1$	13073
13021	2. $\vdash_F E_0[\text{stat } \tau e]$	13074
13022	by <i>inversion</i>	13075
13023	3. QED by the induction hypothesis (2)	13076
13024	CASE $E = v_0 E_1 :$	13077
13025	1. $E[\text{stat } \tau e] = v_0 E_1[\text{stat } \tau e]$	13078
13026	2. $\vdash_F E_1[\text{stat } \tau e]$	13079
13027	by <i>inversion</i>	13080
13028	3. QED by the induction hypothesis (2)	13081
13029	CASE $E = \langle E_0, e_1 \rangle :$	13082
13030	1. $E[\text{stat } \tau e] = \langle E_0[\text{stat } \tau e], e_1 \rangle$	13083
13031	2. $\vdash_F E_0[\text{stat } \tau e]$	13084
13032	by <i>inversion</i>	13085
13033	3. QED by the induction hypothesis (2)	13086
13034	CASE $E = \langle v_0, E_1 \rangle :$	13087
13035		13088

13091	6. QED by (1, 5)	13146
13092	CASE $E^\bullet = v_0 E^\bullet_1 :$	
13093	1. $E^\bullet[e] = v_0 E^\bullet_1[e]$	
13094	$\wedge E^\bullet[e'] = v_0 E^\bullet_1[e']$	
13095	2. $\vdash_F v_0 E^\bullet_1[e] : \tau$	
13096	3. $\vdash_F v_0 : \tau_0$	
13097	$\wedge \vdash_F E^\bullet_1[e] : \tau_1$	
13098	by <i>inversion</i>	
13099	4. $\vdash_F E^\bullet_1[e'] : \tau_1$	
13100	by the induction hypothesis (3)	
13101	5. $\vdash_F v_0 E^\bullet_1[e'] : \tau$	
13102	by (2, 3, 4)	
13103	6. QED by (1, 5)	
13104	CASE $E^\bullet = op^1 E^\bullet_0 :$	
13105	1. $E^\bullet[e] = op^1 E^\bullet_0[e]$	
13106	$\wedge E^\bullet[e'] = op^1 E^\bullet_0[e']$	
13107	2. $\vdash_F op^1 E^\bullet_0[e] : \tau$	
13108	3. $\vdash_F E^\bullet_0[e] : \tau_0$	
13109	by <i>inversion</i>	
13110	4. $\vdash_F E^\bullet_0[e'] : \tau_0$	
13111	by the induction hypothesis (3)	
13112	5. $\vdash_F op^1 E^\bullet_0[e'] : \tau$	
13113	by (2, 3, 4)	
13114	6. QED by (1, 5)	
13115	CASE $E^\bullet = \langle E^\bullet_0, e_1 \rangle :$	
13116	1. $E^\bullet[e] = \langle E^\bullet_0[e], e_1 \rangle$	
13117	$\wedge E^\bullet[e'] = \langle E^\bullet_0[e'], e_1 \rangle$	
13118	2. $\vdash_F \langle E^\bullet_0[e], e_1 \rangle : \tau$	
13119	3. $\vdash_F E^\bullet_0[e] : \tau_0$	
13120	$\wedge \vdash_F e_1 : \tau_1$	
13121	by <i>inversion</i>	
13122	4. $\vdash_F E^\bullet_0[e'] : \tau_0$	
13123	by the induction hypothesis (3)	
13124	5. $\vdash_F \langle E^\bullet_0[e'], e_1 \rangle : \tau$	
13125	by (2, 3, 4)	
13126	6. QED by (1, 5)	
13127	CASE $E^\bullet = \langle v_0, E^\bullet_1 \rangle :$	
13128	1. $E^\bullet[e] = \langle v_0, E^\bullet_1[e] \rangle$	
13129	$\wedge E^\bullet[e'] = \langle v_0, E^\bullet_1[e'] \rangle$	
13130	2. $\vdash_F \langle v_0, E^\bullet_1[e] \rangle : \tau$	
13131	3. $\vdash_F v_0 : \tau_0$	
13132	$\wedge \vdash_F E^\bullet_1[e] : \tau_1$	
13133	by <i>inversion</i>	
13134	4. $\vdash_F E^\bullet_1[e'] : \tau_1$	
13135	by the induction hypothesis (3)	
13136	5. $\vdash_F \langle v_0, E^\bullet_1[e'] \rangle : \tau$	
13137	by (2, 3, 4)	
13138	6. QED by (1, 5)	
13139	CASE $E^\bullet = op^2 E^\bullet_0 e_1 :$	
13140	1. $E^\bullet[e] = op^2 E^\bullet_0[e] e_1$	
13141	$\wedge E^\bullet[e'] = op^2 E^\bullet_0[e'] e_1$	
13142	2. $\vdash_F op^2 E^\bullet_0[e] e_1 : \tau$	
13143		
13144		
13145		
	3. $\vdash_F E^\bullet_0[e] : \tau_0$	13146
	$\wedge \vdash_F e_1 : \tau_1$	13147
	by <i>inversion</i>	13148
	4. $\vdash_F E^\bullet_0[e'] : \tau_0$	13149
	by the induction hypothesis (3)	13150
	5. $\vdash_F op^2 E^\bullet_0[e'] e_1 : \tau$	13151
	by (2, 3, 4)	13152
	6. QED by (1, 5)	13153
	CASE $E^\bullet = op^2 v_0 E^\bullet_1 :$	13154
	1. $E^\bullet[e] = op^2 v_0 E^\bullet_1[e]$	13155
	$\wedge E^\bullet[e'] = op^2 v_0 E^\bullet_1[e']$	13156
	2. $\vdash_F op^2 v_0 E^\bullet_1[e] : \tau$	13157
	3. $\vdash_F v_0 : \tau_0$	13158
	$\wedge \vdash_F E^\bullet_1[e] : \tau_1$	13159
	by <i>inversion</i>	13160
	4. $\vdash_F E^\bullet_1[e'] : \tau_1$	13161
	by the induction hypothesis (3)	13162
	5. $\vdash_F op^2 v_0 E^\bullet_1[e'] : \tau$	13163
	by (2, 3, 4)	13164
	6. QED by (1, 5)	13165
	CASE $E^\bullet = chk \tau'' E^\bullet_0 :$	13166
	1. $E^\bullet[e] = chk \tau'' E^\bullet_0[e]$	13167
	$\wedge E^\bullet[e'] = chk \tau'' E^\bullet_0[e']$	13168
	2. $\vdash_F chk \tau'' E^\bullet_0[e] : \tau$	13169
	3. $\vdash_F E^\bullet_0[e] : \tau_0$	13170
	by <i>inversion</i>	13171
	4. $\vdash_F E^\bullet_0[e'] : \tau_0$	13172
	by the induction hypothesis (3)	13173
	5. $\vdash_F chk \tau'' E^\bullet_0[e'] : \tau$	13174
	by (2, 3, 4)	13175
	6. QED by (1, 5)	13176
	□	13177
	Lemma 6.25 : HF dynamic hole substitution	13178
	If $\vdash_F E^\bullet[e]$ and $\vdash_F e'$ then $\vdash_F E^\bullet[e']$	13179
	<i>Proof:</i>	13180
	By induction on the structure of E^\bullet	13181
	CASE $E^\bullet = [] :$	13182
	1. QED $E^\bullet[e'] = e'$	13183
	CASE $E^\bullet = \langle E^\bullet_0, e_1 \rangle :$	13184
	1. $E^\bullet[e] = \langle E^\bullet_0[e], e_1 \rangle$	13185
	$\wedge E^\bullet[e'] = \langle E^\bullet_0[e'], e_1 \rangle$	13186
	2. $\vdash_F \langle E^\bullet_0[e], e_1 \rangle : \tau$	13187
	3. $\vdash_F E^\bullet_0[e] : \tau_0$	13188
	$\wedge \vdash_F e_1 : \tau_1$	13189
	by <i>inversion</i>	13190
	4. $\vdash_F E^\bullet_0[e'] : \tau_1$	13191
	by the induction hypothesis (3)	13192
	5. $\vdash_F \langle E^\bullet_0[e'], e_1 \rangle : \tau$	13193
	by (3, 4)	13194
	6. QED by (1, 5)	13195
	CASE $E^\bullet = \langle v_0, E^\bullet_1 \rangle :$	13196
	1. $E^\bullet[e] = \langle v_0, E^\bullet_1[e] \rangle$	13197
	$\wedge E^\bullet[e'] = \langle v_0, E^\bullet_1[e'] \rangle$	13198
	2. $\vdash_F \langle v_0, E^\bullet_1[e] \rangle : \tau$	13199
		13200

13201	2. $\vdash_F \langle v_0, E^{\bullet}_1[e] \rangle$	13256
13202	3. $\vdash_F v_0$	13257
13203	$\wedge \vdash_F E^{\bullet}_1[e]$	13258
13204	by <i>inversion</i>	13259
13205	4. $\vdash_F E^{\bullet}_1[e']$	13260
13206	by the induction hypothesis (3)	13261
13207	5. $\vdash_F \langle v_0, E^{\bullet}_1[e'] \rangle$	13262
13208	by (3, 4)	13263
13209	6. QED by (1, 5)	13264
13210	CASE $E^{\bullet} = E^{\bullet}_0 e_1 :$	13265
13211	1. $E^{\bullet}[e] = E^{\bullet}_0[e] e_1$	13266
13212	$\wedge E^{\bullet}[e'] = E^{\bullet}_0[e'] e_1$	13267
13213	2. $\vdash_F E^{\bullet}_0[e] e_1$	13268
13214	3. $\vdash_F E^{\bullet}_0[e]$	13269
13215	$\wedge \vdash_F e_1$	13270
13216	by <i>inversion</i>	13271
13217	4. $\vdash_F E^{\bullet}_0[e']$	13272
13218	by the induction hypothesis (3)	13273
13219	5. $\vdash_F E^{\bullet}_0[e'] e_1$	13274
13220	by (3, 4)	13275
13221	6. QED by (1, 5)	13276
13222	CASE $E^{\bullet} = v_0 E^{\bullet}_1 :$	13277
13223	1. $E^{\bullet}[e] = v_0 E^{\bullet}_1[e]$	13278
13224	$\wedge E^{\bullet}[e'] = v_0 E^{\bullet}_1[e']$	13279
13225	2. $\vdash_F v_0 E^{\bullet}_1[e]$	13280
13226	3. $\vdash_F v_0$	13281
13227	$\wedge \vdash_F E^{\bullet}_1[e]$	13282
13228	by <i>inversion</i>	13283
13229	4. $\vdash_F E^{\bullet}_1[e']$	13284
13230	by the induction hypothesis (3)	13285
13231	5. $\vdash_F v_0 E^{\bullet}_1[e']$	13286
13232	by (3, 4)	13287
13233	6. QED by (1, 5)	13288
13234	CASE $E^{\bullet} = op^1 E^{\bullet}_0 :$	13289
13235	1. $E^{\bullet}[e] = op^1 E^{\bullet}_0[e]$	13290
13236	$\wedge E^{\bullet}[e'] = op^1 E^{\bullet}_0[e']$	13291
13237	2. $\vdash_F op^1 E^{\bullet}_0[e]$	13292
13238	3. $\vdash_F E^{\bullet}_0[e]$	13293
13239	by <i>inversion</i>	13294
13240	4. $\vdash_F E^{\bullet}_0[e']$	13295
13241	by the induction hypothesis (3)	13296
13242	5. $\vdash_F op^1 E^{\bullet}_0[e']$	13297
13243	by (4)	13298
13244	6. QED by (1, 5)	13299
13245	CASE $E^{\bullet} = op^2 E^{\bullet}_0 e_1 :$	13300
13246	1. $E^{\bullet}[e] = op^2 E^{\bullet}_0[e] e_1$	13301
13247	$\wedge E^{\bullet}[e'] = op^2 E^{\bullet}_0[e'] e_1$	13302
13248	2. $\vdash_F op^2 E^{\bullet}_0[e] e_1$	13303
13249	3. $\vdash_F E^{\bullet}_0[e]$	13304
13250	$\wedge \vdash_F e_1$	13305
13251	by <i>inversion</i>	13306
13252	4. $\vdash_F E^{\bullet}_0[e']$	13307
13253	by the induction hypothesis (3)	13308
13254		13309
13255		13310

13311	2. $\vdash_F E_0[e]$	13366
13312	by <i>inversion</i>	13367
13313	3. QED by the induction hypothesis (2)	13368
13314	CASE $E = \langle v_0, E_1 \rangle :$	13369
13315	1. $E[e] = \langle v_0, E_1[e] \rangle$	13370
13316	2. $\vdash_F E_1[e]$	13371
13317	by <i>inversion</i>	13372
13318	3. QED by the induction hypothesis (2)	13373
13319	CASE $E = op^1 E_0 :$	13374
13320	1. $E[e] = op^1 E_0[e]$	13375
13321	2. $\vdash_F E_0[e]$	13376
13322	by <i>inversion</i>	13377
13323	3. QED by the induction hypothesis (2)	13378
13324	CASE $E = op^2 E_0 e_1 :$	13379
13325	1. $E[e] = op^2 E_0[e] e_1$	13380
13326	2. $\vdash_F E_0[e]$	13381
13327	by <i>inversion</i>	13382
13328	3. QED by the induction hypothesis (2)	13383
13329	CASE $E = op^2 v_0 E_1 :$	13384
13330	1. $E[e] = op^2 v_0 E_1[e]$	13385
13331	2. $\vdash_F E_1[e]$	13386
13332	by <i>inversion</i>	13387
13333	3. QED by the induction hypothesis (2)	13388
13334	CASE $E = \text{chk } \tau'' E_0 :$	13389
13335	1. $E[e] = \text{chk } \tau'' E_0[e]$	13390
13336	2. $\vdash_F E_0[e]$	13391
13337	by <i>inversion</i>	13392
13338	3. QED by the induction hypothesis (2)	13393
13339	CASE $E = \text{dyn } \tau'' E_0 :$	13394
13340	1. Contradiction by $\vdash_F E[e]$	13395
13341	CASE $E = \text{stat } \tau_0 E_0 :$	13396
13342	1. $E[e] = \text{stat } \tau_0 E_0[e]$	13397
13343	2. $\vdash_F E_0[e] : \tau_0$	13398
13344	by <i>inversion</i>	13399
13345	3. QED by <i>static context static hole substitution</i> (2)	13400
13346	□	
13347	Lemma 6.28 : HF dynamic context dynamic hole substitution	13401
13348	If $\vdash_F E[e]$ and contains $\vdash_F e : \tau'$, and furthermore $\vdash_F e' : \tau'$,	13402
13349	then $\vdash_F E[e'] : \tau$	13403
13350	<i>Proof:</i>	13403
13351	By induction on the structure of E .	13404
13352	CASE $E \in E^\bullet :$	13405
13353	1. QED by <i>dynamic boundary-free hole substitution</i>	13406
13354	CASE $E = E_0 e_1 :$	13407
13355	1. $E[e] = E_0[e] e_1$	13408
13356	2. $\vdash_F E_0[e]$	13409
13357	by <i>inversion</i>	13410
13358	3. QED by the induction hypothesis (2)	13411
13359	CASE $E = v_0 E_1 :$	13412
13360	1. $E[e] = v_0 E_1[e]$	13413
13361	2. $\vdash_F E_1[e]$	13414
13362	by <i>inversion</i>	13415
13363	3. QED by the induction hypothesis (2)	13416
13364	CASE $E = \langle E_0, e_1 \rangle :$	13417
13365	1. $E[e] = \langle E_0[e], e_1 \rangle$	13418
		13419
	□	13420
13347	Lemma 6.29 : HF static context static hole substitution	13400
13348	If $\vdash_F E[e] : \tau$ and contains $\vdash_F e : \tau'$, and furthermore $\vdash_F e' : \tau'$,	13401
13349	then $\vdash_F E[e'] : \tau$	13402
13350	<i>Proof:</i>	13403
13351	By induction on the structure of E .	13404
13352	CASE $E \in E^\bullet :$	13405
13353	1. QED by <i>static boundary-free hole substitution</i>	13406
13354	CASE $E = E_0 e_1 :$	13407
13355	1. $E[e] = E_0[e] e_1$	13408
13356	2. $\vdash_F E_0[e] : \tau_0$	13409
13357	by <i>inversion</i>	13410
13358	3. QED by the induction hypothesis (2)	13411
13359	CASE $E = v_0 E_1 :$	13412
13360	1. $E[e] = v_0 E_1[e]$	13413
13361	2. $\vdash_F E_1[e] : \tau_1$	13414
13362	by <i>inversion</i>	13415
13363	3. QED by the induction hypothesis (2)	13416
13364	CASE $E = \langle E_0, e_1 \rangle :$	13417
13365	1. $E[e] = \langle E_0[e], e_1 \rangle$	13418
		13419

13421	2. $\vdash_F E_0[e] : \tau_0$	13476
13422	by <i>inversion</i>	13477
13423	3. QED by the induction hypothesis (2)	13478
13424	CASE $E = \langle v_0, E_1 \rangle :$	13479
13425	1. $E[e] = \langle v_0, E_1[e] \rangle$	13480
13426	2. $\vdash_F E_1[e] : \tau_1$	13481
13427	by <i>inversion</i>	13482
13428	3. QED by the induction hypothesis (2)	13483
13429	CASE $E = op^1 E_0 :$	13484
13430	1. $E[e] = op^1 E_0[e]$	13485
13431	2. $\vdash_F E_0[e] : \tau_0$	13486
13432	by <i>inversion</i>	13487
13433	3. QED by the induction hypothesis (2)	13488
13434	CASE $E = op^2 E_0 e_1 :$	13489
13435	1. $E[e] = op^2 E_0[e] e_1$	13490
13436	2. $\vdash_F E_0[e] : \tau_0$	13491
13437	by <i>inversion</i>	13492
13438	3. QED by the induction hypothesis (2)	13493
13439	CASE $E = op^2 v_0 E_1 :$	13494
13440	1. $E[e] = op^2 v_0 E_1[e]$	13495
13441	2. $\vdash_F E_1[e] : \tau_1$	13496
13442	by <i>inversion</i>	13497
13443	3. QED by the induction hypothesis (2)	13498
13444	CASE $E = \text{chk } \tau'' E_0 :$	13499
13445	1. $E[e] = \text{chk } \tau'' E_0[e]$	13500
13446	2. $\vdash_F E_0[e] : \tau_0$	13501
13447	by <i>inversion</i>	13502
13448	3. QED by the induction hypothesis (2)	13503
13449	CASE $E = \text{dyn } \tau_0 E_0 :$	13504
13450	1. $E[e] = \text{dyn } \tau_0 E_0[e]$	13505
13451	2. $\vdash_F E_0[e]$	13506
13452	by <i>inversion</i>	13507
13453	3. QED by <i>static dyn hole typing</i> (2)	13508
13454	CASE $E = \text{stat } \tau_0 E_0 :$	13509
13455	1. Contradiction by $\vdash_F E[e] : \tau$	13510
13456	□	13511
13457	Lemma 6.30 : HF static context dynamic hole substitution	13512
13458	If $\vdash_F E[e] : \tau$ and contains $\vdash_F e$, and furthermore $\vdash_F e'$, then	13513
13459	$\vdash_F E[e'] : \tau$	13514
13460	<i>Proof:</i>	13515
13461	By induction on the structure of E .	13516
13462	CASE $E \in E^\bullet :$	13517
13463	1. Contradiction by $\vdash_F E[e] : \tau$	13518
13464	CASE $E = E_0 e_1 :$	13519
13465	1. $E[e] = E_0[e] e_1$	13520
13466	2. $\vdash_F E_0[e] : \tau_0$	13521
13467	by <i>inversion</i>	13522
13468	3. QED by the induction hypothesis (2)	13523
13469	CASE $E = v_0 E_1 :$	13524
13470	1. $E[e] = v_0 E_1[e]$	13525
13471	2. $\vdash_F E_1[e] : \tau_1$	13526
13472	by <i>inversion</i>	13527
13473	3. QED by the induction hypothesis (2)	13528
13474		13529
13475		13530

- If $\vdash_F x : \tau$ then $(x : \tau') \in \Gamma$ and $\tau' \leqslant \tau$
- If $\vdash_F \lambda(x : \tau'_d). e' : \tau$ then $(x : \tau'_d), \Gamma \vdash_F e' : \tau'_c$ and $\tau'_d \Rightarrow \tau'_c \leqslant \tau$
- If $\vdash_F \langle e_0, e_1 \rangle : \tau_0 \times \tau_1$ then $\Gamma \vdash_F e_0 : \tau'_0$ and $\Gamma \vdash_F e_1 : \tau'_1$ and $\tau'_0 \leqslant \tau_0$ and $\tau'_1 \leqslant \tau_1$
- If $\vdash_F e_0 e_1 : \tau_c$ then $\Gamma \vdash_F e_0 : \tau'_d \Rightarrow \tau'_c$ and $\Gamma \vdash_F e_1 : \tau'_d$ and $\tau'_c \leqslant \tau_c$
- If $\vdash_F \text{fst } e : \tau$ then $\Gamma \vdash_F e : \tau_0 \times \tau_1$ and $\Delta(\text{fst}, \tau_0 \times \tau_1) = \tau_0$ and $\tau_0 \leqslant \tau$
- If $\vdash_F \text{snd } e : \tau$ then $\Gamma \vdash_F e : \tau_0 \times \tau_1$ and $\Delta(\text{snd}, \tau_0 \times \tau_1) = \tau_1$ and $\tau_1 \leqslant \tau$
- If $\vdash_F op^2 e_0 e_1 : \tau$ then $\Gamma \vdash_F e_0 : \tau_0$ and $\Gamma \vdash_F e_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau'$ and $\tau' \leqslant \tau$
- If $\vdash_F \text{mon } \tau'_0 \times \tau'_1 \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$ then either:
 - $\Gamma \vdash_F \langle v_0, v_1 \rangle : \tau$
 - or $\Gamma \vdash_F \langle v_0, v_1 \rangle : \tau$
- If $\vdash_F \text{mon } \tau'_d \Rightarrow \tau'_c \lambda x. e : \tau_d \Rightarrow \tau_c$ then $\Gamma \vdash_F \lambda x. e$ and $\tau'_d \Rightarrow \tau'_c \leqslant \tau_d \Rightarrow \tau_c$
- If $\vdash_F \text{mon } \tau'_d \Rightarrow \tau'_c \lambda(x : \tau''_d). e : \tau_d \Rightarrow \tau_c$ then $\Gamma \vdash_F \lambda(x : \tau_x). e : \tau''_d \Rightarrow \tau'_c$ and $\tau'_d \Rightarrow \tau'_c \leqslant \tau_d \Rightarrow \tau_c$
- If $\vdash_F \text{dyn } \tau' e' : \tau$ then $\Gamma \vdash_F e'$ and $\tau' \leqslant \tau$
- If $\vdash_F \text{chk } \tau e' : \tau$ then $\Gamma \vdash_F e' : \tau'$

Proof:

QED by the definition of $\vdash_F e : \tau$

□

Lemma 6.32 : \vdash_F dynamic inversion

- If $\vdash_F x$ then $x \in \Gamma$
- If $\vdash_F \lambda x. e'$ then $x, \Gamma \vdash_F e'$
- If $\vdash_F \langle e_0, e_1 \rangle$ then $\Gamma \vdash_F e_0$ and $\Gamma \vdash_F e_1$
- If $\vdash_F e_0 e_1$ then $\Gamma \vdash_F e_0$ and $\Gamma \vdash_F e_1$
- If $\vdash_F op^1 e_0$ then $\Gamma \vdash_F e_0$
- If $\vdash_F op^2 e_0 e_1$ then $\Gamma \vdash_F e_0$ and $\Gamma \vdash_F e_1$
- If $\vdash_F \text{mon } (\tau_d \Rightarrow \tau_c) \lambda x. e$ then $\Gamma \vdash_F \lambda x. e$
- If $\vdash_F \text{mon } (\tau_d \Rightarrow \tau_c) \lambda(x : \tau_x). e$ then $\Gamma \vdash_F \lambda(x : \tau_x). e : \tau_x \Rightarrow \tau'_x$
- If $\vdash_F \text{mon } (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle$ then either:
 - $\Gamma \vdash_F \langle v_0, v_1 \rangle : \tau$
 - or $\Gamma \vdash_F \langle v_0, v_1 \rangle : \tau'$
- If $\vdash_F \text{stat } \tau' e'$ then $\Gamma \vdash_F e' : \tau'$

Proof:

QED by the definition of $\vdash_F e$

□

Lemma 6.33 : HF canonical forms

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- If $\vdash_F v : \tau_0 \times \tau_1$ then either:
 - $v = \langle v_0, v_1 \rangle$
 - or $v = \text{mon } \tau'_0 \times \tau'_1 \langle v_0, v_1 \rangle$
 - $\wedge \tau'_0 \times \tau'_1 \leqslant \tau_0 \times \tau_1$
- If $\vdash_F v : \tau_d \Rightarrow \tau_c$ then either:
 - $v = \lambda(x : \tau_x). e'$
 - $\wedge \tau_d \leqslant \tau_x$
 - or $v = \text{mon } (\tau'_d \Rightarrow \tau'_c) (\lambda x. e)$
 - $\wedge \tau'_d \Rightarrow \tau'_c \leqslant \tau_d \Rightarrow \tau_c$
 - or $v = \text{mon } (\tau'_d \Rightarrow \tau'_c) \lambda(x : \tau_x). e$
 - $\wedge \tau'_d \Rightarrow \tau'_c \leqslant \tau_d \Rightarrow \tau_c$
- If $\vdash_F v : \text{Int}$ then $v = i$
- If $\vdash_F v : \text{Nat}$ then $v = i$ and $v \in \mathbb{N}$

Proof:

QED by definition of $\vdash_F \cdot : \tau$

□

Lemma 6.34 : Δ type soundness

If $\vdash_F v_0 : \tau_0$ and $\vdash_F v_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau$ then one of the following holds:

- $\delta(op^2, v_0, v_1) = v$ and $\vdash_F v : \tau$, or
- $\delta(op^2, v_0, v_1) = \text{BndryErr}$

Proof (sketch): Similar to the proof for the higher-order Δ type soundness lemma. □

Lemma 6.35 : δ preservation

- If $\vdash_F v$ and $\delta(op^1, v) = v'$ then $\vdash_F v'$
- If $\vdash_F v_0$ and $\vdash_F v_1$ and $\delta(op^2, v_0, v_1) = v'$ then $\vdash_F v'$

Proof:

Similar to the proof for the higher-order δ preservation lemma.

□

Lemma 6.36 : HF substitution

- If $(x : \tau_x), \Gamma \vdash_F e$ and $\vdash_F v : \tau_x$ then $\Gamma \vdash_F e[x \leftarrow v]$
- If $x, \Gamma \vdash_F e$ and $\vdash_F v$ then $\Gamma \vdash_F e[x \leftarrow v]$
- If $(x : \tau_x), \Gamma \vdash_F e : \tau$ and $\vdash_F v : \tau_x$ then $\Gamma \vdash_F e[x \leftarrow v] : \tau$
- If $x, \Gamma \vdash_F e : \tau$ and $\vdash_F v$ then $\Gamma \vdash_F e[x \leftarrow v] : \tau$

Proof:

Similar to the proof for the higher-order substitution lemma.

□

Lemma 6.37 : weakening

- If $\vdash_F e$ then $x, \Gamma \vdash_F e$
- If $\vdash_F e : \tau$ then $(x : \tau'), \Gamma \vdash_F e : \tau$

Proof:

QED because e is closed under Γ

□

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13641	E.7 Embeddings Summary	13696
13642	The paragraphs in this section summarize the five embeddings with four slogans. Each slogan pertains to one aspect of the embedding:	13697
13643		13698
13644		13699
13645	1. What kinds of checks does the embedding perform when a value reaches a type boundary?	13700
13646	2. When, if ever, does the embedding wrap a value in a monitor?	13701
13647	3. If an ill-typed value reaches a type boundary, when does the embedding signal an error?	13702
13648	4. How do types affect behavior?	13703
13649	These embeddings are ordered on a speculative scale from "most guarantees" to "least guarantees".	13704
13650		13705
13651		13706
13652		13707
13653		13708
13654		13709
13655	Higher-Order embedding	13710
13656	1. recursively check read-only values;	13711
13657	2. monitor functional and mutable values;	13712
13658	3. detect boundary errors as early as possible;	13713
13659	4. types globally constrain behavior.	13714
13660		13715
13661	Co-Natural embedding	13716
13662	1. tag-check all values;	13717
13663	2. monitor all data structures and functions;	13718
13664	3. detect boundary errors as late as possible;	13719
13665	4. types globally constrain behavior	13720
13666		13721
13667	Forgetful embedding	13722
13668	1. tag-check all values;	13723
13669	2. apply at most one monitor to each value;	13724
13670	3. detect boundary errors as late as possible;	13725
13671	4. types (of values) locally constrain behavior	13726
13672	First-Order embedding	13727
13673	1. tag-check all values;	13728
13674	2. never allocate a monitor;	13729
13675	3. detect boundary errors as late as possible;	13730
13676	4. types (of contexts) locally constrain behavior	13731
13677		13732
13678	Erasure embedding	13733
13679	1. never check values;	13734
13680	2. never allocate a monitor;	13735
13681	3. never detect a type boundary error;	13736
13682	4. types do not affect behavior	13737
13683		13738
13684		13739
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13751 **E.8 Simulation Lemmas**

13752 **E.8.1 Definitions**

13753 Combined Language

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 $e = x \mid v \mid \langle e, e \rangle \mid e \ e \mid op^1 \ e \mid op^2 \ e \ e \mid$
 $\text{dyn } \tau \ e \mid \text{stat } \tau \ e \mid \text{Err} \mid \text{chk } K \ e \mid \text{dyn } e \mid \text{stat } e$
 $v = i \mid \langle v, v \rangle \mid \lambda x. \ e \mid \lambda(x:\tau). \ e \mid \text{mon}(\tau \Rightarrow \tau) \ v$
 $E^\bullet = [] \mid E^\bullet \ e \mid v \ E^\bullet \mid \langle E^\bullet, e \rangle \mid \langle v, E^\bullet \rangle \mid$
 $op^1 \ E^\bullet \mid op^2 \ E^\bullet \ e \mid op^2 \ v \ E^\bullet \mid \text{chk } K \ E^\bullet$
 $E = E^\bullet \mid E \ e \mid v \ E \mid \langle E, e \rangle \mid \langle v, E \rangle \mid op^1 \ E \mid$
 $op^2 \ E \ e \mid op^2 \ v \ E \mid \text{dyn } \tau \ E \mid \text{stat } \tau \ E$
 $\text{chk } K \ E \mid \text{dyn } E \mid \text{stat } E$

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$$\boxed{e \underset{1}{\sim}_{\mathcal{E}} e}$$

$\frac{}{\text{Err } \underset{1}{\sim}_{\mathcal{E}} e^E}$	$\frac{e^1 \underset{1}{\sim}_{\mathcal{E}} e^E}{\text{chk } K \ e^1 \underset{1}{\sim}_{\mathcal{E}} e^E}$	$\frac{e^1 \underset{1}{\sim}_{\mathcal{E}} e^E}{\text{dyn } e^1 \underset{1}{\sim}_{\mathcal{E}} e^E}$
		$\frac{e^1 \underset{1}{\sim}_{\mathcal{E}} e^E}{\text{stat } e^1 \underset{1}{\sim}_{\mathcal{E}} e^E}$
	$\frac{}{x \underset{1}{\sim}_{\mathcal{E}} x}$	$\frac{}{i \underset{1}{\sim}_{\mathcal{E}} i}$
		$\frac{e^1 \underset{1}{\sim}_{\mathcal{E}} e^E}{\lambda x. \ e^1 \underset{1}{\sim}_{\mathcal{E}} \lambda x. \ e^E}$
		$\frac{e^1 \underset{1}{\sim}_{\mathcal{E}} e^E}{\lambda(x:\tau). \ e^1 \underset{1}{\sim}_{\mathcal{E}} \lambda(x:\tau). \ e^E}$
	$\frac{e_0^1 \underset{1}{\sim}_{\mathcal{E}} e_0^E \ e_1^1 \underset{1}{\sim}_{\mathcal{E}} e_1^E}{\langle e_0^1, e_1^1 \rangle \underset{1}{\sim}_{\mathcal{E}} \langle e_0^E, e_1^E \rangle}$	$\frac{e_0^1 \underset{1}{\sim}_{\mathcal{E}} e_0^E}{op^1 \ e_0^1 \underset{1}{\sim}_{\mathcal{E}} op^1 \ e_0^E}$
		$\frac{e_0^1 \underset{1}{\sim}_{\mathcal{E}} e_0^E}{op^2 \ e_0^1 \ e_1^1 \underset{1}{\sim}_{\mathcal{E}} op^2 \ e_0^E \ e_1^E}$
		$\frac{e_0^1 \underset{1}{\sim}_{\mathcal{E}} e_0^E}{dyn \ \tau \ e_0^1 \underset{1}{\sim}_{\mathcal{E}} dyn \ \tau \ e_0^E}$
		$\frac{e_0^1 \underset{1}{\sim}_{\mathcal{E}} e_0^E}{stat \ \tau \ e_0^1 \underset{1}{\sim}_{\mathcal{E}} stat \ \tau \ e_0^E}$
		$\frac{}{\text{Err } \underset{1}{\sim}_{\mathcal{E}} \text{Err}}$
$\boxed{E \underset{1}{\sim}_{\mathcal{E}} E}$		
	$\frac{}{[] \underset{1}{\sim}_{\mathcal{E}} []}$	$\frac{E^1 \underset{1}{\sim}_{\mathcal{E}} E^E}{\text{chk } K \ E^1 \underset{1}{\sim}_{\mathcal{E}} E^E}$
		$\frac{E^1 \underset{1}{\sim}_{\mathcal{E}} E^E}{\text{dyn } E^1 \underset{1}{\sim}_{\mathcal{E}} E^E}$
	$\frac{E^1 \underset{1}{\sim}_{\mathcal{E}} E^E}{\text{stat } E^1 \underset{1}{\sim}_{\mathcal{E}} E^E}$	$\frac{E^1 \underset{1}{\sim}_{\mathcal{E}} E^E \ e^1 \underset{1}{\sim}_{\mathcal{E}} e^E}{E^1 \ e^1 \underset{1}{\sim}_{\mathcal{E}} E^E \ e^E}$
	$\frac{v^1 \underset{1}{\sim}_{\mathcal{E}} v^E \ E^1 \underset{1}{\sim}_{\mathcal{E}} E^E}{v^1 \ E^1 \underset{1}{\sim}_{\mathcal{E}} v^E \ E^E}$	$\frac{E^1 \underset{1}{\sim}_{\mathcal{E}} E^E \ e^1 \underset{1}{\sim}_{\mathcal{E}} e^E}{\langle E^1, e^1 \rangle \underset{1}{\sim}_{\mathcal{E}} \langle E^E, e^E \rangle}$
	$\frac{v^1 \underset{1}{\sim}_{\mathcal{E}} v^E \ E^1 \underset{1}{\sim}_{\mathcal{E}} E^E}{\langle v^1, E^1 \rangle \underset{1}{\sim}_{\mathcal{E}} \langle v^E, E^E \rangle}$	$\frac{E^1 \underset{1}{\sim}_{\mathcal{E}} E^E}{op^1 \ E^1 \underset{1}{\sim}_{\mathcal{E}} op^1 \ E^E}$
	$\frac{E^1 \underset{1}{\sim}_{\mathcal{E}} E^E \ e^1 \underset{1}{\sim}_{\mathcal{E}} e^E}{op^2 \ E^1 \ e^1 \underset{1}{\sim}_{\mathcal{E}} op^2 \ E^E \ e^E}$	$\frac{v^1 \underset{1}{\sim}_{\mathcal{E}} v^E \ E^1 \underset{1}{\sim}_{\mathcal{E}} E^E}{op^2 \ v^1 \ E^1 \underset{1}{\sim}_{\mathcal{E}} op^2 \ v^E \ E^E}$
	$\frac{E^1 \underset{1}{\sim}_{\mathcal{E}} E^E}{dyn \ \tau \ E^1 \underset{1}{\sim}_{\mathcal{E}} dyn \ \tau \ E^E}$	$\frac{E^1 \underset{1}{\sim}_{\mathcal{E}} E^E}{stat \ \tau \ E^1 \underset{1}{\sim}_{\mathcal{E}} stat \ \tau \ E^E}$

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13861	$e \text{ H} \lesssim_1 e$		13916
13862	$\frac{e^H \text{ H} \lesssim_1 e^1}{e^H \text{ H} \lesssim_1 \text{chk } K \text{ e}^1}$	$\frac{\text{Err} \text{ H} \lesssim_1 e^1}{\text{dyn } \tau \text{ e}^H \text{ H} \lesssim_1 \text{dyn } \tau \text{ e}^1}$	13917
13863		$\frac{e^H \text{ H} \lesssim_1 e^1}{\text{dyn } \tau \text{ e}^H \text{ H} \lesssim_1 \text{dyn } e^1}$	13918
13864		$\frac{e^H \text{ H} \lesssim_1 e^1}{\text{dyn } \tau_0 (\text{stat } \tau_1 e^H) \text{ H} \lesssim_1 e^1}$	13919
13865			13920
13866	$\frac{e^H \text{ H} \lesssim_1 e^1}{\text{dyn } \tau \text{ e}^H \text{ H} \lesssim_1 \text{dyn } e^1}$	$\frac{e^H \text{ H} \lesssim_1 e^1}{\text{dyn } \tau_0 (\text{stat } \tau_1 e^H) \text{ H} \lesssim_1 e^1}$	13921
13867			13922
13868			13923
13869	$\frac{e^H \text{ H} \lesssim_1 e^1}{\text{stat } \tau \text{ e}^H \text{ H} \lesssim_1 \text{stat } \tau \text{ e}^1}$	$\frac{e^H \text{ H} \lesssim_1 e^1}{\text{stat } \tau \text{ e}^H \text{ H} \lesssim_1 \text{stat } e^1}$	13924
13870			13925
13871			13926
13872			13927
13873	$\frac{e^H \text{ H} \lesssim_1 e^1}{\text{stat } \tau_0 (\text{dyn } \tau_1 e^H) \text{ H} \lesssim_1 e^1}$	$\frac{e_0^H \text{ H} \lesssim_1 e_0^1 \quad e_1^H \text{ H} \lesssim_1 e_1^1}{e_0^H \text{ e}_1^H \text{ H} \lesssim_1 e_0^1 \text{ e}_1^1}$	13928
13874			13929
13875			13930
13876	$\frac{e_0^H \text{ H} \lesssim_1 e_0^1 \quad e_1^H \text{ H} \lesssim_1 e_1^1}{\langle e_0^H, e_1^H \rangle \text{ H} \lesssim_1 \langle e_0^1, e_1^1 \rangle}$	$\frac{e^H \text{ H} \lesssim_1 e^1}{op^1 \text{ e}^H \text{ H} \lesssim_1 op^1 \text{ e}^1}$	13931
13877			13932
13878			13933
13879			13934
13880	$\frac{e_0^H \text{ H} \lesssim_1 e_0^1 \quad e_1^H \text{ H} \lesssim_1 e_1^1}{op^2 \text{ e}_0^H \text{ e}_1^H \text{ H} \lesssim_1 op^2 \text{ e}_0^1 \text{ e}_1^1}$	$\frac{}{x \text{ H} \lesssim_1 x} \quad \frac{}{i \text{ H} \lesssim_1 i}$	13935
13881			13936
13882			13937
13883	$\frac{e^H \text{ H} \lesssim_1 e^1}{\lambda x. e^H \text{ H} \lesssim_1 \lambda x. e^1}$	$\frac{e^H \text{ H} \lesssim_1 e^1}{\lambda(x:\tau). e^H \text{ H} \lesssim_1 \lambda(x:\tau). e^1}$	13938
13884			13939
13885			13940
13886			13941
13887	$\frac{v^H \text{ H} \lesssim_1 v^1}{\text{mon } \tau v^H \text{ H} \lesssim_1 v^1}$	$\frac{}{\text{Err} \text{ H} \lesssim_1 \text{Err}}$	13942
13888			13943
13889	$E \text{ H} \lesssim_1 E$		13944
13890			13945
13891	$\frac{E^H \text{ H} \lesssim_1 E^1}{[] \text{ I} \lesssim_E []}$	$\frac{E^H \text{ H} \lesssim_1 E^1 \quad e^H \text{ H} \lesssim_1 e^1}{E^H \text{ e}^H \text{ H} \lesssim_1 E^1 \text{ e}^1}$	13946
13892			13947
13893			13948
13894	$\frac{v^H \text{ H} \lesssim_1 v^1 \quad E^H \text{ H} \lesssim_1 E^1}{v^H \text{ E}^H \text{ H} \lesssim_1 v^1 \text{ E}^1}$	$\frac{E^H \text{ H} \lesssim_1 E^1 \quad e^H \text{ H} \lesssim_1 e^1}{\langle E^H, e^H \rangle \text{ H} \lesssim_1 \langle E^1, e^1 \rangle}$	13949
13895			13950
13896			13951
13897			13952
13898	$\frac{v^H \text{ H} \lesssim_1 v^1 \quad E^H \text{ H} \lesssim_1 E^1}{\langle v^H, E^H \rangle \text{ H} \lesssim_1 \langle v^1, E^1 \rangle}$	$\frac{E^H \text{ H} \lesssim_1 E^1}{op^1 \text{ E}^H \text{ H} \lesssim_1 op^1 \text{ E}^1}$	13953
13899			13954
13900			13955
13901	$\frac{E^H \text{ H} \lesssim_1 E^1 \quad e^H \text{ H} \lesssim_1 e^1}{op^2 \text{ E}^H \text{ e}^H \text{ H} \lesssim_1 op^2 \text{ E}^1 \text{ e}^1}$	$\frac{v^H \text{ H} \lesssim_1 v^1 \quad E^H \text{ H} \lesssim_1 E^1}{op^2 \text{ v}^H \text{ E}^H \text{ H} \lesssim_1 op^2 \text{ v}^1 \text{ E}^1}$	13956
13902			13957
13903			13958
13904	$\frac{E^H \text{ H} \lesssim_1 E^1}{\text{dyn } \tau \text{ E}^H \text{ H} \lesssim_1 \text{dyn } \tau \text{ E}^1}$	$\frac{E^H \text{ H} \lesssim_1 E^1}{\text{dyn } \tau \text{ E}^H \text{ H} \lesssim_1 \text{dyn } E^1}$	13959
13905			13960
13906			13961
13907			13962
13908	$\frac{E^H \text{ H} \lesssim_1 E^1}{\text{dyn } \tau_0 (\text{stat } \tau_1 E^H) \text{ H} \lesssim_1 E^1}$	$\frac{E^H \text{ H} \lesssim_1 E^1}{\text{stat } \tau \text{ E}^H \text{ H} \lesssim_1 \text{stat } \tau \text{ E}^1}$	13963
13909			13964
13910			13965
13911	$\frac{E^H \text{ H} \lesssim_1 E^1}{\text{stat } \tau \text{ E}^H \text{ H} \lesssim_1 \text{stat } E^1}$	$\frac{E^H \text{ H} \lesssim_1 E^1}{\text{stat } \tau_0 (\text{dyn } \tau_1 E^H) \text{ H} \lesssim_1 E^1}$	13966
13912			13967
13913			13968
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13915			13970

13971 E.8.2 Theorems

13972 **Theorem 8.0 : Err approximation**

13973 If $e \in e_S$ and $\vdash e : \tau$ then the following statements hold:

- 13974 • if $e \xrightarrow{*_{E-S}} \text{Err}$ then $e \xrightarrow{*_{I-S}} \text{Err}$
- 13975 • if $e \xrightarrow{*_{I-S}} \text{Err}$ then $e \xrightarrow{*_{H-S}} \text{Err}$

13976 *Proof:*

13977 QED by 1-E approximation and H-1 approximation.

13978 \square

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14026 E.8.3 Lemmas

14027 **Lemma 8.1 : 1-E approximation**

14028 If $e \in e_S$ and $\vdash e : \tau$ and $e \xrightarrow{*_{E-S}} \text{Err}$ then:

- 14029 • $\vdash e : [\tau] \rightsquigarrow e''$
- 14030 • $e'' \xrightarrow{*_{I-S}} \text{Err}$

14031 *Proof:*

- 14032 • $e'' \xrightarrow{1\sim E} e$
by 1-E static reflexivity
- 14033 • QED by 1-E simulation

14035 \square

14036 **Lemma 8.2 : 1-E static reflexivity**

14037 If $\Gamma \vdash e : \tau$ and $\Gamma \vdash e : \tau \rightsquigarrow e''$ then $e'' \xrightarrow{1\sim E} e$.

14038 *Proof:*

14039 By structural induction on the $\Gamma \vdash e : \tau \rightsquigarrow e''$ judgment.

14040 **CASE** $\boxed{\Gamma \vdash i : \text{Nat} \rightsquigarrow i} :$

14043 1. QED $i \xrightarrow{1\sim E} i$

14044 **CASE** $\boxed{\Gamma \vdash i : \text{Int} \rightsquigarrow i} :$

14046 1. QED $i \xrightarrow{1\sim E} i$

14047 **CASE** $\boxed{\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 : \tau_1 \rightsquigarrow e'_1} :$

14050 1. $e'_0 \xrightarrow{1\sim E} e_0$

14051 $\wedge e'_1 \xrightarrow{1\sim E} e_1$

14052 by the induction hypothesis

14053 2. QED

14054 **CASE** $\boxed{(x:\tau_d), \Gamma \vdash e : \tau_c \rightsquigarrow e'} :$

14055 $\Gamma \vdash \lambda(x:\tau_d). e : \tau_d \Rightarrow \tau_c \rightsquigarrow \lambda(x:\tau_d). e'$

14057 1. $e' \xrightarrow{1\sim E} e$

14058 by the induction hypothesis

14059 2. QED

14060 **CASE** $\boxed{\Gamma \vdash x : \tau \rightsquigarrow x} :$

14061 1. QED $x \xrightarrow{1\sim E} x$

14062 **CASE** $\boxed{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 : \tau_d \rightsquigarrow e'_1 \quad [\tau_c] = K} :$

14063 $\Gamma \vdash e_0 e_1 : \tau_c \rightsquigarrow \text{chk } K(e'_0 e'_1)$

14064 :

14065 1. $e'_0 \xrightarrow{1\sim E} e_0$

14066 $\wedge e'_1 \xrightarrow{1\sim E} e_1$

14067 by the induction hypothesis

14068 2. $e'_0 e'_1 \xrightarrow{1\sim E} e_0 e_1$

14069 3. QED $\text{chk } K e'_0 e'_1 \xrightarrow{1\sim E} e_0 e_1$

14070 **CASE** $\boxed{\Gamma \vdash e : \tau_0 \times \tau_1 \rightsquigarrow e' \quad [\tau_0] = K} :$

14071 $\Gamma \vdash \text{fst } e : \tau_0 \rightsquigarrow \text{chk } K(\text{fst } e')$

14072 1. $e' \xrightarrow{1\sim E} e$

14073 by the induction hypothesis

14074 2. $\text{fst } e' \xrightarrow{1\sim E} \text{fst } e$

14075 3. QED $\text{chk } K \text{fst } e' \xrightarrow{1\sim E} \text{fst } e$

14081	CASE	$\frac{\Gamma \vdash e : \tau_0 \times \tau_1 \rightsquigarrow e' \quad \lfloor \tau_1 \rfloor = K}{\Gamma \vdash \text{snd } e : \tau_1 \rightsquigarrow \text{chk } K(\text{snd } e')}$:	
14082		1. $e' \underset{1\sim E}{\underset{\sim}{\sim}} e$		
14083		by the induction hypothesis		
14084		2. $\text{snd } e' \underset{1\sim E}{\underset{\sim}{\sim}} \text{snd } e$		
14085		3. QED $\text{chk } K \text{ snd } e' \underset{1\sim E}{\underset{\sim}{\sim}} \text{snd } e$		
14086		$\frac{\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 : \tau_1 \rightsquigarrow e'_1}{\Gamma \vdash \text{op}^2 e_0 e_1 : \tau \rightsquigarrow \text{op}^2 e'_0 e'_1}$:	
14087	CASE			
14088		1. $e'_0 \underset{1\sim E}{\underset{\sim}{\sim}} e_0$		
14089		$\wedge e'_1 \underset{1\sim E}{\underset{\sim}{\sim}} e_1$		
14090		by the induction hypothesis		
14091		2. QED		
14092		$\frac{\Gamma \vdash e : \tau' \rightsquigarrow e' \quad \tau' \leqslant \tau}{\Gamma \vdash e : \tau \rightsquigarrow e'}$:	
14093	CASE			
14094		1. QED by the induction hypothesis		
14095	CASE	$\frac{}{\Gamma \vdash \text{Err} : \tau \rightsquigarrow \text{Err}}$:	
14096		1. QED $\text{Err} \underset{1\sim E}{\underset{\sim}{\sim}} \text{Err}$		
14097	CASE			
14098		1. QED $\text{Err} \underset{1\sim E}{\underset{\sim}{\sim}} \text{Err}$		
14099		$\frac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \text{dyn } \tau \ e : \tau \rightsquigarrow \text{dyn } \tau \ e'}$:	
14100		1. $e' \underset{1\sim E}{\underset{\sim}{\sim}} e$		
14101		by the induction hypothesis		
14102		2. QED		
14103		□		
14104	Lemma 8.3 : 1-E dynamic reflexivity			
14105	If $\Gamma \vdash e$ and $\Gamma \vdash e \rightsquigarrow e''$ then $e'' \underset{1\sim E}{\underset{\sim}{\sim}} e$.			
14106	<i>Proof:</i>			
14107	By structural induction on the $\Gamma \vdash e \rightsquigarrow e''$ judgment.			
14108	CASE	$\frac{}{\Gamma \vdash i \rightsquigarrow i}$:	
14109		1. QED $i \underset{1\sim E}{\underset{\sim}{\sim}} i$		
14110		□		
14111	Lemma 8.4 : 1-E simulation			
14112	If $E^1[e_0^1] \underset{1\sim E}{\underset{\sim}{\sim}} E^E[e_0^E]$ and $E^E[e_0^E] \rightarrow_{E-S} E^E[e_1^E]$ then $E^1[e_0^1] \rightarrow_{1-S}^* E^1[e_1^1]$ and $E^1[e_1^1] \underset{1\sim E}{\underset{\sim}{\sim}} E^E[e_1^E]$			
14113	<i>Proof:</i>			
14114	By case analysis on $e_0^E \triangleright_{E-S} e_1^E$.			
14115	CASE	$\frac{}{\Gamma \vdash i \rightsquigarrow i}$:	
14116		1. QED $i \underset{1\sim E}{\underset{\sim}{\sim}} i$		
14117		□		
14118	CASE			
14119		1. $e_0^1 = \text{dyn } \tau_0 \ v_0^E \triangleright_{E-S} v_0^E$		
14120		by definition $\underset{1\sim E}{\underset{\sim}{\sim}}$		
14121		2. $e_0^1 \underset{1\sim E}{\underset{\sim}{\sim}} v_0^E$		
14122		by (1)		
14123		3. $\vdash_1 e_0^1$		
14124		by 1 <i>inversion</i>		
14125		4. $e_0^1 \rightarrow_{1-D}^* v_0^1$ and $v_0^1 \underset{1\sim E}{\underset{\sim}{\sim}} v_0^E$		
14126		$\vee e_0^1 \rightarrow_{1-D}^* \text{BndryErr}$		
14127		by 1-E <i>dynamic value stutter</i>		
14128		5. IF $e_0^1 \rightarrow_{1-D}^* \text{BndryErr}$:		
14129		a. $e_0^1 \rightarrow_{1-S}^* \text{BndryErr}$		
14130		b. $E^1[\text{BndryErr}] \underset{1\sim E}{\underset{\sim}{\sim}} E^E[e_1^E]$		
14131		by 1-E <i>hole substitution</i>		
14132		c. QED		
14133	CASE	$\frac{}{\Gamma \vdash x \rightsquigarrow x}$:	
14134		1. $e' \underset{1\sim E}{\underset{\sim}{\sim}} e$		
14135		by the induction hypothesis		
		2. QED		
		□		

1. QED $x \underset{1\sim E}{\underset{\sim}{\sim}} x$	14136
CASE	14137
$\frac{\Gamma \vdash e_0 \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 \rightsquigarrow e'_1}{\Gamma \vdash e_0 e_1 \rightsquigarrow e'_0 e'_1}$	14138
1. $e'_0 \underset{1\sim E}{\underset{\sim}{\sim}} e_0$	14139
$\wedge e'_1 \underset{1\sim E}{\underset{\sim}{\sim}} e_1$	14140
by the induction hypothesis	14141
2. QED	14142
CASE	14143
$\frac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \text{op}^1 e \rightsquigarrow \text{op}^1 e'}$	14144
1. $e' \underset{1\sim E}{\underset{\sim}{\sim}} e$	14145
by the induction hypothesis	14146
2. QED	14147
CASE	14148
$\frac{\Gamma \vdash e_0 \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 \rightsquigarrow e'_1}{\Gamma \vdash \text{op}^2 e_0 e_1 \rightsquigarrow \text{op}^2 e'_0 e'_1}$	14149
1. $e'_0 \underset{1\sim E}{\underset{\sim}{\sim}} e_0$	14150
$\wedge e'_1 \underset{1\sim E}{\underset{\sim}{\sim}} e_1$	14151
by the induction hypothesis	14152
2. QED	14153
CASE	14154
$\frac{\Gamma \vdash \text{Err} \rightsquigarrow \text{Err}}{\Gamma \vdash \text{Err} \rightsquigarrow \text{Err}}$	14155
1. QED $\text{Err} \underset{1\sim E}{\underset{\sim}{\sim}} \text{Err}$	14156
2. QED	14157
CASE	14158
$\frac{\Gamma \vdash \text{Err} \rightsquigarrow \text{Err}}{\Gamma \vdash \text{Err} \rightsquigarrow \text{Err}}$	14159
1. QED $\text{Err} \underset{1\sim E}{\underset{\sim}{\sim}} \text{Err}$	14160
2. QED	14161
CASE	14162
$\frac{\Gamma \vdash e : \tau \rightsquigarrow e'}{\Gamma \vdash \text{stat } \tau \ e \rightsquigarrow \text{stat } \tau \ e'}$	14163
1. $e' \underset{1\sim E}{\underset{\sim}{\sim}} e$	14164
by 1-E <i>static reflexivity</i>	14165
2. QED	14166
CASE	14167
$\frac{}{\Gamma \vdash x \rightsquigarrow x}$	14168
1. $e' \underset{1\sim E}{\underset{\sim}{\sim}} e$	14169
by structural induction on the $\Gamma \vdash e \rightsquigarrow e''$ judgment.	14170
2. QED	14171
	14172
By case analysis on $e_0^E \triangleright_{E-S} e_1^E$.	14173
CASE	14174
$\frac{}{\Gamma \vdash v_0^E \triangleright_{E-S} v_0^E}$	14175
1. $e_0^1 = \text{dyn } \tau_0 \ v_0^E \triangleright_{E-S} v_0^E$	14176
by definition $\underset{1\sim E}{\underset{\sim}{\sim}}$	14177
2. $e_0^1 \underset{1\sim E}{\underset{\sim}{\sim}} v_0^E$	14178
by (1)	14179
3. $\vdash_1 e_0^1$	14180
by 1 <i>inversion</i>	14181
4. $e_0^1 \rightarrow_{1-D}^* v_0^1$ and $v_0^1 \underset{1\sim E}{\underset{\sim}{\sim}} v_0^E$	14182
$\vee e_0^1 \rightarrow_{1-D}^* \text{BndryErr}$	14183
by 1-E <i>dynamic value stutter</i>	14184
5. IF $e_0^1 \rightarrow_{1-D}^* \text{BndryErr}$:	14185
a. $e_0^1 \rightarrow_{1-S}^* \text{BndryErr}$	14186
b. $E^1[\text{BndryErr}] \underset{1\sim E}{\underset{\sim}{\sim}} E^E[e_1^E]$	14187
by 1-E <i>hole substitution</i>	14188
c. QED	14189
	14190

14191	ELSE $e_{0'}^1 \rightarrow_{1-D}^* v_0^1 :$	14246
14192	a. $e_0^1 \rightarrow_{1-S}^* v_0^1$	14247
14193	b. $E^1[v_0^1] \underset{1-E}{\sim} E^E[v_0^E]$ by 1-E hole substitution	14248
14194	c. QED	14249
14195	CASE stat $\tau_0 v_0^E \triangleright_{E-S} v_0^E :$	14250
14196	1. $e_0^1 = \text{stat } \tau_0 e_{0'}^1$ by definition $\underset{1-E}{\sim}$	14251
14197	2. $e_{0'}^1 \underset{1-E}{\sim} v_0^E$ by (1)	14252
14198	3. $\vdash_1 e_{0'}^1 : [\tau_0]$ by 1 inversion	14253
14199	4. $e_{0'}^1 \rightarrow_{1-S}^* v_0^1$ and $v_0^1 \underset{1-E}{\sim} v_0^E$ $\vee e_{0'}^1 \rightarrow_{1-S}^* \text{BndryErr}$	14254
14200	5. IF $e_{0'}^1 \rightarrow_{1-S}^* \text{BndryErr} :$	14255
14201	a. $e_0^1 \rightarrow_{1-D}^* \text{BndryErr}$	14256
14202	b. $E^1[\text{BndryErr}] \underset{1-E}{\sim} E^E[e_1^E]$ by 1-E dynamic value stutter	14257
14203	c. QED	14258
14204	ELSE $e_{0'}^1 \rightarrow_{1-S}^* v_0^1 :$	14259
14205	a. $e_0^1 \rightarrow_{1-D}^* v_0^1$	14260
14206	b. $E^1[v_0^1] \underset{1-E}{\sim} E^E[v_0^E]$ by 1-E hole substitution	14261
14207	c. QED	14262
14208	CASE $v_0^E v_1^E \triangleright_{E-S} \text{TagErr}$	14263
14209	$\wedge \vdash_1 e_0^1 : K_0 :$	14264
14210	1. $e_0^1 = e_{0'}^1 e_1^1$ by $e_0^1 \underset{1-E}{\sim} e_0^E$	14265
14211	2. $e_{0'}^1 \rightarrow_{1-S}^* v_0^1$ and $v_0^1 \underset{1-E}{\sim} v_0^E$ $\vee e_{0'}^1 \rightarrow_{1-S}^* \text{BndryErr}$	14266
14212	3. IF $e_{0'}^1 \rightarrow_{1-S}^* \text{BndryErr} :$	14267
14213	a. $e_0^1 \rightarrow_{1-D}^* \text{BndryErr}$	14268
14214	b. $E^1[\text{BndryErr}] \underset{1-E}{\sim} E^E[\text{TagErr}]$ by 1-E static value stutter	14269
14215	c. QED	14270
14216	ELSE $e_{0'}^1 \rightarrow_{1-S}^* v_0^1 :$	14271
14217	a. $e_0^1 \rightarrow_{1-D}^* v_0^1$	14272
14218	b. $E^1[v_0^1] \underset{1-E}{\sim} E^E[v_0^E]$ by 1-E hole substitution	14273
14219	c. QED	14274
14220	CASE $v_0^E v_1^E \triangleright_{E-S} \text{TagErr}$	14275
14221	$\wedge \vdash_1 e_0^1 : K_0 :$	14276
14222	1. $e_0^1 = e_{0'}^1 e_1^1$ by $e_0^1 \underset{1-E}{\sim} e_0^E$	14277
14223	2. $e_{0'}^1 \rightarrow_{1-S}^* v_0^1$ and $v_0^1 \underset{1-E}{\sim} v_0^E$ $\vee e_{0'}^1 \rightarrow_{1-S}^* \text{BndryErr}$	14278
14224	3. IF $e_{0'}^1 \rightarrow_{1-S}^* \text{BndryErr} :$	14279
14225	a. $e_0^1 \rightarrow_{1-D}^* \text{BndryErr}$	14280
14226	b. $QED E^1[\text{BndryErr}] \underset{1-E}{\sim} E^E[\text{TagErr}]$	14281
14227	4. $e_{0'}^1 \rightarrow_{1-S}^* v_1^1$ and $v_1^1 \underset{1-E}{\sim} v_1^E$ $\vee e_{0'}^1 \rightarrow_{1-S}^* \text{BndryErr}$	14282
14228	5. IF $e_{0'}^1 \rightarrow_{1-S}^* \text{BndryErr} :$	14283
14229	a. $e_0^1 \rightarrow_{1-D}^* \text{BndryErr}$	14284
14230	b. $QED E^1[\text{BndryErr}] \underset{1-E}{\sim} E^E[\text{TagErr}]$	14285
14231	6. $v_0^E \in \mathbb{Z}$ $\vee v_0^E \in \langle v, v' \rangle$	14286
14232	7. IF $v_0^E \in \mathbb{Z} :$	14287
14233	a. $v_0^1 \in \mathbb{Z}$ by definition \triangleright_{E-S}	14288
14234	b. $\vdash_1 v_0^1 v_1^1 : K_0$ by 1 static preservation	14289
14235	c. Contradiction by (a, b)	14290
14236	ELSE $v_0^E \in \langle v, v' \rangle :$	14291
14237	a. $v_0^1 \in \langle v, v' \rangle$ by (2)	14292
14238	b. $\vdash_1 v_0^1 v_1^1 : K_0$ by 1 static preservation	14293
14239	c. Contradiction by (a, b)	14294
14240	CASE $v_0^E \in \langle v, v' \rangle :$	14295
14241	a. $v_0^1 \in \langle v, v' \rangle$ by (2)	14296
14242	b. $\vdash_1 v_0^1 v_1^1 : K_0$ by 1 static preservation	14297
14243	c. Contradiction by (a, b)	14298
14244	QED	14299
14245	CASE $(\lambda(x:\tau). e_0^E) v_1^E \triangleright_{E-S} e_0^E[x \leftarrow v_1^E] :$	14300

14301	1. $e_0^1 = e_{0'}^1 e_1^1$	14356
14302	by $e_0^1 \underset{1\sim E}{\sim} e_0^E$	14357
14303	2. $e_{0'}^1 \rightarrow_{1-D}^* v_0^1$ and $v_0^1 \underset{1\sim E}{\sim} v_0^E$	14358
14304	$\vee e_{0'}^1 \rightarrow_{1-D}^* \text{BndryErr}$	14359
14305	by 1-E dynamic value stutter	14360
14306	3. IF $e_{0'}^1 \rightarrow_{1-D}^* \text{BndryErr}$:	14361
14307	a. $e^1 \rightarrow_{1-D}^* \text{BndryErr}$	14362
14308	b. QED $E^1[\text{BndryErr}] \underset{1\sim E}{\sim} E^E[\text{TagErr}]$	14363
14309	4. $e_{0'}^1 \rightarrow_{1-D}^* v_1^1$ and $v_1^1 \underset{1\sim E}{\sim} v_1^E$	14364
14310	$\vee e_{0'}^1 \rightarrow_{1-D}^* \text{BndryErr}$	14365
14311	by 1-E static value stutter	14366
14312	5. IF $e_{0'}^1 \rightarrow_{1-D}^* \text{BndryErr}$:	14367
14313	a. $e^1 \rightarrow_{1-D}^* \text{BndryErr}$	14368
14314	b. QED $E^1[\text{BndryErr}] \underset{1\sim E}{\sim} E^E[\text{TagErr}]$	14369
14315	6. $v_0^1 = \lambda(x:\tau). e^1$ and $e^1 \underset{1\sim E}{\sim} e^E$	14370
14316	by (2)	14371
14317	7. $v_0^1 v_1^1 \triangleright_{1-D} e^1[x \leftarrow v_1^1]$	14372
14318	$\vee v_0^1 v_1^1 \triangleright_{1-S} \text{BndryErr}$ and $X(\lfloor \tau \rfloor, v_1^1) = \text{BndryErr}$	14373
14319	$\vee v_0^1 v_1^1 \triangleright_{1-S} e^1[x \leftarrow X(\lfloor \tau \rfloor, v_1^1)]$	14374
14320	8. IF $v_0^1 v_1^1 \triangleright_{1-D} e^1[x \leftarrow v_1^1]$:	14375
14321	a. $e^1[x \leftarrow v_1^1] \underset{1\sim E}{\sim} e^E[x \leftarrow v_1^E]$	14376
14322	by 1-E substitution	14377
14323	b. $E^1[e^1[x \leftarrow v_1^1]] \underset{1\sim E}{\sim} E^E[e^E[x \leftarrow v_1^E]]$	14378
14324	by 1-E hole substitution	14379
14325	c. QED	14380
14326	IF $v_0^1 v_1^1 \triangleright_{1-S} \text{BndryErr}$:	14381
14327	a. $e^1 \rightarrow_{1-S}^* \text{BndryErr}$	14382
14328	b. QED $E^1[\text{BndryErr}] \underset{1\sim E}{\sim} E^E[\text{BndryErr}]$	14383
14329	ELSE $v_0^1 v_1^1 \triangleright_{1-S} e^1[x \leftarrow X(\lfloor \tau \rfloor, v_1^1)]$:	14384
14330	a. $X(\lfloor \tau \rfloor, v_1^1) = v_1^1$	14385
14331	b. $e^1[x \leftarrow v_1^1] \underset{1\sim E}{\sim} e^E[x \leftarrow v_1^E]$	14386
14332	by 1-E substitution	14387
14333	c. $E^1[e^1[x \leftarrow v_1^1]] \underset{1\sim E}{\sim} E^E[e^E[x \leftarrow v_1^E]]$	14388
14334	by 1-E hole substitution	14389
14335	d. QED	14390
14336	CASE $op^1 v_0^E \triangleright_{E-S} \text{TagErr}$:	14391
14337	1. $e_0^1 = op^1 e_{0'}^1$	14392
14338	by $e_0^1 \underset{1\sim E}{\sim} e_0^E$	14393
14339	2. $e_{0'}^1 \rightarrow_{1-D}^* v_0^1$ and $v_0^1 \underset{1\sim E}{\sim} v_0^E$	14394
14340	$\vee e_{0'}^1 \rightarrow_{1-D}^* \text{BndryErr}$	14395
14341	$\vee e_{0'}^1 \rightarrow_{1-S}^* v_0^1$ and $v_0^1 \underset{1\sim E}{\sim} v_0^E$	14396
14342	$\vee e_{0'}^1 \rightarrow_{1-S}^* \text{BndryErr}$	14397
14343	$\vee e_{0'}^1 \rightarrow_{1-S}^* \text{BndryErr}$	14398
14344	by 1-E dynamic value stutter or 1-E static value stutter	14399
14345	3. IF $e_{0'}^1 \rightarrow_{1-D}^* \text{BndryErr}$	14400
14346	$\vee e_{0'}^1 \rightarrow_{1-S}^* \text{BndryErr}$	14401
14347	a. $e^1 \rightarrow_{1-D}^* \text{BndryErr}$	14402
14348	b. QED $E^1[\text{BndryErr}] \underset{1\sim E}{\sim} E^E[\text{TagErr}]$	14403
14349	IF $e_{0'}^1 \rightarrow_{1-D}^* v_0^1$:	14404
14350	a. $v_0^E \notin \langle v, v \rangle$	14405
14351	by definition \triangleright_{E-S}	14406
14352	b. $v_0^E \in \langle v, v \rangle$	14407
14353	by definition \triangleright_{E-S}	14408
14354	c. $op^1 v_0^1 \triangleright_{1-D} \delta(op^1, v_0^1)$	14409
14355	by definition \triangleright_{1-S}	14410

14411	$\vee e_{0'}^1 \rightarrow_{1-S}^* \text{BndryErr}$	14466
14412	by 1-E dynamic value stutter or 1-E static value	
14413	stutter	
14414	3. $e_{1'}^1 \rightarrow_{1-D}^* v_1^1$ and $v_1^1 \sim_E v_1^E$	
14415	$\vee e_{1'}^1 \rightarrow_{1-D}^* \text{BndryErr}$	
14416	$\vee e_{1'}^1 \rightarrow_{1-S}^* v_1^1$ and $v_1^1 \sim_E v_1^E$	
14417	$\vee e_{1'}^1 \rightarrow_{1-S}^* \text{BndryErr}$	
14418	by 1-E dynamic value stutter or 1-E static value	
14419	stutter	
14420	4. IF $e_{0'}^1 \rightarrow_{1-D}^* \text{BndryErr}$	
14421	$\vee e_{0'}^1 \rightarrow_{1-S}^* \text{BndryErr}$	
14422	$\vee e_{1'}^1 \rightarrow_{1-D}^* \text{BndryErr}$	
14423	$\vee e_{1'}^1 \rightarrow_{1-S}^* \text{BndryErr}$:	
14424	a. QED $E^1[\text{BndryErr}] \sim_E E^E[\text{TagErr}]$	
14425	IF $e_{0'}^1 \rightarrow_{1-D}^* v_0^1$	
14426	$\wedge e_{1'}^1 \rightarrow_{1-D}^* v_1^1$:	
14427	a. $v_0^E \notin \mathbb{Z}$	
14428	$\vee v_1^E \notin \mathbb{Z}$	
14429	by definition \triangleright_{E-S}	
14430	b. $v_0^1 \notin \mathbb{Z}$	
14431	$\vee v_1^1 \notin \mathbb{Z}$	
14432	by (a)	
14433	c. $op^2 v_0^1 v_1^1 \triangleright_{1-D} \text{TagErr}$	
14434	by definition \triangleright_{1-D}	
14435	d. QED $E^1[\text{TagErr}] \sim_E E^E[\text{TagErr}]$	
14436	ELSE $e_{0'}^1 \rightarrow_{1-S}^* v_0^1$	
14437	$\wedge e_{1'}^E \rightarrow_{1-S}^* v_1^1$:	
14438	a. $\vdash_1 op^2 e_{0'}^1 e_{1'}^1 : K_0$	
14439	b. $\vdash_1 op^2 v_0^1 v_1^1 : K_0$	
14440	by 1 static preservation	
14441	c. $v_0^E \notin \mathbb{Z}$	
14442	$\vee v_1^E \notin \mathbb{Z}$	
14443	by definition \triangleright_{E-S}	
14444	d. $v_0^1 \notin \mathbb{Z}$	
14445	$\vee v_1^1 \notin \mathbb{Z}$	
14446	by (c)	
14447	e. Contradiction by (b)	
14448	CASE $op^2 v_0^E v_1^E \triangleright_{E-S} \delta(op^2, v_0^E, v_1^E)$:	
14449	1. $e_0^1 = op^2 e_{0'}^1 e_{1'}^1$	
14450	by $e_0^1 \sim_E e_{1'}^E$	
14451	2. $e_{0'}^1 \rightarrow_{1-D}^* v_0^1$ and $v_0^1 \sim_E v_0^E$	
14452	$\vee e_{0'}^1 \rightarrow_{1-D}^* \text{BndryErr}$	
14453	$\vee e_{0'}^1 \rightarrow_{1-S}^* v_0^1$ and $v_0^1 \sim_E v_0^E$	
14454	$\vee e_{0'}^1 \rightarrow_{1-S}^* \text{BndryErr}$	
14455	by 1-E dynamic value stutter or 1-E static value	
14456	stutter	
14457	3. $e_{1'}^1 \rightarrow_{1-D}^* v_1^1$ and $v_1^1 \sim_E v_1^E$	
14458	$\vee e_{1'}^1 \rightarrow_{1-D}^* \text{BndryErr}$	
14459	$\vee e_{1'}^1 \rightarrow_{1-S}^* v_1^1$ and $v_1^1 \sim_E v_1^E$	
14460	$\vee e_{1'}^1 \rightarrow_{1-S}^* \text{BndryErr}$	
14461	by 1-E dynamic value stutter or 1-E static value	
14462	stutter	
14463		
14464		
14465		
4.	IF $e_{0'}^1 \rightarrow_{1-D}^* \text{BndryErr}$	14466
	$\vee e_{0'}^1 \rightarrow_{1-S}^* \text{BndryErr}$	14467
	$\vee e_{1'}^1 \rightarrow_{1-D}^* \text{BndryErr}$	14468
	$\vee e_{1'}^1 \rightarrow_{1-S}^* \text{BndryErr}$:	14469
	a. QED $E^1[\text{BndryErr}] \sim_E E^E[\text{TagErr}]$	14470
	IF $e_{0'}^1 \rightarrow_{1-D}^* v_0^1$	14471
	$\wedge e_{1'}^1 \rightarrow_{1-D}^* v_1^1$:	14472
	a. $v_0^E \in \mathbb{Z}$	14473
	$\wedge v_1^E \in \mathbb{Z}$	14474
	by definition \triangleright_{E-S}	14475
	b. $v_0^1 \in \mathbb{Z}$	14476
	by $v_0^1 \sim_E v_0^E$	14477
	c. $v_1^1 \in \mathbb{Z}$	14478
	by $v_1^1 \sim_E v_1^E$	14479
	d. $op^2 v_0^1 v_1^1 \triangleright_{1-D} \delta(op^2, v_0^1, v_1^1)$	14480
	by definition \triangleright_{1-D}	14481
	e. $\delta(op^2, v_0^1, v_1^1) \sim_E \delta(op^2, v_0^E, v_1^E)$	14482
	by 1-E delta	14483
	f. QED $E^1[\delta(op^2, v_0^1, v_1^1)] \sim_E E^E[\delta(op^2, v_0^E, v_1^E)]$	14484
	ELSE $e_{0'}^1 \rightarrow_{1-S}^* v_0^1$	14485
	$\wedge e_{1'}^E \rightarrow_{1-S}^* v_1^1$:	14486
	a. $v_0^E \in \mathbb{Z}$	14487
	$\wedge v_1^E \in \mathbb{Z}$	14488
	by definition \triangleright_{E-S}	14489
	b. $v_0^1 \in \mathbb{Z}$	14490
	by $v_0^1 \sim_E v_0^E$	14491
	c. $v_1^1 \in \mathbb{Z}$	14492
	by $v_1^1 \sim_E v_1^E$	14493
	d. $op^2 v_0^1 v_1^1 \triangleright_{1-S} \delta(op^2, v_0^1, v_1^1)$	14494
	by definition \triangleright_{1-S}	14495
	e. $\delta(op^2, v_0^1, v_1^1) \sim_E \delta(op^2, v_0^E, v_1^E)$	14496
	by 1-E delta	14497
	f. QED $E^1[\delta(op^2, v_0^1, v_1^1)] \sim_E E^E[\delta(op^2, v_0^E, v_1^E)]$	14498
	□	14500
	Lemma 8.5 : 1-E static value stutter	14501
	If $e^1 \sim_E v^E$ and $\vdash_1 e^1 : K$ then one of the following holds:	14502
	• $e^1 \rightarrow_{1-S}^* v^1$ and $v^1 \sim_E v^E$	14503
	• $e^1 \rightarrow_{1-S}^* \text{BndryErr}$	14504
	<i>Proof:</i>	14505
	By induction on the structure of e^1 , and case analysis on $e^1 \sim_E v^E$.	14506
	CASE e^1 is a value :	14507
	1. QED	14508
	CASE $e^1 = \text{chk } K e_0^1$:	14509
	1. $e_0^1 \sim_E v^E$	14510
	by definition \sim_E	14511
	2. $\vdash_1 e_0^1 : \text{Any}$	14512
	3. $e_0^1 \rightarrow_{1-S}^* v_0^1$ and $v_0^1 \sim_E v^E$	14513
	$\vee e_0^1 \rightarrow_{1-S}^* \text{BndryErr}$	14514
	by the induction hypothesis	14515
	4. IF $e_0^1 \rightarrow_{1-S}^* \text{BndryErr}$:	14516
	a. QED $e^1 \rightarrow_{1-S}^* \text{BndryErr}$	14517
		14518
		14519
		14520

14521 **IF** $e_0^1 \rightarrow_{1-S}^* v_0^1$
 14522 $\wedge X(K, e_0^1) = \text{BndryErr}$:
 14523 a. QED $e^1 \rightarrow_{1-S}^* \text{BndryErr}$
 14524 **ELSE** $e_0^1 \rightarrow_{1-S}^* v_0^1$
 14525 $\wedge X(K, e_0^1) = v_0^1$:
 14526 a. QED $e^1 \rightarrow_{1-S}^* v_0^1$
 14527 **CASE** $e^1 = \text{dyn } e_0^1$:
 14528 1. $e_0^1 \sim_E v^E$
 14529 by definition \sim_E
 14530 2. $\vdash_1 e_0^1$
 14531 3. $e_0^1 \rightarrow_{1-D}^* v_0^1$ and $v_0^1 \sim_E v^E$
 14532 $\vee e_0^1 \rightarrow_{1-D}^* \text{BndryErr}$
 14533 by 1-E dynamic value stutter
 14534 4. **IF** $e_0^1 \rightarrow_{1-D}^* \text{BndryErr}$:
 14535 a. QED $e^1 \rightarrow_{1-S}^* \text{BndryErr}$
 14536 **ELSE** :
 14537 a. QED $e^1 \rightarrow_{1-S}^* v_0^1$
 14538 **CASE** $e^1 = \text{stat } v_0^1$:
 14539 1. Contradiction by $\vdash_1 e^1 : K$
 14540 □

Lemma 8.6 : 1-E dynamic value stutter

If $e^1 \sim_E v^E$ and $\vdash_1 e^1$ then one of the following holds:

- $e^1 \rightarrow_{1-D}^* v^1$ and $v^1 \sim_E v^E$
- $e^1 \rightarrow_{1-D}^* \text{BndryErr}$

Proof:

By induction on the structure of e^1 , and case analysis on $e^1 \sim_E v^E$.

CASE e^1 is a value :

1. QED

CASE $e^1 = \text{chk } K e_0^1$:

1. Contradiction by $\vdash_1 e^1$

CASE $e^1 = \text{dyn } e_0^1$:

1. Contradiction by $\vdash_1 e^1$

CASE $e^1 = \text{stat } v_0^1$:

1. $e_0^1 \sim_E v^E$
by definition \sim_E
2. $\vdash_1 e_0^1$: Any
3. $e_0^1 \rightarrow_{1-S}^* v_0^1$ and $v_0^1 \sim_E v^E$
 $\vee e_0^1 \rightarrow_{1-S}^* \text{BndryErr}$
by 1-E static value stutter
4. **IF** $e_0^1 \rightarrow_{1-S}^* \text{BndryErr}$:
a. QED $e^1 \rightarrow_{1-D}^* \text{BndryErr}$
5. **ELSE** :
a. QED $e^1 \rightarrow_{1-D}^* v_0^1$

Lemma 8.7 : 1-E hole substitution

If $e^1 \sim_E e^E$ and $E^1 \sim_E E^E$ then $E^1[e^1] \sim_E E^E[e^E]$

Proof:

By induction on the structure of E^1 .

CASE $E^1 = []$:
1. $e^E = []$

CASE $E^1 = op^1 E_0^1$:
1. $E^E = op^1 E_0^E$
 $\wedge E_0^1 \sim_E E_0^E$
by definition \sim_E

2. $E_0^1[e^1] \sim_E E_0^E[e^E]$

by the induction hypothesis

3. $E^1[e^1] = op^1 E_0^1[e_0^1]$
4. $E^E[e^E] = op^1 E_0^E[e_0^E]$
5. QED by (1, 2)

CASE $E^1 = op^2 E_0^1 e_1^1$:

1. $E^E = op^2 E_0^E e_1^E$
 $\wedge E_0^1 \sim_E E_0^E$
 $\wedge e_1^1 \sim_E e_1^E$
by definition \sim_E

2. $E_0^1[e^1] \sim_E E_0^E[e^E]$

by the induction hypothesis

3. $E^1[e^1] = op^2 E_0^1[e_0^1]$
4. $E^E[e^E] = op^2 E_0^E[e_0^E]$
5. QED by (1, 2)

CASE $E^1 = E_0^1 e_1^1$:
1. $E^E = E_0^E e_1^E$
 $\wedge E_0^1 \sim_E E_0^E$
 $\wedge e_1^1 \sim_E e_1^E$
by definition \sim_E

2. $E_0^1[e^1] \sim_E E_0^E[e^E]$

by the induction hypothesis

3. $E^1[e^1] = E_0^1[e_0^1] e_1^1$
4. $E^E[e^E] = E_0^E[e_0^E] e_1^E$
5. QED by (1, 2)

CASE $E^1 = v_0^1 E_1^1$:
1. $E^E = v_0^E E_1^E$
 $\wedge e_0^1 \sim_E e_0^E$
 $\wedge E_1^1 \sim_E E_1^E$
by definition \sim_E

2. $E_1^1[e^1] \sim_E E_1^E[e^E]$

by the induction hypothesis

3. $E^1[e^1] = v_0^1 E_1^1[e_1^1]$
4. $E^E[e^E] = v_0^E E_1^E[e_1^E]$
5. QED by (1, 2)

CASE $E^1 = \langle E_0^1, e_1^1 \rangle$:
1. $E^E = \langle E_0^E, e_1^E \rangle$
 $\wedge E_0^1 \sim_E E_0^E$
 $\wedge e_1^1 \sim_E e_1^E$
by definition \sim_E

2. $E_0^1[e^1] \sim_E E_0^E[e^E]$

by the induction hypothesis

3. $E^1[e^1] = \langle E_0^1[e_0^1], e_1^1 \rangle$
4. $E^E[e^E] = \langle E_0^E[e_0^E], e_1^E \rangle$
5. QED by (1, 2)

CASE $E^1 = \langle v_0^1, E_1^1 \rangle$:
1. $E^E = \langle v_0^E, E_1^E \rangle$
 $\wedge e_0^1 \sim_E e_0^E$
 $\wedge E_1^1 \sim_E E_1^E$
by definition \sim_E

2. $E_1^1[e^1] \sim_E E_1^E[e^E]$

by the induction hypothesis

3. $E^1[e^1] = \langle e_0^1, E_1^1[e_1^1] \rangle$
4. $E^E[e^E] = \langle e_0^E, E_1^E[e_1^E] \rangle$
5. QED by (1, 2)

CASE $E^1 = op^1 E_0^1$:
1. $E^E = op^1 E_0^E$
 $\wedge E_0^1 \sim_E E_0^E$
by definition \sim_E

2. $E_0^1[e^1] \sim_E E_0^E[e^E]$

by the induction hypothesis

3. $E^1[e^1] = op^1 E_0^1[e_0^1]$
4. $E^E[e^E] = op^1 E_0^E[e_0^E]$
5. QED by (1, 2)

CASE $E^1 = op^2 E_0^1 e_1^1$:
1. $E^E = op^2 E_0^E e_1^E$
 $\wedge E_0^1 \sim_E E_0^E$
 $\wedge e_1^1 \sim_E e_1^E$
by definition \sim_E

2. $E_0^1[e^1] \sim_E E_0^E[e^E]$

by the induction hypothesis

3. $E^1[e^1] = op^2 E_0^1[e_0^1]$
4. $E^E[e^E] = op^2 E_0^E[e_0^E]$
5. QED by (1, 2)

CASE $E^1 = E_0^1 e_1^1$:
1. $E^E = E_0^E e_1^E$
 $\wedge E_0^1 \sim_E E_0^E$
 $\wedge e_1^1 \sim_E e_1^E$
by definition \sim_E

2. $E_0^1[e^1] \sim_E E_0^E[e^E]$

by the induction hypothesis

3. $E^1[e^1] = E_0^1[e_0^1] e_1^1$
4. $E^E[e^E] = E_0^E[e_0^E] e_1^E$
5. QED by (1, 2)

CASE $E^1 = v_0^1 E_1^1$:
1. $E^E = v_0^E E_1^E$
 $\wedge e_0^1 \sim_E e_0^E$
 $\wedge E_1^1 \sim_E E_1^E$
by definition \sim_E

2. $E_1^1[e^1] \sim_E E_1^E[e^E]$

by the induction hypothesis

3. $E^1[e^1] = v_0^1 E_1^1[e_1^1]$
4. $E^E[e^E] = v_0^E E_1^E[e_1^E]$
5. QED by (1, 2)

CASE $E^1 = \langle v_0^1, E_1^1 \rangle$:
1. $E^E = \langle v_0^E, E_1^E \rangle$
 $\wedge e_0^1 \sim_E e_0^E$
 $\wedge E_1^1 \sim_E E_1^E$
by definition \sim_E

2. $E_1^1[e^1] \sim_E E_1^E[e^E]$

by the induction hypothesis

3. $E^1[e^1] = \langle v_0^1, E_1^1[e_1^1] \rangle$
4. $E^E[e^E] = \langle v_0^E, E_1^E[e_1^E] \rangle$
5. QED by (1, 2)

CASE $E^1 = \langle E_0^1, v_0^1 \rangle$:
1. $E^E = \langle E_0^E, v_0^E \rangle$
 $\wedge E_0^1 \sim_E E_0^E$
 $\wedge v_0^1 \sim_E v_0^E$
by definition \sim_E

2. $E_0^1[e^1] \sim_E E_0^E[e^E]$

by the induction hypothesis

3. $E^1[e^1] = \langle E_0^1[v_0^1], v_0^1 \rangle$
4. $E^E[e^E] = \langle E_0^E[v_0^E], v_0^1 \rangle$
5. QED by (1, 2)

CASE $E^1 = v_0^1 E_0^1$:
1. $E^E = v_0^E E_0^E$
 $\wedge v_0^1 \sim_E v_0^E$
by definition \sim_E

2. $E_0^1[e^1] \sim_E E_0^E[e^E]$

by the induction hypothesis

3. $E^1[e^1] = v_0^1 E_0^1[v_0^1]$
4. $E^E[e^E] = v_0^E E_0^E[v_0^1]$
5. QED by (1, 2)

CASE $E^1 = v_0^1 v_1^1$:
1. $E^E = v_0^E v_1^E$
 $\wedge v_0^1 \sim_E v_0^E$
 $\wedge v_1^1 \sim_E v_1^E$
by definition \sim_E

2. $v_0^1[v_1^1] \sim_E v_0^E[v_1^E]$

by the induction hypothesis

3. $E^1[v_1^1] = v_0^1 v_1^1$
4. $E^E[v_1^E] = v_0^E v_1^E$
5. QED by (1, 2)

CASE $E^1 = v_0^1 v_1^1$:
1. $E^E = v_0^E v_1^E$
 $\wedge v_0^1 \sim_E v_0^E$
 $\wedge v_1^1 \sim_E v_1^E$
by definition \sim_E

2. $v_0^1[v_1^1] \sim_E v_0^E[v_1^E]$

by the induction hypothesis

3. $E^1[v_1^1] = v_0^1 v_1^1$
4. $E^E[v_1^E] = v_0^E v_1^E$
5. QED by (1, 2)

CASE $E^1 = v_0^1 v_1^1$:
1. $E^E = v_0^E v_1^E$
 $\wedge v_0^1 \sim_E v_0^E$
 $\wedge v_1^1 \sim_E v_1^E$
by definition \sim_E

2. $v_0^1[v_1^1] \sim_E v_0^E[v_1^E]$

by the induction hypothesis

3. $E^1[v_1^1] = v_0^1 v_1^1$
4. $E^E[v_1^E] = v_0^E v_1^E$
5. QED by (1, 2)

CASE $E^1 = v_0^1 v_1^1$:
1. $E^E = v_0^E v_1^E$
 $\wedge v_0^1 \sim_E v_0^E$
 $\wedge v_1^1 \sim_E v_1^E$
by definition \sim_E

2. $v_0^1[v_1^1] \sim_E v_0^E[v_1^E]$

by the induction hypothesis

3. $E^1[v_1^1] = v_0^1 v_1^1$
4. $E^E[v_1^E] = v_0^E v_1^E$
5. QED by (1, 2)

CASE $E^1 = v_0^1 v_1^1$:
1. $E^E = v_0^E v_1^E$
 $\wedge v_0^1 \sim_E v_0^E$
 $\wedge v_1^1 \sim_E v_1^E$
by definition \sim_E

2. $v_0^1[v_1^1] \sim_E v_0^E[v_1^E]$

by the induction hypothesis

3. $E^1[v_1^1] = v_0^1 v_1^1$
4. $E^E[v_1^E] = v_0^E v_1^E$
5. QED by (1, 2)

CASE $E^1 = v_0^1 v_1^1$:
1. $E^E = v_0^E v_1^E$
 $\wedge v_0^1 \sim_E v_0^E$
 $\wedge v_1^1 \sim_E v_1^E$
by definition \sim_E

2. $v_0^1[v_1^1] \sim_E v_0^E[v_1^E]$

by the induction hypothesis

3. $E^1[v_1^1] = v_0^1 v_1^1$
4. $E^E[v_1^E] = v_0^E v_1^E$
5. QED by (1, 2)

CASE $E^1 = v_0^1 v_1^1$:
1. $E^E = v_0^E v_1^E$
 $\wedge v_0^1 \sim_E v_0^E$
 $\wedge v_1^1 \sim_E v_1^E$
by definition \sim_E

2. $v_0^1[v_1^1] \sim_E v_0^E[v_1^E]$

by the induction hypothesis

3. $E^1[v_1^1] = v_0^1 v_1^1$
4. $E^E[v_1^E] = v_0^E v_1^E$
5. QED by (1, 2)

CASE $E^1 = v_0^1 v_1^1$:
1. $E^E = v_0^E v_1^E$
 $\wedge v_0^1 \sim_E v_0^E$
 $\wedge v_1^1 \sim_E v_1^E$
by definition \sim_E

2. $v_0^1[v_1^1] \sim_E v_0^E[v_1^E]$

by the induction hypothesis

3. $E^1[v_1^1] = v_0^1 v_1^1$
4. $E^E[v_1^E] = v_0^E v_1^E$
5. QED by (1, 2)

CASE $E^1 = v_0^1 v_1^1$:
1. $E^E = v_0^E v_1^E$
 $\wedge v_0^1 \sim_E v_0^E$
 $\wedge v_1^1 \sim_E v_1^E$
by definition \sim_E

2. $v_0^1[v_1^1] \sim_E v_0^E[v_1^E]$

by the induction hypothesis

3. $E^1[v_1^1] = v_0^1 v_1^1$
4. $E^E[v_1^E] = v_0^E v_1^E$
5. QED by (1, 2)

CASE $E^1 = v_0^1 v_1^1$:
1. $E^E = v_0^E v_1^E$
 $\wedge v_0^1 \sim_E v_0^E$
 $\wedge v_1^1 \sim_E v_1^E$
by definition \sim_E

2. $v_0^1[v_1^1] \sim_E v_0^E[v_1^E]$

by the induction hypothesis

3. $E^1[v_1^1] = v_0^1 v_1^1$
4. $E^E[v_1^E] = v_0^E v_1^E$
5. QED by (1, 2)

CASE $E^1 = v_0^1 v_1^1$:
1. $E^E = v_0^E v_1^E$
 $\wedge v_0^1 \sim_E v_0^E$
 $\wedge v_1^1 \sim_E v_1^E$
by definition \sim_E

2. $v_0^1[v_1^1] \sim_E v_0^E[v_1^E]$

by the induction hypothesis

3. $E^1[v_1^1] = v_0^1 v_1^1$
4. $E^E[v_1^E] = v_0^E v_1^E$
5. QED by (1, 2)

CASE $E^1 = v_0^1 v_1^1$:
1. $E^E = v_0^E v_1^E$
 $\wedge v_0^1 \sim_E v_0^E$
 $\wedge v_1^1 \sim_E v_1^E$
by definition \sim_E

2. $v_0^1[v_1^1] \sim_E v_0^E[v_1^E]$

by the induction hypothesis

3. $E^1[v_1^1] = v_0^1 v_1^1$
4. $E^E[v_1^E] = v_0^E v_1^E$
5. QED by (1, 2)

CASE $E^1 = v_0^1 v_1^1$:
1. $E^E = v_0^E v_1^E$
 $\wedge v_0^1 \sim_E v_0^E$
 $\wedge v_1^1 \sim_E v_1^E$
by definition \sim_E

2. $v_0^1[v_1^1] \sim_E v_0^E[v_1^E]$

by the induction hypothesis

3. $E^1[v_1^1] = v_0^1 v_1^1$
4. $E^E[v_1^E] = v_0^E v_1^E$
5. QED by (1, 2)

CASE $E^1 = v_0^1 v_1^1$:
1. $E^E = v_0^E v_1^E$
 $\wedge v_0^1 \sim_E v_0^E$
 $\wedge v_1^1 \sim_E v_1^E$
by definition \sim_E

2. $v_0^1[v_1^1] \sim_E v_0^E[v_1^E]$

by the induction hypothesis

3. $E^1[v_1^1] = v_0^1 v_1^1$
4. $E^E[v_1^E] = v_0^E v_1^E$
5. QED by (1, 2)

CASE $E^1 = v_0^1 v_1^1$:
1. $E^E = v_0^E v_1^E$
 $\wedge v_0^1 \sim_E v_0^E$
 $\wedge v_1^1 \sim_E v_1^E$
by definition \sim_E

2. $v_0^1[v_1^1] \sim_E v_0^E[v_1^E]$

14631	1. $E^E = op^2 E_0^E e_1^E$	14686
14632	$\wedge E_0^1 \underset{1 \sim E}{\lesssim} E_0^E$	14687
14633	$\wedge e_1^1 \underset{1 \sim E}{\lesssim} e_1^E$	14688
14634	by definition $\underset{1 \sim E}{\lesssim}$	14689
14635	2. $E_0^1[e^1] \underset{1 \sim E}{\lesssim} E_0^E[e^E]$	14690
14636	by the induction hypothesis	14691
14637	3. $E^1[e^1] = op^2 E_0^1[e_0^1] e_1^1$	14692
14638	4. $E^E[e^E] = op^2 E_0^E[e_0^E] e_1^E$	14693
14639	5. QED by (1, 2)	14694
14640	CASE $E^1 = op^2 v_0^1 E_1^1$:	14695
14641	1. $E^E = op^2 e_0^E E_1^E$	14696
14642	$\wedge e_0^1 \underset{1 \sim E}{\lesssim} e_0^E$	14697
14643	$\wedge E_1^1 \underset{1 \sim E}{\lesssim} E_1^E$	14698
14644	by definition $\underset{1 \sim E}{\lesssim}$	14699
14645	2. $E_1^1[e^1] \underset{1 \sim E}{\lesssim} E_1^E[e^E]$	14700
14646	by the induction hypothesis	14701
14647	3. $E^1[e^1] = op^2 e_0^1 E_1^1[e_1^1]$	14702
14648	4. $E^E[e^E] = op^2 e_0^E E_1^E[e_1^E]$	14703
14649	5. QED by (1, 2)	14704
14650	CASE $E^1 = \text{dyn } \tau E_0^1$:	14705
14651	1. $E^E = \text{dyn } \tau E_0^E$	14706
14652	$\wedge E_0^1 \underset{1 \sim E}{\lesssim} E_0^E$	14707
14653	by definition $\underset{1 \sim E}{\lesssim}$	14708
14654	2. $E_0^1[e^1] \underset{1 \sim E}{\lesssim} E_0^E[e^E]$	14709
14655	by the induction hypothesis	14710
14656	3. $E^1[e^1] = \text{dyn } \tau E_0^1[e_0^1]$	14711
14657	4. $E^E[e^E] = \text{dyn } \tau E_0^E[e_0^E]$	14712
14658	5. QED by (1, 2)	14713
14659	CASE $E^1 = \text{stat } \tau E_0^1$:	14714
14660	1. $E^E = \text{stat } \tau E_0^E$	14715
14661	$\wedge E_0^1 \underset{1 \sim E}{\lesssim} E_0^E$	14716
14662	by definition $\underset{1 \sim E}{\lesssim}$	14717
14663	2. $E_0^1[e^1] \underset{1 \sim E}{\lesssim} E_0^E[e^E]$	14718
14664	by the induction hypothesis	14719
14665	3. $E^1[e^1] = \text{stat } \tau E_0^1[e_0^1]$	14720
14666	4. $E^E[e^E] = \text{stat } \tau E_0^E[e_0^E]$	14721
14667	5. QED by (1, 2)	14722
14668	CASE $E^1 = \text{chk } K E_0^1$:	14723
14669	1. $E_0^1 \underset{1 \sim E}{\lesssim} E^E$	14724
14670	2. $E_0^1[e^1] \underset{1 \sim E}{\lesssim} E^E[e^E]$	14725
14671	by the induction hypothesis (1)	14726
14672	3. $E^1[e^1] = \text{chk } K E_0^1[e^1]$	14727
14673	4. QED (2)	14728
14674	CASE $E^1 = \text{dyn } E_0^1$:	14729
14675	1. $E_0^1 \underset{1 \sim E}{\lesssim} E^E$	14730
14676	2. $E_0^1[e^1] \underset{1 \sim E}{\lesssim} E^E[e^E]$	14731
14677	by the induction hypothesis (1)	14732
14678	3. $E^1[e^1] = \text{dyn } E_0^1[e^1]$	14733
14679	4. QED (2)	14734
14680	CASE $E^1 = \text{stat } E_0^1$:	14735
14681	1. $E_0^1 \underset{1 \sim E}{\lesssim} E^E$	14736
14682	2. $E_0^1[e^1] \underset{1 \sim E}{\lesssim} E^E[e^E]$	14737
14683	by the induction hypothesis (1)	14738
14684	3. $E^1[e^1] = \text{stat } E_0^1[e^1]$	14739
14685		14740

14741	2. $e_0^1[x \leftarrow v^1] \underset{\sim}{\underset{E}{\sim}} e_0^E[x \leftarrow v^E]$	14796
14742	by the induction hypothesis (1)	14797
14743	3. $e^1[x \leftarrow v^1] = \lambda(y:\tau). e_0^1[x \leftarrow v^E]$	14798
14744	$\wedge e^E[x \leftarrow v^E] = \lambda(y:\tau). e_0^E[x \leftarrow v^E]$	14799
14745	4. QED (2, 3)	14800
14746	CASE $e^1 = \text{mon } \tau v_0^1$:	14801
14747	1. Contradiction by $e^1 \underset{\sim}{\underset{E}{\sim}} e^E$	14802
14748	CASE $e^1 = \langle e_0^1, e_1^1 \rangle$:	14803
14749	1. $e^E = \langle e_0^E, e_1^E \rangle$	14804
14750	2. $e_0^1[x \leftarrow v^1] \underset{\sim}{\underset{E}{\sim}} e_0^E[x \leftarrow v^E]$	14805
14751	$\wedge e_1^1[x \leftarrow v^1] \underset{\sim}{\underset{E}{\sim}} e_1^E[x \leftarrow v^E]$	14806
14752	by the induction hypothesis	14807
14753	3. $e^1[x \leftarrow v^1] = \langle e_0^1[x \leftarrow v^1], e_1^1[x \leftarrow v^1] \rangle$	14808
14754	$\wedge e^E[x \leftarrow v^E] = \langle e_0^E[x \leftarrow v^E], e_1^E[x \leftarrow v^E] \rangle$	14809
14755	4. QED (2, 3)	14810
14756	CASE $e^1 = e_0^1 e_1^1$:	14811
14757	1. $e^E = e_0^E e_1^E$	14812
14758	2. $e_0^1[x \leftarrow v^1] \underset{\sim}{\underset{E}{\sim}} e_0^E[x \leftarrow v^E]$	14813
14759	$\wedge e_1^1[x \leftarrow v^1] \underset{\sim}{\underset{E}{\sim}} e_1^E[x \leftarrow v^E]$	14814
14760	by the induction hypothesis	14815
14761	3. $e^1[x \leftarrow v^1] = e_0^1[x \leftarrow v^1] e_1^1[x \leftarrow v^1]$	14816
14762	$\wedge e^E[x \leftarrow v^E] = e_0^E[x \leftarrow v^E] e_1^E[x \leftarrow v^E]$	14817
14763	4. QED (2, 3)	14818
14764	CASE $e^1 = op^1 e_0^1$:	14819
14765	1. $e^E = op^1 e_0^E$	14820
14766	2. $e_0^1[x \leftarrow v^1] \underset{\sim}{\underset{E}{\sim}} e_0^E[x \leftarrow v^E]$	14821
14767	by the induction hypothesis	14822
14768	3. $e^1[x \leftarrow v^1] = op^1 e_0^1[x \leftarrow v^1]$	14823
14769	$\wedge e^E[x \leftarrow v^E] = op^1 e_0^E[x \leftarrow v^E]$	14824
14770	4. QED (2, 3)	14825
14771	CASE $e^1 = op^2 e_0^1 e_1^1$:	14826
14772	1. $e^E = op^2 e_0^E e_1^E$	14827
14773	2. $e_0^1[x \leftarrow v^1] \underset{\sim}{\underset{E}{\sim}} e_0^E[x \leftarrow v^E]$	14828
14774	$\wedge e_1^1[x \leftarrow v^1] \underset{\sim}{\underset{E}{\sim}} e_1^E[x \leftarrow v^E]$	14829
14775	by the induction hypothesis	14830
14776	3. $e^1[x \leftarrow v^1] = op^2 e_0^1[x \leftarrow v^1] e_1^1[x \leftarrow v^1]$	14831
14777	$\wedge e^E[x \leftarrow v^E] = op^2 e_0^E[x \leftarrow v^E] e_1^E[x \leftarrow v^E]$	14832
14778	4. QED (2, 3)	14833
14779	CASE $e^1 = \text{dyn } \tau e_0^1$:	14834
14780	1. $e^E = \text{dyn } \tau e_0^E$	14835
14781	2. $e_0^1[x \leftarrow v^1] \underset{\sim}{\underset{E}{\sim}} e_0^E[x \leftarrow v^E]$	14836
14782	by the induction hypothesis	14837
14783	3. $e^1[x \leftarrow v^1] = \text{dyn } \tau e_0^1[x \leftarrow v^1]$	14838
14784	$\wedge e^E[x \leftarrow v^E] = \text{dyn } \tau e_0^E[x \leftarrow v^E]$	14839
14785	4. QED (2, 3)	14840
14786	CASE $e^1 = \text{stat } \tau e_0^1$:	14841
14787	1. $e^E = \text{stat } \tau e_0^E$	14842
14788	2. $e_0^1[x \leftarrow v^1] \underset{\sim}{\underset{E}{\sim}} e_0^E[x \leftarrow v^E]$	14843
14789	by the induction hypothesis	14844
14790	3. $e^1[x \leftarrow v^1] = \text{stat } \tau e_0^1[x \leftarrow v^1]$	14845
14791	$\wedge e^E[x \leftarrow v^E] = \text{stat } \tau e_0^E[x \leftarrow v^E]$	14846
14792	4. QED (2, 3)	14847
14793		14848
14794		14849
14795		14850

14851 **CASE** $op^2 = \text{quotient} :$
 14852 1. $v_0^1 \in \mathbb{Z}$
 14853 $\wedge v_1^1 \in \mathbb{Z}$
 14854 by $\delta(op^2, v_0^1, v_1^1)$ is defined
 14855 2. $v_0^1 = v_0^E$
 14856 $\wedge v_1^1 = v_1^E$
 14857 by \sim_{E}
 14858 3. QED
 14859 □

14906 **Lemma 8.10 : H-1 approximation**

14907 If $e \in e_S$ and $\vdash e : \tau$ and $\vdash e : [\tau] \rightsquigarrow e''$ and $e'' \rightarrow_{1-S}^* \text{Err}$
 14908 then $e \rightarrow_{H-S}^* \text{Err}$

14909 *Proof:*

- $e \sim_{H-1} e''$
 by **H-1 static reflexivity**
- QED by **H-1 simulation**

14913 □

14914 **Lemma 8.11 : H-1 static reflexivity**

14915 If $\Gamma \vdash e : \tau$
 14916 $\wedge \Gamma \vdash e : [\tau] \rightsquigarrow e''$
 14917 then $e \sim_{H-1} e''$.

14918 *Proof:*

14919 By structural induction on the $\Gamma \vdash e : [\tau] \rightsquigarrow e''$ judgment.

14921 **CASE** $\boxed{\Gamma \vdash i : \text{Nat} \rightsquigarrow i} :$

14923 1. QED $i \sim_{H-1} i$

14924 **CASE** $\boxed{\Gamma \vdash i : \text{Int} \rightsquigarrow i} :$

14926 1. QED $i \sim_{H-1} i$

14927 **CASE** $\boxed{\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 : \tau_1 \rightsquigarrow e'_1} :$

14929 1. $e'_0 \sim_{H-1} e_0$
 14931 $\wedge e'_1 \sim_{H-1} e_1$
 14932 by the induction hypothesis

14933 2. QED

14934 **CASE** $\boxed{(x : \tau_d), \Gamma \vdash e : \tau_c \rightsquigarrow e'} :$

14936 $\Gamma \vdash \lambda(x : \tau_d). e : \tau_d \Rightarrow \tau_c \rightsquigarrow \lambda(x : \tau_d). e'$

14937 1. $e' \sim_{H-1} e$

14938 by the induction hypothesis

14939 2. QED

14940 **CASE** $\boxed{\Gamma \vdash x : \tau \rightsquigarrow x} :$

14941 1. QED $x \sim_{H-1} x$

14943 **CASE** $\boxed{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 : \tau_d \rightsquigarrow e'_1 \quad [\tau_c] = K} :$

14944 $\Gamma \vdash e_0 e_1 : \tau_c \rightsquigarrow \text{chk } K(e'_0 e'_1)$

14945 :

14947 1. $e'_0 \sim_{H-1} e_0$
 14948 $\wedge e'_1 \sim_{H-1} e_1$
 14949 by the induction hypothesis

14950 2. $e'_0 e'_1 \sim_{H-1} e_0 e_1$

14951 3. QED $\text{chk } K e'_0 e'_1 \sim_{H-1} e_0 e_1$

14952 **CASE** $\boxed{\Gamma \vdash e : \tau_0 \times \tau_1 \rightsquigarrow e' \quad [\tau_0] = K} :$

14953 $\Gamma \vdash \text{fst } e : \tau_0 \rightsquigarrow \text{chk } K(\text{fst } e')$

14955 1. $e' \sim_{H-1} e$

14956 by the induction hypothesis

14957 2. $\text{fst } e' \sim_{H-1} \text{fst } e$

14958 3. QED $\text{chk } K \text{ fst } e' \sim_{H-1} \text{fst } e$

14961	CASE	$\frac{\Gamma \vdash e : \tau_0 \times \tau_1 \rightsquigarrow e' \quad \lfloor \tau_1 \rfloor = K}{\Gamma \vdash \text{snd } e : \tau_1 \rightsquigarrow \text{chk } K(\text{snd } e')} :$	
14962		1. $e' \text{ H} \leqslant_1 e$ by the induction hypothesis	15016
14963		2. $\text{snd } e' \text{ H} \leqslant_1 \text{snd } e$	15017
14964		3. QED $\text{chk } K \text{ snd } e' \text{ H} \leqslant_1 \text{snd } e$	15018
14965		$\frac{\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 : \tau_1 \rightsquigarrow e'_1}{\Gamma \vdash \text{op}^2 e_0 e_1 : \tau \rightsquigarrow \text{op}^2 e'_0 e'_1} :$	15019
14966	CASE	1. $e'_0 \text{ H} \leqslant_1 e_0$ $\wedge e'_1 \text{ H} \leqslant_1 e_1$ by the induction hypothesis	15020
14967		2. QED	15021
14968		$\frac{\Gamma \vdash e : \tau' \rightsquigarrow e' \quad \tau' \leqslant: \tau}{\Gamma \vdash e : \tau \rightsquigarrow e'} :$	15022
14969	CASE	1. QED by the induction hypothesis	15023
14970		$\frac{\Gamma \vdash \text{Err} : \tau \rightsquigarrow \text{Err}}{\Gamma \vdash \text{Err} \text{ H} \leqslant_1 \text{Err}} :$	15024
14971		1. QED $\text{Err} \text{ H} \leqslant_1 \text{Err}$	15025
14972	CASE	1. $e' \text{ H} \leqslant_1 e$ by the induction hypothesis	15026
14973		2. QED	15027
14974		$\frac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \text{dyn } \tau e : \tau \rightsquigarrow \text{dyn } \tau e'} :$	15028
14975		1. $e' \text{ H} \leqslant_1 e$ by the induction hypothesis	15029
14976		2. QED	15030
14977	CASE	$\frac{\Gamma \vdash e : \tau \rightsquigarrow e'}{\Gamma \vdash \text{Err} \text{ H} \leqslant_1 \text{Err}} :$	15031
14978		1. QED $\text{Err} \text{ H} \leqslant_1 \text{Err}$	15032
14979	CASE	1. QED by the induction hypothesis	15033
14980		$\frac{\Gamma \vdash \text{Err} : \tau \rightsquigarrow \text{Err}}{\Gamma \vdash \text{op}^2 e_0 e_1 \rightsquigarrow \text{op}^2 e'_0 e'_1} :$	15034
14981		1. $e'_0 \text{ H} \leqslant_1 e_0$ $\wedge e'_1 \text{ H} \leqslant_1 e_1$ by the induction hypothesis	15035
14982		2. QED	15036
14983		$\frac{\Gamma \vdash \text{Err} \text{ H} \leqslant_1 \text{Err}}{\Gamma \vdash \text{op}^2 e_0 e_1 \rightsquigarrow \text{op}^2 e'_0 e'_1} :$	15037
14984	CASE	1. $e' \text{ H} \leqslant_1 e$ by the induction hypothesis	15038
14985		2. QED	15039
14986		$\frac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \text{stat } \tau e \rightsquigarrow \text{stat } \tau e'} :$	15040
14987		1. $e' \text{ H} \leqslant_1 e$ by 1-E static reflexivity	15041
14988		2. QED	15042
14989		□	15043
14990		Lemma 8.12 : H-1 dynamic reflexivity	15044
14991		If $\Gamma \vdash e$ $\wedge \Gamma \vdash e \rightsquigarrow e''$ then $e \text{ H} \leqslant_1 e''$.	15045
14992		<i>Proof:</i>	15046
14993		By structural induction on the $\Gamma \vdash e \rightsquigarrow e''$ judgment.	15047
14994		CASE	15048
14995		$\frac{}{\Gamma \vdash i \rightsquigarrow i} :$	15049
14996		1. QED $i \text{ H} \leqslant_1 i$	15050
14997		CASE	15051
14998		$\frac{\Gamma \vdash e_0 \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 \rightsquigarrow e'_1}{\Gamma \vdash \langle e_0, e_1 \rangle \rightsquigarrow \langle e'_0, e'_1 \rangle} :$	15052
14999		1. $e'_0 \text{ H} \leqslant_1 e_0$ $\wedge e'_1 \text{ H} \leqslant_1 e_1$ by the induction hypothesis	15053
15000		2. QED	15054
15001		CASE	15055
15002		$\frac{x, \Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \lambda x. e \rightsquigarrow \lambda x. e'} :$	15056
15003		1. $e' \text{ H} \leqslant_1 e$ by the induction hypothesis	15057
15004		2. QED	15058
15005		CASE	15059
15006		$\frac{\Gamma \vdash \lambda x. e \rightsquigarrow \lambda x. e'}{\Gamma \vdash E^1[\text{dyn } v_0] \rightarrow_{1-S} E^1[v_0]} :$	15060
15007		1. $e_0^H = \text{dyn } \tau_0 e_0^H$ $\wedge e_0^H \text{ H} \leqslant_1 v_0^1$ by $\text{H} \leqslant_1$	15061
15008		2. $\vdash_H e_0^H$ by $\vdash_H E^1[e_0^H] : \tau$	15062
15009		3. $e_0^H \rightarrow_{H-S}^* v_0^H$ and $v_0^H \text{ H} \leqslant_1 v_0^1$ $\vee e_0^H \rightarrow_{H-S}^* \text{BndryErr}$ by H-1 static value stutter (1, 2)	15063
15010		4. IF $e_0^H \rightarrow_{H-S}^* v_0^H$ and $v_0^H \text{ H} \leqslant_1 v_0^1$:	15064
15011		1. $e' \text{ H} \leqslant_1 e$ by the induction hypothesis	15065
15012		2. QED	15066
15013		□	15067
15014			15068
15015			15069

15016	CASE	$\frac{}{\Gamma \vdash x \rightsquigarrow x} :$	15017
15017	1. QED $x \text{ H} \leqslant_1 x$	15018	
15018	CASE	$\frac{\Gamma \vdash e_0 \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 \rightsquigarrow e'_1}{\Gamma \vdash e_0 e_1 \rightsquigarrow e'_0 e'_1} :$	15019
15019	1. $e'_0 \text{ H} \leqslant_1 e_0$ $\wedge e'_1 \text{ H} \leqslant_1 e_1$ by the induction hypothesis	15020	
15020	2. QED	15021	
15021	CASE	$\frac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \text{op}^1 e \rightsquigarrow \text{op}^1 e'} :$	15022
15022	1. $e' \text{ H} \leqslant_1 e$ by the induction hypothesis	15023	
15023	2. QED	15024	
15024	CASE	$\frac{\Gamma \vdash e_0 \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 \rightsquigarrow e'_1}{\Gamma \vdash \text{op}^2 e_0 e_1 \rightsquigarrow \text{op}^2 e'_0 e'_1} :$	15025
15025	1. $e'_0 \text{ H} \leqslant_1 e_0$ $\wedge e'_1 \text{ H} \leqslant_1 e_1$ by the induction hypothesis	15026	
15026	2. QED	15027	
15027	CASE	$\frac{\Gamma \vdash \text{Err} \rightsquigarrow \text{Err}}{\Gamma \vdash \text{Err} \text{ H} \leqslant_1 \text{Err}} :$	15028
15028	1. QED $\text{Err} \text{ H} \leqslant_1 \text{Err}$	15029	
15029	CASE	$\frac{\Gamma \vdash \text{Err} \text{ H} \leqslant_1 \text{Err}}{\Gamma \vdash \text{op}^2 e_0 e_1 \rightsquigarrow \text{op}^2 e'_0 e'_1} :$	15030
15030	1. $e'_0 \text{ H} \leqslant_1 e_0$ $\wedge e'_1 \text{ H} \leqslant_1 e_1$ by the induction hypothesis	15031	
15031	2. QED	15032	
15032	CASE	$\frac{\Gamma \vdash \text{Err} \rightsquigarrow \text{Err}}{\Gamma \vdash \text{op}^2 e_0 e_1 \rightsquigarrow \text{op}^2 e'_0 e'_1} :$	15033
15033	1. $e' \text{ H} \leqslant_1 e$ by the induction hypothesis	15034	
15034	2. QED	15035	
15035	CASE	$\frac{\Gamma \vdash e_0 \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 \rightsquigarrow e'_1}{\Gamma \vdash \text{op}^2 e_0 e_1 \rightsquigarrow \text{op}^2 e'_0 e'_1} :$	15036
15036	1. $e'_0 \text{ H} \leqslant_1 e_0$ $\wedge e'_1 \text{ H} \leqslant_1 e_1$ by the induction hypothesis	15037	
15037	2. QED	15038	
15038	CASE	$\frac{\Gamma \vdash \text{Err} \rightsquigarrow \text{Err}}{\Gamma \vdash \text{stat } \tau e \rightsquigarrow \text{stat } \tau e'} :$	15039
15039	1. $e' \text{ H} \leqslant_1 e$ by 1-E static reflexivity	15040	
15040	2. QED	15041	
15041		□	15042
15042	Lemma 8.13 : H-1 simulation		15043
15043		If $\vdash_H E^H[e_0^H] : \tau$ $\wedge \vdash_H E^1[e_0^1] : \lfloor \tau \rfloor$ $\wedge E^H[e_0^H] \text{ H} \leqslant_1 E^1[e_0^1]$ $\wedge E^1[e_0^1] \rightarrow_{1-S} E^1[e_1^1]$ then $E^H[e_0^H] \rightarrow_{H-S}^* E^H[e_n^H]$ and $E^H[e_n^H] \text{ H} \leqslant_1 E^1[e_1^1]$	15044
15044		<i>Proof:</i>	15045
15045		By case analysis on $E^1[e_0^1] \rightarrow_{1-S} E^1[e_1^1]$	15046
15046		CASE	15047
15047		$\frac{}{\Gamma \vdash E^1[\text{dyn } v_0^1] \rightarrow_{1-S} E^1[v_0^1]} :$	15048
15048		1. $e_0^H = \text{dyn } \tau_0 e_0^H$ $\wedge e_0^H \text{ H} \leqslant_1 v_0^1$ by $\text{H} \leqslant_1$	15049
15049		2. $\vdash_H e_0^H$ by $\vdash_H E^1[e_0^H] : \tau$	15050
15050		3. $e_0^H \rightarrow_{H-S}^* v_0^H$ and $v_0^H \text{ H} \leqslant_1 v_0^1$ $\vee e_0^H \rightarrow_{H-S}^* \text{BndryErr}$ by H-1 static value stutter (1, 2)	15051
15051		4. IF $e_0^H \rightarrow_{H-S}^* v_0^H$ and $v_0^H \text{ H} \leqslant_1 v_0^1$:	15052

15071	a. $\vdash_H v_0^H$	3. $e_{0'}^H \rightarrow_{H-S}^* v_0^H$ and $v_0^H \leq_H 1 v_0^1$	15126
15072	by H static preservation	$\vee e_{0'}^H \rightarrow_{H-S}^* \text{BndryErr}$	15127
15073	b. $\text{dyn } \tau_0 v_0^H \rightarrow_{H-S}^* e_{1'}^H$	by H-1 static value stutter (1, 2)	15128
15074	$\wedge e_{1'}^H \leq_H 1 v_0^1$	4. IF $e_{0'}^H \rightarrow_{H-S}^* v_0^H$ and $v_0^H \leq_H 1 v_0^1$:	15129
15075	$\wedge e_{1'}^H \in v$ or $e_{1'}^H = \text{BndryErr}$	a. $\vdash_H v_0^H$	15130
15076	by H-1 boundary checking	by H static preservation	15131
15077	c. $\text{QED } E^H[e_0^H] \rightarrow_{H-S}^* E^H[e_1^H]$	b. $\text{dyn } \tau_0 v_0^H \rightarrow_{H-S}^* e_{1'}^H$	15132
15078	ELSE $e_{0'}^H \rightarrow_{H-S}^* \text{BndryErr}$:	$\wedge e_{1'}^H \leq_H 1 v_0^1$	15133
15079	a. $\text{QED } E^H[v_0^H] \rightarrow_{H-S}^* \text{BndryErr}$	$\wedge e_{1'}^H \in v$ or $e_{1'}^H = \text{BndryErr}$	15134
15080	CASE $E^1[\text{stat } v_0^1] \rightarrow_{1-S} E^1[v_0^1]$:	by H-1 boundary checking	15135
15081	1. $e_0^H = \text{stat } \tau_0 e_{0'}^H$	c. $\text{QED } E^H[e_0^H] \rightarrow_{H-S}^* E^H[e_1^H]$	15136
15082	$\wedge e_{0'}^H \leq_H 1 v_0^1$	ELSE $e_{0'}^H \rightarrow_{H-S}^* \text{BndryErr}$:	15137
15083	by $\leq_H 1$	a. $\text{QED } E^H[e_0^H] \rightarrow_{H-S}^* \text{BndryErr}$	15138
15084	2. $\vdash_H e_{0'}^H : \tau_0$	CASE $E^1[\text{stat } \tau_0 v_0^1] \rightarrow_{1-S} E^1[v_0^1]$:	15139
15085	by $\vdash_H E^H[e_0^H] : \tau$	1. $e_0^H = \text{stat } \tau_0 e_{0'}^H$	15140
15086	3. $e_{0'}^H \rightarrow_{H-D}^* v_0^H$ and $v_0^H \leq_H 1 v_0^1$	$\wedge e_{0'}^H \leq_H 1 v_0^1$	15141
15087	$\vee e_{0'}^H \rightarrow_{H-D}^* \text{BndryErr}$	by $\leq_H 1$	15142
15088	by H-1 dynamic value stutter (1, 2)	2. $\vdash_H e_{0'}^H : \tau_0$	15143
15089	4. IF $e_{0'}^H \rightarrow_{H-D}^* v_0^H$ and $v_0^H \leq_H 1 v_0^1$:	by $\vdash_H E^H[e_0^H] : \tau$	15144
15090	a. $\vdash_H v_0^H : \tau_0$	3. $e_{0'}^H \rightarrow_{H-D}^* v_0^H$ and $v_0^H \leq_H 1 v_0^1$	15145
15091	by H dynamic preservation	$\vee e_{0'}^H \rightarrow_{H-D}^* \text{BndryErr}$	15146
15092	b. $\text{stat } \tau_0 v_0^H \rightarrow_{H-D}^* e_{1'}^H$	by H-1 dynamic value stutter (1, 2)	15147
15093	$\wedge e_{1'}^H \leq_H 1 v_0^1$	4. IF $e_{0'}^H \rightarrow_{H-D}^* v_0^H$ and $v_0^H \leq_H 1 v_0^1$:	15148
15094	$\wedge e_{1'}^H \in v$ or $e_{1'}^H = \text{BndryErr}$	a. $\vdash_H v_0^H : \tau_0$	15149
15095	by H-1 boundary checking	by H dynamic preservation	15150
15096	c. $\text{QED } E^H[e_0^H] \rightarrow_{H-S}^* E^H[e_1^H]$	b. $\text{stat } \tau_0 v_0^H \rightarrow_{H-D}^* e_{1'}^H$	15151
15097	ELSE $e_{0'}^H \rightarrow_{H-D}^* \text{BndryErr}$:	$\wedge e_{1'}^H \leq_H 1 v_0^1$	15152
15098	a. $\text{QED } E^H[e_0^H] \rightarrow_{H-S}^* \text{BndryErr}$	$\wedge e_{1'}^H \in v$ or $e_{1'}^H = \text{BndryErr}$	15153
15099	CASE $E^1[\text{dyn } \tau_0 v_0^1] \rightarrow_{1-S} E^1[\text{BndryErr}]$:	by H-1 boundary checking	15154
15100	1. $e_0^H = \text{dyn } \tau_0 e_{0'}^H$	c. $\text{QED } E^H[e_0^H] \rightarrow_{H-S}^* E^H[e_1^H]$	15155
15101	$\wedge e_{0'}^H \leq_H 1 v_0^1$	ELSE $e_{0'}^H \rightarrow_{H-D}^* \text{BndryErr}$:	15156
15102	by $\leq_H 1$	a. $\text{QED } E^H[e_0^H] \rightarrow_{H-S}^* \text{BndryErr}$	15157
15103	2. $\vdash_H e_{0'}^H$	CASE $E^1[\text{chk } K_0 v_0^1] \rightarrow_{1-S} E^1[\text{BndryErr}]$:	15158
15104	by $\vdash_H E^H[e_0^H] : \tau$	1. $\vdash_1 \text{chk } K_0 v_0^1 : K_0$	15159
15105	3. $e_{0'}^H \rightarrow_{H-S}^* v_0^H$ and $v_0^H \leq_H 1 v_0^1$	by $\vdash_1 E^1[e_0^1] : K$	15160
15106	$\vee e_{0'}^H \rightarrow_{H-S}^* \text{BndryErr}$	2. $\vdash_H e_0^H : \tau_0$	15161
15107	by H-1 static value stutter (1, 2)	by H-1 static hole typing	15162
15108	4. IF $e_{0'}^H \rightarrow_{H-S}^* v_0^H$ and $v_0^H \leq_H 1 v_0^1$:	3. $e_0^H \rightarrow_{H-S}^* v_0^H$ and $v_0^H \leq_H 1 v_0^1$	15163
15109	a. $X(\lfloor \tau_0 \rfloor, v_0^1) = \text{BndryErr}$	$\vee e_0^H \rightarrow_{H-S}^* \text{BndryErr}$	15164
15110	by $E^1[\text{dyn } \tau_0 v_0^1] \rightarrow_{1-S} E^1[\text{BndryErr}]$	by H-1 static value stutter (2)	15165
15111	b. $\mathcal{D}_H(\tau_0, v_0^H) = \text{BndryErr}$	4. IF $e_0^H \rightarrow_{H-S}^* v_0^H$ and $v_0^H \leq_H 1 v_0^1$:	15166
15112	by X inversion	a. $\vdash_1 v_0^1 : \lfloor \tau_0 \rfloor$	15167
15113	c. $\text{QED } E^H[e_0^H] \rightarrow_{H-S}^* \text{BndryErr}$	by H-1 value inversion (2, 3)	15168
15114	ELSE $e_{0'}^H \rightarrow_{H-S}^* \text{BndryErr}$:	b. $\lfloor \tau_0 \rfloor = K_0$	15169
15115	a. $\text{QED } E^H[e_0^H] \rightarrow_{H-S}^* \text{BndryErr}$	by chk inversion, H-1 static reflexivity, H static preservation, and 1 static preservation	15170
15116	CASE $E^1[\text{dyn } \tau_0 v_0^1] \rightarrow_{1-S} E^1[v_0^1]$:	c. $\text{chk } K_0 v_0^1 \triangleright_{1-S} v_0^1$	15171
15117	1. $e_0^H = \text{dyn } \tau_0 e_{0'}^H$	by definition \triangleright_{1-S}	15172
15118	$\wedge e_{0'}^H \leq_H 1 v_0^1$	d. Contradiction by $E^1[\text{chk } K_0 v_0^1] \rightarrow_{1-S} E^1[\text{BndryErr}]$	15173
15119	by $\leq_H 1$	(b)	15174
15120	2. $\vdash_H e_{0'}^H$	ELSE $e_0^H \rightarrow_{H-S}^* \text{BndryErr}$:	15175
15121	by $\vdash_H E^H[e_0^H] : \tau$	a. $\text{QED } E^H[e_0^H] \rightarrow_{H-S}^* \text{BndryErr}$	15176
15122			15177
15123			15178
15124			15179
15125			15180

15181	CASE $E^1[\text{chk } K_0 v_0^1] \rightarrow_{1-S} E^1[v_0^1]$:		
15182	1. $\vdash_1 \text{chk } K_0 v_0^1 : K_0$		
15183	by $\vdash_1 E^1[e_0^1] : K$		
15184	2. $\vdash_H e_0^H : \tau_0$		
15185	by H-1 static hole typing		
15186	3. $e_0^H \rightarrow_{H-S}^* v_0^H$ and $v_0^H \underset{H}{\lesssim} v_0^1$		
15187	$\vee e_0^H \rightarrow_{H-S}^* \text{BndryErr}$		
15188	by H-1 static value stutter (2)		
15189	4. IF $e_0^H \rightarrow_{H-S}^* v_0^H$ and $v_0^H \underset{H}{\lesssim} v_0^1$		
15190	a. QED $E^H[e_0^H] \rightarrow_{H-S}^* v_0^H$		
15191	ELSE $e_0^H \rightarrow_{H-S}^* \text{BndryErr}$:		
15192	a. QED $E^H[e_0^H] \rightarrow_{H-S}^* \text{BndryErr}$		
15193	CASE $E^1[v_0^1 v_1^1] \rightarrow_{1-S} E^1[e_2^1]$		
15194	$\wedge \vdash_1 v_0^1 v_1^1 : K'$:		
15195	1. $\vdash_H e_0^H : \tau_0$		
15196	by H-1 static hole typing		
15197	2. $e_0^H = e_{0'}^H e_{1'}^H$		
15198	by $\underset{H}{\lesssim} 1$		
15199	3. $e_{0'}^H \rightarrow_{H-S}^* v_0^H$ and $v_0^H \underset{H}{\lesssim} v_0^1$		
15200	$\vee e_{0'}^H \rightarrow_{H-S}^* \text{BndryErr}$		
15201	by H-1 static value stutter (1, 2)		
15202	4. $e_{1'}^H \rightarrow_{H-S}^* v_1^H$ and $v_1^H \underset{H}{\lesssim} v_1^1$		
15203	$\vee e_{1'}^H \rightarrow_{H-S}^* \text{BndryErr}$		
15204	by H-1 static value stutter (1, 2)		
15205	5. IF $e_{0'}^H \rightarrow_{H-S}^* v_0^H$		
15206	$\wedge e_{1'}^H \rightarrow_{H-S}^* v_1^H$:		
15207	a. $v_0^H v_1^H \rightarrow_{H-S}^* e_2^H$		
15208	$\wedge e_2^H \underset{H}{\lesssim} 1$		
15209	by H-1 static application (1)		
15210	b. QED $E^H[e_0^H] \rightarrow_{H-S}^* E^H[e_2^H]$		
15211	IF $e_{0'}^H \rightarrow_{H-S}^* \text{BndryErr}$:		
15212	a. QED $E^H[e_0^H] \rightarrow_{H-S}^* \text{BndryErr}$		
15213	ELSE $e_{1'}^H \rightarrow_{H-S}^* \text{BndryErr}$:		
15214	a. QED $E^H[e_0^H] \rightarrow_{H-S}^* \text{BndryErr}$		
15215	CASE $E^1[v_0^1 v_1^1] \rightarrow_{1-S} E^1[e_2^1]$		
15216	$\wedge \vdash_1 v_0^1 v_1^1 :$		
15217	1. $\vdash_H e_0^H$		
15218	by H-1 static hole typing		
15219	2. $e_0^H = e_{0'}^H e_{1'}^H$		
15220	by $\underset{H}{\lesssim} 1$		
15221	3. $e_{0'}^H \rightarrow_{H-D}^* v_0^H$ and $v_0^H \underset{H}{\lesssim} v_0^1$		
15222	$\vee e_{0'}^H \rightarrow_{H-D}^* \text{BndryErr}$		
15223	by H-1 dynamic value stutter (1, 2)		
15224	4. $e_{1'}^H \rightarrow_{H-D}^* v_1^H$ and $v_1^H \underset{H}{\lesssim} v_1^1$		
15225	$\vee e_{1'}^H \rightarrow_{H-D}^* \text{BndryErr}$		
15226	by H-1 dynamic value stutter (1, 2)		
15227	5. IF $e_{0'}^H \rightarrow_{H-D}^* v_0^H$		
15228	$\wedge e_{1'}^H \rightarrow_{H-D}^* v_1^H$:		
15229	a. $v_0^H v_1^H \rightarrow_{H-D}^* e_2^H$		
15230	$\wedge e_2^H \underset{H}{\lesssim} 1$		
15231	by H-1 dynamic application (1)		
15232	b. QED $E^H[e_0^H] \rightarrow_{H-D}^* E^H[e_2^H]$		
15233	CASE $E^1[op^1 v_0^1] \rightarrow_{1-S} E^1[\delta(op^1, v_0^1)]$		
15234	$\wedge \vdash_1 op^1 v_0^1 : K_0$:		
15235	1. $\vdash_H e_0^H : \tau_0$		
15236	by H-1 static hole typing		
15237	2. $e_0^H = op^1 e_{0'}^H$		
15238	by $\underset{H}{\lesssim} 1$		
15239	3. $\vdash_H e_{0'}^H : \tau_0'$		
15240	by H inversion (1, 2)		
15241	4. $e_{0'}^H \underset{H}{\lesssim} 1$		
15242	by $\underset{H}{\lesssim} 1$		
15243	5. $e_{0'}^H \rightarrow_{H-S}^* v_0^H$ and $v_0^H \underset{H}{\lesssim} v_0^1$		
15244	$\vee e_{0'}^H \rightarrow_{H-S}^* \text{BndryErr}$		
15245	by H-1 static value stutter		
15246	6. IF $e_{0'}^H \rightarrow_{H-S}^* v_0^H$:		
15247	a. $op^1 v_0^H \triangleright_{H-S} \delta(op^1, v_0^1)$		
15248	by \triangleright_{H-S}		
15249	b. $\delta(op^1, v_0^1) \underset{H}{\lesssim} 1 \delta(op^1, v_0^1)$		
15250	by H-1 δ-preservation		
15251	c. QED $E^H[e_0^H] \rightarrow_{H-S}^* E^H[\delta(op^1, v_0^1)]$		
15252	ELSE $e_{0'}^H \rightarrow_{H-S}^* \text{BndryErr}$:		
15253	a. QED $E^H[e_0^H] \rightarrow_{H-S}^* \text{BndryErr}$		
15254	CASE $E^1[op^1 v_0^1] \rightarrow_{1-S} E^1[\delta(op^1, v_0^1)]$		
15255	$\wedge \vdash_1 op^1 v_0^1 :$		
15256	1. $\vdash_H e_0^H$		
15257	by H-1 dynamic hole typing		
15258	2. $e_0^H = op^1 e_{0'}^H$		
15259	by $\underset{H}{\lesssim} 1$		
15260	3. $\vdash_H e_{0'}^H : \tau_0'$		
15261	by H inversion (1, 2)		
15262	4. $e_{0'}^H \underset{H}{\lesssim} 1$		
15263	by $\underset{H}{\lesssim} 1$		
15264	5. $e_{0'}^H \rightarrow_{H-S}^* v_0^1$		
15265	by $\underset{H}{\lesssim} 1$		
15266	6. IF $e_{0'}^H \rightarrow_{H-S}^* v_0^1$:		
15267	a. $op^1 v_0^H \triangleright_{H-S} \delta(op^1, v_0^1)$		
15268	by \triangleright_{H-S}		
15269	b. $\delta(op^1, v_0^1) \underset{H}{\lesssim} 1 \delta(op^1, v_0^1)$		
15270	by H-1 δ-preservation		
15271	c. QED $E^H[e_0^H] \rightarrow_{H-S}^* E^H[\delta(op^1, v_0^1)]$		
15272	ELSE $e_{0'}^H \rightarrow_{H-S}^* \text{BndryErr}$:		
15273	a. QED $E^H[e_0^H] \rightarrow_{H-S}^* \text{BndryErr}$		
15274	CASE $E^1[op^1 v_0^1] \rightarrow_{1-S} E^1[\delta(op^1, v_0^1)]$		
15275	$\wedge \vdash_1 op^1 v_0^1 :$		
15276	1. $\vdash_H e_0^H$		
15277	by H-1 dynamic hole typing		
15278	2. $e_0^H = op^1 e_{0'}^H$		
15279	by $\underset{H}{\lesssim} 1$		
15280	3. $\vdash_H e_{0'}^H : \tau_0'$		
15281	by H inversion (1, 2)		
15282	4. $e_{0'}^H \underset{H}{\lesssim} 1$		
15283	by $\underset{H}{\lesssim} 1$		
15284	5. $e_{0'}^H \rightarrow_{H-D}^* v_0^H$ and $v_0^H \underset{H}{\lesssim} v_0^1$		
15285	$\vee e_{0'}^H \rightarrow_{H-D}^* \text{BndryErr}$		
15286	by H-1 dynamic value stutter		
15287	6. IF $e_{0'}^H \rightarrow_{H-D}^* v_0^H$:		
15288	a. $op^1 v_0^H \triangleright_{H-D} \delta(op^1, v_0^1)$		
15289	by \triangleright_{H-D}		
15290	b. $\delta(op^1, v_0^1) \underset{H}{\lesssim} 1 \delta(op^1, v_0^1)$		
15291	by H-1 δ-preservation		
15292	c. QED $E^H[e_0^H] \rightarrow_{H-D}^* E^H[\delta(op^1, v_0^1)]$		
15293	ELSE $e_{0'}^H \rightarrow_{H-D}^* \text{BndryErr}$:		
15294	a. QED $E^H[e_0^H] \rightarrow_{H-D}^* \text{BndryErr}$		
15295	CASE $E^1[op^1 v_0^1] \rightarrow_{1-S} E^1[\text{TagErr}]$		
15296	$\wedge \vdash_1 op^1 v_0^1 :$		
15297	1. $\vdash_H e_0^H$		
15298	by definition \rightarrow_{1-S}		
15299	2. $\vdash_H e_0^H$		
15300	by H-1 dynamic hole typing (1)		

15291	3. $e_0^H = op^1 e_{0'}^H$	15346
15292	by $\text{H} \lesssim_1$	15347
15293	4. $\vdash_H e_{0'}^H : \tau_{0'}$	15348
15294	by H inversion (2, 3)	15349
15295	5. $e_{0'}^H \text{ H} \lesssim_1 v_0^1$	15350
15296	by $\text{H} \lesssim_1$	15351
15297	6. $e_{0'}^H \xrightarrow{*_{\text{H-D}}} v_0^H$ and $v_0^H \text{ H} \lesssim_1 v_0^1$	15352
15298	$\vee e_{0'}^H \xrightarrow{*_{\text{H-D}}} \text{BndryErr}$	15353
15299	by H-1 dynamic value stutter	
15300	7. IF $e_{0'}^H \xrightarrow{*_{\text{H-D}}} v_0^H$:	
15301	a. $\delta(op^1, v_0^1)$ is undefined	
15302	by definition $\triangleright_{\text{H-D}}$	
15303	b. $v_0^1 \notin \langle v, v \rangle$	
15304	by (a)	
15305	c. $v_0^H \notin \langle v, v \rangle$	
15306	by $v_0^H \text{ H} \lesssim_1 v_0^1$	
15307	d. QED $E^H[e_0^H] \xrightarrow{*_{\text{H-S}}} \text{TagErr}$	
15308	ELSE $e_{0'}^H \xrightarrow{*_{\text{H-D}}} \text{BndryErr}$:	
15309	a. QED $E^H[e_{0'}^H] \xrightarrow{*_{\text{H-S}}} \text{BndryErr}$	
15310	CASE $E^1[op^2 v_0^1 v_1^1] \rightarrow_{\text{1-S}} E^1[\delta(op^2, v_0^1, v_1^1)]$	
15311	$\wedge \vdash_1 op^2 v_0^1 v_1^1 : K_0$:	
15312	1. $\vdash_H e_0^H : \tau_0$	
15313	by H-1 static hole typing	
15314	2. $e_0^H = op^2 e_{0'}^H e_{1'}^H$	
15315	by $\text{H} \lesssim_1$	
15316	3. $\vdash_H e_{0'}^H : \tau_{0'}$	
15317	$\wedge \vdash_H e_{1'}^H : \tau_{1'}$	
15318	by H inversion (1, 2)	
15319	4. $e_{0'}^H \text{ H} \lesssim_1 v_0^1$	
15320	$\wedge e_{1'}^H \text{ H} \lesssim_1 v_1^1$	
15321	by $\text{H} \lesssim_1$	
15322	5. $e_{0'}^H \xrightarrow{*_{\text{H-S}}} v_0^H$ and $v_0^H \text{ H} \lesssim_1 v_0^1$	
15323	$\vee e_{0'}^H \xrightarrow{*_{\text{H-S}}} \text{BndryErr}$	
15324	by H-1 static value stutter	
15325	6. $e_{1'}^H \xrightarrow{*_{\text{H-S}}} v_1^H$ and $v_1^H \text{ H} \lesssim_1 v_1^1$	
15326	$\vee e_{1'}^H \xrightarrow{*_{\text{H-S}}} \text{BndryErr}$	
15327	by H-1 static value stutter	
15328	7. IF $e_{0'}^H \xrightarrow{*_{\text{H-S}}} v_0^H$	
15329	$\wedge e_{1'}^H \xrightarrow{*_{\text{H-S}}} v_1^H$:	
15330	a. $op^2 v_0^H v_1^H \triangleright_{\text{H-S}} \delta(op^2, v_0^H, v_1^H)$	
15331	by $\triangleright_{\text{H-D}}$	
15332	b. $\delta(op^2, v_0^H, v_1^H) \text{ H} \lesssim_1 \delta(op^2, v_0^1, v_1^1)$	
15333	by H-1 δ-preservation	
15334	c. QED $E^H[e_0^H] \xrightarrow{*_{\text{H-S}}} E^H[\delta(op^2, v_0^H, v_1^H)]$	
15335	IF $e_{0'}^H \xrightarrow{*_{\text{H-S}}} \text{BndryErr}$:	
15336	a. QED $E^H[e_0^H] \xrightarrow{*_{\text{H-S}}} \text{BndryErr}$	
15337	ELSE $e_{1'}^H \xrightarrow{*_{\text{H-S}}} \text{BndryErr}$:	
15338	a. QED $E^H[e_0^H] \xrightarrow{*_{\text{H-S}}} \text{BndryErr}$	
15339	CASE $E^1[op^2 v_0^1 v_1^1] \rightarrow_{\text{1-S}} E^1[\delta(op^2, v_0^1, v_1^1)]$	
15340	$\wedge \vdash_1 op^2 v_0^1 v_1^1 :$	
15341	1. $\vdash_H e_0^H$	
15342	by H-1 static hole typing	
15343		
15344		
15345		
	2. $e_0^H = op^2 e_{0'}^H e_{1'}^H$	15346
	by $\text{H} \lesssim_1$	15347
	3. $\vdash_H e_{0'}^H$	15348
	$\wedge \vdash_H e_{1'}^H$	15349
	by H inversion (1, 2)	15350
	4. $e_{0'}^H \text{ H} \lesssim_1 v_0^1$	15351
	$\wedge e_{1'}^H \text{ H} \lesssim_1 v_1^1$	15352
	by $\text{H} \lesssim_1$	15353
	5. $e_{0'}^H \xrightarrow{*_{\text{H-D}}} v_0^H$ and $v_0^H \text{ H} \lesssim_1 v_0^1$	15354
	$\vee e_{0'}^H \xrightarrow{*_{\text{H-D}}} \text{BndryErr}$	15355
	by H-1 dynamic value stutter	15356
	6. $e_{1'}^H \xrightarrow{*_{\text{H-D}}} v_1^H$ and $v_1^H \text{ H} \lesssim_1 v_1^1$	15357
	$\vee e_{1'}^H \xrightarrow{*_{\text{H-D}}} \text{BndryErr}$	15358
	by H-1 dynamic value stutter	15359
	7. IF $e_{0'}^H \xrightarrow{*_{\text{H-D}}} v_0^H$	15360
	$\wedge e_{1'}^H \xrightarrow{*_{\text{H-D}}} v_1^H$:	15361
	a. $op^2 v_0^H v_1^H \triangleright_{\text{H-D}} \delta(op^2, v_0^H, v_1^H)$	15362
	by $\triangleright_{\text{H-D}}$	15363
	b. $\delta(op^2, v_0^H, v_1^H) \text{ H} \lesssim_1 \delta(op^2, v_0^1, v_1^1)$	15364
	by H-1 δ-preservation	15365
	c. QED $E^H[e_0^H] \xrightarrow{*_{\text{H-S}}} E^H[\delta(op^2, v_0^H, v_1^H)]$	15366
	IF $e_{0'}^H \xrightarrow{*_{\text{H-D}}} \text{BndryErr}$:	15367
	a. QED $E^H[e_0^H] \xrightarrow{*_{\text{H-S}}} \text{BndryErr}$	15368
	ELSE $e_{1'}^H \xrightarrow{*_{\text{H-D}}} \text{BndryErr}$:	15369
	a. QED $E^H[e_0^H] \xrightarrow{*_{\text{H-S}}} \text{BndryErr}$	15370
	CASE $E^1[op^2 v_0^1 v_1^1] \rightarrow_{\text{1-S}} E^1[\text{TagErr}] :$	15372
	1. $\vdash_1 op^2 v_0^1 v_1^1$	15373
	by definition $\rightarrow_{\text{1-S}}$	15374
	2. $\vdash_H e_0^H$	15375
	by H-1 dynamic hole typing (1)	15376
	3. $e_0^H = op^2 e_{0'}^H e_{1'}^H$	15377
	by $\text{H} \lesssim_1$	15378
	4. $\vdash_H e_{0'}^H : \tau_{0'}$	15379
	$\wedge \vdash_H e_{1'}^H : \tau_{1'}$	15380
	by H inversion (2, 3)	15381
	5. $e_{0'}^H \text{ H} \lesssim_1 v_0^1$	15382
	$\wedge e_{1'}^H \text{ H} \lesssim_1 v_1^1$	15383
	by $\text{H} \lesssim_1$	15384
	6. $e_{0'}^H \xrightarrow{*_{\text{H-D}}} v_0^H$ and $v_0^H \text{ H} \lesssim_1 v_0^1$	15385
	$\vee e_{0'}^H \xrightarrow{*_{\text{H-D}}} \text{BndryErr}$	15386
	by H-1 dynamic value stutter	15387
	7. $e_{1'}^H \xrightarrow{*_{\text{H-D}}} v_1^H$ and $v_1^H \text{ H} \lesssim_1 v_1^1$	15388
	$\vee e_{1'}^H \xrightarrow{*_{\text{H-D}}} \text{BndryErr}$	15389
	by H-1 dynamic value stutter	15390
	8. IF $e_{0'}^H \xrightarrow{*_{\text{H-D}}} v_0^H$	15391
	$\wedge e_{1'}^H \xrightarrow{*_{\text{H-D}}} v_1^H$:	15392
	a. $\delta(op^2, v_0^1, v_1^1)$ is undefined	15393
	by definition $\triangleright_{\text{H-D}}$	15394
	b. $v_0^1 \notin \mathbb{Z}$	15395
	$\wedge v_1^1 \notin \mathbb{Z}$	15396
	by (a)	15397
		15398
		15399
		15400

15401 c. $v_0^H \notin \mathbb{Z}$
 15402 by $v_0^H \text{ H} \lesssim_1 v_0^1$
 15403 d. $v_1^H \notin \mathbb{Z}$
 15404 by $v_1^H \text{ H} \lesssim_1 v_1^1$
 15405 e. QED $E^H[e_0^H] \rightarrow_{H-S}^* \text{TagErr}$
 15406 **IF** $e_{0'}^H \rightarrow_{H-D}^* \text{BndryErr}$:
 15407 a. QED $E^H[e_0^H] \rightarrow_{H-S}^* \text{BndryErr}$
 15408 **ELSE** $e_{1'}^H \rightarrow_{H-D}^* \text{BndryErr}$:
 15409 a. QED $E^H[e_0^H] \rightarrow_{H-S}^* \text{BndryErr}$
 15410

□

Lemma 8.14 : H-1 static application

If $v_0^H v_1^H \text{ H} \lesssim_1 v_0^1 v_1^1$
 $\wedge \vdash_H v_0^H v_1^H : \tau$
 $\wedge \vdash_H v_0^1 v_1^1 : \text{Any}$
 $\wedge v_0^1 v_1^1 \rightarrow_{1-S} e_2^H$ then $v_0^H v_1^H \rightarrow_{H-S}^* e_2^H$ and $e_2^H \text{ H} \lesssim_1 e_2^1$
 Proof:

By induction on the number of monitors in v_0^H , proceeding by case analysis on $v_0^1 v_1^1 \rightarrow_{1-S} e_2^H$.

CASE $(\lambda(x:\tau_0). e_0^1) v_1^1 \rightarrow_{1-S} \text{BndryErr}$

$\wedge v_0^H = \lambda(x:\tau_0). e_0^H$:

1. $X(\lfloor \tau_0 \rfloor, v_1^1) = \text{BndryErr}$

by definition \triangleright_{1-S}

2. $\vdash_H v_1^H : \tau_1$

$\wedge \tau_1 \leqslant \tau_0$

by **H inversion**

3. $v_1^H \text{ H} \lesssim_1 v_1^1$

by $v_0^H v_1^H \text{ H} \lesssim_1 v_0^1 v_1^1$

4. $\vdash_H v_1^1 : \lfloor \tau_1 \rfloor$

by **H-1 value inversion** (3)

5. $\vdash_H v_1^1 : \lfloor \tau_0 \rfloor$

by **subtyping preservation** (2)

6. $X(\lfloor \tau_0 \rfloor, v_1^1) = v_1^1$

by (5)

7. Contradiction by (1, 6)

CASE $(\lambda(x:\tau_0). e_0^1) v_1^1 \rightarrow_{1-S} \text{BndryErr}$

$\wedge v_0^H = \text{mon}(\tau_d \Rightarrow \tau_c) v_{0'}^H$:

1. $v_{0'}^H \text{ H} \lesssim_1 v_0^1$

$\wedge v_1^H \text{ H} \lesssim_1 v_1^1$

by $v_0^H v_1^H \text{ H} \lesssim_1 v_0^1 v_1^1$

2. $v_0^H v_1^H \triangleright_{H-S} \text{dyn } \tau_c(v_{0'}^H (\text{stat } \tau_d v_1^H))$

by definition \triangleright_{H-S}

3. $\vdash_H \text{stat } \tau_d v_1^H$

by **H static preservation**

4. $\text{stat } \tau_d v_1^H \rightarrow_{H-D}^* v_1^H$ and $v_1^H \text{ H} \lesssim_1 v_1^1$

by **H-1 boundary checking** (1, 3)

5. $v_{0'}^H = \text{mon}(\tau'_d \Rightarrow \tau'_c) v_{0''}^H$ and $v_{0''}^H \text{ H} \lesssim_1 v_0^1$

$\vee v_{0'}^H = \lambda(x:\tau_0). e^H$ and $e^H \text{ H} \lesssim_1 e_0^1$

by (1)

6. **IF** $v_{0'}^H = \lambda(x:\tau_0). e^H$:

a. Contradiction by $\vdash_H v_0^H v_1^H : \tau$

7. $v_{0'}^H v_1^H \triangleright_{H-D} \text{stat } \tau_c(v_{0''}^H (\text{dyn } \tau'_d v_1^H))$

15454

15455

8. $\text{dyn } \tau'_d v_{1'}^H \rightarrow_{H-S}^* v_{1''}^H$ and $v_{1''}^H \text{ H} \lesssim_1 v_1^1$
 $\vee \text{dyn } \tau'_d v_{1'}^H \rightarrow_{H-S}^* \text{BndryErr}$
 by **H-1 boundary checking**
 9. **IF** $\text{dyn } \tau'_d v_{1'}^H \rightarrow_{H-S}^* \text{BndryErr}$:
 a. QED $v_0^H v_1^H \rightarrow_{H-S}^* \text{BndryErr}$
 10. $\text{dyn } \tau_c (\text{stat } \tau_c ([])) \text{ H} \lesssim_1 []$
 by definition $\text{H} \lesssim_1$
 11. $v_{0''}^H v_{1''}^H \text{ H} \lesssim_1 v_0^1 v_1^1$
 12. $\vdash_H v_{0''}^H v_{1''}^H : \tau'_c$
 by **H static preservation**
 13. $v_{0''}^H v_{1''}^H \rightarrow_{H-S}^* e_2^H$
 $\wedge e_2^H \text{ H} \lesssim_1 \text{BndryErr}$
 by the induction hypothesis
 14. QED $v_0^H v_1^H \rightarrow_{H-S}^* \text{BndryErr}$
CASE $(\lambda(x:\tau_0). e_0^1) v_1^1 \rightarrow_{1-S} e_0^1[x \leftarrow X(\lfloor \tau_0 \rfloor, v_1^1)]$
 $\wedge v_0^H = \lambda(x:\tau_0). e_0^H$:
 1. $X(\lfloor \tau_0 \rfloor, v_1^1) = v_1^1$
 by definition \triangleright_{1-S}
 2. $\vdash_H v_1^H : \tau_1$
 $\wedge \tau_1 \leqslant \tau_0$
 by **H inversion**
 3. $v_1^H \text{ H} \lesssim_1 v_1^1$
 $\wedge e_0^H \text{ H} \lesssim_1 e_0^1$
 by $v_0^H v_1^H \text{ H} \lesssim_1 v_0^1 v_1^1$
 4. $(\lambda(x:\tau_0). e_0^H) v_1^H \triangleright_{H-S} e_0^H[x \leftarrow v_1^H]$
 5. $e_0^H[x \leftarrow v_1^H] \text{ H} \lesssim_1 e_0^1[x \leftarrow v_1^1]$
 by **H-1 substitution**
 6. QED $v_0^H v_1^H \rightarrow_{H-S}^* e_0^H[x \leftarrow v_1^H]$
CASE $(\lambda(x:\tau_0). e_0^1) v_1^1 \rightarrow_{1-S} e_0^1[x \leftarrow X(\lfloor \tau_0 \rfloor, v_1^1)]$
 $\wedge v_0^H = \text{mon}(\tau_d \Rightarrow \tau_c) v_{0'}^H$:
 1. $v_{0'}^H \text{ H} \lesssim_1 v_0^1$
 $\wedge v_1^H \text{ H} \lesssim_1 v_1^1$
 by $v_0^H v_1^H \text{ H} \lesssim_1 v_0^1 v_1^1$
 2. $v_0^H v_1^H \triangleright_{H-S} \text{dyn } \tau_c(v_{0'}^H (\text{stat } \tau_d v_1^H))$
 by definition \triangleright_{H-S}
 3. $\vdash_H \text{stat } \tau_d v_1^H$
 by **H static preservation**
 4. $\text{stat } \tau_d v_1^H \rightarrow_{H-D}^* v_1^H$ and $v_1^H \text{ H} \lesssim_1 v_1^1$
 by **H-1 boundary checking** (1, 3)
 5. $v_{0'}^H = \text{mon}(\tau'_d \Rightarrow \tau'_c) v_{0''}^H$ and $v_{0''}^H \text{ H} \lesssim_1 v_0^1$
 $\vee v_{0'}^H = \lambda(x:\tau_0). e^H$ and $e^H \text{ H} \lesssim_1 e_0^1$
 by (1)
 6. **IF** $v_{0'}^H = \lambda(x:\tau_0). e^H$:
 a. $\vdash_H \text{mon}(\tau_d \Rightarrow \tau_c) v_{0'}^H : \tau_d \Rightarrow \tau_c$
 by $\vdash_H v_0^H v_1^H : \tau$
 b. $\vdash_H v_{0'}^H$
 by **H inversion** (a)
 c. Contradiction by $\vdash_H \lambda(x:\tau_0). e^H$
 7. $v_{0'}^H v_{1'}^H \triangleright_{H-D} \text{stat } \tau_c(v_{0''}^H (\text{dyn } \tau'_d v_{1'}^H))$
 8. $\vdash_H \text{dyn } \tau'_d v_{1'}^H : \tau'_d$
 by **H dynamic preservation**

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15511	9. $\text{dyn } \tau'_d v_{1'}^H \rightarrow_{\text{H-S}}^* v_{1''}^H$ and $v_{1''}^H \text{ H}\lesssim_1 v_1^1$	15566
15512	$\vee \text{dyn } \tau'_d v_{1'}^H \rightarrow_{\text{H-S}}^* \text{BndryErr}$	15567
15513	by H-1 boundary checking	15568
15514	10. IF $\text{dyn } \tau'_d v_{1'}^H \rightarrow_{\text{H-S}}^* \text{BndryErr} :$	15569
15515	a. $\text{QED } v_0^H v_1^H \rightarrow_{\text{H-S}}^* \text{BndryErr}$	15570
15516	11. $\text{dyn } \tau_c (\text{stat } \tau'_c [\]) \text{ H}\lesssim_1 [\]$	15571
15517	by definition $\text{H}\lesssim_1$	15572
15518	12. $v_{0''}^H v_{1''}^H \text{ H}\lesssim_1 v_0^1 v_1^1$	15573
15519	13. $\vdash_H v_{0''}^H v_{1''}^H : \tau'_c$	15574
15520	by H static preservation	15575
15521	14. $v_{0''}^H v_{1''}^H \rightarrow_{\text{H-S}}^* e_2^H$	15576
15522	$\wedge e_2^H \text{ H}\lesssim_1 e_2^1$	15577
15523	by the induction hypothesis	15578
15524	15. $\text{QED } v_0^H v_1^H \rightarrow_{\text{H-S}}^* \text{dyn } \tau_c (\text{stat } \tau'_c e_2^H)$	15579
15525	CASE $(\lambda x. e_0^1) v_1^1 \rightarrow_{\text{1-S}} \text{dyn } e_0^1[x \leftarrow v_1^1]$	15580
15526	$\wedge v_0^H = \lambda x. e_0^H :$	15581
15527	1. Contradiction by $\vdash_H v_0^H v_1^H : \tau$	15582
15528	CASE $(\lambda x. e_0^1) v_1^1 \rightarrow_{\text{1-S}} \text{dyn } e_0^1[x \leftarrow v_1^1]$	15583
15529	$\wedge v_0^H = \text{mon}(\tau_d \Rightarrow \tau_c) v_{0'}^H :$	15584
15530	1. $v_{0'}^H \text{ H}\lesssim_1 (\lambda x. e_0^1)$	15585
15531	$\wedge v_1^H \text{ H}\lesssim_1 v_1^1$	15586
15532	by $v_0^H v_1^H \text{ H}\lesssim_1 v_0^1 v_1^1$	15587
15533	2. $v_{0'}^H v_1^H \triangleright_{\text{H-S}} \text{dyn } \tau_c (v_{0'}^H (\text{stat } \tau_d v_1^H))$	15588
15534	by definition $\triangleright_{\text{H-S}}$	15589
15535	3. $\vdash_H \text{stat } \tau_d v_1^H$	15590
15536	by H static preservation	15591
15537	4. $\text{stat } \tau_d v_1^H \rightarrow_{\text{H-D}}^* v_{1'}^H$ and $v_{1'}^H \text{ H}\lesssim_1 v_1^1$	15592
15538	by H-1 boundary checking	15593
15539	5. $\text{dyn } \tau_c [\] \text{ H}\lesssim_1 \text{dyn } [\]$	15594
15540	by definition $\text{H}\lesssim_1$	15595
15541	6. $\vdash_H v_{0'}^H v_{1'}^H$	15596
15542	by H static preservation (2, 3)	15597
15543	7. $v_{0'}^H v_{1'}^H \text{ H}\lesssim_1 v_0^1 v_1^1$	15598
15544	8. $v_{0'}^H v_{1'}^H \rightarrow_{\text{H-D}}^* e_2^H$	15599
15545	$\wedge e_2^H \text{ H}\lesssim_1 e_2^1$	15600
15546	by H-1 dynamic application (6, 7)	15601
15547	9. $\text{QED } v_0^H v_1^H \rightarrow_{\text{H-S}}^* \text{dyn } \tau_c e_2^H$	15602
15548	$\wedge \text{dyn } \tau_c e_2^H \text{ H}\lesssim_1 \text{dyn } e_2^1$	15603
15549	□	15604
15550		15605
15551	Lemma 8.15 : H-1 dynamic application	15606
15552	If $v_0^H v_1^H \text{ H}\lesssim_1 v_0^1 v_1^1$	15607
15553	$\wedge \vdash_H v_0^H v_1^H$	15608
15554	$\wedge \vdash_H v_0^1 v_1^1$	15609
15555	$\wedge v_0^1 v_1^1 \rightarrow_{\text{1-D}} e_2^1$ then $v_0^H v_1^H \rightarrow_{\text{H-D}}^* e_2^H$ and $e_2^H \text{ H}\lesssim_1 e_2^1$	15610
15556	<i>Proof:</i>	15611
15557	By induction on the number of monitors in v_0^H , proceeding by case analysis on $v_0^1 v_1^1 \rightarrow_{\text{1-D}} e_2^1$.	15612
15559	CASE $v_0^1 v_1^1 \triangleright_{\text{1-D}} \text{TagErr} :$	15613
15560	1. $v_0^1 \in \mathbb{Z}$	15614
15561	$\vee v_0^1 \in \langle v, v \rangle$	15615
15562	by definition $\triangleright_{\text{1-D}}$	15616
15563		15617
15564		15618
15565		15619

15621 **CASE** $(\lambda(x:\tau_0). e_0^1) v_1^1 \rightarrow_{1-D} \text{stat } e_0^1[x \leftarrow X(\lfloor \tau_0 \rfloor, v_1^1)]$
 15622 $\wedge v_0^H = \lambda(x:\tau_0). e_0^H :$
 15623 1. $\vdash_H v_0^H$
 by **H inversion**
 15624 2. Contradiction by $\vdash_H \lambda(x:\tau_0). e_0^H$
 15625 **CASE** $(\lambda(x:\tau_0). e_0^1) v_1^1 \rightarrow_{1-D} \text{stat } e_0^1[x \leftarrow X(\lfloor \tau_0 \rfloor, v_1^1)]$
 15626 $\wedge v_0^H = \text{mon } (\tau_d \Rightarrow \tau_c) v_{0'}^H :$
 15627 1. $v_{0'}^H \underset{H}{\lesssim} v_0^1$
 $\wedge v_1^H \underset{H}{\lesssim} v_1^1$
 15628 by $v_0^H v_1^H \underset{H}{\lesssim} v_0^1 v_1^1$
 15629 2. $v_{0'}^H v_1^H \triangleright_{H-D} \text{stat } \tau_c (v_{0'}^H (\text{dyn } \tau_d v_1^H))$
 15630 by definition \triangleright_{H-D}
 15631 3. $\vdash_H \text{dyn } \tau_d v_1^H : \tau_d$
 15632 by **H dynamic preservation**
 15633 4. $\text{dyn } \tau_d v_1^H \xrightarrow{*_{H-S}} v_{1'}^H$ and $v_{1'}^H \underset{H}{\lesssim} v_1^1$
 15634 $\vee \text{dyn } \tau_d v_1^H \xrightarrow{*_{H-S}} \text{BndryErr}$
 15635 by **H-1 boundary checking** (1, 3)
 15636 5. **IF** $\text{dyn } \tau_d v_1^H \xrightarrow{*_{H-S}} \text{BndryErr} :$
 a. QED $v_0^H v_1^H \xrightarrow{*_{H-D}} \text{BndryErr}$
 15637 6. $\text{stat } \tau_c [] \underset{H}{\lesssim} \text{stat } []$
 by definition of $\underset{H}{\lesssim}$
 15638 7. $\vdash_H v_{0'}^H v_1^H : \tau_c$
 by **H dynamic preservation**
 15639 8. $v_{0'}^H v_1^H \underset{H}{\lesssim} v_0^1 v_1^1$
 15640 9. $v_{0'}^H v_1^H \xrightarrow{*_{H-S}} e_2^H$
 $\wedge e_2^H \underset{H}{\lesssim} e_2^1$
 by **H-1 static application** (7, 8)
 15641 10. QED $v_0^H v_1^H \xrightarrow{*_{H-D}} \text{stat } \tau_c e_2^H$
 $\wedge \text{stat } \tau_c e_2^H \underset{H}{\lesssim} \text{stat } e_2^1$
 15642 **CASE** $(\lambda x. e_0^1) v_1^1 \rightarrow_{1-S} e_0^1[x \leftarrow v_1^1]$
 $\wedge v_0^H = \lambda x. e_0^H :$
 15643 1. $e_0^H \underset{H}{\lesssim} e_0^1$
 $\wedge v_1^H \underset{H}{\lesssim} v_1^1$
 15644 by $v_0^H v_1^H \underset{H}{\lesssim} v_0^1 v_1^1$
 15645 2. $v_0^H v_1^H \triangleright_{H-D} e_0^1[x \leftarrow v_1^H]$
 15646 by definition \triangleright_{H-D}
 15647 3. $e_0^1[x \leftarrow v_1^H] \underset{H}{\lesssim} e_0^1[x \leftarrow v_1^1]$
 15648 by **H-1 substitution**
 15649 4. QED
 15650 **CASE** $(\lambda x. e_0^1) v_1^1 \rightarrow_{1-D} e_0^1[x \leftarrow v_1^1]$
 $\wedge v_0^H = \text{mon } (\tau_d \Rightarrow \tau_c) v_{0'}^H :$
 15651 1. $v_{0'}^H \underset{H}{\lesssim} (\lambda x. e_0^1) v_1^H \underset{H}{\lesssim} v_1^1$
 15652 by $v_0^H v_1^H \underset{H}{\lesssim} v_0^1 v_1^1$
 15653 2. $v_0^H v_1^H \triangleright_{H-D} \text{stat } \tau_c (v_{0'}^H (\text{dyn } \tau_d v_1^H))$
 15654 by definition \triangleright_{H-D}
 15655 3. $\vdash_H \text{dyn } \tau_d v_1^H : \tau_d$
 by **H dynamic preservation**
 15656 4. $\text{dyn } \tau_d v_1^H \xrightarrow{*_{H-D}} v_{1'}^H$ and $v_{1'}^H \underset{H}{\lesssim} v_1^1$
 15657 $\vee \text{dyn } \tau_d v_1^H \xrightarrow{*_{H-D}} \text{BndryErr}$
 15658 by **H-1 boundary checking** (1, 3)
 15659 5. **IF** $\text{dyn } \tau_d v_1^H \xrightarrow{*_{H-D}} \text{BndryErr} :$
 a. QED $v_0^H v_1^H \xrightarrow{*_{H-D}} \text{BndryErr}$

15660 6. $v_{0'}^H = \text{mon } (\tau_d \Rightarrow \tau_c) v_{0''}^H$ and $v_{0''}^H \underset{H}{\lesssim} v_0^1$
 $\vee v_{0'}^H = \lambda x. e^H$ and $e^H \underset{H}{\lesssim} e_0^1$
 by (1)
 15661 7. **IF** $v_{0'}^H = \lambda x. e^H :$
 a. $\vdash_H v_{0'}^H : \tau_0$
 $\wedge \tau_0 \leqslant \tau_d \Rightarrow \tau_c$
 by **H inversion**
 b. Contradiction by $\vdash_H \lambda x. e^H : \tau_0$
 15662 8. $v_{0'}^H v_{1'}^H \triangleright_{H-S} \text{dyn } \tau_c (\text{stat } \tau_d v_{1'}^H)$
 by definition \triangleright_{H-S}
 15663 9. $\vdash_H \text{stat } \tau_d v_{1'}^H : \tau_d$
 by **H static preservation**
 15664 10. $\text{stat } \tau_d v_{1'}^H \xrightarrow{*_{H-S}} v_{1''}^H$ and $v_{1''}^H \underset{H}{\lesssim} v_1^1$
 by **H-1 boundary checking**
 15665 11. $\text{stat } \tau_c (\text{dyn } \tau_c []) \underset{H}{\lesssim} []$
 by definition $\underset{H}{\lesssim}$
 15666 12. $v_{0''}^H v_{1''}^H \underset{H}{\lesssim} v_0^1 v_1^1$
 15667 13. $\vdash_H v_{0''}^H v_{1''}^H$
 by **H dynamic preservation**
 15668 14. $v_{0''}^H v_{1''}^H \xrightarrow{*_{H-D}} e_2^H$
 $\wedge e_2^H \underset{H}{\lesssim} e_2^1$
 by the induction hypothesis
 15669 15. QED $v_0^H v_1^H \xrightarrow{*_{H-S}} \text{stat } \tau_c (\text{dyn } \tau_c e_2^H)$

□

Lemma 8.16 : X inversion

If $X(\lfloor \tau \rfloor, v^1) = \text{BndryErr}$ and $v^H \underset{H}{\lesssim} v^1$ then $\mathcal{D}_H(\tau, v^H) = \text{BndryErr}$

Proof:

By case analysis on τ .

CASE $\tau = \text{Nat} :$

- $v^1 \notin \mathbb{N}$
 by $X(\lfloor \tau \rfloor, v^1) = \text{BndryErr}$
- $v^H \notin \mathbb{N}$
 by (1)
- $\mathcal{D}_H(\tau, v^H) = \text{BndryErr}$
 by (2)
- QED

CASE $\tau = \text{Int} :$

- $v^1 \notin \mathbb{Z}$
 by $X(\lfloor \tau \rfloor, v^1) = \text{BndryErr}$
- $v^H \notin \mathbb{Z}$
 by (1)
- $\mathcal{D}_H(\tau, v^H) = \text{BndryErr}$
 by (2)
- QED

CASE $\tau = \tau_0 \times \tau_1 :$

- $v^1 \notin \langle v, v \rangle$
 by $X(\lfloor \tau \rfloor, v^1) = \text{BndryErr}$
- $v^H \notin \langle v, v \rangle$
 by (1)
- $\mathcal{D}_H(\tau, v^H) = \text{BndryErr}$
 by (2)
- QED

15731 **CASE** $\tau = \tau_d \Rightarrow \tau_c :$
 15732 1. $v^1 \in \mathbb{Z}$
 $\vee v^1 \in \langle v, v \rangle$
 15733 by $X(\lfloor \tau \rfloor, v^1) = \text{BndryErr}$
 15734 2. $v^H \in \mathbb{Z}$
 15735 $\vee v^H \in \langle v, v \rangle$
 15736 by (1)
 15737 3. $\mathcal{D}_H(\tau, v^H) = \text{BndryErr}$
 15738 by (2)
 15739 4. QED
 15740 □

Lemma 8.17 : H-1 static value stutter

15743 If $e^H \underset{H-S}{\lesssim} v^1$ and $\vdash_H e^H : \tau$ then one of the following holds:
 15744 • $e^H \xrightarrow{^*_{H-S}} v^H$ and $v^H \underset{H-S}{\lesssim} v^1$
 15745 • $e^H \xrightarrow{^*_{H-S}} \text{BndryErr}$

15746 *Proof:*

15747 By induction on the number of boundary terms in e^H ,
 15748 and case analysis on $e^H \underset{H-S}{\lesssim} v^1$.

15749 **CASE** e^H is a value :

15750 1. QED
 15751 **CASE** $e^H = \text{dyn } \tau (\text{stat } \tau' e_2^H) :$

15752 1. $e_2^H \underset{H-S}{\lesssim} v^1$
 by $\underset{H-S}{\lesssim}$
 15753 2. $\vdash_H \text{stat } \tau' e_2^H$
 $\wedge \vdash_H e_2^H : \tau'$
 15754 by **H inversion** and **H inversion**
 15755 3. $e_2^H \xrightarrow{^*_{H-S}} v_2^H$
 $\wedge v_2^H \underset{H-S}{\lesssim} v^1$
 15756 by the induction hypothesis (1, 2)

15757 4. $\vdash_H \text{stat } \tau' v_2^H$
 by **H static preservation** (2, 3)
 15758 5. $\text{stat } \tau' v_2^H \xrightarrow{^*_{H-S}} v_3^H$ and $v_3^H \underset{H-S}{\lesssim} v^1$
 $\vee \text{stat } \tau' v_2^H \xrightarrow{^*_{H-S}} \text{BndryErr}$
 15759 by **H-1 boundary checking** (4)
 15760 6. **IF** $\text{stat } \tau' v_2^H \xrightarrow{^*_{H-S}} v_3^H :$
 a. $\vdash_H \text{dyn } \tau v_3^H : \tau$
 by **H static preservation** (5)
 b. $\text{dyn } \tau v_3^H \xrightarrow{^*_{H-S}} v_4^H$ and $v_4^H \underset{H-S}{\lesssim} v^1$
 $\vee \text{dyn } \tau v_3^H \xrightarrow{^*_{H-S}} \text{BndryErr}$
 by **H-1 boundary checking** (a)

15761 c. QED (b)

15762 **ELSE** $\text{stat } \tau' v_2^H \xrightarrow{^*_{H-S}} \text{BndryErr} :$
 15763 a. QED $\text{stat } \tau (\text{dyn } \tau' e_2^H) \xrightarrow{^*_{H-S}} \text{BndryErr}$

15764 **CASE** $e^H = \text{stat } \tau (\text{dyn } \tau' e_2^H) :$

15765 1. Contradiction by $\vdash_H e^H : \tau$

15766 □

Lemma 8.18 : H-1 dynamic value stutter

15767 If $e^H \underset{H-S}{\lesssim} v^1$ and $\vdash_H e^H$ then one of the following holds:
 15768 • $e^H \xrightarrow{^*_{H-D}} v^H$ and $v^H \underset{H-S}{\lesssim} v^1$
 15769 • $e^H \xrightarrow{^*_{H-D}} \text{BndryErr}$

15770 *Proof:*

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By induction on the number of boundary terms in e^H ,
 and case analysis on $e^H \underset{H-S}{\lesssim} v^1$.

CASE e^H is a value :

1. QED

CASE $e^H = \text{dyn } \tau (\text{stat } \tau' e_2^H) :$

1. Contradiction by $\vdash_H e^H : \tau$

CASE $e^H = \text{stat } \tau (\text{dyn } \tau' e_2^H) :$

1. $e_2^H \underset{H-S}{\lesssim} v^1$

by $\underset{H-S}{\lesssim}$

2. $\vdash_H \text{dyn } \tau' e_2^H : \tau'$

$\wedge \vdash_H e_2^H$

by **H inversion** and **H inversion**

3. $e_2^H \xrightarrow{^*_{H-D}} v_2^H$

$\wedge v_2^H \underset{H-S}{\lesssim} v^1$

by the induction hypothesis (1, 2)

4. $\vdash_H \text{dyn } \tau' v_2^H$

by **H static preservation** (2, 3)

5. $\text{dyn } \tau' v_2^H \xrightarrow{^*_{H-S}} v_3^H$ and $v_3^H \underset{H-S}{\lesssim} v^1$

$\vee \text{dyn } \tau' v_2^H \xrightarrow{^*_{H-S}} \text{BndryErr}$

by **H-1 boundary checking** (4)

6. **IF** $\text{dyn } \tau' v_2^H \xrightarrow{^*_{H-S}} v_3^H :$

a. $\vdash_H \text{stat } \tau v_3^H : \tau$

by **H static preservation** (5)

b. $\text{stat } \tau v_3^H \xrightarrow{^*_{H-S}} v_4^H$ and $v_4^H \underset{H-S}{\lesssim} v^1$

$\vee \text{stat } \tau v_3^H \xrightarrow{^*_{H-S}} \text{BndryErr}$

by **H-1 boundary checking** (a)

c. QED (b)

ELSE $\text{dyn } \tau' v_2^H \xrightarrow{^*_{H-S}} \text{BndryErr} :$

a. QED $\text{stat } \tau (\text{dyn } \tau' e_2^H) \xrightarrow{^*_{H-S}} \text{BndryErr}$

□

Lemma 8.19 : H-1 static hole typing

If $\vdash_H E^H[e^H] : \tau$

$\wedge \vdash_H E^1[e^1] : K$

$\wedge E^H[e^H] \underset{H-S}{\lesssim} E^1[e^1]$

then one of the following holds:

• $\vdash_H e^H : \tau'$

$\wedge \vdash_H e^1 : K'$

• $\vdash_H e^H$

$\wedge \vdash_H e^1$

Proof:

By induction on the structure of $E^H[e^H] \underset{H-S}{\lesssim} E^1[e^1]$
 judgment.

CASE $[] \underset{H-S}{\lesssim} [] :$

1. $E^H[e^H] = e^H$

2. $E^1[e^1] = e^1$

3. QED

CASE $E^H \underset{H-S}{\lesssim} \text{chk } K E_0^1 :$

1. $E^1[e^1] = \text{chk } K E_0^1[e^1]$

2. $\vdash_H E_0^1[e^1] : \text{Any}$

by **1 inversion**

3. $E^H[e^H] \underset{H-S}{\lesssim} E_0^1[e^1]$

4. QED by the induction hypothesis

CASE $E_0^H e_1^H \underset{H-S}{\lesssim} E_0^1 e_1^1 :$

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15841	1. $E^H[e^H] = E_0^H[e^H] e_1^H$	15896
15842	$\wedge E^1[e^1] = E_0^1[e^1] e_1^1$	15897
15843	2. $\vdash_H E_0^H[e^H] : \tau_d \Rightarrow \tau_c$	15898
15844	by H inversion	15899
15845	3. $\vdash_1 E_0^1[e^1] : \text{Fun}$	15900
15846	4. $E_0^H[e^H] \sim_1 E_0^1[e^1]$	15901
15847	5. QED by the induction hypothesis	15902
15848	CASE $v_0^H E_1^H \sim_1 v_0^1 E_1^1 :$	15903
15849	1. $E^H[e^H] = v_0^H E_1^H[e^H]$	15904
15850	$\wedge E^1[e^1] = v_0^1 E_1^1[e^1]$	15905
15851	2. $\vdash_H E_1^H[e^H] : \tau_d$	15906
15852	by H inversion	15907
15853	3. $\vdash_1 E_1^1[e^1] : \text{Any}$	15908
15854	4. $E_1^H[e^H] \sim_1 E_1^1[e^1]$	15909
15855	5. QED by the induction hypothesis	15910
15856	CASE $\langle E_0^H, e_1^H \rangle \sim_1 \langle E_0^1, e_1^1 \rangle :$	15911
15857	1. $E^H[e^H] = \langle E_0^H[e^H], e_1^H \rangle$	15912
15858	$\wedge E^1[e^1] = \langle E_0^1[e^1], e_1^1 \rangle$	15913
15859	2. $\vdash_H E_0^H[e^H] : \tau_0$	15914
15860	by H inversion	15915
15861	3. $\vdash_1 E_0^1[e^1] : \text{Any}$	15916
15862	4. $E_0^H[e^H] \sim_1 E_0^1[e^1]$	15917
15863	5. QED by the induction hypothesis	15918
15864	CASE $\langle v_0^H, E_1^H \rangle \sim_1 \langle v_0^1, E_1^1 \rangle :$	15919
15865	1. $E^H[e^H] = \langle v_0^H, E_1^H[e^H] \rangle$	15920
15866	$\wedge E^1[e^1] = \langle v_0^1, E_1^1[e^1] \rangle$	15921
15867	2. $\vdash_H E_1^H[e^H] : \tau_1$	15922
15868	by H inversion	15923
15869	3. $\vdash_1 E_1^1[e^1] : \text{Any}$	15924
15870	4. $E_1^H[e^H] \sim_1 E_1^1[e^1]$	15925
15871	5. QED by the induction hypothesis	15926
15872	CASE $op^1 E_0^H \sim_1 op^1 E_0^1 :$	15927
15873	1. $E^H[e^H] = op^1 E_0^H[e^H]$	15928
15874	$\wedge E^1[e^1] = op^1 E_0^1[e^1]$	15929
15875	2. $\vdash_H E_0^H[e^H] : \tau_0 \times \tau_1$	15930
15876	by H inversion	15931
15877	3. $\vdash_1 E_0^1[e^1] : \text{Pair}$	15932
15878	4. $E_0^H[e^H] \sim_1 E_0^1[e^1]$	15933
15879	5. QED by the induction hypothesis	15934
15880	CASE $op^2 E_0^H e_1^H \sim_1 op^2 E_0^1 e_1^1 :$	15935
15881	1. $E^H[e^H] = op^2 E_0^H[e^H] e_1^H$	15936
15882	$\wedge E^1[e^1] = op^2 E_0^1[e^1] e_1^1$	15937
15883	2. $\vdash_H E_0^H[e^H] : \tau_0$	15938
15884	by H inversion	15939
15885	3. $\vdash_1 E_0^1[e^1] : K_0$	15940
15886	4. $E_0^H[e^H] \sim_1 E_0^1[e^1]$	15941
15887	5. QED by the induction hypothesis	15942
15888	CASE $op^2 v_0^H E_1^H \sim_1 op^2 v_0^1 E_1^1 :$	15943
15889	1. $E^1[e^1] = op^2 v_0^H E_1^1[e^1]$	15944
15890	2. $\vdash_H E_1^H[e^H] : \tau_1$	15945
15891	by H inversion	15946
15892	3. $\vdash_1 E_1^1[e^1] : K_1$	15947
15893	4. $E_1^H[e^H] \sim_1 E_1^1[e^1]$	15948
15894	5. QED by the induction hypothesis	15949
15895		15950

15951	3. $\vdash_1 E_0^1[e^1]$	16006
15952	4. $E_0^H[e^H] \mathrel{H\sim 1} E_0^1[e^1]$	16007
15953	5. QED by the induction hypothesis	16008
15954	CASE $v_0^H E_1^H \mathrel{H\sim 1} v_0^1 E_1^1 :$	16009
15955	1. $E^H[e^H] = v_0^H E_1^H[e^H]$	16010
15956	$\wedge E^1[e^1] = v_0^1 E_1^1[e^1]$	16011
15957	2. $\vdash_H E_1^H[e^H]$	16012
15958	by H inversion	16013
15959	3. $\vdash_1 E_1^1[e^1]$	16014
15960	4. $E_1^H[e^H] \mathrel{H\sim 1} E_1^1[e^1]$	16015
15961	5. QED by the induction hypothesis	16016
15962	CASE $\langle E_0^H, e_1^H \rangle \mathrel{H\sim 1} \langle E_0^1, e_1^1 \rangle :$	16017
15963	1. $E^H[e^H] = \langle E_0^H[e^H], e_1^H \rangle$	16018
15964	$\wedge E^1[e^1] = \langle E_0^1[e^1], e_1^1 \rangle$	16019
15965	2. $\vdash_H E_0^H[e^H]$	16020
15966	by H inversion	16021
15967	3. $\vdash_1 E_0^1[e^1]$	16022
15968	4. $E_0^H[e^H] \mathrel{H\sim 1} E_0^1[e^1]$	16023
15969	5. QED by the induction hypothesis	16024
15970	CASE $\langle v_0^H, E_1^H \rangle \mathrel{H\sim 1} \langle v_0^1, E_1^1 \rangle :$	16025
15971	1. $E^H[e^H] = \langle v_0^H, E_1^H[e^H] \rangle$	16026
15972	$\wedge E^1[e^1] = \langle v_0^1, E_1^1[e^1] \rangle$	16027
15973	2. $\vdash_H E_1^H[e^H]$	16028
15974	by H inversion	16029
15975	3. $\vdash_1 E_1^1[e^1]$	16030
15976	4. $E_1^H[e^H] \mathrel{H\sim 1} E_1^1[e^1]$	16031
15977	5. QED by the induction hypothesis	16032
15978	CASE $op^1 E_0^H \mathrel{H\sim 1} op^1 E_0^1 :$	16033
15979	1. $E^H[e^H] = op^1 E_0^H[e^H]$	16034
15980	$\wedge E^1[e^1] = op^1 E_0^1[e^1]$	16035
15981	2. $\vdash_H E_0^H[e^H]$	16036
15982	by H inversion	16037
15983	3. $\vdash_1 E_0^1[e^1]$	16038
15984	4. $E_0^H[e^H] \mathrel{H\sim 1} E_0^1[e^1]$	16039
15985	5. QED by the induction hypothesis	16040
15986	CASE $op^2 E_0^H e_1^H \mathrel{H\sim 1} op^2 E_0^1 e_1^1 :$	16041
15987	1. $E^H[e^H] = op^2 E_0^H[e^H] e_1^H$	16042
15988	$\wedge E^1[e^1] = op^2 E_0^1[e^1] e_1^1$	16043
15989	2. $\vdash_H E_0^H[e^H]$	16044
15990	by H inversion	16045
15991	3. $\vdash_1 E_0^1[e^1]$	16046
15992	4. $E_0^H[e^H] \mathrel{H\sim 1} E_0^1[e^1]$	16047
15993	5. QED by the induction hypothesis	16048
15994	CASE $op^2 v_0^H E_1^H \mathrel{H\sim 1} op^2 v_0^1 E_1^1 :$	16049
15995	1. $E^H[e^H] = op^2 v_0^H E_1^H[e^H]$	16050
15996	$\wedge E^1[e^1] = op^2 v_0^1 E_1^1[e^1]$	16051
15997	2. $\vdash_H E_1^H[e^H]$	16052
15998	by H inversion	16053
15999	3. $\vdash_1 e_0^1$	16054
16000	$\wedge \vdash_1 E_1^1[e^1]$	16055
16001	4. $E_1^H[e^H] \mathrel{H\sim 1} E_1^1[e^1]$	16056
16002	5. QED by the induction hypothesis	16057
16003		16058
16004		16059
16005		16060

16061	mon $\tau v^H \underset{H\sim 1}{\lesssim} v^1$	
16062	by $v^H \underset{H\sim 1}{\lesssim} v^1$ QED by the induction hypothesis	
16063	□	
16064	Lemma 8.22 : H-1 boundary checking	
16065	• If $\vdash_H \text{stat } \tau v^H$ and $v^H \underset{H\sim 1}{\lesssim} v^1$ then $\text{stat } \tau v^H \rightarrow_{1-D}^* v_1^H$ $\wedge v_1^H \underset{H\sim 1}{\lesssim} v^1$	16116
16066		16117
16067	• If $\vdash_H \text{dyn } \tau v^H : \tau$ and $v^H \underset{H\sim 1}{\lesssim} v^1$ then one of the following holds:	16118
16068	– $\text{dyn } \tau v^H \rightarrow_{1-S}^* \text{BndryErr}$	16119
16069	– $\text{dyn } \tau v^H \rightarrow_{1-S}^* v_1^H$ $\wedge v_1^H \underset{H\sim 1}{\lesssim} v^1$	16120
16070		16121
16071		16122
16072		16123
16073	<i>Proof:</i>	16124
16074	By the following two lemmas: H-1 stat checking and H-1 dyn checking .	16125
16075	□	16126
16076	Lemma 8.23 : H-1 stat checking	16127
16077	If $\vdash_H \text{stat } \tau v^H$ $\wedge v^H \underset{H\sim 1}{\lesssim} v^1$	16128
16078	then $\text{stat } \tau v^H \rightarrow_{1-D}^* v_1^H$ $\wedge v_1^H \underset{H\sim 1}{\lesssim} v^1$	16129
16079		16130
16080		16131
16081		16132
16082	<i>Proof:</i>	16133
16083	By induction on the structure of τ .	16134
16084	CASE $\tau = \text{Nat} :$	16135
16085	1. $\text{stat } \tau v^H \triangleright_{H-D} v^H$ by definition \triangleright_{H-D}	16136
16086	2. QED	16137
16087		16138
16088	CASE $\tau = \text{Int} :$	16139
16089	1. $\text{stat } \tau v^H \triangleright_{H-D} v^H$ by definition \triangleright_{H-D}	16140
16090	2. QED	16141
16091		16142
16092	CASE $\tau = \tau_0 \times \tau_1 :$	16143
16093	1. $v^H = \langle v_0^H, v_1^H \rangle$ by H inversion and canonical forms	16144
16094	2. $v^1 = \langle v_0^1, v_1^1 \rangle$ $\wedge v_0^H \underset{H\sim 1}{\lesssim} v_0^1$ $\wedge v_1^H \underset{H\sim 1}{\lesssim} v_1^1$ by $\underset{H\sim 1}{\lesssim}$	16145
16095	3. $\text{stat } \tau v^H \triangleright_{H-D} \langle \text{stat } \tau_0 v_0^H, \text{stat } \tau_1 v_1^H \rangle$ by definition \triangleright_{H-D}	16146
16096	4. $\vdash_H \text{stat } \tau_0 v_0^H$ $\wedge \vdash_H \text{stat } \tau_1 v_1^H$ by H static preservation and H inversion	16147
16097	5. $\text{stat } \tau_0 v_0^H \rightarrow_{1-D}^* v_0^H$ $\wedge v_0^H \underset{H\sim 1}{\lesssim} v_0^1$ $\wedge \text{stat } \tau_1 v_1^H \rightarrow_{1-D}^* v_1^H$ $\wedge v_1^H \underset{H\sim 1}{\lesssim} v_1^1$ by the induction hypothesis (2, 4)	16148
16098	6. $\text{stat } \tau v^H \rightarrow_{H-D}^* \langle v_0^H, v_1^H \rangle$ $\wedge \langle v_0^H, v_1^H \rangle \underset{H\sim 1}{\lesssim} v^1$ by (5)	16149
16099	7. QED	16150
16100		16151
16101		16152
16102		16153
16103		16154
16104		16155
16105		16156
16106		16157
16107		16158
16108		16159
16109		16160
16110		16161
16111		16162
16112		16163
16113	CASE $\tau = \tau_d \Rightarrow \tau_c :$	16164
16114		16165
16115		16166

1.	$\text{stat } \tau v^H \triangleright_{H-D} \text{mon } \tau v^H$	16167
2.	$\text{mon } \tau v^H \underset{H\sim 1}{\lesssim} v^1$ by $v^H \underset{H\sim 1}{\lesssim} v^1$	16168
3.	QED	16169
□		16170
	Lemma 8.24 : H-1 dyn checking	
	If $\vdash_H \text{dyn } \tau v^H : \tau$ $\wedge v^H \underset{H\sim 1}{\lesssim} v^1$	
	then one of the following holds:	
	• $\text{dyn } \tau v^H \rightarrow_{1-S}^* \text{BndryErr}$	16122
	• $\text{dyn } \tau v^H \rightarrow_{1-S}^* v_1^H$ $\wedge v_1^H \underset{H\sim 1}{\lesssim} v^1$	16123
		16124
	<i>Proof:</i>	
	By induction on the structure of τ .	
	CASE $\tau = \text{Nat} :$	
	IF $v^H \in \natural :$	16131
	1. $\text{dyn } \tau v^H \triangleright_{H-S} v^H$ by definition \triangleright_{H-S}	16132
	2. QED	16133
	ELSE $v^H \notin \natural :$	16135
	1. $\text{dyn } \tau v^H \triangleright_{H-S} \text{BndryErr}$	16136
	2. QED	16137
	CASE $\tau = \text{Int} :$	16138
	IF $v^H \in \mathbb{Z} :$	16139
	1. $\text{dyn } \tau v^H \triangleright_{H-S} v^H$ by definition \triangleright_{H-S}	16140
	2. QED	16141
	ELSE $v^H \notin \mathbb{Z} :$	16143
	1. $\text{dyn } \tau v^H \triangleright_{H-S} \text{BndryErr}$	16144
	2. QED	16145
	CASE $\tau = \tau_0 \times \tau_1 :$	16146
	IF $v^H = \langle v_0^H, v_1^H \rangle :$	16147
	1. $v^1 = \langle v_0^1, v_1^1 \rangle$ $\wedge v_0^H \underset{H\sim 1}{\lesssim} v_0^1$ $\wedge v_1^H \underset{H\sim 1}{\lesssim} v_1^1$ by $\underset{H\sim 1}{\lesssim}$	16148
	2. $\text{dyn } \tau v^H \triangleright_{H-S} \langle \text{dyn } \tau_0 v_0^H, \text{dyn } \tau_1 v_1^H \rangle$ by definition \triangleright_{H-S}	16149
	3. $\vdash_H \text{dyn } \tau_0 v_0^H : \tau_0$ $\wedge \vdash_H \text{dyn } \tau_1 v_1^H : \tau_1$ by H static preservation and H inversion	16150
	4. $\text{dyn } \tau_0 v_0^H \rightarrow_{1-D}^* e_0^H$ $\wedge e_0^H \underset{H\sim 1}{\lesssim} v_0^1$ $\wedge e_0^H = v_0^H$ or $e_0^H = \text{BndryErr}$	16151
	$\wedge \text{dyn } \tau_1 v_1^H \rightarrow_{1-D}^* e_1^H$ $\wedge e_1^H \underset{H\sim 1}{\lesssim} v_1^1$ $e_1^H = v_1^H$ or $e_1^H = \text{BndryErr}$	16152
	by the induction hypothesis (2, 4)	16153
	5. $\text{dyn } \tau v^H \rightarrow_{H-S}^* \text{BndryErr}$ $\vee \text{dyn } \tau v^H \rightarrow_{H-S}^* \langle v_0^H, v_1^H \rangle$ and $\langle v_0^H, v_1^H \rangle \underset{H\sim 1}{\lesssim} v^1$ by (5)	16154
	6. QED	16155
	ELSE $v^H \notin \langle v, v \rangle :$	16156

16171 1. $\text{dyn } \tau v^H \triangleright_{H-S} \text{BndryErr}$
 16172 2. QED
CASE $\tau = \tau_d \Rightarrow \tau_c :$
 16173 **IF** $v^H = \lambda x. e^H :$
 16174 1. $\text{dyn } \tau v^H \triangleright_{H-S} \text{mon } \tau v^H$
 16175 2. $\text{mon } \tau v^H \underset{H}{\lesssim} v^1$
 16176 by $v^H \underset{H}{\lesssim} v^1$
 16177 3. QED
 16178 **IF** $v^H = \text{mon } \tau_0 v_0^H :$
 16179 1. $\text{dyn } \tau v^H \triangleright_{H-S} \text{mon } \tau v^H$
 16180 2. $\text{mon } \tau v^H \underset{H}{\lesssim} v^1$
 16181 by $v^H \underset{H}{\lesssim} v^1$
 16182 3. QED
 16183 **ELSE** $v^H \in i \vee v^H \in \langle v, v \rangle :$
 16184 1. $\text{dyn } \tau v^H \triangleright_{H-S} \text{BndryErr}$
 16185 2. QED
 16186 \square
 16187

Lemma 8.25 : H-1 hole substitution

16188 If $E^H \underset{H}{\lesssim} E^1$
 16189 $\wedge e^H \underset{H}{\lesssim} e^1$
 16190 then $E^H[e^H] \underset{H}{\lesssim} E^1[e^1]$
 16191

Proof:

By induction on the structure of the $E^H \underset{H}{\lesssim} E^1$ judgment.

CASE $[] \underset{H}{\lesssim} [] :$

1. $E^H[e^H] = e^H$
2. $E^1[e^1] = e^1$
3. QED

CASE $E^H \underset{H}{\lesssim} \text{chk } K_0 E_0^1 :$

1. $E^H \underset{H}{\lesssim} E_0^1$

2. QED by the induction hypothesis

CASE $E_0^H e_1^H \underset{H}{\lesssim} E_0^1 e_1^1 :$

1. $E_0^H \underset{H}{\lesssim} E_0^1$

2. QED by the induction hypothesis

CASE $v_0^H E_1^H \underset{H}{\lesssim} v_0^1 E_1^1 :$

1. $E_1^H \underset{H}{\lesssim} E_1^1$

2. QED by the induction hypothesis

CASE $\langle E_0^H, e_1^H \rangle \underset{H}{\lesssim} \langle E_0^1, e_1^1 \rangle :$

1. $E_0^H \underset{H}{\lesssim} E_0^1$

2. QED by the induction hypothesis

CASE $\langle v_0^H, E_1^H \rangle \underset{H}{\lesssim} \langle v_0^1, E_1^1 \rangle :$

1. $E_1^H \underset{H}{\lesssim} E_1^1$

2. QED by the induction hypothesis

CASE $op^1 E_0^H \underset{H}{\lesssim} op^1 E_0^1 :$

1. $E_0^H \underset{H}{\lesssim} E_0^1$

2. QED by the induction hypothesis

CASE $op^2 E_0^H e_1^H \underset{H}{\lesssim} op^2 E_0^1 e_1^1 :$

1. $E_0^H \underset{H}{\lesssim} E_0^1$

2. QED by the induction hypothesis

CASE $op^2 v_0^H E_1^H \underset{H}{\lesssim} op^2 v_0^1 E_1^1 :$

1. $E_1^H \underset{H}{\lesssim} E_1^1$

2. QED by the induction hypothesis

CASE $\text{dyn } \tau_0 E_0^H \underset{H}{\lesssim} \text{dyn } \tau_0 E_0^1 :$

16225

1. $E_0^H \underset{H}{\lesssim} E_0^1$
 2. QED by the induction hypothesis
CASE $\text{dyn } \tau_0 E_0^H \underset{H}{\lesssim} \text{dyn } E_0^1 :$
 1. $E_0^H \underset{H}{\lesssim} E_0^1$
 2. QED by the induction hypothesis
CASE $\text{dyn } \tau_0 (\text{stat } \tau_1 E_0^H) \underset{H}{\lesssim} E_0^1 :$
 1. $E_0^H \underset{H}{\lesssim} E_0^1$
 2. QED by the induction hypothesis
CASE $\text{stat } \tau_0 E_0^H \underset{H}{\lesssim} \text{stat } \tau_0 E_0^1 :$
 1. $E_0^H \underset{H}{\lesssim} E_0^1$
 2. QED by the induction hypothesis
CASE $\text{stat } \tau_0 E_0^H \underset{H}{\lesssim} \text{stat } E_0^1 :$
 1. $E_0^H \underset{H}{\lesssim} E_0^1$
 2. QED by the induction hypothesis
CASE $\text{stat } \tau_0 (\text{dyn } \tau_1 E_0^H) \underset{H}{\lesssim} E_0^1 :$
 1. $E_0^H \underset{H}{\lesssim} E_0^1$
 2. QED by the induction hypothesis
CASE $\text{stat } \tau_0 (\text{stat } \tau_1 E_0^H) \underset{H}{\lesssim} E_0^1 :$
 1. $E_0^H \underset{H}{\lesssim} E_0^1$
 2. QED by the induction hypothesis
CASE $\text{chk } K_0 e_0^1[x \leftarrow v^1] = \text{chk } K_0 e_0^1[x \leftarrow v^1]$
 1. $e^H \underset{H}{\lesssim} e^1$
 2. $e^H[x \leftarrow v^H] \underset{H}{\lesssim} e_0^1[x \leftarrow v^1]$
 by the induction hypothesis
 3. $\text{chk } K_0 e_0^1[x \leftarrow v^1] = \text{chk } K_0 e_0^1[x \leftarrow v^1]$
 4. QED
CASE $\text{Err } \underset{H}{\lesssim} e^1 :$
 1. $\text{Err}[x \leftarrow v^H] = \text{Err}$
 2. $\text{QED } \text{Err } \underset{H}{\lesssim} e^1[x \leftarrow v^1]$
CASE $\text{dyn } \tau_0 e_0^H \underset{H}{\lesssim} \text{dyn } \tau_0 e_0^1 :$
 1. $e_0^H \underset{H}{\lesssim} e_0^1$
 2. $e_0^H[x \leftarrow v^H] \underset{H}{\lesssim} e_0^1[x \leftarrow v^1]$
 by the induction hypothesis
 3. $\text{dyn } \tau_0 e_0^H[x \leftarrow v^H] = \text{dyn } \tau_0 e_0^H[x \leftarrow v^H]$
 4. $\text{dyn } \tau_0 e_0^1[x \leftarrow v^1] = \text{dyn } \tau_0 e_0^1[x \leftarrow v^1]$
 5. QED
CASE $\text{dyn } \tau_0 e_0^H \underset{H}{\lesssim} \text{dyn } e_0^1 :$
 1. $e_0^H \underset{H}{\lesssim} e_0^1$
 2. $e_0^H[x \leftarrow v^H] \underset{H}{\lesssim} e_0^1[x \leftarrow v^1]$
 by the induction hypothesis
 3. $\text{dyn } \tau_0 e_0^H[x \leftarrow v^H] = \text{dyn } \tau_0 e_0^H[x \leftarrow v^H]$
 4. $\text{dyn } e_0^1[x \leftarrow v^1] = \text{dyn } e_0^1[x \leftarrow v^1]$
 5. QED
CASE $\text{dyn } \tau_0 (\text{stat } \tau_1 e_0^H) \underset{H}{\lesssim} e^1 :$
 1. $e_0^H \underset{H}{\lesssim} e^1$
 2. $e_0^H[x \leftarrow v^H] \underset{H}{\lesssim} e^1[x \leftarrow v^1]$
 by the induction hypothesis
 3. $\text{dyn } \tau_0 (\text{stat } \tau_1 e_0^H)[x \leftarrow v^H] = \text{dyn } \tau_0 (\text{stat } \tau_1 e_0^H[x \leftarrow v^H])$
 4. QED
CASE $\text{stat } \tau_0 e_0^H \underset{H}{\lesssim} \text{stat } \tau_0 e_0^1 :$
 1. $e_0^H \underset{H}{\lesssim} e_0^1$
 2. QED by the induction hypothesis

16281	1. $e_0^H \text{ H} \lesssim_1 e_0^1$	16336
16282	2. $e_0^H[x \leftarrow v^H] \text{ H} \lesssim_1 e_0^1[x \leftarrow v^1]$	16337
16283	by the induction hypothesis	16338
16284	3. $\text{stat } \tau_0 e_0^H[x \leftarrow v^H] = \text{stat } \tau_0 e_0^H[x \leftarrow v^H]$	16339
16285	4. QED	16340
16286	CASE $\text{stat } \tau_0 e_0^H \text{ H} \lesssim_1 \text{stat } e_0^1 :$	16341
16287	1. $e_0^H \text{ H} \lesssim_1 e_0^1$	16342
16288	2. $e_0^H[x \leftarrow v^H] \text{ H} \lesssim_1 e_0^1[x \leftarrow v^1]$	16343
16289	by the induction hypothesis	16344
16290	3. $\text{stat } \tau_0 e_0^H[x \leftarrow v^H] = \text{stat } \tau_0 e_0^H[x \leftarrow v^H]$	16345
16291	4. $\text{stat } e_0^1[x \leftarrow v^1] = \text{stat } e_0^1[x \leftarrow v^1]$	16346
16292	5. QED	16347
16293	CASE $\text{stat } \tau_0 (\text{dyn } \tau_1 e_0^H) \text{ H} \lesssim_1 e^1 :$	16348
16294	1. $e_0^H \text{ H} \lesssim_1 e^1$	16349
16295	2. $e_0^H[x \leftarrow v^H] \text{ H} \lesssim_1 e^1[x \leftarrow v^1]$	16350
16296	by the induction hypothesis	16351
16297	3. $\text{stat } \tau_0 (\text{dyn } \tau_1 e_0^H)[x \leftarrow v^H] = \text{stat } \tau_0 (\text{dyn } \tau_1 e_0^H[x \leftarrow v^H])$	16352
16298	4. QED	16353
16299	CASE $e_0^H e_1^H \text{ H} \lesssim_1 e_0^1 e_1^1 :$	16354
16300	1. $e_0^H \text{ H} \lesssim_1 e_0^1$	16355
16301	$\wedge e_1^H \text{ H} \lesssim_1 e_1^1$	16356
16302	2. $e_0^H[x \leftarrow v^H] \text{ H} \lesssim_1 e_0^1[x \leftarrow v^1]$	16357
16303	$\wedge e_1^H[x \leftarrow v^H] \text{ H} \lesssim_1 e_1^1[x \leftarrow v^1]$	16358
16304	by the induction hypothesis	16359
16305	3. $e_0^H e_1^H[x \leftarrow v^H] = e_0^H[x \leftarrow v^H] e_1^H[x \leftarrow v^H]$	16360
16306	4. $e_0^1 e_1^1[x \leftarrow v^1] = e_0^1[x \leftarrow v^1] e_1^1[x \leftarrow v^1]$	16361
16307	5. QED	16362
16308	CASE $\langle e_0^H, e_1^H \rangle \text{ H} \lesssim_1 \langle e_0^1, e_1^1 \rangle :$	16363
16309	1. $e_0^H \text{ H} \lesssim_1 e_0^1$	16364
16310	$\wedge e_1^H \text{ H} \lesssim_1 e_1^1$	16365
16311	2. $e_0^H[x \leftarrow v^H] \text{ H} \lesssim_1 e_0^1[x \leftarrow v^1]$	16366
16312	$\wedge e_1^H[x \leftarrow v^H] \text{ H} \lesssim_1 e_1^1[x \leftarrow v^1]$	16367
16313	by the induction hypothesis	16368
16314	3. $\langle e_0^H, e_1^H \rangle[x \leftarrow v^H] = \langle e_0^H[x \leftarrow v^H], e_1^H[x \leftarrow v^H] \rangle$	16369
16315	4. $\langle e_0^1, e_1^1 \rangle[x \leftarrow v^1] = \langle e_0^1[x \leftarrow v^1], e_1^1[x \leftarrow v^1] \rangle$	16370
16316	5. QED	16371
16317	CASE $op^1 e_0^H \text{ H} \lesssim_1 op^1 e_0^1 :$	16372
16318	1. $e_0^H \text{ H} \lesssim_1 e_0^1$	16373
16319	2. $e_0^H[x \leftarrow v^H] \text{ H} \lesssim_1 e_0^1[x \leftarrow v^1]$	16374
16320	by the induction hypothesis	16375
16321	3. $op^1 e_0^H[x \leftarrow v^H] = op^1 e_0^H[x \leftarrow v^H]$	16376
16322	4. $op^1 e_0^1[x \leftarrow v^1] = op^1 e_0^1[x \leftarrow v^1]$	16377
16323	5. QED	16378
16324	CASE $op^2 e_0^H e_1^H \text{ H} \lesssim_1 op^2 e_0^1 e_1^1 :$	16379
16325	1. $e_0^H \text{ H} \lesssim_1 e_0^1$	16380
16326	$\wedge e_1^H \text{ H} \lesssim_1 e_1^1$	16381
16327	2. $e_0^H[x \leftarrow v^H] \text{ H} \lesssim_1 e_0^1[x \leftarrow v^1]$	16382
16328	$\wedge e_1^H[x \leftarrow v^H] \text{ H} \lesssim_1 e_1^1[x \leftarrow v^1]$	16383
16329	by the induction hypothesis	16384
16330	3. $op^2 e_0^H e_1^H[x \leftarrow v^H] = op^2 e_0^H[x \leftarrow v^H] e_1^H[x \leftarrow v^H]$	16385
16331	4. $op^2 e_0^1 e_1^1[x \leftarrow v^1] = op^2 e_0^1[x \leftarrow v^1] e_1^1[x \leftarrow v^1]$	16386
16332	5. QED	16387
16333	CASE $\text{Err} \text{ H} \lesssim_1 \text{Err} :$	16388
16334	1. QED $\text{Err}[x \leftarrow v^H] = \text{Err}$	16389
16335	□	16390

Lemma 8.27 : chk inversion

If $\Gamma \vdash_1 \text{chk } K e' : K$ and $\Gamma \vdash e : \tau \rightsquigarrow \text{chk } K e'$ then $K = \lfloor \tau \rfloor$.
Proof:

By case analysis on \rightsquigarrow .

CASE	$\frac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \rightsquigarrow e'_0 \quad \Gamma \vdash e_1 : \tau_d \rightsquigarrow e'_1 \quad \lfloor \tau_c \rfloor = K}{\Gamma \vdash e_0 e_1 : \tau_c \rightsquigarrow \text{chk } K (e'_0 e'_1)}$
:	1. QED

16391	CASE	$\frac{\Gamma \vdash e : \tau_0 \times \tau_1 \rightsquigarrow e' \quad \lfloor \tau_0 \rfloor = K}{\Gamma \vdash \text{fst } e : \tau_0 \rightsquigarrow \text{chk } K (\text{fst } e')}$: 16446
16392		1. QED	16447
16393	CASE	$\frac{\Gamma \vdash e : \tau_0 \times \tau_1 \rightsquigarrow e' \quad \lfloor \tau_1 \rfloor = K}{\Gamma \vdash \text{snd } e : \tau_1 \rightsquigarrow \text{chk } K (\text{snd } e')}$: 16448
16394		1. QED	16449
16395	CASE	$\frac{\Gamma \vdash e : \tau_0 \times \tau_1 \rightsquigarrow e' \quad \lfloor \tau_1 \rfloor = K}{\Gamma \vdash \text{snd } e : \tau_1 \rightsquigarrow \text{chk } K (\text{snd } e')}$: 16450
16396		1. QED	16451
16397		□	16452
16398			16453
16399	Lemma 8.28 : H-1 δ -preservation		16454
16400	• If $v^H \text{ H}\lesssim_1 v^1$ and $\delta(op^1, v^H)$ is defined then $\delta(op^1, v^H) \text{ H}\lesssim_1 \delta(op^1, v^1)$		16455
16401			16456
16402	• If $v_0^H \text{ H}\lesssim_1 v_0^1$ and $v_1^H \text{ H}\lesssim_1 v_1^1$ and $\delta(op^2, v_0^H, v_1^H)$ is defined then $\delta(op^2, v_0^H, v_1^H) \text{ H}\lesssim_1 \delta(op^2, v_0^1, v_1^1)$		16457
16403			16458
16404			16459
16405	<i>Proof:</i>		16460
16406	CASE $op^1 = \text{fst}$:		16461
16407	1. $v^H = \langle v_0^H, v_1^H \rangle$		16462
16408	by $\delta(\text{fst}, v^H)$ is defined		16463
16409	2. $v^1 = \langle v_0^1, v_1^1 \rangle$		16464
16410	$\wedge v_0^H \text{ H}\lesssim_1 v_0^1$ and $v_1^H \text{ H}\lesssim_1 v_1^1$		16465
16411	by $\text{H}\lesssim_1$		16466
16412	3. $\delta(\text{fst}, v^H) = v_0^H$		16467
16413	$\wedge \delta(\text{fst}, v^1) = v_0^1$		16468
16414	4. QED (2)		16469
16415	CASE $op^1 = \text{snd}$:		16470
16416	1. $v^H = \langle v_0^H, v_1^H \rangle$		16471
16417	by $\delta(\text{snd}, v^H)$ is defined		16472
16418	2. $v^1 = \langle v_0^1, v_1^1 \rangle$		16473
16419	$\wedge v_0^H \text{ H}\lesssim_1 v_0^1$ and $v_1^H \text{ H}\lesssim_1 v_1^1$		16474
16420	by $\text{H}\lesssim_1$		16475
16421	3. $\delta(\text{snd}, v^H) = v_1^H$		16476
16422	$\wedge \delta(\text{snd}, v^1) = v_1^1$		16477
16423	4. QED (2)		16478
16424	CASE $op^2 = \text{sum}$:		16479
16425	1. $v_0^H \in \mathbb{Z}$		16480
16426	$\wedge v_1^H \in \mathbb{Z}$		16481
16427	by $\delta(op^2, v_0^H, v_1^H)$ is defined		16482
16428	2. $v_0^H = v_0^1$		16483
16429	$\wedge v_1^H = v_1^1$		16484
16430	by $\text{H}\lesssim_1$		16485
16431	3. QED		16486
16432	CASE $op^2 = \text{quotient}$:		16487
16433	1. $v_0^H \in \mathbb{Z}$		16488
16434	$\wedge v_1^H \in \mathbb{Z}$		16489
16435	by $\delta(op^2, v_0^H, v_1^H)$ is defined		16490
16436	2. $v_0^H = v_0^1$		16491
16437	$\wedge v_1^H = v_1^1$		16492
16438	by $\text{H}\lesssim_1$		16493
16439	3. QED		16494
16440	□		16495
16441			16496
16442			16497
16443			16498
16444			16499
16445			16500