# A Spectrum of Soundness and Performance

# Supplementary Material

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# A Models

This section contains full definitions of the languages and full proofs of our claims about each language.

Aside from the common notions in section 1.1, the definition and proofs of each model are independent and selfcontained.

### A.1 Preliminaries

# **Definition 1.0**: $\rightarrow^*$ *divergence*

Given a reduction relation  $\rightarrow$ \*, an expression e diverges if for all e' such that  $e \rightarrow$ \* e' there exists an e'' such that  $e' \rightarrow e''$ .

#### **Convention 1.1**: *variable convention*

All  $\lambda$ -bound variables in an expression are distinct from one another, and from any free variables in the expression.

# **Assumption 1.2**: ⊢ permutation

For all typing judgments and properties ⊢:

- If  $x, x', \Gamma \vdash e$  then  $x', x, \Gamma \vdash e$
- If  $(x:\tau), (x':\tau'), \Gamma \vdash e$  then  $(x':\tau'), (x:\tau), \Gamma \vdash e$

#### **Definition 1.3** : ⊢ *boundary-free*

An expression e is boundary free if e does not contain a subterm of the form (dyn  $\tau'$  e'), nor a subterm of the form (stat  $\tau'$  e').

#### Notes:

- The upcoming models use a common surface syntax and typing system, but to keep each model self-contained we reprint this system in each definition.
- The proofs are written in a structured style, typically as a list of basic steps where each step is justified by an assumption, a lemma, or a previous step. Lemma names are *italicized* and hyperlinked to the actual lemma.

# A.2 (H) Higher-Order Embedding

# A.2.1 Higher-Order Definitions

Language H

$$e = x \mid v \mid \langle e, e \rangle \mid e e \mid op^{1} e \mid op^{2} e e \mid$$

$$dyn \tau e \mid stat \tau e \mid Err$$

$$v = i \mid \langle v, v \rangle \mid \lambda x. e \mid \lambda (x:\tau). e \mid$$

$$mon (\tau \Rightarrow \tau) v$$

$$\tau = \text{Nat} \mid \text{Int} \mid \tau \times \tau \mid \tau \Rightarrow \tau$$

$$\Gamma = \cdot \mid x, \Gamma \mid (x:\tau), \Gamma$$

$$Err = \text{BndryErr} \mid \text{TagErr}$$

$$r = v \mid Err$$

$$E^{\bullet} = [] \mid E^{\bullet} e \mid v E^{\bullet} \mid \langle E^{\bullet}, e \rangle \mid \langle v, E^{\bullet} \rangle \mid$$

$$op^{1} E^{\bullet} \mid op^{2} E^{\bullet} e \mid op^{2} v E^{\bullet}$$

$$E = E^{\bullet} \mid E e \mid v E \mid \langle E, e \rangle \mid \langle v, E \rangle \mid op^{1} E \mid$$

$$op^{2} E e \mid op^{2} v E \mid \text{dyn } \tau E \mid \text{stat } \tau E$$

$$\Delta : op^{1} \times \tau \longrightarrow \tau$$

$$\Delta(\operatorname{fst}, \tau_0 \times \tau \longrightarrow t)$$

$$\Delta(\operatorname{fst}, \tau_0 \times \tau_1) = \tau_0$$

$$\Delta(\operatorname{snd}, \tau_0 \times \tau_1) = \tau_1$$

$$\Delta : op^2 \times \tau \times \tau \longrightarrow \tau$$

$$\Delta(op^2, \operatorname{Nat}, \operatorname{Nat}) = \operatorname{Nat}$$

$$\Delta(op^2, \operatorname{Int}, \operatorname{Int}) = \operatorname{Int}$$

$$\frac{\tau \leqslant \tau}{\text{Nat} \leqslant : \text{Int}} \quad \frac{\tau'_d \leqslant : \tau_d \quad \tau_c \leqslant : \tau'_c}{\tau_d \Rightarrow \tau_c \leqslant : \tau'_c \Rightarrow \tau'_c} \quad \frac{\tau_0 \leqslant : \tau'_0 \quad \tau_1 \leqslant : \tau'_1}{\tau_0 \times \tau_1 \leqslant : \tau'_0 \times \tau'_1}$$

$$\frac{\tau \leqslant : \tau' \quad \tau' \leqslant : \tau''}{\tau \leqslant : \tau''}$$

$$\Gamma \vdash e$$

$$\frac{x \in \Gamma}{\Gamma \vdash x} \quad \frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x. e} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}$$

$$\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0 \ e_1} \quad \frac{\Gamma \vdash e}{\Gamma \vdash op^1 \ e} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2 \ e_0 \ e_1} \quad \frac{\Gamma \vdash \text{Err}}{\Gamma \vdash e_0}$$

$$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat} \ \tau \ e}$$

$$\begin{array}{ll} \boxed{\Gamma \vdash e : \tau} \\ & \underbrace{(x \colon \tau) \in \Gamma}_{\Gamma \vdash x \colon \tau} & \underbrace{(x \colon \tau_d), \Gamma \vdash e : \tau_c}_{\Gamma \vdash \lambda(x \colon \tau_d). \ e \colon \tau_d \Rightarrow \tau_c} & \underbrace{i \in \mathbb{N}}_{\Gamma \vdash i \colon \mathrm{Nat}} \\ & \underbrace{\Gamma \vdash e_0 \colon \tau_0}_{\Gamma \vdash e_1 \colon \tau_1} & \underbrace{\Gamma \vdash e_0 \colon \tau_d \Rightarrow \tau_c}_{\Gamma \vdash e_1 \colon \tau_d} \\ & \underbrace{\Gamma \vdash e_1 \colon \tau_1}_{\Gamma \vdash \langle e_0, e_1 \rangle \colon \tau_0 \times \tau_1} & \underbrace{\Gamma \vdash e_0 \colon \tau_d \Rightarrow \tau_c}_{\Gamma \vdash e_0 \colon \tau_d} \end{array}$$

$$\begin{array}{cccc} & \Gamma \vdash e_0 : \tau_0 \\ \Gamma \vdash e_0 : \tau_0 & \Gamma \vdash e_1 : \tau_1 & \Gamma \vdash e : \tau' \\ \underline{\Delta(op^1, \tau_0) = \tau} & \underline{\Delta(op^2, \tau_0, \tau_1) = \tau} & \underline{\tau' \leqslant : \tau} \\ \Gamma \vdash op^1 e_0 : \tau & \overline{\Gamma \vdash op^2 e_0 e_1 : \tau} & \overline{\Gamma \vdash e : \tau} & \overline{\Gamma \vdash Err : \tau} \end{array}$$

$$\frac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn} \ \tau \ e : \tau}$$

$$\frac{\Gamma \vdash_{\!\!\!H} e_0 \quad \Gamma \vdash_{\!\!\!H} e_1}{\Gamma \vdash_{\!\!\!H} e_0 \ e_1} \quad \frac{\Gamma \vdash_{\!\!\!H} e}{\Gamma \vdash_{\!\!\!H} op^1 e} \quad \frac{\Gamma \vdash_{\!\!\!H} e_0 \quad \Gamma \vdash_{\!\!\!H} e_1}{\Gamma \vdash_{\!\!\!H} op^2 e_0 \ e_1} \quad \frac{\Gamma \vdash_{\!\!\!H} Err}{\Gamma \vdash_{\!\!\!H} op^2 e_0 e_1}$$

$$\frac{\Gamma \vdash_{\mathsf{H}} e : \tau}{\left(x : \tau\right) \in \Gamma} \quad \frac{(x : \tau_d), \Gamma \vdash_{\mathsf{H}} e : \tau_c}{\Gamma \vdash_{\mathsf{H}} \lambda(x : \tau_d). e : \tau_d \Rightarrow \tau_c} \quad \frac{i \in \mathbb{N}}{\Gamma \vdash_{\mathsf{H}} i : \mathsf{Nat}}$$

$$\begin{array}{cccc} & \Gamma \vdash_{\!\!\!H} e_0 : \tau_0 \\ \Gamma \vdash_{\!\!\!H} e_0 : \tau_0 & \Gamma \vdash_{\!\!\!H} e_1 : \tau_1 \\ \Delta(op^1,\tau_0) = \tau & \Delta(op^2,\tau_0,\tau_1) = \tau \\ \hline \Gamma \vdash_{\!\!\!H} op^1 e_0 : \tau & \overline{\Gamma} \vdash_{\!\!\!H} op^2 e_0 \ e_1 : \tau & \overline{\Gamma} \vdash_{\!\!\!H} e : \tau' \end{array}$$

$$\frac{\Gamma \vdash_{\mathsf{H}} v}{\Gamma \vdash_{\mathsf{H}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) v : (\tau_d \Rightarrow \tau_c)}$$

```
331
                 \delta(op^2, v, v) = e
332
                  \delta(\text{sum}, i_0, i_1)
                                                          = i_0 + i_1
333
                  \delta(\text{quotient}, i_0, 0) = \text{BndryErr}
334
                  \delta(\text{quotient}, i_0, i_1) = \lfloor i_0/i_1 \rfloor
335
                       if i_1 \neq 0
336
                 \mathcal{D}_{\mathsf{H}}: \tau \times v \longrightarrow e
337
                  \mathcal{D}_{\mathsf{H}}(\tau_d \Rightarrow \tau_c, v)
                                                                = mon(\tau_d \Rightarrow \tau_c) v
338
                       if v = \lambda x. e or v = mon \tau' v'
339
                  \mathcal{D}_{\mathsf{H}}(\tau_0 \times \tau_1, \langle v_0, v_1 \rangle) = \langle \mathsf{dyn} \ \tau_0 \ v_0, \mathsf{dyn} \ \tau_1 \ v_1 \rangle
340
                                                                =i
                  \mathcal{D}_{\mathsf{H}}(\mathsf{Int},i)
341
                  \mathcal{D}_{\mathsf{H}}(\mathsf{Nat},i)
342
                       if i \in \mathbb{N}
343
                  \mathcal{D}_{\mathsf{H}}(\tau,v)
                                                                = BndryErr
344
                       otherwise
345
346
                 S_{\mathsf{H}}: \tau \times v \longrightarrow e
347
                  S_{\rm H}(\tau_d \Rightarrow \tau_c, v)
                                                               = mon(\tau_d \Rightarrow \tau_c) v
348
                  S_{\mathsf{H}}(\tau_0 \times \tau_1, \langle v_0, v_1 \rangle) = \langle \operatorname{stat} \tau_0 \ v_0, \operatorname{stat} \tau_1 \ v_1 \rangle
349
                 \mathcal{S}_{\mathsf{H}}(\mathsf{Int},v)
                                                              = v
350
                                                              =v
                 S_{\mathsf{H}}(\mathsf{Nat},v)
351
                e \triangleright_{\mathsf{H-S}} e
352
                  dyn \tau v
                                                                    \triangleright_{\mathsf{H-S}} \mathcal{D}_{\mathsf{H}}(\tau,v)
353
                  (\text{mon}(\tau_d \Rightarrow \tau_c) v_f) v \triangleright_{H-S} \text{dyn } \tau_c (v_f (\text{stat } \tau_d v))
354
                  (\lambda(x:\tau).e)v
                                                                    \triangleright_{\mathsf{H-S}} e[x \leftarrow v]
355
                  op^1 v
                                                                   \triangleright_{\mathsf{H-S}} \delta(op^1, v)
356
                  op^2 v_0 v_1
                                                                   \triangleright_{\mathsf{H-S}} \delta(op^2, v_0, v_1)
357
                e \rhd_{\mathsf{H-D}} e
358
                  stat \tau v
                                                                \rhd_{\mathsf{H-D}} \mathcal{S}_{\mathsf{H}}(\tau,v)
359
                                                                ⊳<sub>H-D</sub> TagErr
360
                  v_0 v_1
                       if v_0 \in \mathbb{Z} or v_0 = \langle v, v' \rangle
361
362
                  (\text{mon } \tau_d \Rightarrow \tau_c \ v_f) \ v \ \triangleright_{\mathsf{H-D}} \ \text{stat } \tau_c \ (v_f \ (\text{dyn } \tau_d \ v))
                  (\lambda x. e) v
                                                               \triangleright_{\mathsf{H-D}} e[x \leftarrow v]
363
                  op^1 v
364
                                                                \triangleright_{H\text{-}D} TagErr
                       if \delta(op^1, v) is undefined
365
                                                               \triangleright_{\mathsf{H-D}} \delta(\mathit{op}^1, v)
366
                  op^2 v_0 v_1
367
                                                               ⊳<sub>H-D</sub> TagErr
368
                       if \delta(op^2, v_0, v_1) is undefined
                  op^2 v_0 v_1
                                                               \triangleright_{\mathsf{H-D}} \delta(op^2, v_0, v_1)
369
370
                e \rightarrow_{\mathsf{H-S}} e
371
                  E^{\bullet}[e]
                                                       \rightarrow_{\mathsf{H-S}} E^{\bullet}[e']
372
                       if e \rhd_{\mathsf{H-S}} e'
373
                  E[\operatorname{stat} \tau E^{\bullet}[e]] \rightarrow_{\mathsf{H-S}} E[\operatorname{stat} \tau E^{\bullet}[e']]
374
                       if e \triangleright_{\mathsf{H-S}} e'
375
                  E[\operatorname{dyn} \tau E^{\bullet}[e]] \rightarrow_{\mathsf{H-S}} E[\operatorname{dyn} \tau E^{\bullet}[e']]
376
                       if e \triangleright_{\mathsf{H-D}} e'
377
                  E[Err]
                                                      \rightarrow_{\text{H-S}} Err
378
```

```
\begin{array}{c} e \longrightarrow_{\text{H-D}} e \\ E^{\bullet}[e] & \longrightarrow_{\text{H-D}} E^{\bullet}[e'] \\ \text{if } e \rhd_{\text{H-D}} e' \\ E[\text{stat } \tau \ E^{\bullet}[e]] & \longrightarrow_{\text{H-D}} E[\text{stat } \tau \ E^{\bullet}[e']] \\ \text{if } e \rhd_{\text{H-S}} e' \\ E[\text{dyn } \tau \ E^{\bullet}[e]] & \longrightarrow_{\text{H-D}} E[\text{dyn } \tau \ E^{\bullet}[e']] \\ \text{if } e \rhd_{\text{H-D}} e' \\ E[\text{Err}] & \longrightarrow_{\text{H-D}} \text{Err} \\ \hline e \longrightarrow_{\text{H-S}}^* e \\ \text{reflexive, transitive closure of } \longrightarrow_{\text{H-S}} \\ \hline e \longrightarrow_{\text{H-D}}^* e \\ \end{array}
```

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441
          A.2.2 Higher-Order Theorems
                                                                                                                         By static H-soundness and the fact that S_H and \triangleright_{H-S} are
                                                                                                                         subsets of \mathcal{D}'_{H} and \triangleright_{H-D}', respectively.
442
          Theorem 2.0: static H-soundness
443
            If \vdash e : \tau then \vdash_{\vdash} e : \tau and one of the following holds:
444
             • e \to_{\mathsf{H-S}}^* v and \vdash_{\mathsf{H}} v : \tau
445
            • e \to_{\mathsf{H-S}}^{::} E[\mathsf{dyn} \ \tau' \ E^{\bullet}[e']] \ \mathsf{and} \ e' \rhd_{\mathsf{H-D}} \mathsf{TagErr}
446
             • e \rightarrow_{H-S}^* BndryErr
447
             • e diverges
448
          Proof:
449
             1. \vdash_{\mathsf{H}} e : \tau
450
                 by static subset
451
             2. QED by static progress and static preservation.
452
453
          Theorem 2.1: dynamic H-soundness
454
             If \vdash e then \vdash_{\mathsf{H}} e and one of the following holds:
455
             • e \to_{\mathsf{H-D}}^* v and \vdash_{\mathsf{H}} v
456
             • e \rightarrow^*_{H-D} E[e'] and e' \rhd_{H-D} TagErr
457
            • e \rightarrow_{\text{H-D}}^* \text{BndryErr}
458
             • e diverges
459
          Proof:
             1. ⊦<sub>H</sub> e
461
                 by dynamic subset
462
             2. QED by dynamic progress and dynamic preservation.
463
464
          Corollary 2.2: H static soundness
465
            If \vdash e : \tau and e is boundary-free, then one of the following
466
             holds:
467
             • e \to_{\mathsf{H}}^* v and \vdash_{\mathsf{H}} v : \tau
468
             • e \rightarrow_{\mathsf{H}}^* \mathsf{BndryErr}
469
             • e diverges
470
          Proof:
471
              Consequence of the proof for static H-soundness
472
473
          Corollary 2.3: H compilation
474
             If \vdash e : \tau
475
             and \mathcal{D}_H' extends \mathcal{D}_H with a rule to monitor a typed function:
476
               \mathcal{D}'_{\mathsf{H}}(\tau_d \Longrightarrow \tau_c, \lambda(x : \tau). \ e) = \mathsf{mon} \ (\tau_d \Longrightarrow \tau_c) \ (\lambda(x : \tau). \ e)
477
             and \triangleright_{H-D}' extends \triangleright_{H-D} with a rule to apply a typed func-
478
479
              (\lambda(x:\tau). e) v \rhd_{\mathsf{H-D}}' e[x \leftarrow v]
480
             and e \rightarrow_{H-D}' e is defined as:
481
                              \rightarrow_{\mathsf{H-D}}' E[e']
482
                  if e \rhd_{\mathsf{H-D}'} e'
               E[\text{stat } \tau \ v] \rightarrow_{\mathsf{H-D}'} E[\mathcal{D}'_{\mathsf{H}}(\tau, v)]
484
               E[\mathsf{dyn} \ \tau \ v] \ \to_{\mathsf{H-D}}' \ E[\mathcal{D}_{\mathsf{H}}^{"}(\tau,v)]
485
                                \rightarrow_{\mathsf{H-D}}' Err
486
             and \rightarrow_{H-D}^* is the reflexive transitive closure of \rightarrow_{H-D}
487
488
             then one of the following holds:
489
             • e \rightarrow_{\mathsf{H-D}}^* v and \vdash_{\mathsf{H}} v : \tau
490
             • e \rightarrow_{\text{H-D}}^{*}' \text{TagErr}
• e \rightarrow_{\text{H-D}}^{*}' \text{BndryErr}
491
492
             • e diverges
493
          Proof:
494
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551
            A.2.3 Higher-Order Lemmas
                                                                                                                                                 Proof:
552
            Lemma 2.4 : \mathcal{D}_H soundness
553
            If \vdash_{\mathsf{H}} v then \vdash_{\mathsf{H}} \mathcal{D}_{\mathsf{H}}(\tau, v) : \tau
                                                                                                                                                     CASE
554
            Proof:
                                                                                                                                                                     \Gamma \vdash x : \tau
555
                 CASE \mathcal{D}_{\mathsf{H}}(\tau_d \Rightarrow \tau_c, v) = \mathsf{mon}(\tau_d \Rightarrow \tau_c) v:
                                                                                                                                                        1. \Gamma \vdash_{\vdash} x : \tau
556
                    1. \vdash_{\mathsf{H}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) v : \tau_d \Rightarrow \tau_c
557
                         by \vdash_{\mathsf{H}} v
                                                                                                                                                        2. QED
558
                   2. QED
559
                                                                                                                                                     CASE
                 CASE v = \langle v_0, v_1 \rangle
560
                              \wedge \mathcal{D}_{\mathsf{H}}(\tau_0 \times \tau_1, v) = \langle \mathsf{dyn} \ \tau_0 \ v_0, \mathsf{dyn} \ \tau_1 \ v_1 \rangle :
561
                   1. ⊨ v<sub>0</sub>
562
                         \wedge \vdash_{\mathsf{H}} v_1
563
                         by inversion
564
                    2. \vdash_{\mathsf{H}} \mathsf{dyn} \ \tau_0 \ \upsilon_0 : \tau_0
                                                                                                                                                        3. QED
565
                         \land \vdash_{\mathsf{H}} \mathsf{dyn} \ \tau_1 \ v_1 : \tau_1
                                                                                                                                                                         i \in \mathbb{N}
566
                                                                                                                                                     CASE
                         by (1)
567
                   3. QED(2)
568
                                                                                                                                                        1. QED
                 CASE v = i
569
                                                                                                                                                     CASE
                              \wedge \mathcal{D}_{\mathsf{H}}(\mathsf{Int},v) = v:
570
                   1. QED
571
                 case v \in \mathbb{N}
                                                                                                                                                        1. QED
572
                              \wedge \mathcal{D}_{\mathsf{H}}(\mathsf{Nat},v) = v:
                                                                                                                                                     CASE
573
                   1. QED
574
                CASE \mathcal{D}_{H}(\tau, v) = BndryErr:
575
                                                                                                                                                        1. \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0
                    1. QED
576
577
            Lemma 2.5 : S_H soundness
578
            If \vdash_{\mathsf{H}} v : \tau then \vdash_{\mathsf{H}} \mathcal{S}_{\mathsf{H}}(\tau, v)
579
                                                                                                                                                        3. QED
            Proof:
580
                 CASE \vdash_{\mathsf{H}} v : \tau_d \Rightarrow \tau_c
                                                                                                                                                     CASE
581
                              \wedge S_{H}(\tau_{d} \Rightarrow \tau_{c}, v) = \text{mon}(\tau_{d} \Rightarrow \tau_{c}) v:
583
                 CASE \vdash_{\mathsf{H}} v : \tau_0 \times \tau_1
584
                              \wedge \mathcal{S}_{\mathsf{H}}(\tau_0 \times \tau_1, v) = \langle \mathsf{stat} \ \tau_0 \ v_0, \mathsf{stat} \ \tau_1 \ v_1 \rangle :
585
                   1. v = \langle v_0, v_1 \rangle
586
                         by canonical forms
587
                                                                                                                                                        3. QED
                    2. \vdash_{\mathsf{H}} v_0 : \tau_0
588
                                                                                                                                                     CASE
                         \wedge \vdash_{\mathsf{H}} v_1 : \tau_1
                         by inversion (1)
590
                   3. \vdash_{\mathsf{H}} stat \tau_0 \ v_0 : \tau_0
                                                                                                                                                        1. \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0
591
                         by the induction hypothesis (2)
592
                    4. \vdash_{\mathsf{H}} stat \tau_1 \ v_1 : \tau_1
                         by the induction hypothesis (2)
                                                                                                                                                        3. QED
594
                   5. QED
595
                                                                                                                                                     CASE
                 CASE \vdash_{\mathsf{H}} v : \mathsf{Int}
596
                              \wedge S_{\mathsf{H}}(\mathsf{Int},v) = v:
                   1. QED
                                                                                                                                                        1. \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0
598
                 CASE \vdash_{\mathsf{H}} v : \mathsf{Nat}
                              \wedge S_{\mathsf{H}}(\mathsf{Nat},v) = v:
600
                    1. QED
601
                                                                                                                                                        3. QED
602
            Lemma 2.6 : H static subset
603
            If \Gamma \vdash e : \tau then \Gamma \vdash_{\mathsf{H}} e : \tau.
604
605
                                                                                                                                          6
```

```
By structural induction on the derivation of \Gamma \vdash e : \tau.
                 (x:\tau)\in\Gamma
         by (x:\tau) \in \Gamma
                         (x:\tau_d), \Gamma \vdash e:\tau_c
                \overline{\Gamma} \vdash \lambda(x : \tau_d). \ e : \tau_d \Rightarrow \tau_c
   1. (x:\tau_d), \Gamma \vdash_{\vdash} e:\tau_c
         by the induction hypothesis
   2. \Gamma \vdash_{\mathsf{H}} \lambda(x : \tau_d). e : \tau_d \Rightarrow \tau_c
                \Gamma \vdash i : Nat
                \Gamma \vdash i : \mathsf{Int}
                \Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1
                    \Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1
         \wedge \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1
         by the induction hypothesis
   2. \Gamma \vdash_{\mathsf{H}} \langle e_0, e_1 \rangle : \tau_0 \times \tau_1
                \Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash e_1 : \tau_d
                                 \Gamma \vdash e_0 \ e_1 : \tau_c
   1. \Gamma \vdash_{\mathsf{H}} e_0 : \tau_d \Longrightarrow \tau_c
         \wedge \Gamma \vdash_{\mathsf{H}} e_1 : \tau_d
         by the induction hypothesis
   2. \Gamma \vdash_{\mathsf{H}} e_0 \ e_1 : \tau_c
                \Gamma \vdash e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau
                              \Gamma \vdash op^1 e_0 : \tau
         by the induction hypothesis
   2. \Gamma \vdash_{\mathsf{H}} op^1 e_0 : \tau
                \Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau
                                            \Gamma \vdash op^2 e_0 e_1 : \tau
         \wedge \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1
         by the induction hypothesis
   2. \Gamma \vdash_{\mathsf{H}} op^2 e_0 e_1 : \tau
```

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```
\Gamma \vdash e : \tau' \quad \tau' \lessdot \tau' \mid
661
662
663
                  1. \Gamma \vdash_{\mathsf{H}} e : \tau'
664
                       by the induction hypothesis
                  2. \Gamma \vdash_{\mathsf{H}} e : \tau
666
                  3. QED
667
               CASE
668
                             \Gamma \vdash \mathsf{Err} : \tau
669
670
                  1. QED
671
               CASE
672
                             \Gamma \vdash \mathsf{dyn} \ \tau \ e : \tau
673
                  1. Γ <del>| e</del>
674
                       by dynamic subset
675
                  2. \Gamma \vdash_{\mathsf{H}} \mathsf{dyn} \ \tau \ e : \tau
676
                      by (1)
677
                  3. QED
678
679
           Lemma 2.7: H dynamic subset
680
            If \Gamma \vdash e then \Gamma \vdash_{\vdash} e.
681
682
           Proof:
               By structural induction on the derivation of \Gamma \vdash e.
683
                             x \in \Gamma
685
                             \Gamma \vdash x
                 1. \Gamma \vdash_{\mathsf{H}} x
687
                      by x \in \Gamma
688
                  2. QED
689
690
691
692
693
                      by the induction hypothesis
694
                  2. \Gamma \vdash_{\mathsf{H}} \lambda x. e
695
                       by (1)
696
                  3. QED
697
               CASE
698
                  1. QED
700
                             \Gamma \vdash e_0 \quad \Gamma \vdash e_1
701
               CASE
702
                               \Gamma \vdash \langle e_0, e_1 \rangle
                 1. \Gamma \vdash_{\mathsf{H}} e_0
704
                       \wedge \Gamma \vdash_{\mathsf{H}} e_1
705
                       by the induction hypothesis
706
                  2. \Gamma \vdash_{\mathsf{H}} \langle e_0, e_1 \rangle
707
                      by (1)
708
                  3. QED
709
                             \Gamma \vdash e_0 \quad \Gamma \vdash e_1
               CASE
710
                                  \Gamma \vdash e_0 \ e_1
711
712
713
```

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715

```
1. \Gamma \vdash_{\mathsf{H}} e_0
          \wedge \Gamma \vdash_{\mathsf{H}} e_1
          by the induction hypothesis
      2. \Gamma \vdash_{\mathsf{H}} e_0 \ e_1
          by (1)
     3. QED
      1. Γ ⊦<sub>H</sub> e
           by the induction hypothesis
      2. \Gamma \vdash_{\mathsf{H}} op^1 e
          by (1)
      3. QED
                \Gamma \vdash e_0 \quad \Gamma \vdash e_1
   CASE
                 \Gamma \vdash op^2 e_0 e_1
      1. \Gamma \vdash_{\mathsf{H}} e_0
           \land \; \Gamma \vdash_{\mathsf{H}} e_1
           by the induction hypothesis
      2. \Gamma \vdash_{\mathsf{H}} op^2 e_0 e_1
          by (1)
      3. QED
   CASE
      1. QED
                   \Gamma \vdash e : \tau
   CASE
                \Gamma \vdash \mathsf{stat}\ \tau\ e
      1. \Gamma \vdash_{\mathsf{H}} e : \tau
          by static subset
      2. \Gamma \vdash<sub>H</sub> stat \tau e
          by (1)
      3. QED
Lemma 2.8: H static progress
If \vdash_{\mathsf{H}} e : \tau then one of the following holds:
  • e is a value
  • e \in Err
  • e \rightarrow_{H-S} e'
  • e \rightarrow_{H-S} BndryErr
  • e = E[\text{dyn } \tau' E^{\bullet}[e']] \text{ and } e' \triangleright_{H-D} \text{TagErr}
   By the boundary factoring lemma, there are seven possi-
   ble cases.
   CASE e is a value :
     1. QED
   CASE e = E^{\bullet}[v_0 \ v_1]:
      1. \vdash_{\mathsf{H}} v_0 \ v_1 : \tau'
          by static hole typing
      2. \vdash_{\mathsf{H}} v_0 : \tau_d \Rightarrow \tau_c
           \wedge \vdash_{\mathsf{H}} v_1 : \tau_d
          by inversion
```

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771
                 3. v_0 = \lambda(x : \tau_d'). e'
                                                                                                                                   IF e' \rightarrow_{H-D} BndryErr:
                                                                                                                                                                                                                                     826
772
                     \forall v_0 = \min(\tau_d' \Rightarrow \tau_c') v_f
                                                                                                                                     a. QED e \rightarrow_{H-S} E[dyn \tau' BndryErr]
                                                                                                                                                                                                                                     827
773
                     by canonical forms
                                                                                                                                   ELSE e' = E'[e''] and e'' \triangleright_{H-D} \mathsf{TagErr}:
                                                                                                                                                                                                                                     828
                4. IF v_0 = \lambda(x : \tau_d). e':
                                                                                                                                     a. E' \in E^{\bullet}
774
                        a. e \rightarrow_{H-S} E^{\bullet}[e'[x \leftarrow v_1]]
775
                                                                                                                                          by e' is boundary-free
                            by v_0 v_1 \triangleright_{\mathsf{H-S}} e'[x \leftarrow v_1]
                                                                                                                                     b. QED
                                                                                                                                                                                                                                     831
776
777
                                                                                                                            CASE e = E[\text{stat } \tau' e'] and e' is boundary-free:
                       b. oed
                                                                                                                                                                                                                                     832
778
                     ELSE v_0 = \text{mon}(\tau_d' \Rightarrow \tau_c') v_f:
                                                                                                                               1. e' is a value
                                                                                                                                                                                                                                     833
                        a. e \rightarrow_{H-S} E^{\bullet}[\operatorname{dyn} \tau'_{c}(v_{f}(\operatorname{stat} \tau'_{d}v_{1}))]
                                                                                                                                   \forall e' \in Err
                                                                                                                                   \vee e' \rightarrow_{H-S} e''
780
                            by v_0 \ v_1 \rhd_{\mathsf{H-S}} \mathsf{dyn} \ \tau'_c \ (v_f \ (\mathsf{stat} \ \tau'_d \ v_1))
                                                                                                                                                                                                                                     835
781
                        b. QED
                                                                                                                                   \lor e' \rightarrow_{H-S} BndryErr
                                                                                                                                   \vee e' = E''[\mathsf{dyn} \ \tau'' \ E^{\bullet''}[e'']] \ \mathsf{and} \ e'' \rhd_{\mathsf{H-D}} \mathsf{TagErr}
              CASE e = E^{\bullet}[op^1 v]:
782
                                                                                                                                                                                                                                     837
                1. \vdash_{\mathsf{H}} op^1 v : \tau'
783
                                                                                                                                   by static progress
                                                                                                                                                                                                                                     838
                     by static hole typing
                                                                                                                               2. IF e' is a value :
784
                                                                                                                                                                                                                                     839
785
                                                                                                                                     a. QED e \rightarrow_{H-S} E[S_H(\tau', e')]
                                                                                                                                                                                                                                     840
                 2. \vdash_{\mathsf{H}} v : \tau_0 \times \tau_1
786
                     by inversion
                                                                                                                                   IF e' \in Err:
                                                                                                                                                                                                                                     841
787
                3. v = \langle v_0, v_1 \rangle
                                                                                                                                     a. QED e \rightarrow_{H-S} e'
                                                                                                                                                                                                                                     842
                                                                                                                                   IF e' \rightarrow_{H-S} e'':
                     by canonical forms
788
                                                                                                                                                                                                                                     843
                4. IF op^1 = fst:
                                                                                                                                     a. QED e \rightarrow_{H-S} E[\text{stat } \tau' e'']
789
                                                                                                                                                                                                                                     844
790
                        a. \delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0
                                                                                                                                   IF e' \rightarrow_{H-S} BndryErr:
                                                                                                                                                                                                                                     845
                                                                                                                                     a. QED e \rightarrow_{H-S} E[\text{stat } \tau' \text{ BndryErr}]
791
                       b. e \rightarrow_{\mathsf{H-S}} E^{\bullet}[v_0]
                                                                                                                                                                                                                                     846
                            by op^1 v \triangleright_{\mathsf{H-S}} v_0
                                                                                                                                   ELSE e' = E''[\text{dyn } \tau'' E^{\bullet ''}[e'']] and e'' \rhd_{\text{H-D}} \text{TagErr}
                                                                                                                                                                                                                                     847
                        c. QED
                                                                                                                                                                                                                                     848
793
                     ELSE op^1 = \text{snd}:
                                                                                                                                     a. Contradiction by e' is boundary-free
794
                                                                                                                                                                                                                                     849
795
                        a. \delta(\operatorname{snd}, \langle v_0, v_1 \rangle) = v_1
                                                                                                                            CASE e = E[Err]:
                                                                                                                                                                                                                                     850
                        b. e \rightarrow_{\mathsf{H-S}} E^{\bullet}[v_1]
                                                                                                                               1. QED e \rightarrow_{H-S} Err
                                                                                                                                                                                                                                     851
797
                            by op^1 v \triangleright_{\mathsf{H-S}} v_1
                                                                                                                                                                                                                                     852
798
                        c. QED
                                                                                                                        Lemma 2.9 : H dynamic progress
              CASE e = E^{\bullet}[op^2 v_0 v_1]:
799
                                                                                                                                                                                                                                     854
                                                                                                                         If \vdash_{\mathsf{H}} e then one of the following holds:
800
                 1. \vdash_{\mathsf{H}} op^2 v_0 v_1 : \tau'
                                                                                                                                                                                                                                     855
                                                                                                                          • e is a value
801
                     by static hole typing
                                                                                                                          • e \in Err
                                                                                                                                                                                                                                     856
802
                2. \vdash_{\mathsf{H}} v_0 : \tau_0
                                                                                                                          • e \rightarrow_{H-D} e'
                                                                                                                          • e \rightarrow_{H-D} BndryErr
803
                                                                                                                                                                                                                                     858
                     \wedge \vdash_{\mathsf{H}} v_1 : \tau_1
                     \wedge \Delta(op^2, \tau_0, \tau_1) = \tau^{\prime\prime}
804
                                                                                                                                                                                                                                     859
                                                                                                                          • e = E[e'] and e' \triangleright_{H-D} TagErr
                                                                                                                        Proof:
                     by inversion
805
                                                                                                                            By the boundary factoring lemma, there are seven cases.
                3. \delta(op^2, v_0, v_1) = e'
806
                                                                                                                                                                                                                                     861
807
                     by \Delta type soundness (2)
                                                                                                                            CASE e is a value :
                                                                                                                                                                                                                                     862
                4. op^2 v_0 v_1 \rhd_{H-S} e'
                                                                                                                               1. OED
808
                                                                                                                                                                                                                                     863
                                                                                                                            CASE e = E^{\bullet}[v_0 \ v_1]:
                     by (3)
809
                                                                                                                               IF v_0 = \lambda x. e':
                5. QED by e \rightarrow_{H-S} E^{\bullet}[e']
                                                                                                                                                                                                                                     865
810
                                                                                                                                  1. e \rightarrow_{\mathsf{H-D}} E^{\bullet}[e'[x \leftarrow v_1]]
              CASE e = E[\text{dyn } \tau' e'] and e' is boundary-free:
811
                                                                                                                                      by v_0 v_1 \triangleright_{\mathsf{H-D}} e'[x \leftarrow v_1]
812
                 1. e' is a value
                                                                                                                                                                                                                                     867
                     \forall e' \in Err
                                                                                                                                  2. QED
                     \vee e' \rightarrow_{\mathsf{H-D}} e''
                                                                                                                               IF v_0 = \text{mon}(\tau_d \Rightarrow \tau_c) v_f:
814
                                                                                                                                                                                                                                     869
                     \lor e' \rightarrow_{H-D} BndryErr
                                                                                                                                  1. e \rightarrow_{H-D} E^{\bullet}[\text{stat } \tau_c (v_f (\text{dyn } \tau_d v_1))]
                     \lor e' = E'[e''] and e'' \rhd_{H-D} TagErr
                                                                                                                                      by v_0 \ v_1 \rhd_{\mathsf{H-D}} stat \tau_c \ (v_f \ (\mathsf{dyn} \ \tau_d \ v_1))
816
                                                                                                                                                                                                                                     871
                     by dynamic progress
                                                                                                                                  2. QED
                 2. IF e' is a value :
                                                                                                                               ELSE v_0 = i
818
                                                                                                                                                                                                                                     873
                                                                                                                                          \vee v_0 = \langle v, v' \rangle:
819
                        a. QED e \rightarrow_{H-S} E[\mathcal{D}_{H}(\tau', e')]
                                                                                                                                                                                                                                     874
                     IF e' \in Err:
                                                                                                                                  1. e \rightarrow_{H-D} TagErr
820
                                                                                                                                                                                                                                     875
                                                                                                                                      by (v_0 \ v_1) \triangleright_{\mathsf{H-D}} \mathsf{TagErr}
                       a. QED e \rightarrow_{H-S} e'
821
                                                                                                                                                                                                                                     876
                                                                                                                                  2. QED
822
                     IF e' \rightarrow_{H-D} e'':
                                                                                                                                                                                                                                     877
                                                                                                                            CASE e = E^{\bullet}[op^1 v]:
823
                        a. QED e \rightarrow_{H-S} E[dyn \tau' e'']
                                                                                                                                                                                                                                     878
                                                                                                                               IF \delta(op^1, v) = e':
824
                                                                                                                                                                                                                                     879
825
                                                                                                                                                                                                                                     880
                                                                                                                  8
```

```
1. (op^1 v) >_{H-D} e'
881
                                                                                                                          Lemma 2.10: H static preservation
                                                                                                                                                                                                                                        936
882
                    2. QED
                                                                                                                          If \vdash_{\mathsf{H}} e : \tau and e \rightarrow_{\mathsf{H-S}} e' then \vdash_{\mathsf{H}} e' : \tau
                                                                                                                                                                                                                                        937
883
                  ELSE \delta(op^1, v) is undefined:
                                                                                                                          Proof:
                                                                                                                                                                                                                                        938
                     1. e \rightarrow_{H-D} TagErr
884
                                                                                                                              By the boundary factoring lemma there are seven cases.
                                                                                                                                                                                                                                        939
                         by (op^1 v) \triangleright_{H-D} TagErr
                                                                                                                              CASE e is a value :
                                                                                                                                                                                                                                        941
                    2. QED
886
                                                                                                                                1. Contradiction by e \rightarrow_{H-S} e'
              CASE e = E^{\bullet}[op^2 v_0 v_1]:
                                                                                                                                                                                                                                        942
887
                                                                                                                              CASE e = E^{\bullet}[v_0 \ v_1]:
                  IF \delta(op^2, v_0, v_1) = e'':
888
                                                                                                                                 IF v_0 = \lambda(x:\tau_x). e'
                                                                                                                                                                                                                                        943
                     1. op^2 v_0 v_1 \rhd_{H-D} e''
                                                                                                                                       \land e \rightarrow_{\mathsf{H-S}} E^{\bullet}[e'[x \leftarrow v_1]]:
890
                    2. QED
                                                                                                                                    1. \vdash_{\mathsf{H}} v_0 \ v_1 : \tau'
                                                                                                                                                                                                                                        945
                  ELSE \delta(op^2, v_0, v_1) is undefined :
                                                                                                                                        by static hole typing
892
                     1. e \rightarrow_{H-D} TagErr
                                                                                                                                                                                                                                        947
                                                                                                                                    2. \vdash_{\mathsf{H}} v_0 : \tau_d \Rightarrow \tau_c
                         by op^2 v_0 v_1 \triangleright_{H-D} TagErr
893
                                                                                                                                                                                                                                        948
                                                                                                                                        \wedge \vdash_{\mathsf{H}} v_1 : \tau_d
894
                                                                                                                                        \wedge \tau_c \leqslant : \tau'
                                                                                                                                                                                                                                        949
895
              CASE e = E[dyn \tau' e'] and e' is boundary-free :
                                                                                                                                                                                                                                        950
                                                                                                                                        by inversion
896
                 1. e' is a value
                                                                                                                                                                                                                                        951
                                                                                                                                    3. \tau_d \leqslant : \tau_x
                     \forall e' \in Err
897
                                                                                                                                        by canonical forms (2)
                                                                                                                                                                                                                                        952
                      \vee e' \rightarrow_{H-D} e''
898
                                                                                                                                    4. (x:\tau_x) \vdash_{\mathsf{H}} e':\tau_c
                                                                                                                                                                                                                                        953
                      \vee e' \rightarrow_{H-D} BndryErr
899
                                                                                                                                        by inversion (2)
                                                                                                                                                                                                                                        954
                      \vee e' = E[e''] and e'' \triangleright_{H-D} TagErr
                                                                                                                                    5. \vdash_{\mathsf{H}} v_1 : \tau_x
                     by dynamic progress
901
                                                                                                                                        by (2, 3)
                                                                                                                                                                                                                                        956
                 2. IF e' is a value :
                                                                                                                                    6. \vdash_{\mathsf{H}} e'[x \leftarrow v_1] : \tau_c
                                                                                                                                                                                                                                        957
                        a. QED e \rightarrow_{\text{H-D}} E[\mathcal{D}_{\mathsf{H}}(\tau', e')]
                                                                                                                                                                                                                                        958
903
                                                                                                                                        by substitution (4, 5)
                     IF e' \in Err:
                                                                                                                                    7. \vdash_{\mathsf{H}} e'[x \leftarrow v_1] : \tau'
                        a. QED e \rightarrow_{H-D} e'
905
                                                                                                                                        by (2, 6)
                                                                                                                                                                                                                                        960
                     IF e' \rightarrow_{H-D} e'':
                                                                                                                                    8. QED by hole substitution (7)
                        a. QED e \rightarrow_{H-S} E[dyn \tau' e'']
                                                                                                                                 ELSE v_0 = \text{mon}(\tau_d \Rightarrow \tau_c) v_f
907
                                                                                                                                                                                                                                        962
                     IF e' \rightarrow_{H-D} BndryErr:
                                                                                                                                            \land e \rightarrow_{\mathsf{H-S}} E^{\bullet}[\mathsf{dyn} \ \tau_c \ (v_f \ (\mathsf{stat} \ \tau_d \ v_1))] :
                                                                                                                                                                                                                                        963
                        a. QED e \rightarrow_{H-D} E[dyn \tau' BndryErr]
909
                                                                                                                                                                                                                                        964
                                                                                                                                    1. \vdash_{\mathsf{H}} v_0 \ v_1 : \tau'
                     ELSE e' = E[e''] and e'' \triangleright_{H-D} TagErr:
910
                                                                                                                                                                                                                                        965
                                                                                                                                        by static hole typing
                        a. E \in E^{\bullet}
911
                                                                                                                                    2. \vdash_{\mathsf{H}} v_0 : \tau'_d \Rightarrow \tau'_c
                                                                                                                                                                                                                                        966
912
                             by e' is boundary-free
                                                                                                                                                                                                                                        967
                                                                                                                                        \wedge \vdash_{\mathsf{H}} v_1 : \tau'_d
913
                                                                                                                                        \wedge \tau_c' \leqslant : \tau'
                                                                                                                                                                                                                                        968
914
              CASE e = E[\text{stat } \tau' \ e'] \text{ and } e' \text{ is boundary-free} :
                                                                                                                                        by inversion
                                                                                                                                                                                                                                        969
915
                 1. e' is a value
                                                                                                                                    3. \vdash_{\mathsf{H}} v_f
                                                                                                                                                                                                                                        970
                     \forall e' \in Err
916
                                                                                                                                        by inversion (2)
                                                                                                                                                                                                                                        971
917
                      \vee e' \rightarrow_{H-S} e''
                                                                                                                                    4. \tau_d \Rightarrow \tau_c \leqslant : \tau_d' \Rightarrow \tau_c'
                                                                                                                                                                                                                                        972
918
                      \lor e' \rightarrow_{H-S} BndryErr
                                                                                                                                                                                                                                        973
                                                                                                                                        by canonical forms (2)
                     \vee e' = E''[\mathsf{dyn} \ \tau'' \ E^{\bullet''}[e'']] \ \mathsf{and} \ e'' \rhd_{\mathsf{H-D}} \mathsf{TagErr}
                                                                                                                                    5. \tau_d' \leqslant : \tau_d
919
                     by static progress
                                                                                                                                                                                                                                        975
920
                                                                                                                                        \wedge \tau_c \leqslant : \tau'_c
                 2. IF e' is a value :
921
                                                                                                                                        by (4)
922
                        a. QED e \rightarrow_{H-S} E[S_H(\tau', e')]
                                                                                                                                                                                                                                        977
                                                                                                                                    6. \vdash_{\mathsf{H}} v_1 : \tau_d
                     IF e' \in Err:
                                                                                                                                        by (2, 5)
                        a. QED e \rightarrow_{H-S} e'
924
                                                                                                                                                                                                                                        979
                                                                                                                                    7. \vdash_{\mathsf{H}} stat \tau_d \ v_1
                     IF e' \rightarrow_{H-S} e'':
                                                                                                                                                                                                                                        980
                                                                                                                                        by (6)
                        a. QED e \rightarrow_{H-S} E[\text{stat } \tau' e'']
                                                                                                                                    8. \vdash_{\mathsf{H}} v_f \text{ (stat } \tau_d \ v_1)
926
                                                                                                                                                                                                                                        981
                     IF e' \rightarrow_{H-S} BndryErr:
                                                                                                                                                                                                                                        982
                                                                                                                                        by (3, 7)
                        a. QED e \rightarrow_{H-S} E[\text{stat } \tau' \text{ BndryErr}]
928
                                                                                                                                    9. \vdash_{\mathsf{H}} \mathsf{dyn} \ \tau_c \ v_f \ (\mathsf{stat} \ \tau_d \ v_1) : \tau_c
                                                                                                                                                                                                                                        983
                     ELSE e' = E''[\text{dyn } \tau'' E^{\bullet ''}[e'']] and e'' \rhd_{\text{H-D}} \text{TagErr}
                                                                                                                                        by (8)
                                                                                                                                                                                                                                        984
930
                                                                                                                                                                                                                                        985
                                                                                                                                  10. \vdash_{\mathsf{H}} \mathsf{dyn} \ \tau_c \ v_f \ (\mathsf{stat} \ \tau_d \ v_1) : \tau'
931
                        a. Contradiction by e' is boundary-free
                                                                                                                                                                                                                                        986
                                                                                                                                        by (2, 5, 9)
932
              CASE e = E[Err]:
                                                                                                                                  11. QED by hole substitution (10)
                                                                                                                                                                                                                                        987
933
                 1. QED e \rightarrow_{H-D} Err
                                                                                                                              CASE e = E^{\bullet}[op^1 v]:
                                                                                                                                                                                                                                        988
           934
                                                                                                                                                                                                                                        989
935
                                                                                                                                                                                                                                        990
                                                                                                                    9
```

```
991
                  IF v = \langle v_0, v_1 \rangle
                                                                                                                                      2. \vdash_{\mathsf{H}} \mathsf{dyn} \; \tau' \; e' : \tau'
                                                                                                                                                                                                                                           1046
992
                        \wedge op^1 = fst
                                                                                                                                          by boundary hole typing
                                                                                                                                                                                                                                           1047
                                                                                                                                      3. ⊢<sub>H</sub> e′
993
                        \wedge e \rightarrow_{\mathsf{H-S}} E^{\bullet}[v_0]:
                                                                                                                                                                                                                                           1048
994
                     1. \vdash_{\mathsf{H}} \mathsf{fst} \langle v_0, v_1 \rangle : \tau'
                                                                                                                                          by inversion (2)
                                                                                                                                                                                                                                           1049
                         by static hole typing
                                                                                                                                      4. ⊢<sub>H</sub> e''
                                                                                                                                                                                                                                           1050
                                                                                                                                                                                                                                           1051
                     2. \vdash_{\mathsf{H}} \langle v_0, v_1 \rangle : \tau_0 \times \tau_1
                                                                                                                                          by dynamic preservation (3)
996
                                                                                                                                      5. \vdash_{\mathsf{H}} \mathsf{dyn} \; \tau' \; e'' : \tau'
997
                         \wedge \tau_0 \leqslant : \tau'
                                                                                                                                                                                                                                           1052
998
                         by inversion (1)
                                                                                                                                                                                                                                           1053
                                                                                                                                          by (4)
                                                                                                                                      6. QED by hole substitution (5)
                                                                                                                                                                                                                                           1054
                    3. \vdash_{\mathsf{H}} v_0 : \tau_0
                                                                                                                                CASE e = E[\text{stat } \tau' \ e'] \text{ and } e' \text{ is boundary-free} :
1000
                         by inversion (2)
                                                                                                                                                                                                                                           1055
1001
                     4. \vdash_{\mathsf{H}} v_0 : \tau'
                                                                                                                                   IF e' is a value :
                                                                                                                                                                                                                                           1056
                                                                                                                                      1. e \rightarrow_{H-S} E[S_H(\tau', e')]
1002
                         by (2, 3)
                                                                                                                                                                                                                                           1057
1003
                    5. QED by hole substitution (4)
                                                                                                                                      2. \vdash_{\mathsf{H}} stat \tau' e'
                                                                                                                                                                                                                                           1058
1004
                  ELSE v = \langle v_0, v_1 \rangle
                                                                                                                                          by boundary hole typing
                                                                                                                                                                                                                                           1059
1005
                             \wedge op^1 = \text{snd}
                                                                                                                                      3. \vdash_{\mathsf{H}} e' : \tau'
                                                                                                                                                                                                                                           1060
1006
                             \wedge e \rightarrow_{\mathsf{H-S}} E^{\bullet}[v_1]:
                                                                                                                                          by inversion (2)
                                                                                                                                                                                                                                           1061
1007
                     1. \vdash_{\square} snd \langle v_0, v_1 \rangle : \tau'
                                                                                                                                      4. \vdash_{\mathsf{H}} \mathcal{S}_{\mathsf{H}}(\tau',e')
                                                                                                                                                                                                                                           1062
1008
                         by static hole typing
                                                                                                                                          by S_H soundness (3)
                                                                                                                                                                                                                                           1063
                                                                                                                                      5. QED by hole substitution (4)
1009
                    2. \vdash_{\mathsf{H}} \langle v_0, v_1 \rangle : \tau_0 \times \tau_1
                                                                                                                                                                                                                                           1064
1010
                         \wedge \tau_1 \leqslant : \tau'
                                                                                                                                   ELSE e' \rightarrow_{H-S} e'':
                                                                                                                                                                                                                                           1065
1011
                         by inversion (1)
                                                                                                                                      1. e \rightarrow_{H-S} E[\text{stat } \tau' e'']
                                                                                                                                                                                                                                           1066
                                                                                                                                      2. \vdash_{\mathsf{H}} stat \tau' e'
1012
                    3. \vdash_{\mathsf{H}} v_1 : \tau_1
                                                                                                                                                                                                                                           1067
                                                                                                                                          by boundary hole typing
1013
                         by inversion (2)
                                                                                                                                                                                                                                           1068
                                                                                                                                      3. \vdash_{\mathsf{H}} e' : \tau'
1014
                    4. \vdash_{\mathsf{H}} v_1 : \tau'
1015
                         by (2, 3)
                                                                                                                                          by inversion (2)
                                                                                                                                                                                                                                           1070
                                                                                                                                      4. \vdash_{\mathsf{H}} e'' : \tau'
1016
                    5. QED by hole substitution (4)
                                                                                                                                                                                                                                           1071
1017
              CASE e = E^{\bullet}[op^2 v_0 v_1]:
                                                                                                                                          by static preservation (3)
                                                                                                                                                                                                                                           1072
                 1. e \rightarrow_{H-S} E^{\bullet}[\delta(op^2, v_0, v_1)]
1018
                                                                                                                                      5. \vdash<sub>H</sub> stat \tau' e''
                                                                                                                                                                                                                                           1073
                     by e \rightarrow_{H-S} e'
                                                                                                                                          by (4)
1019
                                                                                                                                                                                                                                           1074
                 2. \vdash_{\mathsf{H}} op^2 v_0 v_1 : \tau'
1020
                                                                                                                                      6. QED by hole substitution (5)
                                                                                                                                                                                                                                           1075
1021
                     by static hole typing
                                                                                                                                CASE e = E[Err]:
                                                                                                                                                                                                                                           1076
1022
                                                                                                                                  1. e \rightarrow_{H-S} Err
                                                                                                                                                                                                                                           1077
                 3. \vdash_{\mathsf{H}} v_0 : \tau_0
1023
                                                                                                                                  2. QED by \vdash_{\mathsf{H}} \mathsf{Err} : \tau
                                                                                                                                                                                                                                           1078
                      \wedge \vdash_{\mathsf{H}} v_1 : \tau_1
                      \wedge \Delta(op^2, \tau_0, \tau_1) = \tau^{\prime\prime}
                                                                                                                           1024
                                                                                                                                                                                                                                           1079
                      ∧ τ'' ≤: τ'
1025
                                                                                                                                                                                                                                           1080
                                                                                                                           Lemma 2.11: H dynamic preservation
1026
                     by inversion (1)
                                                                                                                                                                                                                                           1081
                                                                                                                            If \vdash_{\mathsf{H}} e and e \rightarrow_{\mathsf{H}\text{-}\mathsf{D}} e' then \vdash_{\mathsf{H}} e'
1027
                 4. \vdash_{\mathsf{H}} \delta(op^2, v_0, v_1) : \tau''
                                                                                                                                                                                                                                           1082
                                                                                                                            Proof:
1028
                     by \Delta type soundness (2)
                                                                                                                                                                                                                                           1083
                                                                                                                               By the boundary factoring lemma, there are seven cases.
                 5. \vdash_{\mathsf{H}} \delta(op^2, v_0, v_1) : \tau'
1029
                                                                                                                                                                                                                                           1084
                                                                                                                               CASE e is a value :
                     by (2, 3)
                                                                                                                                                                                                                                           1085
1030
                                                                                                                                  1. Contradiction by e \rightarrow_{H-D} e'
                 6. QED by hole substitution (4)
1031
                                                                                                                                                                                                                                           1086
                                                                                                                                CASE e = E^{\bullet}[v_0 \ v_1]:
1032
              CASE e = E[dyn \tau' e'] and e' is boundary-free:
                                                                                                                                                                                                                                           1087
                                                                                                                                   IF v_0 = \lambda x. e'
                  IF e' is a value:
1033
                                                                                                                                                                                                                                           1088
                                                                                                                                         \wedge e \rightarrow_{\mathsf{H-D}} E^{\bullet}[e'[x \leftarrow v_1]]:
1034
                    1. e \rightarrow_{H-S} E[\mathcal{D}_{H}(\tau', e')]
                                                                                                                                                                                                                                           1089
                                                                                                                                      1. \vdash_{\mathsf{H}} v_0 \ v_1
                    2. \vdash<sub>H</sub> dyn \tau' e' : \tau'
1035
                                                                                                                                                                                                                                           1090
                                                                                                                                          by dynamic hole typing
1036
                         by boundary hole typing
                                                                                                                                                                                                                                           1091
                                                                                                                                      2. ⊢<sub>H</sub> v<sub>0</sub>
                    3. ⊢<sub>H</sub> e′
                                                                                                                                                                                                                                           1092
                                                                                                                                          \wedge \vdash_{\mathsf{H}} v_1
1038
                         by inversion (2)
                                                                                                                                                                                                                                           1093
                                                                                                                                          by inversion (1)
1039
                     4. \vdash_{\mathsf{H}} \mathcal{D}_{\mathsf{H}}(\tau', e') : \tau'
                                                                                                                                                                                                                                           1094
                                                                                                                                     3. x \vdash_{\mathsf{H}} e'
1040
                         by \mathcal{D}_{H} soundness (3)
                                                                                                                                                                                                                                           1095
                                                                                                                                          by inversion (2)
1041
                    5. QED by hole substitution (4)
                                                                                                                                                                                                                                           1096
                                                                                                                                      4. \vdash_{\mathsf{H}} e'[x \leftarrow v_1]
1042
                  ELSE e' \rightarrow_{\mathsf{H-D}} e'':
                                                                                                                                                                                                                                           1097
                                                                                                                                          by substitution (2, 3)
1043
                     1. e \rightarrow_{H-S} E[dyn \tau' e'']
                                                                                                                                                                                                                                           1098
                                                                                                                                      5. QED hole substitution (4)
                                                                                                                                                                                                                                           1099
1044
1045
                                                                                                                                                                                                                                           1100
                                                                                                                     10
```

1822	1101	ELSE $v_0 = \text{mon}(\tau_d \Rightarrow \tau_c) v_f$	4. $\vdash_{H} \mathcal{D}_{H}( au',e'): au'$	1156
1996   Dy dynamic hole typing     ESSE $e' \rightarrow +_{12} D e''$ ;   1996   1.6	1102	$\land e \rightarrow_{H-D} E^{\bullet}[stat\ \tau_c\ (v_f\ (dyn\ \tau_d\ v_1))]:$	by $\mathcal{D}_{H}$ soundness (3)	1157
100 2. $\frac{1}{1}$ , $\frac{1}{1}$ 0. $\frac{1}{1}$ 1. $\frac{1}{1}$ 2. $\frac{1}{1}$ , $\frac{1}{1}$ or $\frac{1}{1}$ or $\frac{1}{1}$ 1. $\frac{1}{1}$ 3. $\frac{1}{1}$ , $\frac{1}{1}$ or $\frac{1}{1}$ 2. $\frac{1}{1}$ dyn $\frac{1}{1}$ or $\frac{1}{1}$ 1. $\frac{1}{1}$ 3. $\frac{1}{1}$ or $\frac{1}{1}$ 2. $\frac{1}{1}$ dyn $\frac{1}{1}$ or $\frac{1}{1}$ 1.	1103	1. $\vdash_{H} v_0 \ v_1$	5. QED by hole substitution (4)	1158
168	1104	by dynamic hole typing		1159
100   by inversion (1)   by boundary hole typing   102     100   by inversion (2)   $A \cdot r_i < r' r'$   104     101   $A \cdot r_i + $	1105	$2. \vdash_{H} v_0$	- , -	1160
100	1106	$\wedge \vdash_{H} v_1$	· · ·	1161
1999   by inversion (2)	1107	by inversion (1)	by boundary hole typing	1162
ine         4, $\frac{1}{1}$ , $\frac{1}{1}$ , $a^{1}$ , $a^{2}$ , $a^{2}$ , $a^{2}$ 17         by (2)         4, $\frac{1}{1}$ , $a^{2}$ (1)         116         by (2)         117         15, $\frac{1}{1}$ , $\frac{1}{1}$ , $\frac{1}{1}$ (1)         15, $\frac{1}{1}$ , $\frac{1}{1}$ , $\frac{1}{1}$ (2)         16, $\frac{1}{1}$ , $\frac{1}$	1108	3. $\vdash_{H} v_f : \tau_d \Rightarrow \tau_c$		1163
1112   $by(2)$   $by(3)$   $by(4)$   $by(3)$   $by(4)$	1109	by inversion (2)	$\wedge \tau' \leqslant : \tau''$	1164
112   5. $h_1  b_1  (dyn  \Gamma_d  v_1) :  \Gamma_c$   by dynamic preservation (3)   1125     1136   6. $h_1  \text{stat}  \Gamma_c  v_1  (dyn  \Gamma_d  v_1)$   by (4)   1126     1146   6. $h_1  \text{stat}  \Gamma_c  v_1  (dyn  \Gamma_d  v_1)$   by (4)   1126     115   by (5)   5. $h_1  dyn  r_1  e'' :  r'$   1126     116   7. Opin by hole substitution   CASE $e = E[\text{stat}  r'  e']  \text{and}  e'  \text{is boundary-free} :   1171     117   CASE e = E'[op^2  v] :   1172     118   IF v = (v_0, v_1)   1. e \to (v$	1110	•••		1165
1119 by (3, 4) (100 by (5) (100 by (6) c) (100 by	1111	by (2)		1166
111   6   1,	1112	5. $\vdash_{H} v_f (dyn \ \tau_d \ v_1) : \tau_c$		1167
116   by (5)   6. QED by hole substitution (5)   1170     117   CASE $e = E^*[op^1 v]$ :   117     118   IF $v = (v_0, v_1)$   1172     119   $A \circ op^1 = \text{fst}$   1172     110   $A \circ op^1 = \text{fst}$   1174     110   $A \circ op^1 = \text{fst}$   1175     110   $A \circ op^1 = \text{fst}$   1176     110   $A \circ op^1 = \text{fst}$   1176     1112   $A \circ op^1 = \text{fst}$   1176     1112   $A \circ op^1 = \text{fst}$   1176     1112   $A \circ op^1 = \text{fst}$   1177     1113   $A \circ op^1 = \text{fst}$   1177     1114   $A \circ op^1 = \text{fst}$   1177     1115   $A \circ op^1 = \text{fst}$   1178     1116   $A \circ op^1 = \text{fst}$   1179     1117   $A \circ op^1 = \text{fst}$   1179     1118   $A \circ op^1 = \text{fst}$   1179     1119   $A \circ op^1 = \text{fst}$   1179     1110   $A \circ op^1 = \text{fst}$   1179     1110   $A \circ op^1 = \text{fst}$   1179     1111   $A \circ op^1 = \text{fst}$   1179     1112   $A \circ op^1 = \text{fst}$   1179     1113   $A \circ op^1 = \text{fst}$   1179     1114   $A \circ op^1 = \text{fst}$   1179     1115   $A \circ op^1 = \text{fst}$   1179     1116   $A \circ op^1 = \text{fst}$   1179     1117   $A \circ op^1 = \text{fst}$   1179     1119   $A \circ op^1 = \text{fst}$   1179     1110   $A \circ op^1 = \text{fst}$   1179     1111   $A \circ op^1 = \text{fst}$   1179     1112   $A \circ op^1 = \text{fst}$   1179     1113   $A \circ op^1 = \text{fst}$   1179     1114   $A \circ op^1 = \text{fst}$   1179     1115   $A \circ op^1 = \text{fst}$   1179     1116   $A \circ op^1 = \text{fst}$   1179     1117   $A \circ op^1 = \text{fst}$   1179     1118   $A \circ o$	1113		··	1168
116	1114			1169
117 CASE $e = E^{\dagger}[op^{\dagger} v]$ : 118 IF $v = \langle v_0, v_1 \rangle$ 1. $e \rightarrow_{H \cap P} E[S_H(\tau', e')]$ 1. 172 119	1115	• • •	• •	1170
118   IF $v = \langle v_0, v_1 \rangle$   1. $e \rightarrow_{HD} E[Sh(r', e')]$   1173   1174   1175   11	1116		<b>CASE</b> $e = E[\text{stat } \tau' \ e'] \text{ and } e' \text{ is boundary-free} :$	1171
1150   $\land op^1 = \text{fst}$   2. $\vdash_{i_1}$ stat $r' e'$   1174   1176	1117	CASE $e = E^{\bullet}[op^1 v]$ :	IF $e' \in v$ :	1172
1120	1118	,	1. $e \rightarrow_{\text{H-D}} E[S_{\text{H}}(\tau', e')]$	1173
112   1. $I_{H}$ op $I^{U}$ by dynamic hole typing   by inversion (2)   1177   1178   1179	1119	$\wedge op^1 = fst$	2. ⊢ <sub>H</sub> stat τ' e'	1174
1122   by dynamic hole typing   by inversion (2)   1177   1128   2. $\frac{1}{14}$ , $v$   4. $\frac{1}{14}$ , $S_{11}(r', e')$   1178   1178   1179   1178   1179	1120	$\wedge e \rightarrow_{H-D} E^{\bullet}[v_0]:$	by boundary hole typing	1175
1124 by inversion (1) by S <sub>H</sub> soundness (3) 1172 by S <sub>H</sub> soundness (3) 1173 by S <sub>H</sub> soundness (3) 1174 by inversion (2) 5. QED by hole substitution (5) 1180 by inversion (2) 2. $\frac{1}{1}$ stat $\frac{1}{1}$ conversion (2) 2. $\frac{1}{1}$ stat $\frac{1}{1}$ conversion (3) 1181 1181 1. $\frac{1}{1}$ conversion (2) 1182 by inversion (3) 1183 1. $\frac{1}{1}$ conversion (4) 1183 1. $\frac{1}{1}$ conversion (5) 1184 1185 1. $\frac{1}{1}$ conversion (7) 1185 1185 1. $\frac{1}{1}$ conversion (8) 1185 1. $\frac{1}{1}$ conversion (9) 1185 1186 1. $\frac{1}{1}$ conversion (1) 1187 1187 1188 1189 1189 1189 1189 1189	1121	1. $\vdash_{H} op^1 v$	3. $\vdash_{H} e' : \tau'$	1176
124   by inversion (1)   by $S_H$ soundness (3)   179   1125   3. $I_H v_0$   5. QED by hole substitution (5)   1130   1132   1132   2. QED by hole substitution   1. $e \rightarrow_{H-D} E[stat \ r' \ e'']$   1132   1132   1132   1132   1132   1132   1133   1. $I_{e} \rightarrow_{H-D} E[v_1]$   1132   1133   1. $I_{e} \rightarrow_{H-D} E[v_1]$   1133   1. $I_{e} \cap_{H-D} E[v_1]$   1134   1135   1. $I_{e} \cap_{H-D} E[v_1]$   1135   1136   1137   1138   1. $I_{e} \cap_{H-D} E[v_1]$   1137   1138   1. $I_{e} \cap_{H-D} E[v_1]$   1139   1139   1134   1139   113	1122	by dynamic hole typing	by inversion (2)	1177
115   3. $h_1 v_0$   5. QED by hole substitution (5)   1180     117   4. QED by hole substitution   1. $e \rightarrow_{H-D} E[\text{stat } r' e'']$   1181     118	1123	2. ⊢ <sub>H</sub> <i>v</i>	4. $\vdash_{H} \mathcal{S}_{H}(\tau',e')$	1178
126   by inversion (2)   ELSE $e' \rightarrow_{H-S} e''$ :   1181   127   4. QED by hole substitution   1. $e \rightarrow_{H-D} E[\text{stat } \tau' e']$   1182   1282   $e = e \lor (0, v_1)$   2. $h_1$ stat $t' e'$   1183   1193   $h_1 \lor v_1$   129   $h_2 \lor v_2$   $h_3 \lor v_4$   129   $h_4 \lor v_4$   129   129   129   129   $h_4 \lor v_4$   $h_4 \lor v_1$   $h_4 \lor v_1$   $h_4 \lor v_2$   $h_4 \lor v_4$	1124	by inversion (1)	by $S_H$ soundness (3)	1179
1127 4. QED by hole substitution 1. $e \rightarrow_{H-D} E[\operatorname{stat} \tau' e'']$ 1182 1182 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 2. $v_1$ stat $\tau' e''$ 1183 1183 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 2. $v_1$ stat $\tau' e''$ 1185 1181 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 2. $v_1$ stat $v' e''$ 1185 1181 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1185 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1185 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1185 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1185 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1185 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1185 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1185 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1187 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1187 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1187 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1189 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1191 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1191 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1192 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1193 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1194 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1195 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1196 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1197 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1197 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1199 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1199 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1199 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1199 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1199 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1199 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1199 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1199 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1199 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1199 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1199 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1199 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$ 1190 $e = \operatorname{LSE} v = \langle v_0, v_1 \rangle$	1125	3. $\vdash_{H} v_0$	5. QED by hole substitution (5)	1180
1128   ELSE $v = \langle v_0, v_1 \rangle$   2. $\frac{1}{k_1}$ stat $r'$ $e'$   1183   1184   1185   1185   $0 \wedge op^2 = \text{snd}$   1184   1185   $0 \wedge e \rightarrow h \rightarrow D^*[v_1]$   1187   $0 \wedge e \rightarrow h \rightarrow D^*[v_1]$   $0 \wedge e \rightarrow h \rightarrow D^*[v_1]$   1187   $0 \wedge e \rightarrow h \rightarrow D^*[v_1]$   $0 \wedge h \rightarrow h \rightarrow D^*[v_1]$   $0 \wedge h $	1126	by inversion (2)	ELSE $e' \rightarrow_{H-S} e''$ :	1181
1129	1127	4. QED by hole substitution	1. $e \rightarrow_{H-D} E[\text{stat } \tau' e'']$	1182
1130 $\wedge e \rightarrow_{H-D} E^*[v_1]$ :       3. $\vdash_H e'$ : $\tau'$ 1185         1131       1. $\vdash_H op^{\dagger} v$ by inversion (2)       1185         1132       by dynamic hole typing       4. $\vdash_H e''$ : $\tau'$ 1187         1133       2. $\vdash_H v$ by inversion (1)       1188         1134       by inversion (2)       5. $\vdash_H \text{ stat } \tau' e''$ 1189         1135       3. $\vdash_H v$ by 4)       1190         1136       by inversion (2)       6. QED by hole substitution (5)       1191         1137       4. QED by hole substitution       CASE $e = E[Err]$ :       1192         1138       CASE $e = E^*[op^2 v_0 v_1]$ :       1. $e \rightarrow_{H-D} Err$ 1193         1140       2. $\vdash_H op^* v_0 v_1$ 1. $e \rightarrow_{H-D} Err$ <td>1128</td> <td>ELSE <math>v = \langle v_0, v_1 \rangle</math></td> <td>2. ⊢<sub>H</sub> stat τ' e'</td> <td>1183</td>	1128	ELSE $v = \langle v_0, v_1 \rangle$	2. ⊢ <sub>H</sub> stat τ' e'	1183
1131   1. $\vdash_{H} op^{1} v$   by inversion (2)   1186   1182   by dynamic hole typing   4. $\vdash_{H} e''$ : $r'$   1187   1187   1188   1188   by inversion (1)   5. $\vdash_{H} stat r' e''$   1189   1186   by inversion (2)   6. QED by hole substitution (5)   1191   1195   1. $e \rightarrow_{H-D} E^{+} [\delta(op^{2}, v_{0}, v_{1})]  $   1. $e \rightarrow_{H-D} E^{+} [\delta(op^{2}, v_{0}, v_{1})] $	1129	$\wedge op^1 = snd$	by boundary hole typing	1184
1132 by dynamic hole typing 4. $^{1}$ $^{1}$ $^{1}$ by static preservation (3) 1188 1134 by inversion (1) 5. $^{1}$ $^{1}$ by (4) 1199 1136 by inversion (2) 6. QED by hole substitution (5) 1191 1137 4. QED by hole substitution (5) 1191 1138 CASE $e = E^{*}[op^{2} v_{0} v_{1}]$ : 1. $e \to_{H-D} E^{*}[\delta(op^{2}, v_{0}, v_{1})]$ 2. QED $^{1}$ $^{1}$ $^{1}$ by dynamic hole typing 1193 1. $e \to_{H-D} E^{*}[\delta(op^{2}, v_{0}, v_{1})]$ 2. QED $^{1}$	1130	$\wedge e \rightarrow_{H-D} E^{\bullet}[v_1]:$	3. $\vdash_{H} e' : \tau'$	1185
1133   2. $\frac{1}{H} v$   by static preservation (3)   1188   1134   by inversion (1)   5. $\frac{1}{H}$ stat $t'$ $e''$   1189   1135   3. $\frac{1}{H} v$ 1   by (4)   1190   1190   1191   11	1131	1. $\vdash_{H} op^1 v$	by inversion (2)	1186
1134       by inversion (1)       5. $I_H$ stat $\tau'e''$ 1189         1135       3. $I_H v_1$ by (4)       1190         1136       by inversion (2)       6. QED by hole substitution (5)       1191         1137       4. QED by hole substitution       CASE $e = E[Err]$ :       1192         1138       CASE $e = E^{\bullet}[op^2 v_0 v_1]$ :       1. $e \rightarrow_{H-D} Err$ 1193         1140       2. $I_H op^2 v_0 v_1$ 1194         1141       by dynamic hole typing       Lemma 2.12 : H static boundary factoring       1195         1142       3. $I_H v_0$ If $I_H e : \tau$ then one of the following holds:       1197         1143 $A I_H v_0$ If $I_H e : \tau$ then one of the following holds:       1197         1144       by inversion (1)       e e is a value       e e is a value         1145       4. $I_H \delta(op^2, v_0, v_1)$ e e = $E^{\bullet}[v_0 v_1]$ 1200         1146       by $\delta$ preservation (2)       e e = $E^{\bullet}[op^2 v_0 v_1]$ 1201         1147       5. QED by hole substitution (3)       e e = $E^{\bullet}[op^2 v_0 v_1]$ 1201         1148       CASE $e = E[dyn \tau' e']$ and $e'$ is boundary-free       e e = $E[stat \tau e']$ where $e'$ is boundary-free       e e = $E[stat v_0 v_1]$ 1202         1150       1. $e \rightarrow_$	1132	by dynamic hole typing	$4. \vdash_{H} e'': \tau'$	1187
1135   3. $\vdash_H v_1$   by (4)   1190   1190   1190   1191   119	1133	2. ⊢ <sub>H</sub> <i>v</i>	by static preservation (3)	1188
1135   3. $\vdash_H v_1$   by (4)   1190   1190   1190   1191   119	1134	by inversion (1)	5. $\vdash_{H} stat \ \tau' \ e''$	1189
1136 by inversion (2)  4. QED by hole substitution  CASE $e = E[Err]$ :  1192  1138 CASE $e = E^{\bullet}[op^2 v_0 v_1]$ :  1. $e \rightarrow_{H-D} E^{\bullet}[\delta(op^2, v_0, v_1)]$ 1. $e \rightarrow_{H-D} E^{\bullet}[\delta(op^2, v_0, v_1)]$ 1. $e \rightarrow_{H-D} E^{\bullet}[\delta(op^2, v_0, v_1)]$ 2. QED $^{\downarrow}_{H}$ Err  1194  1195  1196  1197  1198  1199  1	1135	3. ⊢ <sub>H</sub> v <sub>1</sub>		1190
1138 CASE $e = E^{\bullet}[op^2 v_0 v_1]$ :  1. $e \rightarrow_{\text{H-D}} \text{Err}$ 1193  1. $e \rightarrow_{\text{H-D}} E^{\bullet}[\delta(op^2, v_0, v_1)]$ 2. QED $\vdash_{\text{H}} \text{Err}$ 1194  1140 2. $\vdash_{\text{H}} op^2 v_0 v_1$ 1195  1141 by dynamic hole typing  1152 Lemma 2.12: H static boundary factoring 1165  1164 3. $\vdash_{\text{H}} v_1$ 1175  1176 • $e$ is a value 1177  1187 • $e$ is a value 1188  1189 • $e$ is a value 1199  1199  1100 • $e$ is a value 1199  1100 • $e$ is $e$ is $e$ is $e$ value 1100 • $e$ is $e$ value 1100 • $e$ is $e$ value 1101 • $e$ value: 1102 • $e$ is $e$ value: 1103 • $e$ is $e$ value: 1104 • $e$ value: 1105 • $e$ is $e$ value: 1106 • $e$ is $e$ value: 1107 • $e$ value: 1108 • $e$ value: 1109 • $e$ value: 1109 • $e$ value: 1100 • $e$ value: 1100 • $e$ value: 1100 • $e$ value: 1100 • $e$ value: 1101 • $e$ $e$ value: 1102 • $e$ value: 1103 • $e$ value: 1104 • $e$ value: 1105 • $e$ value: 1106 • $e$ value: 1107 • $e$ value: 1108 • $e$ value: 1109 • $e$ value: 1109 • $e$ value: 1100 • $e$ value: 1100 • $e$ value: 1100 • $e$ value: 1101 • $e$ value: 1102 • $e$ value: 1103 • $e$ value: 1104 • $e$ value: 1105 • $e$ value: 1106 • $e$ value: 1107 • $e$ value: 1108 • $e$ value: 1109 • $e$ value: 1109 • $e$ value: 1100 • $e$ value: 1	1136		6. QED by hole substitution (5)	1191
1139 1. $e \rightarrow_{\text{H-D}} E^{\bullet}[\delta(op^2, v_0, v_1)]$ 2. $\text{QED} \vdash_{\text{H}} \text{Err}$ 1194 1194 2. $\vdash_{\text{H}} op^2 v_0 v_1$ 1195 1141 by dynamic hole typing 1196 1142 3. $\vdash_{\text{H}} v_0$ 1197 1143 $\land \vdash_{\text{H}} v_1$ • $e$ is a value 1198 1144 by inversion (1) • $e$ is a value 1199 1145 4. $\vdash_{\text{H}} \delta(op^2, v_0, v_1)$ • $e$ is a value 1199 1146 by $\delta$ preservation (2) • $e$ $E^{\bullet}[op^1 v]$ 1201 1201 1347 5. QED by hole substitution (3) • $e$ $E^{\bullet}[dyn \tau' e']$ and $e'$ is boundary-free: • $e$ $E^{\bullet}[ev]$ $v$ where $e'$ is boundary-free 1202 1149 1F $e'$ is a value: • $e$ $E^{\bullet}[DH(\tau', e')]$ 1706 1151 2. $\vdash_{\text{H}} dyn \tau' e' : \tau'$ 1707 1907 1907 1907 1907 1907 1907 1907	1137	4. QED by hole substitution	CASE $e = E[Err]$ :	1192
1139 1. $e \rightarrow_{\text{H-D}} E^{\bullet}[\delta(op^2, v_0, v_1)]$ 2. $\text{QED} \vdash_{\text{H}} \text{Err}$ 1194 1194 2. $\vdash_{\text{H}} op^2 v_0 v_1$ 1195 1141 by dynamic hole typing 1196 1142 3. $\vdash_{\text{H}} v_0$ 1197 1143 $\land \vdash_{\text{H}} v_1$ • $e$ is a value 1198 1144 by inversion (1) • $e$ is a value 1199 1145 4. $\vdash_{\text{H}} \delta(op^2, v_0, v_1)$ • $e$ is a value 1199 1146 by $\delta$ preservation (2) • $e$ $E^{\bullet}[op^1 v]$ 1201 1201 1347 5. QED by hole substitution (3) • $e$ $E^{\bullet}[dyn \tau' e']$ and $e'$ is boundary-free: • $e$ $E^{\bullet}[ev]$ $v$ where $e'$ is boundary-free 1202 1149 1F $e'$ is a value: • $e$ $E^{\bullet}[DH(\tau', e')]$ 1706 1151 2. $\vdash_{\text{H}} dyn \tau' e' : \tau'$ 1707 1907 1907 1907 1907 1907 1907 1907	1138	CASE $e = E^{\bullet}[op^2 v_0 v_1]$ :	1. $e \rightarrow_{H-D} Err$	1193
1140 2. $\vdash_{\vdash_{\vdash}} op^2 v_0 v_1$	1139			1194
1141 by dynamic hole typing  1142 3. $\vdash_{H} v_0$ 1143 $\land \vdash_{H} v_1$ 1144 by inversion (1)  1145 4. $\vdash_{H} \delta(op^2, v_0, v_1)$ 1146 by $\delta$ preservation (2)  1147 5. QED by hole substitution (3)  1148 <b>CASE</b> $e = E[dyn \ \tau' \ e']$ and $e'$ is boundary-free:  1149 IF $e'$ is a value:  1150 1. $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1151 2. $\vdash_{H} dyn \ \tau' \ e' : \tau'$ 1152 by boundary hole typing  1164 by dynamic hole typing  1175 chapter of the following holds:  1176 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1177 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1188 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1199 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1100 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1101 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1102 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1103 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1104 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1105 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1106 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1107 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1108 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1109 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1100 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1100 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1101 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1102 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1103 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1104 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1105 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1106 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1107 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1109 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1110 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1111 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1110 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1111 $e \to_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 111 $e \to$	1140			1195
1142 3. $\vdash_{H} v_0$	1141		Lemma 2.12 : H static houndary factoring	1196
1143 $\wedge \vdash_{H} v_{1}$	1142			1197
1144 by inversion (1)  1145 4. $\vdash_{H} \delta(op^{2}, v_{0}, v_{1})$ 1146 by $\delta$ preservation (2)  1147 5. QED by hole substitution (3)  1148 CASE $e = E[\operatorname{dyn} \tau' e']$ and $e'$ is boundary-free:  1149 IF $e'$ is a value:  1150 1. $e \rightarrow_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 1151 2. $\vdash_{H} \operatorname{dyn} \tau' e' : \tau'$ 1152 by boundary hole typing  1153 3. $\vdash_{H} e'$ 1154 by inversion (2)  1165 • $e = E^{\bullet}[v_{0} v_{1}]$ 1170 • $e = E^{\bullet}[op^{1} v]$ 1170 • $e = E[\operatorname{dyn} \tau e']$ where $e'$ is boundary-free  1170 • $e = E[\operatorname{dyn} \tau e']$ where $e'$ is boundary-free  1170 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1170 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1170 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1170 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1170 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1170 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1170 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1170 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1170 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1170 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1170 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1170 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1170 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1171 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1171 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1171 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1171 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1171 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1172 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1173 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1174 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1175 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1175 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1175 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1175 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1175 • $e = E[\operatorname{hyn} \tau e']$ where $e'$ is boundary-free  1176 • $e = E[\operatorname{hyn} $	1143	**	··	1198
1145 4. $\vdash_{H} \delta(op^2, v_0, v_1)$ by $\delta$ preservation (2)       146 by $\delta$ preservation (2)       5. QED by hole substitution (3)       148 <b>CASE</b> $e = E[dyn \ \tau' \ e']$ and $e'$ is boundary-free :     149 <b>IF</b> $e'$ is a value :     150       1. $e \rightarrow_{H-D} E[\mathcal{D}_{H}(\tau', e')]$ 151       2. $\vdash_{H} dyn \ \tau' \ e' : \tau'$ 3. $\vdash_{H} e'$ 3. $\vdash_{H} e'$ 4. $\vdash_{H} \delta(op^2, v_0, v_1)$ 1201       4. $\vdash_{H} \delta(op^2, v_0, v_1)$ 1201       6. $\vdash_{H} e = E[op^2 v_0 v_1]$ 1202       6. $\vdash_{H} e = E[dyn \tau' e']$ where $e'$ is boundary-free       1203       1204       1205       1206       1206       1207       1208       1208       1209       1209       1209       1209       1209       1209       1209       1200       1201       1202       1203       1204       1204       1205       1206       1207       1208       1208       1209       1200	1144			1199
1146 by $\delta$ preservation (2)  1147 5. QED by hole substitution (3)  1148 CASE $e = E[\operatorname{dyn} \tau' e']$ and $e'$ is boundary-free:  1149 IF $e'$ is a value:  1150 1. $e \to_{\text{H-D}} E[\mathcal{D}_{\text{H}}(\tau', e')]$ 1151 2. $\vdash_{\text{H}} \operatorname{dyn} \tau' e' : \tau'$ 1152 by boundary hole typing  1153 3. $\vdash_{\text{H}} e'$ 1154 by inversion (2)  1165 $e = E[\operatorname{op}^2 v_0 v_1]$ 1170 $e = E[\operatorname{dyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{dyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-free  1170 $e = E[\operatorname{cyn} \tau e']$ where $e'$ is boundary-	1145	· · · · · · · · · · · · · · · · · · ·		1200
1147 5. QED by hole substitution (3)  1148 CASE $e = E[\operatorname{dyn} \tau' e']$ and $e'$ is boundary-free:  1149 IF $e'$ is a value:  1150 1. $e \to_{\text{H-D}} E[\mathcal{D}_{\text{H}}(\tau', e')]$ 1151 2. $\vdash_{\text{H}} \operatorname{dyn} \tau' e' : \tau'$ 1152 by boundary hole typing  1153 3. $\vdash_{\text{H}} e'$ 1154 by inversion (2)  1165 $e = E[\operatorname{dyn} \tau e']$ where $e'$ is boundary-free  1203 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1204 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1205 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1206 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1207 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1208 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1209 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1209 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1200 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1201 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1202 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1203 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1204 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1205 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1206 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1207 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1208 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1209 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1209 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1209 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1209 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1209 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1209 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1209 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1209 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1209 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1209 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1209 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1209 $\bullet e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free  1209 $\bullet e = E[\operatorname{stat} \tau$	1146			1201
1148 <b>CASE</b> $e = E[\operatorname{dyn} \tau' e']$ and $e'$ is boundary-free :  1149 <b>IF</b> $e'$ is a value :  1150 1. $e \to_{\text{H-D}} E[\mathcal{D}_{\text{H}}(\tau', e')]$ 1151 2. $\vdash_{\text{H}} \operatorname{dyn} \tau' e' : \tau'$ 1152 by boundary hole typing 1153 3. $\vdash_{\text{H}} e'$ 1154 by inversion (2) 1155 1. QED 1156 • $e = E[\operatorname{stat} \tau e']$ where $e'$ is boundary-free 1207 • $e = E[\operatorname{Err}]$ 1208 • $e = E[\operatorname{Err}]$ 1209 1209	1147			1202
1149IF $e'$ is a value :• $e = E[Err]$ 120411501. $e \rightarrow_{H-D} E[\mathcal{D}_H(\tau', e')]$ Proof:120511512. $\vdash_H dyn \tau' e' : \tau'$ By the unique static evaluation contexts lemma, there are12061152by boundary hole typingseven cases.120711533. $\vdash_H e'$ CASE $e$ is a value :12081154by inversion (2)1. QED1209	1148			1203
1150 1. $e \rightarrow_{\text{H-D}} E[\mathcal{D}_{\text{H}}(\tau', e')]$ 1205 1151 2. $\vdash_{\text{H}} \text{dyn } \tau' e' : \tau'$ By the unique static evaluation contexts lemma, there are 1206 1152 by boundary hole typing seven cases. 1207 1153 3. $\vdash_{\text{H}} e'$ CASE $e$ is a value : 1208 1154 by inversion (2) 1. QED 1209	1149			1204
1151 2. $\vdash_{H}$ dyn $\tau'$ $e'$ : $\tau'$ By the unique static evaluation contexts lemma, there are 1206 by boundary hole typing seven cases. 1207 1153 3. $\vdash_{H}$ $e'$ CASE $e$ is a value : 1208 1154 by inversion (2) 1. QED 1209	1150			1205
1152by boundary hole typingseven cases.120711533. $\vdash_{H} e'$ <b>CASE</b> $e$ is a value :12081154by inversion (2)1. QED1209	1151		3	1206
1153       3. $\vdash_{H} e'$ CASE $e$ is a value :       1208         1154       by inversion (2)       1. QED       1209	1152			1207
1154 by inversion (2) 1. QED 1209	1153			1208
	1154			1209
	1155		·-	1210

```
1. Contradiction by \vdash_{\mathsf{H}} e : \tau
1211
            CASE e = E[v_0 \ v_1]:
                                                                                                                                                                                                             1266
1212
               1. E = E^{\bullet}
                                                                                                               CASE e = i
                                                                                                                                                                                                             1267
                                                                                                                         \vee e = \lambda(x : \tau_d). e'
1213
                   \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                                                                                             1268
                   \vee E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                         \vee e = \text{mon}(\tau_d \Rightarrow \tau_c)v:
1214
                                                                                                                                                                                                             1269
1215
                   by inner boundary
                                                                                                                 1. QED e is a value
                                                                                                                                                                                                             1270
               2. IF E = E^{\bullet}:
                                                                                                               CASE e = \langle e_0, e_1 \rangle:
                                                                                                                                                                                                             1271
1216
1217
                     a. QED e = E^{\bullet}[v_0 \ v_1]
                                                                                                                  IF e_0 \notin v:
                                                                                                                                                                                                             1272
                   IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
1218
                                                                                                                    1. \vdash_{\mathsf{H}} e_0 : \tau_0
                                                                                                                                                                                                             1273
                     a. QED e = E'[\mathsf{dyn} \ \tau \ E^{\bullet}[v_0 \ v_1]]
1219
                                                                                                                        by inversion
                                                                                                                                                                                                             1274
                   ELSE E = E'[\text{stat } \tau E^{\bullet}]:
1220
                                                                                                                    2. e_0 = E_0[e'_0]
                                                                                                                                                                                                             1275
                     a. QED e = E'[\text{stat } \tau E^{\bullet}[v_0 \ v_1]]
1221
                                                                                                                        by the induction hypothesis (1)
                                                                                                                                                                                                             1276
             CASE e = E[op^1 v]:
                                                                                                                    3. E = \langle E_0, e_1 \rangle
1222
                                                                                                                                                                                                             1277
               1. E = E^{\bullet}
                                                                                                                    4. QED e = E[e'_0]
                                                                                                                                                                                                             1278
                   \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
1224
                                                                                                                  IF e_0 \in v
                                                                                                                                                                                                             1279
1225
                   \vee E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                             1280
                                                                                                                       \land e_1 \notin v:
1226
                   by inner boundary
                                                                                                                                                                                                             1281
                                                                                                                    1. \vdash_{\mathsf{H}} e_1 : \tau_1
               2. IF E = E^{\bullet}:
1227
                                                                                                                        by inversion
                                                                                                                                                                                                             1282
1228
                     a. OED e = E^{\bullet}[op^1 v]
                                                                                                                    2. e_1 = E_1[e_1']
                                                                                                                                                                                                             1283
                   IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                                        by the induction hypothesis (1)
1229
                                                                                                                                                                                                             1284
                     a. QED e = E'[\text{dyn } \tau E^{\bullet}[op^1 v]]
1230
                                                                                                                    3. E = \langle e_0, E_1 \rangle
                                                                                                                                                                                                             1285
1231
                   ELSE E = E'[\text{stat } \tau E^{\bullet}]:
                                                                                                                    4. QED e = E[e'_1]
                                                                                                                                                                                                             1286
                     a. QED e = E'[\text{stat } \tau E^{\bullet}[op^1 v]]
                                                                                                                  ELSE e_0 \in v
1232
                                                                                                                                                                                                             1287
            CASE e = E[op^2 v_0 v_1]:
1233
                                                                                                                            \land e_1 \in v:
                                                                                                                                                                                                             1288
               1. E = E^{\bullet}
                                                                                                                    1. E = []
1234
                   \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
1235
                                                                                                                    2. QED e is a value
                                                                                                                                                                                                             1290
1236
                   \vee E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                               CASE e = e_0 e_1:
                                                                                                                                                                                                             1291
1237
                   by inner boundary
                                                                                                                  IF e_0 \notin v:
                                                                                                                                                                                                             1292
               2. IF E = E^{\bullet}:
                                                                                                                                                                                                             1293
                                                                                                                    1. \vdash_{\mathsf{H}} e_0 : \tau_0
                     a. QED e = E^{\bullet}[op^2 v_0 v_1]
                                                                                                                        by inversion
1239
                                                                                                                                                                                                             1294
                   IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                                    2. e_0 = E_0[e'_0]
1240
                                                                                                                                                                                                             1295
                     a. QED e = E'[\mathsf{dyn} \ \tau \ E^{\bullet}[\mathsf{op}^2 \ v_0 \ v_1]]
1241
                                                                                                                        by the induction hypothesis (1)
                                                                                                                                                                                                             1296
1242
                   ELSE E = E'[\text{stat } \tau E^{\bullet}]:
                                                                                                                    3. E = E_0 e_1
                                                                                                                                                                                                             1297
                     a. QED e = E'[\text{stat } \tau E^{\bullet}[op^2 v_0 v_1]]
                                                                                                                    4. QED e = E[e'_0]
1243
                                                                                                                                                                                                             1298
1244
             CASE e = E[\mathsf{dyn} \ \tau \ v]:
                                                                                                                  IF e_0 \in v
                                                                                                                                                                                                             1299
1245
               1. QED v is boundary-free
                                                                                                                       \wedge e_1 \notin v:
                                                                                                                                                                                                             1300
            CASE e = E[\text{stat } \tau \ v]:
1246
                                                                                                                    1. \vdash_{\mathsf{H}} e_1 : \tau_1
                                                                                                                                                                                                             1301
1247
               1. QED v is boundary-free
                                                                                                                        by inversion
                                                                                                                                                                                                             1302
            CASE e = E[Err]:
                                                                                                                    2. e_1 = E_1[e_1']
1248
                                                                                                                                                                                                             1303
               1. QED
                                                                                                                        by the induction hypothesis (1)
1249
         3. E = e_0 E_1
1250
                                                                                                                                                                                                             1305
                                                                                                                    4. QED e = E[e'_1]
1251
                                                                                                                                                                                                             1306
         Lemma 2.13: H unique static evaluation contexts
1252
                                                                                                                  ELSE e_0 \in v
                                                                                                                                                                                                             1307
            If \vdash_{\mathsf{H}} e : \tau then one of the following holds:
1253
            • e is a value
                                                                                                                            \land e_1 \in v:
1254
                                                                                                                    1. E = []
            \bullet e = E[v_0 \ v_1]
                                                                                                                                                                                                             1309
1255
            • e = E[op^1 v]
                                                                                                                    2. QED e = E[e_0 \ e_1]
                                                                                                                                                                                                             1310
                                                                                                               CASE e = op^1 e_0:
            • e = E[op^2 v_0 v_1]
1256
                                                                                                                                                                                                             1311
            • e = E[\mathsf{dyn} \ \tau \ v]
                                                                                                                  IF e_0 \notin v:
                                                                                                                                                                                                             1312
1258
            • e = E[\operatorname{stat} \tau v]
                                                                                                                    1. \vdash_{\mathsf{H}} e_0 : \tau_0
                                                                                                                                                                                                             1313
1259
            • e = E[Err]
                                                                                                                        by inversion
                                                                                                                                                                                                             1314
         Proof:
                                                                                                                    2. e_0 = E_0[e'_0]
1260
                                                                                                                                                                                                             1315
            By induction on the structure of e.
                                                                                                                        by the induction hypothesis (1)
1261
                                                                                                                                                                                                             1316
            CASE e = x
1262
                                                                                                                    3. E = op^1 E_0
                                                                                                                                                                                                             1317
                      \vee e = \lambda x. e'
1263
                                                                                                                    4. QED e = E[e'_0]
                                                                                                                                                                                                             1318
                      \vee e = \operatorname{stat} \tau e':
                                                                                                                  ELSE e_0 \in v:
1264
                                                                                                                                                                                                             1319
1265
                                                                                                                                                                                                             1320
                                                                                                      12
```

```
1321
                   1. E = []
                                                                                                                            a. OED E is boundary-free
                                                                                                                                                                                                                     1376
1322
                   2. QED e = E[op^1 e_0]
                                                                                                                          IF E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                                                                                                                                     1377
             CASE e = op^2 e_0 e_1:
1323
                                                                                                                            a. E' = E'_0 e_1
                                                                                                                                                                                                                     1378
                                                                                                                            b. QED E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                IF e_0 \notin v:
1324
                                                                                                                                                                                                                     1379
                                                                                                                          ELSE E_0 = E_0'[\operatorname{stat} \tau E^{\bullet}]:
1325
                   1. \vdash_{\mathsf{H}} e_0 : \tau_0
                                                                                                                                                                                                                     1380
                      by inversion
                                                                                                                            a. E' = E'_0 e_1
                                                                                                                                                                                                                     1381
1326
                                                                                                                            b. QED E = E'[\text{stat } \tau E^{\bullet}]
1327
                   2. e_0 = E_0[e'_0]
                                                                                                                                                                                                                     1382
                                                                                                                   CASE E = v_0 E_1:
                       by the induction hypothesis (1)
1328
                                                                                                                                                                                                                     1383
                  3. E = op^2 E_0 e_1
                                                                                                                      1. E_1 = E^{\bullet}
                   4. QED e = E[e'_0]
1330
                                                                                                                          \vee E_1 = E'_1[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                                                                                                     1385
1331
                IF e_0 \in v
                                                                                                                          \vee E_1 = E_1'[\operatorname{stat} \tau E^{\bullet}]
                                                                                                                                                                                                                     1386
                                                                                                                          by the induction hypothesis
1332
                      \wedge e_1 \notin v:
                                                                                                                                                                                                                     1387
1333
                                                                                                                      2. IF E_1 = E^{\bullet}:
                   1. \vdash_{\mathsf{H}} e_1 : \tau_1
                                                                                                                                                                                                                     1388
1334
                      by inversion
                                                                                                                            a. QED E is boundary-free
                                                                                                                                                                                                                     1389
1335
                   2. e_1 = E_1[e_1']
                                                                                                                          IF E_1 = E'_1[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                                                                                                                                     1390
1336
                       by the induction hypothesis (1)
                                                                                                                            a. E' = v_0 E'_1
                                                                                                                                                                                                                     1391
                   3. E = op^2 e_0 E_1
                                                                                                                            b. QED E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
1337
                                                                                                                                                                                                                     1392
1338
                   4. QED e = E[e'_1]
                                                                                                                          ELSE E_1 = E_1'[\operatorname{stat} \tau E^{\bullet}]:
                                                                                                                                                                                                                     1393
                ELSE e_0 \in v
                                                                                                                            a. E' = v_0 E'_1
1339
                                                                                                                                                                                                                     1394
                                                                                                                            b. QED E = E'[\text{stat } \tau E^{\bullet}]
1340
                          \land e_1 \in v:
1341
                   1. E = []
                                                                                                                    CASE E = \langle E_0, e_1 \rangle:
                                                                                                                                                                                                                     1396
                   2. QED e = E[op^2 e_0 e_1]
                                                                                                                      1. E_0 = E^{\bullet}
1342
                                                                                                                                                                                                                     1397
                                                                                                                          \vee E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]
             CASE e = \text{dyn } \tau \ e_0:
                                                                                                                                                                                                                     1398
1343
                                                                                                                          \vee E_0 = E_0'[\operatorname{stat} \tau E^{\bullet}]
1344
                IF e_0 \notin v:
1345
                   1. \vdash_{\vdash} e_0
                                                                                                                          by the induction hypothesis
                                                                                                                                                                                                                     1400
1346
                      by inversion
                                                                                                                      2. IF E_0 = E^{\bullet}:
                   2. e_0 = E_0[e'_0]
                                                                                                                            a. QED E is boundary-free
1347
                                                                                                                                                                                                                     1402
                       by unique dynamic evaluation contexts (1)
                                                                                                                          IF E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
1348
                                                                                                                                                                                                                     1403
                                                                                                                            a. E' = \langle E'_0, e_1 \rangle
                   3. E = \operatorname{dyn} \tau E_0
1349
                                                                                                                                                                                                                     1404
1350
                   4. QED e = E[e'_0]
                                                                                                                            b. QED E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                                                                                                     1405
                                                                                                                          ELSE E_0 = E_0'[\text{stat } \tau E^{\bullet}]:
1351
                ELSE e_0 \in v:
                                                                                                                                                                                                                     1406
                                                                                                                            a. E' = \langle E'_0, e_1 \rangle
1352
                   1. E = []
                                                                                                                                                                                                                     1407
1353
                  2. QED e = E[dyn \tau e_0]
                                                                                                                            b. QED E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                                     1408
                                                                                                                    CASE E = \langle v_0, E_1 \rangle:
1354
             CASE e = \text{stat } \tau e_0:
                                                                                                                                                                                                                     1409
                Contradiction by \vdash_{\mathsf{H}} e : \tau
1355
                                                                                                                      1. E_1 = E^{\bullet}
                                                                                                                                                                                                                     1410
             CASE e = Err:
                                                                                                                          \vee E_1 = E_1'[\mathsf{dyn} \ \tau \ E^{\bullet}]
1356
                                                                                                                                                                                                                     1411
1357
               1. E = []
                                                                                                                          \vee E_1 = E_1'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                                     1412
1358
               2. QED e = E[Err]
                                                                                                                          by the induction hypothesis
                                                                                                                                                                                                                     1413
1359
                                                                                                                      2. IF E_1 = E^{\bullet}:
                                                                                                                            a. QED E is boundary-free
                                                                                                                                                                                                                     1415
1360
          Lemma 2.14: H inner boundary
                                                                                                                          IF E_1 = E_1'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
1361
            For all contexts E, one of the following holds:
                                                                                                                                                                                                                     1416
1362
            • E = E^{\bullet}
                                                                                                                            a. E' = \langle v_0, E'_1 \rangle
                                                                                                                                                                                                                     1417
                                                                                                                            b. QED E = E'[\text{dyn } \tau E^{\bullet}]
            • E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                          ELSE E_1 = E'_1[\text{stat } \tau E^{\bullet}]:
1364
            • E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                                     1419
                                                                                                                            a. E' = \langle v_0, E'_1 \rangle
1365
                                                                                                                                                                                                                     1420
             By induction on the structure of E.
                                                                                                                            b. QED E = E'[\text{stat } \tau E^{\bullet}]
1366
                                                                                                                                                                                                                     1421
             CASE E = E^{\bullet}:
1367
                                                                                                                   CASE E = op^1 E_0:
                                                                                                                                                                                                                     1422
               1. QED
                                                                                                                      1. E_0 = E^{\bullet}
1368
                                                                                                                                                                                                                     1423
             CASE E = E_0 e_1:
1369
                                                                                                                          \vee E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                                                                                                     1424
               1. E_0 = E^{\bullet}
                                                                                                                          \vee E_0 = E_0'[\operatorname{stat} \tau E^{\bullet}]
1370
                                                                                                                                                                                                                     1425
                    \vee E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]
1371
                                                                                                                          by the induction hypothesis
                                                                                                                                                                                                                     1426
                    \vee E_0 = E_0'[\text{stat } \tau E^{\bullet}]
1372
                                                                                                                      2. IF E_0 = E^{\bullet}:
                                                                                                                                                                                                                     1427
                    by the induction hypothesis
1373
                                                                                                                            a. QED E is boundary-free
                                                                                                                                                                                                                     1428
               2. IF E_0 = E^{\bullet}:
                                                                                                                          IF E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
1374
                                                                                                                                                                                                                     1429
1375
                                                                                                                                                                                                                     1430
                                                                                                          13
```

```
1431
                       a. E' = op^1 E'_0
                                                                                                                                  b. QED E = E'[\mathsf{dyn} \ \tau' \ E^{\bullet}]
                                                                                                                                                                                                                              1486
                      b. QED E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
1432
                                                                                                                               ELSE E_0 = E_0'[\text{stat } \tau' E^{\bullet}]:
                                                                                                                                                                                                                              1487
                    ELSE E_0 = E_0'[\operatorname{stat} \tau E^{\bullet}]:
                                                                                                                                  a. E' = \operatorname{stat} \tau E'_0
1433
                                                                                                                                                                                                                              1488
                       a. E' = op^1 E'_0
                                                                                                                                  b. QED E = E'[\text{stat } \tau' E^{\bullet}]
1434
                                                                                                                                                                                                                              1489
1435
                      b. QED E = E'[\text{stat } \tau E^{\bullet}]
             CASE E = op^2 E_0 e_1:
                                                                                                                                                                                                                              1491
1436
                                                                                                                     Lemma 2.15: H dynamic boundary factoring
1437
                1. E_0 = E^{\bullet}
                                                                                                                                                                                                                              1492
                                                                                                                     If \vdash_{\mathsf{H}} e then one of the following holds:
1438
                    \vee E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                       • e is a value
                                                                                                                                                                                                                              1493
1439
                    \vee E_0 = E_0'[\operatorname{stat} \tau E^{\bullet}]
                                                                                                                       • e = E^{\bullet}[v_0 \ v_1]
                                                                                                                                                                                                                              1494
1440
                    by the induction hypothesis
                                                                                                                       \bullet \ e = E^{\bullet}[op^1 \ v]
                                                                                                                                                                                                                              1495
1441
                2. IF E_0 = E^{\bullet}:
                                                                                                                       • e = E^{\bullet}[op^2 v_0 v_1]
                                                                                                                                                                                                                              1496
                       a. QED E is boundary-free
1442
                                                                                                                       • e = E[dyn \tau e'] where e' is boundary-free
                                                                                                                                                                                                                              1497
1443
                    IF E_0 = E_0'[\text{dyn } \tau \ E^{\bullet}]:
                                                                                                                                                                                                                              1498
                                                                                                                       • e = E[\text{stat } \tau e'] where e' is boundary-free
1444
                       a. E' = op^2 E'_0 e_1
                                                                                                                       • e = E[Err]
                                                                                                                                                                                                                              1499
1445
                       b. QED E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                     Proof:
                                                                                                                                                                                                                              1500
                                                                                                                        By the unique dynamic evaluation contexts lemma, there
1446
                    ELSE E_0 = E_0'[\text{stat } \tau E^{\bullet}]:
                                                                                                                                                                                                                              1501
                       a. E' = op^2 E'_0 e_1
                                                                                                                        are seven cases.
1447
                                                                                                                                                                                                                              1502
                                                                                                                        CASE e is a value :
                       b. QED E = E'[\text{stat } \tau E^{\bullet}]
1448
                                                                                                                                                                                                                              1503
                                                                                                                           1. QED
             CASE E = op^2 v_0 E_1:
                                                                                                                                                                                                                              1504
1449
                                                                                                                        CASE e = E[v_0 \ v_1]:
1450
                1. E_1 = E^{\bullet}
                                                                                                                                                                                                                              1505
                                                                                                                           1. E = E^{\bullet}
1451
                    \vee E_1 = E_1'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                                                                                                              1506
                                                                                                                               \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
1452
                    \vee E_1 = E_1'[\operatorname{stat} \tau E^{\bullet}]
                                                                                                                                                                                                                              1507
                                                                                                                               \vee E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                                              1508
1453
                    by the induction hypothesis
                                                                                                                               by inner boundary
1454
                2. IF E_1 = E^{\bullet}:
                                                                                                                           2. IF E = E^{\bullet}:
1455
                      a. QED E is boundary-free
                                                                                                                                                                                                                              1510
                                                                                                                                  a. QED e = E^{\bullet}[v_0 \ v_1]
1456
                    IF E_1 = E_1'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                                                                                                                                              1511
                                                                                                                               IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                                                                                                                                              1512
1457
                       a. E' = op^2 v_0 E'_1
                                                                                                                                  a. QED e = E'[\mathsf{dyn} \ \tau \ E^{\bullet}[v_0 \ v_1]]
                                                                                                                                                                                                                              1513
                      b. QED E = E'[\text{dyn } \tau E^{\bullet}]
                                                                                                                               ELSE E = E'[\text{stat } \tau E^{\bullet}]:
1459
                    ELSE E_1 = E_1'[\text{stat } \tau E^{\bullet}]:
                                                                                                                                                                                                                              1514
                                                                                                                                  a. QED e = E'[\text{stat } \tau E^{\bullet}[v_0 \ v_1]]
1460
                                                                                                                                                                                                                              1515
                       a. E' = op^2 v_0 E'_1
                                                                                                                        CASE e = E[op^1 v]:
1461
                       b. QED E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                                              1516
                                                                                                                           1. E=E^{\bullet}
1462
                                                                                                                                                                                                                              1517
             CASE E = \text{dyn } \tau E_0:
                                                                                                                               \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
1463
                                                                                                                                                                                                                              1518
                1. E_0 = E^{\bullet}
                                                                                                                               \vee E = E'[\text{stat } \tau E^{\bullet}]
1464
                                                                                                                                                                                                                              1519
                    \vee E_0 = E_0'[\mathsf{dyn} \ \tau' \ E^{\bullet}]
                                                                                                                               by inner boundary
1465
                    \vee E_0 = E_0'[\operatorname{stat} \tau' E^{\bullet}]
                                                                                                                                                                                                                              1520
                                                                                                                           2. IF E = E^{\bullet}:
1466
                    by the induction hypothesis
                                                                                                                                                                                                                              1521
                                                                                                                                 a. QED e = E^{\bullet}[op^1 v]
1467
                2. IF E_0 = E^{\bullet}:
                                                                                                                                                                                                                              1522
                                                                                                                               IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
1468
                                                                                                                                                                                                                              1523
                       a. QED
                                                                                                                                  a. QED e = E'[\mathsf{dyn} \ \tau \ E^{\bullet}[\mathsf{op}^1 \ v]]
1469
                    IF E_0 = E_0' [\text{dyn } \tau' E^{\bullet}]:
                                                                                                                               ELSE E = E'[\text{stat } \tau E^{\bullet}]:
                                                                                                                                                                                                                              1525
1470
                       a. E' = \operatorname{dyn} \tau E'_0
                                                                                                                                  a. QED e = E'[\text{stat } \tau E^{\bullet}[op^1 v]]
1471
                       b. QED E = E'[\mathsf{dyn} \ \tau' \ E^{\bullet}]
                                                                                                                                                                                                                              1526
                                                                                                                        CASE e = E[op^2 v_0 v_1]:
1472
                                                                                                                                                                                                                              1527
                    ELSE E_0 = E_0'[\text{stat } \tau' E^{\bullet}]:
                                                                                                                           1. E = E^{\bullet}
1473
                                                                                                                                                                                                                              1528
                       a. E' = \operatorname{dyn} \tau E'_0
                                                                                                                               \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
1474
                       b. QED E = E'[\text{stat } \tau' E^{\bullet}]
                                                                                                                                                                                                                              1529
                                                                                                                               \vee E = E'[\text{stat } \tau E^{\bullet}]
1475
                                                                                                                                                                                                                              1530
             CASE E = \text{stat } \tau E_0:
                                                                                                                               by inner boundary
1476
                1. E_0 = E^{\bullet}
                                                                                                                                                                                                                              1531
                                                                                                                           2. IF E = E^{\bullet}:
1477
                                                                                                                                                                                                                              1532
                    \vee E_0 = E_0'[\mathsf{dyn} \ \tau' \ E^{\bullet}]
                                                                                                                                  a. QED e = E^{\bullet}[op^2 v_0 v_1]
1478
                    \vee E_0 = E_0'[\operatorname{stat} \tau' E^{\bullet}]
                                                                                                                                                                                                                              1533
                                                                                                                               IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
1479
                                                                                                                                                                                                                              1534
                    by the induction hypothesis
                                                                                                                                  a. QED e = E'[\mathsf{dyn} \ \tau \ E^{\bullet}[\mathsf{op}^2 \ v_0 \ v_1]]
1480
                                                                                                                                                                                                                              1535
                2. IF E_0 = E^{\bullet}:
                                                                                                                               ELSE E = E'[\text{stat } \tau E^{\bullet}]:
1481
                                                                                                                                                                                                                              1536
                      a. QED
                                                                                                                                  a. QED e = E'[\text{stat } \tau E^{\bullet}[op^2 v_0 v_1]]
1482
                    IF E_0 = E_0'[\operatorname{dyn} \tau' E^{\bullet}]:
                                                                                                                                                                                                                              1537
                                                                                                                        CASE e = E[\mathsf{dyn} \ \tau \ v]:
1483
                                                                                                                                                                                                                              1538
                       a. E' = \operatorname{stat} \tau E'_0
                                                                                                                           1. QED v is boundary-free
1484
                                                                                                                                                                                                                              1539
1485
                                                                                                                                                                                                                              1540
                                                                                                              14
```

```
4. QED e = E[e'_0]
1541
           CASE e = E[\text{stat } \tau \ v]:
                                                                                                                                                                                           1596
             1. QED v is boundary-free
1542
                                                                                                        IF e_0 \in v
                                                                                                                                                                                           1597
                                                                                                             \land e_1 \notin v:
1543
           CASE e = E[Err]:
                                                                                                                                                                                           1598
             1. QED
1544
                                                                                                          1. \vdash_{\mathsf{H}} e_1
                                                                                                                                                                                           1599
1545
                                                                                                             by inversion
                                                                                                                                                                                           1600
                                                                                                          2. e_1 = E_1[e_1']
                                                                                                                                                                                           1601
1546
        Lemma 2.16: H unique dynamic evaluation contexts
1547
                                                                                                             by the induction hypothesis (1)
                                                                                                                                                                                           1602
          If \vdash_{\mathsf{H}} e then one of the following holds:
1548
          • e is a value
                                                                                                          3. E = e_0 E_1
                                                                                                                                                                                           1603
1549
          • e = E[v_0 \ v_1]
                                                                                                          4. QED e = E[e'_1]
1550
          \bullet e = E[op^1 v]
                                                                                                        ELSE e_0 \in v
                                                                                                                                                                                           1605
          \bullet \ e = E[op^2 \ v_0 \ v_1]
1551
                                                                                                                 \land e_1 \in v:
                                                                                                                                                                                           1606
                                                                                                          1. E = []
1552
          • e = E[\mathsf{dyn} \ \tau \ v]
                                                                                                                                                                                           1607
1553
                                                                                                          2. QED e = E[e_0 \ e_1]
          • e = E[\operatorname{stat} \tau v]
                                                                                                                                                                                           1608
1554
          • e = E[Err]
                                                                                                     CASE e = op^1 e_0:
                                                                                                                                                                                           1609
1555
        Proof:
                                                                                                        IF e_0 \notin v:
                                                                                                                                                                                           1610
           By induction on the structure of e.
1556
                                                                                                                                                                                           1611
                                                                                                          1. \vdash_{\mathsf{H}} e_0
           CASE e = x
1557
                                                                                                             by inversion
                                                                                                                                                                                           1612
                    \vee e = \lambda(x:\tau).e'
1558
                                                                                                          2. e_0 = E_0[e_0']
                                                                                                                                                                                           1613
                    \vee e = \operatorname{dyn} \tau e':
                                                                                                             by the induction hypothesis (1)
1559
                                                                                                                                                                                           1614
             1. Contradiction by \vdash_{\!\!\!\perp} e
1560
                                                                                                          3. E = op^1 E_0
                                                                                                                                                                                           1615
           CASE e = i
1561
                                                                                                          4. QED e = E[e'_0]
                                                                                                                                                                                           1616
                    \vee e = \lambda x. e'
                                                                                                        ELSE e_0 \in v:
1562
                                                                                                                                                                                           1617
                    \vee e = mon(\tau_d \Rightarrow \tau_c)v:
                                                                                                          1. E = []
                                                                                                                                                                                           1618
1563
             1. QED e is a value
                                                                                                          2. QED e = E[op^1 e_0]
1564
                                                                                                                                                                                           1619
           CASE e = Err:
                                                                                                     CASE e = op^2 e_0 e_1:
1565
                                                                                                                                                                                           1620
             1. E = []
1566
                                                                                                        IF e_0 \notin v:
                                                                                                                                                                                           1621
             2. QED e = E[Err]
1567
                                                                                                          1. \vdash_{\mathsf{H}} e_0
                                                                                                                                                                                           1622
           CASE e = \langle e_0, e_1 \rangle:
1568
                                                                                                             by inversion
                                                                                                                                                                                           1623
              IF e_0 \notin v:
1569
                                                                                                          2. e_0 = E_0[e'_0]
                                                                                                                                                                                           1624
                1. ⊢<sub>H</sub> e<sub>0</sub>
                                                                                                             by the induction hypothesis (1)
1570
                                                                                                                                                                                           1625
                    by inversion
                                                                                                          3. E = op^2 E_0 e_1
1571
                                                                                                                                                                                           1626
                2. e_0 = E_0[e_0']
                                                                                                          4. QED e = E[e'_0]
                                                                                                                                                                                           1627
                    by the induction hypothesis (1)
1573
                                                                                                        IF e_0 \in v
                                                                                                                                                                                           1628
                3. E = \langle E_0, e_1 \rangle
1574
                                                                                                             \land e_1 \notin v:
                                                                                                                                                                                           1629
                4. QED e = E[e'_0]
1575
                                                                                                          1. \vdash_{\mathsf{H}} e_1
                                                                                                                                                                                           1630
              IF e_0 \in v
                                                                                                             by inversion
1576
                                                                                                                                                                                           1631
1577
                   \wedge e_1 \notin v:
                                                                                                          2. e_1 = E_1[e_1']
                                                                                                                                                                                           1632
1578
                1. ⊢<sub>H</sub> e<sub>1</sub>
                                                                                                             by the induction hypothesis (1)
                                                                                                                                                                                           1633
                    by inversion
                                                                                                          3. E = op^2 e_0 E_1
                2. e_1 = E_1[e_1']
                                                                                                          4. QED e = E[e'_1]
                                                                                                                                                                                           1635
1580
                    by the induction hypothesis (1)
                                                                                                        ELSE e_0 \in v
1581
                                                                                                                                                                                           1636
                3. E = \langle e_0, E_1 \rangle
1582
                                                                                                                 \land e_1 \in v:
                                                                                                                                                                                           1637
                4. QED e = E[e'_1]
                                                                                                          1. E = []
              ELSE e_0 \in v
                                                                                                          2. QED e = E[op^2 e_0 e_1]
1584
                                                                                                                                                                                           1639
                       \land e_1 \in v:
1585
                                                                                                     CASE e = \text{stat } \tau \ e_0:
                                                                                                                                                                                           1640
                1. E = []
1586
                                                                                                        IF e_0 \notin v:
                                                                                                                                                                                           1641
                2. QED e is a value
                                                                                                                                                                                           1642
                                                                                                          1. \vdash_{\mathsf{H}} e_0
           CASE e = e_0 e_1:
1588
                                                                                                             by inversion
                                                                                                                                                                                           1643
              IF e_0 \notin v:
1589
                                                                                                          2. e_0 = E_0[e'_0]
                                                                                                                                                                                           1644
                1. ⊢<sub>H</sub> e<sub>0</sub>
1590
                                                                                                             by unique static evaluation contexts (1)
                                                                                                                                                                                           1645
                    by inversion
1591
                                                                                                          3. E = \operatorname{stat} \tau E_0
                                                                                                                                                                                           1646
                2. e_0 = E_0[e'_0]
                                                                                                          4. QED e = E[e'_0]
1592
                                                                                                                                                                                           1647
                    by the induction hypothesis (1)
1593
                                                                                                        ELSE e_0 \in v:
                                                                                                                                                                                           1648
                3. E = E_0 e_1
1594
                                                                                                          1. E = []
1595
                                                                                                                                                                                           1650
                                                                                             15
```

```
1651
                   2. QED e = E[\text{stat } \tau \ e_0]
                                                                                                                            2. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e]
                                                                                                                                                                                                                                1706
          1652
                                                                                                                                by inversion
                                                                                                                                                                                                                                1707
1653
                                                                                                                            3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                                1708
          Lemma 2.17 : H static hole typing
                                                                                                                         CASE E^{\bullet} = v_0 E^{\bullet}_1:
1654
          If \vdash_{\mathsf{H}} E^{\bullet}[e] : \tau then the derivation contains a sub-term \vdash_{\mathsf{H}} e : \tau'
                                                                                                                                                                                                                                1709
1655
                                                                                                                            1. E^{\bullet}[e] = v_0 E^{\bullet}_{1}[e]
                                                                                                                                                                                                                                1710
                                                                                                                            2. \vdash_{\mathsf{H}} E^{\bullet}_{1}[e]
                                                                                                                                                                                                                                1711
1656
             By induction on the structure of E^{\bullet}.
                                                                                                                                by inversion
1657
                                                                                                                                                                                                                                1712
             CASE E^{\bullet} = []:
                                                                                                                            3. QED by the induction hypothesis (2)
1658
                                                                                                                                                                                                                                1713
                1. QED E^{\bullet}[e] = e
1659
                                                                                                                         CASE E^{\bullet} = \langle E^{\bullet}_{0}, e_{1} \rangle:
                                                                                                                                                                                                                                1714
              CASE E^{\bullet} = E^{\bullet}_{0} e_{1}:
                                                                                                                            1. E^{\bullet}[e] = \langle E^{\bullet}_{0}[e], e_{1} \rangle
1660
                                                                                                                                                                                                                                1715
                1. E^{\bullet}[e] = E^{\bullet}_{0}[e] e_{1}
1661
                                                                                                                            2. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e]
                2. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e] : \tau_{d} \Rightarrow \tau_{c}
                                                                                                                                by inversion
1662
                                                                                                                                                                                                                                1717
                    by inversion
                                                                                                                            3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                                1718
1663
                3. QED by the induction hypothesis (2)
                                                                                                                         CASE E^{\bullet} = \langle v_0, E^{\bullet}_1 \rangle:
1664
              CASE E^{\bullet} = v_0 E^{\bullet}_1:
                                                                                                                                                                                                                                1719
1665
                                                                                                                            1. E^{\bullet}[e] = \langle v_0, E^{\bullet}_1[e] \rangle
                                                                                                                                                                                                                                1720
                1. E^{\bullet}[e] = v_0 E^{\bullet}[e]
1666
                                                                                                                            2. \vdash_{\mathsf{H}} E^{\bullet}{}_{1}[e]
                                                                                                                                                                                                                                1721
                2. \vdash_{\mathsf{H}} E^{\bullet}_{1}[e] : \tau_{d}
1667
                                                                                                                                by inversion
                                                                                                                                                                                                                                1722
                    by inversion
                                                                                                                            3. QED by the induction hypothesis (2)
1668
                                                                                                                                                                                                                                1723
                3. QED by the induction hypothesis (2)
                                                                                                                         CASE E^{\bullet} = op^1 E^{\bullet}_0:
                                                                                                                                                                                                                                1724
1669
              CASE E^{\bullet} = \langle E^{\bullet}_{0}, e_{1} \rangle:
                                                                                                                            1. E^{\bullet}[e] = op^1 E^{\bullet}_{0}[e]
1670
                                                                                                                                                                                                                                1725
                1. E^{\bullet}[e] = \langle E^{\bullet}_{0}[e], e_{1} \rangle
1671
                                                                                                                            2. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e]
                                                                                                                                                                                                                                1726
                2. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e] : \tau_{0}
                                                                                                                                by inversion
1672
                                                                                                                                                                                                                                1727
                    by inversion
                                                                                                                            3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                                1728
1673
                3. QED by the induction hypothesis (2)
                                                                                                                         CASE E^{\bullet} = op^2 E^{\bullet}_0 e_1:
1674
                                                                                                                                                                                                                                1729
              CASE E^{\bullet} = \langle v_0, E^{\bullet}_1 \rangle:
                                                                                                                            1. E^{\bullet}[e] = op^2 E^{\bullet}_{0}[e] e_1
1675
                1. E^{\bullet}[e] = \langle v_0, E^{\bullet}_1[e] \rangle
                                                                                                                                                                                                                                1730
                                                                                                                            2. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e]
                                                                                                                                                                                                                                1731
                2. \vdash_{\mathsf{H}} E^{\bullet}_{1}[e] : \tau_{1}
                                                                                                                                by inversion
                                                                                                                                                                                                                                1732
1677
                    by inversion
                                                                                                                            3. QED by the induction hypothesis (2)
1678
                                                                                                                                                                                                                                1733
                3. QED by the induction hypothesis (2)
                                                                                                                         CASE E^{\bullet} = op^2 v_0 E^{\bullet}_1:
1679
                                                                                                                                                                                                                                1734
              CASE E^{\bullet} = op^1 E^{\bullet}_0:
                                                                                                                            1. E^{\bullet}[e] = op^2 v_0 E^{\bullet}_1[e]
1680
                                                                                                                                                                                                                                1735
                1. E^{\bullet}[e] = op^1 E^{\bullet}_{0}[e]
1681
                                                                                                                            2. \vdash_{\mathsf{H}} E^{\bullet}_{1}[e]
                                                                                                                                                                                                                                1736
                2. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e] : \tau_{0}
1682
                                                                                                                                by inversion
                                                                                                                                                                                                                                1737
                    by inversion
1683
                                                                                                                            3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                                1738
                3. QED by the induction hypothesis (2)
                                                                                                                      П
1684
                                                                                                                                                                                                                                1739
              CASE E^{\bullet} = op^2 E^{\bullet}_0 e_1:
                                                                                                                                                                                                                                1740
1685
                1. E^{\bullet}[e] = op^2 E^{\bullet}_{0}[e] e_1
                                                                                                                      Lemma 2.19 : H boundary hole typing
1686
                                                                                                                                                                                                                                1741
                2. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e] : \tau_{0}
                                                                                                                        • If \vdash_{\mathsf{H}} E[\mathsf{dyn} \ \tau \ e] : \tau' then the derivation contains a sub-term
1687
                                                                                                                            \vdash_{\mathsf{H}} \mathsf{dyn} \; \tau \; e : \tau
                    by inversion
1688
                                                                                                                        • If \vdash_{\mathsf{H}} E[\mathsf{dyn} \ \tau \ e] then the derivation contains a sub-term
                                                                                                                                                                                                                                1743
                3. QED by the induction hypothesis (2)
                                                                                                                            \vdash_{\mathsf{H}} \mathsf{dyn} \; \tau \; e : \tau
              CASE E^{\bullet} = op^2 v_0 E^{\bullet}_1:
                                                                                                                                                                                                                                1745
                                                                                                                        • If \vdash_{\mathsf{H}} E[\mathsf{stat} \ \tau \ e] : \tau' then the derivation contains a sub-term
1690
                1. E^{\bullet}[e] = op^2 v_0 E^{\bullet}_1[e]
1691
                                                                                                                            \vdash_{\mathsf{H}} stat \tau e
                2. \vdash_{\mathsf{H}} E^{\bullet}_{1}[e] : \tau_{1}
1692
                                                                                                                        • If \vdash_{\mathsf{H}} E[\mathsf{stat}\ \tau\ e] then the derivation contains a sub-term
                                                                                                                                                                                                                                1747
                    by inversion
1693
                                                                                                                                                                                                                                1748
                                                                                                                            \vdash_{\mathsf{H}} stat \tau e
                3. QED by the induction hypothesis (2)
1694
                                                                                                                                                                                                                                1749
                                                                                                                      Proof:
1695
                                                                                                                         By the following four lemmas: static dyn hole typing,
                                                                                                                                                                                                                                1750
          Lemma 2.18: H dynamic hole typing
                                                                                                                         dynamic dyn hole typing, static stat hole typing, and
1696
                                                                                                                                                                                                                                1751
          If \vdash_{\mathsf{H}} E^{\bullet}[e] then the derivation contains a sub-term \vdash_{\mathsf{H}} e
1697
                                                                                                                         dynamic stat hole typing.
                                                                                                                                                                                                                                1752
          Proof:
1698
                                                                                                                                                                                                                                1753
             By induction on the structure of E^{\bullet}.
1699
                                                                                                                                                                                                                                1754
                                                                                                                      Lemma 2.20: H static dyn hole typing
             CASE E^{\bullet} = []:
1700
                                                                                                                                                                                                                                1755
                                                                                                                      If \vdash_{\mathsf{L}} E[\mathsf{dyn} \ \tau \ e] : \tau' then the derivation contains a sub-term
                1. QED E^{\bullet}[e] = e
1701
                                                                                                                       \vdash_{\mathsf{H}} \mathsf{dyn} \ \tau \ e : \tau.
                                                                                                                                                                                                                                1756
              CASE E^{\bullet} = E^{\bullet}_{0} e_{1}:
1702
                                                                                                                      Proof:
                                                                                                                                                                                                                                1757
                1. E^{\bullet}[e] = E^{\bullet}_{0}[e] e_{1}
1703
                                                                                                                                                                                                                                1758
                                                                                                                         By induction on the structure of E.
1704
                                                                                                                                                                                                                                1759
1705
```

1761	case $E \in E^{\bullet}$ :	By induction on the structure of <i>E</i> .	18
1762	1. $\vdash_{H} dyn \ \tau \ e : \tau''$	CASE $E \in E^{\bullet}$ :	18
1763	by static hole typing	1. Contradiction by $\vdash_{H} E[dyn \ \tau \ e]$	18
1764	2. $\vdash_{H} dyn \ \tau \ e : \tau$	<b>CASE</b> $E = E_0 e_1$ :	18
1765	by inversion (1)	1. $E[dyn \ \tau \ e] = E_0[dyn \ \tau \ e] \ e_1$	182
1766	3. QED	2. $\vdash_{H} E_0[dyn \ \tau \ e]$	182
1767	<b>CASE</b> $E = E_0 e_1$ :	by inversion	182
1768	1. $E[dyn \ \tau \ e] = E_0[dyn \ \tau \ e] \ e_1$	3. QED by the induction hypothesis (2)	182
1769	2. $\vdash_{H} E_0[dyn \ \tau \ e] : \tau_0$	CASE $E = v_0 E_1$ :	182
1770	by inversion	1. $E[\operatorname{dyn} \tau \ e] = v_0 E_1[\operatorname{dyn} \tau \ e]$	182
1771	3. QED by the induction hypothesis (2)	2. $\vdash_H E_1[dyn\  au\ e]$	182
1772	$\mathbf{CASE} \ E = v_0 \ E_1 :$	by inversion	182
1773	1. $E[\operatorname{dyn} \tau \ e] = v_0 \ E_1[\operatorname{dyn} \tau \ e]$	3. QED by the induction hypothesis (2)	182
1774	2. $\vdash_{H} E_1[dyn\ \tau\ e]: \tau_1$	CASE $E = \langle E_0, e_1 \rangle$ :	182
1775	by inversion	1. $E[\operatorname{dyn} \tau \ e] = \langle E_0[\operatorname{dyn} \tau \ e], e_1 \rangle$	183
1776	3. QED by the induction hypothesis (2)	2. $\vdash_H E_0[dyn\ \tau\ e]$	183
1777	CASE $E = \langle E_0, e_1 \rangle$ :	by inversion	183
1778	1. $E[\operatorname{dyn} \tau \ e] = \langle E_0[\operatorname{dyn} \tau \ e], e_1 \rangle$	3. QED by the induction hypothesis (2)	183
1779	2. $\vdash_{H} E_0[dyn \ \tau \ e] : \tau_0$	$\mathbf{CASE} \ E = \langle v_0, E_1 \rangle :$	183
1780	by inversion	1. $E[\operatorname{dyn} \tau \ e] = \langle v_0, E_1[\operatorname{dyn} \tau \ e] \rangle$	183
1781	3. QED by the induction hypothesis (2)	2. $\vdash_H E_1[dyn\ \tau\ e]$	183
1782	CASE $E = \langle v_0, E_1 \rangle$ :	by inversion	183
1783	1. $E[\operatorname{dyn} \tau \ e] = \langle v_0, E_1[\operatorname{dyn} \tau \ e] \rangle$	3. QED by the induction hypothesis (2)	183
1784	2. $\vdash_{H} E_1[dyn\ \tau\ e]: \tau_1$	$\mathbf{CASE} \ E = op^1 E_0:$	183
1785	by inversion	1. $E[\operatorname{dyn} \tau \ e] = op^1 E_0[\operatorname{dyn} \tau \ e]$	184
1786	3. QED by the induction hypothesis (2)	2. $\vdash_H E_0[dyn\ \tau\ e]$	184
1787	$\mathbf{CASE} \ E = op^1 E_0 :$	by inversion	184
1788	1. $E[\operatorname{dyn} \tau \ e] = op^1 E_0[\operatorname{dyn} \tau \ e]$	3. QED by the induction hypothesis (2)	184
1789	2. $\vdash_{H} E_0[dyn\ \tau\ e] : \tau_0$	<b>CASE</b> $E = op^2 E_0 e_1$ :	184
1790	by inversion	1. $E[\operatorname{dyn} \tau \ e] = op^2 E_0[\operatorname{dyn} \tau \ e] e_1$	184
1791	3. QED by the induction hypothesis (2)	2. $\vdash_{H} E_0[dyn\ \tau\ e]$	184
1792	$\mathbf{CASE} \ E = op^2 E_0 \ e_1 :$	by inversion	184
1793	1. $E[\operatorname{dyn} \tau \ e] = op^2 E_0[\operatorname{dyn} \tau \ e] e_1$	3. QED by the induction hypothesis (2)	184
1794	2. $\vdash_{H} E_0[dyn \ \tau \ e] : \tau_0$	$\mathbf{CASE} \ E = op^2 \ v_0 \ E_1 :$	184
1795	by inversion	1. $E[\operatorname{dyn} \tau \ e] = op^2 \ v_0 \ E_1[\operatorname{dyn} \tau \ e]$	185
1796	3. QED by the induction hypothesis (2)	2. $\vdash_{H} E_1[dyn \ \tau \ e]$	185
1797	$\mathbf{CASE} \ E = op^2 \ v_0 \ E_1 :$	by inversion	185
1798	1. $E[\operatorname{dyn} \tau e] = op^2 v_0 E_1[\operatorname{dyn} \tau e]$	3. QED by the induction hypothesis (2)	185
1799	2. $\vdash_{H} E_1[dyn \ \tau \ e] : \tau_1$	<b>CASE</b> $E = \operatorname{dyn} \tau E_0$ :	185
1800	by inversion	1. Contradiction by $\vdash_{H} E[dyn \ \tau \ e]$	185
1801	3. QED by the induction hypothesis (2)	<b>CASE</b> $E = \operatorname{stat} \tau_0 E_0$ :	185
1802	$\mathbf{case} \ E = dyn \ \tau_0 \ E_0 :$	1. $E[dyn \ \tau \ e] = stat \ \tau_0 \ E_0[dyn \ \tau \ e]$	185
1803	1. $E[dyn \ \tau \ e] = dyn \ \tau_0 \ E_0[dyn \ \tau \ e]$	2. $\vdash_H E_0[dyn\ \tau\ e]: \tau_0$	185
1804	2. $\vdash_{H} E_0[dyn \ \tau \ e]$	by inversion	185
1805	by inversion	3. QED by static dyn hole typing (2)	180
1806	3. QED by dynamic dyn hole typing (2)	_	180
1807	<b>CASE</b> $E = \text{stat } \tau_0 E_0$ :	Lemma 2.22 : H static stat hole typing	180
1808	1. Contradiction by $\vdash_{H} E[dyn \ \tau \ e] : \tau'$	If $\vdash_H E[\text{stat } \tau \ e] : \tau'$ then the derivation contains a sub-term	186 186
1809		$\vdash_{H} stat \ \tau \ e.$	18
1810	Lemma 2.21 : H dynamic dyn hole typing	Proof:	
1811	If $\vdash_H E[dyn \ \tau \ e]$ then the derivation contains a sub-term	By induction on the structure of $E$ .	18
1812 1813	$\vdash_{H} dyn \ \tau \ e : \tau.$	<b>CASE</b> $E \in E^{\bullet}$ :  1. Controduction by $E[\text{ctot } \sigma, a] \cdot \sigma'$	18 18
1814	Proof:	1. Contradiction by $\vdash_{H} E[stat \ \tau \ e] : \tau'$	18
1815	17		18

1871	<b>CASE</b> $E = E_0 e_1$ :	2. $\vdash_H E_0[\operatorname{stat} \tau \ e]$	1926
1872	1. $E[\operatorname{stat} \tau \ e] = E_0[\operatorname{stat} \tau \ e] \ e_1$	by inversion	1927
1873	2. $\vdash_{H} E_0[stat\ \tau\ e]: \tau_0$	3. QED by the induction hypothesis (2)	1928
1874	by inversion	CASE $E = v_0 E_1$ :	1929
1875	3. QED by the induction hypothesis (2)	1. $E[\operatorname{stat} \tau e] = v_0 E_1[\operatorname{stat} \tau e]$	1930
1876	CASE $E = v_0 E_1$ :	2. $\vdash_{H} E_1[stat \ \tau \ e]$	1931
1877	1. $E[\operatorname{stat} \tau \ e] = v_0 E_1[\operatorname{stat} \tau \ e]$	by inversion	1932
1878	2. $\vdash_{H} E_1[stat\ \tau\ e]: \tau_1$	3. QED by the induction hypothesis (2)	1933
1879	by inversion	<b>CASE</b> $E = \langle E_0, e_1 \rangle$ :	1934
1880	3. QED by the induction hypothesis (2)	1. $E[\operatorname{stat} \tau e] = \langle E_0[\operatorname{stat} \tau e], e_1 \rangle$	1935
1881	<b>CASE</b> $E = \langle E_0, e_1 \rangle$ :	2. $\vdash_{H} E_0[stat \ \tau \ e]$	1936
1882	1. $E[\operatorname{stat} \tau \ e] = \langle E_0[\operatorname{stat} \tau \ e], e_1 \rangle$	by inversion	1937
1883	2. $\vdash_{H} E_0[stat\ \tau\ e]: \tau_0$	3. QED by the induction hypothesis (2)	1938
1884	by inversion	CASE $E = \langle v_0, E_1 \rangle$ :	1939
1885	3. QED by the induction hypothesis (2)	1. $E[\operatorname{stat} \tau e] = \langle v_0, E_1[\operatorname{stat} \tau e] \rangle$	1940
1886	CASE $E = \langle v_0, E_1 \rangle$ :	2. $\vdash_{H} E_1[stat \ \tau \ e]$	1941
1887	1. $E[\operatorname{stat} \tau \ e] = \langle v_0, E_1[\operatorname{stat} \tau \ e] \rangle$	by inversion	1942
1888	2. $\vdash_{H} E_1[stat\ \tau\ e]: \tau_1$	3. QED by the induction hypothesis (2)	1943
1889	by inversion	CASE $E = op^1 E_0$ :	1944
1890	3. QED by the induction hypothesis (2)	1. $E[\operatorname{stat} \tau e] = op^1 E_0[\operatorname{stat} \tau e]$	1945
1891	CASE $E = op^1 E_0$ :	2. $\vdash_{H} E_0[stat \ \tau \ e]$	1946
1892	1. $E[\operatorname{stat} \tau e] = op^1 E_0[\operatorname{stat} \tau e]$	by inversion	1947
1893	2. $\vdash_{H} E_0[stat\ \tau\ e]: \tau_0$	3. QED by the induction hypothesis (2)	1948
1894	by inversion	<b>CASE</b> $E = op^2 E_0 e_1$ :	1949
1895	3. QED by the induction hypothesis (2)	1. $E[\operatorname{stat} \tau e] = op^2 E_0[\operatorname{stat} \tau e] e_1$	1950
1896	<b>CASE</b> $E = op^2 E_0 e_1$ :	2. $\vdash_{H} E_0[stat \ \tau \ e]$	1951
1897	1. $E[\operatorname{stat} \tau \ e] = op^2 E_0[\operatorname{stat} \tau \ e] e_1$	by inversion	1952
1898	2. $\vdash_{H} E_0[stat \ \tau \ e] : \tau_0$	3. QED by the induction hypothesis (2)	1953
1899	by inversion	$\mathbf{CASE} \ E = op^2 \ v_0 \ E_1 :$	1954
1900	3. QED by the induction hypothesis (2)	1. $E[\operatorname{stat} \tau \ e] = op^2 \ v_0 \ E_1[\operatorname{stat} \tau \ e]$	1955
1901	$\mathbf{CASE} \ E = op^2 \ v_0 \ E_1 :$	2. $\vdash_{H} E_1[stat \ \tau \ e]$	1956
1902	1. $E[\operatorname{stat} \tau \ e] = op^2 \ v_0 \ E_1[\operatorname{stat} \tau \ e]$	by inversion	1957
1903	2. $\vdash_{H} E_1[stat\ \tau\ e] : \tau_1$	3. QED by the induction hypothesis (2)	1958
1904	by inversion	CASE $E = \text{dyn } \tau E_0$ :	1959
1905	3. QED by the induction hypothesis (2)	1. Contradiction by $\vdash_H E[\text{stat } \tau \ e]$	1960
1906	CASE $E = \text{dyn } \tau_0 E_0$ :	<b>CASE</b> $E = \text{stat } \tau_0 E_0$ :	1961
1907	1. $E[\operatorname{stat} \tau \ e] = \operatorname{dyn} \tau_0 E_0[\operatorname{stat} \tau \ e]$	1. $E[\operatorname{stat} \tau \ e] = \operatorname{stat} \tau_0 \ E_0[\operatorname{stat} \tau \ e]$	1962
1908	2. $\vdash_{H} E_0[stat \ \tau \ e]$	2. $\vdash_{H} E_0[stat\ \tau\ e] : \tau_0$	1963
1909	by inversion	by inversion	1964
1910	3. QED by dynamic stat hole typing (2)	3. QED by static stat hole typing (2)	1965
1911	CASE $E = \text{stat } \tau_0 E_0$ :		1966
1912	1. Contradiction by $\vdash_{H} E[stat \ \tau \ e] : \tau'$	<b>Lemma 2.24</b> : H static boundary-free hole substitution	1967
1913		If $\vdash_{H} E^{\bullet}[e] : \tau$ and the derivation contains a sub-term $\vdash_{H} e : \tau'$	1968
1914	<b>Lemma 2.23</b> : H dynamic stat hole typing	and $\vdash_{H} e' : \tau'$ then $\vdash_{H} E^{\bullet}[e'] : \tau$ .	1969
1915	If $\vdash_{H} E[stat \ \tau \ e]$ then the derivation contains a sub-term	Proof:	1970
1916	$\vdash_{H} stat \ \tau \ e.$	By induction on the structure of $E^{\bullet}$	1971
1917	Proof:	CASE $E^{\bullet} = []$ :	1972
1918	By induction on the structure of <i>E</i> .	1. $E^{\bullet}[e] = e$	1973
1919	CASE $E \in E^{\bullet}$ :	$\wedge E^{\bullet}[e'] = e'$	1974
1920	1. QED by dynamic hole typing	2. <sub>H</sub> e: τ	1975
1921	CASE $E = E_0 e_1$ :	by (1)	1976
1922	1. $E[\operatorname{stat} \tau \ e] = E_0[\operatorname{stat} \tau \ e] \ e_1$	3. $\tau' = \tau$	1977
1923		$4.  \vdash_{H} e' : \tau$	1978
1924			1979

```
1981
                    5. QED by (1, 4)
                                                                                                                                                                                                                                                                                            2036
                                                                                                                                                             3. \vdash_{\mathsf{H}} v_0 : \tau_0
1982
                 CASE E^{\bullet} = E^{\bullet}_{0} e_{1}:
                                                                                                                                                                   \wedge \vdash_{\mathsf{H}} E^{\bullet}_{1}[e] : \tau_{1}
                                                                                                                                                                                                                                                                                            2037
1983
                    1. E^{\bullet}[e] = E^{\bullet}_{0}[e] e_{1}
                                                                                                                                                                   by inversion
                                                                                                                                                                                                                                                                                            2038
                           \wedge E^{\bullet}[e'] = E^{\bullet}_{0}[e'] e_{1}
                                                                                                                                                             4. \vdash_{\mathsf{H}} E^{\bullet}_{1}[e'] : \tau_{1}
1984
                                                                                                                                                                                                                                                                                            2039
1985
                    2. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e] e_{1} : \tau
                                                                                                                                                                   by the induction hypothesis (3)
                                                                                                                                                                                                                                                                                            2040
                    3. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e] : \tau_{0}
                                                                                                                                                             5. \vdash_{\mathsf{H}} \langle v_0, E^{\bullet}_1[e'] \rangle : \tau
                                                                                                                                                                                                                                                                                            2041
1986
                                                                                                                                                                   by (2, 3, 4)
1987
                          \land \vdash_{\mathsf{H}} e_1 : \tau_1
                                                                                                                                                                                                                                                                                            2042
                                                                                                                                                             6. QED by (1, 5)
1988
                          by inversion
                                                                                                                                                                                                                                                                                            2043
                    4. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e'] : \tau_{0}
                                                                                                                                                          CASE E^{\bullet} = op^2 E^{\bullet}_0 e_1:
                                                                                                                                                                                                                                                                                            2044
                          by the induction hypothesis (3)
                                                                                                                                                             1. E^{\bullet}[e] = op^2 E^{\bullet}_{0}[e] e_1
1990
                                                                                                                                                                                                                                                                                            2045
                     5. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e'] e_{1} : \tau
                                                                                                                                                                   \wedge E^{\bullet}[e'] = op^2 E^{\bullet}_{0}[e'] e_1
                                                                                                                                                                                                                                                                                            2046
                                                                                                                                                             2. \vdash_{\mathsf{H}} op^2 E^{\bullet}_{0}[e] e_1 : \tau
                          by (2, 3, 4)
1992
                                                                                                                                                                                                                                                                                            2047
1993
                    6. QED by (1, 5)
                                                                                                                                                             3. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e] : \tau_{0}
                                                                                                                                                                                                                                                                                            2048
                  CASE E^{\bullet} = v_0 E^{\bullet}_1:
1994
                                                                                                                                                                   \land \vdash_{\mathsf{H}} e_1 : \tau_1
                                                                                                                                                                                                                                                                                            2049
1995
                    1. E^{\bullet}[e] = v_0 E^{\bullet}_{1}[e]
                                                                                                                                                                   by inversion
                                                                                                                                                                                                                                                                                            2050
1996
                           \wedge E^{\bullet}[e'] = v_0 E^{\bullet}_{1}[e']
                                                                                                                                                             4. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e'] : \tau_{0}
                                                                                                                                                                                                                                                                                            2051
1997
                                                                                                                                                                   by the induction hypothesis (3)
                    2. \vdash_{\mathsf{H}} v_0 E^{\bullet}_{1}[e] : \tau
                                                                                                                                                                                                                                                                                            2052
                    3. \vdash_{\mathsf{H}} \upsilon_0 : \tau_0
1998
                                                                                                                                                             5. \vdash_{\mathsf{H}} op^2 E^{\bullet}_{0}[e'] e_1 : \tau
                                                                                                                                                                                                                                                                                            2053
                           \wedge \vdash_{\mathsf{H}} E^{\bullet}_{1}[e] : \tau_{1}
                                                                                                                                                                   by (2, 3, 4)
1999
                                                                                                                                                                                                                                                                                            2054
2000
                          by inversion
                                                                                                                                                             6. QED by (1, 5)
                                                                                                                                                                                                                                                                                            2055
2001
                    4. \vdash_{\mathsf{H}} E^{\bullet}_{1}[e'] : \tau_{1}
                                                                                                                                                          CASE E^{\bullet} = op^2 v_0 E^{\bullet}_1:
                                                                                                                                                                                                                                                                                            2056
                          by the induction hypothesis (3)
                                                                                                                                                             1. E^{\bullet}[e] = op^2 v_0 E^{\bullet}[e]
                                                                                                                                                                                                                                                                                            2057
                    5. \vdash_{\mathsf{H}} v_0 E^{\bullet}_1[e'] : \tau
                                                                                                                                                                   \wedge E^{\bullet}[e'] = op^2 v_0 E^{\bullet}_{1}[e']
                                                                                                                                                                                                                                                                                            2058
2003
                                                                                                                                                             2. \vdash_{\mathsf{H}} op^2 v_0 E^{\bullet}_{1}[e] : \tau
                          by (2, 3, 4)
2004
                                                                                                                                                                                                                                                                                            2059
                    6. QED by (1, 5)
2005
                                                                                                                                                             3. \vdash_{\mathsf{H}} v_0 : \tau_0
                                                                                                                                                                                                                                                                                            2060
                  CASE E^{\bullet} = op^1 E^{\bullet}_0:
                                                                                                                                                                   \wedge \vdash_{\mathsf{H}} E^{\bullet}_{1}[e] : \tau_{1}
                                                                                                                                                                                                                                                                                            2061
2007
                    1. E^{\bullet}[e] = op^1 E^{\bullet}_{0}[e]
                                                                                                                                                                   by inversion
                                                                                                                                                                                                                                                                                            2062
                          \wedge E^{\bullet}[e'] = op^1 E^{\bullet}_{0}[e']
2008
                                                                                                                                                             4. \vdash_{\mathsf{H}} E^{\bullet}_{1}[e'] : \tau_{1}
                                                                                                                                                                                                                                                                                            2063
                     2. \vdash_{\Box} op^1 E^{\bullet}_{0}[e] : \tau
                                                                                                                                                                   by the induction hypothesis (3)
2009
                                                                                                                                                                                                                                                                                            2064
2010
                    3. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e] : \tau_{0}
                                                                                                                                                             5. \vdash_{\mathsf{H}} op^2 v_0 E^{\bullet}_1[e'] : \tau
                                                                                                                                                                                                                                                                                            2065
2011
                          by inversion
                                                                                                                                                                   by (2, 3, 4)
                                                                                                                                                                                                                                                                                            2066
2012
                    4. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e'] : \tau_{0}
                                                                                                                                                             6. QED by (1, 5)
                                                                                                                                                                                                                                                                                            2067
2013
                          by the induction hypothesis (3)
                                                                                                                                                                                                                                                                                            2068
2014
                    5. \vdash_{\mathsf{H}} op^1 E^{\bullet}_{0}[e'] : \tau
                                                                                                                                                                                                                                                                                            2069
                                                                                                                                                     Lemma 2.25 : H dynamic hole substitution
2015
                          by (2, 3, 4)
                                                                                                                                                                                                                                                                                            2070
                                                                                                                                                      If \vdash_{\mathsf{H}} E^{\bullet}[e] and \vdash_{\mathsf{H}} e' then \vdash_{\mathsf{H}} E^{\bullet}[e']
                    6. QED by (1, 5)
2016
                                                                                                                                                                                                                                                                                            2071
                                                                                                                                                      Proof:
2017
                  CASE E^{\bullet} = \langle E^{\bullet}_{0}, e_{1} \rangle:
                                                                                                                                                                                                                                                                                            2072
                                                                                                                                                          By induction on the structure of E^{\bullet}
2018
                    1. E^{\bullet}[e] = \langle E^{\bullet}_{0}[e], e_{1} \rangle
                                                                                                                                                                                                                                                                                            2073
                                                                                                                                                          CASE E^{\bullet} = []:
                           \wedge E^{\bullet}[e'] = \langle E^{\bullet}_{0}[e'], e_{1} \rangle
2019
                                                                                                                                                                                                                                                                                            2074
                                                                                                                                                             1. QED E^{\bullet}[e'] = e'
                    2. \vdash_{\mathsf{H}} \langle E^{\bullet}{}_{0}[e], e_{1} \rangle : \tau
                                                                                                                                                                                                                                                                                            2075
2020
                                                                                                                                                          CASE E^{\bullet} = \langle E^{\bullet}_{0}, e_{1} \rangle:
                    3. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e] : \tau_{0}
2021
                                                                                                                                                             1. E^{\bullet}[e] = \langle E^{\bullet}_{0}[e], e_{1} \rangle
                                                                                                                                                                                                                                                                                            2076
2022
                           \land \vdash_{\mathsf{H}} e_1 : \tau_1
                                                                                                                                                                                                                                                                                            2077
                                                                                                                                                                   \wedge E^{\bullet}[e'] = \langle E^{\bullet}_{0}[e'], e_{1} \rangle
2023
                          by inversion
                                                                                                                                                                                                                                                                                            2078
                                                                                                                                                             2. \vdash_{\mathsf{H}} \langle E^{\bullet}_{0}[e], e_{1} \rangle
2024
                    4. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e'] : \tau_{0}
                                                                                                                                                                                                                                                                                            2079
                                                                                                                                                             3. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e]
                          by the induction hypothesis (3)
2025
                                                                                                                                                                                                                                                                                            2080
                                                                                                                                                                   \land \vdash_{\mathsf{H}} e_1
                     5. \vdash_{\sqcup} \langle E^{\bullet}_{0}[e'], e_{1} \rangle : \tau
2026
                                                                                                                                                                                                                                                                                            2081
                                                                                                                                                                   by inversion
                          by (2, 3, 4)
                                                                                                                                                                                                                                                                                            2082
                                                                                                                                                             4. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e']
                    6. QED by (1, 5)
2028
                                                                                                                                                                                                                                                                                            2083
                                                                                                                                                                   by the induction hypothesis (3)
2029
                  CASE E^{\bullet} = \langle v_0, E^{\bullet}_1 \rangle:
                                                                                                                                                                                                                                                                                            2084
                                                                                                                                                             5. \vdash_{\mathsf{H}} \langle E^{\bullet}_{0}[e'], e_{1} \rangle
                    1. E^{\bullet}[e] = \langle v_0, E^{\bullet}_1[e] \rangle
2030
                                                                                                                                                                                                                                                                                            2085
                                                                                                                                                                   by (3, 4)
2031
                           \wedge E^{\bullet}[e'] = \langle v_0, E^{\bullet}_1[e'] \rangle
                                                                                                                                                                                                                                                                                            2086
                                                                                                                                                             6. QED by (1, 5)
2032
                    2. \vdash_{\mathsf{H}} \langle v_0, E^{\bullet}_1[e] \rangle : \tau
                                                                                                                                                                                                                                                                                            2087
                                                                                                                                                          CASE E^{\bullet} = \langle v_0, E^{\bullet}_1 \rangle:
2033
                                                                                                                                                                                                                                                                                            2088
2034
                                                                                                                                                                                                                                                                                            2089
2035
                                                                                                                                                                                                                                                                                            2090
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2091
                 1. E^{\bullet}[e] = \langle v_0, E^{\bullet}_1[e] \rangle
                                                                                                                                      4. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e']
                                                                                                                                                                                                                                                   2146
                       \wedge E^{\bullet}[e'] = \langle v_0, E^{\bullet}_1[e'] \rangle
                                                                                                                                           by the induction hypothesis (3)
2092
                                                                                                                                                                                                                                                   2147
                 2. \vdash_{\mathsf{H}} \langle v_0, E^{\bullet}_1[e] \rangle
2093
                                                                                                                                      5. \vdash_{\mathsf{H}} op^2 E^{\bullet}{}_0[e'] e_1
                                                                                                                                                                                                                                                   2148
                                                                                                                                           by (3, 4)
2094
                  3. \vdash_{\mathsf{H}} v_0
                                                                                                                                                                                                                                                   2149
                      \wedge \vdash_{\mathsf{H}} E^{\bullet}_{1}[e]
                                                                                                                                      6. QED by (1, 5)
                                                                                                                                                                                                                                                   2150
                                                                                                                                    CASE E^{\bullet} = op^2 v_0 E^{\bullet}_1:
                      by inversion
                                                                                                                                                                                                                                                   2151
2096
                                                                                                                                      1. E^{\bullet}[e] = op^2 v_0 E^{\bullet}[e]
2097
                  4. \vdash_{\mathsf{H}} E^{\bullet}_{1}[e']
                                                                                                                                                                                                                                                   2152
                                                                                                                                           \wedge E^{\bullet}[e'] = op^2 v_0 E^{\bullet}_{1}[e']
2098
                      by the induction hypothesis (3)
                                                                                                                                                                                                                                                   2153
                 5. \vdash_{\mathsf{H}} \langle v_0, E^{\bullet}_1[e'] \rangle
                                                                                                                                      2. \vdash_{\mathsf{H}} op^2 v_0 E^{\bullet}_{1}[e]
                                                                                                                                                                                                                                                   2154
2100
                      by (3, 4)
                                                                                                                                      3. \vdash_{\mathsf{H}} v_0
                                                                                                                                                                                                                                                   2155
2101
                 6. QED by (1, 5)
                                                                                                                                           \wedge \vdash_{\mathsf{H}} E^{\bullet}_{1}[e]
                                                                                                                                                                                                                                                   2156
               CASE E^{\bullet} = E^{\bullet}_{0} e_{1}:
                                                                                                                                           by inversion
2102
                                                                                                                                                                                                                                                   2157
                 1. E^{\bullet}[e] = E^{\bullet}_{0}[e] e_{1}
2103
                                                                                                                                      4. \vdash_{\mathsf{H}} E^{\bullet}_{1}[e']
                                                                                                                                                                                                                                                   2158
                                                                                                                                           by the induction hypothesis (3)
2104
                      \wedge E^{\bullet}[e'] = E^{\bullet}_{0}[e'] e_{1}
                                                                                                                                                                                                                                                   2159
2105
                                                                                                                                      5. \vdash_{\mathsf{H}} op^2 v_0 E^{\bullet}_{1}[e']
                  2. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e] e_{1}
                                                                                                                                                                                                                                                   2160
2106
                 3. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e]
                                                                                                                                           by (3, 4)
                                                                                                                                                                                                                                                   2161
                                                                                                                                      6. QED by (1, 5)
2107
                                                                                                                                                                                                                                                   2162
                       \wedge \vdash_{\mathsf{H}} e_1
2108
                      by inversion
                                                                                                                                                                                                                                                   2163
2109
                  4. \vdash_{\mathsf{H}} E^{\bullet}_{0}[e']
                                                                                                                                                                                                                                                   2164
                                                                                                                                Lemma 2.26: H hole substitution
2110
                      by the induction hypothesis (3)
                                                                                                                                  • If \vdash_{\mathsf{H}} E[e] and the derivation contains a sub-term \vdash_{\mathsf{H}} e : \tau'
                                                                                                                                                                                                                                                   2165
2111
                  5. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e'] e_{1}
                                                                                                                                      and \vdash_{\mathsf{H}} e' : \tau' then \vdash_{\mathsf{H}} E[e'].
                                                                                                                                                                                                                                                   2166
                      by (3, 4)
2112
                                                                                                                                  • If \vdash_{\mathsf{H}} E[e] and the derivation contains a sub-term \vdash_{\mathsf{H}} e and
                 6. QED by (1, 5)
2113
                                                                                                                                                                                                                                                   2168
                                                                                                                                      \vdash_{\mathsf{H}} e' \text{ then } \vdash_{\mathsf{H}} E[e'].
               CASE E^{\bullet} = v_0 E^{\bullet}_1:
2114
                                                                                                                                                                                                                                                   2169
                                                                                                                                  • If \vdash_{\vdash} E[e] : \tau and the derivation contains a sub-term \vdash_{\vdash} e : \tau
                  1. E^{\bullet}[e] = v_0 E^{\bullet}_{1}[e]
                                                                                                                                                                                                                                                   2170
2115
                                                                                                                                      \tau' and \vdash_{\mathsf{H}} e' : \tau' then \vdash_{\mathsf{H}} E[e'] : \tau.
                      \wedge E^{\bullet}[e'] = v_0 E^{\bullet}_{1}[e']
                                                                                                                                                                                                                                                   2171
                                                                                                                                  • If \vdash_{\mathsf{H}} E[e] : \tau and the derivation contains a sub-term \vdash_{\mathsf{H}} e
2117
                  2. \vdash_{\mathsf{H}} v_0 E^{\bullet}_1[e]
                                                                                                                                                                                                                                                   2172
                                                                                                                                      and \vdash_{\mathsf{H}} e' then \vdash_{\mathsf{H}} E[e'] : \tau.
2118
                 3. \vdash_{\mathsf{H}} v_0
                                                                                                                                                                                                                                                   2173
                                                                                                                                Proof:
                       \wedge \vdash_{\mathsf{H}} E^{\bullet}_{1}[e]
                                                                                                                                                                                                                                                   2174
2119
                                                                                                                                   By the following four lemmas: dynamic context static hole
2120
                      by inversion
                                                                                                                                    substitution, dynamic context dynamic hole substitution,
                                                                                                                                                                                                                                                   2175
2121
                  4. \vdash_{\perp} E^{\bullet}_{1}[e']
                                                                                                                                    static context static hole substitution, and static context
                                                                                                                                                                                                                                                   2176
                      by the induction hypothesis (3)
                                                                                                                                    dynamic hole substitution.
                                                                                                                                                                                                                                                   2177
                  5. \vdash_{\mathsf{H}} \upsilon_0 E^{\bullet}_1[e']
2123
                                                                                                                                                                                                                                                   2178
                                                                                                                                                                                                                                                   2179
2124
                      by (3, 4)
                                                                                                                                Lemma 2.27 : H dynamic context static hole substitution
2125
                 6. QED by (1, 5)
                                                                                                                                                                                                                                                   2180
                                                                                                                                If \vdash_{\mathsf{H}} E[e] and contains \vdash_{\mathsf{H}} e : \tau', and furthermore \vdash_{\mathsf{H}} e' : \tau',
               CASE E^{\bullet} = op^1 E^{\bullet}_0:
2126
                                                                                                                                                                                                                                                   2181
                                                                                                                                 then \vdash_{\mathsf{H}} E[e']
2127
                 1. E^{\bullet}[e] = op^1 E^{\bullet}_{0}[e]
                                                                                                                                Proof:
                                                                                                                                                                                                                                                   2182
2128
                      \wedge E^{\bullet}[e'] = op^1 E^{\bullet}_0[e']
                                                                                                                                   By induction on the structure of E.
                                                                                                                                                                                                                                                   2183
                 2. \vdash_{\mathsf{H}} op^1 E^{\bullet}_{0}[e]
2129
                                                                                                                                   case E \in E^{\bullet}:
                                                                                                                                                                                                                                                   2184
                 3. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e]
                                                                                                                                      1. Contradiction by \vdash_{\!\!\!\perp} E[e]
                                                                                                                                                                                                                                                   2185
2130
                      by inversion
2131
                                                                                                                                    CASE E = E_0 e_1:
                                                                                                                                                                                                                                                   2186
2132
                  4. \vdash_{\mathsf{H}} E^{\bullet}_{0}[e']
                                                                                                                                                                                                                                                   2187
                                                                                                                                      1. E[e] = E_0[e] e_1
                      by the induction hypothesis (3)
2133
                                                                                                                                      2. \vdash_{\mathsf{H}} E_0[e]
                                                                                                                                                                                                                                                   2188
2134
                  5. \vdash_{\mathsf{H}} op^1 E^{\bullet}_{0}[e']
                                                                                                                                           by inversion
                                                                                                                                                                                                                                                   2189
                      by (4)
2135
                                                                                                                                      3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                                                   2190
                 6. QED by (1, 5)
2136
                                                                                                                                    CASE E = v_0 E_1:
                                                                                                                                                                                                                                                   2191
               CASE E^{\bullet} = op^2 E^{\bullet}_0 e_1:
2137
                                                                                                                                      1. E[e] = v_0 E_1[e]
                                                                                                                                                                                                                                                   2192
                 1. E^{\bullet}[e] = op^2 E^{\bullet}_{0}[e] e_1
2138
                                                                                                                                      2. \vdash_{\mathsf{H}} E_1[e]
                                                                                                                                                                                                                                                   2193
                      \wedge E^{\bullet}[e'] = op^2 E^{\bullet}_{0}[e'] e_1
2139
                                                                                                                                           by inversion
                                                                                                                                                                                                                                                   2194
                  2. \vdash_{\mathsf{H}} op^2 E^{\bullet}{}_0[e] e_1
2140
                                                                                                                                      3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                                                   2195
                 3. \vdash_{\mathsf{H}} E^{\bullet}{}_{0}[e]
2141
                                                                                                                                    CASE E = \langle E_0, e_1 \rangle:
                                                                                                                                                                                                                                                   2196
2142
                      \wedge \vdash_{\vdash} e_1
                                                                                                                                      1. E[e] = \langle E_0[e], e_1 \rangle
                                                                                                                                                                                                                                                   2197
2143
                      by inversion
                                                                                                                                                                                                                                                   2198
2144
                                                                                                                                                                                                                                                   2199
2145
                                                                                                                                                                                                                                                   2200
                                                                                                                         20
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2201
                                                                                                   CASE E = \langle v_0, E_1 \rangle:
                                                                                                                                                                                       2256
             2. \vdash_{\mathsf{H}} E_0[e]
                 by inversion
                                                                                                     1. E[e] = \langle v_0, E_1[e] \rangle
2202
                                                                                                                                                                                       2257
2203
             3. QED by the induction hypothesis (2)
                                                                                                     2. \vdash_{\mathsf{H}} E_1[e]
                                                                                                                                                                                       2258
           CASE E = \langle v_0, E_1 \rangle:
                                                                                                         by inversion
2204
                                                                                                                                                                                       2259
             1. E[e] = \langle v_0, E_1[e] \rangle
                                                                                                     3. QED by the induction hypothesis (2)
                                                                                                                                                                                       2260
             E_1[e]
                                                                                                   CASE E = op^1 E_0:
                                                                                                                                                                                       2261
2206
                 by inversion
                                                                                                     1. E[e] = op^1 E_0[e]
2207
                                                                                                                                                                                       2262
             3. QED by the induction hypothesis (2)
2208
                                                                                                     2. \vdash_{\mathsf{H}} E_0[e]
                                                                                                                                                                                       2263
2209
           CASE E = op^1 E_0:
                                                                                                         by inversion
                                                                                                                                                                                       2264
             1. E[e] = op^1 E_0[e]
                                                                                                     3. QED by the induction hypothesis (2)
2210
                                                                                                                                                                                       2265
2211
             2. \vdash_{\mathsf{H}} E_0[e]
                                                                                                   CASE E = op^2 E_0 e_1:
                                                                                                                                                                                       2266
                                                                                                     1. E[e] = op^2 E_0[e] e_1
                 by inversion
2212
                                                                                                                                                                                       2267
2213
             3. QED by the induction hypothesis (2)
                                                                                                     2. \vdash_{\mathsf{H}} E_0[e]
                                                                                                                                                                                       2268
           CASE E = op^2 E_0 e_1:
2214
                                                                                                         by inversion
                                                                                                                                                                                       2269
2215
             1. E[e] = op^2 E_0[e] e_1
                                                                                                     3. QED by the induction hypothesis (2)
                                                                                                                                                                                       2270
2216
             2. \vdash_{\mathsf{H}} E_0[e]
                                                                                                   CASE E = op^2 v_0 E_1:
                                                                                                                                                                                       2271
                                                                                                     1. E[e] = op^2 v_0 E_1[e]
2217
                 by inversion
                                                                                                                                                                                       2272
2218
             3. QED by the induction hypothesis (2)
                                                                                                     E_1[e]
                                                                                                                                                                                       2273
           CASE E = op^2 v_0 E_1:
                                                                                                         by inversion
2219
                                                                                                                                                                                       2274
             1. E[e] = op^2 v_0 E_1[e]
                                                                                                     3. QED by the induction hypothesis (2)
                                                                                                                                                                                       2275
2221
             2. \vdash_{\mathsf{H}} E_1[e]
                                                                                                   CASE E = \text{dyn } \tau'' E_0:
                                                                                                                                                                                       2276
                 by inversion
                                                                                                     1. Contradiction by \vdash_{\mathsf{H}} E[e]
2222
                                                                                                                                                                                       2277
             3. QED by the induction hypothesis (2)
                                                                                                   CASE E = \text{stat } \tau_0 E_0:
                                                                                                                                                                                       2278
2223
           CASE E = \text{dyn } \tau^{\prime\prime} E_0:
                                                                                                     1. E[e] = \text{stat } \tau_0 E_0[e]
2224
                                                                                                                                                                                       2279
             1. Contradiction by \vdash_{\mathsf{H}} E[e]
                                                                                                     2. \vdash_{\mathsf{H}} E_0[e] : \tau_0
2225
                                                                                                                                                                                       2280
2226
           CASE E = \text{stat } \tau_0 E_0:
                                                                                                         by inversion
             1. E[e] = \text{stat } \tau_0 E_0[e]
                                                                                                     3. QED by static context dynamic hole substitution (2)
2227
                                                                                                                                                                                       2282
             2. \vdash_{\mathsf{H}} E_0[e] : \tau_0
                                                                                                                                                                                       2283
                 by inversion
2229
                                                                                                Lemma 2.29: H static context static hole substitution
                                                                                                                                                                                       2284
2230
             3. QED by static context static hole substitution (2)
                                                                                                                                                                                       2285
                                                                                                  If \vdash_{\mathsf{H}} E[e] : \tau and contains \vdash_{\mathsf{H}} e : \tau', and furthermore \vdash_{\mathsf{H}} e' : \tau',
2231
                                                                                                  then \vdash_{\mathsf{H}} E[e'] : \tau
                                                                                                                                                                                       2286
2232
                                                                                                Proof:
                                                                                                                                                                                       2287
        Lemma 2.28: H dynamic context dynamic hole substitution
2233
                                                                                                   By induction on the structure of E.
                                                                                                                                                                                       2288
         If \vdash_{\mathsf{H}} E[e] and contains \vdash_{\mathsf{H}} e, and furthermore \vdash_{\mathsf{H}} e', then
2234
                                                                                                   case E \in E^{\bullet}:
                                                                                                                                                                                       2289
          2235
        Proof:
                                                                                                     1. QED by static boundary-free hole substitution
                                                                                                                                                                                       2290
2236
           By induction on the structure of E.
                                                                                                   CASE E = E_0 e_1:
                                                                                                                                                                                       2291
2237
           CASE E \in E^{\bullet}:
                                                                                                     1. E[e] = E_0[e] e_1
                                                                                                                                                                                       2292
2238
             1. QED by dynamic boundary-free hole substitution
                                                                                                     2. \vdash_{\mathsf{H}} E_0[e] : \tau_0
                                                                                                                                                                                       2293
2239
           CASE E = E_0 e_1:
                                                                                                         by inversion
             1. E[e] = E_0[e] e_1
                                                                                                     3. QED by the induction hypothesis (2)
                                                                                                                                                                                       2295
2240
2241
             2. \vdash_{\mathsf{H}} E_0[e]
                                                                                                   CASE E = v_0 E_1:
                                                                                                                                                                                       2296
2242
                 by inversion
                                                                                                     1. E[e] = v_0 E_1[e]
                                                                                                                                                                                       2297
             3. QED by the induction hypothesis (2)
                                                                                                     2. \vdash_{\mathsf{H}} E_1[e] : \tau_1
                                                                                                                                                                                       2298
           CASE E = v_0 E_1:
                                                                                                         by inversion
2244
                                                                                                                                                                                       2299
2245
             1. E[e] = v_0 E_1[e]
                                                                                                     3. QED by the induction hypothesis (2)
                                                                                                                                                                                       2300
                                                                                                   CASE E = \langle E_0, e_1 \rangle:
2246
             2. \vdash_{\mathsf{H}} E_1[e]
                                                                                                                                                                                       2301
                 by inversion
                                                                                                     1. E[e] = \langle E_0[e], e_1 \rangle
                                                                                                                                                                                       2302
2248
             3. QED by the induction hypothesis (2)
                                                                                                     2. \vdash_{\mathsf{H}} E_0[e] : \tau_0
                                                                                                                                                                                       2303
2249
           CASE E = \langle E_0, e_1 \rangle:
                                                                                                         by inversion
                                                                                                                                                                                       2304
2250
             1. E[e] = \langle E_0[e], e_1 \rangle
                                                                                                     3. QED by the induction hypothesis (2)
                                                                                                                                                                                       2305
                                                                                                   CASE E = \langle v_0, E_1 \rangle:
2251
             2. \vdash_{\mathsf{H}} E_0[e]
                                                                                                                                                                                       2306
2252
                 by inversion
                                                                                                     1. E[e] = \langle v_0, E_1[e] \rangle
                                                                                                                                                                                       2307
2253
             3. QED by the induction hypothesis (2)
                                                                                                                                                                                       2308
2254
                                                                                                                                                                                       2309
2255
                                                                                                                                                                                       2310
```

```
CASE E = op^1 E_0:
2311
                  2. \vdash_{\mathsf{H}} E_1[e] : \tau_1
                                                                                                                                                                                                                                                     2366
                       by inversion
                                                                                                                                        1. E[e] = op^1 E_0[e]
2312
                                                                                                                                                                                                                                                     2367
                                                                                                                                       2. \vdash_{\mathsf{H}} E_0[e] : \tau_0
2313
                  3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                                                     2368
               CASE E = op^1 E_0:
2314
                                                                                                                                            by inversion
                                                                                                                                                                                                                                                     2369
2315
                  1. E[e] = op^1 E_0[e]
                                                                                                                                        3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                                                     2370
                  2. \vdash_{\mathsf{H}} E_0[e] : \tau_0
                                                                                                                                     CASE E = op^2 E_0 e_1:
                                                                                                                                                                                                                                                     2371
2316
                       by inversion
                                                                                                                                        1. E[e] = op^2 E_0[e] e_1
2317
                                                                                                                                                                                                                                                     2372
2318
                  3. QED by the induction hypothesis (2)
                                                                                                                                       2. \vdash_{\mathsf{H}} E_0[e] : \tau_0
                                                                                                                                                                                                                                                     2373
2319
               CASE E = op^2 E_0 e_1:
                                                                                                                                            by inversion
                                                                                                                                                                                                                                                     2374
                  1. E[e] = op^2 E_0[e] e_1
                                                                                                                                        3. QED by the induction hypothesis (2)
2320
                                                                                                                                                                                                                                                     2375
2321
                  2. \vdash_{\mathsf{H}} E_0[e] : \tau_0
                                                                                                                                     CASE E = op^2 v_0 E_1:
                                                                                                                                                                                                                                                     2376
                      by inversion
                                                                                                                                        1. E[e] = op^2 v_0 E_1[e]
2322
                                                                                                                                                                                                                                                     2377
                  3. QED by the induction hypothesis (2)
                                                                                                                                       2. \vdash_{\mathsf{H}} E_1[e] : \tau_1
                                                                                                                                                                                                                                                     2378
               CASE E = op^2 v_0 E_1:
                                                                                                                                            by inversion
2324
                                                                                                                                                                                                                                                     2379
2325
                  1. E[e] = op^2 v_0 E_1[e]
                                                                                                                                        3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                                                     2380
2326
                  2. \vdash_{\mathsf{H}} E_1[e] : \tau_1
                                                                                                                                     CASE E = \text{dyn } \tau_0 E_0:
                                                                                                                                                                                                                                                     2381
2327
                       by inversion
                                                                                                                                        1. E[e] = \text{dyn } \tau_0 E_0[e]
                                                                                                                                                                                                                                                     2382
2328
                  3. QED by the induction hypothesis (2)
                                                                                                                                        2. \vdash_{\mathsf{H}} E_0[e]
                                                                                                                                                                                                                                                     2383
               CASE E = \text{dyn } \tau_0 E_0:
2329
                                                                                                                                            by inversion
                                                                                                                                                                                                                                                     2384
2330
                  1. E[e] = \text{dyn } \tau_0 E_0[e]
                                                                                                                                       3. QED by dynamic stat hole typing (2)
2331
                  2. \vdash_{\mathsf{H}} E_0[e]
                                                                                                                                     CASE E = \text{stat } \tau_0 E_0:
                                                                                                                                                                                                                                                     2386
                       by inversion
                                                                                                                                        1. Contradiction by \vdash_{\mathsf{H}} E[e] : \tau
2332
                                                                                                                                                                                                                                                     2387
                  3. QED by static dyn hole typing (2)
                                                                                                                                                                                                                                                     2388
2333
2334
               CASE E = \text{stat } \tau_0 E_0:
                                                                                                                                                                                                                                                     2389
                                                                                                                                 Lemma 2.31: \vdash_H static inversion
2335
                  1. Contradiction by \vdash_{\mathsf{H}} E[e] : \tau
                                                                                                                                   • If \Gamma \vdash_{\mathsf{H}} x : \tau then (x : \tau') \in \Gamma and \tau' \leqslant \tau
                                                                                                                                                                                                                                                     2390
2336
                                                                                                                                   • If \Gamma \vdash_{\mathsf{H}} \lambda(x : \tau'_d). e' : \tau then (x : \tau'_d), \Gamma \vdash_{\mathsf{H}} e' : \tau'_c and
2337
                                                                                                                                                                                                                                                     2392
                                                                                                                                        \tau_d \Rightarrow \tau_c \leqslant \tau
           Lemma 2.30: H static context dynamic hole substitution
2338
                                                                                                                                                                                                                                                     2393
                                                                                                                                   • If \Gamma \vdash_{\mathsf{H}} \langle e_0, e_1 \rangle : \tau then \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0 and \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1 and
             If \vdash_{\mathsf{H}} E[e] : \tau and contains \vdash_{\mathsf{H}} e, and furthermore \vdash_{\mathsf{H}} e', then
2339
                                                                                                                                                                                                                                                     2394
              \vdash_{\mathsf{H}} E[e'] : \tau
2340
                                                                                                                                                                                                                                                     2395
           Proof:
                                                                                                                                   • If \Gamma \vdash_{\mathsf{H}} e_0 \ e_1 : \tau_c then \Gamma \vdash_{\mathsf{H}} e_0 : \tau_d' \Rightarrow \tau_c' and \Gamma \vdash_{\mathsf{H}} e_1 : \tau_d' and
2341
               By induction on the structure of E.
                                                                                                                                                                                                                                                     2396
2342
               case E \in E^{\bullet}:
                                                                                                                                                                                                                                                     2397
                                                                                                                                   • If \Gamma \vdash_{\mathsf{H}} \mathsf{fst} \ e : \tau \mathsf{ then } \Gamma \vdash_{\mathsf{H}} e : \tau_0 \times \tau_1 \mathsf{ and } \Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0
2343
                  1. Contradiction by \vdash_{\mathsf{H}} E[e] : \tau
                                                                                                                                                                                                                                                     2398
                                                                                                                                        and \tau_0 \leqslant : \tau
2344
                                                                                                                                                                                                                                                     2399
               CASE E = E_0 e_1:
                                                                                                                                   • If \Gamma \vdash_{\mathsf{H}} \mathsf{snd} \ e : \tau \mathsf{ then } \Gamma \vdash_{\mathsf{H}} e : \tau_0 \times \tau_1 \mathsf{ and } \Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1
2345
                                                                                                                                                                                                                                                     2400
                  1. E[e] = E_0[e] e_1
                                                                                                                                       and \tau_1 \leqslant : \tau
2346
                                                                                                                                   • If \Gamma \vdash_{\mathsf{H}} op^2 e_0 e_1 : \tau then \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0 and \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1 and
                                                                                                                                                                                                                                                     2401
                  2. \vdash_{\mathsf{H}} E_0[e] : \tau_0
2347
                       by inversion
                                                                                                                                                                                                                                                     2402
                                                                                                                                        \Delta(op^2, \tau_0, \tau_1) = \tau' and \tau' \leqslant \tau
2348
                                                                                                                                   • If \Gamma \vdash_{\mathsf{H}} \mathsf{mon} \, \tau_d \Rightarrow \tau_c \, \upsilon' : \tau \, \mathsf{then} \, \Gamma \vdash_{\mathsf{H}} \upsilon' \, \mathsf{and} \, \tau_d \Rightarrow \tau_c \leqslant : \tau
                                                                                                                                                                                                                                                     2403
                  3. QED by the induction hypothesis (2)
               CASE E = v_0 E_1:
                                                                                                                                                                                                                                                     2404
                                                                                                                                   • If \Gamma \vdash_{\vdash} dyn \tau' e' : \tau then \Gamma \vdash_{\vdash} e' and \tau' \leqslant \tau
                                                                                                                                                                                                                                                     2405
2350
                  1. E[e] = v_0 E_1[e]
2351
                  2. \vdash_{\mathsf{H}} E_1[e] : \tau_1
                                                                                                                                                                                                                                                     2406
                                                                                                                                     QED by the definition of \Gamma \vdash_{\mathsf{H}} e : \tau
2352
                                                                                                                                                                                                                                                     2407
                       by inversion
2353
                  3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                                                     2408
                                                                                                                                 Lemma 2.32 : \vdash_H dynamic inversion
2354
               CASE E = \langle E_0, e_1 \rangle:
                                                                                                                                                                                                                                                     2409
                                                                                                                                   • If \Gamma \vdash_{\mathsf{H}} x then x \in \Gamma
2355
                                                                                                                                                                                                                                                     2410
                  1. E[e] = \langle E_0[e], e_1 \rangle
                                                                                                                                    • If \Gamma \vdash_{\mathsf{H}} \lambda x. e' then x, \Gamma \vdash_{\mathsf{H}} e'
2356
                  2. \vdash_{\mathsf{H}} E_0[e] : \tau_0
                                                                                                                                                                                                                                                     2411
                                                                                                                                   • If \Gamma \vdash_{\mathsf{H}} \langle e_0, e_1 \rangle then \Gamma \vdash_{\mathsf{H}} e_0 and \Gamma \vdash_{\mathsf{H}} e_1
                                                                                                                                                                                                                                                     2412
                       by inversion
                                                                                                                                   • If \Gamma \vdash_{\mathsf{H}} e_0 \ e_1 then \Gamma \vdash_{\mathsf{H}} e_0 and \Gamma \vdash_{\mathsf{H}} e_1
2358
                  3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                                                     2413
                                                                                                                                   • If \Gamma \vdash_{\mathsf{H}} op^1 e_0 then \Gamma \vdash_{\mathsf{H}} e_0
2359
                                                                                                                                                                                                                                                     2414
               CASE E = \langle v_0, E_1 \rangle:
                                                                                                                                   • If \Gamma \vdash_{\mathsf{H}} op^2 e_0 e_1 then \Gamma \vdash_{\mathsf{H}} e_0 and \Gamma \vdash_{\mathsf{H}} e_1
2360
                                                                                                                                                                                                                                                     2415
                  1. E[e] = \langle v_0, E_1[e] \rangle
                                                                                                                                   • If \Gamma \vdash_{\mathsf{H}} \mathsf{mon} \, \tau_d \Rightarrow \tau_c \, v' \, \mathsf{then} \, \Gamma \vdash_{\mathsf{H}} v' : \tau_d \Rightarrow \tau_c
2361
                                                                                                                                                                                                                                                     2416
                  2. \vdash_{\!\!\!\!\perp} E_1[e] : \tau_1
                                                                                                                                   • If \Gamma \vdash_{\mathsf{H}} \mathsf{stat} \ \tau' \ e' \ \mathsf{then} \ \Gamma \vdash_{\mathsf{H}} e' : \tau'
2362
                       by inversion
                                                                                                                                                                                                                                                     2417
2363
                                                                                                                                                                                                                                                     2418
                  3. QED by the induction hypothesis (2)
                                                                                                                                     QED by the definition of \Gamma \vdash_{\mathsf{H}} e
2364
                                                                                                                                                                                                                                                     2419
2365
                                                                                                                                                                                                                                                     2420
                                                                                                                          22
```

```
• If \vdash_{\mathsf{H}} v and \delta(op^1, v) = e then \vdash_{\mathsf{H}} e
2421
                                                                                                                                                                                                                                    2476
                                                                                                                          • If \vdash_{\mathsf{H}} v_0 and \vdash_{\mathsf{H}} v_1 and \delta(op^2, v_0, v_1) = e then \vdash_{\mathsf{H}} e
2422
                                                                                                                                                                                                                                    2477
          Lemma 2.33: H canonical forms
2423
                                                                                                                                                                                                                                    2478
             • If \vdash_{\mathsf{H}} v : \tau_0 \times \tau_1 then v = \langle v_0, v_1 \rangle
                                                                                                                           CASE \delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0:
2424
             • If \vdash_{\mathsf{H}} v : \tau_d \Rightarrow \tau_c then either:
                                                                                                                                                                                                                                    2479
                                                                                                                              1. ⊢<sub>H</sub> υ<sub>0</sub>
2425
                -\upsilon = \lambda(x:\tau_x).e'
                                                                                                                                   by inversion
2426
                                                                                                                                                                                                                                    2481
                    \wedge \tau_d \leqslant : \tau_x
                                                                                                                              2. OED
2427
                - or v = \text{mon}(\tau'_d \Rightarrow \tau'_c)v'
                                                                                                                                                                                                                                    2482
                                                                                                                            CASE \delta(\operatorname{snd}, \langle v_0, v_1 \rangle) = v_1:
2428
                    \wedge \tau_d' \Rightarrow \tau_c' \leqslant : \tau_d \Rightarrow \tau_c
                                                                                                                                                                                                                                    2483
                                                                                                                              1. \vdash_{\mathsf{H}} v_1
2429
             • If \vdash_{\mathsf{H}} v: Int then v \in i
                                                                                                                                                                                                                                    2484
                                                                                                                                   by inversion
2430
             • If \vdash_{\mathsf{H}} v: Nat then v \in \mathbb{N}
                                                                                                                                                                                                                                    2485
                                                                                                                              2. OED
2431
          Proof:
                                                                                                                                                                                                                                    2486
                                                                                                                            CASE \delta(\text{sum}, v_0, v_1) = v_0 + v_1:
              QED by definition of \vdash_{\mathsf{H}} e : \tau
2432
                                                                                                                                                                                                                                    2487
2433
                                                                                                                                                                                                                                    2488
                                                                                                                            CASE \delta(\text{quotient}, v_0, v_1) = \lfloor v_0/v_1 \rfloor:
2434
                                                                                                                                                                                                                                    2489
          Lemma 2.34 : \Delta type soundness
2435
                                                                                                                                                                                                                                    2490
             If \vdash_{\mathsf{H}} v_0 : \tau_0 and \vdash_{\mathsf{H}} v_1 : \tau_1 and \Delta(op^2, \tau_0, \tau_1) = \tau then \vdash_{\mathsf{H}}
                                                                                                                            CASE \delta(op^2, v_0, v_1) = \text{BndryErr}:
2436
                                                                                                                                                                                                                                    2491
             \delta(op^2, v_0, v_1) : \tau.
                                                                                                                              1. QED
2437
                                                                                                                                                                                                                                    2492
2438
                                                                                                                                                                                                                                    2493
              By case analysis on the definition of \Delta.
                                                                                                                        Lemma 2.36: H substitution
2439
                                                                                                                                                                                                                                    2494
              CASE \Delta(sum, Nat, Nat) = Nat :
                                                                                                                          • If (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e and \vdash_{\mathsf{H}} v:\tau_x then \Gamma \vdash_{\mathsf{H}} e[x \leftarrow v]
2440
                                                                                                                                                                                                                                    2495
                1. v_0 = i_0, i_0 \in \mathbb{N}
                                                                                                                          • If x, \Gamma \vdash_{\mathsf{H}} e and \vdash_{\mathsf{H}} v then \Gamma \vdash_{\mathsf{H}} e[x \leftarrow v]
2441
                                                                                                                                                                                                                                    2496
                     \wedge v_1 = i_1, i_1 \in \mathbb{N}
                                                                                                                          • If (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e:\tau and \vdash_{\mathsf{H}} v:\tau_x then \Gamma \vdash_{\mathsf{H}} e[x \leftarrow v]:\tau
2442
                                                                                                                                                                                                                                    2497
                     by canonical forms
                                                                                                                          • If x, \Gamma \vdash_{\mathsf{H}} e : \tau and \vdash_{\mathsf{H}} v then \Gamma \vdash_{\mathsf{H}} e[x \leftarrow v] : \tau
                                                                                                                                                                                                                                    2498
2443
                2. \delta(\text{sum}, i_0, i_1) = i_0 + i_1 \in \mathbb{N}
                                                                                                                                                                                                                                    2499
2444
                3. QED
                                                                                                                           By the following four lemmas: dynamic context static
2445
              CASE \Delta(\text{sum}, \text{Int}, \text{Int}) = \text{Int}:
                                                                                                                                                                                                                                    2500
                                                                                                                            value substitution, dynamic context dynamic value substi-
                                                                                                                                                                                                                                    2501
                1. v_0 = i_0
                                                                                                                            tution, static context static value substitution, and static
2447
                                                                                                                                                                                                                                    2502
                     \wedge v_1 = i_1
                                                                                                                            context dynamic value substitution.
                     by canonical forms
2449
                                                                                                                                                                                                                                    2504
                2. \delta(\text{sum}, i_0, i_1) = i_0 + i_1 \in i
                                                                                                                        Lemma 2.37: H dynamic-static substitution
2450
                                                                                                                                                                                                                                    2505
                3. QED
                                                                                                                        If (x:\tau_x), \Gamma \vdash_{\vdash} e and \vdash_{\vdash} v:\tau_x then \Gamma \vdash_{\vdash} e[x \leftarrow v]
2451
                                                                                                                                                                                                                                    2506
              CASE \Delta(quotient, Nat, Nat) = Nat :
                                                                                                                        Proof:
                                                                                                                                                                                                                                    2507
                1. v_0 = i_0, i_0 \in \mathbb{N}
2453
                                                                                                                           By induction on the structure of e.
                                                                                                                                                                                                                                    2508
                     \wedge v_1 = i_1, i_1 \in \mathbb{N}
2454
                                                                                                                                                                                                                                    2509
                                                                                                                           CASE e = x:
                     by canonical forms
                                                                                                                              1. Contradiction by (x:\tau_x), \Gamma \vdash_{\!\!\!\perp} x
                                                                                                                                                                                                                                    2510
2455
                2. IF i_1 = 0:
2456
                                                                                                                            CASE e = x':
                                                                                                                                                                                                                                    2511
                       a. \delta(\text{quotient}, i_0, i_1) = \text{BndryErr}
2457
                                                                                                                              1. QED by (x'[x \leftarrow v]) = x'
                                                                                                                                                                                                                                    2512
                       b. QED by \vdash_{\mathsf{H}} \mathsf{BndryErr} : \tau
                                                                                                                            CASE e = i:
2458
                                                                                                                                                                                                                                    2513
                     ELSE i_1 \neq 0:
                                                                                                                              1. QED by i[x \leftarrow v] = i
2459
                       a. \delta(\text{quotient}, i_0, i_1) = \lfloor i_0/i_1 \rfloor \in \mathbb{N}
                                                                                                                            CASE e = \lambda x' \cdot e':
                                                                                                                                                                                                                                    2515
2460
                       b. oed
                                                                                                                              1. e[x \leftarrow v] = \lambda x' \cdot (e'[x \leftarrow v])
2461
                                                                                                                                                                                                                                    2516
              CASE \Delta(quotient, lnt, lnt) = lnt :
2462
                                                                                                                              2. x', (x:\tau_x), \Gamma \vdash_{\vdash} e'
                                                                                                                                                                                                                                    2517
                1. v_0 = i_0
                                                                                                                                  by inversion
                                                                                                                                                                                                                                    2518
                     \wedge v_1 = i_1
                                                                                                                              3. x', \Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v]
                                                                                                                                                                                                                                    2519
2464
                     by canonical forms
2465
                                                                                                                                   by the induction hypothesis (2)
                                                                                                                                                                                                                                    2520
                 2. IF i_1 = 0:
                                                                                                                              4. \Gamma \vdash_{\mathsf{H}} \lambda x'. (e'[x \leftarrow v])
2466
                                                                                                                                                                                                                                    2521
                       a. \delta(\text{quotient}, i_0, i_1) = \text{BndryErr}
2467
                                                                                                                                   by (3)
                       b. QED by \vdash_{\mathsf{H}} \mathsf{BndryErr} : \tau
2468
                                                                                                                              5. QED
                                                                                                                                                                                                                                    2523
                     ELSE i_1 \neq 0:
2469
                                                                                                                            CASE e = \lambda(x':\tau').e':
                                                                                                                                                                                                                                    2524
                       a. \delta(\text{quotient}, i_0, i_1) = \lfloor i_0/i_1 \rfloor \in i
                                                                                                                              1. Contradiction by (x:\tau_x), \Gamma \vdash_H e
2470
                                                                                                                                                                                                                                    2525
                       b. QED
2471
                                                                                                                            CASE e = mon(\tau_d \Rightarrow \tau_c)v':
                                                                                                                                                                                                                                    2526
                                                                                                                              1. e[x \leftarrow v] = \text{mon}(\tau_d \Rightarrow \tau_c) v'[x \leftarrow v]
2472
                                                                                                                                                                                                                                    2527
          Lemma 2.35 : \delta preservation
2473
                                                                                                                                                                                                                                    2528
2474
                                                                                                                                                                                                                                    2529
```

```
2531
                  2. (x:\tau_x), \Gamma \vdash_{\mathsf{H}} v': \tau_d \Rightarrow \tau_c
                                                                                                                                          3. \Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v] : \tau'
                                                                                                                                                                                                                                                          2586
                       by inversion
                                                                                                                                               by static context static value substitution (2)
2532
                                                                                                                                                                                                                                                          2587
2533
                  3. \Gamma \vdash_{\mathsf{H}} v'[x \leftarrow v] : \tau_d \Rightarrow \tau_c
                                                                                                                                          4. \Gamma \vdash_{\mathsf{H}} \operatorname{stat} \tau' e'[x \leftarrow v]
                                                                                                                                                                                                                                                          2588
                       by static context static value substitution (2)
                                                                                                                                               by (3)
2534
                                                                                                                                                                                                                                                          2589
                  4. \Gamma \vdash_{\mathsf{H}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) v'[x \leftarrow v]
                                                                                                                                          5. QED
                                                                                                                                                                                                                                                          2590
                       by (3)
                                                                                                                                        CASE e = Err:
                                                                                                                                                                                                                                                          2591
2536
2537
                                                                                                                                          1. QED Err = \text{Err}[x \leftarrow v]
                  5. OED
                                                                                                                                                                                                                                                          2592
2538
                CASE e = \langle e_0, e_1 \rangle:
                                                                                                                                                                                                                                                          2593
                  1. e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
                                                                                                                                                                                                                                                          2594
                                                                                                                                   Lemma 2.38: H dynamic-dynamic substitution
2540
                  2. (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e_0
                                                                                                                                                                                                                                                          2595
                                                                                                                                    If x, \Gamma \vdash_{\mathsf{H}} e and \vdash_{\mathsf{H}} v then \Gamma \vdash_{\mathsf{H}} e[x \leftarrow v]
                       \land (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e_1
                       by inversion
2542
                                                                                                                                                                                                                                                          2597
                                                                                                                                       By induction on the structure of e
2543
                  3. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]
                                                                                                                                                                                                                                                          2598
                                                                                                                                       CASE e = x:
2544
                       \wedge \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v]
                                                                                                                                                                                                                                                          2599
                                                                                                                                          1. e[x \leftarrow v] = v
2545
                       by the induction hypothesis (2)
                                                                                                                                                                                                                                                          2600
                                                                                                                                          2. \Gamma \vdash_{\mathsf{H}} v
2546
                  4. \Gamma \vdash_{\mathsf{H}} \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
                                                                                                                                                                                                                                                          2601
                                                                                                                                               by weakening
                       by (3)
                                                                                                                                                                                                                                                          2602
                                                                                                                                          3. QED
                  5. OED
2548
                                                                                                                                                                                                                                                          2603
                                                                                                                                        CASE e = x':
                CASE e = e_0 \ e_1:
2549
                                                                                                                                          1. QED by (x'[x \leftarrow v]) = x'
                                                                                                                                                                                                                                                          2604
                  1. e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                                                                                                                                                          2605
                                                                                                                                        CASE e = i:
2551
                  2. (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e_0
                                                                                                                                                                                                                                                          2606
                                                                                                                                          1. QED by i[x \leftarrow v] = i
                       \land (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e_1
2552
                                                                                                                                                                                                                                                          2607
                                                                                                                                        CASE e = \lambda x' \cdot e':
                       by inversion
2553
                                                                                                                                                                                                                                                          2608
                                                                                                                                          1. e[x \leftarrow v] = \lambda x' \cdot (e'[x \leftarrow v])
2554
                  3. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]
                                                                                                                                                                                                                                                          2609
                                                                                                                                          2. x', x, \Gamma \vdash_{\mathsf{H}} e'
2555
                       \wedge \Gamma \vdash_{\vdash} e_1 | x \leftarrow v |
                                                                                                                                                                                                                                                          2610
                                                                                                                                               by inversion
                       by the induction hypothesis (2)
                                                                                                                                                                                                                                                          2611
                                                                                                                                          3. x', \Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v]
2557
                  4. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                                                                                                                                                          2612
                                                                                                                                               by the induction hypothesis (2)
                       by (3)
                                                                                                                                                                                                                                                          2613
                                                                                                                                          4. \Gamma \vdash_{\mathsf{H}} \lambda x' . (e'[x \leftarrow v])
2559
                  5. QED
                                                                                                                                                                                                                                                          2614
                                                                                                                                               by (3)
                CASE e = op^1 e_0:
2560
                                                                                                                                                                                                                                                          2615
                                                                                                                                          5. QED
                  1. e[x \leftarrow v] = op^1 e_0[x \leftarrow v]
2561
                                                                                                                                                                                                                                                          2616
                                                                                                                                        CASE e = \lambda(x':\tau'). e':
                  2. (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e_0
                                                                                                                                                                                                                                                          2617
                                                                                                                                          1. Contradiction by x, \Gamma \vdash_{\!\!\!\perp} e
2563
                       by inversion
                                                                                                                                                                                                                                                          2618
                                                                                                                                        CASE e = mon(\tau_d \Rightarrow \tau_c)v':
2564
                  3. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]
                                                                                                                                                                                                                                                          2619
                                                                                                                                          1. e[x \leftarrow v] = mon(\tau_d \Rightarrow \tau_c)v'[x \leftarrow v]
                       by the induction hypothesis (2)
                                                                                                                                                                                                                                                          2620
2565
                                                                                                                                          2. x, \Gamma \vdash_{\mathsf{H}} v' : \tau_d \Rightarrow \tau_c
2566
                  4. \Gamma \vdash_{\mathsf{H}} op^1 e_0[x \leftarrow v]
                                                                                                                                                                                                                                                          2621
                                                                                                                                               by inversion
2567
                       by (3)
                                                                                                                                                                                                                                                          2622
                                                                                                                                          3. \Gamma \vdash_{\mathsf{H}} v'[x \leftarrow v] : \tau_d \Rightarrow \tau_c
2568
                  5. QED
                                                                                                                                                                                                                                                          2623
                                                                                                                                               by static context dynamic value substitution (2)
                CASE e = op^2 e_0 e_1:
                                                                                                                                          4. \Gamma \vdash_{\mathsf{H}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) v'[x \leftarrow v]
                   1. e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                                                                                                                                                          2625
2570
                                                                                                                                               by (3)
2571
                  2. (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e_0
                                                                                                                                                                                                                                                          2626
                                                                                                                                          5. QED
2572
                       \land (x:\tau_x), \Gamma \vdash_{\vdash} e_1
                                                                                                                                                                                                                                                          2627
                                                                                                                                        CASE e = \langle e_0, e_1 \rangle:
                       by inversion
                                                                                                                                                                                                                                                          2628
                                                                                                                                          1. e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
2574
                  3. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]
                                                                                                                                                                                                                                                          2629
                                                                                                                                          2. x, \Gamma \vdash_{\mathsf{H}} e_0
2575
                       \wedge \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v]
                                                                                                                                                                                                                                                          2630
                                                                                                                                               \land x, \Gamma \vdash_{\mathsf{H}} e_1
                       by the induction hypothesis (2)
2576
                                                                                                                                                                                                                                                          2631
                                                                                                                                               by inversion
                  4. \Gamma \vdash_{\mathsf{H}} op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                                                                                                                                                          2632
                                                                                                                                          3. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]
2578
                       by (3)
                                                                                                                                                                                                                                                          2633
                                                                                                                                               \wedge \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v]
                  5. QED
                                                                                                                                                                                                                                                          2634
                                                                                                                                               by the induction hypothesis (2)
2580
                CASE e = \text{stat } \tau' e':
                                                                                                                                                                                                                                                          2635
                                                                                                                                          4. \Gamma \vdash_{\mathsf{H}} \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
                  1. e[x \leftarrow v] = \text{stat } \tau' e'[x \leftarrow v]
2581
                                                                                                                                                                                                                                                          2636
                                                                                                                                               by (3)
2582
                  2. (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e':\tau'
                                                                                                                                                                                                                                                          2637
                                                                                                                                          5. QED
2583
                       by inversion
                                                                                                                                                                                                                                                          2638
                                                                                                                                        CASE e = e_0 \ e_1:
2584
                                                                                                                                                                                                                                                          2639
2585
                                                                                                                                                                                                                                                          2640
                                                                                                                            24
```

```
1. e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                                         5. \Gamma \vdash_{\mathsf{H}} \upsilon : \tau
2641
                                                                                                                                                                                                                                                       2696
2642
                  2. x, \Gamma \vdash_{\mathsf{H}} e_0
                                                                                                                                              by weakening
                                                                                                                                                                                                                                                       2697
2643
                       \land x, \Gamma \vdash_{\vdash} e_1
                                                                                                                                         6. QED
                                                                                                                                                                                                                                                       2698
                       by inversion
                                                                                                                                      CASE e = x':
2644
                                                                                                                                                                                                                                                       2699
                  3. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]
                                                                                                                                         1. QED by (x'[x \leftarrow v]) = x'
                       \wedge \Gamma \vdash_{\vdash} e_1[x \leftarrow v]
                                                                                                                                                                                                                                                       2701
                                                                                                                                      CASE e = i:
2646
                       by the induction hypothesis (2)
2647
                                                                                                                                         1. QED by i[x \leftarrow v] = i
                                                                                                                                                                                                                                                       2702
2648
                  4. \Gamma \vdash_{\vdash} e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                                      CASE e = \lambda x' \cdot e':
                                                                                                                                                                                                                                                       2703
                       by (3)
                                                                                                                                         1. Contradiction by (x:\tau_x), \Gamma \vdash_H e:\tau
                                                                                                                                                                                                                                                       2704
2650
                  5. QED
                                                                                                                                      CASE e = \lambda(x':\tau').e':
                                                                                                                                                                                                                                                       2705
2651
               CASE e = op^1 e_0:
                                                                                                                                         1. e[x \leftarrow v] = \lambda(x':\tau').(e'[x \leftarrow v])
                                                                                                                                                                                                                                                       2706
                                                                                                                                         2. (x':\tau'), (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e' : \tau'_c
                  1. e[x \leftarrow v] = op^1 e_0[x \leftarrow v]
2652
                                                                                                                                                                                                                                                       2707
2653
                                                                                                                                              \wedge \tau' \Rightarrow \tau'_c \leqslant : \tau
                                                                                                                                                                                                                                                       2708
                  2. x, \Gamma \vdash_{\mathsf{H}} e_0
                                                                                                                                         3. (x':\tau'), \Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v] : \tau'_c
2654
                       by inversion
                                                                                                                                                                                                                                                       2709
2655
                                                                                                                                              by the induction hypothesis (2)
                                                                                                                                                                                                                                                       2710
                  3. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]
2656
                       by the induction hypothesis (2)
                                                                                                                                         4. \Gamma \vdash_{\mathsf{H}} \lambda(x' : \tau'). e' : \tau
                                                                                                                                                                                                                                                       2711
2657
                  4. \Gamma \vdash_{\mathsf{H}} op^1 e_0[x \leftarrow v]
                                                                                                                                                                                                                                                       2712
                                                                                                                                         5. QED
                       by (3)
                                                                                                                                      CASE e = mon(\tau_d \Rightarrow \tau_c)v':
2658
                                                                                                                                                                                                                                                       2713
                                                                                                                                         1. e[x \leftarrow v] = \text{mon}(\tau_d \Rightarrow \tau_c)v'[x \leftarrow v]
2659
                  5. QED
                                                                                                                                                                                                                                                       2714
               CASE e = op^2 e_0 e_1:
                                                                                                                                         2. (x:\tau_x), \Gamma \vdash_{\mathsf{H}} v'
                                                                                                                                                                                                                                                       2715
                  1. e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                                             by inversion
                                                                                                                                                                                                                                                       2716
2661
                  2. x, \Gamma \vdash_{\mathsf{H}} e_0
                                                                                                                                         3. \Gamma \vdash_{\mathsf{H}} v'[x \leftarrow v]
                                                                                                                                                                                                                                                       2717
                                                                                                                                              by dynamic context static value substitution (2)
                                                                                                                                                                                                                                                       2718
                       \land x, \Gamma \vdash_{\vdash} e_1
2663
                                                                                                                                         4. \Gamma \vdash_{\mathsf{H}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) v'[x \leftarrow v] : \tau
                       by inversion
                                                                                                                                                                                                                                                       2719
                  3. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]
2665
                                                                                                                                             by (3)
                                                                                                                                                                                                                                                       2720
                       \wedge \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v]
                                                                                                                                         5. QED
                                                                                                                                                                                                                                                       2721
                       by the induction hypothesis (2)
                                                                                                                                      CASE e = \langle e_0, e_1 \rangle:
                                                                                                                                                                                                                                                       2722
2667
                                                                                                                                         1. e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
                  4. \Gamma \vdash_{\mathsf{H}} op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                                                                                                                                                       2723
                       by (3)
2669
                                                                                                                                         2. (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0
                                                                                                                                                                                                                                                       2724
2670
                  5. QED
                                                                                                                                              \land (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e_1:\tau_1
                                                                                                                                                                                                                                                       2725
               CASE e = \text{stat } \tau' e':
2671
                                                                                                                                              by inversion
                                                                                                                                                                                                                                                       2726
                  1. e[x \leftarrow v] = \text{stat } \tau' e'[x \leftarrow v]
                                                                                                                                         3. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0
                                                                                                                                                                                                                                                       2727
2673
                  2. x, \Gamma \vdash_{\mathsf{H}} e' : \tau'
                                                                                                                                              \wedge \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v] : \tau_1
                                                                                                                                                                                                                                                       2728
2674
                       by inversion
                                                                                                                                              by the induction hypothesis (2)
                                                                                                                                                                                                                                                       2729
                  3. \Gamma \vdash_{\vdash} e'[x \leftarrow v] : \tau'
                                                                                                                                         4. \Gamma \vdash_{\mathsf{H}} \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle : \tau
                                                                                                                                                                                                                                                       2730
2675
                       by static context static value substitution (2)
2676
                                                                                                                                             by (3)
                                                                                                                                                                                                                                                       2731
                  4. \Gamma \vdash_{\mathsf{H}} \mathsf{stat} \ \tau' \ e'[x \leftarrow v]
2677
                                                                                                                                         5. QED
                                                                                                                                                                                                                                                       2732
2678
                       by (3)
                                                                                                                                      CASE e = e_0 \ e_1:
                                                                                                                                                                                                                                                       2733
                                                                                                                                         1. e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]
2679
                  5. QED
               CASE e = Err:
                                                                                                                                         2. (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0
                                                                                                                                                                                                                                                       2735
2680
                  1. QED Err = Err[x \leftarrow v]
2681
                                                                                                                                              \land (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e_1:\tau_1
                                                                                                                                                                                                                                                       2736
2682
                                                                                                                                              by inversion
                                                                                                                                                                                                                                                       2737
                                                                                                                                         3. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0
                                                                                                                                                                                                                                                       2738
           Lemma 2.39 : H static-static substitution
                                                                                                                                              \wedge \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v] : \tau_1
                                                                                                                                                                                                                                                       2739
2684
            If (x:\tau_x), \Gamma \vdash_{\vdash} e:\tau and \vdash_{\vdash} v:\tau_x then \Gamma \vdash_{\vdash} e[x \leftarrow v]:\tau
2685
                                                                                                                                              by the induction hypothesis (2)
                                                                                                                                         4. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] e_1[x \leftarrow v] : \tau
2686
                                                                                                                                                                                                                                                       2741
               By induction on the structure of e.
                                                                                                                                             by (3)
                                                                                                                                                                                                                                                       2742
               CASE e = x:
2688
                                                                                                                                         5. QED
                                                                                                                                                                                                                                                       2743
                  1. e[x \leftarrow v] = v
2689
                                                                                                                                      CASE e = op^1 e_0:
                                                                                                                                                                                                                                                       2744
                  2. (x:\tau_x), \Gamma \vdash_{\mathsf{H}} x:\tau
2690
                                                                                                                                         1. e[x \leftarrow v] = op^1 e_0[x \leftarrow v]
                                                                                                                                                                                                                                                       2745
                  3. \tau_x \leqslant : \tau
2691
                                                                                                                                         2. (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0
                                                                                                                                                                                                                                                       2746
                       by inversion
2692
                                                                                                                                              by inversion
                                                                                                                                                                                                                                                       2747
                  4. \vdash_{\mathsf{H}} v : \tau
2693
                                                                                                                                         3. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0
                                                                                                                                                                                                                                                       2748
                       by (3)
                                                                                                                                              by the induction hypothesis (2)
2694
                                                                                                                                                                                                                                                       2749
2695
                                                                                                                                                                                                                                                       2750
                                                                                                                           25
```

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2751
                  4. \Gamma \vdash_{\mathsf{H}} op^1 e_0[x \leftarrow v] : \tau
                                                                                                                                         4. \Gamma \vdash_{\mathsf{H}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) v'[x \leftarrow v] : \tau
                                                                                                                                                                                                                                                       2806
2752
                      by (3)
                                                                                                                                             by (3)
                                                                                                                                                                                                                                                       2807
2753
                  5. QED
                                                                                                                                         5. QED
                                                                                                                                                                                                                                                       2808
               CASE e = op^2 e_0 e_1:
2754
                                                                                                                                      CASE e = \langle e_0, e_1 \rangle:
                                                                                                                                                                                                                                                       2809
2755
                  1. e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                                         1. e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
                                                                                                                                                                                                                                                       2810
                  2. (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0
                                                                                                                                         2. x, \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0
                                                                                                                                                                                                                                                       2811
2756
                       \wedge (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e_1:\tau_1
2757
                                                                                                                                              \wedge x, \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1
                                                                                                                                                                                                                                                       2812
2758
                       by inversion
                                                                                                                                              by inversion
                                                                                                                                                                                                                                                       2813
2759
                  3. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0
                                                                                                                                         3. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0
                                                                                                                                                                                                                                                       2814
2760
                       \wedge \Gamma \vdash_{\vdash} e_1[x \leftarrow v] : \tau_1
                                                                                                                                              \wedge \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v] : \tau_1
                                                                                                                                                                                                                                                       2815
2761
                       by the induction hypothesis (2)
                                                                                                                                              by the induction hypothesis (2)
                                                                                                                                                                                                                                                       2816
                  4. \Gamma \vdash_{\mathsf{L}} op^2 e_0[x \leftarrow v] e_1[x \leftarrow v] : \tau
                                                                                                                                         4. \Gamma \vdash_{\sqcup} \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle : \tau
2762
                                                                                                                                                                                                                                                       2817
2763
                       by (3)
                                                                                                                                             by (3)
                                                                                                                                                                                                                                                       2818
2764
                  5. QED
                                                                                                                                         5. QED
                                                                                                                                                                                                                                                       2819
2765
               CASE e = \text{dyn } \tau' e':
                                                                                                                                                                                                                                                       2820
                                                                                                                                      CASE e = e_0 \ e_1:
                                                                                                                                         1. e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]
2766
                  1. e[x \leftarrow v] = \text{dyn } \tau' e'[x \leftarrow v]
                                                                                                                                                                                                                                                       2821
                  2. (x:\tau_x), \Gamma \vdash_{\mathsf{H}} e'
2767
                                                                                                                                         2. x, \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0
                                                                                                                                                                                                                                                       2822
                                                                                                                                              \wedge x, \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1
                       by inversion
2768
                                                                                                                                                                                                                                                       2823
                  3. \Gamma \vdash_{\vdash} e'[x \leftarrow v]
2769
                                                                                                                                              by inversion
                                                                                                                                                                                                                                                       2824
2770
                       by dynamic context static value substitution (2)
                                                                                                                                         3. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0
                                                                                                                                                                                                                                                       2825
2771
                  4. \Gamma \vdash_{\mathsf{H}} \mathsf{dyn} \ \tau' \ e'[x \leftarrow v] : \tau
                                                                                                                                              \wedge \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v] : \tau_1
                                                                                                                                                                                                                                                       2826
                       by (3)
                                                                                                                                              by the induction hypothesis (2)
2772
                                                                                                                                                                                                                                                       2827
                                                                                                                                         4. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] e_1[x \leftarrow v] : \tau
                                                                                                                                                                                                                                                       2828
2773
                  5. QED
2774
               CASE e = Err:
                                                                                                                                              by (3)
                                                                                                                                                                                                                                                       2829
2775
                  1. QED by Err = Err[x \leftarrow v]
                                                                                                                                         5. QED
                                                                                                                                                                                                                                                       2830
                                                                                                                                      CASE e = op^1 e_0:
2776
                                                                                                                                                                                                                                                       2831
2777
                                                                                                                                         1. e[x \leftarrow v] = op^1 e_0[x \leftarrow v]
                                                                                                                                                                                                                                                       2832
           Lemma 2.40: H static-dynamic substitution
2778
                                                                                                                                         2. x, \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0
                                                                                                                                                                                                                                                       2833
           If x, \Gamma \vdash_{\mathsf{H}} e : \tau and \vdash_{\mathsf{H}} v then \Gamma \vdash_{\mathsf{H}} e[x \leftarrow v] : \tau
2779
                                                                                                                                              by inversion
                                                                                                                                                                                                                                                       2834
2780
                                                                                                                                         3. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0
                                                                                                                                                                                                                                                       2835
               By induction on the structure of e.
2781
                                                                                                                                              by the induction hypothesis (2)
                                                                                                                                                                                                                                                       2836
               CASE e = x:
2782
                                                                                                                                         4. \Gamma \vdash_{\mathsf{H}} op^1 e_0[x \leftarrow v] : \tau
                                                                                                                                                                                                                                                       2837
                  1. Contradiction by x, \Gamma \vdash_{\mathsf{H}} x : \tau
2783
                                                                                                                                              by (3)
                                                                                                                                                                                                                                                       2838
               CASE e = x':
2784
                                                                                                                                                                                                                                                       2839
                                                                                                                                         5. QED
                  1. QED by (x'[x \leftarrow v]) = x'
                                                                                                                                      CASE e = op^2 e_0 e_1:
2785
                                                                                                                                                                                                                                                       2840
               CASE e = i:
                                                                                                                                         1. e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
2786
                                                                                                                                                                                                                                                       2841
                  1. QED by i[x \leftarrow v] = i
2787
                                                                                                                                         2. x, \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0
                                                                                                                                                                                                                                                       2842
               CASE e = \lambda x'. e':
2788
                                                                                                                                              \land x, \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1
                                                                                                                                                                                                                                                       2843
                  1. Contradiction by (x:\tau_x), \Gamma \vdash_H e:\tau
                                                                                                                                              by inversion
2789
               CASE e = \lambda(x':\tau').e':
                                                                                                                                         3. \Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0
                                                                                                                                                                                                                                                       2845
2790
                  1. e[x \leftarrow v] = \lambda(x':\tau'). (e'[x \leftarrow v])
2791
                  2. (x':\tau'), x, \Gamma \vdash_{\mathsf{H}} e' : \tau'_c
                                                                                                                                              \wedge \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v] : \tau_1
                                                                                                                                                                                                                                                       2846
2792
                                                                                                                                              by the induction hypothesis (2)
                                                                                                                                                                                                                                                       2847
                       \wedge \tau' \Rightarrow \tau'_c \leqslant \tau
                                                                                                                                         4. \Gamma \vdash_{\mathsf{H}} op^2 e_0[x \leftarrow v] e_1[x \leftarrow v] : \tau
                  3. (x':\tau'), \Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v] : \tau'_{c}
2794
                                                                                                                                             by (3)
                                                                                                                                                                                                                                                       2849
                       by the induction hypothesis (2)
2795
                                                                                                                                         5. OED
                                                                                                                                                                                                                                                       2850
                  4. \Gamma \vdash_{\mathsf{H}} \lambda(x' : \tau'). e' : \tau
                                                                                                                                      CASE e = \text{dyn } \tau' e':
2796
                                                                                                                                                                                                                                                       2851
                  5. QED
                                                                                                                                         1. e[x \leftarrow v] = \text{dyn } \tau' e'[x \leftarrow v]
                                                                                                                                                                                                                                                       2852
               CASE e = mon(\tau_d \Rightarrow \tau_c)v':
2798
                                                                                                                                         2. x, \Gamma \vdash_{\mathsf{H}} e'
                                                                                                                                                                                                                                                       2853
                  1. e[x \leftarrow v] = \text{mon}(\tau_d \Rightarrow \tau_c)v'[x \leftarrow v]
2799
                                                                                                                                             by inversion
                                                                                                                                                                                                                                                       2854
                  2. x, \Gamma \vdash_{\mathsf{H}} v'
2800
                                                                                                                                         3. \Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v]
                                                                                                                                                                                                                                                       2855
                       by inversion
                                                                                                                                              by dynamic context dynamic value substitution (2)
2801
                                                                                                                                                                                                                                                       2856
                  3. \Gamma \vdash_{\mathsf{H}} v'[x \leftarrow v]
                                                                                                                                         4. \Gamma \vdash_{\mathsf{H}} \mathsf{dyn} \ \tau' \ e'[x \leftarrow v] : \tau
2802
                                                                                                                                                                                                                                                       2857
                       by dynamic context dynamic value substitution (2)
2803
                                                                                                                                              by (3)
                                                                                                                                                                                                                                                       2858
                                                                                                                                         5. QED
                                                                                                                                                                                                                                                       2859
2804
2805
```

```
CASE e = Err:
2861
2862
                1. QED by Err = Err[x \leftarrow v]
2863
2864
          Lemma 2.41: weakening
2865
            • If \Gamma \vdash_{\mathsf{H}} e then x, \Gamma \vdash_{\mathsf{H}} e
           • If \Gamma \vdash_{\mathsf{H}} e : \tau then (x : \tau'), \Gamma \vdash_{\mathsf{H}} e : \tau
2866
2867
2868
             ullet e is closed under \Gamma
2869
                by \Gamma \vdash_{\mathsf{H}} e
2870
                \vee \ \Gamma \vdash_{\mathsf{H}} e : \tau
2871
             • QED x is unused in the derivation
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```

#### A.3 (E) Erasure Embedding A.3.1 Erasure Definitions Language E $e = x \mid v \mid \langle e, e \rangle \mid e \mid op^1 \mid e \mid op^2 \mid e \mid e \mid$ dyn $\tau e \mid \text{stat } \tau e \mid \text{Err}$ $= i | \langle v, v \rangle | \lambda x. e | \lambda(x:\tau). e$ = Nat | Int | $\tau \times \tau$ | $\tau \Rightarrow \tau$ $\Gamma = \cdot \mid x, \Gamma \mid (x:\tau), \Gamma$ Err = BndryErr | TagErr $r = v \mid Err$ $E^{\bullet} = [] | E^{\bullet} e | v E^{\bullet} | \langle E^{\bullet}, e \rangle | \langle v, E^{\bullet} \rangle |$ $op^1 E^{\bullet} \mid op^2 E^{\bullet} e \mid op^2 v E^{\bullet}$ $E = E^{\bullet} \mid E \mid v \mid E \mid \langle E, e \rangle \mid \langle v, E \rangle \mid op^{1} \mid E \mid$ $op^2 E e \mid op^2 v E \mid dyn \tau E \mid stat \tau E$ $\Delta: op^1 \times \tau \longrightarrow \tau$ $\Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0$ $\Delta(\operatorname{snd}, \tau_0 \times \tau_1) = \tau_1$ $\Delta: op^2 \times \tau \times \tau \longrightarrow \tau$ $\Delta(op^2, Nat, Nat) = Nat$ $\Delta(op^2, Int, Int) = Int$ $\tau \leqslant : \tau$ $\frac{\tau_d' \leqslant : \tau_d \quad \tau_c \leqslant : \tau_c'}{\tau_d \Rightarrow \tau_c \leqslant : \tau_d' \Rightarrow \tau_c'} \quad \frac{\tau_0 \leqslant : \tau_0' \quad \tau_1 \leqslant : \tau_1'}{\tau_0 \times \tau_1 \leqslant : \tau_0' \times \tau_1'}$ $\frac{\tau \leqslant : \tau' \quad \tau' \leqslant : \tau''}{\tau \leqslant : \tau''}$ $\Gamma \vdash e$ $\frac{x \in \Gamma}{\Gamma \vdash x} \quad \frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x. \, e} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, \, e_1 \rangle}$ $\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0 \ e_1} \quad \frac{\Gamma \vdash e}{\Gamma \vdash op^1 \ e} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2 \ e_0 \ e_1} \quad \frac{\Gamma \vdash \mathsf{Err}}{\Gamma \vdash \mathsf{Err}}$ $\Gamma \vdash e : \tau$ $\Gamma \vdash \operatorname{stat} \tau e$

```
\Gamma \vdash e : \tau
               (x:\tau)\in\Gamma
                                                          (x:\tau_d), \Gamma \vdash e:\tau_c \qquad \qquad i \in \mathbb{N}
                \Gamma \vdash x : \tau \Gamma \vdash \lambda(x : \tau_d). e : \tau_d \Rightarrow \tau_c \Gamma \vdash i : Nat
                                                                    \Gamma \vdash e_0 : \tau_0 \qquad \qquad \Gamma \vdash e_0 : \tau_d \Longrightarrow \tau_c
              \frac{\Gamma \vdash e_1 : \tau_1}{\Gamma \vdash i : \mathsf{Int}} \quad \frac{\Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1} \quad \frac{\Gamma \vdash e_1 : \tau_d}{\Gamma \vdash e_0 \; e_1 : \tau_c}
                                                                \Gamma \vdash e_0 : \tau_0
      \Gamma \vdash e_0 : \tau_0
                                                               \Gamma \vdash e_1 : \tau_1 \qquad \Gamma \vdash e : \tau'
 \frac{\Delta(op^1,\tau_0)=\tau}{\Gamma\vdash op^1\ e_0:\tau} \quad \frac{\Delta(op^2,\tau_0,\tau_1)=\tau}{\Gamma\vdash op^2\ e_0\ e_1:\tau} \quad \frac{\tau'\leqslant \tau}{\Gamma\vdash e:\tau} \quad \frac{\Gamma\vdash \mathsf{Err}:\tau}{\Gamma\vdash \mathsf{Err}:\tau}
                                                                     \frac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn} \ \tau \ e : \tau}
 \Gamma \vdash_{\mathsf{F}} e
      \frac{\Gamma \vdash_{\!\! E} e_0 \quad \Gamma \vdash_{\!\! E} e_1}{\Gamma \vdash_{\!\! E} \langle e_0, e_1 \rangle} \quad \frac{\Gamma \vdash_{\!\! E} e_0 \quad \Gamma \vdash_{\!\! E} e_1}{\Gamma \vdash_{\!\! E} e_0 \ e_1} \quad \frac{\Gamma \vdash_{\!\! E} e}{\Gamma \vdash_{\!\! E} op^1 \ e}
      \frac{\Gamma \vdash_{\!\!\!E} e_0 \quad \Gamma \vdash_{\!\!\!E} e_1}{\Gamma \vdash_{\!\!\!E} op^2 e_0 \; e_1} \quad \frac{\Gamma \vdash_{\!\!\!E} e}{\Gamma \vdash_{\!\!\!E} \mathsf{Err}} \quad \frac{\Gamma \vdash_{\!\!\!E} e}{\Gamma \vdash_{\!\!\!E} \mathsf{dyn} \; \tau \; e} \quad \frac{\Gamma \vdash_{\!\!\!E} e}{\Gamma \vdash_{\!\!\!E} \mathsf{stat} \; \tau \; e}
  \delta(op^1, v) = e
    \delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0
    \delta(\mathsf{snd}, \langle v_0, v_1 \rangle) = v_1
  \delta(op^2, v, v) = e
    \overline{\delta}(\text{sum}, i_0, i_1) = i_0 + i_1
    \delta(\text{quotient}, i_0, 0) = \text{BndryErr}
    \delta(\text{quotient}, i_0, i_1) = \lfloor i_0/i_1 \rfloor
        if i_1 \neq 0
  \mathcal{D}_{\mathsf{E}}: \tau \times v \longrightarrow e
    \mathcal{D}_{\mathsf{F}}(\tau,\upsilon) = \upsilon
  S_{\mathsf{E}}: \tau \times v \longrightarrow e
    S_{\mathsf{F}}(\tau, v) = v
```

```
e \rhd_{\mathsf{E-S}} e
3081
3082
               dyn \tau v
                                         \rhd_{\mathsf{E-S}} \ \mathcal{D}_{\mathsf{E}}(\tau,v)
3083
                                         \rhd_{\mathsf{E-S}} \mathcal{S}_{\mathsf{E}}(\tau, v)
               stat \tau v
3084
                                         ⊳<sub>E-S</sub> TagErr
               v_0 v_1
3085
                   if v_0 \in \mathbb{Z} or v_0 = \langle v, v' \rangle
3086
               (\lambda(x:\tau).e)v \triangleright_{\mathsf{E-S}} e[x \leftarrow v]
3087
               (\lambda x. e) v
                                         \triangleright_{\mathsf{E-S}} e[x \leftarrow v]
3088
               op^1 v
                                         ⊳<sub>E-S</sub> TagErr
3089
                   if \delta(op^1, v) is undefined
3090
               op^1 v
                                         \triangleright_{\mathsf{E-S}} \delta(op^1, v)
3091
               op^2 v_0 v_1
                                         ⊳<sub>E-S</sub> TagErr
3092
                   if \delta(op^2, v_0, v_1) is undefined
3093
               op^2 v_0 v_1
                                         \triangleright_{\mathsf{E-S}} \delta(op^2, v_0, v_1)
3094
              e \rhd_{\mathsf{E-D}} e
3095
               stat \tau v
                                         \rhd_{\mathsf{E-D}} \mathcal{S}_{\mathsf{E}}(\tau, v)
3096
                                         \triangleright_{\mathsf{E-D}} \mathcal{D}_{\mathsf{E}}(\tau,v)
               dyn \tau v
3097
               v_0 v_1
                                         ⊳<sub>E-D</sub> TagErr
3098
                   if v_0 \in \mathbb{Z} or v_0 = \langle v, v' \rangle
3099
               (\lambda(x:\tau).e) v \triangleright_{\mathsf{E-D}} e[x \leftarrow v]
3100
               (\lambda x. e) v
                                        \triangleright_{\mathsf{E-D}} e[x \leftarrow v]
3101
               op^1 v
                                         ⊳<sub>E-D</sub> TagErr
3102
                   if \delta(op^1, v) is undefined
3103
               op^1 v
                                      \triangleright_{\mathsf{E-D}} \delta(op^1, v)
3104
               op^2 v_0 v_1
                                        ⊳<sub>E-D</sub> TagErr
3105
                   if \delta(op^2, v_0, v_1) is undefined
3106
                                        \triangleright_{\mathsf{E-D}} \delta(op^2, v_0, v_1)
               op^2 v_0 v_1
3107
              e \rightarrow_{\mathsf{E-S}} e
3108
               E[e] \longrightarrow_{\mathsf{E-S}} E[e']
3109
                   if e \triangleright_{\mathsf{E-S}} e'
3110
               E[Err] \rightarrow_{E-S} Err
3111
              e \rightarrow_{\mathsf{E-D}} e
3112
               E[e] \longrightarrow_{\mathsf{E-D}} E[e']
3113
3114
                   if e \rhd_{\mathsf{E-D}} e'
               E[\mathsf{Err}] \to_{\mathsf{E-D}} \mathsf{Err}
3115
3116
              e \rightarrow_{E-S}^* e reflexive, transitive closure of \rightarrow_{E-S}
3117
3118
              e \rightarrow_{E-D}^{*} e | reflexive, transitive closure of \rightarrow_{E-D}
3119
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```

3191	A.3.2 Erasure Theorems	3246
3192	<b>Theorem 3.0</b> : static E-soundness	3247
3193	If $\vdash e : \tau$ then $\vdash_{E} e$ and one of the following holds:	3248
3194	$\bullet \ e \rightarrow_{E-S}^* v \text{ and } \vdash_{E} v$	3249
3195	• $e \rightarrow_{E-S}^* TagErr$	3250
3196	• $e \rightarrow_{E-S}^* BndryErr$	3251
3197	• <i>e</i> diverges	3252
3198	Proof:	3253
3199	1. <sub>E</sub> e	3254 3255
3200 3201	by static subset	3256
3202	2. QED by progress and preservation	3257
3203		3258
3204	Theorem 3.1: dynamic E-soundness	3259
3205	If $\vdash$ e then $\vdash$ <sub>E</sub> e and one of the following holds:	3260
3206	• $e \to_{E-D}^* v$ and $\vdash_{E} v$	3261
3207	• $e \rightarrow_{E-D}^* TagErr$ • $e \rightarrow_{E-D}^* BndryErr$	3262
3208	• $e$ diverges	3263
3209	Proof:	3264
3210	$1. \xrightarrow{*}_{E-D}^{*} = \xrightarrow{*}_{E-S}^{*}$	3265
3211	by definition	3266
3212	2. QED by static E soundness	3267
3213		3268
3214	Remark 3.2 : E-compilation	3269
3215	The $\rightarrow_{E-S}^*$ and $\rightarrow_{E-D}^*$ relations are identical. In practice, uses	3270
3216	of $\rightarrow_{F-S}^*$ may be replaced with $\rightarrow_{F-D}^*$ .	3271
3217		3272
3218		3273
3219	Theorem 3.3: boundary-free E-soundness  If Late 7 and a is boundary free then one of the following	3274
3220	If $\vdash e : \tau$ and $e$ is boundary-free then one of the following holds:	3275
3221 3222	• $e \to_{E-S}^* v$ and $\vdash v : \tau$	3276 3277
3223	• $e \rightarrow_{E-S}^{E-S}$ BndryErr	3278
3224	• e diverges	3279
3225	Proof:	3280
3226	QED by boundary-free progress and broundary-free preser-	3281
3227	_vation.	3282
3228		3283
3229		3284
3230		3285
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3242 3243		3297
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3245	30	3300

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3301
            A.3.3 Erasure Lemmas
3302
            Lemma 3.4 : \mathcal{D}_{E} soundness
3303
            If \vdash_{\mathsf{E}} v then \vdash_{\mathsf{E}} \mathcal{D}_{\mathsf{E}}(\tau, v).
3304
            Proof:
3305
               CASE \mathcal{D}_{\mathsf{E}}(\tau, v) = v:
3306
                  1. QED
3307
3308
            Lemma 3.5 : S_E soundness
3309
            If \vdash_{\mathsf{F}} v then \vdash_{\mathsf{F}} \mathcal{S}_{\mathsf{E}}(\tau, v).
3310
            Proof:
3311
               CASE S_{E}(\tau, v) = v:
3312
                  1. QED
3313
3314
            Lemma 3.6: static subset
3315
            If \Gamma \vdash e : \tau then \Gamma \vdash_{\mathsf{F}} e.
3316
3317
               By structural induction on the typing relation.
3318
3319
                              (x:\tau)\in\Gamma
3320
                              \Gamma \vdash x : \tau
3321
                  1. (x:\tau) \in \Gamma
3322
                  2. \Gamma \vdash_{\mathsf{F}} x
3323
                       by (1)
3324
                  3. QED
3325
                             \frac{(x\!:\!\tau_d),\Gamma\vdash e:\tau_c}{\Gamma\vdash \lambda(x\!:\!\tau_d).\,e:\tau_d\Rightarrow\tau_c}
3326
                CASE
3327
3328
                  1. (x:\tau_d), \Gamma \vdash_{\mathsf{E}} e
3329
                        by the induction hypothesis
3330
                  2. \Gamma \vdash_{\mathsf{E}} \lambda(x : \tau_d). e
3331
                        by (1)
3332
                  3. QED
3333
3334
                CASE
3335
3336
                  1. QED
3337
                CASE
3338
                              \Gamma \vdash_{\!\!E} i : \mathsf{Int}
3339
                  1. QED
3340
                              \Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1
                CASE
3341
                                 \Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1
3342
3343
                  1. \Gamma \vdash_{\mathsf{E}} e_0
3344
                        \wedge \Gamma \vdash_{\mathsf{F}} e_1
3345
                        by the induction hypothesis
3346
                  2. \Gamma \vdash_{\mathsf{F}} \langle e_0, e_1 \rangle
3347
                        by (1)
3348
                  3. QED
3349
                              \Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash e_1 : \tau_d
                CASE
3350
                                            \Gamma \vdash e_0 \; e_1 : \tau_c
3351
                  1. \Gamma \vdash_{\overline{E}} e_0
3352
3353
                        \wedge \Gamma \vdash_{\mathsf{F}} e_1
                        by the induction hypothesis
3354
3355
```

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2. \Gamma \vdash_{\mathsf{E}} e_0 \ e_1
           by (1)
      3. QED
                 \Gamma \vdash e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau \mid_{:}
    CASE
                             \Gamma \vdash op^1 e_0 : \tau
       1. \Gamma \vdash_{\mathsf{E}} e_0
           by the induction hypothesis
      2. \Gamma \vdash_{\mathsf{F}} op^1 e_0
           by (1)
      3. QED
                  \Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau
                                         \Gamma \vdash op^2 e_0 e_1 : \tau
       1. \Gamma \vdash_{\mathsf{F}} e_0
            \wedge \Gamma \vdash_{\mathsf{F}} e_1
           by the induction hypothesis
       2. \Gamma \vdash_{\mathsf{F}} op^2 e_0 e_1
           by (1)
      3. QED
                 \Gamma \vdash e : \tau' \quad \tau' \mathrel{<:} \tau
      1. Γ <del>|</del><sub>F</sub> e
           by the induction hypothesis
       2. QED
    CASE
                  \Gamma \vdash \mathsf{Err} : \tau
      1. QED
    CASE
                  \Gamma \vdash \mathsf{dyn} \ \tau \ e : \tau
      1. Γ <del>| e</del>
           by dynamic subset
       2. \Gamma \vdash_{\mathsf{E}} \mathsf{dyn} \ \tau \ e
           by (1)
       3. QED
Lemma 3.7 : dynamic subset
If \Gamma \vdash e then \Gamma \vdash_{\mathsf{F}} e.
Proof:
    By structural induction on the ⊢ relation.
                 x \in \Gamma
    CASE
                  \Gamma \vdash x
      1. x \in \Gamma
       2. \Gamma \vdash_{\mathsf{E}} x
           by (1)
      3. QED
    CASE
            by the induction hypothesis
```

3520

```
3411
                  2. \Gamma \vdash_{\mathsf{E}} \lambda x. e
3412
                       by (1)
3413
                  3. QED
3414
                CASE
3415
                              \Gamma \vdash i
3416
                  1. QED
3417
                              \Gamma \vdash e_0 \quad \Gamma \vdash e_1
                CASE
3418
                                \Gamma \vdash \langle e_0, e_1 \rangle
3419
                  1. \Gamma \vdash_{\mathsf{F}} e_0
3420
3421
                       \wedge \Gamma \vdash_{\mathsf{F}} e_1
                       by the induction hypothesis
3422
                  2. \Gamma \vdash_{\mathsf{E}} \langle e_0, e_1 \rangle
3423
3424
                       by (1)
3425
                  3. QED
3426
                              \Gamma \vdash e_0 \quad \Gamma \vdash e_1
                CASE
3427
                                   \Gamma \vdash e_0 \ e_1
3428
                  1. \Gamma \vdash_{\mathsf{E}} e_0
3429
                       \wedge \Gamma \vdash_{\mathsf{F}} e_1
3430
                       by the induction hypothesis
3431
                  2. \Gamma \vdash_{\mathsf{F}} e_0 \ e_1
3432
                       by (1)
3433
                  3. QED
3434
3435
                CASE
3436
                              \Gamma \vdash op^1 e
3437
                  1. \Gamma \vdash_{\mathsf{F}} e_0
3438
                       by the induction hypothesis
3439
                  2. \Gamma \vdash_{\mathsf{F}} op^1 e_0
3440
                       by (1)
3441
                  3. QED
3442
                              \Gamma \vdash e_0 \quad \Gamma \vdash e_1.
                CASE
3443
                               \Gamma \vdash op^2 e_0 e_1
3444
3445
                  1. \Gamma \vdash_{\mathsf{E}} e_0
3446
                       \wedge \Gamma \vdash_{\mathsf{F}} e_1
3447
                       by the induction hypothesis
3448
                  2. \Gamma \vdash_{\mathsf{E}} op^2 e_0 e_1
3449
                       by (1)
3450
                  3. QED
3451
                CASE
3452
                              \Gamma \vdash \mathsf{Err}
3453
                  1. QED
3454
                                \Gamma \vdash e : \tau
                CASE
3455
                              \Gamma \vdash \mathsf{stat} \ \tau \ e
3456
3457
                  1. Γ ⊢<sub>F</sub> e
3458
                       by static subset
3459
                  2. \Gamma ⊢ stat \tau e
3460
                       by (1)
3461
                  3. QED
3462
3463
            Lemma 3.8 : E progress
3464
```

```
If \vdash_{\mathsf{F}} e then one of the following holds:
                                                                                                       3466
  • e is a value
                                                                                                       3467
  • e \in Err
                                                                                                       3468
  • e \rightarrow_{E-S} e'
                                                                                                       3469
  • e \rightarrow_{E-S} TagErr
                                                                                                       3470
  • e \rightarrow_{E-S} BndryErr
                                                                                                       3471
Proof:
                                                                                                       3472
   By the unique evaluation contexts lemma, there are seven
                                                                                                       3473
   possible cases.
                                                                                                       3474
   CASE e is a value :
                                                                                                       3475
     1. QED
                                                                                                       3476
   CASE e = E[v_0 \ v_1]:
                                                                                                       3477
      IF v_0 = \lambda x. e':
                                                                                                       3478
         1. e \rightarrow_{E-S} E[e'[x \leftarrow v_1]]
                                                                                                       3479
             by v_0 \ v_1 \rhd_{\mathsf{E-S}} e'[x \leftarrow v_1]
                                                                                                       3480
         2. QED
                                                                                                       3481
      IF v_0 = \lambda(x:\tau).e':
                                                                                                       3482
         1. e \rightarrow_{E-S} E[e'[x \leftarrow v_1]]
                                                                                                       3483
             by v_0 \ v_1 \rhd_{\mathsf{E-S}} e'[x \leftarrow v_1]
                                                                                                       3484
        2. QED
                                                                                                       3485
      ELSE v_0 = i
                                                                                                       3486
                 \vee v_0 = \langle v, v' \rangle:
                                                                                                       3487
         1. e \rightarrow_{E-S} TagErr
                                                                                                       3488
             by v_0 v_1 \triangleright_{\mathsf{E-S}} \mathsf{TagErr}
                                                                                                       3489
         2. QED
                                                                                                       3490
   CASE e = E[op^1 v]:
                                                                                                       3491
      IF \delta(op^1, v) = e'':
                                                                                                       3492
        1. e \rightarrow_{\mathsf{E-S}} E[e'']
                                                                                                       3493
             by (op^1 v) \triangleright_{\mathsf{E-S}} e''
                                                                                                       3494
        2. QED
                                                                                                       3495
      ELSE \delta(op^1, v) is undefined :
                                                                                                       3496
         1. e \rightarrow_{E-S} TagErr
                                                                                                       3497
             by (op^1 v) \triangleright_{E-S} \mathsf{TagErr}
                                                                                                       3498
         2. QED
                                                                                                       3499
   CASE e = E[op^2 v_0 v_1]:
                                                                                                       3500
      IF \delta(op^2, v_0, v_1) = e'':
                                                                                                       3501
         1. e \rightarrow_{\mathsf{E-S}} E[e'']
                                                                                                       3502
             by (op^2 v_0 v_1) \triangleright_{\mathsf{E-S}} e''
                                                                                                       3503
         2. QED
                                                                                                       3504
      IF \delta(op^2, v_0, v_1) = \text{BndryErr}:
                                                                                                       3505
         1. e \rightarrow_{E-S} BndryErr
                                                                                                       3506
             by (op^2 v_0 v_1) \triangleright_{\mathsf{E-S}} \mathsf{BndryErr}
                                                                                                       3507
         2. QED
                                                                                                       3508
      ELSE \delta(op^2, v_0, v_1) is undefined :
                                                                                                       3509
         1. e \rightarrow_{E-S} TagErr
                                                                                                       3510
             by (op^2 v_0 v_1) \triangleright_{\mathsf{F-S}} \mathsf{TagErr}
                                                                                                       3511
         2. QED
                                                                                                       3512
   CASE e = E[\mathsf{dyn} \ \tau \ v]:
                                                                                                       3513
     1. e \rightarrow_{E-S} E[\mathcal{D}_{E}(\tau, v)]
                                                                                                       3514
     2. QED
                                                                                                       3515
   CASE e = E[\operatorname{stat} \tau v]:
                                                                                                       3516
     1. e \rightarrow_{\mathsf{E-S}} E[\mathcal{S}_{\mathsf{E}}(\tau, v)]
                                                                                                       3517
     2. QED
                                                                                                       3518
   CASE eE[Err]:
```

```
3521
               1. e \rightarrow_{E-S} Err
                                                                                                                   1. E[\text{stat } \tau \ v] \rightarrow_{E-S} S_E(\tau, v)
                                                                                                                                                                                                                3576
3522
               2. QED
                                                                                                                   2. \vdash_{\mathsf{F}} stat \tau \ v
                                                                                                                                                                                                                3577
3523
                                                                                                                       by hole typing
                                                                                                                                                                                                                3578
3524
         Lemma 3.9 : E preservation
                                                                                                                   3. ⊢ υ
                                                                                                                                                                                                                3579
3525
                                                                                                                       by inversion (2)
         If \vdash_{\mathsf{F}} e and e \to_{\mathsf{E-S}} e' then \vdash_{\mathsf{E}} e'.
                                                                                                                                                                                                                3581
                                                                                                                   4. \vdash_{\mathsf{E}} \mathcal{S}_{\mathsf{E}}(\tau, v)
3526
         Proof:
3527
                                                                                                                       by S_F soundness (3)
                                                                                                                                                                                                                3582
            By unique evaluation contexts there are seven cases.
3528
                                                                                                                   5. QED by hole substitution (4)
                                                                                                                                                                                                                3583
             CASE e is a value :
3529
                                                                                                                 CASE e = E[Err]:
                                                                                                                                                                                                                3584
               1. Contradiction by e \rightarrow_{F-S} e'
3530
                                                                                                                   1. E[Err] \rightarrow_{E-S} Err
                                                                                                                                                                                                                3585
             CASE e = E[v_0 \ v_1]:
3531
                                                                                                                   2. QED
                                                                                                                                                                                                                3586
               1. v_0 = \lambda x. e' or v_0 = \lambda(x:\tau). e'
                                                                                                             3532
                                                                                                                                                                                                                3587
                   \wedge E[v_0 \ v_1] \rightarrow_{\mathsf{E-S}} E[e'[x \leftarrow v_1]]
3533
                                                                                                                                                                                                                3588
               2. \vdash_{\mathsf{F}} v_0 \ v_1
                                                                                                             Lemma 3.10: E boundary-free progress
                                                                                                               If \vdash e : \tau and e is boundary-free, then one of the following
3534
                                                                                                                                                                                                                3589
                   by hole typing
3535
                                                                                                                                                                                                                3590
                                                                                                               holds:
               3. \vdash_{\mathsf{F}} v_0
3536
                                                                                                                                                                                                                3591
                                                                                                               • e is a value
                   \wedge \vdash_{\mathsf{F}} v_1
3537
                                                                                                                                                                                                                3592
                                                                                                               • e \in Err
                   by inversion (2)
3538
                                                                                                                                                                                                                3593
                                                                                                               • e \rightarrow_{E-S} e'
               4. x \vdash_{\mathsf{E}} e'
3539
                                                                                                               • e \rightarrow_{E-S} BndryErr
                                                                                                                                                                                                                3594
                   by inversion (3)
                                                                                                             Proof:
3540
                                                                                                                                                                                                                3595
               5. \vdash_{\mathsf{F}} e'[x \leftarrow v_1]
3541
                                                                                                                 By the unique static evaluation contexts lemma, there are
                                                                                                                                                                                                                3596
                   by substitution (3, 4)
                                                                                                                 five cases:
3542
                                                                                                                                                                                                                3597
               6. QED by hole substitution (5)
                                                                                                                 CASE e = v:
3543
                                                                                                                                                                                                                3598
             CASE e = E[op^1 v]:
                                                                                                                   1. QED
                                                                                                                                                                                                                3599
               1. E[op^1 v] \rightarrow_{E-S} E[v']
                                                                                                                 CASE e = E[v_0 \ v_1]:
3545
                   \wedge \delta(op^1, v) = e''
                                                                                                                                                                                                                3600
                                                                                                                    IF v_0 = \lambda(x:\tau').e':
                                                                                                                                                                                                                3601
               2. \vdash_{\mathsf{F}} op^1 v
3547
                                                                                                                      1. e \rightarrow_{\mathsf{E-S}} E[e'[x \leftarrow v_1]]
                                                                                                                                                                                                                3602
                   by hole typing
                                                                                                                          by v_0 \ v_1 \triangleright_{\mathsf{E-S}} e'[x \leftarrow v_1]
                                                                                                                                                                                                                3603
               3. F v
                                                                                                                      2. QED
3549
                                                                                                                                                                                                                3604
                   by inversion (2)
                                                                                                                    ELSE v_0 = \lambda x. e'
3550
                                                                                                                                                                                                                3605
               4. ⊢ e''
                                                                                                                              \vee v_0 = i
3551
                                                                                                                                                                                                                3606
                   by \delta preservation (1,3)
                                                                                                                              \vee v_0 = \langle v, v' \rangle:
                                                                                                                                                                                                                3607
               5. QED by hole substitution (4)
                                                                                                                      1. Contradiction by \vdash e : \tau
3553
                                                                                                                                                                                                                3608
             CASE e = E[op^2 v_0 v_1]:
                                                                                                                 CASE e = E[op^1 v]:
3554
                                                                                                                                                                                                                3609
               1. E[op^2 v_0 v_1] \to_{E-S} E[v']
                                                                                                                    IF \delta(op^1, v) = e'':
3555
                                                                                                                                                                                                                3610
                   \wedge \delta(op^2, v_0, v_1) = e^{\prime\prime}
                                                                                                                      1. e \rightarrow_{\mathsf{F-S}} E[e'']
3556
                                                                                                                                                                                                                3611
               2. \vdash_{\mathsf{F}} op^2 v_0 v_1
                                                                                                                          by (op^1 v) \triangleright_{E-S} e''
3557
                                                                                                                                                                                                                3612
                   by hole typing
3558
                                                                                                                                                                                                                3613
               3. \vdash_{\mathsf{F}} v_0
                                                                                                                    ELSE \delta(op^1, v) is undefined :
                                                                                                                                                                                                                3614
                   \wedge \vdash_{\mathsf{F}} v_1
                                                                                                                      1. Contradiction by \vdash e : \tau
                                                                                                                                                                                                                3615
3560
                   by inversion (2)
                                                                                                                 CASE e = E[op^2 v_0 v_1]:
3561
                                                                                                                                                                                                                3616
               4. ⊢ e"
                                                                                                                    IF \delta(op^2, v_0, v_1) = e'':
3562
                                                                                                                                                                                                                3617
                   by \delta preservation (3)
                                                                                                                      1. e \rightarrow_{E-S} E[e'']
                                                                                                                                                                                                                3618
               5. QED by hole substitution (4)
                                                                                                                          by (op^2 v_0 v_1) \triangleright_{\mathsf{E-S}} e''
                                                                                                                                                                                                                3619
3564
             CASE e = E[\mathsf{dyn} \ \tau \ v]:
3565
                                                                                                                                                                                                                3620
               1. E[\mathsf{dyn} \ \tau \ v] \rightarrow_{\mathsf{E-S}} E[\mathcal{D}_{\mathsf{E}}(\tau, v)]
                                                                                                                    IF \delta(op^2, v_0, v_1) = \text{BndryErr}:
3566
                                                                                                                                                                                                                3621
               2. \vdash_{\mathsf{F}} \mathsf{dyn} \, \tau \, v
                                                                                                                      1. e \rightarrow_{E-S} BndryErr
                                                                                                                                                                                                                3622
                   by hole typing
                                                                                                                          by (op^2 v_0 v_1) \triangleright_{E-S} BndryErr
3568
                                                                                                                                                                                                                3623
               3. \ <sub>F</sub> v
3569
                                                                                                                      2. QED
                                                                                                                                                                                                                3624
                   by inversion (2)
                                                                                                                    ELSE \delta(op^2, v_0, v_1) is undefined :
3570
                                                                                                                                                                                                                3625
               4. \vdash_{\mathsf{E}} \mathcal{D}_{\mathsf{E}}(\tau, v)
                                                                                                                      1. Contradiction by \vdash e : \tau
3571
                                                                                                                                                                                                                3626
                   by \mathcal{D}_{E} soundness (3)
                                                                                                                 CASE e = E[Err]:
3572
                                                                                                                                                                                                                3627
               5. QED by hole substitution (4)
                                                                                                                   1. E[Err] \rightarrow_{E-S} Err
3573
                                                                                                                                                                                                                3628
             CASE e = E[\text{stat } \tau \ v]:
3574
                                                                                                                   2. QED
                                                                                                                                                                                                                3629
3575
                                                                                                                                                                                                                3630
                                                                                                       33
```

```
If \vdash_{\mathsf{F}} e then one of the following holds:
         3631
                                                                                                                                                                                                 3686
                                                                                                        • e is a value
3632
                                                                                                                                                                                                 3687
         Lemma 3.11: E boundary-free preservation
                                                                                                        • e = E[v_0 \ v_1]
3633
         If \vdash e : \tau and e is boundary-free and e \rightarrow_{E-S} e' then \vdash e' : \tau
                                                                                                                                                                                                 3688
                                                                                                        • e = E[op^1 v]
3634
           and e' is boundary-free.
                                                                                                                                                                                                 3689
                                                                                                        • e = E[op^2 v_0 v_1]
         Proof:
                                                                                                        • e = E[\mathsf{dyn} \ \tau \ v]
           By the unique static evaluation contexts lemma, there are
                                                                                                                                                                                                 3691
3636
                                                                                                        • e = E[\text{stat } \tau \ v]
3637
            five cases.
                                                                                                                                                                                                 3692
                                                                                                        • e = E[Err]
            CASE e is a value :
3638
                                                                                                                                                                                                 3693
                                                                                                      Proof:
              1. Contradiction by e \rightarrow_{F-S} e'
3639
                                                                                                                                                                                                 3694
                                                                                                         By induction on the structure of e.
            CASE e = E[v_0 \ v_1]:
3640
                                                                                                                                                                                                 3695
                                                                                                         CASE e = x:
               IF v_0 = \lambda(x:\tau_d).e':
3641
                                                                                                                                                                                                 3696
                                                                                                           1. Contradiction by \vdash_{E} e
                 1. E[v_0 \ v_1] \rightarrow_{\mathsf{E-S}} E[e'[x \leftarrow v_1]]
3642
                                                                                                                                                                                                 3697
                                                                                                         CASE e = i
                 2. \vdash v_0 \ v_1 : \tau_c
3643
                                                                                                                                                                                                 3698
                                                                                                                  \vee e = \lambda x. e'
                 3. \vdash v_0 : \tau_d \Rightarrow \tau_c
3644
                                                                                                                                                                                                 3699
                                                                                                                  \vee e = \lambda(x:\tau_d).e':
3645
                     \wedge \vdash v_1 : \tau_d
                                                                                                                                                                                                 3700
                                                                                                           1. QED
3646
                     by (2)
                                                                                                                                                                                                 3701
                                                                                                         CASE e = \langle e_0, e_1 \rangle:
                 4. (x:\tau_d) \vdash e' : \tau_c
3647
                                                                                                                                                                                                 3702
                                                                                                            IF e_0 \notin v:
                    by (3)
3648
                                                                                                                                                                                                 3703
                                                                                                              1. \vdash_{\mathsf{F}} e_0
                 5. \vdash e'[x \leftarrow v_1] : \tau_c
                                                                                                                                                                                                 3704
3649
                                                                                                                  by inversion
                    by substitution (3, 4)
                                                                                                                                                                                                 3705
                                                                                                              2. e_0 = E_0[e_0']
                 6. e'[x \leftarrow v_1] is boundary-free
3651
                                                                                                                                                                                                 3706
                                                                                                                  by the induction hypothesis
                     by e' and v_1 are boundary-free
3652
                                                                                                                                                                                                 3707
                                                                                                              3. E = \langle E_0, e_1 \rangle
                7. QED
                                                                                                                                                                                                 3708
3653
                                                                                                              4. QED e = E[e'_0]
               ELSE:
3654
                                                                                                                                                                                                 3709
                                                                                                            IF e_0 \in v
                 1. Contradiction by \vdash e : \tau
3655
                                                                                                                                                                                                 3710
                                                                                                                 \land e_1 \notin v:
            CASE e = E[op^1 v]:
3656
                                                                                                                                                                                                 3711
                                                                                                              1. \vdash_{\mathsf{F}} e_1
              1. E[op^1 v] \rightarrow_{\mathsf{E-S}} E[v']
                                                                                                                                                                                                 3712
3657
                                                                                                                  by inversion
                  \wedge \delta(op^1, v) = e^{\prime\prime}
                                                                                                                                                                                                 3713
                                                                                                              2. e_1 = E_1[e_1]
              2. \vdash op^1 \upsilon : \tau'
3659
                                                                                                                                                                                                 3714
                                                                                                                  by the induction hypothesis
              3. \vdash v : \tau_0
3660
                                                                                                                                                                                                 3715
                                                                                                              3. E = \langle e_0, E_1 \rangle
              4. ⊢ e'' : τ'
3661
                                                                                                                                                                                                 3716
                                                                                                              4. QED e = E[e'_1]
                  by \delta preservation (3)
                                                                                                                                                                                                 3717
                                                                                                            ELSE e_0 \in v
              5. QED
3663
                                                                                                                                                                                                 3718
                                                                                                                     \land e_1 \in v:
            CASE e = E[op^2 v_0 v_1]:
                                                                                                                                                                                                 3719
3664
                                                                                                              1. QED
              1. E[op^2 v_0 v_1] \rightarrow_{\mathsf{E-S}} E[v']
                                                                                                                                                                                                 3720
3665
                                                                                                         CASE e = e_0 e_1:
                  \wedge \delta(op^2, v_0, v_1) = e^{\prime\prime}
3666
                                                                                                                                                                                                 3721
                                                                                                            IF e_0 \notin v:
              2. \vdash op^2 v_0 v_1 : \tau'
3667
                                                                                                                                                                                                 3722
                                                                                                              1. \vdash<sub>F</sub> e_0
              3. \vdash v_0 : \tau_0
3668
                                                                                                                                                                                                 3723
                                                                                                                  by inversion
                  \wedge \vdash v_1 : \tau_1
                                                                                                              2. e_0 = E_0[e_0']
              4. \vdash e'' : τ'
                                                                                                                                                                                                 3725
3670
                                                                                                                  by the induction hypothesis
                  by \delta preservation (3)
3671
                                                                                                                                                                                                 3726
                                                                                                              3. E = E_0 e_1
3672
              5. QED
                                                                                                                                                                                                 3727
                                                                                                              4. QED e = E[e'_0]
            CASE e = E[Err]:
3673
                                                                                                                                                                                                 3728
                                                                                                            IF e_0 \in v
              1. E[Err] \rightarrow_{E-S} Err
3674
                                                                                                                                                                                                 3729
                                                                                                                 \land e_1 \notin v:
              2. QED by \vdash Err : \tau
3675
                                                                                                                                                                                                 3730
                                                                                                              1. \vdash_{\mathsf{F}} e_1
         3676
                                                                                                                                                                                                 3731
                                                                                                                  by inversion
3677
                                                                                                                                                                                                 3732
         Lemma 3.12 : E unique evaluation contexts
                                                                                                              2. e_1 = E_1[e_1]
3678
                                                                                                                                                                                                 3733
                                                                                                                  by the induction hypothesis
3679
                                                                                                                                                                                                 3734
                                                                                                              3. E = e_0 E_1
3680
                                                                                                                                                                                                 3735
                                                                                                              4. QED e = E[e'_1]
                                                                                                                                                                                                 3736
3681
                                                                                                            ELSE e_0 \in v
3682
                                                                                                                                                                                                 3737
                                                                                                                     \land e_1 \in v:
3683
                                                                                                                                                                                                 3738
                                                                                                              1. E = []
3684
                                                                                                                                                                                                 3739
3685
                                                                                                                                                                                                 3740
                                                                                                34
```

3741	2. QED	<b>CASE</b> $e = Err$ :	3796
3742	CASE $e = op^1 e_0$ :	1. E = []	3797
3743	IF $e_0 \notin v$ :	2. QED	3798
3744	1. $\vdash_{E} e_0$		3799
3745	by inversion	<b>Lemma 3.13</b> : E hole typing	3800
3746	2. $e_0 = E_0[e'_0]$	If $\vdash_{E} E[e]$ then the derivation contains a sub-term $\vdash_{E} e$	3801
3747	by the induction hypothesis	Proof:	3802
3748	3. $E = op^1 E_0$	By induction on the structure of <i>E</i> .	3803
3749	4. QED $e = E[e'_0]$	CASE $E = []$ :	3804
3750	ELSE $e_0 \in v$ :	1. QED $E[e] = e$	3805
3751	1. $E = []$	<b>CASE</b> $E = E_0 e_1$ :	3806
3752	2. QED	1. $E[e] = E_0[e] e_1$	3807
3753	$\mathbf{CASE} \ \ e = op^2 \ e_0 \ e_1 :$	$2. \vdash_{E} E_0[e]$	3808
3754	<b>IF</b> $e_0 \notin v$ :	by inversion	3809
3755	1. ⊢ <sub>E</sub> e <sub>0</sub>	3. QED by the induction hypothesis (2)	3810
3756	by inversion	CASE $E = v_0 E_1$ :	3811
3757	2. $e_0 = E_0[e'_0]$	1. $E[e] = v_0 E_1[e]$	3812
3758	by the induction hypothesis	2. $\vdash_{E} E_1[e]$	3813
3759	3. $E = op^2 E_0 e_1$	by inversion	3814
3760	4. QED $e = E[e'_0]$	3. QED by the induction hypothesis (2)	3815
3761	IF $e_0 \in v$	CASE $E = \langle E_0, e_1 \rangle$ :	3816
3762	$\wedge e_1 \notin v$ :	1. $E[e] = \langle E_0[e], e_1 \rangle$	3817
3763	1. $\vdash_{E} e_1$	2. $\vdash_{E} E_0[e]$	3818
3764	by inversion	by inversion	3819 3820
3765	2. $e_1 = E_1[e'_1]$	3. QED by the induction hypothesis (2)	
3766	by the induction hypothesis $\frac{F}{2} = \frac{F}{2} = \frac{F}{2}$	$\mathbf{CASE} \ E = \langle v_0, E_1 \rangle :$	3821 3822
3767 3768	3. $E = op^2 e_0 E_1$	1. $E[e] = \langle v_0, E_1[e] \rangle$	3823
3769	4. QED $e=E[e_1']$ ELSE $e_0\in v$	$2. \vdash_{E} E_1[e]$	3824
3770	$\wedge e_1 \in v:$	by inversion	3825
3771	1. $E = \begin{bmatrix} 1 \end{bmatrix}$	3. QED by the induction hypothesis (2)	3826
3772	2. QED	$\mathbf{CASE} \ E = op^1 E_0 :$	3827
3773	<b>CASE</b> $e = \text{dyn } \tau e_0$ :	1. $E[e] = op^1 E_0[e]$	3828
3774	<b>IF</b> $e_0 \notin v$ :	2. $\vdash_{E} E_0[e]$	3829
3775	1. $\vdash_{E} e_0$	by inversion	3830
3776	by inversion	3. QED by the induction hypothesis (2)	3831
3777	2. $e_0 = E_0[e'_0]$	<b>CASE</b> $E = op^2 E_0 e_1$ : 1. $E[e] = op^2 E_0[e] e_1$	3832
3778	by the induction hypothesis	1. $E[e] = op E_0[e] e_1$ 2. $\vdash_{\mathbb{E}} E_0[e]$	3833
3779	3. $E = \operatorname{dyn} \tau E_0$	by inversion	3834
3780	4. QED $e = E[e'_0]$	3. QED by the induction hypothesis (2)	3835
3781	ELSE $e_0 \in v$ :	<b>CASE</b> $E = op^2 v_0 E_1$ :	3836
3782	1. $E = []$	1. $E[e] = op^2 v_0 E_1[e]$	3837
3783	2. QED	2. $\vdash_{F} E_1[e]$	3838
3784	<b>CASE</b> $e = \operatorname{stat} \tau e_0$ :	by inversion	3839
3785	IF $e_0 \notin v$ :	3. QED by the induction hypothesis (2)	3840
3786	1. ⊢ <sub>E</sub> e <sub>0</sub>	case $E = \text{dyn } \tau E_0$ :	3841
3787	by inversion	1. $E[e] = \operatorname{dyn} \tau E_0[e]$	3842
3788	2. $e_0 = E_0[e_0']$	2. $\vdash_{F} E_0[e]$	3843
3789	by the induction hypothesis	by inversion	3844
3790	3. $E = \operatorname{stat} \tau E_0$	3. QED by the induction hypothesis (2)	3845
3791	4. QED $e = E[e'_0]$	<b>CASE</b> $E = \operatorname{stat} \tau E_0$ :	3846
3792	ELSE $e_0 \in v$ :	1. $E[e] = \operatorname{stat} \tau E_0[e]$	3847
3793	1. $E = []$	r 1 of.1	3848
3794	2. QED		3849
3795		35	3850

3851	2. $\vdash_{E} E_0[e]$	$4. \vdash_{E} E_1[e']$	3906
3852	by inversion	by the induction hypothesis (3)	3907
3853	3. QED by the induction hypothesis (2)	5. $\vdash_{E} v_0 E_1[e']$	3908
3854		by (3, 4)	3909
3855	<b>Lemma 3.14</b> : E hole substitution	6. QED by (1, 5)	3910
3856	If $\vdash_{\overline{E}} E[e]$ and $\vdash_{\overline{E}} e'$ then $\vdash_{\overline{E}} E[e']$	CASE $E = op^1 E_0$ :	3911
3857 3858	Proof:	1. $E[e] = op^1 E_0[e]$ $\wedge E[e'] = op^1 E_0[e']$	3912 3913
3859	By induction on the structure of <i>E</i> .		3914
3860	CASE $E = []:$	2. $r_{E}$ op $L_{0}[e]$ 3. $r_{F}$ $E_{0}[e]$	3915
3861	1. QED $E[e'] = e'$	by inversion	3916
3862	CASE $E = \langle E_0, e_1 \rangle$ : 1. $E[e] = \langle E_0[e], e_1 \rangle$	4. $\vdash_{F} E_0[e']$	3917
3863	1. $E[e] = \langle E_0[e], e_1 \rangle$ $\wedge E[e'] = \langle E_0[e'], e_1 \rangle$	by the induction hypothesis (3)	3918
3864	$ \begin{array}{l} A E[e] - \langle E_0[e], e_1 \rangle \\ 2. \vdash_{F} \langle E_0[e], e_1 \rangle \end{array} $	5. $\vdash_{F} op^1 E_0[e']$	3919
3865	3. $\vdash_{E} E_0[e]$	by (3, 4)	3920
3866	$\wedge \vdash_{F} e_1$	6. QED by (1, 5)	3921
3867	by inversion	<b>CASE</b> $E = op^2 E_0 e_1$ :	3922
3868	4. $\vdash_{F} E_0[e']$	1. $E[e] = op^2 E_0[e] e_1$	3923
3869	by the induction hypothesis (3)	$\wedge E[e'] = op^2 E_0[e'] e_1$	3924
3870	5. $\vdash_{E} \langle E_0[e'], e_1 \rangle$	$2. \vdash_{E} op^2 E_0[e] e_1$	3925
3871	by (3, 4)	3. $\vdash_{E} E_0[e]$	3926
3872	6. QED by (1, 5)	$\land \vdash_{E} e_1$	3927
3873	CASE $E = \langle v_0, E_1 \rangle$ :	by inversion	3928
3874	1. $E[e] = \langle v_0, E_1[e] \rangle$	4. $\vdash_{E} E_0[e']$	3929
3875	$\wedge E[e'] = \langle v_0, E_1[e'] \rangle$	by the induction hypothesis (3)	3930
3876	2. $\vdash_{E} \langle v_0, E_1[e] \rangle$	5. $\vdash_{E} op^2 E_0[e'] e_1$	3931
3877	3. $\vdash_{E} v_0$	by (3, 4)	3932
3878	$\wedge \vdash_{E} E_1[e]$	6. QED by (1, 5)	3933
3879	by inversion	$\mathbf{CASE} \ E = op^2 \ v_0 \ E_1 :$	3934
3880	4. $\vdash_{E} E_1[e']$	1. $E[e] = op^2 v_0 E_1[e]$	3935
3881 3882	by the induction hypothesis (3)		3936 3937
3883	5. $\vdash_{E} \langle v_0, E_1[e'] \rangle$	<del>-</del>	3938
3884	by (3, 4)	3. $\vdash_{E} v_0$	3939
3885	6. QED by (1, 5)	$\wedge \vdash_{E} E_1[e]$ by inversion	3940
3886	CASE $E = E_0 e_1$ :	4. $\vdash_{E} E_1[e']$	3941
3887	1. $E[e] = E_0[e] e_1$	by the induction hypothesis (3)	3942
3888		5. $\vdash_{F} op^2 v_0 E_1[e']$	3943
3889	2. F <sub>E</sub> E <sub>0</sub> [e] e <sub>1</sub> 3. F <sub>E</sub> E <sub>0</sub> [e]	by (3, 4)	3944
3890	3. 'Ε΄ 120[ε] Λ 'Ε΄ ε1	6. QED by (1, 5)	3945
3891	by inversion	$\mathbf{CASE} \ E = dyn \ \tau \ E_0 :$	3946
3892	4. ⊢ <sub>E</sub> E <sub>0</sub> [e']	1. $E[e] = \operatorname{dyn} \tau E_0[e]$	3947
3893	by the induction hypothesis (3)	$\wedge E[e'] = \operatorname{dyn} \tau E_0[e']$	3948
3894	5. $\vdash_{F} E_0[e'] e_1$	2. $\vdash_{F} dyn \ \tau \ E_0[e]$	3949
3895	by (3, 4)	3. $\vdash_{E} E_0[e]$	3950
3896	6. QED by (1, 5)	by inversion	3951
3897	CASE $E = v_0 E_1$ :	4. $\vdash_{E} E_0[e']$	3952
3898	1. $E[e] = v_0 E_1[e]$	by the induction hypothesis (3)	3953
3899	$\wedge E[e'] = v_0 E_1[e']$	5. $\vdash_{E} dyn \ \tau \ E_0[e']$	3954
3900	$2. \vdash_{E} v_0 E_1[e]$	by (3, 4)	3955
3901	$3. \vdash_{E} v_0$	6. QED by (1, 5)	3956
3902	$\wedge \vdash_{E} E_1[e]$	<b>CASE</b> $E = \operatorname{stat} \tau E_0$ :	3957
3903	by inversion	1. $E[e] = \operatorname{stat} \tau E_0[e]$	3958
3904		$\wedge E[e'] = \operatorname{stat} \tau E_0[e']$	3959
3905		36	3960

```
3961
                                                                                                                                                 4. \Gamma \vdash_{\mathsf{E}} \lambda(x' : \tau'). (e'[x \leftarrow v])
                                                                                                                                                                                                                                                                      4016
                   2. \vdash_{\mathsf{E}} \mathsf{stat} \ \tau \ E_0[e]
3962
                   3. \vdash_{\mathsf{F}} E_0[e]
                                                                                                                                                      by (3)
                                                                                                                                                                                                                                                                      4017
3963
                        by inversion
                                                                                                                                                 5. QED
                                                                                                                                                                                                                                                                      4018
                   4. \vdash_{\mathsf{F}} E_0[e']
3964
                                                                                                                                              CASE e = \langle e_0, e_1 \rangle:
                                                                                                                                                                                                                                                                      4019
                        by the induction hypothesis (3)
                                                                                                                                                 1. e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
                                                                                                                                                 2. x, \Gamma \vdash_{\mathsf{F}} e_0
                   5. \vdash_{\mathsf{E}} stat \tau E_0[e']
                                                                                                                                                                                                                                                                      4021
3966
3967
                        by (3, 4)
                                                                                                                                                      \land x, \Gamma \vdash_{\mathsf{F}} e_1
                                                                                                                                                                                                                                                                      4022
3968
                   6. QED by (1, 5)
                                                                                                                                                      by inversion
                                                                                                                                                                                                                                                                      4023
3969
                                                                                                                                                 3. \Gamma \vdash_{\mathsf{F}} e_0[x \leftarrow v]
                                                                                                                                                                                                                                                                      4024
3970
                                                                                                                                                      \wedge \Gamma \vdash_{\mathsf{F}} e_1[x \leftarrow v]
                                                                                                                                                                                                                                                                      4025
            Lemma 3.15 : \vdash_{\scriptscriptstyle E} inversion
3971
                                                                                                                                                      by the induction hypothesis (2)
                                                                                                                                                                                                                                                                      4026
               • If \Gamma \vdash_{\mathsf{F}} e_0 \ e_1 then \Gamma \vdash_{\mathsf{F}} e_0 and \Gamma \vdash_{\mathsf{F}} e_1
3972
                                                                                                                                                 4. \Gamma \vdash_{\mathsf{E}} \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
                                                                                                                                                                                                                                                                      4027
               • If \Gamma \vdash_{\mathsf{F}} \lambda x. e then x, \Gamma \vdash_{\mathsf{F}} e
3973
                                                                                                                                                      by (3)
                                                                                                                                                                                                                                                                      4028
               • If \Gamma \vdash_{\mathsf{F}} \lambda(x : \tau). e then (x : \tau), \Gamma \vdash_{\mathsf{F}} e
3974
               • If \Gamma \vdash_{\mathsf{F}} op^1 e then \Gamma \vdash_{\mathsf{F}} e
                                                                                                                                                 5. QED
                                                                                                                                                                                                                                                                      4029
3975
                                                                                                                                                                                                                                                                      4030
               • If \Gamma \vdash_{\mathsf{F}} op^2 e_0 e_1 then \Gamma \vdash_{\mathsf{F}} e_0 and \Gamma \vdash_{\mathsf{F}} e_1
                                                                                                                                              CASE e = e_0 \ e_1:
3976
                                                                                                                                                 1. e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                                                                                                                                                                      4031
               • If \Gamma \vdash_{\mathsf{F}} \mathsf{dyn} \ \tau \ e \ \mathsf{then} \ \Gamma \vdash_{\mathsf{F}} e
                                                                                                                                                 2. x, \Gamma \vdash_{\mathsf{E}} e_0
3977
                                                                                                                                                                                                                                                                      4032
               • If \Gamma \vdash_{\mathsf{F}} \mathsf{stat} \ \tau \ e \ \mathsf{then} \ \Gamma \vdash_{\mathsf{F}} e
3978
            Proof:
                                                                                                                                                      \land x, \Gamma \vdash_{\mathsf{F}} e_1
                                                                                                                                                                                                                                                                      4033
3979
                                                                                                                                                      by inversion
                                                                                                                                                                                                                                                                      4034
                QED by the definition of \vdash_{\mathsf{E}} e.
3980
                                                                                                                                                 3. \Gamma \vdash_{\mathsf{F}} e_0[x \leftarrow v]
                                                                                                                                                                                                                                                                      4035
3981
                                                                                                                                                      \wedge \Gamma \vdash_{\mathsf{F}} e_1[x \leftarrow v]
                                                                                                                                                                                                                                                                      4036
            Lemma 3.16 : E substitution
                                                                                                                                                      by the induction hypothesis (2)
3982
                                                                                                                                                                                                                                                                      4037
            If x, \Gamma \vdash_{\mathsf{E}} e or (x : \tau), \Gamma \vdash_{\mathsf{E}} e, and \vdash_{\mathsf{E}} v then \Gamma \vdash_{\mathsf{E}} e[x \leftarrow v]
                                                                                                                                                 4. \Gamma \vdash_{\mathsf{F}} e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                                                                                                                                                                      4038
3983
3984
                                                                                                                                                      by (3)
                                                                                                                                                                                                                                                                      4039
                By induction on the structure of e.
3985
                                                                                                                                                 5. QED
                                                                                                                                                                                                                                                                      4040
                CASE e = x:
                                                                                                                                              CASE e = op^1 e_0:
                                                                                                                                                                                                                                                                      4041
                   1. e[x \leftarrow v] = v
                                                                                                                                                 1. e[x \leftarrow v] = op^1 e_0[x \leftarrow v]
3987
                                                                                                                                                                                                                                                                      4042
                   2. \Gamma \vdash_{\mathsf{E}} v
                                                                                                                                                 2. x, \Gamma \vdash_{\mathsf{E}} e_0
                                                                                                                                                                                                                                                                      4043
                        by weakening
3989
                                                                                                                                                      by inversion
                                                                                                                                                                                                                                                                      4044
                   3. QED
3990
                                                                                                                                                                                                                                                                      4045
                                                                                                                                                 3. \Gamma \vdash_{\mathsf{F}} e_0[x \leftarrow v]
                CASE e = x':
3991
                                                                                                                                                      by the induction hypothesis (2)
                                                                                                                                                                                                                                                                      4046
                   1. QED by x'[x \leftarrow v] = x'
3992
                                                                                                                                                 4. \Gamma \vdash_{\mathsf{F}} op^1 e_0[x \leftarrow v]
                                                                                                                                                                                                                                                                      4047
                CASE e = i:
3993
                                                                                                                                                      by (3)
                                                                                                                                                                                                                                                                      4048
                   1. QED by i[x \leftarrow v] = i
3994
                                                                                                                                                                                                                                                                      4049
                                                                                                                                                 5. QED
                CASE e = \lambda x. e':
3995
                                                                                                                                              CASE e = op^2 e_0 e_1:
                                                                                                                                                                                                                                                                      4050
                   1. QED by (\lambda x. e')[x \leftarrow v] = \lambda x. e'
                                                                                                                                                 1. e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
3996
                                                                                                                                                                                                                                                                      4051
                CASE e = \lambda(x:\tau').e':
3997
                                                                                                                                                 2. x, \Gamma \vdash_{\mathsf{F}} e_0
                                                                                                                                                                                                                                                                      4052
                   1. QED by (\lambda(x:\tau').e')[x \leftarrow v] = \lambda(x:\tau').e'
3998
                                                                                                                                                      \land x, \Gamma \vdash_{\mathsf{F}} e_1
                                                                                                                                                                                                                                                                      4053
                CASE e = \lambda x' \cdot e':
                                                                                                                                                      by inversion
                   1. e[x \leftarrow v] = \lambda x' \cdot (e'[x \leftarrow v])
                                                                                                                                                3. \Gamma \vdash_{\mathsf{F}} e_0[x \leftarrow v]
                                                                                                                                                                                                                                                                      4055
4000
                   2. x', x, \Gamma \vdash_{\mathsf{F}} e'
4001
                                                                                                                                                      \wedge \Gamma \vdash_{\mathsf{F}} e_1[x \leftarrow v]
                                                                                                                                                                                                                                                                      4056
                        by inversion
4002
                                                                                                                                                      by the induction hypothesis (2)
                                                                                                                                                                                                                                                                      4057
                   3. x', \Gamma \vdash_{\mathsf{F}} e'[x \leftarrow v]
                                                                                                                                                 4. \Gamma \vdash_{\mathsf{F}} op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                                                                                                                                                                      4058
                        by the induction hypothesis (2)
4004
                                                                                                                                                      by (3)
                                                                                                                                                                                                                                                                      4059
                   4. \Gamma \vdash_{\mathsf{F}} \lambda x' . e'[x \leftarrow v]
4005
                                                                                                                                                 5. OED
                                                                                                                                                                                                                                                                      4060
                        by (3)
                                                                                                                                              CASE e = \text{dyn } \tau' e':
4006
                                                                                                                                                                                                                                                                      4061
                   5. QED
4007
                                                                                                                                                 1. e[x \leftarrow v] = \text{dyn } \tau' e'[x \leftarrow v]
                                                                                                                                                                                                                                                                      4062
                CASE e = \lambda(x':\tau').e':
4008
                                                                                                                                                 2. x, \Gamma \vdash_{\mathsf{F}} e'
                                                                                                                                                                                                                                                                      4063
                   1. e[x \leftarrow v] = \lambda(x':\tau').(e'[x \leftarrow v])
4009
                                                                                                                                                      by inversion
                                                                                                                                                                                                                                                                      4064
                   2. (x':\tau'), x, \Gamma \vdash_{\scriptscriptstyle E} e'
4010
                                                                                                                                                 3. \Gamma \vdash_{\mathsf{F}} e'[x \leftarrow v]
                                                                                                                                                                                                                                                                      4065
                        by inversion
4011
                                                                                                                                                      by the induction hypothesis (2)
                                                                                                                                                                                                                                                                      4066
                   3. (x':\tau'), \Gamma \vdash_{\mathsf{F}} e'[x \leftarrow v]
4012
                                                                                                                                                 4. \Gamma \vdash_{\mathsf{F}} \mathsf{dyn} \ \tau' \ (e'[x \leftarrow v])
                                                                                                                                                                                                                                                                      4067
                        by the induction hypothesis (2)
4013
                                                                                                                                                      by (3)
                                                                                                                                                                                                                                                                      4068
                                                                                                                                                 5. QED
4014
                                                                                                                                                                                                                                                                      4069
4015
                                                                                                                                                                                                                                                                      4070
```

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4071
            CASE e = \text{stat } \tau' e':
                                                                                                          CASE e = \langle e_0, e_1 \rangle:
                                                                                                                                                                                                    4126
              1. e[x \leftarrow v] = \text{stat } \tau' e'[x \leftarrow v]
4072
                                                                                                             IF e_0 \notin v:
                                                                                                                                                                                                    4127
4073
              2. x, \Gamma \vdash_{\mathsf{F}} e'
                                                                                                               1. e_0 = E_0[e'_0]
                                                                                                                                                                                                    4128
                  by inversion
                                                                                                                   by the induction hypothesis
4074
                                                                                                                                                                                                    4129
              3. \Gamma \vdash_{\mathsf{E}} e'[x \leftarrow v]
                                                                                                               2. E = \langle E_0, e_1 \rangle
                  by the induction hypothesis (2)
                                                                                                               3. QED by e = E[e'_0]
                                                                                                                                                                                                    4131
4076
              4. \Gamma \vdash_{\mathsf{F}} \mathsf{stat} \ \tau' \ (e'[x \leftarrow v])
4077
                                                                                                             IF e_0 \in v
                                                                                                                                                                                                    4132
4078
                  by (3)
                                                                                                                  \land e_1 \notin v:
                                                                                                                                                                                                    4133
4079
              5. QED
                                                                                                               1. e_1 = E_1[e_1']
                                                                                                                                                                                                    4134
                                                                                                                   by the induction hypothesis
4080
            CASE e = Err:
                                                                                                                                                                                                    4135
              1. QED by Err[x \leftarrow v] = Err
                                                                                                               2. E = \langle e_0, E_1 \rangle
                                                                                                                                                                                                    4136
4082
                                                                                                               3. QED by e = E[e'_1]
                                                                                                                                                                                                    4137
4083
                                                                                                             ELSE e_0 \in v
                                                                                                                                                                                                    4138
         Lemma 3.17 : \delta preservation
4084
           • If \vdash_{\mathsf{F}} v and \delta(op^1, v) = e' then \vdash_{\mathsf{F}} e'
                                                                                                                       \land e_1 \in v:
                                                                                                                                                                                                    4139
4085
                                                                                                               1. E = []
                                                                                                                                                                                                    4140
           • If \vdash_{\mathsf{F}} v_0 and \vdash_{\mathsf{F}} v_1 and \delta(op^2, v_0, v_1) = e' then \vdash_{\mathsf{F}} v'
4086
                                                                                                               2. QED e = E[\langle e_0, e_1 \rangle]
                                                                                                                                                                                                    4141
         Proof:
                                                                                                          CASE e = e_0 e_1:
4087
                                                                                                                                                                                                    4142
            CASE \delta(\text{fst}, \langle v_0, v_1 \rangle) = v_0:
4088
                                                                                                             IF e_0 \notin v:
                                                                                                                                                                                                    4143
              1. \vdash_{\mathsf{F}} v_0
4089
                                                                                                               1. e_0 = E_0[e'_0]
                  by inversion
                                                                                                                                                                                                    4144
4090
              2. QED
                                                                                                                   by the induction hypothesis
                                                                                                                                                                                                    4145
4091
                                                                                                               2. E = E_0 e_1
            CASE \delta(\operatorname{snd}, \langle v_0, v_1 \rangle) = v_1:
                                                                                                                                                                                                    4146
                                                                                                               3. QED by e = E[e'_0]
4092
                                                                                                                                                                                                    4147
              1. \vdash_{\mathsf{F}} v_1
4093
                                                                                                             If e_0 \in v
                                                                                                                                                                                                    4148
                  by inversion
4094
                                                                                                                  \land e_1 \notin v:
                                                                                                                                                                                                    4149
              2. QED
4095
            CASE \delta(\text{sum}, v_0, v_1) = v_0 + v_1:
                                                                                                               1. e_1 = E_1[e_1']
                                                                                                                                                                                                    4150
                                                                                                                   by the induction hypothesis
                                                                                                                                                                                                    4151
                                                                                                               2. E = e_0 E_1
                                                                                                                                                                                                    4152
4097
            CASE \delta(\text{quotient}, v_0, v_1) = |v_0/v_1|:
                                                                                                               3. QED by e = E[e'_1]
                                                                                                                                                                                                    4153
4099
                                                                                                             ELSE e_0 \in v
                                                                                                                                                                                                    4154
            CASE \delta(\text{quotient}, v_0, v_1) = \text{BndryErr}:
4100
                                                                                                                       \land e_1 \in v:
                                                                                                                                                                                                    4155
                                                                                                               1. E = []
4101
                                                                                                                                                                                                    4156
4102
                                                                                                               2. QED e = E[e_0 \ e_1]
                                                                                                                                                                                                    4157
         Lemma 3.18: weakening
4103
                                                                                                          CASE e = op^1 e_0:
                                                                                                                                                                                                    4158
           • If \Gamma \vdash_{\mathsf{F}} e then x, \Gamma \vdash_{\mathsf{F}} e
4104
                                                                                                             IF e_0 \notin v:
                                                                                                                                                                                                    4159
           • If \Gamma \vdash_{\mathsf{F}} e then (x:\tau), \Gamma \vdash_{\mathsf{F}} e
                                                                                                               1. e_0 = E_0[e_0']
4105
                                                                                                                                                                                                    4160
                                                                                                                   by the induction hypothesis
4106
                                                                                                                                                                                                    4161
            QED because e is closed under \Gamma
4107
                                                                                                               2. E = op^1 E_0
                                                                                                                                                                                                    4162
                                                                                                               3. QED e = E[e'_0]
4108
                                                                                                                                                                                                    4163
         Lemma 3.19: unique static evaluation contexts
                                                                                                             ELSE e_0 \in v:
4109
                                                                                                                                                                                                    4164
         If \vdash e : \tau then one of the following holds:
                                                                                                               1. E = []
4110
                                                                                                                                                                                                    4165
           • e is a value
4111
                                                                                                               2. QED e = E[op^1 e_0]
                                                                                                                                                                                                    4166
           \bullet e = E[v_0 \ v_1]
                                                                                                          CASE e = op^2 e_0 e_1:
4112
                                                                                                                                                                                                    4167
           • e = E[op^1 v]
4113
                                                                                                             IF e_0 \notin v:
                                                                                                                                                                                                    4168
           \bullet \ e = E[op^2 \ v_0 \ v_1]
4114
                                                                                                               1. e_0 = E_0[e'_0]
                                                                                                                                                                                                    4169
           • e = E[Err]
                                                                                                                   by the induction hypothesis
4115
         Proof:
                                                                                                                                                                                                    4170
                                                                                                               2. E = op^2 E_0 e_1
4116
            By induction on the structure of e.
                                                                                                                                                                                                    4171
            CASE e = x:
                                                                                                               3. QED e = E[e'_0]
                                                                                                                                                                                                    4172
4118
              1. Contradiction by \vdash e : \tau
                                                                                                             IF e_0 \in v
                                                                                                                                                                                                    4173
4119
            CASE e = i
                                                                                                                  \land e_1 \notin v:
                                                                                                                                                                                                    4174
4120
                     \vee e = \lambda(x:\tau_d).e':
                                                                                                               1. e_1 = E_1[e_1']
                                                                                                                                                                                                    4175
                                                                                                                   by the induction hypothesis
4121
              1. QED e is a value
                                                                                                                                                                                                    4176
4122
            CASE e = \text{stat } \tau e':
                                                                                                               2. E = op^2 e_0 E_1
                                                                                                                                                                                                    4177
4123
              1. Contradiction by \vdash_1 e : K
                                                                                                               3. QED e = E[e'_1]
                                                                                                                                                                                                    4178
                                                                                                                                                                                                    4179
4124
4125
                                                                                                                                                                                                    4180
                                                                                                 38
```

```
4181
                                                                                                                           CASE e = \lambda x. e':
                                                                                                                                                                                                                                   4236
                 ELSE e_0 \in v
                                                                                                                              1. Contradiction by (x:\tau_x), \Gamma \vdash e:\tau
4182
                            \land e_1 \in v:
                                                                                                                                                                                                                                   4237
4183
                    1. E = []
                                                                                                                           CASE e = \lambda(x:\tau').e':
                                                                                                                                                                                                                                   4238
                    2. QED e = E[op^2 e_0 e_1]
                                                                                                                              1. QED by (\lambda(x:\tau').e')[x \leftarrow v] = \lambda(x:\tau').e'
4184
                                                                                                                                                                                                                                   4239
              CASE e = dyn \tau e_0:
                                                                                                                           CASE e = \lambda(x':\tau').e':
                                                                                                                                                                                                                                   4240
                1. Contradiction by e is boundary-free
                                                                                                                              1. e[x \leftarrow v] = \lambda(x':\tau'). (e'[x \leftarrow v])
                                                                                                                                                                                                                                   4241
4186
                                                                                                                             2. (x':\tau'), x, \Gamma \vdash e'
4187
              CASE e = \text{stat } \tau e_0:
                                                                                                                                                                                                                                   4242
4188
                 1. Contradiction by \vdash e : \tau
                                                                                                                                  by static inversion forms
                                                                                                                                                                                                                                   4243
4189
              CASE e = Err:
                                                                                                                             3. (x':\tau'), \Gamma \vdash e'[x \leftarrow v]
                                                                                                                                                                                                                                   4244
4190
                1. E = []
                                                                                                                                  by the induction hypothesis (2)
                                                                                                                                                                                                                                   4245
                 2. QED e = E[Err]
                                                                                                                              4. \Gamma \vdash \lambda(x':\tau'). (e'[x \leftarrow v])
4192
                                                                                                                                  by (3)
                                                                                                                                                                                                                                   4247
4193
                                                                                                                              5. QED
                                                                                                                                                                                                                                   4248
          Lemma 3.20: ⊢ static inversion
4194
             • If \Gamma \vdash x : \tau then (x : \tau') \in \Gamma and \tau' \leqslant \tau
                                                                                                                           CASE e = \langle e_0, e_1 \rangle:
                                                                                                                                                                                                                                   4249
4195
                                                                                                                              1. e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
                                                                                                                                                                                                                                   4250
             • If \Gamma \vdash \lambda(x : \tau'_d). e' : \tau then (x : \tau'_d), \Gamma \vdash e' : \tau'_c and \tau'_d \Rightarrow \tau'_c \leqslant \varepsilon
4196
                                                                                                                              2. x, \Gamma \vdash e_0
                                                                                                                                                                                                                                   4251
4197
             • If \Gamma \vdash \langle e_0, e_1 \rangle : \tau then \Gamma \vdash e_0 : \tau_0 and \Gamma \vdash e_1 : \tau_1 and
                                                                                                                                  \land x, \Gamma \vdash e_1
                                                                                                                                                                                                                                   4252
4198
                                                                                                                                  by static inversion forms
                                                                                                                                                                                                                                   4253
                \tau_0 \times \tau_1 \leqslant : \tau
                                                                                                                             3. \Gamma \vdash e_0[x \leftarrow v]
4199
             • If \Gamma \vdash e_0 \ e_1 : \tau_c then \Gamma \vdash e_0 : \tau_d' \Rightarrow \tau_c' and \Gamma \vdash e_1 : \tau_d' and
                                                                                                                                                                                                                                   4254
4200
                                                                                                                                  \wedge \Gamma \vdash e_1[x \leftarrow v]
                                                                                                                                                                                                                                   4255
4201
             • If \Gamma \vdash fst e : \tau then \Gamma \vdash e : \tau_0 \times \tau_1 and \Delta(\text{fst}, \tau_0 \times \tau_1) = \tau_0 and
                                                                                                                                  by the induction hypothesis (2)
                                                                                                                                                                                                                                   4256
                                                                                                                             4. \Gamma \vdash \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
4202
                                                                                                                                                                                                                                   4257
                \tau_0 \leqslant : \tau
                                                                                                                                  by (3)
                                                                                                                                                                                                                                   4258
4203
             • If \Gamma \vdash \text{snd } e : \tau \text{ then } \Gamma \vdash e : \tau_0 \times \tau_1 \text{ and } \Delta(\text{snd}, \tau_0 \times \tau_1) = \tau_1
4204
                and \tau_1 \leqslant : \tau
                                                                                                                              5. QED
                                                                                                                                                                                                                                   4259
4205
             • If \Gamma \vdash op^2 e_0 e_1 : \tau then \Gamma \vdash e_0 : \tau_0 and \Gamma \vdash e_1 : \tau_1 and
                                                                                                                           CASE e = e_0 e_1:
                                                                                                                                                                                                                                   4260
4206
                \Delta(op^2, \tau_0, \tau_1) = \tau' and \tau' \leqslant \tau
                                                                                                                              1. e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                                                                                                                                   4261
4207
                                                                                                                              2. x, \Gamma \vdash e_0
                                                                                                                                                                                                                                   4262
             • If \Gamma \vdash \text{dyn } \tau' \ e' : \tau \text{ then } \Gamma \vdash e' \text{ and } \tau' \leqslant \tau
                                                                                                                                  \land x, \Gamma \vdash e_1
                                                                                                                                                                                                                                   4263
              QED by the definition of \Gamma \vdash e : \tau
                                                                                                                                  by static inversion forms
4209
                                                                                                                                                                                                                                   4264
4210
                                                                                                                             3. \Gamma \vdash e_0[x \leftarrow v]
                                                                                                                                                                                                                                   4265
4211
          Lemma 3.21: canonical forms
                                                                                                                                  \wedge \Gamma \vdash e_1[x \leftarrow v]
                                                                                                                                                                                                                                   4266
4212
             • If \vdash \upsilon : \tau_0 \times \tau_1 then \upsilon = \langle \upsilon_0, \upsilon_1 \rangle
                                                                                                                                  by the induction hypothesis (2)
                                                                                                                                                                                                                                   4267
4213
             • If \vdash \upsilon : \tau_d \Rightarrow \tau_c then \upsilon = \lambda(x : \tau_x) \cdot e'
                                                                                                                              4. \Gamma \vdash e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                                                                                                                                   4268
4214
                                                                                                                                  by (3)
                                                                                                                                                                                                                                   4269
                 \wedge \tau_d \leqslant : \tau_x
4215
             • If \vdash v: Int then v = i
                                                                                                                              5. OED
                                                                                                                                                                                                                                   4270
                                                                                                                           CASE e = op^1 e_0:
4216
             • If \vdash v: Nat then v = i and v \in \mathbb{N}
                                                                                                                                                                                                                                   4271
4217
                                                                                                                             1. e[x \leftarrow v] = op^1 e_0[x \leftarrow v]
                                                                                                                                                                                                                                   4272
              QED by definition of \vdash e : \tau
4218
                                                                                                                             2. x, \Gamma \vdash e_0
                                                                                                                                                                                                                                   4273
                                                                                                                                  by static inversion forms
4219
                                                                                                                                                                                                                                   4274
          Lemma 3.22: substitution
                                                                                                                             3. \Gamma \vdash e_0[x \leftarrow v]
                                                                                                                                                                                                                                   4275
4220
             If (x:\tau_x), \Gamma \vdash e:\tau, and e is boundary-free and \vdash v:\tau_x then
                                                                                                                                  by the induction hypothesis (2)
4221
                                                                                                                                                                                                                                   4276
            \Gamma \vdash e[x \leftarrow v] : \tau
4222
                                                                                                                              4. \Gamma \vdash op^1 e_0[x \leftarrow v]
                                                                                                                                                                                                                                   4277
          Proof:
4223
                                                                                                                                  by (3)
                                                                                                                                                                                                                                   4278
              By induction on the structure of e.
4224
                                                                                                                             5. QED
                                                                                                                                                                                                                                   4279
              CASE e = x:
                                                                                                                           CASE e = op^2 e_0 e_1:
4225
                1. e[x \leftarrow v] = v
                                                                                                                              1. e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
4226
                                                                                                                                                                                                                                   4281
                2. \tau_x = \tau
                                                                                                                             2. x, \Gamma \vdash e_0
                                                                                                                                                                                                                                   4282
                3. \Gamma \vdash \upsilon : \tau
4228
                                                                                                                                  \wedge x, \Gamma \vdash e_1
                                                                                                                                                                                                                                   4283
                     by weakening
                                                                                                                                  by static inversion forms
                                                                                                                                                                                                                                   4284
                4. QED
4230
                                                                                                                              3. \Gamma \vdash e_0[x \leftarrow v]
                                                                                                                                                                                                                                   4285
              CASE e = x':
4231
                                                                                                                                  \wedge \Gamma \vdash e_1[x \leftarrow v]
                                                                                                                                                                                                                                   4286
                1. QED by x'[x \leftarrow v] = x'
                                                                                                                                  by the induction hypothesis (2)
4232
                                                                                                                                                                                                                                   4287
              CASE e = i:
4233
                                                                                                                              4. \Gamma \vdash op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                                                                                                                                   4288
                1. QED by i[x \leftarrow v] = i
4234
                                                                                                                                  by (3)
                                                                                                                                                                                                                                   4289
4235
                                                                                                                                                                                                                                   4290
                                                                                                                 39
```

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4291
              5. OED
4292
           CASE e = \text{dyn } \tau' e':
              1. Contradiction by e is boundary-free
4293
           CASE e = \text{stat } \tau' e':
4294
4295
              1. Contradiction by e is boundary-free
           CASE e = Err:
4296
4297
              1. QED \text{Err}[x \leftarrow v] = \text{Err}
4298
4299
         Lemma 3.23 : \delta preservation
4300
           • If \vdash v and \delta(op^1, v) = v' then \vdash e'
4301
           • If \vdash v_0 and \vdash v_1 and \delta(op^2, v_0, v_1) = e' then \vdash v'
4302
4303
           CASE \delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0:
4304
4305
                 by static inversion forms
4306
              2. QED
4307
           CASE \delta(\operatorname{snd}, \langle v_0, v_1 \rangle) = v_1:
4308
              1. \vdash v_1
4309
                 by static inversion forms
4310
              2. QED
4311
           CASE \delta(\text{sum}, v_0, v_1) = v_0 + v_1:
4312
             1. QED
4313
           CASE \delta(quotient, v_0, v_1) = |v_0/v_1|:
4314
              1. QED
4315
            CASE \delta(\text{quotient}, v_0, v_1) = \text{BndryErr}:
4316
              1. QED
4317
4318
         Lemma 3.24: weakening
4319
           • If \Gamma \vdash e then x, \Gamma \vdash e
4320
           • If \Gamma \vdash e then (x : \tau), \Gamma \vdash e
4321
        Proof:
4322
            QED because e is closed under \Gamma
4323
4324
4325
4326
4327
4328
4330
4331
4332
4333
4334
4335
4336
4338
4339
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4342
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4344
4345
                                                                                             40
```

## A.4 (1) First-Order Embedding

## A.4.1 First-Order Definitions

## Language 1 dyn $\tau$ e | stat $\tau$ e | Err | chk K e | dyn e | stat e $i \mid \langle v, v \rangle \mid \lambda x. e \mid \lambda(x:\tau). e$ = Nat | Int | $\tau \times \tau$ | $\tau \Rightarrow \tau$ = Nat | Int | Pair | Fun | Any $= \cdot \mid x, \Gamma \mid (x:\tau), \Gamma$ = BndryErr | TagErr $= v \mid Err$ $= \left[ \right] \mid E^{\bullet} e \mid v E^{\bullet} \mid \langle E^{\bullet}, e \rangle \mid \langle v, E^{\bullet} \rangle \mid$ $op^1 E^{\bullet} \mid op^2 E^{\bullet} e \mid op^2 v E^{\bullet} \mid \operatorname{chk} K E^{\bullet}$ $= E^{\bullet} \mid E e \mid v E \mid \langle E, e \rangle \mid \langle v, E \rangle \mid op^{1} E \mid$

 $op^2 E e \mid op^2 v E \mid dyn \tau E \mid stat \tau E \mid$ 

 $chk K E \mid dyn E \mid stat E$ 

$$\Delta: op^{1} \times \tau \longrightarrow \tau$$

$$\Delta(fst, \tau_{0} \times \tau_{1}) = \tau_{0}$$

$$\Delta(snd, \tau_{0} \times \tau_{1}) = \tau_{1}$$

$$\Delta: op^{2} \times \tau \times \tau \longrightarrow \tau$$

$$\Delta(op^2, \text{Nat}, \text{Nat}) = \text{Nat}$$
  
 $\Delta(op^2, \text{Int}, \text{Int}) = \text{Int}$ 

$$\tau \leqslant : \tau$$

$$\frac{\tau_d' \leqslant: \tau_d \quad \tau_c \leqslant: \tau_c'}{\mathsf{Nat} \leqslant: \mathsf{Int}} \quad \frac{\tau_d' \leqslant: \tau_d \quad \tau_c \leqslant: \tau_c'}{\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'} \quad \frac{\tau_0 \leqslant: \tau_0' \quad \tau_1 \leqslant: \tau_1'}{\tau_0 \times \tau_1 \leqslant: \tau_0' \times \tau_1'}$$

$$\frac{\tau\leqslant \tau}{\tau\leqslant \tau} \quad \frac{\tau\leqslant \tau' \quad \tau'\leqslant \tau''}{\tau\leqslant \tau''}$$

 $\Gamma \vdash e$ 

$$\frac{x \in \Gamma}{\Gamma \vdash x} \quad \frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x. e} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash i} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}$$

$$\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0 \ e_1} \quad \frac{\Gamma \vdash e}{\Gamma \vdash op^1 \ e} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2 \ e_0 \ e_1} \quad \frac{\Gamma \vdash \text{Err}}{\Gamma \vdash op^2 \ e_0 \ e_1}$$

$$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat} \ \tau \ e}$$

$$\frac{\left[\Gamma \vdash e : \tau\right]}{\left[\Gamma \vdash x : \tau\right]} \frac{(x : \tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash x : \tau} \frac{i \in \mathbb{N}}{\Gamma \vdash \lambda(x : \tau_d), e : \tau_d \Rightarrow \tau_c} \frac{i \in \mathbb{N}}{\Gamma \vdash i : \text{Nat}}$$

$$\frac{\Gamma \vdash e_0 : \tau_0}{\Gamma \vdash i : \mathsf{Int}} \quad \frac{\Gamma \vdash e_0 : \tau_0}{\Gamma \vdash e_1 : \tau_1} \quad \frac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c}{\Gamma \vdash e_1 : \tau_d} \\ \frac{\Gamma \vdash e_1 : \tau_d}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1} \quad \frac{\Gamma \vdash e_0 : \tau_d}{\Gamma \vdash e_0 \; e_1 : \tau_c}$$

$$\begin{array}{cccc} & \Gamma \vdash e_0 : \tau_0 \\ \Gamma \vdash e_0 : \tau_0 & \Gamma \vdash e_1 : \tau_1 & \Gamma \vdash e : \tau' \\ \frac{\Delta(op^1, \tau_0) = \tau}{\Gamma \vdash op^1 \; e_0 : \tau} & \frac{\Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash op^2 \; e_0 \; e_1 : \tau} & \frac{\tau' \leqslant : \tau}{\Gamma \vdash e : \tau} & \frac{\Gamma \vdash \operatorname{Err} : \tau}{\Gamma \vdash \operatorname{Err} : \tau} \end{array}$$

$$\frac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn} \ \tau \ e : \tau}$$

$$K \leqslant : K$$

$$\frac{}{K \leqslant : \mathsf{Any}} \quad \frac{}{\mathsf{Nat} \leqslant : \mathsf{Int}} \quad \frac{K \leqslant : K' \quad K' \leqslant : K''}{K \leqslant : K''}$$

$$\begin{bmatrix} \lfloor \tau \rfloor = K \end{bmatrix} = \text{Nat} 
 \lfloor \text{Int} \rfloor = \text{Int} 
 \lfloor \tau_0 \times \tau_1 \rfloor = \text{Pair} 
 \lfloor \tau_d \Rightarrow \tau_c \rfloor = \text{Fun} 
 \boxed{\Gamma \vdash e \leadsto e}$$

$$\frac{\Gamma \vdash e_0 \leadsto e_0'}{\Gamma \vdash e_1 \leadsto e_1'} \qquad \frac{x, \Gamma \vdash e \leadsto e'}{\Gamma \vdash \langle e_0, e_1 \rangle \leadsto \langle e_0', e_1' \rangle} \qquad \frac{x, \Gamma \vdash e \leadsto e'}{\Gamma \vdash \lambda x. \, e \leadsto \lambda x. \, e'}$$

$$\frac{\Gamma \vdash e_0 \leadsto e_0'}{\Gamma \vdash e_1 \leadsto e_1'} - \frac{\Gamma \vdash e \leadsto e'}{\Gamma \vdash e_0 e_1 \leadsto e_0' e_1'} - \frac{\Gamma \vdash e \leadsto e'}{\Gamma \vdash op^1 e \leadsto op^1 e'}$$

$$\frac{ \begin{array}{c} \Gamma \vdash e_0 \leadsto e_0' \\ \Gamma \vdash e_1 \leadsto e_1' \\ \hline \Gamma \vdash op^2 e_0 \ e_1 \leadsto op^2 \ e_0' \ e_1' \end{array} }{ \Gamma \vdash \operatorname{Err} \leadsto \operatorname{Err} }$$

$$\frac{\Gamma \vdash e : \tau \leadsto e'}{\Gamma \vdash \operatorname{stat} \tau \ e \leadsto \operatorname{stat} \tau \ e'}$$

```
\Gamma \vdash e : \tau \leadsto e
4511
4512
                                                                  \Gamma \vdash i : \text{Nat} \leadsto i \Gamma \vdash i : \text{Int} \leadsto i
4513
4514
                                                               \frac{\Gamma \vdash e_0 : \tau_0 \leadsto e_0' \quad \Gamma \vdash e_1 : \tau_1 \leadsto e_1'}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1 \leadsto \langle e_0', e_1' \rangle}
 4515
4516
4517
4518
                              \frac{(x\!:\!\tau_d),\Gamma\vdash e:\tau_c\leadsto e'}{\Gamma\vdash \lambda(x\!:\!\tau_d).\,e:\tau_d\!\Rightarrow\!\tau_c\leadsto \lambda(x\!:\!\tau_d).\,e'}\quad \frac{}{\Gamma\vdash x:\tau\leadsto x}
4519
4520
 4521
                                    \frac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \leadsto e_0' \quad \Gamma \vdash e_1 : \tau_d \leadsto e_1' \quad \lfloor \tau_c \rfloor = K}{\Gamma \vdash e_0 e_1 : \tau_c \leadsto \operatorname{chk} K (e_0' e_1')}
4522
 4523
4524
                                                                      \Gamma \vdash e : \tau_0 \times \tau_1 \leadsto e' \quad \lfloor \tau_0 \rfloor = K
 4525
                                                                       \Gamma \vdash \text{fst } e : \tau_0 \leadsto \text{chk } K \text{ (fst } e')
 4526
 4527
                                                                      \Gamma \vdash e : \tau_0 \times \tau_1 \leadsto e' \quad \lfloor \tau_1 \rfloor = K
 4528
4529
                                                                    \Gamma \vdash \operatorname{snd} e : \tau_1 \leadsto \operatorname{chk} K (\operatorname{snd} e')
 4530
4531
                                            \frac{\Gamma \vdash e_0 : \tau_0 \leadsto e_0'}{\Gamma \vdash e_1 : \tau_1 \leadsto e_1'} \qquad \frac{\Gamma \vdash e : \tau' \leadsto e'}{\tau' \leqslant : \tau}
\frac{\Gamma \vdash op^2 e_0 e_1 : \tau \leadsto op^2 e_0' e_1'}{\Gamma \vdash e : \tau \leadsto e'}
 4532
4533
4534
4535
4536
                                          \frac{\Gamma \vdash \mathsf{Err} : \tau \leadsto \mathsf{Err}}{\Gamma \vdash \mathsf{dyn} \ \tau \ e : \tau \leadsto \mathsf{dyn} \ \tau \ e'}
4537
 4538
4539
                            4540
4541
 4542
4543
                                       \frac{x \in \Gamma}{\Gamma \vdash_1 x} \quad \frac{(x : \tau) \in \Gamma}{\Gamma \vdash_1 x} \quad \frac{\Gamma \vdash_1 e_0 \quad \Gamma \vdash_1 e_1}{\Gamma \vdash_1 e_0 e_1} \quad \frac{\Gamma \vdash_1 e}{\Gamma \vdash_1 op^1 e}
 4544
 4545
4546
                              \frac{\Gamma \vdash_{\!\!\!\!1} e_0 \quad \Gamma \vdash_{\!\!\!1} e_1}{\Gamma \vdash_{\!\!\!1} op^2 e_0 \ e_1} \quad \frac{\Gamma \vdash_{\!\!\!1} \mathsf{Err}}{\Gamma \vdash_{\!\!\!1} \mathsf{Err}} \quad \frac{\Gamma \vdash_{\!\!\!1} e : \lfloor \tau \rfloor}{\Gamma \vdash_{\!\!\!1} \mathsf{stat} \ \tau \ e} \quad \frac{\Gamma \vdash_{\!\!\!1} e : \mathsf{Any}}{\Gamma \vdash_{\!\!\!1} \mathsf{stat} \ e}
 4547
4548
4549
```

```
\Gamma \vdash_{\mathbf{1}} e : K
                                                                                                                            \Gamma \vdash_{\mathbf{1}} e_0 : \mathsf{Any}
                        \frac{i \in \mathbb{N}}{\Gamma \vdash_{1} i : \mathsf{Nat}} \quad \frac{1 \vdash_{1} e_{0} : \mathsf{Any}}{\Gamma \vdash_{1} i : \mathsf{Int}} \quad \frac{\Gamma \vdash_{1} e_{1} : \mathsf{Any}}{\Gamma \vdash_{1} \langle e_{0}, e_{1} \rangle : \mathsf{Pair}}
           \frac{x,\Gamma \vdash_{\!\!\!\!\!\! 1} e}{\Gamma \vdash_{\!\!\! 1} \lambda x.\, e : \mathsf{Fun}} \quad \frac{(x\!:\!\tau),\Gamma \vdash_{\!\!\! 1} e : \mathsf{Any}}{\Gamma \vdash_{\!\!\! 1} \lambda (x\!:\!\tau).\, e : \mathsf{Fun}} \quad \frac{x \in \Gamma}{\Gamma \vdash_{\!\!\! 1} x : \mathsf{Any}}
                       (x:\tau) \in \Gamma  \Gamma \vdash_1 e_0 : \operatorname{Fun}
                       \frac{\lfloor \tau \rfloor = K}{\Gamma \vdash_{1} x : K} \qquad \frac{\Gamma \vdash_{1} e_{1} : \mathsf{Any}}{\Gamma \vdash_{1} e_{0} e_{1} : \mathsf{Any}} \qquad \frac{\Gamma \vdash_{1} e : \mathsf{Pair}}{\Gamma \vdash_{1} \mathsf{fst} e : \mathsf{Any}}
                                                                                           \Gamma \vdash_{\scriptscriptstyle{1}} e_0 : K_0 \quad \Gamma \vdash_{\scriptscriptstyle{1}} e_1 : K_1
                          \frac{\Gamma \vdash_{1} e : Pair}{\Gamma \vdash_{1} snd e : Any} \frac{\Delta(op^{2}, K_{0}, K_{1}) = K}{\Gamma \vdash_{1} op^{2} e_{0} e_{1} : K}
        \frac{\Gamma \vdash_{\!\!\!1} e: K' \quad K' \leqslant: K}{\Gamma \vdash_{\!\!\!1} e: K} \quad \frac{\Gamma \vdash_{\!\!\!1} e \quad \lfloor \tau \rfloor = K}{\Gamma \vdash_{\!\!\!1} \mathsf{Err}: K} \quad \frac{\Gamma \vdash_{\!\!\!1} e \quad \lfloor \tau \rfloor = K}{\Gamma \vdash_{\!\!\!1} \mathsf{dyn} \; \tau \; e: K}
                                        \frac{\Gamma \vdash_{\!\!\!1} e}{\Gamma \vdash_{\!\!\!1} \mathsf{dyn} \ e : \mathsf{Any}} \quad \frac{\Gamma \vdash_{\!\!\!1} e : \mathsf{Any}}{\Gamma \vdash_{\!\!\!1} \mathsf{chk} \ K \ e : K}
 \delta(op^1, v) = e
   \delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0
   \delta(\operatorname{snd}, \langle v_0, v_1 \rangle) = v_1
 \delta(op^2, v, v) = e
   \delta(\mathsf{sum},i_0,i_1)
                                                             = i_0 + i_1
   \delta(\text{quotient}, i_0, 0) = \text{BndryErr}
   \delta(\text{quotient}, i_0, i_1) = \lfloor i_0/i_1 \rfloor
           if i_1 \neq 0
 \mathcal{D}_1: \tau \times v \longrightarrow v
   \mathcal{D}_1(\tau, v) = \mathcal{X}(|\tau|, v)
 S_1: \tau \times v \longrightarrow v
   S_1(\tau, v) = v
 X: K \times v \longrightarrow v
   X(\operatorname{Fun}, \lambda x. e)
                                                               =\lambda x. e
   X(\operatorname{Fun}, \lambda(x : \tau). e) = \lambda(x : \tau). e
   \mathcal{X}(\mathsf{Pair}, \langle v_0, v_1 \rangle) = \langle v_0, v_1 \rangle
   \mathcal{X}(\mathsf{Int},i)
                                                                =i
   X(Nat, i)
                                                                =i
           if i \in \mathbb{N}
                                                                = BndryErr
   X(K,v)
           otherwise
```

```
4621
              e \rhd_{1-S} e
4622
               dyn v
                                         >1-S v
4623
                                         \rhd_{1-S} \mathcal{D}(\tau,v)
               dyn \tau v
4624
               \operatorname{chk} K v
                                         \triangleright_{1-S} X(K,v)
4625
               (\lambda(x:\tau).e) v \triangleright_{1-S} BndryErr
4626
                   if X(\lfloor \tau \rfloor, v) = BndryErr
4627
               (\lambda(x:\tau).e) v \triangleright_{1-S} e[x \leftarrow \mathcal{X}(\lfloor \tau \rfloor, v)]
4628
               (\lambda x. e) v
                                         \triangleright_{1-S} \operatorname{dyn} (e[x \leftarrow v])
4629
               op^1 v
                                         \triangleright_{1-S} \delta(op^1, v)
4630
               op^2 v_0 v_1
                                         \triangleright_{1-S} \delta(op^2, v_0, v_1)
4631
              e \rhd_{1-D} e
4632
               stat v
                                         ⊳<sub>1-D</sub> v
                                         \triangleright_{1-D} \mathcal{S}(\tau,v)
               stat \tau v
4634
                                         ⊳<sub>1-D</sub> TagErr
               v_0 v_1
4635
                   if v_0 \in \mathbb{Z} or v_0 = \langle v, v' \rangle
4636
               (\lambda(x:\tau).e)v \triangleright_{1-D} BndryErr
4637
                   if X(|\tau|, v) = BndryErr
4638
               (\lambda(x:\tau).e) v \triangleright_{1-D} \text{ stat } (e[x \leftarrow \mathcal{X}(\lfloor \tau \rfloor, v)])
4639
               (\lambda x. e) v
                                         \triangleright_{1-D} e[x \leftarrow v]
4640
               op^1 v
                                         ⊳<sub>1-D</sub> TagErr
4641
                   if \delta(op^1, v) is undefined
4642
               op^1 v
                                        \triangleright_{1-D} \delta(op^1, v)
4643
               op^2 v_0 v_1
                                         ⊳<sub>1-D</sub> TagErr
4644
                   if \delta(op^2, v_0, v_1) is undefined
4645
                                        \triangleright_{1-D} \delta(op^2, v_0, v_1)
               op^2 v_0 v_1
4646
              e \rightarrow_{1-S} e
4647
               E^{\bullet}[e]
                                            \rightarrow_{1-S} E^{\bullet}[e']
4648
                   if e \triangleright_{1-S} e'
4649
               E[\operatorname{stat} \tau E^{\bullet}[e]] \rightarrow_{1-S} E[\operatorname{stat} \tau E^{\bullet}[e']]
4650
                   if e \triangleright_{1-S} e'
4651
               E[\operatorname{dyn} \tau E^{\bullet}[e]] \rightarrow_{1-S} E[\operatorname{dyn} \tau E^{\bullet}[e']]
4652
                   if e \triangleright_{1-D} e'
4653
               E[Err]
                                             \rightarrow_{1-S} Err
4654
              e \rightarrow_{1-D} e
4655
                                             \rightarrow_{1-D} E^{\bullet}[e']
               E^{\bullet}[e]
4656
4657
                   if e \triangleright_{1-D} e'
4658
               E[\operatorname{stat} \tau E^{\bullet}[e]] \rightarrow_{1-D} E[\operatorname{stat} \tau E^{\bullet}[e']]
4659
                   if e \triangleright_{1-S} e'
               E[\operatorname{dyn} \tau E^{\bullet}[e]] \rightarrow_{1-D} E[\operatorname{dyn} \tau E^{\bullet}[e']]
4660
4661
                   if e \triangleright_{1-D} e'
4662
               E[Err]
                                              \rightarrow_{1-D} Err
              e \rightarrow_{1-S}^{*} e reflexive, transitive closure of \rightarrow_{1-S}
4664
4665
              e \rightarrow_{1-D}^{*} e | reflexive, transitive closure of \rightarrow_{1-D}
4666
4667
4668
4669
4670
4671
4672
4673
4674
```

```
Definition 4.0: 1 boundary-free
                                                                               4676
 An expression e is boundary free if e does not contain a
                                                                               4677
 subterm of the form:
                                                                               4678
 • (dyn \tau' e'),
                                                                               4679
 • (stat \tau' e'),
                                                                               4680
 • (dyn e′), or
                                                                               4681
 • (stat e′).
                                                                               4682
                                                                               4683
                                                                               4684
                                                                               4685
                                                                               4686
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                                                                               4728
```

4731	A.4.2 First-Order Theorems	_5. QED
4732	<b>Theorem 4.1</b> : static 1-soundness	
4733	If $\vdash e : \tau$ then $\vdash e : \tau \leadsto e''$ and $\vdash_1 e'' : \lfloor \tau \rfloor$ and one of the	Corollary 4.5: 1 compilation
4734	following holds:	If $\vdash e : \tau$
4735	• $e'' \rightarrow_{1-S}^* v$ and $\vdash_1 v : \lfloor \tau \rfloor$	and $\vdash e : \tau \leadsto e''$
4736	• $e'' \rightarrow_{1-S}^{*} E[\text{dyn } \tau' E^{\bullet}[e']] \text{ and } e' \triangleright_{1-D} \text{ TagErr}$	and $\vdash_1 e'' : \lfloor \tau \rfloor$
4737	• $c'' \rightarrow_{1-S} E[dyn F \bullet [a']] \text{ and } c' \triangleright_{1-D} \text{ rager}$	and $\triangleright_{1-D}'$ is similar to $\triangleright_{1-D}$ but without the no-op bound-
4738	• $e'' \rightarrow_{1-S}^* E[\text{dyn } E^{\bullet}[e']]$ and $e' \triangleright_{1-D} \text{TagErr}$	aries, as follows:
4739	• $e'' \rightarrow_{1-S}^*$ BndryErr	$\operatorname{chk} K v \qquad \rhd_{1-D'} X(K,v)$
4740	• e'' diverges  Proof:	$v_0 v_1 >_{1-D}' TagErr$
4741		if $v_0 \in \mathbb{Z}$ or $v_0 = \langle v, v' \rangle$
4742	1. $\vdash_1 e : \tau \leadsto e''$	$(\lambda(x:\tau).e) \ v \ \triangleright_{1-D}' \ BndryErr$
4743	$\wedge \vdash_1 e'' : \lfloor \tau \rfloor$	$if X(\lfloor \tau \rfloor, v) = BndryErr$
4744	by → static soundness	$(\lambda(x:\tau).e) \ v \ \triangleright_{1-D}' \ e[x \leftarrow \mathcal{X}(\lfloor \tau \rfloor, v)]$
4745	2. QED by static progress and static preservation	
		$(\lambda x. e) v \qquad \triangleright_{1-D}' e[x \leftarrow v]$
4746	<b>Theorem 4.2</b> : dynamic 1-soundness	$op^1 v  ightharpoonup_{1-D}' TagErr$
4747	If $\vdash e$ then $\vdash e \leadsto e''$ and $\vdash_1 e''$ and one of the following holds:	if $\delta(op^1, v)$ is undefined
4748	• $e'' \rightarrow_{1-D}^* v$ and $\vdash_1 v$	$op^1 v \qquad \triangleright_{1-D'} \delta(op^1, v)$
4749	• $e'' \rightarrow_{1-D}^* E[e']$ and $e' \triangleright_{1-D} TagErr$	$op^2 v_0 v_1 \qquad \rhd_{1-D}' TagErr$
4750	• $e'' \rightarrow_{\text{1-D}}^* \text{BndryErr}$	if $\delta(op^2, v_0, v_1)$ is undefined
4751	• e" diverges	$op^2 v_0 v_1 \qquad \rhd_{1-D}' \delta(op^2, v_0, v_1)$
4752	Proof:	and $e \rightarrow_{1-D}' e$ is defined as:
4753	1. $\vdash_1 e \rightsquigarrow e''$	$E[e] \longrightarrow_{1-D}' E[e']$
4754	^ <b>⊢</b> <sub>1</sub> e''	if $e \triangleright_{1-D}' e'$
4755	by → dynamic soundness	$E[\operatorname{stat} \tau \ v] \rightarrow_{1-D}' E[\mathcal{D}_1(\tau, v)]$
4756	2. QED by dynamic progress and dynamic preservation	$E[dyn \ \tau \ v] \rightarrow_{1-D}' E[\mathcal{D}_1(\tau, v)]$
4757		$E[Err] \longrightarrow_{1-D}' Err$
4758	<b>Theorem 4.3</b> : boundary-free 1-soundness	and $\rightarrow_{1-D}^*$ is the reflexive transitive closure of $\rightarrow_{1-D}$
4759	If $\vdash e : \tau$ and $e$ is boundary-free then one of the following	
4760	holds:	then one of the following holds:
4761	• $e \rightarrow_{1-S}^* v$ and $+ v : \tau$	• $e'' \rightarrow_{1-D}^* v$ and $\vdash_1 v : \lfloor \tau \rfloor$
4762	• $e \rightarrow_{1-S}^* BndryErr$	• $e'' \rightarrow_{1-D}^{*D}' \text{TagErr}$
4763	• <i>e</i> diverges	• $e'' \rightarrow_{1-D}^{*}'$ BndryErr
4764	Proof:	• e diverges
4765	QED by progress and preservation	<i>Proof</i> ( <i>sketch</i> ): By <i>static</i> 1- <i>soundness</i> and the fact that $\triangleright_{1-S}$ is a
4766		subset of $\triangleright_{1-D}'$ (modulo the dyn $e$ and stat $e$ boundaries).
4767	<b>Theorem 4.4</b> : H/1 base type equivalence	
4768	If $\vdash e : \tau$ and all boundary terms in $e$ are of the following	
4769	four forms:	
4770		
4771	• dyn Int e'	
4772	• stat Int e'	
4773	• stat Nat e'	
4774	• dyn Nat e'	
4774	and $\vdash e : \tau \leadsto e''$ , then $e \to_{H-S}^* v$ if and only if $e'' \to_{1-S}^* v$ .	
	Proof:	
4776	1. $\mathcal{D}_{H}(Int,v) = \mathcal{D}_{1}(Int,v)$	
4777	by by definition	
4778	2. $\mathcal{D}_{H}(Nat,v) = \mathcal{D}_{1}(Nat,v)$	
4779	by by definition	
4780	3. $S_{H}(Int,v) = S_1(Int,v)$	
4781	by by definition	
4782	4. $S_H(Nat, v) = S_1(Nat, v)$	
4783	by by definition	
4784		
4785	44	

```
4841
           A.4.3 First-Order Lemmas
4842
           Lemma 4.6 : \mathcal{D}_1 soundness
4843
           If \vdash_1 v then \vdash_1 \mathcal{D}_1(\tau, v) : \lfloor \tau \rfloor.
4844
           Proof:
4845
              • \mathcal{D}_1(\tau, v) = \mathcal{X}(\lfloor \tau \rfloor, v)
4846
              • QED by check soundness
4847
4848
           Lemma 4.7 : S_1 soundness
4849
           If \vdash_1 v : \tau then \vdash_1 S_1(\tau, v).
4850
           Proof:
4851
              • S_1(\tau, v) = X(\lfloor \tau \rfloor, v)
4852
              • QED check soundness
4853
4854
           Lemma 4.8 : → static soundness
4855
           If \Gamma \vdash e : \tau then \Gamma \vdash e : \tau \leadsto e' and \Gamma \vdash_{\tau} e' : |\tau|.
4856
4857
              By induction on the structure of \Gamma \vdash e : \tau.
4858
4859
                             (x:\tau) \in \Gamma
               CASE
4860
                             \Gamma \vdash x : \tau
4861
                 1. \Gamma \vdash x \rightsquigarrow x
4862
                 2. \Gamma \vdash_{\mathbf{1}} x : |\tau|
4863
                      by (x:\tau) \in \Gamma
4864
                 3. QED
4865
                                   (x:\tau_d), \Gamma \vdash e:\tau_c
               CASE
4867
                            \Gamma \vdash \lambda(x:\tau_d). e: \tau_d \Rightarrow \tau_c
4868
                 1. \Gamma \vdash e : \tau_c \leadsto e'
4869
                       \land (x:\tau_d), \Gamma \vdash e': |\tau_c|
4870
                      by the induction hypothesis
4871
                 2. (x:\tau_d), \Gamma \vdash e' : Any
4872
                      by \lfloor \tau_c \rfloor <: Any
4873
                 3. \lambda(x:\tau_d). e: \tau_d \Rightarrow \tau_c \rightsquigarrow \lambda(x:\tau_d). e'
4874
                 4. \Gamma \vdash_1 \lambda(x : \tau_d). e' : \operatorname{Fun}
4875
                      by (2)
4876
                 5. QED(3, 4)
4877
                                i \in \mathbb{N}
4878
               CASE
                            \Gamma \vdash i : \mathsf{Nat}
4879
4880
                 1. \Gamma \vdash i : \text{Nat} \leadsto i
4881
                 2. QED by \Gamma \vdash_1 i: Nat
4882
               CASE
                            \Gamma \vdash i: \mathsf{Int}
4884
                 1. \Gamma \vdash i : Int \leadsto i
4885
                 2. QED by \Gamma \vdash_1 i: Int
4886
                            \Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1
4887
               CASE
4888
                               \Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1
4889
                 1. \Gamma \vdash e_0 : \tau_0 \leadsto e'_0
4890
                       \wedge \Gamma \vdash e'_0 : \lfloor \tau_0 \rfloor
4891
                      by the induction hypothesis
4892
4893
```

4895

```
2. \Gamma \vdash e_1 : \tau_1 \leadsto e'_1
                                                                                                                                       4896
        \wedge \Gamma \vdash e'_1 : \lfloor \tau_1 \rfloor
                                                                                                                                       4897
        by the induction hypothesis
                                                                                                                                       4898
  3. \Gamma \vdash_{1} e_{0}: Any
                                                                                                                                       4899
        by \lfloor \tau_0 \rfloor <: Any
  4. \Gamma ⊢<sub>1</sub> e_1 : Any
                                                                                                                                       4901
        by \lfloor \tau_1 \rfloor <: Any
                                                                                                                                       4902
   5. \Gamma \vdash \langle e_0, e_1 \rangle : \tau \leadsto \langle e'_0, e'_1 \rangle
                                                                                                                                       4903
        by (1, 2)
  6. \Gamma \vdash_1 \langle e'_0, e'_1 \rangle: Pair
                                                                                                                                       4905
        by (3, 4)
                                                                                                                                       4906
   7. QED by (5, 6)
                                                                                                                                       4907
                                                                                                                                       4908
               \Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash e_1 : \tau_d
CASE
                                                                                                                                       4909
                                 \Gamma \vdash e_0 \ e_1 : \tau_c
                                                                                                                                       4910
  1. \Gamma \vdash e_0 : \tau_d \Longrightarrow \tau_c \leadsto e'_0
                                                                                                                                       4911
        \wedge \Gamma \vdash_{1} e'_{0} : \lfloor \tau_{d} \Rightarrow \tau_{c} \rfloor
                                                                                                                                       4912
        by the induction hypothesis
                                                                                                                                       4913
   2. \Gamma \vdash e_1 : \tau_d \leadsto e'_1
                                                                                                                                       4914
        \wedge \Gamma \vdash_{1} e'_{1} : \lfloor \tau_{c} \rfloor
                                                                                                                                       4915
        by the induction hypothesis
                                                                                                                                       4916
  3. \Gamma \vdash_{1} e'_{0}: Fun
                                                                                                                                       4917
        by \lfloor \tau_d \Rightarrow \tau_c \rfloor = \operatorname{\mathsf{Fun}}
                                                                                                                                       4918
   4. \Gamma ⊢<sub>1</sub> e'_1: Any
                                                                                                                                       4919
        by \lfloor \tau_c \rfloor <: Any
                                                                                                                                       4920
  5. \Gamma \vdash e_0 \ e_1 : \tau_c \leadsto \operatorname{chk} \left[\tau_c\right] \left(e_0' \ e_1'\right)
                                                                                                                                       4921
        by (1, 2)
                                                                                                                                       4922
  6. \Gamma \vdash_{1} \operatorname{chk} \lfloor \tau_{c} \rfloor (e'_{0} e'_{1}) : \lfloor \tau_{c} \rfloor
                                                                                                                                       4923
        by (3, 4)
                                                                                                                                       4924
   7. QED by (5, 6)
                                                                                                                                       4925
               \Gamma \vdash e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau
                                                                                                                                       4926
CASE
                                                                                                                                       4927
                             \Gamma \vdash op^1 e_0 : \tau
                                                                                                                                       4928
    IF op^1 = \overline{\mathsf{fst}}:
                                                                                                                                       4929
       1. \Delta(\mathsf{fst}, \tau_0) = \tau
                                                                                                                                       4930
       2. \tau_0 = \tau \times \tau'
                                                                                                                                       4931
             by \Delta inversion
                                                                                                                                       4932
       3. \Gamma \vdash e_0 : \tau \times \tau' \leadsto e'_0
                                                                                                                                       4933
             \land \Gamma \vdash_{\!\!\!\!1} e_0' : \lfloor \tau \times \tau' \rfloor
                                                                                                                                       4934
             by the induction hypothesis
                                                                                                                                       4935
       4. \Gamma ⊢<sub>1</sub> e'_0: Pair
                                                                                                                                       4936
             by \lfloor \tau \times \tau' \rfloor = Pair
                                                                                                                                       4937
       5. \Gamma \vdash \mathsf{fst} \ e_0 : \tau \leadsto \mathsf{chk} \ \lfloor \tau \rfloor \ (\mathsf{fst} \ e_0')
                                                                                                                                       4938
             by (2)
                                                                                                                                       4939
       6. \Gamma \vdash_{1} \operatorname{chk} \lfloor \tau \rfloor (\operatorname{fst} e'_{0}) : \lfloor \tau \rfloor
             by (3)
                                                                                                                                       4941
       7. QED by 4,5
                                                                                                                                       4942
    ELSE op^1 = \text{snd}:
                                                                                                                                       4943
       1. \Delta(\operatorname{snd}, \tau_0) = \tau
       2. \tau_0 = \tau' \times \tau
                                                                                                                                       4945
             by \Delta inversion
                                                                                                                                       4946
       3. \Gamma \vdash e_0 : \tau' \times \tau \leadsto e'_0
                                                                                                                                       4947
             \wedge \Gamma \vdash_{1} e'_{0} : \lfloor \tau' \times \tau \rfloor
                                                                                                                                       4948
             by the induction hypothesis
                                                                                                                                       4949
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4951
                        4. \Gamma \vdash_1 e'_0: Pair
4952
                             by \lfloor \tau' \times \tau \rfloor = Pair
4953
                        5. \Gamma \vdash \operatorname{snd} e_0 : \tau \leadsto \operatorname{chk} \lfloor \tau \rfloor (\operatorname{snd} e_0')
4954
                             by (2)
4955
                        6. \Gamma \vdash_1 \operatorname{chk} \lfloor \tau \rfloor (\operatorname{snd} e_0') : \lfloor \tau \rfloor
                             by (3)
4956
4957
                        7. QED by 4,5
                                \Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau \mid_{:}
4958
                 CASE
4959
                                                         \Gamma \vdash op^2 e_0 e_1 : \tau
4960
                    1. \Gamma \vdash e_0 : \tau_0 \leadsto e'_0
4961
                         \land \Gamma \vdash_{\!\!\!\!1} e_0' : \lfloor \tau_0 \rfloor
4962
                         by the induction hypothesis
4963
                    2. \Gamma \vdash e_1 : \tau_1 \leadsto e'_1
4964
                          \wedge \Gamma \vdash_{1} e'_{1} : \lfloor \tau_{1} \rfloor
4965
                         by the induction hypothesis
4966
                    3. \Delta(op^2, \lfloor \tau_0 \rfloor, \lfloor \tau_1 \rfloor) = \lfloor \tau \rfloor
4967
                         by \Delta tag preservation
4968
                    4. \Gamma \vdash op^2 e_0 e_1 : \tau \leadsto op^2 e'_0 e'_1
4969
                         by (1, 2)
4970
                    5. \Gamma \vdash_1 op^2 e'_0 e'_1 : \lfloor \tau \rfloor
4971
                         by (1, 2, 3)
4972
                    6. QED by (5, 6)
4973
4974
                                \Gamma \vdash e : \tau' \quad \tau' \lessdot: \tau
                 CASE
4975
4976
                   1. \Gamma \vdash e : \tau' \leadsto e'
4977
                         \wedge \Gamma \vdash_{\mathbf{1}} e' : |\tau'|
                         by the induction hypothesis
4979
                    2. |\tau'| \leq |\tau|
4980
                         by subtyping preservation
4981
                    3. \Gamma \vdash_{1} e' : \lfloor \tau \rfloor
4982
                         by (2)
4983
                    4. QED by (1, 3)
4984
4985
                                \Gamma \vdash \mathsf{Err} : \tau
4986
4987
                    1. \Gamma \vdash \mathsf{Err} : \tau \leadsto \mathsf{Err}
4988
                    2. \Gamma \vdash_{\mathbf{1}} \mathsf{Err} : \tau
                    3. QED
4989
4990
                                         \Gamma \vdash e
                 CASE
4991
                                \Gamma \vdash \mathsf{dyn} \ \tau \ e : \tau
4992
                    1. \Gamma \vdash e \leadsto e'
4993
                         \wedge \Gamma \vdash_{1} e'
4994
                         by \rightsquigarrow dynamic soundness
                    2. \Gamma \vdash \mathsf{dyn} \ \tau \ e : \tau \leadsto \mathsf{dyn} \ \tau \ e'
4996
                         by (1)
                    3. \Gamma \vdash_1 \operatorname{dyn} \tau \ e' : \lfloor \tau \rfloor
4998
                         by (1)
                    4. QED by (2, 3)
5000
5001
             Lemma 4.9: → dynamic soundness
5002
             If \Gamma \vdash e then \Gamma \vdash e \leadsto e' and \Gamma \vdash_1 e'
5003
             Proof:
5004
```

```
By induction on the structure of \Gamma \vdash e.
              x\in \Gamma
CASE
              \Gamma \vdash x
  1. \Gamma \vdash x \rightsquigarrow x
  2. \Gamma \vdash_{1} x
        by x \in \Gamma
  3. QED
                x, \Gamma \vdash e
CASE
              \Gamma \vdash \lambda x. e
   1. x, \overline{\Gamma \vdash e \leadsto e'}
        \land x, \Gamma \vdash_{1} e'
        by the induction hypothesis
  2. \Gamma \vdash \lambda x. e \rightsquigarrow \lambda x. e'
        by (1)
  3. \Gamma \vdash_1 \lambda x. e'
        by (1)
   4. QED by (2, 3)
CASE
  1. \Gamma \vdash i \leadsto i
   2. \Gamma \vdash_{1} i
   3. QED
              \Gamma \vdash e_0 \quad \Gamma \vdash e_1
CASE
                 \Gamma \vdash \langle e_0, e_1 \rangle
  1. \Gamma \vdash e_0 \leadsto e'_0
        \wedge \Gamma \vdash_{1} e'_{0}
        by the induction hypothesis
  2. \Gamma \vdash e_1 \leadsto e'_1
        \wedge \Gamma \vdash_{1} e'_{1}
        by the induction hypothesis
  3. \Gamma \vdash \langle e_0, e_1 \rangle \rightsquigarrow \langle e'_0, e'_1 \rangle
        by (1, 2)
   4. \Gamma \vdash_1 \langle e'_0, e'_1 \rangle
       by (1, 2)
  5. QED by (3, 4)
              \Gamma \vdash e_0 \quad \Gamma \vdash e_1
CASE
                    \Gamma \vdash e_0 \ e_1
   1. \Gamma \vdash e_0 \leadsto e'_0
        \wedge \Gamma \vdash_{1} e'_{0}
        by the induction hypothesis
   2. \Gamma \vdash e_1 \leadsto e'_1
        \wedge \Gamma \vdash_{1} e'_{1}
        by the induction hypothesis
  3. \Gamma \vdash e_0 \ e_1 \leadsto e'_0 \ e'_1
        by (1, 2)
  4. \Gamma \vdash_1 e'_0 e'_1
        by (1, 2)
   5. QED by (3, 4)
```

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5061
                                                                                                                          CASE e = E^{\bullet}[v_0 \ v_1]:
              CASE
5062
                                                                                                                             1. \vdash_1 v_0 v_1 : K'
                          \Gamma \vdash op^1 e
5063
                                                                                                                                 by static hole typing
                1. \Gamma \vdash e \leadsto e'
                                                                                                                            2. \vdash_{\mathbf{1}} v_0 : \mathsf{Fun}
5064
                     \wedge \Gamma \vdash_{1} e'
                                                                                                                                 by inversion (1)
                     by the induction hypothesis
                                                                                                                            3. v_0 = \lambda x. e'
5066
                2. \Gamma \vdash op^1 e \leadsto op^1 e'
                                                                                                                                 \vee v_0 = \lambda(x:\tau_d). e'
5067
                     by (1)
                                                                                                                                 by canonical forms (2)
5068
                3. \Gamma \vdash_1 op^1 e'
                                                                                                                            4. IF v_0 = \lambda x. e':
                     by (1)
                                                                                                                                   a. e \rightarrow_{1-S} E^{\bullet}[\mathsf{dyn} (e'[x \leftarrow v_1])]
5070
                4. QED by (2, 3)
5071
                                                                                                                                        by (\lambda x. e') v_1 \triangleright_{1-S} (\text{dyn} (e'[x \leftarrow v_1]))
                          \Gamma \vdash e_0 \quad \Gamma \vdash e_1
5072
                                                                                                                                   b. QED
5073
                                                                                                                                 IF v_0 = \lambda(x:\tau_d). e'
                           \Gamma \vdash op^2 e_0 e_1
5074
                                                                                                                                       \wedge \mathcal{X}(|\tau_d|, v_1) = v_1:
                1. \Gamma \vdash e_0 \leadsto e'_0
5075
                                                                                                                                   a. e \rightarrow_{1-S} E^{\bullet}[e'[x \leftarrow v_1]]
                     \wedge \Gamma \vdash_{1} e'_{0}
5076
                                                                                                                                        by (\lambda(x:\tau_d). e') v_1 \triangleright_{1-S} e'[x \leftarrow v_1]
                     by the induction hypothesis
5077
                                                                                                                                   b. QED
                2. \Gamma \vdash e_1 \leadsto e'_1
5078
                                                                                                                                 ELSE v_0 = \lambda(x:\tau_d).e'
                     \wedge \Gamma \vdash_{1} e'_{1}
                                                                                                                                           \wedge \mathcal{X}(\lfloor \tau_d \rfloor, v_1) = \text{BndryErr}:
5079
                     by the induction hypothesis
5080
                                                                                                                                   a. e \rightarrow_{1-S} E^{\bullet}[BndryErr]
                3. \Gamma \vdash op^2 e_0 e_1 \rightsquigarrow op^2 e'_0 e'_1
5081
                                                                                                                                        by (\lambda(x:\tau_d).e') v_1 \triangleright_{1-S} BndryErr
                     by (1, 2)
5082
                                                                                                                                   b. QED
                4. \Gamma \vdash_1 op^2 e'_0 e'_1
                                                                                                                          CASE e = E^{\bullet}[op^1 v]:
5083
                    by (1, 2)
                                                                                                                             1. op^{1} = fst
5084
                5. QED by 3,4
                                                                                                                                 \vee op^1 = snd
5085
              CASE
5086
                                                                                                                            2. \vdash_{1} op^{1} v : K'
                          \Gamma \vdash \mathsf{Err}
5087
                                                                                                                                 by static hole typing
                1. \Gamma \vdash Err \leadsto Err
                                                                                                                            3. \vdash_1 v: Pair
                2. Γ ⊦₁ Err
                                                                                                                                 by inversion (2)
5089
                3. QED
5090
                                                                                                                            4. v = \langle v_0, v_1 \rangle
5091
                            \Gamma \vdash e : \tau
                                                                                                                                 by canonical forms (3)
              CASE
                                                                                                                             5. \delta(op^1, v) = v_i where i \in \{0, 1\}
                          \Gamma \vdash \text{stat } \tau e
5093
                                                                                                                                 by (1, 3)
                1. \Gamma \vdash e : \tau \leadsto e'
5094
                                                                                                                             6. e \rightarrow_{1-S} E^{\bullet}[v_i]
                     \wedge \Gamma \vdash_{\!\!\!1} e' : |\tau|
5095
                                                                                                                                 by (op^1 v) \triangleright_{1-S} v_i
                     by → static soundness
5096
                                                                                                                            7. QED
                2. \Gamma \vdash \operatorname{stat} \tau e \rightsquigarrow \operatorname{stat} \tau e'
5097
                                                                                                                          CASE e = E^{\bullet}[op^2 v_0 v_1]:
                     by (1)
5098
                                                                                                                             1. \vdash_1 op^2 v_0 v_1 : K'
                3. \Gamma \vdash_1 stat \tau e
                                                                                                                                 by static hole typing
5099
                     by (1)
5100
                                                                                                                            2. \vdash_1 v_0 : K_0
                4. QED by (2,3)
5101
                                                                                                                                 \wedge \vdash_1 v_1 : K_1
5102
                                                                                                                                 \wedge \Delta(op^2, K_0, K_1) = K_2
          Lemma 4.10: 1 static progress
5103
                                                                                                                                 by inversion (1)
            If \vdash_1 e : K then one of the following holds:
5104
                                                                                                                            3. \delta(op^2, v_0, v_1) = e''
             • e is a value
5105
                                                                                                                                 by \Delta tag soundness
             • e \in Err
5106
                                                                                                                             4. QED by e \rightarrow_{1-S} E^{\bullet}[e'']
             • e \rightarrow_{1-S} e'
5107
                                                                                                                          CASE e = E^{\bullet}[\operatorname{chk} K v_0]:
             • e \rightarrow_{1-S} BndryErr
5108
                                                                                                                            1. e \rightarrow_{1-S} E^{\bullet}[X(K,v)]
             • e = E[\text{dyn } \tau' E^{\bullet}[e']] \text{ and } e' \rightarrow_{1-D} \text{TagErr}
5109
                                                                                                                             2. OED
             • e = E[\text{dyn } E^{\bullet}[e']] \text{ and } e' \rightarrow_{1-D} \text{TagErr}
                                                                                                                          CASE e = E[dyn e'] where e' is boundary-free:
5110
5111
                                                                                                                            1. e' is a value
             By the boundary factoring lemma, there are ten cases.
5112
                                                                                                                                 \forall e' \in Err
              CASE e is a value :
5113
                                                                                                                                 \vee e' \rightarrow_{1-D} e''
                1. QED
5114
5115
```

```
a. E' \in E^{\bullet}
5171
                    \vee e' \rightarrow_{1-D} BndryErr
                                                                                                                                                                                                                   5226
5172
                    \vee e' = E'[e''] and e'' \triangleright_{1-D} \mathsf{TagErr}
                                                                                                                               by e' is boundary-free
                                                                                                                                                                                                                   5227
5173
                   by dynamic progress
                                                                                                                                                                                                                   5228
               2. IF e' is a value :
                                                                                                                  CASE e = E[\text{stat } \tau \ e'] where e' is boundary-free :
5174
                                                                                                                                                                                                                   5229
5175
                      a. QED e \rightarrow_{1-S} E[v]
                                                                                                                     1. e' is a value
                                                                                                                                                                                                                   5230
                   IF e' \in Err:
                                                                                                                         \forall e' \in Err
                                                                                                                                                                                                                   5231
5176
5177
                     a. QED e \rightarrow_{1-S} e'
                                                                                                                         \vee e' \rightarrow_{1-S} e''
                                                                                                                                                                                                                   5232
                   IF e' \rightarrow_{1-D} e'':
                                                                                                                         \vee e' \rightarrow_{1-S} BndryErr
5178
                                                                                                                                                                                                                   5233
5179
                      a. QED e \rightarrow_{1-S} E[dyn e'']
                                                                                                                         \vee e' = E''[\mathsf{dyn} \ \tau'' \ E^{\bullet}''[e'']] \text{ and } e'' \rhd_{1-D} \mathsf{TagErr}
                                                                                                                                                                                                                   5234
                                                                                                                         \vee e' = E''[\mathsf{dyn}\ E^{\bullet}''[e'']] and e'' \rhd_{1-D} \mathsf{TagErr}
5180
                   IF e' \rightarrow_{1-D} BndryErr:
                                                                                                                                                                                                                   5235
5181
                     a. QED e \rightarrow_{1-S} E[\text{dyn BndryErr}]
                                                                                                                         by static progress
                                                                                                                                                                                                                   5236
                    ELSE e' = E'[e''] and e'' \triangleright_{1-D} TagErr:
                                                                                                                    2. IF e' is a value :
5182
                                                                                                                                                                                                                   5237
5183
                     a. E' \in E^{\bullet}
                                                                                                                           a. QED e \rightarrow_{1-S} E[S_1(\tau',e')]
                                                                                                                                                                                                                   5238
                                                                                                                         IF e' \in Err:
5184
                          by e' is boundary-free
                                                                                                                                                                                                                   5239
5185
                                                                                                                           a. QED e \rightarrow_{1-S} e'
                                                                                                                                                                                                                   5240
                     b. oed
5186
             CASE e = E[\text{stat } e'] where e' is boundary-free:
                                                                                                                         IF e' \rightarrow_{1-S} e'':
                                                                                                                                                                                                                   5241
               1. e' is a value
5187
                                                                                                                           a. QED e \rightarrow_{1-S} E[\text{stat } \tau' e'']
                                                                                                                                                                                                                   5242
                   \vee e' \in Err
                                                                                                                         IF e' \rightarrow_{1-S} BndryErr:
5188
                                                                                                                                                                                                                   5243
                    \vee e' \rightarrow_{1-S} e''
                                                                                                                           a. QED e \rightarrow_{1-S} E[\text{stat } \tau' \text{ BndryErr}]
5189
                                                                                                                                                                                                                   5244
                                                                                                                         IF e' = E''[\text{dyn } \tau'' E^{\bullet''}[e'']] and e'' \triangleright_{1\text{-D}} \text{TagErr}
5190
                    \vee e' \rightarrow_{1-S} BndryErr
                                                                                                                                                                                                                   5245
5191
                    \vee e' = E''[\mathsf{dyn} \ \tau'' \ E^{\bullet''}[e'']] \text{ and } e'' \rhd_{1-D} \mathsf{TagErr}
                                                                                                                                                                                                                   5246
                    \vee e' = E''[\mathsf{dyn} \ E^{\bullet''}[e'']] \text{ and } e'' \rhd_{1-D} \mathsf{TagErr}
                                                                                                                           a. Contradiction by e' is boundary-free
5192
                                                                                                                                                                                                                   5247
                                                                                                                         ELSE e' = E''[\mathsf{dyn}\ E^{\bullet''}[e'']] and e'' \rhd_{1-D} \mathsf{TagErr}
                   by static progress
                                                                                                                                                                                                                   5248
5193
               2. IF e' is a value :
5194
                                                                                                                                                                                                                   5249
                     a. QED e \rightarrow_{1-S} E[e']
5195
                                                                                                                           a. Contradiction by e' is boundary-free
                                                                                                                                                                                                                   5250
                   IF e' \in Err:
                                                                                                                  CASE e = E[Err]:
                                                                                                                                                                                                                   5251
5197
                      a. QED e \rightarrow_{1-S} e'
                                                                                                                     1. QED e \rightarrow_{1-S} Err
                                                                                                                                                                                                                   5252
                   IF e' \rightarrow_{1-S} e'':
                                                                                                                                                                                                                   5253
                     a. QED e \rightarrow_{1-S} E[\text{stat } e'']
5199
                                                                                                                                                                                                                   5254
                                                                                                               Lemma 4.11: 1 dynamic progress
                   IF e' \rightarrow_{1-S} BndryErr:
5200
                                                                                                                                                                                                                   5255
                                                                                                                 If \vdash_1 e : K then one of the following holds:
5201
                      a. QED e \rightarrow_{1-S} E[\text{stat BndryErr}]
                                                                                                                 • e is a value
                                                                                                                                                                                                                   5256
                   IF e' = E''[\text{dyn } \tau'' E^{\bullet''}[e'']] and e'' \triangleright_{1-D} \text{TagErr}
                                                                                                                 • e \in Err
                                                                                                                                                                                                                   5257
5203
                                                                                                                                                                                                                   5258
                                                                                                                 • e \rightarrow_{1-D} e'
5204
                      a. Contradiction by e' is boundary-free
                                                                                                                                                                                                                   5259
                                                                                                                 • e \rightarrow_{1-D} BndryErr
5205
                   ELSE e' = E''[\mathsf{dyn}\ E^{\bullet''}[e'']] and e'' \rhd_{1-D} \mathsf{TagErr}
                                                                                                                 • e = E[e'] and e' \rightarrow_{1-D} TagErr
                                                                                                                                                                                                                   5260
                                                                                                               Proof:
5206
                                                                                                                                                                                                                   5261
5207
                      a. Contradiction by e' is boundary-free
                                                                                                                  By the boundary factoring lemma, there are nine cases.
                                                                                                                                                                                                                   5262
             CASE e = E[dyn \tau e'] where e' is boundary-free :
                                                                                                                  CASE e = v:
5208
                                                                                                                                                                                                                   5263
               1. e' is a value
                                                                                                                     1. QED
5209
                                                                                                                                                                                                                   5264
                    \forall e' \in Err
                                                                                                                  CASE e = E^{\bullet}[v_0 \ v_1]:
                                                                                                                                                                                                                   5265
5210
                    \vee e' \rightarrow_{1-D} e''
                                                                                                                     IF v_0 = \lambda x. e_0:
5211
                                                                                                                                                                                                                   5266
                    \vee e' \rightarrow_{1-D} BndryErr
                                                                                                                        1. e \rightarrow_{1-D} E^{\bullet}[e_0[x \leftarrow v_1]]
5212
                                                                                                                                                                                                                   5267
                    \vee e' = E'[e''] and e'' \triangleright_{1-D} \mathsf{TagErr}
                                                                                                                            by (\lambda x. e_0) v_1 \triangleright_{1-D} e_0[x \leftarrow v_1]
5213
                                                                                                                                                                                                                   5268
                   by dynamic progress
                                                                                                                        2. QED
5214
                                                                                                                                                                                                                   5269
                                                                                                                     IF v_0 = \lambda(x:\tau_d). e_0
5215
               2. IF e' is a value :
                                                                                                                                                                                                                   5270
                      a. QED e \rightarrow_{1-S} E[\mathcal{D}_1(\tau', e')]
                                                                                                                           \wedge \mathcal{X}(\lfloor \tau_d \rfloor, v_1) = v_1:
5216
                                                                                                                                                                                                                   5271
                                                                                                                        1. e \rightarrow_{1-D} E^{\bullet}[\text{stat}(e_0[x \leftarrow v_1])]
                   IF e' \in Err:
                                                                                                                                                                                                                   5272
                      a. QED e \rightarrow_{1-S} e'
                                                                                                                            by (\lambda(x:\tau_d). e_0) v_1 \triangleright_{1-D} (\text{stat } e_0[x \leftarrow v_1])
5218
                                                                                                                                                                                                                   5273
5219
                   IF e' \rightarrow_{1-D} e'':
                                                                                                                       2. OED
                                                                                                                                                                                                                   5274
                                                                                                                     IF v_0 = \lambda(x:\tau_d). e_0
5220
                      a. QED e \rightarrow_{1-S} E[\mathsf{dyn} \ \tau' \ e'']
                                                                                                                                                                                                                   5275
                                                                                                                           \wedge \mathcal{X}(\lfloor \tau_d \rfloor, v_1) = \text{BndryErr}:
5221
                   IF e' \rightarrow_{1-D} BndryErr:
                                                                                                                                                                                                                   5276
                                                                                                                        1. e \rightarrow_{1-D} E^{\bullet}[BndryErr]
5222
                     a. QED e \rightarrow_{1-S} E[dyn \tau' BndryErr]
                                                                                                                                                                                                                   5277
                   ELSE e' = E'[e''] and e'' \triangleright_{1-D} \mathsf{TagErr}:
5223
                                                                                                                            by (\lambda(x:\tau_d). e_0) v_1 \triangleright_{1-D} BndryErr
                                                                                                                                                                                                                   5278
                                                                                                                        2. QED
5224
                                                                                                                                                                                                                   5279
5225
                                                                                                                                                                                                                   5280
                                                                                                         48
```

```
IF e' \rightarrow_{1-S} e'':
5281
                ELSE v_0 = i
                                                                                                                                                                                                             5336
                         \vee v_0 = \langle v, v' \rangle:
                                                                                                                       a. QED e \rightarrow_{1-S} E[\text{stat } e'']
5282
                                                                                                                                                                                                             5337
5283
                  1. e \rightarrow_{1-D} E^{\bullet}[TagErr]
                                                                                                                     IF e' \rightarrow_{1-S} BndryErr:
                                                                                                                                                                                                             5338
                                                                                                                       a. QED e \rightarrow_{1-S} E[\text{stat BndryErr}]
                      by v_0 \ v_1 \rhd_{1-D} \mathsf{TagErr}
                                                                                                                                                                                                             5339
5284
                                                                                                                     IF e' = E''[\text{dyn } \tau'' E^{\bullet ''}[e'']] and e'' \rhd_{1-D} \text{TagErr}
5285
                  2. QED
                                                                                                                                                                                                             5340
            CASE e = E^{\bullet}[op^1 v]:
                                                                                                                                                                                                             5341
5286
               IF \delta(op^1, v) = v':
                                                                                                                       a. Contradiction by e' is boundary-free
5287
                                                                                                                                                                                                             5342
                  1. e \rightarrow_{1-D} E^{\bullet}[v']
                                                                                                                     ELSE e' = E''[\mathsf{dyn}\ E^{\bullet''}[e'']] and e'' \triangleright_{1-D} \mathsf{TagErr}
                                                                                                                                                                                                             5343
5288
5289
                      by (op^1 v) \triangleright_{1-D} v'
                                                                                                                                                                                                             5344
                                                                                                                        a. Contradiction by e' is boundary-free
5290
                  2. QED
                                                                                                                                                                                                             5345
5291
                ELSE \delta(op^1, v) is undefined :
                                                                                                               CASE e = E[dyn \tau e'] where e' is boundary-free :
                                                                                                                                                                                                             5346
                  1. e \rightarrow_{1-D} TagErr
                                                                                                                 1. e' is a value
5292
                                                                                                                                                                                                             5347
                                                                                                                     \forall e' \in Err
                      by (op^1 v) \triangleright_{1-D} \mathsf{TagErr}
                                                                                                                                                                                                             5348
                                                                                                                     \vee e' \rightarrow_{1-D} e''
5294
                  2. OED
                                                                                                                                                                                                             5349
             CASE e = E^{\bullet}[op^2 v_0 v_1]:
5295
                                                                                                                     \vee e' \rightarrow_{1-D} BndryErr
                                                                                                                                                                                                             5350
5296
               IF \delta(op^2, v_0, v_1) = e'':
                                                                                                                     \vee e' = E[e''] and e'' \triangleright_{1-D} TagErr
                                                                                                                                                                                                             5351
                  1. QED by e \rightarrow_{1-D} E[e'']
                                                                                                                     by dynamic progress
5297
                                                                                                                                                                                                             5352
5298
                ELSE \delta(op^2, v_0, v_1) is undefined:
                                                                                                                 2. IF e' is a value :
                                                                                                                                                                                                             5353
                  1. e \rightarrow_{1-D} E^{\bullet}[TagErr]
                                                                                                                        a. QED e \rightarrow_{1-D} E[\mathcal{D}_1(\tau', e')]
5299
                                                                                                                                                                                                             5354
                      by (op^2 v_0 v_1) \triangleright_{1-D} TagErr
5300
                                                                                                                     IF e' \in Err:
                                                                                                                                                                                                             5355
5301
                  2. QED
                                                                                                                       a. QED e \rightarrow_{1-D} e'
                                                                                                                                                                                                             5356
             CASE e = E^{\bullet}[\operatorname{chk} K v_0]:
                                                                                                                     IF e' \rightarrow_{1-D} e'':
5302
                                                                                                                                                                                                             5357
                                                                                                                        a. QED e \rightarrow_{1-S} E[dyn \tau' e'']
               1. Contradiction by \vdash_1 e
                                                                                                                                                                                                             5358
5303
             CASE e = E[dyn v] where e' is boundary-free:
                                                                                                                     IF e' \rightarrow_{1-D} BndryErr:
5304
                                                                                                                                                                                                             5359
               1. e' is a value
                                                                                                                        a. QED e \rightarrow_{1-D} E[dyn \tau' BndryErr]
5305
                                                                                                                                                                                                             5360
5306
                   \forall e' \in Err
                                                                                                                     ELSE e' = E[e''] and e'' >_{1-D} TagErr:
                                                                                                                                                                                                             5361
5307
                   \vee e' \rightarrow_{1-D} e''
                                                                                                                        a. E \in E^{\bullet}
                                                                                                                                                                                                             5362
                   \vee e' \rightarrow_{1-D} BndryErr
                                                                                                                            by e' is boundary-free
                                                                                                                                                                                                             5363
                   \vee e' = E'[e''] and e'' \triangleright_{1-D} TagErr
5309
                                                                                                                        b. QED
                                                                                                                                                                                                             5364
5310
                   by dynamic progress
                                                                                                               CASE e = E[\text{stat } \tau \ e'] where e' is boundary-free:
                                                                                                                                                                                                             5365
5311
               2. IF e' is a value :
                                                                                                                 1. e' is a value
                                                                                                                                                                                                             5366
                     a. QED e \rightarrow_{1-S} E[v]
                                                                                                                     \forall e' \in Err
                                                                                                                                                                                                             5367
                                                                                                                     \vee e' \rightarrow_{1-S} e''
5313
                   IF e' \in Err:
                                                                                                                                                                                                             5368
                                                                                                                     \vee e' \rightarrow_{1-S} BndryErr
5314
                     a. QED e \rightarrow_{1-S} e'
                                                                                                                                                                                                             5369
                                                                                                                     \vee e' = E''[\mathsf{dyn} \ \tau'' \ E^{\bullet \prime \prime}[e'']] \text{ and } e'' \rhd_{1-D} \mathsf{TagErr}
5315
                   IF e' \rightarrow_{1-D} e'':
                                                                                                                                                                                                             5370
                                                                                                                     \vee e' = E''[\text{dyn } \tau'' E^{\bullet''}[e'']] \text{ and } e'' \rhd_{1-D} \text{TagErr}
                     a. QED e \rightarrow_{1-S} E[dyn e'']
5316
                                                                                                                                                                                                             5371
5317
                   IF e' \rightarrow_{1-D} BndryErr:
                                                                                                                     by static progress
                                                                                                                                                                                                             5372
5318
                     a. QED e \rightarrow_{1-S} E[\text{dyn BndryErr}]
                                                                                                                 2. IF e' is a value :
                                                                                                                                                                                                             5373
                   ELSE e' = E'[e''] and e'' >_{1-D} TagErr:
                                                                                                                        a. QED e \rightarrow_{1-S} E[S_1(\tau', e')]
5319
                                                                                                                                                                                                             5374
                     a. E' \in E^{\bullet}
                                                                                                                     IF e' \in Err:
                                                                                                                                                                                                             5375
5320
                         by e' is boundary-free
                                                                                                                       a. QED e \rightarrow_{1-S} e'
5321
                                                                                                                                                                                                             5376
5322
                     b. oed
                                                                                                                     IF e' \rightarrow_{1-S} e'':
                                                                                                                                                                                                             5377
             CASE e = E[\text{stat } e'] where e' is boundary-free:
5323
                                                                                                                       a. QED e \rightarrow_{1-S} E[\text{stat } \tau' e'']
                                                                                                                                                                                                             5378
                                                                                                                     IF e' \rightarrow_{1-S} BndryErr:
5324
               1. e' is a value
                                                                                                                                                                                                             5379
                   \forall e' \in Err
                                                                                                                       a. QED e \rightarrow_{1-S} E[\text{stat } \tau' \text{ BndryErr}]
5325
                                                                                                                                                                                                             5380
                                                                                                                     IF e' = E''[\text{dyn } \tau'' E^{\bullet''}[e'']] and e'' \triangleright_{1\text{-D}} \text{TagErr}
                   \vee e' \rightarrow_{1-S} e''
5326
                                                                                                                                                                                                             5381
                   \vee e' \rightarrow_{1-S} BndryErr
                                                                                                                                                                                                             5382
                   \vee e' = E''[\mathsf{dyn} \ \tau'' \ E^{\bullet''}[e'']] \text{ and } e'' \rhd_{1-D} \mathsf{TagErr}
                                                                                                                        a. Contradiction by e' is boundary-free
5328
                                                                                                                                                                                                             5383
                   \vee e' = E''[\mathsf{dyn}\ E^{\bullet}''[e'']] and e'' \rhd_{1-D} \mathsf{TagErr}
                                                                                                                     ELSE e' = E''[\mathsf{dyn}\ E^{\bullet}''[e'']] and e'' \rhd_{1-D} \mathsf{TagErr}
5329
                                                                                                                                                                                                             5384
5330
                   by static progress
                                                                                                                                                                                                             5385
5331
               2. IF e' is a value :
                                                                                                                        a. Contradiction by e' is boundary-free
                                                                                                                                                                                                             5386
5332
                     a. QED e \rightarrow_{1-S} E[e']
                                                                                                               CASE e = E[Err]:
                                                                                                                                                                                                             5387
5333
                   IF e' \in Err:
                                                                                                                  1. e \rightarrow_{1-D} Err
                                                                                                                                                                                                             5388
5334
                     a. QED e \rightarrow_{1-S} e'
                                                                                                                  2. QED
                                                                                                                                                                                                             5389
5335
                                                                                                                                                                                                             5390
                                                                                                      49
```

5391		3. $v = \langle v_0, v_1 \rangle$	5446
5392	Lemma 4.12: 1 static preservation	by canonical forms	5447
5393	If $\vdash_1 e : K$ and $e \rightarrow_{1-S} e'$ then $\vdash_1 e' : K$	4. $\vdash_1 v_0$ : Any	5448
5394	Proof:	$\wedge \vdash_1 v_1 : Any$	5449
5395	By the <i>boundary factoring</i> lemma, there are ten cases to	by inversion (2, 3)	5450
5396	consider.	5. $v' = v_0$	5451
5397	<b>CASE</b> $e$ is a value :	$\vee v' = v_1$	5452
5398	1. Contradiction by $e \rightarrow_{1-S} e'$	by $\delta(fst, v) = v_0$	5453
5399	CASE $e = E^{\bullet}[v_0 \ v_1]$ :	$\wedge \ \delta(snd, v) = v_1$	5454
5400	IF $v_0 = \lambda x. e'$	6. QED by hole substitution (5)	5455
5401	$\land e \rightarrow_{1-S} E^{\bullet}[dyn\ e'[x \leftarrow v_1]]:$	CASE $e = E^{\bullet}[op^2 \ v_0 \ v_1]$	5456
5402	1. $\vdash_1 v_0 v_1 : Any$	$\wedge  \delta(op^2, v_0, v_1) = e^{\prime\prime}$	5457
5403	by static hole typing	$\wedge e \rightarrow_{1-S} E^{\bullet}[e'']:$	5458
5404	2. $\vdash_1 v_0$ : Fun	1. $\vdash_1 op^2 v_0 v_1 : K'$	5459
5405	$\wedge \vdash_1 v_1 : Any$	by static hole typing	5460
5406	by inversion	2. $\vdash_1 v_0 : K_0$	5461
5407	3. $x \vdash_1 e'$	$\wedge \vdash_1 v_1 : K_1$	5462
5408	by inversion (2)	$\wedge \Delta(op^2, K_0, K_1) = K''$	5463
5409	$4. \vdash_1 v_1$	$\wedge K^{\prime\prime} <: K^{\prime}$	5464
5410	by static value inversion (2)	by inversion (1)	5465
5411	5. $\vdash_1 e'[x \leftarrow v_1]$	3. $\vdash_1 e'' : K''$	5466
5412	by substitution (3, 4)	by $\Delta$ tag soundness (3)	5467
5413	6. $\vdash_1 \operatorname{dyn} (e'[x \leftarrow v_1]) : \operatorname{Any}$	$4. \vdash_1 e'' : K'$	5468
5414	by (5)	by (2, 3)	5469
5415	7. QED by hole substitution	5. QED by hole substitution (4)	5470
5416	<b>IF</b> $v_0 = \lambda(x:\tau). e'$	CASE $e = E^{\bullet}[\operatorname{chk} K_0 v_0]$ :	5471
5417	$\land \mathcal{X}(\lfloor \tau \rfloor, v_1) = BndryErr$	1. $E^{\bullet}[\operatorname{chk} K_0 \ v_0] \rightarrow_{1-S} E^{\bullet}[X(K_0, v_0)]$	5472
5418	$\wedge e \rightarrow_{1-D} E^{\bullet}[BndryErr]:$	2. $\vdash_1 \operatorname{chk} K_0 v : K''$	5473
5419	1. $\vdash_1 v_0 v_1$ : Any	by static hole typing	5474
5420	by static hole typing	3. $K_0 \leqslant K''$	5475
5421	2. 🔓 BndryErr : Any	by inversion	5476
5422	3. QED by hole substitution (2)	4. $\vdash_1 X(K_0, v_0) : K_0$	5477
5423	ELSE $v_0 = \lambda(x:\tau). e'$	by check soundness	5478
5424	$\wedge e \to_{1-S} E^{\bullet}[e'[x \leftarrow X(\lfloor \tau \rfloor, v_1)]]:$	5. QED by (3, 4, hole substitution)	5479
5425	1. $\vdash_1 v_0 v_1 : Any$	<b>CASE</b> $e = E[dyn e']$ where $e'$ is boundary-free :	5480
5426	by static hole typing	IF $e'$ is a value:	5481
5427	2. $\vdash_1 v_0$ : Fun	1. $e \rightarrow_{1-S} E[e']$	5482
5428	$\land \vdash_1 v_1 : Any$	2. $\vdash_1 \operatorname{dyn} e'$ : Any	5483
5429	by inversion (1)	by boundary hole typing	5484
5430	3. $(x:\tau) \vdash_1 e' : Any$	3. $\vdash_1 e'$	5485
5431	by inversion (2)	by inversion (2)	5486
5432	$4. \vdash_{1} X(\lfloor \tau \rfloor, v_{1}) : \lfloor \tau \rfloor$	4. $\vdash_1 e'$ : Any	5487
5433	by check soundness (2)	by dynamic value inversion (3)	5488
5434	5. $\vdash_1 e[x \leftarrow \mathcal{X}(\lfloor \tau \rfloor, v_1)]$ : Any	5. QED by hole substitution (4)	5489 5490
5435	by substitution (3, 4)	ELSE $e' \rightarrow_{1-D} e''$ :	
5436	6. QED by hole substitution	1. $e \rightarrow_{1-S} E[dyn \ e'']$	5491 5492
5437	$\mathbf{CASE} \ \ e = E^{\bullet}[op^1 \ v]$	2. $\vdash_1 \text{dyn } e' : \text{Any}$	5492
5438	$\wedge  \delta(op^1, v) = v'$	by boundary hole typing	5494
5439 5440	$\wedge e \to_{1-S} E^{\bullet}[v']:$	3. ⊢ <sub>1</sub> e'	5494
	1. $\vdash_1 op^1 v$ : Any	by inversion (2)	5495
5441 5442	by static hole typing	4. $\vdash_1 e''$	5496
5443	2. ⊢ <sub>1</sub> v : Pair	by dynamic preservation (3)	5497
5444	by inversion		5499
5445	F	0	5500
5 243	3	v	3300

```
5501
                 5. \vdash₁ dyn e'': Any
                                                                                                                 5. QED by hole substitution (4)
                                                                                                                                                                                                       5556
                                                                                                               ELSE e' \rightarrow_{1-S} e'':
5502
                     by (4)
                                                                                                                                                                                                       5557
5503
                 6. QED by hole substitution (5)
                                                                                                                 1. e \rightarrow_{1-S} E[\text{stat } \tau' e'']
                                                                                                                                                                                                       5558
            CASE e = E[\text{stat } e'] where e' is boundary-free :
                                                                                                                 2. \vdash₁ stat \tau' e'
5504
                                                                                                                                                                                                       5559
               IF e' is a value :
                                                                                                                     by boundary hole typing
                                                                                                                                                                                                       5560
                  1. e \rightarrow_{1-S} E[e']
                                                                                                                 3. \vdash_1 e' : |\tau'|
                                                                                                                                                                                                       5561
5506
                 2. ⊦₁ stat e′
5507
                                                                                                                     by inversion (2)
                                                                                                                                                                                                       5562
                                                                                                                 4. \vdash_1 e'' : [\tau']
5508
                     by boundary hole typing
                                                                                                                                                                                                       5563
5509
                 3. \vdash_{1} e' : Any
                                                                                                                     by static preservation (3)
                                                                                                                                                                                                       5564
                                                                                                                 5. \vdash<sub>1</sub> stat \tau' e''
5510
                     by inversion (2)
                                                                                                                                                                                                       5565
5511
                  4. ⊦₁ e'
                                                                                                                     by (4)
                                                                                                                                                                                                       5566
                     by static value inversion (3)
                                                                                                                 6. QED by hole substitution (5)
5512
                                                                                                                                                                                                       5567
5513
                 5. QED by hole substitution (4)
                                                                                                            CASE e = E[Err]:
                                                                                                                                                                                                       5568
               ELSE e' \rightarrow_{1-S} e'':
                                                                                                              1. e \rightarrow_{1-S} Err
5514
                                                                                                                                                                                                       5569
5515
                  1. e \rightarrow_{1-S} E[\text{stat } e'']
                                                                                                              2. QED \vdash<sub>1</sub> Err : K
                                                                                                                                                                                                       5570
5516
                 2. \vdash_1 stat e'
                                                                                                                                                                                                       5571
5517
                     by boundary hole typing
                                                                                                                                                                                                       5572
                                                                                                        Lemma 4.13: 1 dynamic preservation
5518
                 3. \vdash_1 e': Any
                                                                                                                                                                                                       5573
                                                                                                         If \vdash_1 e and e \rightarrow_{1-D} e' then \vdash_1 e'
5519
                     by inversion (2)
                                                                                                                                                                                                       5574
                                                                                                         Proof:
                 4. \vdash_1 e'': Any
                                                                                                                                                                                                       5575
                                                                                                            By boundary factoring there are nine cases.
5521
                     by static preservation (3)
                                                                                                                                                                                                       5576
                                                                                                            CASE e is a value :
                 5. ⊦₁ stat e"
5522
                                                                                                                                                                                                       5577
                                                                                                              1. Contradiction by e \rightarrow_{1-D} e'
                     by (4)
                                                                                                                                                                                                       5578
5523
                                                                                                            CASE e = E^{\bullet}[v_0 \ v_1]:
                 6. QED by hole substitution (5)
5524
                                                                                                                                                                                                       5579
                                                                                                               IF v_0 = \lambda x. e'
            CASE e = E[dyn \tau e'] where e' is boundary-free :
5525
                                                                                                                                                                                                       5580
                                                                                                                    \wedge e \rightarrow_{1-D} E^{\bullet}[e'[x \leftarrow v_1]]:
5526
               IF e' is a value:
                                                                                                                                                                                                       5581
                                                                                                                 1. \vdash_1 v_0 v_1
5527
                  1. e \rightarrow_{1-S} E[\mathcal{D}_1(\tau', e')]
                                                                                                                                                                                                       5582
                                                                                                                     by dynamic hole typing
                 2. \vdash<sub>1</sub> dyn \tau' e' : \lfloor \tau' \rfloor
                                                                                                                                                                                                       5583
                                                                                                                 2. \vdash_{1} v_0
                     by boundary hole typing
5529
                                                                                                                                                                                                       5584
                                                                                                                     \wedge \vdash_1 v_1
5530
                 3. ⊦₁ e'
                                                                                                                                                                                                       5585
                                                                                                                     by inversion (1)
5531
                     by inversion (2)
                                                                                                                                                                                                       5586
                                                                                                                 3. x \vdash_{1} e'
                 4. \vdash_1 \mathcal{D}_1(\tau', e') : \lfloor \tau' \rfloor
                                                                                                                                                                                                       5587
                                                                                                                     by inversion (2)
5533
                     by \mathcal{D}_1 soundness (3)
                                                                                                                                                                                                       5588
                                                                                                                 4. \vdash_1 e'[x \leftarrow v_1]
5534
                 5. QED by hole substitution (4)
                                                                                                                                                                                                       5589
                                                                                                                     by substitution (2, 3)
5535
               ELSE e' \rightarrow_{1-D} e'':
                                                                                                                                                                                                       5590
                                                                                                                 5. QED by hole substitution
                 1. e \rightarrow_{1-S} E[\mathsf{dyn} \ \tau' \ e'']
5536
                                                                                                                                                                                                       5591
                                                                                                               IF v_0 = \lambda(x:\tau).e'
5537
                 2. \vdash_1 \operatorname{dyn} \tau' e' : \lfloor \tau' \rfloor
                                                                                                                                                                                                       5592
                                                                                                                        \wedge \mathcal{X}(|\tau|, v_1) = \text{BndryErr}
5538
                     by boundary hole typing
                                                                                                                                                                                                       5593
                                                                                                                        \wedge e \rightarrow_{1-D} E^{\bullet}[BndryErr]:
                 3. \vdash_1 e'
5539
                                                                                                                 1. \vdash_1 v_0 v_1
                                                                                                                                                                                                       5595
                     by inversion (2)
5540
                                                                                                                     by dynamic hole typing
                 4. ⊦₁ e''
5541
                                                                                                                                                                                                       5596
                                                                                                                 2. ⊢<sub>1</sub> BndryErr
5542
                     by dynamic preservation (3)
                                                                                                                                                                                                       5597
                                                                                                                 3. QED by hole substitution (2)
                 5. \vdash_1 dyn \tau' e'' : [\tau']
                                                                                                                                                                                                       5598
                                                                                                               ELSE v_0 = \lambda(x:\tau). e'
5544
                     by (4)
                                                                                                                                                                                                       5599
                                                                                                                        \land e \rightarrow_{1-D} E^{\bullet}[\text{stat}(e'[x \leftarrow \mathcal{X}(|\tau|, v_1)])]:
                 6. QED by hole substitution (5)
5545
                                                                                                                                                                                                       5600
                                                                                                                 1. \vdash_1 v_0 v_1
            CASE e = E[\text{stat } \tau \ e'] where e' is boundary-free:
5546
                                                                                                                                                                                                       5601
                                                                                                                     by dynamic hole typing
5547
               IF e' is a value :
                                                                                                                                                                                                       5602
                                                                                                                 2. \vdash_1 v_0
                 1. e \rightarrow_{1-S} E[S_1(\tau', e')]
5548
                                                                                                                                                                                                       5603
                                                                                                                     \wedge \vdash_{_{1}} v_{1}
5549
                 2. \vdash_1 stat \tau' e'
                                                                                                                                                                                                       5604
                                                                                                                     by inversion (1)
5550
                     by boundary hole typing
                                                                                                                                                                                                       5605
                                                                                                                 3. (x:\tau) \vdash_{1} e : Any
5551
                 3. \vdash_{1} e' : \lfloor \tau' \rfloor
                                                                                                                                                                                                       5606
                                                                                                                     by inversion (2)
5552
                     by inversion (2)
                                                                                                                                                                                                       5607
                                                                                                                 4. \vdash_{1} X(\lfloor \tau \rfloor, v_1) : \lfloor \tau \rfloor
5553
                 4. \vdash_1 S_1(\tau', e')
                                                                                                                                                                                                       5608
                                                                                                                     by check soundness (2)
                     by S_1 soundness (3)
5554
                                                                                                                                                                                                       5609
5555
                                                                                                                                                                                                       5610
                                                                                                   51
```

5611	5. $\vdash_1 e[x \leftarrow X(\lfloor \tau \rfloor, v_1)]$ : Any	4. ⊦₁ <i>e'</i>	5666
5612	by substitution (3, 4)	by static value inversion (3)	5667
5613	6. $\vdash_1$ stat $(e[x \leftarrow X(\lfloor \tau \rfloor, v_1)])$	5. QED by hole substitution (5)	5668
5614	by (5)	ELSE $e' \rightarrow_{1-S} e''$ :	5669
5615	7. QED by hole substitution (6)	1. $e \rightarrow_{1-D} E[\text{stat } e'']$	5670
5616	$\mathbf{CASE} \ \ e = E^{\bullet}[op^1 \ v]$	2. ⊢₁ stat e'	5671
5617	$\wedge  \delta(op^1, v) = v'$	by boundary hole typing	5672
5618	$\wedge e \rightarrow_{\text{1-D}} E^{\bullet}[v']$ :	3. $\vdash_1 e'$ : Any	5673
5619	1. $\vdash_1 op^1 v$	by inversion (2)	5674
5620	by dynamic hole typing	4. ⊢ <sub>1</sub> <i>e</i> ′′ : Any	5675
5621	2. ⊢ <sub>1</sub> <i>v</i>	by static preservation (3)	5676
5622	by inversion (1)	5. ⊢₁ stat <i>e''</i>	5677
5623	3. ⊢ <sub>1</sub> v′	by (4)	5678
5624	by $\delta$ preservation (2)	6. QED by hole substitution (5)	5679
5625	4. QED by hole substitution (3)	<b>CASE</b> $e = E[\text{dyn } \tau \ e']$ where $e'$ is boundary-free :	5680
5626	$\mathbf{CASE} \ \ e = E^{\bullet}[op^2 \ v_0 \ v_1]$	IF $e'$ is a value :	5681
5627	$\wedge  \delta(op^2, v_0, v_1) = e^{\prime\prime}$	1. $e \rightarrow_{1-D} E[\mathcal{D}_1(\tau', e')]$	5682
5628	$\wedge e \rightarrow_{1-D} E^{\bullet}[e^{\prime\prime}]:$	2. ⊢₁ dyn τ′ e′ : [τ′]	5683
5629	1. $\vdash_1 op^2 v_0 v_1$	by boundary hole typing	5684
5630	by dynamic hole typing	3. ⊢ <sub>1</sub> <i>e</i> ′	5685
5631	$2. \vdash_1 v_0$	by inversion (2)	5686
5632	$\land \vdash_{1} v_{1}$	$4. \vdash_{1} \mathcal{D}_{1}(\tau', e') : \lfloor \tau' \rfloor$	5687
5633	by inversion (1)	by $\mathcal{D}_1$ soundness (3)	5688
5634	3. ⊢ <sub>1</sub> e''	5. QED by hole substitution (4)	5689
5635	by $\delta$ preservation (2)	ELSE $e' \rightarrow_{1-D} e''$ :	5690
5636	4. QED by hole substitution (3)	1. $e \rightarrow_{\text{1-D}} E[\text{dyn } \tau' e'']$	5691
5637	<b>CASE</b> $e = E[dyn e']$ where $e'$ is boundary-free:	2. $\vdash_1 dyn \ \tau' \ e' : \lfloor \tau' \rfloor$	5692
5638	IF $e'$ is a value:	by boundary hole typing	5693
5639	$1. e \rightarrow_{1-D} E[e']$	3. ⊢ <sub>1</sub> <i>e′</i>	5694
5640	2. ⊢ <sub>1</sub> dyn e' : Any	$\wedge \tau' \leqslant : \tau''$	5695
5641	by boundary hole typing	by inversion (2)	5696
5642	$3. \vdash_1 e'$	4. ⊢ <sub>1</sub> e''	5697
5643	by inversion (2)	by dynamic preservation (3)	5698
5644	$4. \vdash_1 e' : Any$	5. $\vdash_1 \operatorname{dyn} \tau' e'' : \lfloor \tau' \rfloor$	5699
5645	by $\mathcal{D}_1$ soundness (3)	by (4)	5700
5646	5. QED by hole substitution (4)	6. QED by hole substitution (5)	5701
5647	ELSE $e' \rightarrow_{1-D} e''$ :	<b>CASE</b> $e = E[\text{stat } \tau \ e']$ where $e'$ is boundary-free :	5702
5648	1. $e \rightarrow_{\text{1-D}} E[\text{dyn } e'']$	IF $e' \in v$ :	5703
5649	2. ⊢₁ dyn <i>e'</i> : Any	1. $e \rightarrow_{1-D} E[S_1(\tau', e')]$	5704
5650	by boundary hole typing	2. $\vdash_1$ stat $\tau'$ $e'$	5705
5651	3. ⊢ <sub>1</sub> e'	by boundary hole typing	5706
5652	by inversion (2)	3. $\vdash_1 e' : \lfloor \tau' \rfloor$	5707
5653	4. ⊢ <sub>1</sub> e"	by inversion (2)	5708
5654	by dynamic preservation (3)	$4. \vdash_{1} S_{1}(\tau',e')$	5709
5655	5. ⊢₁ dyn e'' : Any	by $S_1$ soundness (3)	5710
5656	by (4)	5. QED by hole substitution (5)	5711
5657	6. QED by hole substitution (5)	ELSE $e' \rightarrow_{1-S} e''$ :	5712
5658	<b>CASE</b> $e = E[\text{stat } e']$ where $e'$ is boundary-free :	1. $e \rightarrow_{1-D} E[\text{stat } \tau' e'']$	5713
5659	IF $e' \in v$ :	2. $\vdash_1$ stat $\tau'$ $e'$	5714
5660	1. $e \rightarrow_{1-D} E[e']$	by boundary hole typing	5715
5661	2. ⊢₁ stat e'	3. $\vdash_1 e' : \lfloor \tau' \rfloor$	5716
5662	by boundary hole typing	by inversion (2)	5717
5663	3. + <sub>1</sub> e' : Any	4. $\vdash_1 e'' : [\tau']$	5718
5664	by inversion (2)	by static preservation (3)	5719
5665		52	5720

```
5. \vdash<sub>1</sub> stat \tau' e''
5721
                                                                                                            By the L unique static evaluation contexts lemma, there
                                                                                                                                                                                                        5776
5722
                                                                                                            are five cases.
                                                                                                                                                                                                        5777
                     by (4)
                                                                                                            CASE e is a value :
5723
                 6. QED by hole substitution (5)
                                                                                                                                                                                                        5778
            CASE e = E[Err]:
                                                                                                              1. Contradiction by e \rightarrow_{1-S} e'
                                                                                                                                                                                                        5779
5724
5725
              1. e \rightarrow_{1-D} Err
                                                                                                            CASE e = E^{\bullet}[v_0 \ v_1]:
                                                                                                                                                                                                        5780
              2. QED ⊦₁ Err
                                                                                                               IF v_0 = \lambda(x:\tau_d). e':
                                                                                                                                                                                                        5781
5726
5727
                                                                                                                 1. E^{\bullet}[v_0 \ v_1] \rightarrow_{1-S} E^{\bullet}[e'[x \leftarrow v_1]]
                                                                                                                                                                                                        5782
5728
                                                                                                                 2. \vdash v_0 \ v_1 : \tau_c
                                                                                                                                                                                                        5783
         Lemma 4.14: boundary-free progress
5729
                                                                                                                 3. \vdash v_0 : \tau_d \Rightarrow \tau_c
                                                                                                                                                                                                        5784
          If \vdash e : \tau and e is boundary-free, then one of the following
5730
                                                                                                                     \wedge \vdash v_1 : \tau_d
                                                                                                                                                                                                        5785
           holds:
5731
                                                                                                                     by (2)
                                                                                                                                                                                                        5786
           • e is a value
5732
                                                                                                                 4. (x:\tau_d) \vdash e' : \tau_c
                                                                                                                                                                                                        5787
           • e \rightarrow_{1-S} e'
5733
                                                                                                                     by (3)
                                                                                                                                                                                                        5788
           • e \rightarrow_{1-S} BndryErr
5734
                                                                                                                 5. \vdash e'[x \leftarrow v_1] : \tau_c
                                                                                                                                                                                                        5789
5735
            By the L unique static evaluation contexts lemma, there
                                                                                                                     by substitution (3, 4)
                                                                                                                                                                                                        5790
5736
            are five cases:
                                                                                                                 6. e'[x \leftarrow v_1] is boundary-free
                                                                                                                                                                                                        5791
5737
                                                                                                                     by e' and v_1 are boundary-free
            CASE e = v:
                                                                                                                                                                                                        5792
                                                                                                                                                                                                       5793
5738
              1. QED
                                                                                                                 7. OED
            CASE e = E^{\bullet}[v_0 \ v_1]:
                                                                                                               ELSE :
5739
                                                                                                                                                                                                        5794
5740
               IF v_0 = \lambda(x:\tau').e':
                                                                                                                 1. Contradiction by \vdash e : \tau
                                                                                                                                                                                                        5795
5741
                 1. e \rightarrow_{1-S} E^{\bullet}[e'[x \leftarrow v_1]]
                                                                                                            CASE e = E^{\bullet}[op^1 v]:
                                                                                                                                                                                                        5796
                     by v_0 \ v_1 \rhd_{1-S} e'[x \leftarrow v_1]
                                                                                                              1. E^{\bullet}[op^1 v] \rightarrow_{1-S} E^{\bullet}[v']
5742
                                                                                                                                                                                                        5797
                                                                                                                  \wedge \delta(op^1, v) = e''
                 2. QED
                                                                                                                                                                                                        5798
5743
               ELSE v_0 = \lambda x. e'
                                                                                                              2. \vdash op^1 \upsilon : \tau'
5744
                                                                                                                                                                                                        5799
                                                                                                              3. \vdash \upsilon : \tau_0
5745
                         \vee v_0 = i
                                                                                                                                                                                                        5800
5746
                         \vee v_0 = \langle v, v' \rangle:
                                                                                                              4. \vdash e'' : \tau'
                                                                                                                                                                                                        5801
5747
                  1. Contradiction by \vdash e : \tau
                                                                                                                  by \delta preservation (3)
                                                                                                                                                                                                        5802
            CASE e = E^{\bullet}[op^1 v]:
5748
                                                                                                              5. OED
                                                                                                                                                                                                        5803
               IF \delta(op^1, v) = e'':
                                                                                                            CASE e = E^{\bullet}[op^2 v_0 v_1]:
5749
                                                                                                                                                                                                        5804
                 1. e \rightarrow_{1-S} E^{\bullet}[e^{\prime\prime}]
                                                                                                              1. E^{\bullet}[op^2 v_0 v_1] \rightarrow_{1-S} E^{\bullet}[v']
5750
                                                                                                                                                                                                        5805
                                                                                                                  \wedge \delta(op^2, v_0, v_1) = e^{\prime\prime}
                     by (op^1 v) \triangleright_{1-S} e''
5751
                                                                                                                                                                                                        5806
                 2. QED
                                                                                                              2. \vdash op^2 v_0 v_1 : \tau'
                                                                                                                                                                                                        5807
               ELSE \delta(op^1, v) is undefined :
5753
                                                                                                              3. \vdash v_0 : \tau_0
                                                                                                                                                                                                        5808
5754
                  1. Contradiction by \vdash e : \tau
                                                                                                                  \wedge \vdash v_1 : \tau_1
                                                                                                                                                                                                        5809
                                                                                                              4. \vdash e^{\prime\prime} : \tau^{\prime}
            CASE e = E^{\bullet}[op^2 v_0 v_1]:
5755
                                                                                                                                                                                                        5810
               IF \delta(op^2, v_0, v_1) = e'':
                                                                                                                  by \delta preservation (3)
5756
                                                                                                                                                                                                        5811
5757
                 1. e \rightarrow_{1-S} E^{\bullet}[e'']
                                                                                                              5. QED
                                                                                                                                                                                                        5812
5758
                     by (op^2 v_0 v_1) >_{1-S} e''
                                                                                                            CASE e = E^{\bullet}[Err]:
                                                                                                                                                                                                        5813
                 2. QED
                                                                                                              1. E^{\bullet}[Err] \rightarrow_{1-S} Err
5759
               IF \delta(op^2, v_0, v_1) = \text{BndryErr}:
                                                                                                              2. QED by \vdash Err : \tau
                                                                                                                                                                                                        5815
5760
                                                                                                         1. e \rightarrow_{1-S} BndryErr
5761
                                                                                                                                                                                                        5816
5762
                     by (op^2 v_0 v_1) \triangleright_{1-S} BndryErr
                                                                                                                                                                                                        5817
                                                                                                         Lemma 4.16: X soundness
5763
                 2. OED
                                                                                                                                                                                                        5818
                                                                                                         For all K and v, \vdash_{\mathsf{I}} X(K, v) : K.
5764
               ELSE \delta(op^2, v_0, v_1) is undefined :
                                                                                                                                                                                                        5819
                                                                                                         Proof:
5765
                  1. Contradiction by \vdash e : \tau
                                                                                                                                                                                                        5820
                                                                                                            CASE \vdash_1 v : K :
            CASE e = E^{\bullet}[Err]:
5766
                                                                                                                                                                                                        5821
                                                                                                              1. X(K, v) = v
5767
              1. E^{\bullet}[Err] \rightarrow_{1-S} Err
                                                                                                                                                                                                        5822
                                                                                                              2. QED
              2. QED
5768
                                                                                                                                                                                                        5823
                                                                                                            CASE \not\vdash_1 v : K :
5769
                                                                                                                                                                                                        5824
                                                                                                              1. X(K, v) = BndryErr
5770
                                                                                                                                                                                                        5825
         Lemma 4.15: 1 boundary-free preservation
5771
                                                                                                         5826
         If \vdash e : \tau and e is boundary-free and e \rightarrow_{1-S} e' then \vdash e' : \tau
5772
          and e' is boundary-free.
                                                                                                                                                                                                        5827
                                                                                                         Lemma 4.17: 1 static boundary factoring
5773
         Proof:
                                                                                                                                                                                                        5828
5774
                                                                                                                                                                                                        5829
5775
                                                                                                                                                                                                        5830
```

```
If \vdash_1 e : K then one of the following holds:
5831
                                                                                                                       \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                                                                                                5886
           • e is a value
5832
                                                                                                                       \vee E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                                5887
           • e = E^{\bullet}[v_0 \ v_1]
5833
                                                                                                                       by inner boundary
                                                                                                                                                                                                                5888
           \bullet e = E^{\bullet}[op^1 v]
                                                                                                                  2. IF E = E^{\bullet}:
5834
                                                                                                                                                                                                                5889
           • e = E^{\bullet}[op^2 v_0 v_1]
                                                                                                                         a. QED e = E^{\bullet}[op^2 v_0 v_1]
5835
           • e = E^{\bullet}[\operatorname{chk} K v]
                                                                                                                       IF E = E'[\mathsf{dyn}\,E^{\bullet}]:
                                                                                                                                                                                                                5891
5836
           • e = E[dyn e'] where e' is boundary-free
5837
                                                                                                                         a. QED e = E'[\mathsf{dyn} \ E^{\bullet}[op^2 \ v_0 \ v_1]]
                                                                                                                                                                                                                5892
           • e = E[\text{stat } e'] where e' is boundary-free
                                                                                                                       IF E = E'[\operatorname{stat} E^{\bullet}]:
5838
                                                                                                                                                                                                                5893
           • e = E[dyn \tau e'] where e' is boundary-free
5839
                                                                                                                         a. QED e = E'[\text{stat } E^{\bullet}[op^2 v_0 v_1]]
           • e = E[\text{stat } \tau \ e'] where e' is boundary-free
                                                                                                                       IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
5840
                                                                                                                                                                                                                5895
           • e = E[Err]
5841
                                                                                                                         a. QED e = E'[\operatorname{dyn} \tau E^{\bullet}[op^2 v_0 v_1]]
                                                                                                                                                                                                                5896
         Proof:
                                                                                                                       ELSE E = E'[\text{stat } \tau E^{\bullet}]:
5842
                                                                                                                                                                                                                5897
            By the unique evaluation contexts lemma, there are ten
                                                                                                                         a. QED e = E'[\text{stat } \tau E^{\bullet}[op^2 v_0 v_1]]
                                                                                                                                                                                                                5898
                                                                                                                CASE e = E[\mathsf{dyn} \ v]:
5844
                                                                                                                                                                                                                5899
            CASE e is a value :
5845
                                                                                                                   1. QED v is boundary-free
                                                                                                                                                                                                                5900
               1. QED
5846
                                                                                                                CASE e = E[\text{stat } v]:
                                                                                                                                                                                                                5901
            CASE e = E[v_0 \ v_1]:
                                                                                                                   1. QED v is boundary-free
5847
                                                                                                                                                                                                                5902
               1. E=E^{\bullet}
5848
                                                                                                                CASE e = E[\mathsf{dyn} \ \tau \ v]:
                                                                                                                                                                                                                5903
                   \vee E = E'[\mathsf{dyn}\ E^{\bullet}]
                                                                                                                  1. QED v is boundary-free
5849
                                                                                                                                                                                                                5904
                   \vee E = E'[\operatorname{stat} E^{\bullet}]
5850
                                                                                                                CASE e = E[\text{stat } \tau \ v]:
                                                                                                                                                                                                                5905
                   \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
5851
                                                                                                                   1. QED v is boundary-free
                                                                                                                                                                                                                5906
                   \vee E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                CASE e = E[Err]:
5852
                                                                                                                                                                                                                5907
                   by inner boundary
                                                                                                                                                                                                                5908
                                                                                                                   1. QED
5853
               2. IF E = E^{\bullet}:
5854
                                                                                                                                                                                                                5909
                     a. QED e = E^{\bullet}[v_0 \ v_1]
5855
                                                                                                             Lemma 4.18: 1 unique static evaluation contexts
                                                                                                                                                                                                                5910
                   IF E = E'[\mathsf{dyn}\ E^{\bullet}]:
5856
                                                                                                              If \vdash_1 e : K then one of the following holds:
                                                                                                                                                                                                                5911
                     a. QED e = E'[\mathsf{dyn}\ E^{\bullet}[v_0\ v_1]]
5857
                                                                                                                                                                                                                5912
                                                                                                               • e is a value
                   IF E = E'[\operatorname{stat} E^{\bullet}]:
                                                                                                                                                                                                                5913
                                                                                                               \bullet e = E[v_0 \ v_1]
                     a. QED e = E'[\text{stat } E^{\bullet}[v_0 \ v_1]]
5859
                                                                                                               \bullet \ e = E[op^1 \ v]
                                                                                                                                                                                                                5914
                   IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                               • e = E[op^2 v_0 v_1]
5860
                                                                                                                                                                                                                5915
                     a. QED e = E'[\text{dyn } \tau E^{\bullet}[v_0 \ v_1]]
5861
                                                                                                               • e = E[\operatorname{chk} K v]
                                                                                                                                                                                                                5916
                   ELSE E = E'[\text{stat } \tau E^{\bullet}]:
                                                                                                               • e = E[\mathsf{dyn} \ v]
                                                                                                                                                                                                                5917
                     a. QED e = E'[\text{stat } \tau E^{\bullet}[v_0 \ v_1]]
5863
                                                                                                               • e = E[\text{stat } v]
                                                                                                                                                                                                                5918
            CASE e = E[op^1 v]:
5864
                                                                                                               • e = E[\mathsf{dyn} \ \tau \ \upsilon]
                                                                                                                                                                                                                5919
               1. E = E^{\bullet}
5865
                                                                                                                                                                                                                5920
                                                                                                               • e = E[\text{stat } \tau \ v]
                   \vee E = E'[\mathsf{dyn}\ E^{\bullet}]
5866
                                                                                                               • e = E[Err]
                                                                                                                                                                                                                5921
                   \vee E = E'[\text{stat } E^{\bullet}]
5867
                                                                                                             Proof:
                                                                                                                                                                                                                5922
                   \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                By induction on the structure of e.
5868
                                                                                                                                                                                                                5923
                   \vee E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                CASE e = x:
                   by inner boundary
                                                                                                                   1. Contradiction by \vdash_1 e : K
                                                                                                                                                                                                                5925
5870
               2. IF E = E^{\bullet}:
                                                                                                                CASE e = i
5871
                                                                                                                                                                                                                5926
                     a. QED e = E^{\bullet}[op^1 v]
                                                                                                                          \vee e = \lambda x. e'
5872
                                                                                                                                                                                                                5927
                   IF E = E'[\mathsf{dyn}\ E^{\bullet}]:
                                                                                                                          \vee e = \lambda(x:\tau_d).e':
5873
                                                                                                                                                                                                                5928
                     a. QED e = E'[\mathsf{dyn} \ E^{\bullet}[\mathit{op}^1 \ v]]
                                                                                                                   1. OED e is a value
5874
                                                                                                                                                                                                                5929
                   IF E = E'[\operatorname{stat} E^{\bullet}]:
                                                                                                                CASE e = \langle e_0, e_1 \rangle:
5875
                                                                                                                                                                                                                5930
                     a. QED e = E'[\text{stat } E^{\bullet}[op^1 v]]
                                                                                                                   IF e_0 \notin v:
5876
                                                                                                                                                                                                                5931
                   IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                                      1. e_0 = E_0[e_0']
                                                                                                                                                                                                                5932
                     a. QED e = E'[\text{dyn } \tau E^{\bullet}[op^1 v]]
                                                                                                                          by the induction hypothesis
5878
                                                                                                                                                                                                                5933
                   ELSE E = E'[\text{stat } \tau E^{\bullet}]:
                                                                                                                      2. E = \langle E_0, e_1 \rangle
5879
                                                                                                                                                                                                                5934
                     a. QED e = E'[\text{stat } \tau E^{\bullet}[op^1 v]]
                                                                                                                      3. QED by e = E[e'_0]
5880
                                                                                                                                                                                                                5935
            CASE e = E[op^2 v_0 v_1]:
                                                                                                                    IF e_0 \in v
5881
                                                                                                                                                                                                                5936
               1. E = E^{\bullet}
                                                                                                                         \land e_1 \notin v:
5882
                                                                                                                                                                                                                5937
                   \vee E = E'[\mathsf{dyn}\ E^{\bullet}]
5883
                                                                                                                      1. e_1 = E_1[e_1]
                                                                                                                                                                                                                5938
                   \vee E = E'[\operatorname{stat} E^{\bullet}]
                                                                                                                          by the induction hypothesis
5884
                                                                                                                                                                                                                5939
5885
                                                                                                                                                                                                                5940
                                                                                                       54
```

```
5941
                2. E = \langle e_0, E_1 \rangle
                                                                                                        1. E = []
                                                                                                                                                                                       5996
5942
                3. QED by e = E[e'_1]
                                                                                                        2. QED e = E[\operatorname{chk} K e_0]
                                                                                                                                                                                       5997
              ELSE e_0 \in v
5943
                                                                                                   CASE e = \text{dyn } e_0:
                                                                                                                                                                                       5998
                      \land e_1 \in v:
5944
                                                                                                      IF e_0 \notin v:
                                                                                                                                                                                       5999
                1. E = []
                                                                                                        1. \vdash_{1} e_{0}
                                                                                                                                                                                       6000
                2. QED e = E[\langle e_0, e_1 \rangle]
                                                                                                           by inversion
                                                                                                                                                                                       6001
5946
5947
           CASE e = e_0 e_1:
                                                                                                        2. e_0 = E_0[e'_0]
                                                                                                                                                                                       6002
                                                                                                           by unique evaluation contexts (1)
5948
              IF e_0 \notin v:
                                                                                                                                                                                       6003
                1. e_0 = E_0[e'_0]
                                                                                                        3. E = dyn E_0
                   by the induction hypothesis
5950
                                                                                                        4. QED e = E[e'_0]
                                                                                                                                                                                       6005
5951
                2. E = E_0 e_1
                                                                                                      ELSE e_0 \in v:
                                                                                                                                                                                       6006
                3. QED by e = E[e'_0]
                                                                                                        1. E = []
5952
                                                                                                                                                                                       6007
5953
              IF e_0 \in v
                                                                                                        2. QED e = E[\mathsf{dyn}\ e_0]
                                                                                                                                                                                       6008
5954
                  \wedge e_1 \notin v:
                                                                                                   CASE e = \text{stat } e_0:
                                                                                                                                                                                       6009
5955
                1. e_1 = E_1[e_1']
                                                                                                     1. Contradiction by \vdash_1 e : K
                                                                                                                                                                                       6010
5956
                   by the induction hypothesis
                                                                                                   CASE e = \text{dyn } \tau e_0:
                                                                                                                                                                                       6011
                2. E = e_0 E_1
5957
                                                                                                      IF e_0 \notin v:
                                                                                                                                                                                       6012
5958
                3. QED by e = E[e'_1]
                                                                                                                                                                                       6013
                                                                                                        1. \vdash_1 e_0
5959
              ELSE e_0 \in v
                                                                                                           by inversion
                                                                                                                                                                                       6014
5960
                      \land e_1 \in v:
                                                                                                        2. e_0 = E_0[e'_0]
                                                                                                                                                                                       6015
5961
                1. E = []
                                                                                                           by unique evaluation contexts (1)
                                                                                                                                                                                       6016
                2. QED e = E[e_0 \ e_1]
                                                                                                        3. E = dyn \tau E_0
5962
                                                                                                                                                                                       6017
           CASE e = op^1 e_0:
                                                                                                        4. QED e = E[e'_0]
                                                                                                                                                                                       6018
5963
                                                                                                      ELSE e_0 \in v:
5964
             1. IF e_0 \notin v:
                                                                                                                                                                                       6019
5965
                  a. e_0 = E_0[e'_0]
                                                                                                        1. E = []
                                                                                                                                                                                       6020
                      by the induction hypothesis
                                                                                                        2. QED e = E[\text{dyn } \tau e_0]
                                                                                                                                                                                       6021
5967
                  b. E = op^1 E_0
                                                                                                   CASE e = \operatorname{stat} K' e_0:
                                                                                                                                                                                       6022
5968
                   c. QED e = E[e'_0]
                                                                                                     1. Contradiction by \vdash_1 e : K
                                                                                                                                                                                       6023
             2. ELSE e_0 \in v:
5969
                                                                                                   CASE e = Err:
                                                                                                                                                                                       6024
                   a. E = []
5970
                                                                                                     1. E = []
                                                                                                                                                                                       6025
5971
                   b. QED e = E[op^1 e_0]
                                                                                                     2. QED e = E[Err]
                                                                                                                                                                                       6026
                                                                                                5972
           CASE e = op^2 e_0 e_1:
                                                                                                                                                                                       6027
5973
              IF e_0 \notin v:
                                                                                                                                                                                       6028
                                                                                                Lemma 4.19: 1 inner boundary
5974
                1. e_0 = E_0[e_0']
                                                                                                                                                                                       6029
                                                                                                  For all contexts E, one of the following holds:
5975
                   by the induction hypothesis
                                                                                                                                                                                       6030
                                                                                                  \bullet E = E^{\bullet}
                2. E = op^2 E_0 e_1
5976
                                                                                                  • E = E'[\mathsf{dyn}\ v]
                                                                                                                                                                                       6031
5977
                3. QED e = E[e'_0]
                                                                                                  • E = E'[\text{stat } v]
                                                                                                                                                                                       6032
5978
              IF e_0 \in v
                                                                                                  • E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                                                                       6033
                   \land e_1 \notin v:
                                                                                                  • E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                       6034
                                                                                                Proof:
                                                                                                                                                                                       6035
                1. e_1 = E_1[e_1']
5980
                                                                                                   By induction on the structure of E.
                   by the induction hypothesis
5981
                                                                                                                                                                                       6036
                                                                                                   CASE E = E^{\bullet}:
5982
                2. E = op^2 e_0 E_1
                                                                                                                                                                                       6037
                3. QED e = E[e'_1]
                                                                                                     1. QED
                                                                                                                                                                                       6038
                                                                                                   CASE E = E_0 e_1:
5984
              ELSE e_0 \in v
                                                                                                                                                                                       6039
                                                                                                     1. E_0 = E^{\bullet}
                      \land e_1 \in v:
5985
                                                                                                                                                                                       6040
                                                                                                         \vee E_0 = E_0'[\mathsf{dyn}\ E^{\bullet}]
                1. E = []
5986
                                                                                                                                                                                       6041
                                                                                                         \vee E_0 = E_0'[\operatorname{stat} E^{\bullet}]
                2. QED e = E[op^2 e_0 e_1]
5987
                                                                                                                                                                                       6042
                                                                                                         \vee E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]
5988
           CASE e = \operatorname{chk} K e_0:
                                                                                                                                                                                       6043
                                                                                                         \vee E_0 = E_0'[\text{stat } \tau E^{\bullet}]
5989
              IF e_0 \notin v:
                                                                                                                                                                                       6044
                                                                                                         by the induction hypothesis
5990
                1. e_0 = E_0[e'_0]
                                                                                                                                                                                       6045
                                                                                                     2. IF E_0 = E^{\bullet}:
5991
                   by the induction hypothesis
                                                                                                                                                                                       6046
                                                                                                          a. QED E is boundary-free
5992
                2. E = \operatorname{chk} K E_0
                                                                                                                                                                                       6047
                                                                                                         IF E_0 = E_0'[dyn E^{\bullet}]:
5993
                3. QED e = E[e'_0]
                                                                                                                                                                                       6048
                                                                                                           a. E' = E'_0 e_1
5994
              ELSE e_0 \in v:
                                                                                                                                                                                       6049
5995
                                                                                                                                                                                       6050
                                                                                           55
```

6051	b. QED $E = E'[dyn E^{\bullet}]$	1. $E_1=E^{\bullet}$	6106
6052	IF $E_0 = E_0'[\operatorname{stat} E^{\bullet}]$ :	$\vee E_0 = E_0'[\operatorname{dyn} E^{ullet}]$	6107
6053	a. $E' = E'_0 e_1$	$\vee E_0 = E_0'[\operatorname{stat} E^{\bullet}]$	6108
6054	b. QED $E = E'[\text{stat } E^{\bullet}]$	$\vee E_0 = E_0'[\operatorname{dyn} \tau E^{\bullet}]$	6109
6055	IF $E_0 = E'_0[\operatorname{dyn} \tau E^{\bullet}]$ :	$\vee E_0 = E_0'[\operatorname{stat} \tau E^{\bullet}]$	6110
6056	a. $E' = E'_0 e_1$	by the induction hypothesis	6111
6057	b. QED $E = E'[dyn \ \tau \ E^{\bullet}]$	2. <b>IF</b> $E_1 = E^{\bullet}$ :	6112
6058	<b>ELSE</b> $E_0 = E_0'[\operatorname{stat} \tau E^{\bullet}]:$	a. QED $E$ is boundary-free	6113
6059	$a. E' = E'_0 e_1$	IF $E_1 = E_1'[\operatorname{dyn} E^{\bullet}]:$	6114
6060	b. QED $E = E'[\text{stat } \tau E^{\bullet}]$	a. $E' = \langle v_0, E_1' \rangle$	6115
6061	$\mathbf{CASE} \ E = v_0 \ E_1 :$	b. QED $E = E'[\text{dyn } E^{\bullet}]$	6116
6062	1. $E_1 = E^{\bullet}$	IF $E_1 = E_1'[\operatorname{stat} E^{\bullet}]:$	6117
6063	$\vee E_1 = E_1'[dyn\ E^{\bullet}]$	a. $E' = \langle v_0, E_1' \rangle$	6118
6064	$\forall E_1 = E_1'[stat\ E^{\bullet}]$	b. QED $E = E'[\text{stat } E^{\bullet}]$	6119
6065	$\forall E_1 = E_1'[dyn \ \tau \ E^{\bullet}]$	IF $E_1 = E_1'[\operatorname{dyn} \tau E^{\bullet}]:$	6120
6066	$\vee E_1 = E_1'[\operatorname{stat} \tau E^{\bullet}]$	a. $E' = \langle v_0, E'_1 \rangle$	6121
6067	by the induction hypothesis	b. QED $E = E'[\text{dyn } \tau E^{\bullet}]$	6122
6068	2. IF $E_1 = E^{\bullet}$ :	<b>ELSE</b> $E_1 = E_1'[\operatorname{stat} \tau E^{\bullet}]:$	6123
6069	a. QED $E$ is boundary-free	a. $E' = \langle v_0, E'_1 \rangle$	6124
6070	IF $E_1 = E_1'[\operatorname{dyn} E^{\bullet}]:$	b. QED $E = E'[\text{stat } \tau E^{\bullet}]$	6125
6071	a. $E' = v_0 E'_1$ b. QED $E = E'[dyn\ E^{\bullet}]$	<b>CASE</b> $E = op^1 E_0$ : 1. $E_0 = E^{\bullet}$	6126
6072 6073	i. QED $E = E$ [dyn $E$ ]  IF $E_1 = E'_1[\text{stat } E^{\bullet}]$ :	1. $E_0 = E$ $\vee E_0 = E'_0 [\operatorname{dyn} E^{\bullet}]$	6127 6128
6074	a. $E' = v_0 E'_1$	$\forall E_0 = E_0[dyn E]$ $\forall E_0 = E_0'[stat E^{\bullet}]$	6129
6075	a. $E = \mathcal{O}_0 E_1$ b. QED $E = E'[\text{stat } E^{\bullet}]$	$\forall E_0 = E_0[\operatorname{Stat} E]$ $\forall E_0 = E'_0[\operatorname{dyn} \tau E^{\bullet}]$	6130
6076	IF $E_1 = E_1'[\operatorname{dyn} \tau E^{\bullet}]$ :	$\forall E_0 = E_0[uyn \ \tau \ E]$ $\forall E_0 = E_0'[stat \ \tau \ E^{\bullet}]$	6131
6077	a. $E' = v_0 E'_1$	by the induction hypothesis	6132
6078	a. $E = \mathcal{O}_0 E_1$ b. QED $E = E'[dyn \ \tau \ E^{\bullet}]$	2. IF $E_0 = E^{\bullet}$ :	6133
6079	ELSE $E_1 = E_1'[\operatorname{stat} \tau E^{\bullet}]$ :	a. QED E is boundary-free	6134
6080	a. $E' = v_0 E'_1$	a. QEB E is boundary free  IF $E_0 = E'_0[\text{dyn } E^{\bullet}]$ :	6135
6081	b. QED $E = E'[\text{stat } \tau E^{\bullet}]$	a. $E' = op^1 E'_0$	6136
6082	<b>CASE</b> $E = \langle E_0, e_1 \rangle$ :	b. QED $E = E'[dyn\ E^{\bullet}]$	6137
6083	1. $E_0 = E^{\bullet}$	$\mathbf{IF} \ E_0 = E'_0[\operatorname{stat} E^{\bullet}]:$	6138
6084	$\forall E_0 = E_0'[dyn\ E^{\bullet}]$	a. $E' = op^1 E'_0$	6139
6085	$\vee E_0 = E_0'[\operatorname{stat} E^{\bullet}]$	b. QED $E = E'[\text{stat } E^{\bullet}]$	6140
6086	$\vee E_0 = E'_0[\operatorname{dyn} \tau E^{\bullet}]$	$\mathbf{IF} \ E_0 = E'_0[dyn \ \tau \ E^{\bullet}]:$	6141
6087	$\forall E_0 = E_0'[stat \ \tau \ E^{\bullet}]$	a. $E' = op^1 E'_0$	6142
6088	by the induction hypothesis	b. QED $E = E'[\operatorname{dyn} \tau E^{\bullet}]$	6143
6089	2. IF $E_0 = E^{\bullet}$ :	ELSE $E_0 = E'_0[\operatorname{stat} \tau \ E^{\bullet}]$ :	6144
6090	a. QED $E$ is boundary-free	a. $E' = op^1 E'_0$	6145
6091	IF $E_0 = E_0'[\operatorname{dyn} E^{\bullet}]$ :	b. QED $E = E'[\text{stat } \tau E^{\bullet}]$	6146
6092	a. $E' = \langle E'_0, e_1 \rangle$	CASE $E = op^2 E_0 e_1$ :	6147
6093	b. QED $E = E'[dyn\ E^{\bullet}]$	1. $E_0 = E^{\bullet}$	6148
6094	IF $E_0 = E_0'[\operatorname{stat} E^{\bullet}]$ :	$\vee E_0 = E_0'[dyn\ E^{\bullet}]$	6149
6095	a. $E' = \langle E'_0, e_1 \rangle$	$V E_0 = E_0'[stat \ E^{\bullet}]$	6150
6096	b. QED $E = E'[\text{stat } E^{\bullet}]$	$\forall E_0 = E_0'[dyn \ \tau \ E^{\bullet}]$	6151
6097	IF $E_0 = E_0'[\operatorname{dyn} \tau E^{\bullet}]$ :	$V E_0 = E_0'[\operatorname{stat} \tau E^{\bullet}]$	6152
6098	a. $E' = \langle E'_0, e_1 \rangle$	by the induction hypothesis	6153
6099	b. QED $E = E'[dyn \ \tau \ E^{\bullet}]$	2. IF $E_0 = E^{\bullet}$ :	6154
6100	ELSE $E_0 = E_0'[\operatorname{stat} \tau E^{\bullet}]$ :	a. QED $E$ is boundary-free	6155
6101	a. $E' = \langle E'_0, e_1 \rangle$	IF $E_0 = E'_0[\operatorname{dyn} E^{\bullet}]$ :	6156
6102	b. QED $E = E'[\text{stat } \tau E^{\bullet}]$	a. $E' = op^2 E'_0 e_1$	6157
6103	CASE $E = \langle v_0, E_1 \rangle$ :	b. QED $E = E'[dyn\ E^{\bullet}]$	6158
6104		~, ,	6159
6105		56	6160

```
6161
                       IF E_0 = E_0'[\operatorname{stat} E^{\bullet}]:
                                                                                                                                         1. E_0 = E^{\bullet}
                                                                                                                                                                                                                                                        6216
                                                                                                                                              \vee E_0 = E_0'[\mathsf{dyn}\ E^{\bullet}]
6162
                          a. E' = op^2 E'_0 e_1
                                                                                                                                                                                                                                                        6217
                         b. QED E = E'[\text{stat } E^{\bullet}]
                                                                                                                                              \vee E_0 = E_0'[\operatorname{stat} E^{\bullet}]
6163
                                                                                                                                                                                                                                                        6218
                                                                                                                                              \vee E_0 = E_0'[\operatorname{dyn} \tau' E^{\bullet}]
                       IF E_0 = E_0'[\operatorname{dyn} \tau E^{\bullet}]:
6164
                                                                                                                                                                                                                                                        6219
6165
                         a. E' = op^2 E'_0 e_1
                                                                                                                                              \vee E_0 = E_0'[\operatorname{stat} \tau' E^{\bullet}]
                                                                                                                                                                                                                                                        6220
                         b. QED E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                              by the induction hypothesis
                                                                                                                                                                                                                                                        6221
6166
                                                                                                                                         2. IF E_0 = E^{\bullet}:
6167
                       ELSE E_0 = E_0'[\text{stat } \tau E^{\bullet}]:
                                                                                                                                                                                                                                                        6222
6168
                          a. E' = op^2 E'_0 e_1
                                                                                                                                                a. QED
                                                                                                                                                                                                                                                        6223
6169
                         b. QED E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                              IF E_0 = E'_0[\mathsf{dyn}\ E^{\bullet}]:
6170
               CASE E = op^2 v_0 E_1:
                                                                                                                                                a. E' = \operatorname{stat} E'_0
                                                                                                                                                                                                                                                        6225
6171
                                                                                                                                                b. QED E = E'[\mathsf{dyn}\ E^{\bullet}]
                                                                                                                                                                                                                                                        6226
                  1. E_1 = E^{\bullet}
                                                                                                                                              IF E_0 = E_0'[\operatorname{stat} E^{\bullet}]:
6172
                       \vee E_1 = E_1'[\mathsf{dyn}\ E^{\bullet}]
                                                                                                                                                                                                                                                        6227
6173
                                                                                                                                                a. E' = \operatorname{stat} E'_0
                                                                                                                                                                                                                                                        6228
                       \vee E_1 = E_1'[\operatorname{stat} E^{\bullet}]
                                                                                                                                                b. QED E = E'[\text{stat } E^{\bullet}]
6174
                       \vee E_1 = E'_1[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                                                                                                                                        6229
6175
                                                                                                                                              IF E_0 = E_0'[\text{dyn } \tau' E^{\bullet}]:
                       \vee E_1 = E_1'[\operatorname{stat} \tau E^{\bullet}]
                                                                                                                                                                                                                                                        6230
6176
                                                                                                                                                a. E' = \operatorname{stat} E'_0
                                                                                                                                                                                                                                                        6231
                       by the induction hypothesis
                                                                                                                                                b. QED E = E'[\mathsf{dyn} \ \tau' \ E^{\bullet}]
6177
                  2. IF E_1 = E^{\bullet}:
                                                                                                                                                                                                                                                        6232
                                                                                                                                              ELSE E_0 = E_0'[\operatorname{stat} \tau' E^{\bullet}]:
6178
                          a. QED E is boundary-free
                                                                                                                                                                                                                                                        6233
                                                                                                                                                a. E' = \text{stat } E'_0
6179
                       IF E_1 = E_1'[\mathsf{dyn} \ E^{\bullet}]:
                                                                                                                                                                                                                                                        6234
                         a. E' = op^2 v_0 E'_1
                                                                                                                                                b. QED E = E'[\text{stat } \tau' E^{\bullet}]
                                                                                                                                                                                                                                                        6235
                                                                                                                                      CASE E = \text{dyn } \tau E_0:
6181
                         b. QED E = E'[\mathsf{dyn}\ E^{\bullet}]
                                                                                                                                                                                                                                                        6236
                                                                                                                                         1. E_0 = E^{\bullet}
6182
                       IF E_1 = E_1'[\operatorname{stat} E^{\bullet}]:
                                                                                                                                                                                                                                                        6237
                                                                                                                                              \vee E_0 = E_0'[\mathsf{dyn}\ E^{\bullet}]
                                                                                                                                                                                                                                                        6238
6183
                          a. E' = op^2 v_0 E'_1
                                                                                                                                              \vee E_0 = E_0'[\operatorname{stat} E^{\bullet}]
                         b. QED E = E'[\text{stat } E^{\bullet}]
6185
                       IF E_1 = E_1'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                                                              \vee E_0 = E_0'[\mathsf{dyn} \ \tau' \ E^{\bullet}]
                                                                                                                                                                                                                                                        6240
                                                                                                                                              \vee E_0 = E_0'[\operatorname{stat} \tau' E^{\bullet}]
                                                                                                                                                                                                                                                        6241
                          a. E' = op^2 v_0 E'_1
                                                                                                                                              by the induction hypothesis
6187
                                                                                                                                                                                                                                                        6242
                         b. QED E = E'[\text{dyn } \tau E^{\bullet}]
6188
                                                                                                                                         2. IF E_0 = E^{\bullet}:
                                                                                                                                                                                                                                                        6243
                       ELSE E_1 = E_1'[\text{stat } \tau E^{\bullet}]:
                                                                                                                                                a. QED
6189
                                                                                                                                                                                                                                                        6244
                          a. E' = op^2 v_0 E'_1
6190
                                                                                                                                              IF E_0 = E'_0[\mathsf{dyn}\ E^{\bullet}]:
                                                                                                                                                                                                                                                        6245
                          b. QED E = E'[\text{stat } \tau E^{\bullet}]
6191
                                                                                                                                                a. E' = \operatorname{dyn} \tau E'_0
                                                                                                                                                                                                                                                        6246
               CASE E = \text{dyn } E_0:
6192
                                                                                                                                                b. QED E = E'[\mathsf{dyn}\ E^{\bullet}]
                                                                                                                                                                                                                                                        6247
                  1. E_0 = E^{\bullet}
6193
                                                                                                                                              IF E_0 = E_0'[\operatorname{stat} E^{\bullet}]:
                                                                                                                                                                                                                                                        6248
                       \vee E_0 = E_0'[\mathsf{dyn}\ E^{\bullet}]
6194
                                                                                                                                                a. E' = \operatorname{dyn} \tau E'_0
                                                                                                                                                                                                                                                        6249
                       \vee E_0 = E_0'[\operatorname{stat} E^{\bullet}]
6195
                                                                                                                                                b. QED E = E'[\text{stat } E^{\bullet}]
                                                                                                                                                                                                                                                        6250
                       \vee E_0 = E_0'[\mathsf{dyn} \ \tau' \ E^{\bullet}]
                                                                                                                                              IF E_0 = E_0'[\text{dyn } \tau' E^{\bullet}]:
6196
                                                                                                                                                                                                                                                        6251
                       \vee E_0 = E_0'[\operatorname{stat} \tau' E^{\bullet}]
6197
                                                                                                                                                a. E' = \operatorname{dyn} \tau E'_0
                                                                                                                                                                                                                                                        6252
                       by the induction hypothesis
                                                                                                                                                b. QED E = E'[\text{dyn } \tau' E^{\bullet}]
6198
                                                                                                                                                                                                                                                        6253
                  2. IF E_0 = E^{\bullet}:
                                                                                                                                              ELSE E_0 = E_0'[\operatorname{stat} \tau' E^{\bullet}]:
                                                                                                                                                                                                                                                        6254
                         a. QED
                                                                                                                                                a. E' = \operatorname{dyn} \tau E'_0
                                                                                                                                                                                                                                                        6255
6200
                       IF E_0 = E'_0[\mathsf{dyn} \ E^{\bullet}]:
                                                                                                                                                b. QED E = E'[\text{stat } \tau' E^{\bullet}]
6201
                          a. E' = \operatorname{dyn} E'_0
                                                                                                                                                                                                                                                        6256
                                                                                                                                      CASE E = \text{stat } \tau E_0:
                                                                                                                                                                                                                                                        6257
6202
                          b. QED E = E'[\mathsf{dyn}\ E^{\bullet}]
                                                                                                                                         1. E_0 = E^{\bullet}
                                                                                                                                                                                                                                                        6258
                       IF E_0 = E_0'[\operatorname{stat} E^{\bullet}]:
                                                                                                                                              \vee E_0 = E_0'[\mathsf{dyn} E^{\bullet}]
6204
                                                                                                                                                                                                                                                        6259
                          a. E' = \operatorname{dyn} E'_0
                                                                                                                                              \vee E_0 = E_0'[\operatorname{stat} E^{\bullet}]
6205
                                                                                                                                                                                                                                                        6260
                          b. QED E = E'[\text{stat } E^{\bullet}]
                                                                                                                                              \vee E_0 = E_0'[\mathsf{dyn} \ \tau' \ E^{\bullet}]
6206
                       IF E_0 = E_0'[\text{dyn } \tau' E^{\bullet}]:
                                                                                                                                                                                                                                                        6261
6207
                                                                                                                                              \vee E_0 = E_0'[\operatorname{stat} \tau' E^{\bullet}]
                                                                                                                                                                                                                                                        6262
                          a. E' = \operatorname{dyn} E'_0
6208
                         b. QED E = E'[\mathsf{dyn} \ \tau' \ E^{\bullet}]
                                                                                                                                              by the induction hypothesis
                                                                                                                                                                                                                                                        6263
6209
                                                                                                                                         2. IF E_0 = E^{\bullet}:
                                                                                                                                                                                                                                                        6264
                       ELSE E_0 = E_0'[\text{stat } \tau' E^{\bullet}]:
6210
                                                                                                                                                a. QED
                                                                                                                                                                                                                                                        6265
                         a. E' = \operatorname{dyn} E'_0
                                                                                                                                              IF E_0 = E_0'[\mathsf{dyn}\ E^{\bullet}]:
6211
                         b. QED E = E'[\text{stat } \tau' E^{\bullet}]
                                                                                                                                                                                                                                                        6266
6212
                                                                                                                                                a. E' = \operatorname{stat} \tau E'_0
                                                                                                                                                                                                                                                        6267
               CASE E = \text{stat } E_0:
6213
                                                                                                                                                b. QED E = E'[\mathsf{dyn}\ E^{\bullet}]
                                                                                                                                                                                                                                                        6268
                                                                                                                                              IF E_0 = E_0'[\text{stat } E^{\bullet}]:
6214
                                                                                                                                                                                                                                                        6269
                                                                                                                                                                                                                                                        6270
6215
                                                                                                                           57
```

```
a. E' = \text{stat } \tau E'_0
6271
                                                                                                                                     a. OED e = E^{\bullet}[v_0 \ v_1]
                                                                                                                                                                                                                                    6326
                       b. QED E = E'[\text{stat } E^{\bullet}]
                                                                                                                                   IF E = E'[\mathsf{dyn}\ E^{\bullet}]:
6272
                                                                                                                                                                                                                                    6327
                                                                                                                                     a. QED e = E'[\mathsf{dyn} \ E^{\bullet}[v_0 \ v_1]]
6273
                     IF E_0 = E_0'[\text{dyn } \tau' E^{\bullet}]:
                                                                                                                                                                                                                                    6328
                                                                                                                                   IF E = E'[\text{stat } E^{\bullet}]:
                       a. E' = \operatorname{stat} \tau E'_0
6274
                                                                                                                                                                                                                                    6329
                       b. QED E = E'[\mathsf{dyn} \ \tau' \ E^{\bullet}]
                                                                                                                                     a. QED e = E'[\text{stat } E^{\bullet}[v_0 \ v_1]]
                                                                                                                                                                                                                                    6330
                     ELSE E_0 = E_0'[\text{stat } \tau' E^{\bullet}]:
                                                                                                                                   IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                                                                                                                                                    6331
6276
                                                                                                                                     a. QED e = E'[\mathsf{dyn} \ \tau \ E^{\bullet}[v_0 \ v_1]]
                       a. E' = \operatorname{stat} \tau E'_0
6277
                                                                                                                                                                                                                                    6332
                       b. QED E = E'[\text{stat } \tau' E^{\bullet}]
                                                                                                                                   ELSE E = E'[\text{stat } \tau E^{\bullet}]:
6278
                                                                                                                                                                                                                                    6333
6279
              CASE E = \operatorname{chk} K_0 E_0:
                                                                                                                                     a. QED e = E'[\text{stat } \tau E^{\bullet}[v_0 \ v_1]]
                                                                                                                                                                                                                                    6334
                1. E_0 = E^{\bullet}
6280
                                                                                                                           CASE e = E[op^1 v]:
                                                                                                                                                                                                                                    6335
6281
                     \vee E_0 = E_0'[\mathsf{dyn} E^{\bullet}]
                                                                                                                              1. E = E^{\bullet}
                                                                                                                                                                                                                                    6336
                     \vee E_0 = E_0'[\operatorname{stat} E^{\bullet}]
                                                                                                                                   \vee E = E'[\mathsf{dyn}\ E^{\bullet}]
6282
                                                                                                                                                                                                                                    6337
                     \vee E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]
6283
                                                                                                                                   \vee E = E'[\text{stat } E^{\bullet}]
                                                                                                                                                                                                                                    6338
                     \vee E_0 = E_0'[\operatorname{stat} \tau E^{\bullet}]
                                                                                                                                   \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
6284
                                                                                                                                                                                                                                    6339
6285
                     by the induction hypothesis
                                                                                                                                   \vee E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                                                    6340
6286
                 2. IF E_0 = E^{\bullet}:
                                                                                                                                   by inner boundary
                                                                                                                                                                                                                                    6341
                       a. QED E is boundary-free
                                                                                                                              2. IF E = E^{\bullet}:
6287
                                                                                                                                                                                                                                    6342
                     IF E_0 = E'_0[\mathsf{dyn} \ E^{\bullet}]:
                                                                                                                                     a. OED e = E^{\bullet}[op^1 v]
6288
                                                                                                                                                                                                                                    6343
                       a. E' = \operatorname{chk} K_0 E'_0
                                                                                                                                   IF E = E'[\mathsf{dyn}\ E^{\bullet}]:
6289
                                                                                                                                                                                                                                    6344
                       b. QED E = E'[\mathsf{dyn}\ E^{\bullet}]
                                                                                                                                     a. QED e = E'[\mathsf{dyn} E^{\bullet}[\mathsf{op}^1 v]]
6291
                     IF E_0 = E_0'[\operatorname{stat} E^{\bullet}]:
                                                                                                                                   IF E = E'[\text{stat } E^{\bullet}]:
                                                                                                                                                                                                                                    6346
                       a. E' = \text{chk } K_0 E'_0
                                                                                                                                     a. QED e = E'[\text{stat } E^{\bullet}[op^1 v]]
6292
                                                                                                                                                                                                                                    6347
                       b. QED E = E'[\text{stat } E^{\bullet}]
                                                                                                                                   IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                                                                                                                                                    6348
6293
                     IF E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                                                     a. QED e = E'[\text{dyn } \tau E^{\bullet}[op^1 v]]
                       a. E' = \operatorname{chk} K_0 E'_0
                                                                                                                                   ELSE E = E'[\text{stat } \tau E^{\bullet}]:
6295
                                                                                                                                                                                                                                    6350
                       b. QED E = E'[\text{dyn } \tau E^{\bullet}]
                                                                                                                                     a. QED e = E'[\text{stat } \tau E^{\bullet}[op^1 v]]
                                                                                                                                                                                                                                    6351
                                                                                                                           CASE e = E[op^2 v_0 v_1]:
                     ELSE E_0 = E_0'[\text{stat } \tau E^{\bullet}]:
6297
                                                                                                                                                                                                                                    6352
                       a. E' = \operatorname{chk} K_0 E'_0
                                                                                                                              1. E = E^{\bullet}
                                                                                                                                                                                                                                    6353
                       b. QED E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                   \vee E = E'[\mathsf{dyn}\ E^{\bullet}]
6299
                                                                                                                                                                                                                                    6354
6300
                                                                                                                                   \vee E = E'[\text{stat } E^{\bullet}]
                                                                                                                                                                                                                                    6355
6301
                                                                                                                                   \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                                                                                                                    6356
          Lemma 4.20: 1 dynamic boundary factoring
6302
                                                                                                                                   \vee E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                                                    6357
             If \vdash_1 e then one of the following holds:
6303
                                                                                                                                   by inner boundary
                                                                                                                                                                                                                                    6358
             • e is a value
6304
             • e = E^{\bullet}[v_0 \ v_1]
                                                                                                                              2. IF E = E^{\bullet}:
                                                                                                                                                                                                                                    6359
                                                                                                                                     a. OED e = E^{\bullet}[op^2 v_0 v_1]
6305
             \bullet \ e = E^{\bullet}[op^1 \ v]
                                                                                                                                                                                                                                    6360
                                                                                                                                   IF E = E'[\mathsf{dyn}\ E^{\bullet}]:
6306
             • e = E^{\bullet}[op^2 v_0 v_1]
                                                                                                                                                                                                                                    6361
6307
             • e = E[dyn e'] where e' is boundary-free
                                                                                                                                     a. QED e = E'[\mathsf{dyn} \ E^{\bullet}[op^2 \ v_0 \ v_1]]
                                                                                                                                                                                                                                    6362
6308
                                                                                                                                   IF E = E'[\text{stat } E^{\bullet}]:
             • e = E[\text{stat } e'] where e' is boundary-free
                                                                                                                                                                                                                                    6363
                                                                                                                                     a. QED e = E'[\text{stat } E^{\bullet}[op^2 v_0 v_1]]
6309
             • e = E[dyn \tau e'] where e' is boundary-free
                                                                                                                                   IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
             • e = E[\text{stat } \tau e'] where e' is boundary-free
                                                                                                                                                                                                                                    6365
6310
                                                                                                                                     a. QED e = E'[\text{dyn } \tau E^{\bullet}[op^2 v_0 v_1]]
6311
             • e = E[Err]
                                                                                                                                                                                                                                    6366
          Proof:
                                                                                                                                   ELSE E = E'[\text{stat } \tau E^{\bullet}]:
6312
                                                                                                                                                                                                                                    6367
              By the unique evaluation contexts lemma, there are ten
                                                                                                                                     a. QED e = E'[\text{stat } \tau E^{\bullet}[op^2 v_0 v_1]]
6313
                                                                                                                                                                                                                                    6368
              cases.
                                                                                                                            CASE e = E[\operatorname{chk} K' v]:
6314
                                                                                                                                                                                                                                    6369
              CASE e is a value :
                                                                                                                              1. E = E^{\bullet}
6315
                                                                                                                                                                                                                                    6370
                1. OED
                                                                                                                                   \vee E = E'[\mathsf{dyn} E^{\bullet}]
6316
                                                                                                                                                                                                                                    6371
              CASE e = E[v_0 \ v_1]:
6317
                                                                                                                                   \vee E = E'[\text{stat } E^{\bullet}]
                                                                                                                                                                                                                                    6372
                1. E = E^{\bullet}
                                                                                                                                   \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
6318
                                                                                                                                                                                                                                    6373
                     \vee E = E'[\mathsf{dyn}\ E^{\bullet}]
6319
                                                                                                                                   \vee E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                                                    6374
                     \vee E = E'[\operatorname{stat} E^{\bullet}]
6320
                                                                                                                                   by inner boundary
                                                                                                                                                                                                                                    6375
                     \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
6321
                                                                                                                              2. IF E = E^{\bullet}:
                                                                                                                                                                                                                                    6376
                     \vee E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                     a. QED e = E^{\bullet}[\operatorname{chk} K' v]
6322
                                                                                                                                                                                                                                    6377
                     by inner boundary
6323
                                                                                                                                   IF E = E'[\mathsf{dyn}\ E^{\bullet}]:
                                                                                                                                                                                                                                    6378
                2. IF E = E^{\bullet}:
                                                                                                                                     a. QED e = E'[\mathsf{dyn} \ E^{\bullet}[\mathsf{chk} \ K' \ v]]
6324
                                                                                                                                                                                                                                    6379
6325
                                                                                                                                                                                                                                    6380
                                                                                                                 58
```

```
IF E = E'[\text{stat } E^{\bullet}]:
6381
                                                                                                     IF e_0 \notin v:
                                                                                                                                                                                     6436
                   a. QED e = E'[\text{stat } E^{\bullet}[\text{chk } K' v]]
                                                                                                       1. e_0 = E_0[e_0']
6382
                                                                                                                                                                                     6437
                 IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                          by the induction hypothesis
6383
                                                                                                                                                                                     6438
                  a. QED e = E'[\text{dyn } \tau E^{\bullet}[\text{chk } K' \upsilon]]
                                                                                                       2. E = E_0 e_1
6384
                                                                                                                                                                                     6439
                                                                                                       3. QED e = E[e'_0]
                 ELSE E = E'[\text{stat } \tau E^{\bullet}]:
                  a. QED e = E'[\text{stat } \tau E^{\bullet}[\text{chk } K' \upsilon]]
                                                                                                     IF e_0 \in v
                                                                                                                                                                                     6441
6386
           CASE e = E[dyn v]:
6387
                                                                                                         \land e_1 \notin v:
                                                                                                                                                                                     6442
             1. QED v is boundary-free
                                                                                                       1. e_1 = E_1[e_1]
6388
                                                                                                                                                                                     6443
                                                                                                          by the induction hypothesis
           CASE e = E[\text{stat } v]:
                                                                                                       2. E = e_0 E_1
6390
             1. QED v is boundary-free
                                                                                                                                                                                     6445
6391
           CASE e = E[dyn \tau v]:
                                                                                                      3. QED e = E[e'_1]
                                                                                                                                                                                     6446
             1. QED v is boundary-free
6392
                                                                                                     ELSE e_0 \in v
                                                                                                                                                                                     6447
6393
           CASE e = E[\text{stat } \tau \ v]:
                                                                                                             \land e_1 \in v:
                                                                                                                                                                                     6448
6394
             1. QED v is boundary-free
                                                                                                      1. E = []
                                                                                                                                                                                     6449
6395
           CASE e = E[Err]:
                                                                                                       2. QED e = E[e_0 \ e_1]
                                                                                                                                                                                     6450
6396
             1. QED
                                                                                                  CASE e = op^1 e_0:
                                                                                                                                                                                     6451
        6397
                                                                                                    IF e_0 \notin v:
                                                                                                                                                                                     6452
                                                                                                       1. e_0 = E_0[e_0']
6398
                                                                                                                                                                                     6453
        Lemma 4.21: 1 unique dynamic evaluation contexts
                                                                                                          by the induction hypothesis
6399
                                                                                                                                                                                     6454
          If \vdash e then one of the following holds:
6400
          • e is a value
                                                                                                       2. E = op^1 E_0
6401
          • e = E[v_0 \ v_1]
                                                                                                      3. QED e = E[e'_0]
                                                                                                                                                                                     6456
          • e = E[op^1 v]
                                                                                                     ELSE e_0 \in v:
6402
                                                                                                                                                                                     6457
          • e = E[op^2 v_0 v_1]
                                                                                                       1. E = []
6403
                                                                                                                                                                                     6458
                                                                                                       2. QED e = E[op^1 e_0]
6404
          • e = E[\operatorname{chk} K v]
                                                                                                                                                                                     6459
                                                                                                  CASE e = op^2 e_0 e_1:
6405
          • e = E[\mathsf{dyn} \ v]
                                                                                                                                                                                     6460
6406
          • e = E[\text{stat } v]
                                                                                                    IF e_0 \notin v:
6407
                                                                                                       1. e_0 = E_0[e'_0]
          • e = E[\mathsf{dyn} \ \tau \ v]
                                                                                                                                                                                     6462
                                                                                                          by the induction hypothesis
                                                                                                                                                                                     6463
          • e = E[\operatorname{stat} \tau v]
                                                                                                       2. E = op^2 E_0 e_1
6409
          • e = E[Err]
                                                                                                                                                                                     6464
        Proof:
6410
                                                                                                      3. QED e = E[e'_0]
                                                                                                                                                                                     6465
           By induction on the structure of e.
6411
                                                                                                     IF e_0 \in v
                                                                                                                                                                                     6466
6412
           CASE e = x:
                                                                                                         \land e_1 \notin v:
                                                                                                                                                                                     6467
             1. Contradiction by \vdash_1 e
6413
                                                                                                       1. e_1 = E_1[e_1']
                                                                                                                                                                                     6468
           CASE e = i
6414
                                                                                                          by the induction hypothesis
                                                                                                                                                                                     6469
                   \vee e = \lambda x. e'
6415
                                                                                                       2. E = op^2 e_0 E_1
                                                                                                                                                                                     6470
                   \vee e = \lambda(x:\tau_d).e':
                                                                                                      3. QED e = E[e'_1]
6416
                                                                                                                                                                                     6471
             1. QED e is a value
6417
                                                                                                     ELSE e_0 \in v
                                                                                                                                                                                     6472
           CASE e = \langle e_0, e_1 \rangle:
6418
                                                                                                             \land e_1 \in v:
                                                                                                                                                                                     6473
             IF e_0 \notin v:
                                                                                                       1. E = []
                1. e_0 = E_0[e'_0]
                                                                                                       2. QED e = E[op^2 e_0 e_1]
                                                                                                                                                                                     6475
6420
                   by the induction hypothesis
                                                                                                  CASE e = \operatorname{chk} K e':
6421
                                                                                                                                                                                     6476
                2. E = \langle E_0, e_1 \rangle
6422
                                                                                                    1. Contradiction by \vdash_i e
                                                                                                                                                                                     6477
                3. QED e = E[e'_0]
6423
                                                                                                  CASE e = \text{dyn } e_0:
              IF e_0 \in v
6424
                                                                                                    1. Contradiction by \vdash_1 e
                                                                                                                                                                                     6479
                  \land e_1 \notin v:
6425
                                                                                                  CASE e = \text{stat } e_0:
                                                                                                                                                                                     6480
                1. e_1 = E_1[e_1']
6426
                                                                                                    IF e_0 \notin v:
                                                                                                                                                                                     6481
                   by the induction hypothesis
                                                                                                       1. \vdash_{1} e_{0}
                                                                                                                                                                                     6482
                2. E = \langle e_0, E_1 \rangle
6428
                                                                                                          by inversion
                                                                                                                                                                                     6483
               3. QED e = E[e'_1]
6429
                                                                                                       2. e_0 = E_0[e'_0]
                                                                                                                                                                                     6484
              ELSE e_0 \in v
6430
                                                                                                          by unique evaluation contexts (1)
                                                                                                                                                                                     6485
                      \land e_1 \in v:
6431
                                                                                                      3. E = \operatorname{stat} E_0
                                                                                                                                                                                     6486
                1. E = []
                                                                                                       4. QED e = E[e'_0]
6432
                                                                                                                                                                                     6487
                2. QED e = E[\langle e_0, e_1 \rangle]
6433
                                                                                                     ELSE e_0 \in v:
                                                                                                                                                                                     6488
           CASE e = e_0 e_1:
                                                                                                       1. E = []
6434
                                                                                                                                                                                     6489
6435
                                                                                                                                                                                     6490
                                                                                          59
```

```
1. E^{\bullet}[e] = op^2 v_0 E^{\bullet}_1[e]
6491
                   2. QED e = E[\text{stat } e_0]
                                                                                                                                                                                                                       6546
             CASE e = \text{dyn } e_0:
                                                                                                                       2. \vdash_{1} E^{\bullet}_{1}[e] : K_{1}
6492
                                                                                                                                                                                                                       6547
6493
                 Contradiction by \vdash_{1} e
                                                                                                                           by inversion
                                                                                                                                                                                                                       6548
                                                                                                                       3. QED the induction hypothesis (2)
             CASE e = \text{stat } K_0 e_0:
6494
                                                                                                                                                                                                                       6549
                IF e_0 \notin v:
                                                                                                                     CASE E^{\bullet} = \operatorname{chk} K E^{\bullet}_{0}:
                                                                                                                       1. E^{\bullet}[e] = \operatorname{chk} K E^{\bullet}_{0}[e]
                   1. \vdash_{1} e_{0}
                                                                                                                                                                                                                       6551
6496
                                                                                                                       2. \vdash_{1} E^{\bullet}_{0}[e] : Any
6497
                       by inversion
                                                                                                                                                                                                                       6552
6498
                   2. e_0 = E_0[e'_0]
                                                                                                                           by inversion
                                                                                                                                                                                                                       6553
                       by unique evaluation contexts (1)
                                                                                                                       3. QED the induction hypothesis (2)
                                                                                                                                                                                                                       6554
6500
                   3. E = \operatorname{stat} \tau E_0
                                                                                                                                                                                                                       6555
6501
                   4. QED e = E[e'_0]
                                                                                                                                                                                                                       6556
                                                                                                                 Lemma 4.23: 1 dynamic hole typing
                 ELSE e_0 \in v:
6502
                                                                                                                                                                                                                       6557
                                                                                                                 If \vdash_1 E^{\bullet}[e] then the derivation contains a sub-term \vdash_1 e
6503
                   1. E = []
                                                                                                                                                                                                                       6558
                   2. QED e = E[\text{stat } \tau e_0]
6504
                                                                                                                                                                                                                       6559
                                                                                                                    By induction on the structure of E^{\bullet}.
6505
                                                                                                                                                                                                                       6560
                                                                                                                    CASE E^{\bullet} = []:
6506
         Lemma 4.22: 1 static hole typing
                                                                                                                                                                                                                       6561
                                                                                                                       1. QED E^{\bullet}[e] = e
6507
          If \vdash_1 E^{\bullet}[e] : K then the typing derivation contains a sub-term
                                                                                                                                                                                                                       6562
                                                                                                                     CASE E^{\bullet} = E^{\bullet}_{0} e_{1}:
6508
            \vdash_1 e : K' \text{ for some } K'.
                                                                                                                                                                                                                       6563
                                                                                                                       1. E^{\bullet}[e] = E^{\bullet}_{0}[e] e_{1}
6509
          Proof:
                                                                                                                                                                                                                       6564
                                                                                                                       2. \vdash_1 E^{\bullet}_0[e]
6510
             By induction on the structure of E^{\bullet}.
                                                                                                                           by inversion
             CASE E^{\bullet} = []:
6511
                                                                                                                                                                                                                       6566
                                                                                                                       3. QED the induction hypothesis (2)
               1. QED E^{\bullet}[e] = e
6512
                                                                                                                                                                                                                       6567
                                                                                                                     CASE E^{\bullet} = v_0 E^{\bullet}_1:
             CASE E^{\bullet} = E^{\bullet}_{0} e_{1}:
6513
                                                                                                                                                                                                                       6568
                                                                                                                       1. E^{\bullet}[e] = v_0 E^{\bullet}_{1}[e]
               1. E^{\bullet}[e] = E^{\bullet}_{0}[e] e_{1}
6514
                                                                                                                       2. \vdash_{1} E^{\bullet}_{1}[e]
               2. \vdash_{1} E^{\bullet}_{0}[e] : \text{Fun}
6515
                                                                                                                                                                                                                       6570
                                                                                                                           by inversion
6516
                    by inversion
                                                                                                                       3. QED the induction hypothesis (2)
6517
               3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                       6572
                                                                                                                     CASE E^{\bullet} = \langle E^{\bullet}_{0}, e_{1} \rangle:
             CASE E^{\bullet} = v_0 E^{\bullet}_1:
6518
                                                                                                                                                                                                                       6573
                                                                                                                       1. E^{\bullet}[e] = \langle E^{\bullet}_{0}[e], e_{1} \rangle
               1. E^{\bullet}[e] = v_0 E^{\bullet}[e]
6519
                                                                                                                                                                                                                       6574
                                                                                                                       2. \vdash_1 E^{\bullet}_0[e]
6520
               2. \vdash_{1} E^{\bullet}_{1}[e] : Any
                                                                                                                                                                                                                       6575
                                                                                                                           by inversion
6521
                    by inversion
                                                                                                                                                                                                                       6576
                                                                                                                       3. QED the induction hypothesis (2)
               3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                       6577
                                                                                                                     CASE E^{\bullet} = \langle v_0, E^{\bullet}_1 \rangle:
             CASE E^{\bullet} = \langle E^{\bullet}_{0}, e_{1} \rangle:
6523
                                                                                                                                                                                                                       6578
                                                                                                                       1. E^{\bullet}[e] = \langle v_0, E^{\bullet}_1[e] \rangle
               1. E^{\bullet}[e] = \langle E^{\bullet}_{0}[e], e_{1} \rangle
6524
                                                                                                                                                                                                                       6579
                                                                                                                       2. \vdash_1 E^{\bullet}_1[e]
               2. \vdash_{1} E^{\bullet}_{0}[e] : Any
6525
                                                                                                                           by inversion
6526
                   by inversion
                                                                                                                                                                                                                       6581
                                                                                                                       3. QED the induction hypothesis (2)
6527
               3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                       6582
                                                                                                                     CASE E^{\bullet} = op^1 E^{\bullet}_0:
6528
             CASE E^{\bullet} = \langle v_0, E^{\bullet}_1 \rangle:
                                                                                                                                                                                                                       6583
                                                                                                                       1. E^{\bullet}[e] = op^1 E^{\bullet}_{0}[e]
               1. E^{\bullet}[e] = \langle v_0, E^{\bullet}_1[e] \rangle
                                                                                                                       2. \vdash_1 E^{\bullet}_0[e]
               2. \vdash_{1} E^{\bullet}_{1}[e] : Any
                                                                                                                                                                                                                       6585
6530
                                                                                                                           by inversion
                    by inversion
6531
                                                                                                                                                                                                                       6586
                                                                                                                       3. QED the induction hypothesis (2)
6532
               3. QED the induction hypothesis (2)
                                                                                                                                                                                                                       6587
                                                                                                                     CASE E^{\bullet} = op^2 E^{\bullet}_0 e_1:
             CASE E^{\bullet} = op^1 E^{\bullet}_0:
                                                                                                                       1. E^{\bullet}[e] = op^2 E^{\bullet}_{0}[e] e_1
               1. E^{\bullet}[e] = op^1 E^{\bullet}_{0}[e]
6534
                                                                                                                                                                                                                       6589
                                                                                                                      2. \vdash_1 E^{\bullet}_0[e]
               2. \vdash_1 E^{\bullet}_0[e]: Pair
6535
                                                                                                                                                                                                                       6590
                                                                                                                           by inversion
                    by inversion
6536
                                                                                                                                                                                                                       6591
                                                                                                                       3. QED the induction hypothesis (2)
               3. QED the induction hypothesis (2)
                                                                                                                                                                                                                       6592
                                                                                                                     CASE E^{\bullet} = op^2 v_0 E^{\bullet}_1:
             CASE E^{\bullet} = op^2 E^{\bullet}_0 e_1:
6538
                                                                                                                       1. E^{\bullet}[e] = op^2 v_0 E^{\bullet}_{1}[e]
                                                                                                                                                                                                                       6593
               1. E^{\bullet}[e] = op^2 E^{\bullet}_{0}[e] e_1
6539
                                                                                                                                                                                                                       6594
                                                                                                                       2. \vdash_1 E^{\bullet}_1[e]
               2. \vdash_{1} E^{\bullet}_{0}[e] : K_{0}
6540
                                                                                                                                                                                                                       6595
                                                                                                                           by inversion
                   by inversion
6541
                                                                                                                                                                                                                       6596
                                                                                                                       3. QED the induction hypothesis (2)
6542
               3. QED the induction hypothesis (2)
                                                                                                                                                                                                                       6597
                                                                                                                     CASE E^{\bullet} = \operatorname{chk} K E^{\bullet}_{0}:
6543
             CASE E^{\bullet} = op^2 v_0 E^{\bullet}_1:
                                                                                                                                                                                                                       6598
                                                                                                                       1. Contradiction by \vdash_{\mathbf{1}} E^{\bullet}[e]
6544
                                                                                                                                                                                                                       6599
6545
                                                                                                                                                                                                                       6600
```

601		3. QED by the induction hypothesis (2)	66
602	<b>Lemma 4.24</b> : 1 boundary hole typing	CASE $E = op^1 E_0$ :	665
603	• If $\vdash_1 E[dyn\ e]$ then the derivation contains a sub-term $\vdash_1$	1. $E[\operatorname{dyn} \tau e] = op^1 E_0[\operatorname{dyn} \tau e]$	665
604	dyn e : Any	2. $\vdash_1 E_0[dyn\ \tau\ e]: K_0$	665
605	• If $\vdash_1 E[dyn \ e] : K'$ then the derivation contains a sub-term	by inversion	666
606	ել dyn e : Any	3. QED by the induction hypothesis (2)	666
607	• If $\vdash_1 E[\text{stat } e]$ then the derivation contains a sub-term $\vdash_1$	$\mathbf{CASE} \ E = op^2 \ E_0 \ e_1 :$	666
608	stat e	1. $E[\operatorname{dyn} \tau e] = op^2 E_0[\operatorname{dyn} \tau e] e_1$	660
609	• If $\vdash_1 E[\text{stat } e] : K'$ then the derivation contains a sub-term	2. $\vdash_1 E_0[\text{dyn } \tau \ e] : K_0$	666
610	+, stat e	by inversion	666
611	• If $\vdash_1 E[dyn \ \tau \ e]$ then the derivation contains a sub-term	3. QED by the induction hypothesis (2)	666
612	$\vdash_1 \operatorname{dyn} \tau \ e : \lfloor \tau \rfloor$	CASE $E = op^2 v_0 E_1$ :	660
613	• If $\vdash_1 E[\text{dyn } \tau \ e] : K'$ then the derivation contains a sub-term	1. $E[\operatorname{dyn} \tau \ e] = op^2 v_0 E_1[\operatorname{dyn} \tau \ e]$	666
614	$\vdash_1 \operatorname{dyn} \tau \ e : [\tau]$	2. $\vdash_1 E_1[\text{dyn } \tau  e]: K_1$	666
615	• If $\vdash_{\Gamma} E[\text{stat } \tau \ e]$ then the derivation contains a sub-term	by inversion	667
616	$\vdash_1$ stat $\tau$ e	3. QED by the induction hypothesis (2)	66
617	· ·	case $E = \text{dyn } E_0$ :	667
618	• If $\vdash_{T} E[stat \ \tau \ e] : K'$ then the derivation contains a sub-term	1. $E[\operatorname{dyn} \tau \ e] = \operatorname{dyn} E_0[\operatorname{dyn} \tau \ e]$	667
619	$\vdash_1$ stat $\tau$ e		66
	Proof:	2. $\vdash_1 E_0[dyn \ \tau \ e]$	
620	By the following four lemmas: static dyn hole typing,	by inversion	66
621	dynamic dyn hole typing, static stat hole typing, and	3. QED by dynamic dyn hole typing (2)	667
622	dynamic stat hole typing. □	<b>CASE</b> $E = \text{stat } E_0$ :	66
623	_	1. Contradiction by $\vdash_1 E[dyn \ \tau \ e] : \tau'$	66
624	Lemma 4.25: 1 static dyn hole typing	<b>CASE</b> $E = \operatorname{dyn} \tau_0 E_0$ :	66
625	If $\vdash_1 E[dyn \ \tau \ e] : K'$ then the derivation contains a sub-term	1. $E[\operatorname{dyn} \tau \ e] = \operatorname{dyn} \tau_0 \ E_0[\operatorname{dyn} \tau \ e]$	668
626	$\vdash_1 dyn\ \tau\ e: \lfloor \tau \rfloor.$	2. $\vdash_1 E_0[dyn\ \tau\ e]$	668
627	Proof:	by inversion	668
628	By induction on the structure of <i>E</i> .	3. QED by dynamic dyn hole typing (2)	668
629	CASE $E \in E^{\bullet}$ :	CASE $E = \text{stat } \tau_0 E_0$ :	668
630	1. $\vdash$ <sub>1</sub> dyn $\tau$ <i>e</i> : $K''$	1. Contradiction by $\vdash_1 E[dyn \ \tau \ e] : \tau'$	668
631	by static hole typing	CASE $E = \operatorname{chk} K_0 E_0$ :	668
632	2. ⊢₁ dyn <i>τ e</i> : [ <i>τ</i> ]	1. $E[dyn \ \tau \ e] = chk \ K_0 \ E_0[dyn \ \tau \ e]$	668
633	by inversion (1)	2. $\vdash_1 E_0[dyn\ \tau\ e]: Any$	668
634	3. QED	by inversion	668
635	<b>CASE</b> $E = E_0 e_1$ :	3. QED by the induction hypothesis (2)	669
636	1. $E[dyn \ \tau \ e] = E_0[dyn \ \tau \ e] \ e_1$		669
637	2. $\vdash_1 E_0[dyn \ \tau \ e] : K_0$	<b>Lemma 4.26</b> : 1 <i>dynamic</i> dyn <i>hole typing</i>	669
638	by inversion	If $\vdash_1 E[\text{dyn } \tau \ e]$ then the derivation contains a sub-term $\vdash_1$	669
639	3. QED by the induction hypothesis (2)	dyn $\tau$ $e$ : $\lfloor \tau \rfloor$ .	669
640	CASE $E = v_0 E_1$ :	Proof:	669
641	1. $E[dyn \ \tau \ e] = \upsilon_0 \ E_1[dyn \ \tau \ e]$	By induction on the structure of <i>E</i> .	669
642	2. $\vdash_1 E_1[dyn \ \tau \ e] : K_1$	case $E \in E^{\bullet}$ :	669
643	by inversion	1. Contradiction by $\vdash_{I} E[dyn \ \tau \ e]$	669
644	3. QED by the induction hypothesis (2)	<b>CASE</b> $E = E_0 \ e_1 :$	669
645	<b>CASE</b> $E = \langle E_0, e_1 \rangle$ :	1. $E[dyn \ \tau \ e] = E_0[dyn \ \tau \ e] \ e_1$	670
646	1. $E[dyn \ \tau \ e] = \langle E_0[dyn \ \tau \ e], e_1 \rangle$	2. ⊢₁ E₀[dyn τ e]	670
647	2. $\vdash_1 E_0[dyn \ \tau \ e] : K_0$	by inversion	670
648	by inversion	3. QED by the induction hypothesis (2)	670
649	3. QED by the induction hypothesis (2)	CASE $E = v_0 E_1$ :	670
650	CASE $E = \langle v_0, E_1 \rangle$ :	1. $E[dyn \ \tau \ e] = v_0 \ E_1[dyn \ \tau \ e]$	670
651	1. $E[dyn \ \tau \ e] = \langle v_0, E_1[dyn \ \tau \ e] \rangle$	2. $\vdash_1 E_1[dyn \ \tau \ e]$	670
652	2. $\vdash_1 E_1[\text{dyn } \tau \text{ e}] : K_1$	by inversion	670
653	by inversion	3. QED by the induction hypothesis (2)	670
654	-, ····	CASE $E = \langle E_0, e_1 \rangle$ :	670
655	61	( 0/ 1/	67

6711	1 $F[dyn = a] = /F[dyn = a] a$	CAOR F - C F .	6766
6712	1. $E[dyn \ \tau \ e] = \langle E_0[dyn \ \tau \ e], e_1 \rangle$ 2. $\vdash_1 E_0[dyn \ \tau \ e]$	CASE $E = v_0 E_1$ : 1. $E[\text{stat } \tau e] = v_0 E_1[\text{stat } \tau e]$	6767
6713	by inversion	2. $\vdash_1 E_1[\operatorname{stat} \tau  e] = O_0  E_1[\operatorname{stat} \tau  e]$	6768
6714	3. QED by the induction hypothesis (2)	by inversion by $I$	6769
	· · · · · · · · · · · · · · · · · · ·	·	6770
6715	<b>CASE</b> $E = \langle v_0, E_1 \rangle$ :	3. QED by the induction hypothesis (2) CASE $E = \langle E_0, e_1 \rangle$ :	6771
6716	1. $E[dyn \ \tau \ e] = \langle v_0, E_1[dyn \ \tau \ e] \rangle$	·	6772
6717	2. $\vdash_1 E_1[dyn \ \tau \ e]$	1. $E[\operatorname{stat} \tau e] = \langle E_0[\operatorname{stat} \tau e], e_1 \rangle$	
6718	by inversion	2. $\vdash_1 E_0[\operatorname{stat} \tau \ e] : K_0$	6773
6719	3. QED by the induction hypothesis (2)	by inversion	6774
6720	$\mathbf{CASE} \ E = op^1 E_0:$	3. QED by the induction hypothesis (2)	6775
6721	1. $E[dyn \ \tau \ e] = op^1 E_0[dyn \ \tau \ e]$	<b>CASE</b> $E = \langle v_0, E_1 \rangle$ :	6776
6722	2. $\vdash_1 E_0[dyn \ \tau \ e]$	1. $E[\operatorname{stat} \tau e] = \langle v_0, E_1[\operatorname{stat} \tau e] \rangle$	6777
6723	by inversion	$2. \vdash_{I} E_1[stat \ \tau \ e] : K_1$	6778
6724	3. QED by the induction hypothesis (2)	by inversion	6779
6725	$\mathbf{CASE} \ E = op^2 E_0 \ e_1 :$	3. QED by the induction hypothesis (2)	6780
6726	1. $E[\operatorname{dyn} \tau \ e] = op^2 E_0[\operatorname{dyn} \tau \ e] e_1$	$\mathbf{CASE} \ E = op^1 E_0:$	6781
6727	2. $\vdash_1 E_0[dyn \ \tau \ e]$	1. $E[\operatorname{stat} \tau \ e] = op^1 E_0[\operatorname{stat} \tau \ e]$	6782
6728	by inversion	2. $\vdash_1 E_0[\text{stat } \tau \ e] : K_0$	6783
6729	3. QED by the induction hypothesis (2)	by inversion	6784
6730	CASE $E = op^2 v_0 E_1$ :	3. QED by the induction hypothesis (2)	6785
6731	1. $E[dyn \  au \ e] = op^2 \ v_0 \ E_1[dyn \  au \ e]$	CASE $E = op^2 E_0 e_1$ :	6786
6732	2. $\vdash_1 E_1[dyn\ \tau\ e]$	1. $E[\operatorname{stat} \tau \ e] = op^2 E_0[\operatorname{stat} \tau \ e] e_1$	6787
6733	by inversion	2. $\vdash_1 E_0[\text{stat } \tau \ e] : K_0$	6788
6734	3. QED by the induction hypothesis (2)	by inversion	6789
6735	<b>CASE</b> $E = \text{dyn } E_0$ :	3. QED by the induction hypothesis (2)	6790
6736	1. Contradiction by $\vdash_{L} E[dyn \ \tau \ e]$	CASE $E = op^2 v_0 E_1$ :	6791
6737	<b>CASE</b> $E = \text{stat } E_0$ :	1. $E[\operatorname{stat} \tau e] = op^2 v_0 E_1[\operatorname{stat} \tau e]$	6792
6738	1. $E[dyn \tau e] = stat E_0[dyn \tau e]$	2. $\vdash_1 E_1[stat\ \tau\ e]: K_1$	6793
6739	2. $\vdash_1 E_0[dyn \ \tau \ e] : \lfloor \tau_0 \rfloor$	by inversion	6794
6740	by inversion	3. QED by the induction hypothesis (2)	6795
6741	3. QED by static dyn hole typing (2)	CASE $E = \text{dyn } E_0$ :	6796
6742	CASE $E = \text{dyn } \tau E_0$ :	1. $E[\text{stat } \tau \ e] = \text{dyn } E_0[\text{stat } \tau \ e]$	6797
6743	1. Contradiction by $\vdash_1 E[dyn \ \tau \ e]$	2. $\vdash_1 E_0[\operatorname{stat} \tau \ e]$	6798
6744	<b>CASE</b> $E = \text{stat } \tau_0 E_0$ :	by inversion	6799
6745	1. $E[dyn \ \tau \ e] = stat \ \tau_0 \ E_0[dyn \ \tau \ e]$	3. QED by dynamic stat hole typing (2)	6800
6746	2. $\vdash_1 E_0[\text{dyn } \tau \ e] : [\tau_0]$	<b>CASE</b> $E = \operatorname{stat} E_0$ :	6801
6747	by inversion	1. Contradiction by $\vdash_1 E[\text{stat } \tau \ e] : \tau'$	6802
6748	3. QED by static dyn hole typing (2)	<b>CASE</b> $E = \operatorname{dyn} \tau_0 E_0$ :	6803
6749	<b>CASE</b> $E = \operatorname{chk} K_0 E_0$ :	1. $E[\operatorname{stat} \tau \ e] = \operatorname{dyn} \tau_0 E_0[\operatorname{stat} \tau \ e]$	6804
6750	1. Contradiction by $\vdash_1 E[\text{dyn } \tau \ e]$	2. $\vdash_1 E_0[\text{stat } \tau  e]$	6805
6751		by inversion	6806
6752		3. QED by dynamic stat hole typing (2)	6807
6753	Lemma 4.27: 1 static stat hole typing	CASE $E = \text{stat } \tau_0 E_0$ :	6808
6754	If $\vdash_1 E[\text{stat } \tau \ e] : K'$ then the derivation contains a sub-term	1. Contradiction by $\vdash_1 E[stat \ \tau \ e] : \tau'$	6809
6755	$\vdash_1$ stat $\tau$ e.	CASE $E = \operatorname{chk} K_0 E_0$ :	6810
	Proof:	1. $E[\operatorname{stat} \tau \ e] = \operatorname{chk} K_0 E_0[\operatorname{stat} \tau \ e]$	6811
6756	By induction on the structure of $E$ .		
6757	CASE $E \in E^{\bullet}$ :	2. $\vdash_1 E_0[\text{stat } \tau \ e] : \text{Any}$	6812 6813
6758	1. Contradiction by $\vdash_1 E[\text{stat } \tau \ e] : \tau'$	by inversion	
6759	CASE $E = E_0 e_1$ :	3. QED by the induction hypothesis (2) □	6814
6760	1. $E[\operatorname{stat} \tau \ e] = E_0[\operatorname{stat} \tau \ e] \ e_1$		6815
6761	2. $\vdash_1 E_0[\operatorname{stat} \tau \ e] : K_0$	Lemma 4.28: 1 dynamic stat hole typing	6816
6762	by inversion	If $\vdash_1 E[\text{stat } \tau \ e]$ then the derivation contains a sub-term $\vdash_1$	6817
6763	3. QED by the induction hypothesis (2)	stat $\tau$ e.	6818
6764		Proof:	6819
6765	62		6820

6821	By induction on the structure of <i>E</i> .		6876
6822	Case $E \in E^{\bullet}$ :	<b>Lemma 4.29</b> : 1 static hole substitution	6877
6823	1. QED by dynamic hole typing	If $\vdash_1 E^{\bullet}[e] : K$ and the derivation contains a sub-term $\vdash_1 e : K'$	6878
6824	CASE $E = E_0 e_1$ :	and $\vdash_1 e': K'$ , then $\vdash_1 E^{\bullet}[e']: K$	6879
6825	1. $E[\operatorname{stat} \tau \ e] = E_0[\operatorname{stat} \tau \ e] \ e_1$	Proof:	6880
6826	2. $\vdash_1 E_0[stat \ \tau \ e]$	By induction on the structure of $E^{\bullet}$ .	6881
6827	by inversion	CASE $E^{\bullet} = []$ :	6882
6828	3. QED by the induction hypothesis (2)	$1. E^{\bullet}[e] = e$	6883
6829	CASE $E = v_0 E_1$ :	$\wedge E^{\bullet}[e'] = e'$	6884
6830	1. $E[\operatorname{stat} \tau \ e] = v_0 \ E_1[\operatorname{stat} \tau \ e]$	$2. \vdash_1 e : K$	6885
6831	2. $\vdash_1 E_1[stat \ \tau \ e]$	by (1)	6886
6832	by inversion	3. K' = K	6887
6833	3. QED by the induction hypothesis (2)	4. QED	6888
6834	<b>CASE</b> $E = \langle E_0, e_1 \rangle$ :	<b>CASE</b> $E^{\bullet} = \langle E^{\bullet}_{0}, e_{1} \rangle$ :	6889
6835	1. $E[\operatorname{stat} \tau \ e] = \langle E_0[\operatorname{stat} \tau \ e], e_1 \rangle$	1. $E^{\bullet}[e] = \langle E^{\bullet}_{0}[e], e_{1} \rangle$	6890
6836	2. $\vdash_1 E_0[stat \ \tau \ e]$	$\wedge E^{\bullet}[e'] = \langle E^{\bullet}_{0}[e'], e_{1} \rangle$	6891
6837	by inversion	2. $\vdash_1 \langle E^{\bullet}_0[e], e_1 \rangle : K$	6892
6838	3. QED by the induction hypothesis (2)	3. $\vdash_1 E^{\bullet}_0[e] : K_0$	6893
6839	CASE $E = \langle v_0, E_1 \rangle$ :	$\wedge \vdash_{\!$	6894
6840	1. $E[\operatorname{stat} \tau \ e] = \langle v_0, E_1[\operatorname{stat} \tau \ e] \rangle$	by inversion	6895
6841	2. $\vdash_1 E_1[stat \ \tau \ e]$	4. $\vdash_1 E^{\bullet}_0[e']: K_0$	6896
6842	by inversion	by the induction hypothesis (3)	6897
6843	3. QED by the induction hypothesis (2)	$5. \vdash_{1} \langle E^{\bullet}_{0}[e'], e_{1} \rangle : K$	6898
6844	CASE $E = op^1 E_0$ :	by (2, 3, 4)	6899
6845	1. $E[\operatorname{stat} \tau \ e] = op^1 E_0[\operatorname{stat} \tau \ e]$	6. QED by (1, 5)	6900
6846	2. $\vdash_1 E_0[stat \ \tau \ e]$	<b>CASE</b> $E^{\bullet} = \langle v_0, E^{\bullet}_1 \rangle$ :	6901
6847	by inversion	1. $E^{\bullet}[e] = \langle v_0, E^{\bullet}_1[e] \rangle$	6902
6848	3. QED by the induction hypothesis (2)	$\wedge E^{\bullet}[e'] = \langle v_0, E^{\bullet}_{1}[e'] \rangle$	6903
6849	CASE $E = op^2 E_0 e_1$ :	$2. \vdash_1 \langle v_0, E^{\bullet}_1[e] \rangle : K$	6904
6850	1. $E[\operatorname{stat} \tau \ e] = op^2 E_0[\operatorname{stat} \tau \ e] e_1$	3. $\vdash_1 v_0 : K_0$	6905
6851	2. $\vdash_1 E_0[stat\ \tau\ e]$	$\wedge \vdash_{\!$	6906
6852	by inversion	by inversion	6907
6853	3. QED by the induction hypothesis (2)	$4. \vdash_{1} E^{\bullet}_{1}[e'] : K_{1}$	6908
6854	$\mathbf{CASE} \ E = op^2 \ v_0 \ E_1 :$	by the induction hypothesis (3)	6909
6855	1. $E[\operatorname{stat} \tau \ e] = op^2 \ v_0 \ E_1[\operatorname{stat} \tau \ e]$	$5. \vdash_1 \langle v_0, E^{\bullet}_1[e'] \rangle : K$	6910
6856	2. $\vdash_1 E_1[stat\ \tau\ e]$	by (2, 3, 4)	6911
6857	by inversion	6. QED by (1, 5)	6912
6858	3. QED by the induction hypothesis (2)	CASE $E^{\bullet} = E^{\bullet}_{0} e_{1}$ :	6913
6859	<b>CASE</b> $E = \text{dyn } E_0$ :	$1. E^{\bullet}[e] = E^{\bullet}_{0}[e] e_{1}$	6914
6860	1. Contradiction by $\vdash_1 E[\text{stat } \tau \ e]$	$\wedge E^{\bullet}[e'] = E^{\bullet}_{0}[e'] e_{1}$	6915
6861	<b>CASE</b> $E = \text{stat } E_0$ :	$2. \vdash_{1} E^{\bullet}_{0}[e] e_{1} : K$	6916
6862	1. $E[\operatorname{stat} \tau \ e] = \operatorname{stat} E_0[\operatorname{stat} \tau \ e]$	3. $\vdash_1 E^{\bullet}_0[e] : K_0$	6917
6863	2. $\vdash_1 E_0[\operatorname{stat} \tau \ e] : \lfloor \tau_0 \rfloor$	$\wedge \vdash_{\!\!\!\!\! 1} e_1 : K_1$	6918
6864	by inversion	by inversion	6919
6865	3. QED by static stat hole typing (2)	4. $\vdash_1 E^{\bullet}_0[e']: K_0$	6920
6866	CASE $E = \text{dyn } \tau E_0$ :	by the induction hypothesis (3)	6921
6867	1. Contradiction by $\vdash_1 E[\text{stat } \tau \ e]$	$5. \vdash_{1} E^{\bullet}_{0}[e'] e_{1} : K$	6922
6868	<b>CASE</b> $E = \text{stat } \tau_0 E_0$ :	by (2, 3, 4)	6923
6869	1. $E[\text{stat } \tau \ e] = \text{stat } \tau_0 \ E_0[\text{stat } \tau \ e]$	6. QED by (1, 5)	6924
6870	2. $\vdash_1 E_0[\operatorname{stat} \tau \ e] : \lfloor \tau_0 \rfloor$	CASE $E^{\bullet} = v_0 E^{\bullet}_1$ :	6925
6871	by inversion	1. $E^{\bullet}[e] = v_0 E^{\bullet}_{1}[e]$	6926
6872	3. QED by static stat hole typing (2)	$\wedge E^{\bullet}[e'] = v_0 E^{\bullet}_{1}[e']$	6927
6873	$\mathbf{CASE} \ E = \mathbf{chk} \ K_0 \ E_0 :$	2. $\vdash_1 v_0 E^{\bullet}_1[e] : K$	6928
6874	1. Contradiction by $\vdash_1 E[\text{stat } \tau \ e]$		6929
6875		63	6930

6931	3. $\vdash_1 v_0 : K_0$		6986
6932	$\wedge \vdash_1 E^{ullet}_1[e]: K_1$	<b>Lemma 4.30</b> : 1 dynamic hole substitution	6987
6933	by inversion	If $\vdash_1 E^{\bullet}[e]$ and $\vdash_1 e'$ then $\vdash_1 E^{\bullet}[e']$	6988
6934	$4. \vdash_{1} E^{\bullet}_{1}[e'] : K_{1}$	Proof:	6989
6935	by the induction hypothesis (3)	By induction on the structure of $E^{\bullet}$ .	6990
6936	5. $\vdash_1 v_0 E^{\bullet}_1[e'] : K$	CASE $E^{\bullet} = []$ :	6991
6937	by (2, 3, 4)	1. QED $E^{\bullet}[e'] = e'$	6992
6938	6. QED by (1, 5)	<b>CASE</b> $E^{\bullet} = \langle E^{\bullet}_{0}, e_{1} \rangle$ :	6993
6939	$\mathbf{CASE} \ E^{\bullet} = op^1 E^{\bullet}_{0} :$	1. $E^{\bullet}[e] = \langle E^{\bullet}_{0}[e], e_{1} \rangle$	6994
6940	1. $E^{\bullet}[e] = op^1 E^{\bullet}_{0}[e]$	$\wedge E^{\bullet}[e'] = \langle E^{\bullet}_{0}[e'], e_{1} \rangle$	6995
6941	$\wedge E^{\bullet}[e'] = op^1 E^{\bullet}_{0}[e']$	2. $\vdash_1 \langle E^{\bullet}_{0}[e], e_1 \rangle$	6996
6942	2. $\vdash_1 op^1 E^{\bullet}_0[e] : K$	3. $\vdash_1 E^{\bullet}_0[e]$	6997
6943	3. $\vdash_1 E^{\bullet}_0[e] : K_0$	$\wedge \vdash_{1} e_{1}$	6998
6944	by inversion	by inversion	6999
6945	4. $\vdash_1 E^{\bullet}_0[e']: K_0$	4. $\vdash_1 E^{\bullet}_0[e']$	7000
6946	by the induction hypothesis (3)	by the induction hypothesis (3)	7001
6947	5. $\vdash_1 op^1 E^{\bullet}_0[e'] : K$	5. $\vdash_1 \langle E^{\bullet}_0[e'], e_1 \rangle$	7002
6948	by (2, 3, 4)	by (3, 4)	7003
6949	6. QED by (1, 5)	6. QED by (1, 5)	7004
6950	CASE $E^{\bullet} = op^2 E^{\bullet}_0 e_1$ :	CASE $E^{\bullet} = \langle v_0, E^{\bullet}_1 \rangle$ :	7005
6951	1. $E^{\bullet}[e] = op^2 E^{\bullet}_{0}[e] e_1$	1. $E^{\bullet}[e] = \langle v_0, E^{\bullet}_1[e] \rangle$	7006
6952	$\wedge E^{\bullet}[e'] = op^2 E^{\bullet}_{0}[e'] e_1$	$\wedge E^{\bullet}[e'] = \langle v_0, E^{\bullet}_1[e'] \rangle$	7007
6953	2. $\vdash_1 op^2 E^{\bullet}_0[e] e_1 : K$	2. $\vdash_1 \langle v_0, E^{\bullet}_1[e] \rangle$	7008
6954	3. $\vdash_1 E^{\bullet}_0[e] : K_0$	$3. \vdash_1 v_0$	7009
6955	$\land \vdash_{1} e_{1}: K_{1}$	$\wedge \vdash_{1} E^{\bullet}_{1}[e]$	7010
6956	by inversion	by inversion	7011
6957	4. $\vdash_1 E^{\bullet}_0[e']: K_0$	$4. \vdash_1 E^{\bullet}_1[e']$	7012
6958	by the induction hypothesis (3)	by the induction hypothesis (3)	7013
6959	$5. \vdash_1 op^2 E^{\bullet}_0[e'] e_1 : K$	5. $\vdash_1 \langle v_0, E^{\bullet}_1[e'] \rangle$	7014
6960	by (2, 3, 4)	by (3, 4)	7015
6961	6. QED by (1, 5)	6. QED by (1, 5)	7016
6962	$\mathbf{CASE} \ E^{\bullet} = op^2 \ v_0 \ E^{\bullet}_1 :$	CASE $E^{\bullet} = E^{\bullet}_{0} e_{1}$ :	7017
6963	1. $E^{\bullet}[e] = op^2 v_0 E^{\bullet}_1[e]$	1. $E^{\bullet}[e] = E^{\bullet}_{0}[e] e_{1}$	7018
6964	$\wedge E^{\bullet}[e'] = op^2 \ v_0 \ E^{\bullet}_{1}[e']$	$\wedge E^{\bullet}[e'] = E^{\bullet}_{0}[e'] e_{1}$	7019
6965	$2. \vdash_{1} op^{2} v_{0} E^{\bullet}_{1}[e] : K$	2. $\vdash_1 E^{\bullet}_0[e] e_1$	7020
6966	3. $\vdash_1 v_0 : K_0$	3. $\vdash_1 E^{\bullet}_0[e]$	7021
6967	$\wedge \vdash_{1} E^{\bullet}_{1}[e] : K_{1}$	$\wedge \vdash_1 e_1$	7022
6968	by inversion	by inversion	7023
6969	$4. \vdash_{1} E^{\bullet}_{1}[e']: K_{1}$	$4. \vdash_1 E^{\bullet}_0[e']$	7024
6970	by the induction hypothesis (3)	by the induction hypothesis (3)	7025
6971	$5. \vdash_{1} op^{2} v_{0} E^{\bullet}_{1}[e'] : K$	5. $\vdash_1 E^{\bullet}_0[e'] e_1$	7026
6972	by (2, 3, 4)	by (3, 4)	7027
6973	6. QED by (1, 5)	6. QED by (1, 5)	7028
6974	$\mathbf{CASE} \ E^{\bullet} = chk \ K_c \ E^{\bullet}_{0} :$	$\mathbf{CASE} \ E^{\bullet} = v_0 E^{\bullet}_{1}:$	7029
6975	1. $E^{\bullet}[e] = \operatorname{chk} K_c E^{\bullet}_{0}[e]$	1. $E^{\bullet}[e] = v_0 E^{\bullet}_1[e]$	7030
6976	$\wedge E^{\bullet}[e'] = \operatorname{chk} K_c E^{\bullet}_{0}[e']$	$\wedge E^{\bullet}[e'] = \upsilon_0 E^{\bullet}_{1}[e']$	7031
6977	2. $\vdash_1 \operatorname{chk} K_c E^{\bullet}_0[e] : K$	$2. \vdash_{1} v_{0} E^{\bullet}_{1}[e]$	7032
6978	3. $\vdash_{1} E^{\bullet}_{0}[e] : K_{0}$	3. $\vdash_1 v_0$	7033
6979	by inversion	$\wedge \vdash_{1} E^{\bullet}{}_{1}[e]$	7034
6980	$4. \vdash_{1} E^{\bullet}_{0}[e'] : K_{0}$	by inversion	7035
6981	by the induction hypothesis (3)	$4. \vdash_{1} E^{\bullet}_{1}[e']$	7036
6982	5. $\vdash_1 \operatorname{chk} K_c E^{\bullet}_{0}[e'] : K$	by the induction hypothesis (3)	7037
6983	by (2, 3, 4)		7038
6984	6. QED by (1, 5)		7039
6985		64	7040

7041	5. $\vdash_1 v_0 E^{\bullet}_1[e']$	static context static hole substitution, and static context	7096
7042	by (3, 4)	dynamic hole substitution. □	7097 7098
7043 7044	6. QED by $(1, 5)$ <b>CASE</b> $E^{\bullet} = op^1 E^{\bullet}_0$ :		7098
7045	1. $E^{\bullet}[e] = op^1 E^{\bullet}_0[e]$	Lemma 4.32: 1 dynamic context static hole substitution	7100
7046		If $\vdash_1 E[e]$ and contains $\vdash_1 e : K'$ , and furthermore $\vdash_1 e' : K'$ ,	7101
7047	2. $\vdash_1 op^1 E^{\bullet}_{0}[e]$	then $\vdash_1 E[e']$ Proof:	7102
7048	3. ⊢ <sub>1</sub> E• <sub>0</sub> [e]	By induction on the structure of <i>E</i> .	7103
7049	by inversion	case $E \in E^{\bullet}$ :	7104
7050	4. $\vdash_1 E^{\bullet}_0[e']$	1. Contradiction by $\vdash_1 E[e]$	7105
7051	by the induction hypothesis (3)	CASE $E = E_0 e_1$ :	7106
7052	5. $\vdash_1 op^1 E^{\bullet}_0[e']$	1. $E[e] = E_0[e] e_1$	7107
7053	by (3, 4)	2. $\vdash_1 E_0[e]$	7108
7054	6. QED by (1, 5)	by inversion	7109
7055	CASE $E^{\bullet} = op^2 E^{\bullet}_0 e_1$ :	3. QED by the induction hypothesis (2)	7110
7056	1. $E^{\bullet}[e] = op^2 E^{\bullet}[e] e_1$	CASE $E = v_0 E_1$ :	7111
7057	$\wedge E^{\bullet}[e'] = op^2 E^{\bullet}_{0}[e'] e_1$	1. $E[e] = v_0 E_1[e]$	7112
7058	2. $\vdash_1 op^2 E^{\bullet}_0[e] e_1$	2. $\vdash_1 E_1[e]$	7113
7059	3. $\vdash_1 E^{\bullet}_0[e]$	by inversion	7114
7060	$\wedge \vdash_1 e_1$	3. QED by the induction hypothesis (2)	7115
7061	by inversion	<b>CASE</b> $E = \langle E_0, e_1 \rangle$ :	7116
7062	$4. \vdash_{1} E^{\bullet}_{0}[e']$	1. $E[e] = \langle E_0[e], e_1 \rangle$	7117
7063	by the induction hypothesis (3)	2. $\vdash_1 E_0[e]$	7118
7064	5. $\vdash_1 op^2 E^{\bullet}_0[e'] e_1$	by inversion	7119
7065	by (3, 4)	3. QED by the induction hypothesis (2)	7120
7066	6. QED by (1, 5)	<b>CASE</b> $E = \langle v_0, E_1 \rangle$ :	7121
7067	CASE $E^{\bullet} = op^2 v_0 E^{\bullet}_1$ :	1. $E[e] = \langle v_0, E_1[e] \rangle$	7122
7068	1. $E^{\bullet}[e] = op^2 v_0 E^{\bullet}_{1}[e]$	$2. \vdash_1 E_1[e]$	7123
7069	$\wedge E^{\bullet}[e'] = op^2 v_0 E^{\bullet}_{1}[e']$	by inversion	7124
7070	$2. \vdash_1 op^2 v_0 E^{\bullet}_1[e]$	3. QED by the induction hypothesis (2)	7125
7071	3. $\vdash_1 v_0$	CASE $E = op^1 E_0$ :	7126
7072	$\wedge \vdash_{1} E^{\bullet}_{1}[e]$	1. $E[e] = op^1 E_0[e]$	7127
7073	by inversion	2. $\vdash_1 E_0[e]$	7128
7074	$4. \vdash_{1} E^{\bullet}_{1}[e']$	by inversion	7129
7075	by the induction hypothesis (3)	3. QED by the induction hypothesis (2)	7130
7076	5. $\vdash_1 op^2 v_0 E^{\bullet}_1[e']$	<b>CASE</b> $E = op^2 E_0 e_1$ :	7131
7077	by (3, 4)	1. $E[e] = op^2 E_0[e] e_1$	7132
7078	6. QED by (1, 5)	2. $\vdash_1 E_0[e]$	7133
7079	$\mathbf{CASE} \ E^{\bullet} = chk \ K_c \ E^{\bullet}{}_0 :$	by inversion	7134
7080	1. Contradiction by $\vdash_1 E^{\bullet}[e]$	3. QED by the induction hypothesis (2)	7135
7081		$\mathbf{CASE} \ E = op^2 \ v_0 \ E_1 :$	7136
7082	Lemma 4.31: 1 hole substitution	1. $E[e] = op^2 v_0 E_1[e]$	7137
7083	• If $\vdash_1 E[e] : K$ and the derivation contains a sub-term $\vdash_1 e :$	$2. \vdash_1 E_1[e]$	7138
7084	$K'$ and $\vdash_1 e' : K'$ then $\vdash_1 E[e'] : K$ .	by inversion	7139
7085	• If $\vdash_1 E[e] : K$ and the derivation contains a sub-term $\vdash_1 e$	3. QED by the induction hypothesis (2)	7140
7086	and $\vdash_1 e'$ then $\vdash_1 E[e'] : K$ .	<b>CASE</b> $E = \text{dyn } E_0$ :	7141
7087	• If $\vdash_1 E[e]$ and the derivation contains a sub-term $\vdash_1 e : K'$	1. Contradiction by $\vdash_1 E[e]$	7142
7088	and $\vdash_1 e' : K'$ then $\vdash_1 E[e']$ .	<b>CASE</b> $E = \operatorname{stat} E_0$ :	7143
7089	• If $\vdash_1 E[e]$ and the derivation contains a sub-term $\vdash_1 e$ and	1. $E[e] = \operatorname{stat} E_0[e]$	7144
7090	$\vdash_1 e' \text{ then } \vdash_1 E[e'].$	2. $\vdash_1 E_0[e]$ : Any	7145 7146
7091	Proof:	by inversion  3. OFF by static context static help substitution (2)	7146
7092 7093	By the following four lemmas: <i>dynamic context static hole</i>	3. QED by static context static hole substitution (2)	7147
7093	substitution, dynamic context dynamic hole substitution,	CASE $E = \operatorname{dyn} \tau^{\prime\prime} E_0$ :	7149
7094	45		7150

7151	1. Contradiction by $\vdash_{\mathbf{i}} E[e]$	1. $E[e] = \text{stat } E_0[e]$	7206
7152	<b>CASE</b> $E = \text{stat } \tau_0 E_0$ :	2. $\vdash_1 E_0[e]$ : Any	7207
7153	1. $E[e] = \text{stat } \tau_0 E_0[e]$	by inversion	7208
7154	$2. \vdash_{I} E_0[e] : \lfloor \tau_0 \rfloor$	3. QED by static context dynamic hole substitution (2)	7209
7155	by inversion	CASE $E = \operatorname{dyn} \tau'' E_0$ :	7210
7156	3. QED by static context static hole substitution (2)	1. Contradiction by $\vdash_1 E[e]$	7211
7157	CASE $E = \operatorname{chk} K_0 E_0$ :	CASE $E = \operatorname{stat} \tau_0 E_0$ :	7212
7158	1. Contradiction by $\vdash_1 E[e]$	1. $E[e] = \text{stat } \tau_0 E_0[e]$	7213
7159		$2. \vdash_1 E_0[e] : \lfloor \tau_0 \rfloor$	7214
7160	<b>Lemma 4.33</b> : 1 dynamic context dynamic hole substitution	by inversion	7215
7161	If $\vdash_1 E[e]$ and contains $\vdash_1 e$ , and furthermore $\vdash_1 e'$ , then $\vdash_1 E[e']$	3. QED by static context dynamic hole substitution (2)	7216
7162	Proof:	CASE $E = \operatorname{chk} K_0 E_0$ :	7217
7163	By induction on the structure of <i>E</i> .	1. Contradiction by $\vdash_1 E[e]$	7218
7164	case $E \in E^{\bullet}$ :		7219
7165	1. QED by dynamic boundary-free hole substitution	<b>Lemma 4.34</b> : 1 static context static hole substitution	7220
7166	CASE $E = E_0 e_1$ :	If $\vdash_{\Gamma} E[e] : K$ and contains $\vdash_{\Gamma} e : K'$ , and furthermore $\vdash_{\Gamma} e' : K'$ ,	7221
7167	1. $E[e] = E_0[e] e_1$	then $\vdash_1 E[e'] : K$	7222
7168	2. $\vdash_1 E_0[e]$	Proof:	7223
7169	by inversion	By induction on the structure of <i>E</i> .	7224
7170	3. QED by the induction hypothesis (2)	<b>CASE</b> $E \in E^{\bullet}$ :	7225
7171	case $E = v_0 E_1$ :	1. QED by static boundary-free hole substitution	7226
7172	1. $E[e] = v_0 E_1[e]$	<b>CASE</b> $E = E_0 e_1$ :	7227
7173		1. $E[e] = E_0[e] e_1$	7228
7174	2. $\vdash_1 E_1[e]$ by inversion	2. $\vdash_1 E_0[e] : K_0$	7229
7175	3. QED by the induction hypothesis (2)	by inversion	7230
7176	case $E = \langle E_0, e_1 \rangle$ :	3. QED by the induction hypothesis (2)	7231
7177	CASE $E = \langle E_0, e_1 \rangle$ 1. $E[e] = \langle E_0[e], e_1 \rangle$	<b>CASE</b> $E = v_0 E_1$ :	7232
7178		1. $E[e] = v_0 E_1[e]$	7233
7179	2. $\vdash_1 E_0[e]$	2. $\vdash_1 E_1[e] : K_1$	7234
7180	by inversion  3. OND by the industion hymothesis (2)	by inversion	7235
7181	3. QED by the induction hypothesis (2)	3. QED by the induction hypothesis (2)	7236
7182	CASE $E = \langle v_0, E_1 \rangle$ :	<b>CASE</b> $E = \langle E_0, e_1 \rangle$ :	7237
7183	1. $E[e] = \langle v_0, E_1[e] \rangle$	1. $E[e] = \langle E_0[e], e_1 \rangle$	7238
7184	2. $\vdash_1 E_1[e]$ by inversion	2. $\vdash_1 E_0[e] : K_0$	7239
7185	3. QED by the induction hypothesis (2)	by inversion	7240
7186		3. QED by the induction hypothesis (2)	7241
7187	CASE $E = op^1 E_0$ :	case $E = \langle v_0, E_1 \rangle$ :	7242
7188	1. $E[e] = op^1 E_0[e]$	1. $E[e] = \langle v_0, E_1[e] \rangle$	7243
7189	2. $\vdash_1 E_0[e]$	2. $\vdash_1 E_1[e] : K_1$	7244
7190	by inversion  2. One by the industion hymothesis (2)	by inversion	7245
7191	3. QED by the induction hypothesis (2)	3. QED by the induction hypothesis (2)	7246
7192	CASE $E = op^2 E_0 e_1$ :	CASE $E = op^1 E_0$ :	7247
7193	1. $E[e] = op^2 E_0[e] e_1$	1. $E[e] = op^{1} E_{0}[e]$	7248
7194	2. $\vdash_1 E_0[e]$	2. $\vdash_1 E_0[e] : K_0$	7249
7195	by inversion	by inversion	7250
7196	3. QED by the induction hypothesis (2)	3. QED by the induction hypothesis (2)	7251
7197	CASE $E = op^2 v_0 E_1$ :	CASE $E = op^2 E_0 e_1$ :	7252
7198	1. $E[e] = op^2 v_0 E_1[e]$	1. $E[e] = op^2 E_0[e] e_1$	7253
7199	2. ⊢ <sub>1</sub> E <sub>1</sub> [e]	1. $E[e] = op \ E_0[e] \ e_1$ 2. $\vdash_1 E_0[e] : K_0$	7254
7200	by inversion	by inversion	7255
7201	3. QED by the induction hypothesis (2)	3. QED by the induction hypothesis (2)	7256
7202	<b>CASE</b> $E = \text{dyn } E_0$ :	case $E = op^2 v_0 E_1$ :	7257
7203	1. Contradiction by $\vdash_1 E[e]$	<b>CASE</b> $E = op^2 v_0 E_1$ : 1. $E[e] = op^2 v_0 E_1[e]$	7258
7204	<b>CASE</b> $E = \text{stat } E_0$ :	1. $L[e] = op  v_0  E_1[e]$	7259
7205	66		7260
	00	-	

```
2. \vdash_1 E_1[e] : K_1
7261
                                                                                                       2. \vdash_{1} E_{0}[e] : K_{0}
                                                                                                                                                                                         7316
                                                                                                          by inversion
7262
                 by inversion
                                                                                                                                                                                         7317
7263
             3. QED by the induction hypothesis (2)
                                                                                                      3. QED by the induction hypothesis (2)
                                                                                                                                                                                         7318
                                                                                                     CASE E = op^2 E_0 e_1:
           CASE E = \text{dyn } E_0:
7264
                                                                                                                                                                                         7319
                                                                                                       1. E[e] = op^2 E_0[e] e_1
7265
             1. E[e] = \text{dyn } E_0[e]
                                                                                                                                                                                         7320
                                                                                                       2. \vdash_{1} E_{0}[e] : K_{0}
             2. \vdash_{1} E_{0}[e]
                                                                                                                                                                                         7321
7266
                                                                                                          by inversion
7267
                 by inversion
                                                                                                                                                                                         7322
                                                                                                      3. QED by the induction hypothesis (2)
7268
             3. QED by static dyn hole typing (2)
                                                                                                                                                                                         7323
7269
           CASE E = \text{stat } E_0:
                                                                                                     CASE E = op^2 v_0 E_1:
                                                                                                                                                                                         7324
                                                                                                       1. E[e] = op^2 v_0 E_1[e]
7270
             1. Contradiction by \vdash_1 E[e] : K
                                                                                                                                                                                         7325
7271
           CASE E = \text{dyn } \tau_0 E_0:
                                                                                                      2. \vdash_1 E_1[e] : K_1
                                                                                                                                                                                         7326
             1. E[e] = \text{dyn } \tau_0 E_0[e]
                                                                                                          by inversion
7272
                                                                                                                                                                                         7327
7273
             2. \vdash_{1} E_{0}[e]
                                                                                                       3. QED by the induction hypothesis (2)
                                                                                                                                                                                         7328
                                                                                                     CASE E = \text{dyn } E_0:
7274
                 by inversion
                                                                                                                                                                                         7329
7275
             3. QED by static dyn hole typing (2)
                                                                                                       1. E[e] = \text{dyn } E_0[e]
                                                                                                                                                                                         7330
7276
           CASE E = \text{stat } \tau_0 E_0:
                                                                                                       2. \vdash_{1} E_{0}[e]
                                                                                                                                                                                         7331
7277
             1. Contradiction by \vdash_1 E[e] : K
                                                                                                          by inversion
                                                                                                                                                                                         7332
           CASE E = \operatorname{chk} K_0 E_0:
                                                                                                       3. QED by dynamic stat hole typing (2)
7278
                                                                                                                                                                                         7333
                                                                                                     CASE E = \text{stat } E_0:
7279
             1. E[e] = \operatorname{chk} K_0 E_0[e]
                                                                                                                                                                                         7334
7280
             2. \vdash_{1} E_{0}[e]: Any
                                                                                                       1. Contradiction by \vdash_{i} E[e] : K
                                                                                                                                                                                         7335
7281
                 by inversion
                                                                                                     CASE E = \text{dyn } \tau_0 E_0:
                                                                                                                                                                                         7336
             3. QED by the induction hypothesis (2)
                                                                                                       1. E[e] = \text{dyn } \tau_0 E_0[e]
7282
                                                                                                                                                                                         7337
                                                                                                                                                                                         7338
7283
                                                                                                      2. \vdash_{1} E_{0}[e]
7284
                                                                                                          by inversion
        Lemma 4.35: 1 static context dynamic hole substitution
7285
         If \vdash_1 E[e] : K and contains \vdash_1 e, and furthermore \vdash_1 e', then
                                                                                                       3. QED by dynamic stat hole typing (2)
                                                                                                                                                                                         7340
7286
                                                                                                     CASE E = \text{stat } \tau_0 E_0:
          \vdash_{\mathsf{L}} E[e'] : K
7287
                                                                                                       1. Contradiction by \vdash_{\mathsf{L}} E[e] : K
        Proof:
                                                                                                                                                                                         7342
7288
           By induction on the structure of E.
                                                                                                     CASE E = \operatorname{chk} K_0 E_0:
                                                                                                                                                                                         7343
           Case E \in E^{\bullet}:
                                                                                                       1. E[e] = \operatorname{chk} K_0 E_0[e]
7289
                                                                                                                                                                                         7344
7290
             1. Contradiction by \vdash_1 E[e] : K
                                                                                                      2. \vdash_{1} E_{0}[e] : Any
                                                                                                                                                                                         7345
7291
           CASE E = E_0 e_1:
                                                                                                          by inversion
                                                                                                                                                                                         7346
7292
             1. E[e] = E_0[e] e_1
                                                                                                       3. QED by the induction hypothesis (2)
                                                                                                                                                                                         7347
7293
             2. \vdash_1 E_0[e] : K_0
                                                                                                                                                                                         7348
7294
                 by inversion
                                                                                                                                                                                         7349
                                                                                                 Lemma 4.36: 1 static inversion
7295
             3. QED by the induction hypothesis (2)
                                                                                                   • If \vdash_1 \langle e_0, e_1 \rangle : K then \vdash_1 e_0 : Any and \vdash_1 e_1 : Any
                                                                                                                                                                                         7350
7296
           CASE E = v_0 E_1:
                                                                                                                                                                                         7351
                                                                                                   • If \vdash_1 \lambda x. e : K then x \vdash_1 e
7297
             1. E[e] = v_0 E_1[e]
                                                                                                    • If \vdash_1 \lambda(x : \tau). e : K then (x : \tau) \vdash_1 e : Any
7298
             2. \vdash_1 E_1[e] : K_1
                                                                                                                                                                                         7353
                                                                                                    • If \vdash_1 e_0 e_1 : K then K = \text{Any and } \vdash_1 e_0 : \text{Fun and } \vdash_1 e_1 : \text{Any}
                 by inversion
                                                                                                   • If \vdash_1 op^1 e_0 : K then K = Any and \vdash_1 e_0 : Pair
                                                                                                                                                                                         7354
                                                                                                                                                                                         7355
             3. QED by the induction hypothesis (2)
7300
                                                                                                   • If \vdash_1 op^2 e_0 e_1 : K \text{ then } \vdash_1 e_0 : K_0 \text{ and } \vdash_1 e_1 : K_1 \text{ and }
7301
           CASE E = \langle E_0, e_1 \rangle:
                                                                                                       \Delta(op^2, K_0, K_1) = K' \text{ and } K' <: K
7302
             1. E[e] = \langle E_0[e], e_1 \rangle
                                                                                                                                                                                         7357
                                                                                                   • If \vdash_1 dyn \tau e : K then \vdash_1 e and \lfloor \tau \rfloor \leqslant K
7303
             2. \vdash_1 E_0[e] : K_0
                                                                                                                                                                                         7358
                                                                                                   • If \vdash chk K' e_0 : K then \vdash e_0 : Any and K' \leqslant K
7304
                                                                                                                                                                                         7359
                 by inversion
7305
             3. QED by the induction hypothesis (2)
                                                                                                                                                                                         7360
                                                                                                     QED by the definition of \Gamma \vdash_1 e : \tau
           CASE E = \langle v_0, E_1 \rangle:
7306
                                                                                                                                                                                         7361
7307
             1. E[e] = \langle v_0, E_1[e] \rangle
                                                                                                                                                                                         7362
                                                                                                 Lemma 4.37: 1 dynamic inversion
7308
             2. \vdash_1 E_1[e] : K_1
                                                                                                                                                                                         7363
7309
                 by inversion
                                                                                                                                                                                         7364
7310
             3. QED by the induction hypothesis (2)
                                                                                                                                                                                         7365
7311
           CASE E = op^1 E_0:
                                                                                                                                                                                         7366
7312
             1. E[e] = op^1 E_0[e]
                                                                                                                                                                                         7367
7313
                                                                                                                                                                                         7368
7314
7315
                                                                                                                                                                                         7370
```

```
• If \vdash_1 \langle e_0, e_1 \rangle then \vdash_1 e_0 and \vdash_1 e_1
7371
                                                                                                                               a. \delta(\text{quotient}, i_0, i_1) = |i_0/i_1| \in i
                                                                                                                                                                                                                          7426
            • If \vdash_1 \lambda x. e then x \vdash_1 e
7372
                                                                                                                               b. QED
                                                                                                                                                                                                                          7427
            • If \vdash_1 \lambda(x : \tau). e then (x : \tau) \vdash_1 e: Any
                                                                                                                   7373
                                                                                                                                                                                                                          7428
            • If \vdash_1 e_0 e_1 then \vdash_1 e_0 and \vdash_1 e_1
7374
                                                                                                                   Lemma 4.40 : \delta preservation
                                                                                                                                                                                                                          7429
            • If \vdash_1 op^1 e_0 then \vdash_1 e_0
7375
                                                                                                                                                                                                                          7430
                                                                                                                   • If \vdash_1 v and \delta(op^1, v) = e then \vdash_1 e
            • If \vdash_1 op^2 e_0 e_1 then \vdash_1 e_0 and \vdash_1 e_1
                                                                                                                                                                                                                          7431
7376
                                                                                                                     • If \vdash_1 v_0 and \vdash_1 v_1 and \delta(op^2, v_0, v_1) = e then \vdash_1 e
            • If \vdash_1 stat \tau e then \vdash_1 e : \lfloor \tau \rfloor
7377
                                                                                                                                                                                                                          7432
                                                                                                                   Proof:
            • If \vdash_1 stat e then \vdash_1 e: Any
7378
                                                                                                                                                                                                                          7433
                                                                                                                      CASE \delta(\text{fst}, \langle v_0, v_1 \rangle) = v_0:
7379
                                                                                                                                                                                                                          7434
                                                                                                                        1. \vdash_1 v_0
             QED by the definition of \vdash_1 e.
7380
                                                                                                                                                                                                                          7435
                                                                                                                             by inversion
7381
                                                                                                                                                                                                                          7436
                                                                                                                         2. QED
          Lemma 4.38: 1 canonical forms
7382
                                                                                                                      CASE \delta(\text{snd}, \langle v_0, v_1 \rangle) = v_1:
                                                                                                                                                                                                                          7437
            • If \vdash_1 v: Pair then v = \langle v_0, v_1 \rangle
7383
                                                                                                                                                                                                                          7438
                                                                                                                         1. \vdash_1 v_1
            • If \vdash_1 v: Fun then v = \lambda x. e' or v = \lambda(x : \tau_d). e'
7384
                                                                                                                             by inversion
                                                                                                                                                                                                                          7439
7385
            • If \vdash_i v: Int then v = i
                                                                                                                                                                                                                          7440
                                                                                                                        2. QED
            • If \vdash_1 v: Nat then v \in \mathbb{N}
7386
                                                                                                                      CASE \delta(\text{sum}, v_0, v_1) = v_0 + v_1:
                                                                                                                                                                                                                          7441
          Proof:
7387
                                                                                                                                                                                                                          7442
             QED by definition of \vdash_1 \cdot : K
7388
                                                                                                                                                                                                                          7443
                                                                                                                      CASE \delta(\text{quotient}, v_0, v_1) = \lfloor v_0/v_1 \rfloor:
7389
                                                                                                                         1. QED
                                                                                                                                                                                                                          7444
7390
          Lemma 4.39 : \Delta tag soundness
                                                                                                                      CASE \delta(op^2, v_0, v_1) = \text{BndryErr}:
7391
            If \vdash_1 v_0 : K_0 and \vdash_1 v_1 : K_1 and \Delta(op^2, K_0, K_1) = K then
                                                                                                                                                                                                                          7446
                                                                                                                         1. QED
7392
            \vdash_{1} \delta(op^{2}, v_{0}, v_{1}) : K.
                                                                                                                                                                                                                          7447
          Proof:
                                                                                                                                                                                                                          7448
7393
                                                                                                                   Lemma 4.41 : \Delta preservation
             By case analysis on \Delta.
7394
                                                                                                                                                                                                                          7449
                                                                                                                   If \Delta(op^2, \tau_0, \tau_1) = \tau then \Delta(op^2, \lfloor \tau_0 \rfloor, \lfloor \tau_1 \rfloor) = \lfloor \tau \rfloor.
             CASE \Delta(\text{sum}, \text{Nat}, \text{Nat}) = \text{Nat}:
7395
                                                                                                                                                                                                                          7450
                                                                                                                   Proof:
                1. v_0 = i_0, i_0 \in \mathbb{N}
                                                                                                                                                                                                                          7451
                                                                                                                      By case analysis on the definition of \Delta
                    \wedge v_1 = i_1, i_1 \in \mathbb{N}
                                                                                                                                                                                                                          7452
7397
                                                                                                                      CASE \Delta(op^2, \text{Nat}, \text{Nat}) = \text{Nat}:
                    by canonical forms
                                                                                                                                                                                                                          7453
                                                                                                                        1. QED by [Nat] = Nat
                2. \delta(\text{sum}, i_0, i_1) = i_0 + i_1 \in \mathbb{N}
7399
                                                                                                                                                                                                                          7454
                                                                                                                      CASE \Delta(op^2, \text{Int}, \text{Int}) = \text{Int}:
7400
                3. QED
                                                                                                                                                                                                                          7455
                                                                                                                         1. QED by \lfloor Int \rfloor = Int
7401
             CASE \Delta(\text{sum}, \text{Int}, \text{Int}) = \text{Int}:
                                                                                                                                                                                                                          7456
7402
                1. v_0 = i_0
                                                                                                                                                                                                                          7457
                                                                                                                   Lemma 4.42 : \Delta inversion
7403
                    \wedge v_1 = i_1
                                                                                                                                                                                                                          7458
                                                                                                                   • If \Delta(\mathsf{fst}, \tau) = \tau' then \tau = \tau_0 \times \tau_1 and \tau' = \tau_0
                    by canonical forms
7404
                                                                                                                                                                                                                          7459
                                                                                                                     • If \Delta(\operatorname{snd}, \tau) = \tau' then \tau = \tau_0 \times \tau_1 and \tau' = \tau_1
7405
                2. \delta(\text{sum}, i_0, i_1) = i_0 + i_1 \in i
                                                                                                                                                                                                                          7460
                                                                                                                   Proof:
7406
                3. OED
                                                                                                                                                                                                                          7461
                                                                                                                      QED by the definition of \Delta
7407
              CASE \Delta(quotient, Nat, Nat) = Nat :
                                                                                                                                                                                                                          7462
                1. v_0 = i_0, i_0 \in \mathbb{N}
7408
                                                                                                                                                                                                                          7463
                                                                                                                   Lemma 4.43: <: preservation
                    \wedge v_1 = i_1, i_1 \in \mathbb{N}
7409
                                                                                                                   If \tau <: \tau' then \lfloor \tau \rfloor \leqslant : \lfloor \tau' \rfloor
                    by canonical forms
                                                                                                                                                                                                                          7465
7410
                                                                                                                   Proof:
                2. IF i_1 = 0:
7411
                                                                                                                                                                                                                          7466
                                                                                                                      By case analysis on the last rule used to show \tau <: \tau'.
7412
                      a. \delta(\text{quotient}, i_0, i_1) = \text{BndryErr}
                                                                                                                                                                                                                          7467
                                                                                                                      CASE Nat <: Int:
                      b. QED by \vdash_1 BndryErr : K
7413
                                                                                                                         1. QED [Nat] <: [Int]
7414
                    ELSE i_1 \neq 0:
                                                                                                                                                                                                                          7469
                                                                                                                      CASE \tau_d \Rightarrow \tau_c <: \tau'_d \Rightarrow \tau'_c :
7415
                      a. \delta(\text{quotient}, i_0, i_1) = \lfloor i_0/i_1 \rfloor \in \mathbb{N}
                                                                                                                                                                                                                          7470
                                                                                                                        1. \lfloor \tau_d \Rightarrow \tau_c \rfloor = \operatorname{Fun}
7416
                                                                                                                                                                                                                          7471
                                                                                                                             \wedge \lfloor \tau_d' \Rightarrow \tau_c' \rfloor = \operatorname{Fun}
             CASE \Delta(quotient, Int, Int) = Int :
7417
                                                                                                                                                                                                                          7472
                                                                                                                         2. QED
7418
                1. v_0 = i_0
                                                                                                                                                                                                                          7473
                                                                                                                      CASE \tau_0 \times \tau_1 <: \tau_0' \times \tau_1' :
7419
                    \wedge v_1 = i_1
                                                                                                                                                                                                                          7474
                                                                                                                         1. \lfloor \tau_0 \times \tau_1 \rfloor = Pair
                    by canonical forms
7420
                                                                                                                                                                                                                          7475
                                                                                                                             \wedge \left\lfloor \tau_0' \times \tau_1' \right\rfloor = \mathsf{Pair}
7421
                2. IF i_1 = 0:
                                                                                                                                                                                                                          7476
                                                                                                                         2. QED
7422
                      a. \delta(\text{quotient}, i_0, i_1) = \text{BndryErr}
                                                                                                                                                                                                                          7477
7423
                      b. QED by \vdash_1 BndryErr : K
                                                                                                                                                                                                                          7478
                                                                                                                   Lemma 4.44: 1 static value inversion
                    ELSE i_1 \neq 0:
7424
                                                                                                                                                                                                                          7479
7425
                                                                                                                                                                                                                          7480
                                                                                                             68
```

```
• If (x:\tau), \Gamma \vdash_1 e and \vdash_1 v : \lfloor \tau \rfloor then \Gamma \vdash_1 e[x \leftarrow v]
         If \vdash_1 v : Any then \vdash_1 v
7481
                                                                                                                                                                                                        7536
                                                                                                           • If x, \Gamma \vdash_1 e and \vdash_1 v then \Gamma \vdash_1 e[x \leftarrow v]
         Proof:
7482
                                                                                                                                                                                                        7537
                                                                                                           • If (x:\tau_x), \Gamma \vdash_1 e: K and \vdash_1 v: \lfloor \tau_x \rfloor then \Gamma \vdash_1 e[x \leftarrow v]: K
            By induction on the structure of v.
7483
                                                                                                                                                                                                        7538
                                                                                                           • If x, \Gamma \vdash_1 e : K and \vdash_1 v then \Gamma \vdash_1 e[x \leftarrow v] : K
            CASE v = i:
                                                                                                                                                                                                        7539
7484
              1. QED by ⊦₁ v
                                                                                                            By the following four lemmas: dynamic context static
                                                                                                                                                                                                        7541
            CASE v = \langle v_0, v_1 \rangle:
7486
                                                                                                             value substitution, dynamic context dynamic value substi-
                                                                                                                                                                                                        7542
7487
              1. \vdash_{1} v_{0}: Any
                                                                                                             tution, static context static value substitution, and static
                                                                                                                                                                                                        7543
7488
                   \land \vdash_{_{1}} v_{1} : \mathsf{Any}
                                                                                                             context dynamic value substitution.
                  by inversion
7490
               2. \vdash_1 v_0
                                                                                                                                                                                                        7545
                                                                                                         Lemma 4.47: 1 dynamic-static substitution
7491
                  \wedge \vdash_{1} v_{1}
                                                                                                                                                                                                        7546
                                                                                                          If (x:\tau), \Gamma \vdash_1 e and \vdash_1 v : \lfloor \tau \rfloor then \Gamma \vdash_1 e[x \leftarrow v]
                  by the induction hypothesis
7492
                                                                                                                                                                                                        7547
                                                                                                         Proof:
              3. QED by (2)
7493
                                                                                                                                                                                                        7548
                                                                                                            By induction on the structure of e.
7494
            CASE v = \lambda x. e:
                                                                                                                                                                                                        7549
7495
                                                                                                            CASE e = x:
                                                                                                                                                                                                        7550
              1. x \vdash_1 e
                                                                                                               1. e[x \leftarrow v] = v
7496
                  by inversion
                                                                                                                                                                                                        7551
                                                                                                               2. \vdash₁ \upsilon : Any
7497
              2. QED
                                                                                                                                                                                                        7552
                                                                                                                  by \lfloor \tau \rfloor <: Any
7498
            CASE v = \lambda(x:\tau).e:
                                                                                                                                                                                                        7553
                                                                                                               3. ⊦₁ v
              1. (x:\tau) ⊢<sub>1</sub> e : Any
7499
                                                                                                                                                                                                        7554
                                                                                                                  by static value inversion (2)
7500
                  by inversion
7501
              2. QED
                                                                                                               4. Γ ⊢ v
                                                                                                                                                                                                        7556
                                                                                                                  by weakening (3)
7502
                                                                                                                                                                                                        7557
                                                                                                               5. OED
                                                                                                                                                                                                        7558
7503
         Lemma 4.45: 1 dynamic value inversion
                                                                                                             CASE e = x':
7504
         If \vdash_1 v then \vdash_1 v: Any
                                                                                                               1. QED by x'[x \leftarrow v] = x'
7505
                                                                                                                                                                                                        7560
         Proof:
                                                                                                             CASE e = i:
7506
            By induction on the structure of v.
7507
                                                                                                               1. QED by i[x \leftarrow v] = i
                                                                                                                                                                                                        7562
            CASE v = i:
                                                                                                             CASE e = \lambda x. e':
7508
                                                                                                                                                                                                        7563
              1. \vdash_1 v: Int
                                                                                                               1. QED by (\lambda x. e')[x \leftarrow v] = \lambda x. e'
7509
                                                                                                                                                                                                        7564
              2. QED by Int <: Any
                                                                                                             CASE e = \lambda(x:\tau').e':
7510
                                                                                                                                                                                                        7565
            CASE v = \langle v_0, v_1 \rangle:
                                                                                                               1. QED by (\lambda(x:\tau').e')[x \leftarrow v] = \lambda(x:\tau').e'
7511
                                                                                                                                                                                                        7566
              1. \vdash_1 v_0
7512
                                                                                                             CASE e = \lambda x' \cdot e':
                                                                                                                                                                                                        7567
                   \wedge \vdash_1 v_1
                                                                                                               1. e[x \leftarrow v] = \lambda x' \cdot (e'[x \leftarrow v])
7513
                                                                                                                                                                                                        7568
                  by inversion
                                                                                                               2. x', (x:\tau), \Gamma \vdash_{1} e'
7514
                                                                                                                                                                                                        7569
              2. \vdash_1 v_0 : Any
                                                                                                                  by inversion
                                                                                                                                                                                                        7570
7515
                  \wedge \vdash_{_{1}} v_{1} : Any
                                                                                                               3. x', \Gamma \vdash_1 e'[x \leftarrow v]
7516
                                                                                                                                                                                                        7571
                  by the induction hypothesis
                                                                                                                  by dynamic context static value substitution
7517
                                                                                                                                                                                                        7572
              3. \vdash_1 \langle v_0, v_1 \rangle: Pair
                                                                                                               4. \Gamma \vdash_1 \lambda x'. e'[x \leftarrow v]
7518
                                                                                                                                                                                                        7573
                  by (2)
                                                                                                                  by (3)
7519
              4. QED by Pair <: Any
                                                                                                               5. OED
                                                                                                                                                                                                        7575
7520
            CASE v = \lambda x. e:
                                                                                                             CASE e = \lambda(x':\tau'). e':
7521
                                                                                                                                                                                                        7576
              1. x \vdash_{1} e
                                                                                                               1. e[x \leftarrow v] = \lambda(x':\tau').(e'[x \leftarrow v])
7522
                                                                                                                                                                                                        7577
                  by inversion
                                                                                                               2. (x':\tau'), (x:\tau), \Gamma \vdash_{1} e' : Any
7523
                                                                                                                                                                                                        7578
              2. \vdash<sub>1</sub> \lambda x. e: Fun
                                                                                                                  by inversion
7524
                                                                                                                                                                                                        7579
                  by (1)
                                                                                                               3. (x':\tau'), \Gamma \vdash_1 e'[x \leftarrow v] : Any
7525
                                                                                                                                                                                                        7580
              3. QED by Fun <: Any
                                                                                                                   by static context static value substitution
7526
                                                                                                                                                                                                        7581
            CASE v = \lambda(x : \tau). e:
                                                                                                               4. \Gamma \vdash_1 \lambda(x' : \tau'). (e'[x \leftarrow v])
7527
                                                                                                                                                                                                        7582
              1. (x:\tau) \vdash_{1} e : Any
                                                                                                               5. QED
7528
                                                                                                                                                                                                        7583
                  by inversion
                                                                                                             CASE e = \langle e_0, e_1 \rangle:
7529
                                                                                                                                                                                                        7584
              2. \vdash_1 \lambda(x:\tau). e: Fun
                                                                                                               1. e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
7530
                                                                                                                                                                                                        7585
                  by (1)
                                                                                                               2. (x:\tau), \Gamma \vdash_{1} e_{0}
7531
                                                                                                                                                                                                        7586
              3. QED by Fun <: Any
                                                                                                                   \land (x:\tau), \Gamma \vdash_{1} e_{1}
7532
                                                                                                                                                                                                        7587
                                                                                                                   by inversion
7533
                                                                                                                                                                                                        7588
         Lemma 4.46: 1 substitution
7534
7535
                                                                                                                                                                                                        7590
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7591
                3. \Gamma \vdash_1 e_0[x \leftarrow v]
                                                                                                                            3. \Gamma \vdash_{1} e'[x \leftarrow v]: Any
                                                                                                                                by static context static value substitution (2)
7592
                     \wedge \Gamma \vdash_{1} e_{1}[x \leftarrow v]
7593
                     by the induction hypothesis (2)
                                                                                                                            4. \Gamma \vdash_1 \text{stat} (e'[x \leftarrow v])
                4. \Gamma \vdash_1 \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
                                                                                                                                by (3)
7594
7595
                    by (3)
                                                                                                                            5. QED
                                                                                                                      5. QED
7596
7597
              CASE e = e_0 e_1:
                                                                                                                      Lemma 4.48: 1 dynamic-dynamic substitution
                1. e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]
7598
                                                                                                                      If x, \Gamma \vdash_1 e and \vdash_1 v then \Gamma \vdash_1 e[x \leftarrow v]
                2. (x:\tau), \Gamma \vdash_1 e_0
                                                                                                                      Proof:
7600
                     \land (x:\tau), \Gamma \vdash_{1} e_{1}
                                                                                                                         By induction on the structure of e.
7601
                     by inversion
                                                                                                                         CASE e = x:
                3. \Gamma \vdash_1 e_0[x \leftarrow v]
7602
                                                                                                                            1. e[x \leftarrow v] = v
7603
                     \wedge \Gamma \vdash_{1} e_{1}[x \leftarrow v]
                                                                                                                            2. Γ ⊦<sub>1</sub> υ
                     by the induction hypothesis (2)
7604
                                                                                                                                by weakening (3)
7605
                4. \Gamma \vdash_1 e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                            3. QED
7606
                     by (3)
                                                                                                                          CASE e = x':
7607
                5. QED
                                                                                                                            1. QED by x'[x \leftarrow v] = x'
7608
              CASE e = op^1 e_0:
                                                                                                                          CASE e = i:
7609
                1. e[x \leftarrow v] = op^1 e_0[x \leftarrow v]
                                                                                                                            1. QED by i[x \leftarrow v] = i
7610
                2. (x:\tau), \Gamma \vdash_1 e_0
                                                                                                                          CASE e = \lambda x. e':
7611
                    by inversion
                                                                                                                            1. QED by (\lambda x. e')[x \leftarrow v] = \lambda x. e'
                3. \Gamma \vdash_1 e_0[x \leftarrow v]
7612
                                                                                                                          CASE e = \lambda(x:\tau').e':
                     by the induction hypothesis (2)
7613
                                                                                                                            1. QED by (\lambda(x:\tau').e')[x \leftarrow v] = \lambda(x:\tau').e'
                4. \Gamma \vdash_1 op^1 e_0[x \leftarrow v]
7614
                                                                                                                          CASE e = \lambda x'. e':
7615
                     by (3)
                                                                                                                            1. e[x \leftarrow v] = \lambda x' \cdot (e'[x \leftarrow v])
7616
                5. QED
                                                                                                                            2. x', x, \Gamma \vdash_{1} e'
7617
              CASE e = op^2 e_0 e_1:
                                                                                                                                by inversion
                1. e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
7618
                                                                                                                           3. x', \Gamma \vdash_1 e'[x \leftarrow v]
                2. (x:\tau), \Gamma \vdash_{1} e_{0}
7619
                                                                                                                                by the induction hypothesis (2)
7620
                     \land (x:\tau), \Gamma \vdash_{1} e_{1}
                                                                                                                            4. \Gamma \vdash_1 \lambda x' . e'[x \leftarrow v]
7621
                     by inversion
                                                                                                                                by (3)
7622
                3. \Gamma \vdash_1 e_0[x \leftarrow v]
                                                                                                                            5. QED
7623
                     \wedge \Gamma \vdash_{1} e_{1}[x \leftarrow v]
                                                                                                                          CASE e = \lambda(x':\tau').e':
                     by the induction hypothesis (2)
7624
                                                                                                                            1. e[x \leftarrow v] = \lambda(x' : \tau') \cdot (e'[x \leftarrow v])
                4. \Gamma \vdash_1 op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
7625
                                                                                                                            2. (x':\tau'), x, \Gamma \vdash<sub>1</sub> e' : Any
                    by (3)
7626
                                                                                                                                by inversion
7627
                5. QED
                                                                                                                            3. (x':\tau'), \Gamma \vdash_1 e'[x \leftarrow v]: Any
              CASE e = \text{dyn } \tau' e'
7628
                                                                                                                                by static context dynamic value substitution
                        \vee e = dyn e'
                                                                                                                            4. \Gamma \vdash_1 \lambda(x' : \tau'). (e'[x \leftarrow v])
                        \vee e = \operatorname{chk} K' e':
7630
                                                                                                                            5. QED
                1. Contradiction by (x:\tau), \Gamma \vdash_i e
7631
                                                                                                                         CASE e = \langle e_0, e_1 \rangle:
7632
              CASE e = \text{stat } \tau' e':
                                                                                                                            1. e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
                1. e[x \leftarrow v] = \text{stat } \tau' e'[x \leftarrow v]
                                                                                                                            2. x, \Gamma \vdash_{1} e_0
                2. (x:\tau), \Gamma \vdash_1 e' : \lfloor \tau' \rfloor
7634
                                                                                                                                \land x, \Gamma \vdash_{1} e_{1}
7635
                     by inversion
                                                                                                                                by inversion
                3. \Gamma \vdash_{1} e'[x \leftarrow v] : |\tau'|
7636
                                                                                                                            3. \Gamma \vdash_1 e_0[x \leftarrow v]
7637
                     by static context static value substitution (2)
                                                                                                                                \wedge \Gamma \vdash_{1} e_{1}[x \leftarrow v]
7638
                4. \Gamma \vdash_1 \operatorname{stat} \tau' (e'[x \leftarrow v])
                                                                                                                                by the induction hypothesis (2)
7639
                    by (3)
                                                                                                                            4. \Gamma \vdash_1 \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
7640
                5. QED
                                                                                                                                by (3)
              CASE e = \text{stat } e':
7641
                                                                                                                            5. QED
                1. e[x \leftarrow v] = \text{stat } e'[x \leftarrow v]
7642
                                                                                                                          CASE e = e_0 e_1:
7643
                2. (x:\tau), \Gamma \vdash_1 e': Any
                                                                                                                            1. e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]
                     by inversion
7644
7645
                                                                                                               70
```

```
If (x:\tau), \Gamma \vdash_{i} e: K and \vdash_{i} v: |\tau| then \Gamma \vdash_{i} e[x \leftarrow v]: K
7701
                2. x, \Gamma \vdash_{1} e_{0}
                                                                                                                                                                                                                                 7756
                                                                                                                      Proof:
7702
                     \land x, \Gamma \vdash_{1} e_{1}
                                                                                                                                                                                                                                 7757
                                                                                                                          By induction on the structure of e.
7703
                     by inversion
                                                                                                                                                                                                                                 7758
                3. \Gamma \vdash_1 e_0[x \leftarrow v]
                                                                                                                          CASE e = x:
7704
                                                                                                                                                                                                                                 7759
7705
                     \wedge \Gamma \vdash_{1} e_{1}[x \leftarrow v]
                                                                                                                            1. |\tau| \leqslant K
                                                                                                                                                                                                                                 7760
                     by the induction hypothesis (2)
                                                                                                                                by (x:\tau), \Gamma \vdash_1 x:K
                                                                                                                                                                                                                                 7761
7706
                4. \Gamma \vdash_1 e_0[x \leftarrow v] e_1[x \leftarrow v]
7707
                                                                                                                            2. e[x \leftarrow v] = v
                                                                                                                                                                                                                                 7762
7708
                     by (3)
                                                                                                                            3. \vdash_{\scriptscriptstyle 1} v : K
                                                                                                                                                                                                                                 7763
7709
                5. QED
                                                                                                                                by (1)
                                                                                                                                                                                                                                 7764
              CASE e = op^1 e_0:
7710
                                                                                                                            4. \Gamma \vdash_{1} v : K
                                                                                                                                                                                                                                 7765
7711
                1. e[x \leftarrow v] = op^1 e_0[x \leftarrow v]
                                                                                                                                by weakening (3)
                                                                                                                                                                                                                                 7766
                2. x, \Gamma \vdash_{1} e_{0}
7712
                                                                                                                            5. QED
                                                                                                                                                                                                                                 7767
7713
                     by inversion
                                                                                                                          CASE e = x':
                                                                                                                                                                                                                                 7768
                                                                                                                            1. QED by (x'[x \leftarrow v]) = x'
7714
                3. \Gamma \vdash_{1} e_{0}[x \leftarrow v]
                                                                                                                                                                                                                                 7769
7715
                     by the induction hypothesis (2)
                                                                                                                          CASE e = i:
                                                                                                                                                                                                                                 7770
                4. \Gamma \vdash_{1} op^{1} e_{0}[x \leftarrow v]
7716
                                                                                                                            1. QED by i[x \leftarrow v] = i
                                                                                                                                                                                                                                 7771
7717
                     by (3)
                                                                                                                          CASE e = \lambda x. e':
                                                                                                                                                                                                                                 7772
7718
                5. OED
                                                                                                                            1. QED by (\lambda x. e')[x \leftarrow v] = \lambda x. e'
                                                                                                                                                                                                                                 7773
              CASE e = op^2 e_0 e_1:
                                                                                                                          CASE e = \lambda x'. e':
7719
                                                                                                                                                                                                                                 7774
                1. e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
7720
                                                                                                                            1. e[x \leftarrow v] = \lambda x' \cdot (e'[x \leftarrow v])
                                                                                                                                                                                                                                 7775
7721
                2. x, \Gamma \vdash_{1} e_0
                                                                                                                            2. x', (x:\tau), \Gamma \vdash_{1} e'
                                                                                                                                                                                                                                 7776
                     \land x, \Gamma \vdash_{\!\!\!\!1} e_1
                                                                                                                                 by inversion
7722
                                                                                                                                                                                                                                 7777
                                                                                                                            3. x', \Gamma \vdash_1 e'[x \leftarrow v]
                                                                                                                                                                                                                                 7778
7723
                     by inversion
                                                                                                                                 by dynamic context static value substitution
7724
                3. \Gamma \vdash_1 e_0[x \leftarrow v]
                                                                                                                                                                                                                                 7779
                                                                                                                            4. \Gamma \vdash_1 \lambda x' . e'[x \leftarrow v] : K
7725
                     \wedge \Gamma \vdash_{1} e_{1}[x \leftarrow v]
                                                                                                                                                                                                                                 7780
                     by the induction hypothesis (2)
                                                                                                                                by (3)
                                                                                                                                                                                                                                 7781
7727
                4. \Gamma \vdash_1 op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                            5. QED
                                                                                                                                                                                                                                 7782
                                                                                                                          CASE e = \lambda(x':\tau').e':
7728
                     by (3)
                                                                                                                                                                                                                                 7783
                5. QED
                                                                                                                            1. e[x \leftarrow v] = \lambda(x':\tau').(e'[x \leftarrow v])
7729
                                                                                                                                                                                                                                 7784
7730
              CASE e = \text{dyn } \tau' e'
                                                                                                                            2. (x':\tau'), (x:\tau), \Gamma \vdash_{1} e' : Any
                                                                                                                                                                                                                                 7785
7731
                        \vee e = dyn e'
                                                                                                                                 by inversion
                                                                                                                                                                                                                                 7786
7732
                        \vee e = \operatorname{chk} K' e':
                                                                                                                            3. (x':\tau'), \Gamma \vdash_1 e'[x \leftarrow v]: Any
                                                                                                                                                                                                                                 7787
7733
                1. Contradiction by \Gamma \vdash_{1} e
                                                                                                                                 by the induction hypothesis (2)
                                                                                                                                                                                                                                 7788
                                                                                                                            4. \Gamma \vdash_{\mathbf{1}} \lambda(x' : \tau'). (e'[x \leftarrow v]) : K
7734
              CASE e = \text{stat } \tau' e':
                                                                                                                                                                                                                                 7789
7735
                1. e[x \leftarrow v] = \text{stat } \tau' e'[x \leftarrow v]
                                                                                                                                                                                                                                 7790
                                                                                                                            5. QED
                2. x, \Gamma \vdash_{1} e' : \lfloor \tau' \rfloor
7736
                                                                                                                          CASE e = \langle e_0, e_1 \rangle:
                                                                                                                                                                                                                                 7791
7737
                    by inversion
                                                                                                                            1. e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
                                                                                                                                                                                                                                 7792
7738
                3. \Gamma \vdash_{1} e'[x \leftarrow v] : \lfloor \tau' \rfloor
                                                                                                                            2. (x:\tau), \Gamma \vdash_{1} e_{0} : Any
                                                                                                                                                                                                                                 7793
                     by static context dynamic value substitution (2)
                                                                                                                                 \land (x:\tau), \Gamma \vdash_1 e_1 : Any
7739
                4. \Gamma \vdash_1 \operatorname{stat} \tau' (e'[x \leftarrow v])
                                                                                                                                 by inversion
                                                                                                                                                                                                                                 7795
7740
                                                                                                                            3. \Gamma \vdash_1 e_0[x \leftarrow v]: Any
7741
                     by (3)
                                                                                                                                                                                                                                 7796
7742
                5. QED
                                                                                                                                 \wedge \Gamma \vdash_{_{1}} e_{1}[x \leftarrow v] : Any
                                                                                                                                                                                                                                 7797
7743
              CASE e = \text{stat } e':
                                                                                                                                 by the induction hypothesis (2)
                1. e[x \leftarrow v] = \text{stat } e'[x \leftarrow v]
7744
                                                                                                                            4. \Gamma \vdash_1 \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle : K
                                                                                                                                                                                                                                 7799
7745
                2. x, \Gamma \vdash_{1} e' : Any
                                                                                                                                 by (3)
                                                                                                                                                                                                                                 7800
7746
                     by inversion
                                                                                                                            5. QED
                                                                                                                                                                                                                                 7801
7747
                3. \Gamma \vdash_{1} e'[x \leftarrow v]: Any
                                                                                                                          CASE e = e_0 e_1:
                                                                                                                                                                                                                                 7802
                     by static context dynamic value substitution (2)
                                                                                                                            1. e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]
7748
                                                                                                                                                                                                                                 7803
7749
                4. \Gamma \vdash_1 \operatorname{stat} (e'[x \leftarrow v])
                                                                                                                            2. (x:\tau), \Gamma \vdash_{1} e_{0} : K_{0}
                                                                                                                                                                                                                                 7804
7750
                     by (3)
                                                                                                                                 \land (x:\tau), \Gamma \vdash_1 e_1 : K_1
                                                                                                                                                                                                                                 7805
7751
                5. QED
                                                                                                                                 by inversion
                                                                                                                                                                                                                                 7806
7752
                                                                                                                            3. \Gamma \vdash_{1} e_{0}[x \leftarrow v] : K_{0}
                                                                                                                                                                                                                                 7807
7753
                                                                                                                                 \wedge \Gamma \vdash_{1} e_{1}[x \leftarrow v] : K_{1}
                                                                                                                                                                                                                                 7808
          Lemma 4.49: 1 static-static substitution
7754
                                                                                                                                 by the induction hypothesis (2)
                                                                                                                                                                                                                                 7809
7755
                                                                                                                71
                                                                                                                                                                                                                                 7810
```

```
7811
                4. \Gamma \vdash_1 e_0[x \leftarrow v] e_1[x \leftarrow v] : K
                                                                                                                                                                                                                          7866
                                                                                                                                                                                                                          7867
7812
                    by (3)
                                                                                                                   Lemma 4.50: 1 static-dynamic substitution
7813
                5. QED
                                                                                                                                                                                                                          7868
                                                                                                                   If x, \Gamma \vdash_1 e : K and \vdash_1 v then \Gamma \vdash_1 e[x \leftarrow v] : K
              CASE e = op^1 e_0:
7814
                                                                                                                                                                                                                          7869
                                                                                                                   Proof:
7815
                1. e[x \leftarrow v] = op^1 e_0[x \leftarrow v]
                                                                                                                                                                                                                          7870
                                                                                                                      By induction on the structure of e.
                2. (x:\tau), \Gamma \vdash_{1} e_{0} : K_{0}
                                                                                                                                                                                                                          7871
7816
                                                                                                                      CASE e = x:
7817
                    by inversion
                                                                                                                                                                                                                          7872
                                                                                                                         1. K = Any
7818
                3. \Gamma \vdash_{1} e_{0}[x \leftarrow v] : K_{0}
                                                                                                                                                                                                                          7873
                                                                                                                             by x, \Gamma \vdash_1 x : K
                    by the induction hypothesis (2)
7819
                                                                                                                         2. e[x \leftarrow v] = v
7820
                4. \Gamma \vdash_1 op^1 e_0[x \leftarrow v] : K
                                                                                                                                                                                                                          7875
                                                                                                                         3. \vdash_1 v : K
7821
                    by (3)
                                                                                                                                                                                                                          7876
                                                                                                                             by dynamic value inversion
                5. QED
7822
                                                                                                                                                                                                                          7877
                                                                                                                         4. \vdash_1 v: Any
              CASE e = op^2 e_0 e_1:
7823
                                                                                                                                                                                                                          7878
                                                                                                                             by K \leqslant : Any
                1. e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
7824
                                                                                                                                                                                                                          7879
                                                                                                                         5. \Gamma \vdash_{\mathbf{1}} v : \mathsf{Any}
7825
                2. (x:\tau), \Gamma \vdash_{1} e_{0} : K_{0}
                                                                                                                                                                                                                          7880
                                                                                                                             by weakening (3)
7826
                    \land (x:\tau), \Gamma \vdash_{1} e_{1}: K_{1}
                                                                                                                                                                                                                          7881
                                                                                                                         6. QED
7827
                    by inversion
                                                                                                                                                                                                                          7882
                                                                                                                       CASE e = x':
                3. \Gamma \vdash_{1} e_{0}[x \leftarrow v] : K_{0}
7828
                                                                                                                                                                                                                          7883
                                                                                                                         1. QED by x'[x \leftarrow v] = x'
                    \wedge \Gamma \vdash_{1} e_{1}[x \leftarrow v] : K_{1}
7829
                                                                                                                                                                                                                          7884
                                                                                                                       CASE e = i:
7830
                    by the induction hypothesis (2)
                                                                                                                         1. QED by i[x \leftarrow v] = i
7831
                4. \Gamma \vdash_1 op^2 e_0[x \leftarrow v] e_1[x \leftarrow v] : K
                                                                                                                                                                                                                          7886
                                                                                                                       CASE e = \lambda x. e':
                    by (3)
7832
                                                                                                                                                                                                                          7887
                                                                                                                         1. QED by (\lambda x. e')[x \leftarrow v] = \lambda x. e'
                                                                                                                                                                                                                          7888
7833
                5. QED
                                                                                                                       CASE e = \lambda(x:\tau').e':
              CASE e = \text{dvn } \tau' e':
7834
                                                                                                                         1. QED by (\lambda(x:\tau').e')[x \leftarrow v] = \lambda(x:\tau').e'
                1. e[x \leftarrow v] = \text{dyn } \tau' e'[x \leftarrow v]
7835
                                                                                                                       CASE e = \lambda x' . e':
                                                                                                                                                                                                                          7890
                2. (x:\tau), \Gamma \vdash_{1} e'
                                                                                                                                                                                                                          7891
                                                                                                                         1. e[x \leftarrow v] = \lambda x' \cdot (e'[x \leftarrow v])
7837
                    by inversion
                                                                                                                                                                                                                          7892
                                                                                                                         2. x', x, \Gamma \vdash_{\iota} e'
                3. \Gamma \vdash_1 e'[x \leftarrow v]
                                                                                                                                                                                                                          7893
                                                                                                                             by inversion
                    by dynamic context static value substitution (2)
7839
                                                                                                                                                                                                                          7894
                                                                                                                         3. x', \Gamma \vdash_1 e'[x \leftarrow v]
7840
                4. \Gamma \vdash_{1} \operatorname{dyn} \tau' (e'[x \leftarrow v]) : K
                                                                                                                                                                                                                          7895
                                                                                                                             by dynamic context dynamic value substitution
7841
                    by (3)
                                                                                                                                                                                                                          7896
                                                                                                                         4. \Gamma \vdash_1 \lambda x' . e'[x \leftarrow v] : K
7842
                5. QED
                                                                                                                                                                                                                          7897
                                                                                                                             by (3)
7843
              CASE e = \text{dyn } e':
                                                                                                                                                                                                                          7898
                                                                                                                         5. QED
                1. e[x \leftarrow v] = \text{dyn } e'[x \leftarrow v]
7844
                                                                                                                                                                                                                          7899
                                                                                                                       CASE e = \lambda(x':\tau').e':
                2. (x:\tau), \Gamma \vdash_{\iota} e'
                                                                                                                                                                                                                          7900
7845
                                                                                                                         1. e[x \leftarrow v] = \lambda(x':\tau').(e'[x \leftarrow v])
                    by inversion
7846
                                                                                                                                                                                                                          7901
                                                                                                                         2. (x':\tau'), x, \Gamma ⊢<sub>1</sub> e': Any
7847
                3. \Gamma \vdash_{1} e'[x \leftarrow v]
                                                                                                                                                                                                                          7902
                                                                                                                             by inversion
                    by dynamic context static value substitution (2)
7848
                                                                                                                                                                                                                          7903
                                                                                                                         3. (x':\tau'), \Gamma \vdash_{1} e'[x \leftarrow v] : Any
                4. \Gamma \vdash_1 \operatorname{dyn} (e'[x \leftarrow v]) : K
7849
                                                                                                                             by static context dynamic value substitution
                    by (3)
                                                                                                                                                                                                                          7905
7850
                                                                                                                         4. \Gamma \vdash_1 \lambda(x' : \tau'). (e'[x \leftarrow v]) : K
7851
                5. QED
                                                                                                                                                                                                                          7906
                                                                                                                         5. QED
7852
              CASE e = \operatorname{chk} K' e':
                                                                                                                                                                                                                          7907
                                                                                                                       CASE e = \langle e_0, e_1 \rangle:
                1. e[x \leftarrow v] = \operatorname{chk} K'(e'[x \leftarrow v])
7853
                                                                                                                         1. e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
7854
                2. (x:\tau), \Gamma \vdash_1 e': Any
                                                                                                                                                                                                                          7909
                                                                                                                         2. x, \Gamma \vdash_1 e_0: Any
7855
                    by inversion
                                                                                                                                                                                                                          7910
                                                                                                                             \land x, \Gamma \vdash_{1} e_{1} : Any
                3. \Gamma \vdash_{1} e'[x \leftarrow v] : Any
7856
                                                                                                                                                                                                                          7911
                                                                                                                             by inversion
                    by the induction hypothesis (2)
                                                                                                                                                                                                                          7912
                                                                                                                         3. \Gamma \vdash_1 e_0[x \leftarrow v]: Any
7858
                4. \Gamma \vdash_{\mathbf{1}} \mathsf{chk} \; K' \; (e'[x \leftarrow v]) : K
                                                                                                                                                                                                                          7913
                                                                                                                             \wedge \Gamma \vdash_{1} e_{1}[x \leftarrow v] : Any
7859
                    by (3)
                                                                                                                                                                                                                          7914
                                                                                                                             by the induction hypothesis (2)
7860
                5. QED
                                                                                                                                                                                                                          7915
                                                                                                                         4. \Gamma \vdash_1 \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle : K
              CASE e = \text{stat } \tau' e'
7861
                                                                                                                                                                                                                          7916
                                                                                                                             by (3)
7862
                        \vee e = \operatorname{stat} e':
                                                                                                                                                                                                                          7917
                                                                                                                         5. QED
                1. Contradiction by \Gamma \vdash_{\scriptscriptstyle 1} e : K
7863
                                                                                                                                                                                                                          7918
                                                                                                                       CASE e = e_0 \ e_1:
                                                                                                                                                                                                                          7919
7864
7865
                                                                                                                                                                                                                          7920
                                                                                                             72
```

```
1. e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]
7921
                                                                                                                         4. \Gamma \vdash_1 \operatorname{chk} K' (e'[x \leftarrow v]) : K
                                                                                                                                                                                                                           7976
7922
                2. x, \Gamma \vdash_{1} e_{0} : K_{0}
                                                                                                                             by (3)
                                                                                                                                                                                                                           7977
7923
                    \land x, \Gamma \vdash_{_{\!\!1}} e_1 : K_1
                                                                                                                         5. QED
                                                                                                                                                                                                                           7978
                                                                                                                      CASE e = \text{stat } \tau' e'
                    by inversion
7924
                                                                                                                                                                                                                           7979
7925
                3. \Gamma \vdash_1 e_0[x \leftarrow v] : K_0
                                                                                                                                 \vee e = \operatorname{stat} e':
                    \wedge \Gamma \vdash_{1} e_{1}[x \leftarrow v] : K_{1}
                                                                                                                         1. Contradiction by \Gamma \vdash_{\iota} e : K
                                                                                                                                                                                                                           7981
7926
                                                                                                                   by the induction hypothesis (2)
7927
                                                                                                                                                                                                                           7982
7928
                4. \Gamma \vdash_1 e_0[x \leftarrow v] e_1[x \leftarrow v] : K
                                                                                                                                                                                                                           7983
                                                                                                                   Lemma 4.51: weakening
                    by (3)
                                                                                                                    • If \Gamma \vdash_1 e then x, \Gamma \vdash_1 e
7930
                5. QED
                                                                                                                                                                                                                           7985
                                                                                                                     • If \Gamma \vdash_1 e : \tau then (x : \tau'), \Gamma \vdash_1 e : \tau
7931
              CASE e = op^1 e_0:
                                                                                                                   Proof:
                1. e[x \leftarrow v] = op^1 e_0[x \leftarrow v]
7932
                                                                                                                      • e is closed under \Gamma
                                                                                                                                                                                                                           7987
7933
                2. x, \Gamma \vdash_{1} e_{0} : K_{0}
                                                                                                                                                                                                                           7988
                                                                                                                          by \Gamma \vdash_{1} e
                    by inversion
7934
                                                                                                                          \vee \Gamma \vdash_{\!\!\!1} e : \tau \text{ QED}
                                                                                                                                                                                                                           7989
7935
                                                                                                                                                                                                                           7990
                3. \Gamma \vdash_1 e_0[x \leftarrow v] : K_0
7936
                    by the induction hypothesis (2)
                                                                                                                                                                                                                           7991
                                                                                                                   Lemma 4.52 : unique static evaluation contexts
                4. \Gamma \vdash_1 op^1 e_0[x \leftarrow v] : K
7937
                                                                                                                                                                                                                           7992
                                                                                                                    If \vdash e : \tau and e is boundary-free then one of the following
7938
                    by (3)
                                                                                                                                                                                                                           7993
                                                                                                                     holds:
7939
                5. QED
                                                                                                                                                                                                                           7994
                                                                                                                     • e is a value
              CASE e = op^2 e_0 e_1:
7940
                                                                                                                                                                                                                           7995
                                                                                                                     \bullet e = E^{\bullet}[v_0 \ v_1]
7941
                1. e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
                                                                                                                                                                                                                           7996
                                                                                                                     • e = E^{\bullet}[op^1 v]
                2. x, \Gamma \vdash_{1} e_{0} : K_{0}
7942
                                                                                                                                                                                                                           7997
                                                                                                                     \bullet e = E^{\bullet}[op^2 v_0 v_1]
                    \land x, \Gamma \vdash_{\!\!\!\!\!1} e_1 : K_1
                                                                                                                                                                                                                           7998
7943
                                                                                                                     • e = E^{\bullet}[Err]
                    by inversion
7944
                                                                                                                   Proof:
                3. \Gamma \vdash_1 e_0[x \leftarrow v] : K_0
7945
                                                                                                                      By induction on the structure of e.
                                                                                                                                                                                                                           8000
                    \wedge \Gamma \vdash_{1} e_{1}[x \leftarrow v] : K_{1}
                                                                                                                      CASE e = x:
                                                                                                                                                                                                                           8001
7947
                    by the induction hypothesis (2)
                                                                                                                         1. Contradiction by \vdash e : \tau
                                                                                                                                                                                                                           8002
                4. \Gamma \vdash_1 op^2 e_0[x \leftarrow v] e_1[x \leftarrow v] : K
                                                                                                                                                                                                                           8003
                                                                                                                       CASE e = i
                    by (3)
7949
                                                                                                                                 \vee e = \lambda(x:\tau_d).e':
                                                                                                                                                                                                                           8004
7950
                5. QED
                                                                                                                                                                                                                           8005
                                                                                                                         1. QED e is a value
7951
              CASE e = \text{dyn } \tau' e':
                                                                                                                       CASE e = \langle e_0, e_1 \rangle:
                                                                                                                                                                                                                           8006
                1. e[x \leftarrow v] = \text{dyn } \tau' e'[x \leftarrow v]
                                                                                                                          IF e_0 \notin v:
                                                                                                                                                                                                                           8007
7953
                2. x, \Gamma \vdash_{1} e'
                                                                                                                            1. e_0 = E^{\bullet}_0[e'_0]
                                                                                                                                                                                                                           8008
                    by inversion
7954
                                                                                                                                 by the induction hypothesis
                                                                                                                                                                                                                           8009
7955
                3. \Gamma \vdash_{1} e'[x \leftarrow v]
                                                                                                                            2. E^{\bullet} = \langle E^{\bullet}_{0}, e_{1} \rangle
                                                                                                                                                                                                                           8010
                    by dynamic context dynamic value substitution (2)
7956
                                                                                                                            3. QED by e = E^{\bullet}[e'_0]
                                                                                                                                                                                                                           8011
7957
                4. \Gamma \vdash_1 \operatorname{dyn} \tau' (e'[x \leftarrow v]) : K
                                                                                                                          IF e_0 \in v
                                                                                                                                                                                                                           8012
7958
                    by (3)
                                                                                                                                                                                                                           8013
                                                                                                                                \land e_1 \notin v:
                                                                                                                            1. e_1 = E^{\bullet}_1[e'_1]
                5. QED
              CASE e = dyn e':
                                                                                                                                 by the induction hypothesis
                                                                                                                                                                                                                           8015
7960
                1. e[x \leftarrow v] = \text{dyn } e'[x \leftarrow v]
7961
                                                                                                                            2. E^{\bullet} = \langle e_0, E^{\bullet}_1 \rangle
                                                                                                                                                                                                                           8016
                2. x, \Gamma \vdash_{\scriptscriptstyle{1}} e'
7962
                                                                                                                                                                                                                           8017
                                                                                                                            3. QED by e = E^{\bullet}[e'_1]
                                                                                                                          ELSE e_0 \in v
                    by inversion
                                                                                                                                                                                                                           8018
                3. \Gamma \vdash_1 e'[x \leftarrow v]
7964
                                                                                                                                     \land e_1 \in v:
                                                                                                                                                                                                                           8019
                    by dynamic context dynamic value substitution (2)
7965
                                                                                                                            1. E^{\bullet} = []
                                                                                                                                                                                                                           8020
                4. \Gamma \vdash_1 \operatorname{dyn} (e'[x \leftarrow v]) : K
7966
                                                                                                                            2. QED e = E^{\bullet}[\langle e_0, e_1 \rangle]
                                                                                                                                                                                                                           8021
7967
                    by (3)
                                                                                                                                                                                                                           8022
                                                                                                                      CASE e = e_0 e_1:
7968
                5. QED
                                                                                                                          IF e_0 \notin v:
                                                                                                                                                                                                                           8023
7969
              CASE e = \operatorname{chk} K' e':
                                                                                                                            1. e_0 = E^{\bullet}_0[e'_0]
                                                                                                                                                                                                                           8024
7970
                1. e[x \leftarrow v] = \operatorname{chk} K' (e'[x \leftarrow v])
                                                                                                                                 by the induction hypothesis
                                                                                                                                                                                                                           8025
7971
                2. x, \Gamma \vdash_1 e' : Any
                                                                                                                            2. E^{\bullet} = E^{\bullet}_{0} e_{1}
                                                                                                                                                                                                                           8026
7972
                    by inversion
                                                                                                                            3. QED by e = E^{\bullet}[e'_0]
                                                                                                                                                                                                                           8027
7973
                3. \Gamma \vdash_{1} e'[x \leftarrow v] : Any
                                                                                                                                                                                                                           8028
                    by the induction hypothesis (2)
7974
                                                                                                                                                                                                                           8029
7975
                                                                                                                                                                                                                           8030
                                                                                                             73
```

```
• If \Gamma \vdash x : \tau then (x : \tau') \in \Gamma and \tau' \leqslant \tau
8031
                 IF e_0 \in v
                                                                                                                                                                                                                           8086
                                                                                                                     • If \Gamma \vdash \lambda(x : \tau'_d). e' : \tau then (x : \tau'_d), \Gamma \vdash e' : \tau'_c and \tau'_d \Rightarrow \tau'_c \leqslant \varepsilon
8032
                                                                                                                                                                                                                           8087
                      \land e_1 \notin v:
8033
                    1. e_1 = E_1^{\bullet}[e_1]
                                                                                                                                                                                                                           8088
                                                                                                                     • If \Gamma \vdash \langle e_0, e_1 \rangle : \tau then \Gamma \vdash e_0 : \tau_0 and \Gamma \vdash e_1 : \tau_1 and
                       by the induction hypothesis
                                                                                                                                                                                                                           8089
8034
8035
                   2. E^{\bullet} = e_0 E^{\bullet}_1
                                                                                                                                                                                                                           8090
                                                                                                                     • If \Gamma \vdash e_0 \ e_1 : \tau_c then \Gamma \vdash e_0 : \tau_d' \Rightarrow \tau_c' and \Gamma \vdash e_1 : \tau_d' and
                   3. QED by e = E^{\bullet}[e'_1]
                                                                                                                                                                                                                           8091
8036
                                                                                                                         \tau_c' \leqslant : \tau_c
                 ELSE e_0 \in v
                                                                                                                                                                                                                           8092
8037
                                                                                                                     • If \Gamma \vdash fst e : \tau then \Gamma \vdash e : \tau_0 \times \tau_1 and \Delta(\text{fst}, \tau_0 \times \tau_1) = \tau_0 and
                                                                                                                                                                                                                           8093
8038
                           \land e_1 \in v:
8039
                   1. E^{\bullet} = []
                                                                                                                                                                                                                           8094
                                                                                                                     • If \Gamma \vdash \text{snd } e : \tau \text{ then } \Gamma \vdash e : \tau_0 \times \tau_1 \text{ and } \Delta(\text{snd}, \tau_0 \times \tau_1) = \tau_1
                   2. QED e = E^{\bullet}[e_0 \ e_1]
8040
                                                                                                                                                                                                                           8095
                                                                                                                         and \tau_1 \leqslant : \tau
8041
              CASE e = op^1 e_0:
                                                                                                                     • If \Gamma \vdash op^2 e_0 e_1 : \tau then \Gamma \vdash e_0 : \tau_0 and \Gamma \vdash e_1 : \tau_1 and
                 IF e_0 \notin v:
                                                                                                                                                                                                                           8097
8042
                                                                                                                         \Delta(op^2, \tau_0, \tau_1) = \tau' and \tau' \leqslant \tau
8043
                   1. e_0 = E_0^{\bullet}[e_0']
                                                                                                                                                                                                                           8098
                                                                                                                     • If \Gamma \vdash \text{dyn } \tau' \ e' : \tau \text{ then } \Gamma \vdash e' \text{ and } \tau' \leqslant \tau
                       by the induction hypothesis
8044
                                                                                                                                                                                                                           8099
                                                                                                                   Proof:
8045
                   2. E^{\bullet} = op^1 E^{\bullet}_0
                                                                                                                                                                                                                           8100
                                                                                                                      QED by the definition of \Gamma \vdash e : \tau
8046
                   3. QED e = E^{\bullet}[e'_0]
                                                                                                                                                                                                                           8101
                 ELSE e_0 \in v:
8047
                                                                                                                                                                                                                           8102
                                                                                                                   Lemma 4.54: canonical forms
                   1. E^{\bullet} = []
                                                                                                                                                                                                                           8103
8048
                                                                                                                     • If \vdash v : \tau_0 \times \tau_1 then v = \langle v_0, v_1 \rangle
                   2. QED e = E^{\bullet}[op^1 e_0]
8049
                                                                                                                                                                                                                           8104
                                                                                                                     • If \vdash v : \tau_d \Rightarrow \tau_c then v = \lambda(x : \tau_x) \cdot e'
8050
              CASE e = op^2 e_0 e_1:
                                                                                                                                                                                                                           8105
                                                                                                                         \wedge \tau_d \leqslant : \tau_x
8051
                 IF e_0 \notin v:
                                                                                                                                                                                                                           8106
                                                                                                                     • If \vdash v: Int then v = i
                    1. e_0 = E^{\bullet}_0[e'_0]
8052
                                                                                                                                                                                                                           8107
                                                                                                                     • If \vdash v: Nat then v = i and v \in \mathbb{N}
                       by the induction hypothesis
                                                                                                                                                                                                                           8108
8053
                                                                                                                   Proof:
                   2. E^{\bullet} = op^2 E^{\bullet}_{0} e_1
8054
                                                                                                                                                                                                                           8109
                                                                                                                       QED by definition of \vdash e : \tau
                   3. QED e = E^{\bullet}[e'_0]
8055
                                                                                                                                                                                                                           8110
                 IF e_0 \in v
                                                                                                                                                                                                                           8111
                                                                                                                   Lemma 4.55 : substitution
                      \land e_1 \notin v:
                                                                                                                                                                                                                           8112
8057
                                                                                                                    If (x:\tau_x), \Gamma \vdash e:\tau, and e is boundary-free and \vdash v:\tau_x then
                   1. e_1 = E_1^{\bullet}[e_1]
                                                                                                                                                                                                                           8113
                                                                                                                     \Gamma \vdash e[x \leftarrow v] : \tau
                       by the induction hypothesis
                                                                                                                                                                                                                           8114
8059
                                                                                                                   Proof:
                   2. E^{\bullet} = op^2 e_0 E^{\bullet}_1
8060
                                                                                                                                                                                                                           8115
                                                                                                                      By induction on the structure of e.
8061
                   3. QED e = E^{\bullet}[e'_1]
                                                                                                                                                                                                                           8116
                                                                                                                      CASE e = x:
                 ELSE e_0 \in v
                                                                                                                                                                                                                           8117
                                                                                                                         1. e[x \leftarrow v] = v
8063
                           \land e_1 \in v:
                                                                                                                                                                                                                           8118
                                                                                                                         2. \tau_x = \tau
                    1. E^{\bullet} = []
                                                                                                                                                                                                                           8119
8064
                                                                                                                         3. \Gamma \vdash \upsilon : \tau
                   2. OED e = E^{\bullet}[op^2 e_0 e_1]
                                                                                                                                                                                                                           8120
8065
                                                                                                                             by weakening
             CASE e = \operatorname{chk} K' e':
8066
                                                                                                                                                                                                                           8121
                                                                                                                         4. QED
8067
                1. Contradiction by \vdash e : \tau
                                                                                                                                                                                                                           8122
                                                                                                                       CASE e = x':
             CASE e = dyn e_0:
8068
                                                                                                                                                                                                                           8123
                                                                                                                         1. QED by x'[x \leftarrow v] = x'
                1. Contradiction by \vdash e : \tau
8069
                                                                                                                                                                                                                           8124
                                                                                                                       CASE e = i:
             CASE e = \text{stat } e':
                                                                                                                                                                                                                           8125
8070
                                                                                                                         1. QED by i[x \leftarrow v] = i
                1. Contradiction by \vdash e : \tau
8071
                                                                                                                                                                                                                           8126
                                                                                                                       CASE e = \lambda x. e':
8072
             CASE e = \text{dyn } \tau e_0:
                                                                                                                                                                                                                           8127
                                                                                                                         1. Contradiction by (x:\tau_x), \Gamma \vdash e:\tau
                1. QED e is boundary-free
8073
                                                                                                                                                                                                                           8128
                                                                                                                       CASE e = \lambda(x:\tau').e':
8074
              CASE e = \text{stat } \tau e':
                                                                                                                                                                                                                           8129
                                                                                                                         1. QED by (\lambda(x:\tau'), e')[x \leftarrow v] = \lambda(x:\tau'), e'
                1. Contradiction by \vdash e : \tau
8075
                                                                                                                                                                                                                           8130
                                                                                                                       CASE e = \lambda(x':\tau').e':
              CASE e = Err:
8076
                                                                                                                                                                                                                           8131
                                                                                                                         1. e[x \leftarrow v] = \lambda(x' : \tau') \cdot (e'[x \leftarrow v])
8077
                1. E^{\bullet} = []
                                                                                                                                                                                                                           8132
                                                                                                                         2. (x':\tau'), x, \Gamma \vdash e'
8078
                2. QED
                                                                                                                                                                                                                           8133
                                                                                                                             by static inversion forms
8079
                                                                                                                                                                                                                           8134
                                                                                                                         3. (x':\tau'), \Gamma \vdash e'[x \leftarrow v]
8080
                                                                                                                                                                                                                           8135
          Lemma 4.53: ⊢ static inversion
                                                                                                                             by the induction hypothesis (2)
8081
                                                                                                                                                                                                                           8136
                                                                                                                         4. \Gamma \vdash \lambda(x':\tau'). (e'[x \leftarrow v])
8082
                                                                                                                                                                                                                           8137
                                                                                                                             by (3)
8083
                                                                                                                                                                                                                           8138
                                                                                                                         5. QED
8084
                                                                                                                                                                                                                           8139
8085
                                                                                                             74
                                                                                                                                                                                                                           8140
```

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8141
             CASE e = \langle e_0, e_1 \rangle:
               1. e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
8142
8143
               2. x, \Gamma \vdash e_0
                    \land x, \Gamma \vdash e_1
8144
                    by static inversion forms
               3. \Gamma \vdash e_0[x \leftarrow v]
8146
8147
                    \wedge \Gamma \vdash e_1[x \leftarrow v]
8148
                    by the induction hypothesis (2)
8149
               4. \Gamma \vdash \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle
8150
                    by (3)
8151
               5. QED
             CASE e = e_0 \ e_1:
8152
               1. e[x \leftarrow v] = e_0[x \leftarrow v] e_1[x \leftarrow v]
8153
               2. x, \Gamma \vdash e_0
8154
8155
                    \land x, \Gamma \vdash e_1
8156
                    by static inversion forms
               3. \Gamma \vdash e_0[x \leftarrow v]
8157
8158
                    \wedge \Gamma \vdash e_1[x \leftarrow v]
                    by the induction hypothesis (2)
8159
               4. \Gamma \vdash e_0[x \leftarrow v] e_1[x \leftarrow v]
8161
                   by (3)
               5. QED
8162
             CASE e = op^1 e_0:
8163
               1. e[x \leftarrow v] = op^1 e_0[x \leftarrow v]
8164
8165
               2. x, \Gamma \vdash e_0
                    by static inversion forms
8167
               3. \Gamma \vdash e_0[x \leftarrow v]
                    by the induction hypothesis (2)
8168
               4. \Gamma \vdash op^1 e_0[x \leftarrow v]
8169
8170
                    by (3)
8171
               5. QED
8172
             CASE e = op^2 e_0 e_1:
               1. e[x \leftarrow v] = op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
8173
8174
               2. x, \Gamma \vdash e_0
8175
                    \land x, \Gamma \vdash e_1
                    by static inversion forms
8176
8177
               3. \Gamma \vdash e_0[x \leftarrow v]
8178
                    \wedge \Gamma \vdash e_1[x \leftarrow v]
                    by the induction hypothesis (2)
8179
               4. \Gamma \vdash op^2 e_0[x \leftarrow v] e_1[x \leftarrow v]
8180
                    by (3)
8181
8182
               5. QED
             CASE e = \operatorname{chk} K' e':
8183
               1. Contradiction by \vdash e : \tau
8184
             CASE e = \text{dyn } e':
8185
               1. Contradiction by \vdash e : \tau
8186
8187
             CASE e = \text{stat } e':
               1. Contradiction by \vdash e : \tau
8188
8189
             CASE e = \text{dyn } \tau' e':
8190
               1. Contradiction by e is boundary-free
8191
             CASE e = \text{stat } \tau' e':
               1. Contradiction by \vdash e : \tau
8192
8193
             CASE e = Err:
               1. QED \text{Err}[x \leftarrow v] = \text{Err}
8194
```

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8196
                                                                                             8197
Lemma 4.56 : \delta preservation
                                                                                             8198
 • If \vdash v and \delta(op^1, v) = v' then \vdash e'
  • If \vdash v_0 and \vdash v_1 and \delta(op^2, v_0, v_1) = e' then \vdash v'
                                                                                             8199
Proof:
                                                                                             8200
   CASE \delta(\text{fst}, \langle v_0, v_1 \rangle) = v_0:
                                                                                             8201
                                                                                             8202
     1. \vdash v_0
         by static inversion forms
                                                                                             8203
     2. QED
                                                                                             8204
   CASE \delta(\operatorname{snd}, \langle v_0, v_1 \rangle) = v_1:
                                                                                             8205
                                                                                             8206
     1. \vdash v_1
         by static inversion forms
                                                                                             8207
                                                                                             8208
     2. QED
   CASE \delta(\text{sum}, v_0, v_1) = v_0 + v_1:
                                                                                             8209
                                                                                             8210
     1. QED
   CASE \delta(\text{quotient}, v_0, v_1) = \lfloor v_0/v_1 \rfloor:
                                                                                             8211
                                                                                             8212
   CASE \delta(quotient, v_0, v_1) = BndryErr :
                                                                                             8213
                                                                                             8214
     1. QED
                                                                                             8215
                                                                                             8216
Lemma 4.57: weakening
                                                                                             8217
• If \Gamma \vdash e then x, \Gamma \vdash e
                                                                                             8218
  • If \Gamma \vdash e then (x:\tau), \Gamma \vdash e
Proof:
                                                                                             8219
   QED because e is closed under \Gamma
                                                                                             8220
                                                                                             8221
                                                                                             8222
                                                                                             8223
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                                                                                             8225
                                                                                             8226
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### A.5 (HC) Co-Natural Embedding A.5.1 Co-Natural Definitions Language HC $e = x \mid v \mid \langle e, e \rangle \mid e \mid op^1 \mid e \mid op^2 \mid e \mid e \mid$ dyn $\tau e \mid \text{stat } \tau e \mid \text{Err}$ $= i | \langle v, v \rangle | \lambda x. e | \lambda(x:\tau). e$ $mon(\tau \Rightarrow \tau)v \mid mon(\tau \times \tau)v$ = Nat | Int | $\tau \times \tau$ | $\tau \Rightarrow \tau$ $\Gamma = \cdot \mid x, \Gamma \mid (x:\tau), \Gamma$ Err = BndryErr | TagErr $r = v \mid Err$ $E^{\bullet} = [] | E^{\bullet} e | v E^{\bullet} | \langle E^{\bullet}, e \rangle | \langle v, E^{\bullet} \rangle |$ $op^1 E^{\bullet} \mid op^2 E^{\bullet} e \mid op^2 v E^{\bullet}$ $E = E^{\bullet} \mid E \mid v \mid E \mid \langle E, e \rangle \mid \langle v, E \rangle \mid op^{1} \mid E \mid$ $op^2 E e \mid op^2 v E \mid dyn \tau E \mid stat \tau E$ $\Delta: op^1 \times \tau \longrightarrow \tau$ $\Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0$ $\Delta(\operatorname{snd}, \tau_0 \times \tau_1) = \tau_1$ $\Delta: op^2 \times \tau \times \tau \longrightarrow \tau$ $\Delta(op^2, Nat, Nat) = Nat$ $\Delta(op^2, Int, Int) = Int$ $\frac{\tau_d' \leqslant : \tau_d \quad \tau_c \leqslant : \tau_c'}{\mathsf{Nat} \leqslant : \mathsf{Int}} \quad \frac{\tau_d' \leqslant : \tau_d \quad \tau_c \leqslant : \tau_c'}{\tau_d \Rightarrow \tau_c \leqslant : \tau_d' \Rightarrow \tau_c'} \quad \frac{\tau_0 \leqslant : \tau_0' \quad \tau_1 \leqslant : \tau_1'}{\tau_0 \times \tau_1 \leqslant : \tau_0' \times \tau_1'}$ $\frac{\tau \leqslant : \tau' \quad \tau' \leqslant : \tau''}{\tau \leqslant : \tau''}$ $\Gamma \vdash e$ $\frac{x \in \Gamma}{\Gamma \vdash x} \quad \frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x. e} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}$ $\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0 \ e_1} \quad \frac{\Gamma \vdash e}{\Gamma \vdash op^1 \ e} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2 \ e_0 \ e_1} \quad \frac{\Gamma \vdash \mathsf{Err}}{\Gamma \vdash \mathsf{Err}}$ $\Gamma$ + stat $\tau$ e

```
\Gamma \vdash e : \tau
                (x:\tau)\in\Gamma
                                                            (x:\tau_d), \Gamma \vdash e:\tau_c \qquad \qquad i \in \mathbb{N}
                 \Gamma \vdash x : \tau \Gamma \vdash \lambda(x : \tau_d) \cdot e : \tau_d \Rightarrow \tau_c \Gamma \vdash i : \text{Nat}
                                                                       \Gamma \vdash e_0 : \tau_0 \qquad \qquad \Gamma \vdash e_0 : \tau_d \Longrightarrow \tau_c
               \frac{\Gamma \vdash e_1 : \tau_1}{\Gamma \vdash i : \mathsf{Int}} \quad \frac{\Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1} \quad \frac{\Gamma \vdash e_1 : \tau_d}{\Gamma \vdash e_0 \; e_1 : \tau_c}
                                                                   \Gamma \vdash e_0 : \tau_0
       \Gamma \vdash e_0 : \tau_0
                                                                  \Gamma \vdash e_1 : \tau_1 \qquad \Gamma \vdash e : \tau'
 \frac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn} \ \tau \ e : \tau}
 \Gamma \vdash_{\mathsf{C}} e
                       \frac{x \in \Gamma}{\Gamma \vdash_{\Gamma} x} \quad \frac{x, \Gamma \vdash_{\Gamma} e}{\Gamma \vdash_{\Gamma} \lambda x. e} \quad \frac{\Gamma \vdash_{\Gamma} e_0 \quad \Gamma \vdash_{\Gamma} e_1}{\Gamma \vdash_{\Gamma} \langle e_0, e_1 \rangle}
   \Gamma \vdash_{\Gamma} e : \tau
                                                            (x:\tau_d), \Gamma \vdash_{\mathbb{C}} e:\tau_c
              \overline{\Gamma \vdash_{C} x : \tau} \qquad \overline{\Gamma \vdash_{C} \lambda(x : \tau_{d}). e : \tau_{d} \Rightarrow \tau_{c}} \qquad \overline{\Gamma \vdash_{C} i : \text{Nat}}
           \frac{\Gamma \vdash_{\mathbb{C}} e_0 : \tau_0}{\Gamma \vdash_{\mathbb{C}} i : \mathsf{Int}} \quad \frac{\Gamma \vdash_{\mathbb{C}} e_0 : \tau_0}{\Gamma \vdash_{\mathbb{C}} e_0, e_1 \rangle : \tau_0 \times \tau_1} \quad \frac{\Gamma \vdash_{\mathbb{C}} e_0 : \tau_d \Rightarrow \tau_c}{\Gamma \vdash_{\mathbb{C}} e_1 : \tau_d} \\ \frac{\Gamma \vdash_{\mathbb{C}} e_0 : \tau_d \Rightarrow \tau_c}{\Gamma \vdash_{\mathbb{C}} e_0 : \tau_c}
                     \begin{array}{cccc} & \Gamma \vdash_{\mathbb{C}} e_0 : \tau_0 \\ \Gamma \vdash_{\mathbb{C}} e_0 : \tau_0 & \Gamma \vdash_{\mathbb{C}} e_1 : \tau_1 & \Gamma \vdash_{\mathbb{C}} e : \tau' \\ \Delta(op^1, \tau_0) = \tau & \Delta(op^2, \tau_0, \tau_1) = \tau & \tau' <: \tau \end{array}
                      \overline{\Gamma \vdash_{\mathbb{C}} op^{1} e_{0} : \tau} \quad \overline{\Gamma \vdash_{\mathbb{C}} op^{2} e_{0} e_{1} : \tau} \quad \overline{\Gamma \vdash_{\mathbb{C}} e : \tau}
                                                \frac{\Gamma \vdash_{\mathbb{C}} \mathsf{Err} : \tau}{\Gamma \vdash_{\mathbb{C}} \mathsf{Err} : \tau} \quad \frac{\Gamma \vdash_{\mathbb{C}} e}{\Gamma \vdash_{\mathbb{C}} \mathsf{dyn} \; \tau \; e : \tau}
                                                  \frac{\Gamma \vdash_{\mathbb{C}} v}{\Gamma \vdash_{\mathbb{C}} \mathsf{mon} \left(\tau_0 \times \tau_1\right) v : \left(\tau_0 \times \tau_1\right)}
                                              \frac{\Gamma \vdash_{\mathbb{C}} v}{\Gamma \vdash_{\mathbb{C}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) v : (\tau_d \Rightarrow \tau_c)}
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8361
               \delta(op^1, v) = e
8362
                \delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0
8363
                \delta(\operatorname{snd}, \langle v_0, v_1 \rangle) = v_1
8364
               \delta(op^2, v, v) = e
8365
                \delta(\text{sum}, i_0, i_1)
                                                    = i_0 + i_1
8366
8367
                \delta(\text{quotient}, i_0, 0) = \text{BndryErr}
                \delta(\text{quotient}, i_0, i_1) = \lfloor i_0/i_1 \rfloor
8368
                     if i_1 \neq 0
8369
8370
               \mathcal{D}_{\mathsf{C}}: \tau \times v \longrightarrow e
8371
                \mathcal{D}_{\mathsf{C}}(\tau_d \Rightarrow \tau_c, v) = \mathsf{mon}\left(\tau_d \Rightarrow \tau_c\right) v
8372
                    if v = \lambda x. e or v = mon(\tau'_d \Rightarrow \tau'_c)v'
8373
                \mathcal{D}_{\mathsf{C}}(\tau_0 \times \tau_1, v) = \mathsf{mon}(\tau_0 \times \tau_1) v
8374
                     if v = \langle v_0, v_1 \rangle or v = \text{mon}(\tau_0' \times \tau_1') v'
8375
                \mathcal{D}_{\mathsf{C}}(\mathsf{Int},i)
                                               = i
8376
                \mathcal{D}_{C}(Nat, i)
8377
                    if i \in \mathbb{N}
8378
                \mathcal{D}_{\mathsf{C}}(\tau,v)
                                                = BndryErr
8379
                     otherwise
8380
               S_{C}: \tau \times v \longrightarrow e
8381
8382
                S_{\rm C}(\tau_d \Rightarrow \tau_c, v) = \text{mon}(\tau_d \Rightarrow \tau_c) v
                S_{C}(\tau_0 \times \tau_1, v) = mon(\tau_0 \times \tau_1) v
8383
8384
                S_{\rm C}(\tau,v)
                                               = v
                     otherwise
8385
8386
               e \triangleright_{S-C} e
8387
                dyn \tau v
                                                             \triangleright_{S-C} \mathcal{D}_C(\tau,v)
                (\text{mon}(\tau_d \Rightarrow \tau_c) v_f) v \triangleright_{S-C} \text{dyn } \tau_c (v_f e')
8389
                     where e' = \text{stat } \tau_d \ v
8390
                (\lambda(x:\tau).e)v
                                                            \triangleright_{S-C} e[x \leftarrow v]
8391
                fst (mon (\tau_0 \times \tau_1) v) >_{S-C} dyn \tau_0 (fst v)
                \operatorname{snd}(\operatorname{mon}(\tau_0 \times \tau_1) v) \triangleright_{S-C} \operatorname{dyn} \tau_1(\operatorname{snd} v)
8393
                op^1 v
                                                            \triangleright_{S-C} \delta(op^1, v)
8394
                op^2 v_0 v_1
                                                            \triangleright_{S-C} \delta(op^2, v_0, v_1)
8395
               e \rhd_{\mathsf{D-C}} e
8396
                                                             \rhd_{\mathsf{D-C}} \mathcal{S}_{\mathsf{C}}(\tau,v)
               stat \tau v
8397
                v_0 v_1
                                                             ⊳<sub>D-C</sub> TagErr
8398
                     if v_0 \in \mathbb{Z} or v_0 = \langle v, v' \rangle
8399
                (\text{mon}(\tau_d \Rightarrow \tau_c) v_f) v \triangleright_{D-C} \text{stat } \tau_c (v_f e')
8400
                     where e' = \text{dyn } \tau_d v
8401
                (\lambda x. e) v
                                                             \triangleright_{D-C} e[x \leftarrow v]
8402
                fst (mon (\tau_0 \times \tau_1) v) >_{D-C} stat \tau_0 (fst v)
8403
               \operatorname{snd} \left( \operatorname{mon} \left( \tau_0 \times \tau_1 \right) v \right) >_{\mathsf{D-C}} \operatorname{stat} \tau_1 \left( \operatorname{snd} v \right)
8404
                op^1 v
                                                            \rhd_{D\text{-}C} TagErr
8405
                     if \delta(op^1, v) is undefined
8406
                op^1 v
                                                            \triangleright_{\mathsf{D-C}} \delta(\mathit{op}^1, v)
8407
                op^2 v_0 v_1
                                                            ⊳<sub>D-C</sub> TagErr
8408
                    if \delta(op^2, v_0, v_1) is undefined
                op^2 v_0 v_1
                                                            \triangleright_{\mathsf{D-C}} \delta(op^2, v_0, v_1)
8410
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e \rightarrow_{C-S} e
 E^{\bullet}[e]
                                      \rightarrow_{\text{C-S}} E^{\bullet}[e']
       if e \triangleright_{S-C} e'
 E[\operatorname{stat} \tau E^{\bullet}[e]] \rightarrow_{C-S} E[\operatorname{stat} \tau E^{\bullet}[e']]
       if e \triangleright_{S-C} e'
 E[\operatorname{dyn} \tau E^{\bullet}[e]] \rightarrow_{C-S} E[\operatorname{dyn} \tau E^{\bullet}[e']]
        if e \triangleright_{D-C} e'
  E[Err]
                                       \rightarrow_{C-S} Err
e \rightarrow_{\text{C-D}} e
  E^{\bullet}[e]
                                       \rightarrow_{C-D} E^{\bullet}[e']
       if e \triangleright_{\mathsf{D-C}} e'
  E[\operatorname{stat} \tau E^{\bullet}[e]] \rightarrow_{\mathsf{C-D}} E[\operatorname{stat} \tau E^{\bullet}[e']]
        if e \triangleright_{S-C} e'
  E[\operatorname{dyn} \tau E^{\bullet}[e]] \rightarrow_{C-D} E[\operatorname{dyn} \tau E^{\bullet}[e']]
       if e \triangleright_{\mathsf{D-C}} e'
  E[Err]
                                       \rightarrow_{C-D} Err
 e \rightarrow_{C-S}^* e reflexive, transitive closure of \rightarrow_{C-S}
e \rightarrow_{\text{C-D}}^* e | reflexive, transitive closure of \rightarrow_{\text{C-D}}
```

8471	A.5.2 Co-Natural Theorems	852
8472	<b>Theorem 5.0</b> : static HC soundness	852
8473	If $\vdash e : \tau$ then $\vdash_{C} e : \tau$ and one of the following holds:	852
8474	$\bullet \ e \rightarrow_{C-S}^* v \text{ and } \vdash_C v : \tau$	852
8475	• $e \to_{C-S}^* E[dyn \tau' e']$ and $e' \rhd_{D-C} TagErr$	853
8476 8477	• $e \rightarrow_{C-S}^* BndryErr$	853 853
8478	• e diverges	853
8479	Proof:	853
8480	1. $\vdash_{C} e : \tau$	853
8481	by static subset 2. QED by static progress and static preservation.	853
8482		853
8483	Theorem 5.1: dynamic HC-soundness	853
8484	If $\vdash$ <i>e</i> then $\vdash$ <sub>C</sub> <i>e</i> and one of the following holds:	853
8485	• $e \to_{C-D}^* v$ and $\vdash_C v$	854
8486	• $e \rightarrow_{\text{C-D}}^{\text{-B}} E[e']$ and $e' \triangleright_{\text{D-C}} \text{TagErr}$	854
8487	• $e \rightarrow_{C-D}^{C-D}$ BndryErr	854
8488	• e diverges	854
8489	Proof:	854
8490	1. ├ <sub>C</sub> e	854
8491	by dynamic subset	854 854
8492 8493	2. QED by dynamic progress and dynamic preservation.  □	854
8494		854
8495	<b>Corollary 5.2</b> : HC <i>static soundness</i> If $\vdash e : \tau$ and $e$ is boundary-free, then one of the following	855
8496	holds:	855
8497	• $e \to_{C-S}^* v$ and $\vdash_{C} v : \tau$	855
8498	• $e \rightarrow_{C-S}^{C-S}$ BndryErr	855
8499	• e diverges	855
8500	Proof:	855
8501	Consequence of the proof for <i>static</i> HC- <i>soundness</i>	855
8502		855
8503 8504		855 855
8505		856
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8515 8516		857 857
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8525	78	858

```
If \vdash_{C} e : \tau then one of the following holds:
8581
           A.5.3 Co-Natural Lemmas
                                                                                                                                                                                                                                             8636
                                                                                                                               • e is a value
8582
                                                                                                                                                                                                                                             8637
          Lemma 5.3 : \mathcal{D}_C soundness
                                                                                                                               • e \in Err
8583
                                                                                                                                                                                                                                             8638
           If \vdash_{\mathsf{C}} v then \vdash_{\mathsf{C}} \mathcal{D}_{\mathsf{C}}(\tau, v) : \tau
                                                                                                                               • e \rightarrow_{C-S} e'
8584
                                                                                                                                                                                                                                             8639
           Proof:
                                                                                                                               • e \rightarrow_{C-S} BndryErr
8585
              CASE \mathcal{D}_{C}(\tau_{d} \Rightarrow \tau_{c}, v) = \text{mon}(\tau_{d} \Rightarrow \tau_{c}) v:
                                                                                                                               • e = E[\text{dyn } \tau' e'] \text{ and } e' \rightarrow_{\text{C-D}} \text{TagErr}
                                                                                                                                                                                                                                             8641
8586
                 1. \vdash_{C} \text{mon} (\tau_d \Rightarrow \tau_c) v : \tau_d \Rightarrow \tau_c
8587
                     by \vdash_{\mathbb{C}} v
                                                                                                                                By the boundary factoring lemma, there are seven possi-
8588
                                                                                                                                                                                                                                             8643
                                                                                                                                ble cases.
               CASE \mathcal{D}_{C}(\tau_0 \times \tau_1, v) = \text{mon}(\tau_0 \times \tau_1) v:
                                                                                                                                CASE e is a value :
8590
                                                                                                                                                                                                                                             8645
                 1. \vdash_{\mathsf{C}} \mathsf{mon} (\tau_0 \times \tau_1) v : \tau_0 \times \tau_1
                                                                                                                                   1. QED
                                                                                                                                                                                                                                             8646
                     by \vdash_{\mathbb{C}} v
                                                                                                                                CASE e = E^{\bullet}[v_0 \ v_1]:
8592
                                                                                                                                                                                                                                             8647
                 2. QED
                                                                                                                                   1. \vdash_{C} v_0 \ v_1 : \tau'
8593
                                                                                                                                                                                                                                             8648
              CASE v = i
                                                                                                                                       by static hole typing
8594
                                                                                                                                                                                                                                             8649
                          \wedge \mathcal{D}_{\mathsf{C}}(\mathsf{Int},v) = v:
                                                                                                                                   2. \vdash_{\mathsf{C}} v_0 : \tau_d \Rightarrow \tau_c
8595
                                                                                                                                                                                                                                             8650
                 1. QED
                                                                                                                                        \wedge \vdash_{\mathsf{C}} v_1 : \tau_d
8596
                                                                                                                                                                                                                                             8651
               case v \in \mathbb{N}
                                                                                                                                       by inversion
8597
                                                                                                                                                                                                                                             8652
                          \wedge \mathcal{D}_{\mathsf{C}}(\mathsf{Nat}, v) = v:
                                                                                                                                   3. v_0 = \lambda(x : \tau_d). e'
8598
                                                                                                                                                                                                                                             8653
                                                                                                                                        \forall v_0 = \text{mon}(\tau_d' \Rightarrow \tau_c') v_f
8599
                                                                                                                                                                                                                                             8654
              CASE \mathcal{D}_{C}(\tau, v) = BndryErr:
                                                                                                                                        by canonical forms
8600
                 1. QED
                                                                                                                                   4. IF v_0 = \lambda(x : \tau_d). e':
8601
                                                                                                                                                                                                                                             8656
                                                                                                                                          a. e \rightarrow_{\text{C-S}} E^{\bullet}[e'[x \leftarrow v_1]]
8602
                                                                                                                                                                                                                                             8657
           Lemma 5.4 : S_C soundness
                                                                                                                                               by v_0 \ v_1 \triangleright_{S-C} e'[x \leftarrow v_1]
8603
                                                                                                                                                                                                                                             8658
           If \vdash_{\mathsf{C}} v : \tau then \vdash_{\mathsf{C}} \mathcal{S}_{\mathsf{C}}(\tau, v)
                                                                                                                                          b. QED
8604
                                                                                                                                        ELSE v_0 = \text{mon}(\tau_d' \Rightarrow \tau_c') v_f:
           Proof:
8605
                                                                                                                                                                                                                                             8660
                                                                                                                                          a. e \rightarrow_{C-S} E^{\bullet}[\operatorname{dyn} \tau'_c (v_f (\operatorname{stat} \tau'_d v_1))]
              CASE \vdash_{C} v : \tau_{d} \Rightarrow \tau_{c}
8606
                          \wedge S_{C}(\tau_{d} \Rightarrow \tau_{c}, v) = \text{mon}(\tau_{d} \Rightarrow \tau_{c}) v:
                                                                                                                                               by v_0 \ v_1 \rhd_{S-C} \operatorname{dyn} \tau'_c (v_f (\operatorname{stat} \tau'_d v_1))
8607
                                                                                                                                                                                                                                             8662
                                                                                                                                          b. QED
                 1. QED
8608
                                                                                                                                                                                                                                             8663
                                                                                                                                CASE e = E^{\bullet}[op^1 v]:
              CASE \vdash_{C} v : \tau_0 \times \tau_1
8609
                                                                                                                                                                                                                                             8664
                                                                                                                                   1. \vdash_{\mathsf{C}} op^1 \ v : \tau'
                          \wedge \mathcal{S}_{\mathsf{C}}(\tau_0 \times \tau_1, v) = \mathsf{mon}(\tau_0 \times \tau_1) v :
8610
                                                                                                                                                                                                                                             8665
                 1. QED
                                                                                                                                       by static hole typing
8611
                                                                                                                                                                                                                                             8666
               CASE \vdash_{C} v : Int
                                                                                                                                   2. \vdash_{\mathsf{C}} v : \tau_0 \times \tau_1
                                                                                                                                                                                                                                             8667
                          \wedge S_{\mathcal{C}}(\mathsf{Int},v) = v:
                                                                                                                                        by inversion
8613
                                                                                                                                                                                                                                             8668
                                                                                                                                   3. v = \langle v_0, v_1 \rangle
                 1. QED
8614
                                                                                                                                                                                                                                             8669
                                                                                                                                        \forall v = mon(\tau_0 \times \tau_1)v'
              CASE \vdash_{\mathbf{C}} v : \mathsf{Nat}
8615
                                                                                                                                                                                                                                             8670
                          \wedge S_{C}(Nat, v) = v:
                                                                                                                                        by canonical forms
8616
                                                                                                                                                                                                                                             8671
                                                                                                                                   4. IF v = \langle v_0, v_1 \rangle
                 1. QED
8617
                                                                                                                                                                                                                                             8672
                                                                                                                                              \wedge op^1 = fst:
8618
                                                                                                                                                                                                                                             8673
                                                                                                                                          a. \delta(op^1, \langle v_0, v_1 \rangle) = v_0
           Corollary 5.5 : HC static subset
8619
                                                                                                                                               by definition
           If \Gamma \vdash e : \tau then \Gamma \vdash_{\Gamma} e : \tau.
8620
                                                                                                                                                                                                                                             8675
                                                                                                                                          b. e \rightarrow_{\mathsf{C-S}} E^{\bullet}[v_0]
           Proof:
8621
                                                                                                                                                                                                                                             8676
                                                                                                                                               by op^1 v \triangleright_{S-C} v_0
              Consequence of the proof for the higher-order static
8622
                                                                                                                                                                                                                                             8677
                                                                                                                                          c. QED
              subset lemma; both \vdash_{C} and \vdash_{H} have the same typing rules
                                                                                                                                        IF v = \langle v_0, v_1 \rangle
              for surface-language expressions.
8624
                                                                                                                                                                                                                                             8679
                                                                                                                                              \wedge op^1 = \text{snd}:
8625
                                                                                                                                                                                                                                             8680
                                                                                                                                          a. \delta(op^1, \langle v_0, v_1 \rangle) = v_1
           Corollary 5.6: HC dynamic subset
8626
                                                                                                                                                                                                                                             8681
                                                                                                                                               by definition
8627
           If \Gamma \vdash e then \Gamma \vdash_{\overline{C}} e.
                                                                                                                                                                                                                                             8682
                                                                                                                                          b. e \rightarrow_{C-S} E^{\bullet}[v_1]
8628
           Proof:
                                                                                                                                                                                                                                             8683
                                                                                                                                               by op^1 v \triangleright_{S-C} v_1
8629
              Consequence of the proof for the higher-order dynamic
                                                                                                                                                                                                                                             8684
                                                                                                                                          c. QED
8630
                                                                                                                                                                                                                                             8685
              subset lemma.
                                                                                                                                        IF v = mon(\tau_0 \times \tau_1) v'
8631
                                                                                                                                                                                                                                             8686
                                                                                                                                              \wedge op^1 = fst:
8632
                                                                                                                                                                                                                                             8687
           Lemma 5.7: HC static progress
                                                                                                                                          a. e \rightarrow_{C-S} E^{\bullet}[\operatorname{dyn} \tau_0 (op^1 v')]
8633
                                                                                                                                                                                                                                             8688
                                                                                                                                               by definition
8634
                                                                                                                                                                                                                                             8689
8635
                                                                                                                                                                                                                                             8690
                                                                                                                      79
```

```
ELSE e' = E''[\text{dyn } \tau'' E^{\bullet ''}[e'']] \text{ and } e'' \triangleright_{D-C} \text{TagErr}
8691
                      b. oed
                                                                                                                                                                                                                        8746
                    ELSE v = mon(\tau_0 \times \tau_1) v'
8692
                                                                                                                                                                                                                        8747
                              \wedge op^1 = \text{snd}:
8693
                                                                                                                              a. Contradiction by e' is boundary-free
                                                                                                                                                                                                                        8748
                      a. e \rightarrow_{C-S} E^{\bullet}[\mathsf{dyn} \ \tau_1 \ (op^1 \ v')]
                                                                                                                     CASE e = E[Err]:
                                                                                                                                                                                                                        8749
8694
8695
                          by definition
                                                                                                                       1. QED e \rightarrow_{C-S} Err
                                                                                                                                                                                                                        8750
                                                                                                                 П
                      b. QED
                                                                                                                                                                                                                        8751
8696
             CASE e = E^{\bullet}[op^2 v_0 v_1]:
8697
                                                                                                                                                                                                                        8752
                                                                                                                 Lemma 5.8: HC dynamic progress
                1. \vdash_{C} op^{2} v_{0} v_{1} : \tau'
                                                                                                                                                                                                                        8753
8698
                                                                                                                    If \vdash_{C} e then one of the following holds:
                    by static hole typing
                                                                                                                                                                                                                        8754
                                                                                                                    • e is a value
8700
                2. \vdash_{C} v_0 : \tau_0
                                                                                                                    • e \in Err
                                                                                                                                                                                                                        8755
8701
                    \wedge \vdash_{\mathsf{C}} v_1 : \tau_1
                                                                                                                                                                                                                        8756
                                                                                                                    • e \rightarrow_{C-D} e'
                    \wedge \Delta(op^2, \tau_0, \tau_1) = \tau^{\prime\prime}
                                                                                                                    • e \rightarrow_{C-D} BndryErr
8702
                                                                                                                                                                                                                        8757
8703
                    by inversion
                                                                                                                                                                                                                        8758
                                                                                                                    • e \rightarrow_{C-D} TagErr
                                                                                                                  Proof:
               3. \delta(op^2, v_0, v_1) = e'
8704
                                                                                                                                                                                                                        8759
8705
                    by \Delta type soundness (2)
                                                                                                                     By the boundary factoring lemma, there are seven cases.
                                                                                                                                                                                                                        8760
8706
                4. op^2 v_0 v_1 \triangleright_{S-C} e'
                                                                                                                     CASE e is a value :
                                                                                                                                                                                                                        8761
                                                                                                                       1. QED
8707
                    by (3)
                                                                                                                                                                                                                        8762
                                                                                                                     CASE e = E^{\bullet}[v_0 \ v_1]:
8708
               5. QED by e \rightarrow_{C-S} E^{\bullet}[e']
                                                                                                                                                                                                                        8763
                                                                                                                        IF v_0 = \lambda x. e':
             CASE e = E[dyn \tau' e'] and e' is boundary-free :
8709
                                                                                                                                                                                                                        8764
                                                                                                                           1. e \rightarrow_{C-D} E^{\bullet}[e'[x \leftarrow v_1]]
8710
               1. e' is a value
                                                                                                                                                                                                                        8765
8711
                    \forall e' \in Err
                                                                                                                               by v_0 v_1 \triangleright_{\mathsf{D-C}} e'[x \leftarrow v_1]
                                                                                                                                                                                                                        8766
                    \vee e' \rightarrow_{C-D} e''
                                                                                                                           2. QED
8712
                                                                                                                                                                                                                        8767
                                                                                                                        IF v_0 = \text{mon}(\tau_d \Rightarrow \tau_c) v_f:
                    \lor e' \rightarrow_{C-D} BndryErr
                                                                                                                                                                                                                        8768
8713
8714
                    \vee e' = E'[e''] and e'' \triangleright_{D-C} \mathsf{TagErr}
                                                                                                                           1. e \rightarrow_{C-D} E^{\bullet}[\text{stat } \tau_c (v_f (\text{dyn } \tau_d v_1))]
                                                                                                                                                                                                                        8769
                    by dynamic progress
                                                                                                                               by v_0 \ v_1 \rhd_{D-C} stat \tau_c \ (v_f \ (dyn \ \tau_d \ v_1))
8715
                                                                                                                                                                                                                        8770
8716
               2. IF e' is a value :
                                                                                                                          2. OED
                                                                                                                                                                                                                        8771
8717
                      a. QED e \rightarrow_{C-S} E[\mathcal{D}_C(\tau', e')]
                                                                                                                        ELSE v_0 = i
                                                                                                                                                                                                                        8772
                                                                                                                                   \vee v_0 = \langle v, v' \rangle:
                    IF e' \in Err:
                                                                                                                                                                                                                        8773
                                                                                                                           1. e \rightarrow_{C-D} TagErr
                      a. QED e \rightarrow_{C-S} e'
8719
                                                                                                                                                                                                                        8774
                                                                                                                               by (v_0 \ v_1) \triangleright_{D-C} \mathsf{TagErr}
8720
                    IF e' \rightarrow_{C-D} e'':
                                                                                                                                                                                                                        8775
8721
                      a. QED e \rightarrow_{C-S} E[dyn \tau' e'']
                                                                                                                           2. QED
                                                                                                                                                                                                                        8776
                    IF e' \rightarrow_{C-D} BndryErr :
                                                                                                                     CASE e = E^{\bullet}[op^1 v]:
                                                                                                                                                                                                                        8777
                                                                                                                        IF v = mon(\tau_0 \times \tau_1)v'
8723
                      a. QED e \rightarrow_{C-S} E[dyn \tau' BndryErr]
                                                                                                                                                                                                                        8778
                                                                                                                              \wedge op^1 = fst :
                    ELSE e' = E'[e''] and e'' \triangleright_{D-C} \mathsf{TagErr}:
8724
                                                                                                                                                                                                                        8779
                                                                                                                           1. e \rightarrow_{C-D} E^{\bullet}[\text{stat } \tau_0 \ op^1 \ v']
                      a. E' \in E^{\bullet}
8725
                                                                                                                                                                                                                        8780
                                                                                                                               by op^1 v \triangleright_{D-C} stat \tau_0 op^1 v'
8726
                          by e' is boundary-free
                                                                                                                                                                                                                        8781
8727
                                                                                                                           2. QED
                                                                                                                                                                                                                        8782
8728
             CASE e = E[\text{stat } \tau' \ e'] \text{ and } e' \text{ is boundary-free} :
                                                                                                                        IF v = mon(\tau_0 \times \tau_1) v'
                                                                                                                                                                                                                        8783
                                                                                                                              \wedge op^1 = \text{snd}:
               1. e' is a value
                    \forall e' \in Err
                                                                                                                           1. e \rightarrow_{C-D} E^{\bullet}[\text{stat } \tau_1 \ op^1 \ v']
                                                                                                                                                                                                                        8785
8730
                                                                                                                               by op^1 v \triangleright_{D-C} \operatorname{stat} \tau_1 op^1 v'
                    \vee e' \rightarrow_{C-S} e''
8731
                                                                                                                                                                                                                        8786
8732
                    \vee e' \rightarrow_{C-S} BndryErr
                                                                                                                          2. OED
                                                                                                                                                                                                                        8787
                    \vee e' = E''[\mathsf{dyn} \ \tau'' \ E^{\bullet''}[e'']] \ \mathsf{and} \ e'' \rhd_{\mathsf{D-C}} \mathsf{TagErr}
                                                                                                                        IF \delta(op^1, v) = e':
8733
                                                                                                                                                                                                                        8788
                                                                                                                          1. (op^1 v) \triangleright_{D-C} e'
8734
                    by static progress
                                                                                                                                                                                                                        8789
                                                                                                                           2. QED
8735
               2. IF e' is a value :
                                                                                                                        ELSE \delta(op^1, v) is undefined :
                      a. QED e \rightarrow_{C-S} E[S_C(\tau', e')]
8736
                                                                                                                                                                                                                        8791
                                                                                                                           1. e \rightarrow_{C-D} TagErr
                    IF e' \in Err:
                                                                                                                                                                                                                        8792
                                                                                                                               by (op^1 v) \triangleright_{D-C} TagErr
8738
                      a. QED e \rightarrow_{C-S} e'
                                                                                                                                                                                                                        8793
8739
                    IF e' \rightarrow_{C-S} e'':
                                                                                                                           2. QED
                                                                                                                                                                                                                        8794
                                                                                                                     CASE e = E^{\bullet}[op^2 v_0 v_1]:
8740
                      a. QED e \rightarrow_{C-S} E[\text{stat } \tau' e'']
                                                                                                                                                                                                                        8795
                                                                                                                        IF \delta(op^2, v_0, v_1) = e'':
                    IF e' \rightarrow_{C-S} BndryErr:
8741
                                                                                                                                                                                                                        8796
                                                                                                                          1. op^2 v_0 v_1 \rhd_{D-C} e''
8742
                      a. QED e \rightarrow_{C-S} E[\text{stat } \tau' \text{ BndryErr}]
                                                                                                                                                                                                                        8797
8743
                                                                                                                           2. QED
                                                                                                                                                                                                                        8798
                                                                                                                        ELSE \delta(op^2, v_0, v_1) is undefined :
8744
                                                                                                                                                                                                                        8799
8745
                                                                                                           80
                                                                                                                                                                                                                        8800
```

```
1. \vdash_{\mathsf{C}} v_0 \ v_1 : \tau'
8801
                     1. e \rightarrow_{C-D} TagErr
                                                                                                                                                                                                                                       8856
                         by op^2 v_0 v_1 \triangleright_{D-C} TagErr
                                                                                                                                        by static hole typing
8802
                                                                                                                                                                                                                                       8857
8803
                                                                                                                                   2. \vdash_{\mathsf{C}} v_0 : \tau_d \Rightarrow \tau_c
                                                                                                                                                                                                                                       8858
              CASE e = E[\text{dyn } \tau' \ e'] \text{ and } e' \text{ is boundary-free} :
8804
                                                                                                                                        \wedge \vdash_{\mathsf{C}} v_1 : \tau_d
                                                                                                                                                                                                                                       8859
                 1. e' is a value
                                                                                                                                        \wedge \tau_c \leqslant : \tau'
                     \forall e' \in Err
                                                                                                                                        by inversion
                                                                                                                                                                                                                                       8861
8806
                      \vee e' \rightarrow_{C-D} e''
8807
                                                                                                                                   3. \tau_d \leqslant : \tau_x
                                                                                                                                                                                                                                       8862
                      \lor e' \rightarrow_{C-D} BndryErr
                                                                                                                                        by canonical forms (2)
8808
                                                                                                                                                                                                                                       8863
                      \vee e' = E[e''] and e'' \triangleright_{D-C} TagErr
                                                                                                                                   4. (x:\tau_x) \vdash_C e' : \tau_c
8810
                     by dynamic progress
                                                                                                                                        by inversion (2)
                                                                                                                                                                                                                                       8865
8811
                 2. IF e' is a value :
                                                                                                                                   5. \vdash_{C} v_{1} : \tau_{x}
                                                                                                                                                                                                                                       8866
                       a. QED e \rightarrow_{C-D} E[\mathcal{D}_C(\tau', e')]
8812
                                                                                                                                        by (2, 3)
                                                                                                                                                                                                                                       8867
                     IF e' \in Err:
8813
                                                                                                                                   6. \vdash_{\mathcal{C}} e'[x \leftarrow v_1] : \tau_c
                                                                                                                                                                                                                                       8868
                       a. QED e \rightarrow_{C-D} e'
8814
                                                                                                                                        by substitution (4, 5)
                                                                                                                                                                                                                                       8869
8815
                     IF e' \rightarrow_{C-D} e'':
                                                                                                                                   7. \vdash_{\mathsf{C}} e'[x \leftarrow v_1] : \tau'
                                                                                                                                                                                                                                       8870
8816
                        a. QED e \rightarrow_{C-S} E[dyn \tau' e'']
                                                                                                                                        by (2, 6)
                                                                                                                                                                                                                                       8871
                     IF e' \rightarrow_{C-D} BndryErr :
8817
                                                                                                                                   8. QED by hole substitution (7)
                                                                                                                                                                                                                                       8872
                                                                                                                                 ELSE v_0 = mon(\tau_d \Rightarrow \tau_c) v_f
                       a. QED e \rightarrow_{C-D} E[dyn \tau' BndryErr]
8818
                                                                                                                                                                                                                                       8873
                     ELSE e' = E[e''] and e'' \triangleright_{D-C} TagErr:
                                                                                                                                            \land e \rightarrow_{\text{C-S}} E^{\bullet}[\text{dyn } \tau_c (v_f (\text{stat } \tau_d v_1))]:
8819
                                                                                                                                                                                                                                       8874
                        a. E \in E^{\bullet}
                                                                                                                                   1. \vdash_{C} v_0 \ v_1 : \tau'
                                                                                                                                                                                                                                       8875
8821
                            by e' is boundary-free
                                                                                                                                        by static hole typing
                                                                                                                                                                                                                                       8876
                                                                                                                                   2. \vdash_{\mathsf{C}} v_0 : \tau_d' \Rightarrow \tau_c'
8822
                                                                                                                                                                                                                                       8877
              CASE e = E[\text{stat } \tau' \ e'] \text{ and } e' \text{ is boundary-free} :
                                                                                                                                       8878
8823
                 1. e' is a value
                     \forall e' \in Err
8825
                                                                                                                                        by inversion
                                                                                                                                                                                                                                       8880
                      \vee e' \rightarrow_{C-S} e''
                                                                                                                                   3. ⊢<sub>C</sub> v<sub>f</sub>
                                                                                                                                                                                                                                       8881
                      \vee e' \rightarrow_{C-S} BndryErr
                                                                                                                                        by inversion (2)
8827
                                                                                                                                                                                                                                       8882
                      \vee e' = E''[\mathsf{dyn} \ \tau'' \ E^{\bullet''}[e'']] \ \mathsf{and} \ e'' \rhd_{\mathsf{D-C}} \mathsf{TagErr}
                                                                                                                                   4. \tau_d \Rightarrow \tau_c \leqslant : \tau_d' \Rightarrow \tau_c'
                     by static progress
                                                                                                                                        by canonical forms (2)
8829
                                                                                                                                                                                                                                       8884
8830
                 2. IF e' is a value :
                                                                                                                                                                                                                                       8885
                                                                                                                                   5. \tau'_d \leqslant : \tau_d
8831
                        a. QED e \rightarrow_{C-S} E[S_C(\tau', e')]
                                                                                                                                        \wedge \tau_c \leqslant : \tau'_c
                                                                                                                                                                                                                                       8886
                     IF e' \in Err:
                                                                                                                                        by (4)
8833
                       a. QED e \rightarrow_{C-S} e'
                                                                                                                                                                                                                                       8888
                                                                                                                                   6. \vdash_{\mathsf{C}} v_1 : \tau_d
                     IF e' \rightarrow_{C-S} e'':
8834
                                                                                                                                        by (2, 5)
                                                                                                                                                                                                                                       8889
                                                                                                                                   7. \vdash_{\mathsf{C}} stat \tau_d \ v_1
                       a. QED e \rightarrow_{C-S} E[\text{stat } \tau' e'']
8835
                                                                                                                                                                                                                                       8890
                     IF e' \rightarrow_{C-S} BndryErr :
8836
                                                                                                                                       by (6)
                                                                                                                                                                                                                                       8891
8837
                       a. QED e \rightarrow_{C-S} E[\text{stat } \tau' \text{ BndryErr}]
                                                                                                                                   8. \vdash_{\mathsf{C}} v_f (\text{stat } \tau_d \ v_1)
                                                                                                                                                                                                                                       8892
                     ELSE e' = E''[\text{dyn } \tau'' E^{\bullet ''}[e'']] \text{ and } e'' \rhd_{D-C} \text{TagErr}
                                                                                                                                        by (3, 7)
8838
                                                                                                                                                                                                                                       8893
                                                                                                                                   9. \vdash_{\mathsf{C}} \mathsf{dyn} \ \tau_c \ (v_f \ (\mathsf{stat} \ \tau_d \ v_1)) : \tau_c
                       a. Contradiction by e' is boundary-free
                                                                                                                                        by (8)
                                                                                                                                                                                                                                       8895
8840
                                                                                                                                  10. \vdash_{\mathbb{C}} \mathsf{dyn} \ \tau_c \ (v_f \ (\mathsf{stat} \ \tau_d \ v_1)) : \tau'
              CASE e = E[Err]:
8841
                                                                                                                                                                                                                                       8896
8842
                 1. QED e \rightarrow_{C-D} Err
                                                                                                                                        by (2, 5, 9)
                                                                                                                                                                                                                                       8897
8843
                                                                                                                                 11. QED by hole substitution (10)
                                                                                                                             CASE e = E^{\bullet}[op^1 v]:
8844
                                                                                                                                                                                                                                       8899
           Lemma 5.9: HC static preservation
8845
                                                                                                                                 IF v = mon(\tau_0 \times \tau_1) v'
                                                                                                                                                                                                                                       8900
           If \vdash_{C} e : \tau and e \rightarrow_{C-S} e' then \vdash_{C} e' : \tau
                                                                                                                                       \wedge op^1 = fst
8846
                                                                                                                                                                                                                                       8901
           Proof:
8847
                                                                                                                                       \land e \rightarrow_{\text{C-S}} E^{\bullet}[\text{dyn } \tau_0 \text{ (fst } v')]:
                                                                                                                                                                                                                                       8902
              By the boundary factoring lemma there are seven cases.
8848
                                                                                                                                   1. \vdash_{\mathsf{C}} \mathsf{fst}\,v:\tau'
                                                                                                                                                                                                                                       8903
              CASE e is a value :
8849
                                                                                                                                        by static hole typing
                                                                                                                                                                                                                                       8904
                 1. Contradiction by e \rightarrow_{C-S} e'
8850
                                                                                                                                   2. \vdash_{\mathbf{C}} v : \tau_0' \times \tau_1'
                                                                                                                                                                                                                                       8905
              CASE e = E^{\bullet}[v_0 \ v_1]:
                                                                                                                                        \wedge \tau_0' <: \tau'
8851
                                                                                                                                                                                                                                       8906
                 IF v_0 = \lambda(x:\tau_x). e'
8852
                                                                                                                                        by inversion
                                                                                                                                                                                                                                       8907
                       \wedge e \rightarrow_{\text{C-S}} E^{\bullet}[e'[x \leftarrow v_1]]:
8853
                                                                                                                                   3. ⊢ v'
                                                                                                                                                                                                                                       8908
                                                                                                                                        by inversion (2)
8854
                                                                                                                                                                                                                                       8909
                                                                                                                                                                                                                                       8910
8855
                                                                                                                   81
```

8911	4. $\tau_0 \times \tau_1 \leqslant \tau_0' \times \tau_1'$	$4. \vdash_{C} v_1 : \tau'$	8966
8912	by canonical forms (2)	by (2, 3)	8967
8913	5. $\tau_0 \leqslant \tau_0'$	5. QED by hole substitution (4)	8968
8914	6. $\vdash_{\mathbf{C}} fst v'$	CASE $e = E^{\bullet}[op^2 v_0 v_1]$ :	8969
8915	by (3)	1. $e \to_{C-S} E^{\bullet}[\delta(op^2, v_0, v_1)]$	8970
8916	7. $\vdash_{C} dyn \; \tau_0 \; (fst \; v') : \tau_0$	by $e \rightarrow_{C-S} e'$	8971
8917	by (6)	$2. \vdash_{C} op^2 v_0 v_1 : \tau'$	8972
8918	8. $\vdash_{C} dyn \ \tau_0 \ (fst \ v') : \tau'$	by static hole typing	8973
8919	by (2, 5, 7)	3. $\vdash_{\mathbf{C}} v_0 : \tau_0$	8974
8920	9. QED by hole substitution	$\wedge \vdash_{C} v_1 : \tau_1$	8975
8921	IF $v = \min(\tau_0 \times \tau_1) \langle v_0, v_1 \rangle$	$\wedge \Delta(op^2, \tau_0, \tau_1) = \tau''$	8976
8922	$\wedge op^1 = \text{snd}$	$\wedge \tau'' \leqslant \tau'$	8977
8923	$\wedge e \rightarrow_{C-S} E^{\bullet}[dyn \ \tau_0 \ (snd \ v')] :$	by inversion (1) $A = \sum_{i=1}^{\infty} (a_i a_i^2 + a_i a_i) + \frac{\pi}{2} (a_i a_i^2 + a_i) + \frac{\pi}{2} (a_i a_$	8978
8924 8925	1. $\vdash_{C} snd  v : \tau'$	4. $\vdash_{C} \delta(op^2, v_0, v_1) : \tau''$	8979 8980
8926	by static hole typing	by $\Delta$ type soundness (2) 5. $\vdash_{C} \delta(op^2, v_0, v_1) : \tau'$	8981
8927	2. $\vdash_{\mathcal{C}} \upsilon : \tau'_0 \times \tau'_1$ $\land \tau'_1 <: \tau'$	by $(2, 3)$	8982
8928	by inversion	6. QED by hole substitution (4)	8983
8929	3. $\vdash_{\mathcal{C}} v'$	<b>CASE</b> $e = E[\text{dyn } \tau' e']$ and $e'$ is boundary-free:	8984
8930	by inversion (2)	<b>IF</b> $e'$ is a value:	8985
8931	4. $\tau_0 \times \tau_1 \leqslant : \tau_0' \times \tau_1'$	1. $e \rightarrow_{C-S} E[\mathcal{D}_C(\tau', e')]$	8986
8932	by canonical forms (2)	$2. \vdash_{C} dyn \ \tau' \ e' : \tau'$	8987
8933	5. $\tau_1 \leqslant : \tau_1'$	by boundary hole typing	8988
8934	6. $\vdash_{C} snd v'$	3. F <sub>C</sub> e'	8989
8935	by (3)	by inversion (2)	8990
8936	7. $\vdash_{\mathbf{C}} \operatorname{dyn} \tau_0 (\operatorname{snd} v') : \tau_0$	4. $\vdash_{C} \mathcal{D}_{C}(\tau', e') : \tau'$	8991
8937	by (5)	by $\mathcal{D}_{C}$ soundness (3)	8992
8938	8. $\vdash_{C} dyn \ \tau_0 \ (snd \ v') : \tau'$	5. QED by hole substitution (4)	8993
8939	by (2, 5, 7)	ELSE $e' \rightarrow_{C-D} e''$ :	8994
8940	9. QED by hole substitution	1. $e \rightarrow_{C-S} E[dyn \tau' e'']$	8995
8941	$\mathbf{IF} \ \ v = \langle v_0, v_1 \rangle$	$2. \vdash_{C} dyn  \tau'  e' : \tau'$	8996
8942	$\wedge op^1 = fst$	by boundary hole typing	8997
8943	$\wedge e \rightarrow_{C-S} E^{\bullet}[v_0]:$	3. ⊦ <sub>C</sub> e'	8998
8944	1. $\vdash_{C} fst \langle v_0, v_1 \rangle : \tau'$	by inversion (2)	8999
8945	by static hole typing	4. ⊢ <sub>C</sub> e"	9000
8946	2. $\vdash_{C} \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$	by dynamic preservation (3)	9001
8947	$\wedge \tau_0 \leqslant : \tau'$	5. $\vdash_{\mathbf{C}} \operatorname{dyn} \tau' e'' : \tau'$	9002
8948	by inversion (1)	by (4)	9003
8949	3. $\vdash_{C} v_0 : \tau_0$	6. QED by hole substitution (5)	9004
8950	by inversion (2)	<b>CASE</b> $e = E[\text{stat } \tau' \ e'] \text{ and } e' \text{ is boundary-free} :$	9005
8951	$4. \vdash_{C} \upsilon_0 : \tau'$	IF $e'$ is a value :	9006
8952	by (2, 3)	1. $e \rightarrow_{C-S} E[S_C(\tau', e')]$	9007
8953	5. QED by hole substitution (4)	2. $\vdash_{C}$ stat $\tau'$ $e'$	9008
8954	ELSE $v = \langle v_0, v_1 \rangle$	by boundary hole typing	9009
8955	$\wedge op^1 = snd$	$3. \vdash_{C} e' : \tau'$	9010
8956	$\wedge e \rightarrow_{\text{C-S}} E^{\bullet}[v_1]:$	by inversion (2)	9011
8957	1. $\vdash_{C} snd \langle v_0, v_1 \rangle : \tau'$	$A. \vdash_{C} S_{C}(\tau',e')$	9012
8958	by static hole typing	by $S_{C}$ soundness (3)	9013
8959	2. $\vdash_{\mathbb{C}} \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$	5. QED by hole substitution (4)	9014
8960	$\wedge \tau_1 \leqslant : \tau'$	ELSE $e' \rightarrow_{C-S} e''$ :	9015
8961	by inversion (1)	1. $e \rightarrow_{\text{C-S}} E[\text{stat } \tau' e'']$	9016
8962	3. $\vdash_{C} v_1 : \tau_1$	2. $\vdash_{\mathbb{C}}$ stat $\tau'$ $e'$	9017
8963	by inversion (2)	by boundary hole typing	9018
8964		22	9019
8965		82	9020

```
2. \vdash_{\mathbb{C}} \text{mon} (\tau_0 \times \tau_1) v'
9021
                    3. \vdash_{\mathbf{C}} e' : \tau'
                                                                                                                                                                                                                                        9076
9022
                         by inversion (2)
                                                                                                                                        by inversion (1)
                                                                                                                                                                                                                                        9077
                                                                                                                                    3. \vdash_{\mathsf{C}} v' : \tau_0 \times \tau_1
9023
                    4. \vdash_{C} e'' : \tau'
                                                                                                                                                                                                                                        9078
                         by static preservation (3)
                                                                                                                                        by inversion (2)
9024
                                                                                                                                                                                                                                        9079
                    5. \vdash_{\mathbb{C}} stat \tau' e''
9025
                                                                                                                                    4. \vdash_{\mathsf{C}} fst v' : \tau_0
                        by (4)
                                                                                                                                        by (3)
                                                                                                                                                                                                                                        9081
9026
                                                                                                                                    5. \vdash_{\mathsf{C}} \mathsf{stat} \ \tau_0 \ (\mathsf{fst} \ v')
9027
                    6. QED by hole substitution (5)
                                                                                                                                                                                                                                        9082
9028
              CASE e = E[Err]:
                                                                                                                                        by (4)
                                                                                                                                                                                                                                        9083
                 1. e \rightarrow_{C-S} Err
                                                                                                                                   6. QED by hole substitution
                                                                                                                                                                                                                                        9084
9030
                 2. QED by \vdash_{\mathbb{C}} \mathsf{Err} : \tau
                                                                                                                                 IF v = mon(\tau_0 \times \tau_1) v'
                                                                                                                                                                                                                                        9085
          \wedge op^1 = \text{snd}
9031
                                                                                                                                                                                                                                        9086
                                                                                                                                            \land e \rightarrow_{C-D} E[\text{stat } \tau_0 (\text{snd } v')]:
9032
                                                                                                                                                                                                                                        9087
          Lemma 5.10: HC dynamic preservation
                                                                                                                                    1. \vdash_{\mathbb{C}} op^1 v
9033
                                                                                                                                                                                                                                        9088
          If \vdash_{\mathsf{C}} e and e \to_{\mathsf{C-D}} e' then \vdash_{\mathsf{C}} e'
9034
                                                                                                                                        by dynamic hole typing
                                                                                                                                                                                                                                        9089
          Proof:
9035
                                                                                                                                    2. \vdash_{\mathsf{C}} \mathsf{mon} (\tau_0 \times \tau_1) v'
                                                                                                                                                                                                                                        9090
             By the boundary factoring lemma, there are seven cases.
9036
                                                                                                                                        by inversion (1)
                                                                                                                                                                                                                                        9091
              CASE e is a value :
9037
                                                                                                                                    3. \vdash_{\mathcal{C}} v' : \tau_0 \times \tau_1
                                                                                                                                                                                                                                        9092
                 1. Contradiction by e \rightarrow_{C-D} e'
9038
                                                                                                                                        by inversion (2)
                                                                                                                                                                                                                                        9093
              CASE e = E^{\bullet}[v_0 \ v_1]:
                                                                                                                                    4. \vdash_{C} snd v' : \tau_1
9039
                                                                                                                                                                                                                                        9094
                 IF v_0 = \lambda x. e'
9040
                                                                                                                                        by (3)
                                                                                                                                                                                                                                        9095
                        \wedge e \rightarrow_{\mathsf{C-D}} E^{\bullet}[e'[x \leftarrow v_1]] :
9041
                                                                                                                                    5. \vdash_{\mathsf{C}} \operatorname{stat} \tau_1 (\operatorname{snd} v')
                                                                                                                                                                                                                                        9096
                    1. \vdash_{\mathcal{C}} v_0 \ v_1
                                                                                                                                        by (4)
9042
                                                                                                                                                                                                                                        9097
                         by dynamic hole typing
                                                                                                                                    6. QED by hole substitution
                                                                                                                                                                                                                                        9098
9043
                    2. \vdash_{\mathbb{C}} v_0
9044
                                                                                                                                 IF v = \langle v_0, v_1 \rangle
                                                                                                                                                                                                                                        9099
                         \wedge \vdash_{\mathsf{C}} v_1
                                                                                                                                       \wedge op^1 = fst
9045
                                                                                                                                                                                                                                        9100
                         by inversion (1)
                                                                                                                                       \wedge e \rightarrow_{\mathsf{C-D}} E^{\bullet}[v_0]:
                                                                                                                                                                                                                                        9101
                    3. x \vdash_{C} e'
9047
                                                                                                                                    1. \vdash_{\mathbb{C}} op^1 v
                                                                                                                                                                                                                                        9102
                         by inversion (2)
9048
                                                                                                                                        by dynamic hole typing
                                                                                                                                                                                                                                        9103
                    4. \vdash_{C} e'[x \leftarrow v_1]
9049
                                                                                                                                    2. ⊢ v
                                                                                                                                                                                                                                        9104
                         by substitution (2, 3)
9050
                                                                                                                                        by inversion (1)
                                                                                                                                                                                                                                        9105
                    5. QED hole substitution (4)
9051
                                                                                                                                    3. \vdash_{\mathcal{C}} v_0
                                                                                                                                                                                                                                        9106
                  ELSE v_0 = \text{mon}(\tau_d \Rightarrow \tau_c) v_f
                                                                                                                                        by inversion (2)
                                                                                                                                                                                                                                        9107
                            \wedge e \rightarrow_{C-D} E^{\bullet}[\text{stat } \tau_c (v_f (\text{dyn } \tau_d v_1))]:
9053
                                                                                                                                    4. QED by hole substitution
                                                                                                                                                                                                                                        9108
                    1. \vdash_{\mathcal{C}} v_0 \ v_1
                                                                                                                                 ELSE v = \langle v_0, v_1 \rangle
9054
                                                                                                                                                                                                                                        9109
                         by dynamic hole typing
9055
                                                                                                                                            \wedge op^1 = \text{snd}
                                                                                                                                                                                                                                        9110
                    2. ⊢<sub>C</sub> v<sub>0</sub>
                                                                                                                                            \wedge e \rightarrow_{\mathsf{C-D}} E^{\bullet}[v_1]:
9056
                                                                                                                                                                                                                                        9111
                         \wedge \vdash_{C} v_{1}
9057
                                                                                                                                    1. \vdash_{\mathsf{C}} op^1 v
                                                                                                                                                                                                                                        9112
                        by inversion (1)
9058
                                                                                                                                        by dynamic hole typing
                                                                                                                                                                                                                                        9113
                    3. \vdash_{\mathsf{C}} v_f : \tau_d \Rightarrow \tau_c
                                                                                                                                    2. ⊢ v
                                                                                                                                                                                                                                        9114
                         by inversion (2)
                                                                                                                                                                                                                                        9115
                                                                                                                                        by inversion (1)
9060
                    4. \vdash_{\mathsf{C}} \mathsf{dyn} \ \tau_d \ v_1 : \tau_d
                                                                                                                                                                                                                                        9116
9061
                                                                                                                                    3. \vdash_{\mathsf{C}} v_1
                        by (2)
9062
                                                                                                                                        by inversion (2)
                                                                                                                                                                                                                                        9117
                    5. \vdash_{C} v_f (\text{dyn } \tau_d \ v_1) : \tau_c
                                                                                                                                    4. QED by hole substitution
                                                                                                                                                                                                                                        9118
                         by (3, 4)
                                                                                                                              CASE e = E^{\bullet}[op^2 v_0 v_1]:
                                                                                                                                                                                                                                        9119
9064
                    6. \vdash_{C} stat \tau_{c} (v_{f} (dyn \tau_{d} v_{1}))
                                                                                                                                1. e \to_{\text{C-D}} E^{\bullet}[\delta(op^2, v_0, v_1)]
9065
                                                                                                                                                                                                                                        9120
                         by (5)
                                                                                                                                2. \vdash_{\mathcal{C}} op^2 v_0 v_1
9066
                                                                                                                                                                                                                                        9121
                    7. QED by hole substitution
9067
                                                                                                                                     by dynamic hole typing
                                                                                                                                                                                                                                        9122
              CASE e = E^{\bullet}[op^1 v]:
9068
                                                                                                                                3. \vdash_{\mathbf{C}} v_0
                                                                                                                                                                                                                                        9123
                  IF v = mon(\tau_0 \times \tau_1) v'
9069
                                                                                                                                                                                                                                        9124
                                                                                                                                     \wedge \vdash_{\mathbb{C}} v_1
                        \wedge op^1 = fst
9070
                                                                                                                                     by inversion (1)
                                                                                                                                                                                                                                        9125
                        \land e \rightarrow_{\mathsf{C-D}} E[\mathsf{stat} \ \tau_0 \ (\mathsf{fst} \ v')] :
                                                                                                                                4. \vdash_{\mathsf{C}} \delta(op^2, v_0, v_1)
9071
                                                                                                                                                                                                                                        9126
                    1. \vdash_{\mathbb{C}} op^1 v
9072
                                                                                                                                    by \delta preservation (2)
                                                                                                                                                                                                                                        9127
                         by dynamic hole typing
9073
                                                                                                                                5. QED by hole substitution (3)
                                                                                                                                                                                                                                        9128
                                                                                                                              CASE e = E[dyn \tau' e'] and e' is boundary-free:
                                                                                                                                                                                                                                        9129
9074
9075
                                                                                                                                                                                                                                        9130
                                                                                                                   83
```

9131	v o' is a value .	If $\vdash_{C} e : \tau$ then one of the following holds:	9186
9131	IF $e'$ is a value:	• e is a value	9187
	1. $e \rightarrow_{C-D} E[\mathcal{D}_C(\tau', e')]$	$\bullet$ $e = E^{\bullet}[v_0 \ v_1]$	
9133	2. $\vdash_{C} dyn \ \tau' \ e' : \tau'$	$\bullet \ e = E^{\bullet}[op^1 \ v]$	9188
9134	by boundary hole typing	$\bullet$ $e = E^{\bullet}[op^2 v_0 v_1]$	9189
9135	3. ⊢ <sub>C</sub> e'	• $e = E[\text{dyn } \tau \ e']$ where $e'$ is boundary-free	9190 9191
9136	by inversion (2) $(-1)^{-1}$	• $e = E[\text{stat } \tau e']$ where $e'$ is boundary-free	
9137	4. $\vdash_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(\tau',e'):\tau'$	• $e = E[Stat \ t \ t]$ where $t$ is boundary free	9192
9138	by $\mathcal{D}_{C}$ soundness (3)	Proof:	9193
9139	5. QED by hole substitution (4)	By the <i>boundary factoring</i> lemma for the higher-order	9194
9140	ELSE $e' \rightarrow_{C-D} e''$ :	embedding. (The only difference is the meaning of <i>e</i> is <i>a</i>	9195
9141	1. $e \rightarrow_{C-D} E[\operatorname{dyn} \tau' e'']$	value.)	9196
9142	2. $\vdash_{C} dyn \ \tau' \ e' : \tau'$		9197
9143	by boundary hole typing	<b>Lemma 5.12</b> : HC dynamic boundary factoring	9198
9144	3. ⊢ <sub>C</sub> e'	If $\vdash_{C} e$ then one of the following holds:	9199
9145	$\land \tau' \leqslant : \tau''$	• <i>e</i> is a value	9200
9146	by inversion (2)	$\bullet$ $e = E^{\bullet}[v_0 \ v_1]$	9201
9147	4. ⊢ <sub>C</sub> e''	$\bullet \ e = E \ [o_0 \ o_1]$ $\bullet \ e = E^{\bullet} [op^1 \ v]$	9202
9148	by dynamic preservation (3)	7 4 7	9203
9149	5. $\vdash_{C} dyn \; \tau' \; e'' : \tau'$	• $e = E^{\bullet}[op^2 \ v_0 \ v_1]$	9204
9150	by (4)	• $e = E[\text{dyn } \tau \ e']$ where $e'$ is boundary-free	9205
9151	6. QED by hole substitution (5)	• $e = E[\text{stat } \tau \ e']$ where $e'$ is boundary-free	9206
9152	<b>CASE</b> $e = E[\text{stat } \tau' \ e']$ and $e'$ is boundary-free:	• <i>e</i> = <i>E</i> [Err] <i>Proof</i> :	9207
9153	IF $e' \in v$ :	By the <i>boundary factoring</i> lemma for the higher-order	9208
9154	1. $e \rightarrow_{C-D} E[\mathcal{S}_{C}(\tau', e')]$	embedding.	9209
9155	2. $\vdash_{C} stat \ \tau' \ e'$	embedding.  □	9210
9156	by boundary hole typing	T	9211
9157	3. $\vdash_{C} e' : \tau'$	Lemma 5.13 : HC static hole typing	9212
9158	by inversion (2)	If $\vdash_C E^{\bullet}[e] : \tau$ then the derivation contains a sub-term $\vdash_C e : \tau'$	9213
9159	4. $\vdash_{C} \mathcal{S}_{C}(\tau',e')$	<i>Proof (sketch)</i> : Similar to the <i>static hole typing</i> lemma for the	9214
9160	by $S_{C}$ soundness (3)	higher-order embedding. $\Box$	9215
9161	5. QED by hole substitution (5)	<b>Lemma 5.14</b> : HC dynamic hole typing	9216
9162	ELSE $e' \rightarrow_{C-S} e''$ :	If $\vdash_{\mathbb{C}} E^{\bullet}[e]$ then the derivation contains a sub-term $\vdash_{\mathbb{C}} e$	9217
9163	1. $e \rightarrow_{C-D} E[\text{stat } \tau' e'']$	<i>Proof (sketch)</i> : Similar to the <i>static hole typing</i> lemma for the	9218
9164	2. ⊢ <sub>C</sub> stat τ' e'	higher-order embedding. $\Box$	9219
9165	by boundary hole typing	<b>Lemma 5.15</b> : HC boundary hole typing	9220
9166	3. $\vdash_{C} e' : \tau'$	• If $\vdash_{\mathbb{C}} E[dyn \ \tau \ e] : \tau'$ then the derivation contains a sub-term	9221
9167	by inversion (2)	$\vdash_{C} dyn  \tau  e : \tau$	9222
9168	4. $\vdash_{C} e'' : \tau'$	• If $\vdash_{C} E[\text{dyn } \tau \ e]$ then the derivation contains a sub-term	9223
9169	by static preservation (3)	$\vdash_{C} dyn \ \tau \ e : \tau$	9224
9170	5. $\vdash_{C} stat \ \tau' \ e''$	• If $\vdash_{C} E[\text{stat } \tau \ e] : \tau'$ then the derivation contains a sub-term	9225
9171	by (4)	$\vdash_{C} stat \; \tau \; e$	9226
9172	6. QED by hole substitution (5)	• If $\vdash_{\mathbb{C}} E[\text{stat } \tau \ e]$ then the derivation contains a sub-term	9227
9173	CASE $e = E[Err]$ :	$+_{c}$ stat $\tau$ e	9228
9174	1. $e \rightarrow_{C-D} Err$	<i>Proof (sketch)</i> : Similar to the proof for the higher-order <i>bound</i> -	9229
9175	2. QED ⊢ Err	ary hole typing lemma. □	9230
9176		Lemma 5.16 : HC hole substitution	9231
9177	Lamma F. 11 . LIC static houndary factoring	• If $\vdash_{\Gamma} E[e]$ and the derivation contains a sub-term $\vdash_{\Gamma} e : \tau'$	9232
9178	<b>Lemma 5.11</b> : HC static boundary factoring	and $\vdash_{C} e' : \tau'$ then $\vdash_{C} E[e']$ .	9233
9179		• If $\vdash_{\mathbb{C}} E[e]$ and the derivation contains a sub-term $\vdash_{\mathbb{C}} e$ and	9234
9180		$\vdash_{C} e'$ then $\vdash_{C} E[e']$ .	9235
9181		• If $\vdash_{C} E[e] : \tau$ and the derivation contains a sub-term $\vdash_{C} e : \tau'$	9236
9182		and $\vdash_C e' : \tau'$ then $\vdash_C E[e'] : \tau$ .	9237
9183		• If $\vdash_{\mathbb{C}} E[e] : \tau$ and the derivation contains a sub-term $\vdash_{\mathbb{C}} e$	9238
9184			9239
9184		and $\vdash_{\mathbb{C}} e'$ then $\vdash_{\mathbb{C}} E[e'] : \tau$ .	9239
/ IUJ		84	7240

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If \vdash_{\mathbb{C}} v_0 : \tau_0 and \vdash_{\mathbb{C}} v_1 : \tau_1 and \Delta(op^2, \tau_0, \tau_1) = \tau then one of
9241
             Proof (sketch): Similar to the proof of the higher-order hole
                                                                                                                                                                                                                                                                                    9296
                                                                                                                                                    the following holds:
9242
             substitution lemma, just replacing \vdash_{\mathsf{H}} with \vdash_{\mathsf{C}}.
                                                                                                                                                                                                                                                                                    9297
                                                                                                                                                    • \delta(op^2, v_0, v_1) = v and \vdash_{\mathcal{C}} v : \tau, or
9243
                                                                                                                                                                                                                                                                                    9298
             Lemma 5.17 : \vdash_{C} static inversion
                                                                                                                                                    • \delta(op^2, v_0, v_1) = \text{BndryErr}
9244
                • If \Gamma \vdash_{\mathbb{C}} x : \tau then (x : \tau') \in \Gamma and \tau' \leqslant : \tau
                                                                                                                                                                                                                                                                                    9299
                                                                                                                                                 Proof (sketch): Similar to the proof for the higher-order \Delta
9245
                • If \Gamma \vdash_{\mathbb{C}} \lambda(x:\tau'_d).e':\tau then (x:\tau'_d),\Gamma \vdash_{\mathbb{C}} e':\tau'_c and
                                                                                                                                                                                                                                                                                    9300
                                                                                                                                                 type soundness lemma.
                                                                                                                                                                                                                                                                                    9301
9246
                    \tau_d \Rightarrow \tau_c \leqslant \tau
                                                                                                                                                 Lemma 5.21 : \delta preservation
9247
                                                                                                                                                                                                                                                                                    9302
                • If \Gamma \vdash_{\mathsf{C}} \langle e_0, e_1 \rangle : \tau_0 \times \tau_1 then \Gamma \vdash_{\mathsf{C}} e_0 : \tau_0' and \Gamma \vdash_{\mathsf{C}} e_1 : \tau_1' and
                                                                                                                                                   • If \vdash_{\mathcal{C}} v and \delta(op^1, v) = v' then \vdash_{\mathcal{C}} v'
9248
                                                                                                                                                                                                                                                                                    9303
                    \tau_0' \leqslant : \tau_0 \text{ and } \tau_1' \leqslant : \tau_1
                                                                                                                                                    • If \vdash_{C} v_0 and \vdash_{C} v_1 and \delta(op^2, v_0, v_1) = v' then \vdash_{C} v'
9249
                • If \Gamma \vdash_{\Gamma} e_0 \ e_1 : \tau_c then \Gamma \vdash_{\Gamma} e_0 : \tau_d' \Rightarrow \tau_c' and \Gamma \vdash_{\Gamma} e_1 : \tau_d' and
                                                                                                                                                 Proof (sketch): Similar to the proof for the higher-order \delta
9250
                                                                                                                                                                                                                                                                                    9305
                                                                                                                                                 preservation lemma.
                                                                                                                                                                                                                                                                     9251
                                                                                                                                                                                                                                                                                    9306
                • If \Gamma \vdash_{\Gamma} fst e : \tau then \Gamma \vdash_{\Gamma} e : \tau_0 \times \tau_1 and \Delta(\text{fst}, \tau_0 \times \tau_1) = \tau_0
                                                                                                                                                 Lemma 5.22 : HC substitution
9252
                                                                                                                                                                                                                                                                                    9307
                    and \tau_0 \leqslant : \tau
                                                                                                                                                    • If (x:\tau_x), \Gamma \vdash_{\mathbb{C}} e and \vdash_{\mathbb{C}} v:\tau_x then \Gamma \vdash_{\mathbb{C}} e[x \leftarrow v]
9253
                                                                                                                                                                                                                                                                                    9308
                • If \Gamma \vdash_{\Gamma} \text{snd } e : \tau \text{ then } \Gamma \vdash_{\Gamma} e : \tau_0 \times \tau_1 \text{ and } \Delta(\text{snd}, \tau_0 \times \tau_1) = \tau_1
                                                                                                                                                    • If x, \Gamma \vdash_{C} e and \vdash_{C} v then \Gamma \vdash_{C} e[x \leftarrow v]
9254
                    and \tau_1 \leqslant : \tau
                                                                                                                                                                                                                                                                                    9309
9255
                                                                                                                                                    • If (x:\tau_x), \Gamma \vdash_{\Gamma} e:\tau and \vdash_{\Gamma} v:\tau_x then \Gamma \vdash_{\Gamma} e[x \leftarrow v]:\tau
                                                                                                                                                                                                                                                                                    9310
                • If \Gamma \vdash_{\Gamma} op^2 e_0 e_1 : \tau then \Gamma \vdash_{\Gamma} e_0 : \tau_0 and \Gamma \vdash_{\Gamma} e_1 : \tau_1 and
                                                                                                                                                    • If x, \Gamma \vdash_{\mathsf{C}} e : \tau and \vdash_{\mathsf{C}} v then \Gamma \vdash_{\mathsf{C}} e[x \leftarrow v] : \tau
9256
                    \Delta(op^2, \tau_0, \tau_1) = \tau' \text{ and } \tau' \leqslant \tau'
                                                                                                                                                                                                                                                                                    9311
                                                                                                                                                 Proof (sketch): Similar to the proof for the higher-order sub-
                                                                                                                                                                                                                                                                                    9312
9257
                • If \Gamma \vdash_{\Gamma} \text{mon } \tau_0' \times \tau_1' \ v' : \tau_0 \times \tau_1 \text{ then } \Gamma \vdash_{\Gamma} v' \text{ and } \tau_0' \times \tau_1' \leqslant : \tau_0 \times \tau_1
                                                                                                                                                 stitution lemma.
                                                                                                                                                                                                                                                                                    9313
9258
                • If \Gamma \vdash_{\Gamma} \text{mon } \tau'_d \Rightarrow \tau'_c v' : \tau_d \Rightarrow \tau_c \text{ then } \Gamma \vdash_{\Gamma} v' \text{ and } \tau'_d \Rightarrow \tau'_c \leqslant :
9259
                                                                                                                                                 Lemma 5.23: weakening
                                                                                                                                                                                                                                                                                    9314
                • If \Gamma \vdash_{\Gamma} \operatorname{dyn} \tau' e' : \tau then \Gamma \vdash_{\Gamma} e' and \tau' \leqslant \tau
                                                                                                                                                    • If \Gamma \vdash_{\Gamma} e then x, \Gamma \vdash_{\Gamma} e
                                                                                                                                                                                                                                                                                    9315
                                                                                                                                                    • If \Gamma \vdash_{\mathbb{C}} e : \tau then (x : \tau'), \Gamma \vdash_{\mathbb{C}} e : \tau
                                                                                                                                                                                                                                                                                    9316
9261
                 QED by the definition of \Gamma \vdash_{\mathbb{C}} e : \tau
9262
                                                                                                                                                                                                                                                                                    9317
                                                                                                                                                      QED because e is closed under \Gamma
9263
                                                                                                                                                                                                                                                                                    9318
                                                                                                                                                 9319
             Lemma 5.18: \vdash_C dynamic inversion
9265
                                                                                                                                                                                                                                                                                    9320
                • If \Gamma \vdash_{\mathbb{C}} x then x \in \Gamma
9266
                • If \Gamma \vdash_{\Gamma} \lambda x. e' then x, \Gamma \vdash_{\Gamma} e'
9267
                                                                                                                                                                                                                                                                                    9322
                • If \Gamma \vdash_{\Gamma} \langle e_0, e_1 \rangle then \Gamma \vdash_{\Gamma} e_0 and \Gamma \vdash_{\Gamma} e_1
9268
                                                                                                                                                                                                                                                                                    9323
                • If \Gamma \vdash_{\Gamma} e_0 \ e_1 then \Gamma \vdash_{\Gamma} e_0 and \Gamma \vdash_{\Gamma} e_1
9269
                                                                                                                                                                                                                                                                                    9324
                • If \Gamma \vdash_{\Gamma} op^1 e_0 then \Gamma \vdash_{\Gamma} e_0
9270
                                                                                                                                                                                                                                                                                    9325
                • If \Gamma \vdash_{\Gamma} op^2 e_0 e_1 then \Gamma \vdash_{\Gamma} e_0 and \Gamma \vdash_{\Gamma} e_1
9271
                                                                                                                                                                                                                                                                                    9326
                • If \Gamma \vdash_{\Gamma} \text{mon } \tau_d \Rightarrow \tau_c \ v' \text{ then } \Gamma \vdash_{\Gamma} v' : \tau_d \Rightarrow \tau_c
9272
                                                                                                                                                                                                                                                                                    9327
                • If \Gamma \vdash_{\Gamma} \text{mon } \tau_0 \times \tau_1 \ v' \text{ then } \Gamma \vdash_{\Gamma} v' : \tau_0 \times \tau_1
9273
                                                                                                                                                                                                                                                                                    9328
                • If \Gamma \vdash_{\mathbb{C}} stat \tau' e' then \Gamma \vdash_{\mathbb{C}} e' : \tau'
9274
                                                                                                                                                                                                                                                                                    9329
9275
                 QED by the definition of \Gamma \vdash_{C} e
9276
                                                                                                                                                                                                                                                                                    9331
9277
                                                                                                                                                                                                                                                                                    9332
             Lemma 5.19: HC canonical forms
9278
                                                                                                                                                                                                                                                                                    9333
                • If \vdash_{C} v : \tau_0 \times \tau_1 then either:
                   - \upsilon = \langle \upsilon_0, \upsilon_1 \rangle
9280
                                                                                                                                                                                                                                                                                    9335
                    - or v = \text{mon}(\tau_0' \times \tau_1') v'
9281
                                                                                                                                                                                                                                                                                    9336
                        \wedge \tau_0' \times \tau_1' \leqslant : \tau_0 \times \tau_1
9282
                                                                                                                                                                                                                                                                                    9337
                • If \vdash_{C} v : \tau_d \Rightarrow \tau_c then either:
                   - \upsilon = \lambda(x : \tau_x). e'
9284
                                                                                                                                                                                                                                                                                    9339
                        \wedge \tau_d \leqslant : \tau_x
9285
                                                                                                                                                                                                                                                                                    9340
                    - or v = \text{mon}(\tau_d' \Rightarrow \tau_c') v'
9286
                                                                                                                                                                                                                                                                                    9341
                        \wedge \tau_d' \Rightarrow \tau_c' \leqslant : \tau_d \Rightarrow \tau_c
9287
                                                                                                                                                                                                                                                                                    9342
                • If \vdash_{\mathbf{C}} v: Int then v = i
9288
                                                                                                                                                                                                                                                                                    9343
                • If \vdash_{\mathcal{C}} v: Nat then v = i and v \in \mathbb{N}
9289
                                                                                                                                                                                                                                                                                    9344
             Proof:
9290
                                                                                                                                                                                                                                                                                    9345
                 QED by definition of \vdash_{C} e : \tau
9291
                                                                                                                                                                                                                                                                                    9346
9292
                                                                                                                                                                                                                                                                                    9347
             Lemma 5.20 : \triangle type soundness
9293
                                                                                                                                                                                                                                                                                    9348
```

#### A.6 (HF) Forgetful Embedding A.6.1 Forgetful Definitions Language HF $e = x \mid v \mid \langle e, e \rangle \mid e \mid op^1 \mid op^2 \mid e \mid op^2 \mid op^2$ dyn $\tau e$ | stat $\tau e$ | Err | chk $\tau e$ $v = i \mid \langle v, v \rangle \mid \lambda x. e \mid \lambda(x:\tau). e \mid$ $mon(\tau \Rightarrow \tau)(\lambda x. e) \mid mon(\tau \Rightarrow \tau)(\lambda(x:\tau). e) \mid$ $mon(\tau \times \tau) \langle v, v \rangle$ = Nat | Int | $\tau \times \tau$ | $\tau \Rightarrow \tau$ $\Gamma = \cdot \mid x, \Gamma \mid (x:\tau), \Gamma$ Err = BndryErr | TagErr $r = v \mid Err$ $E^{\bullet} = [] | E^{\bullet} e | v E^{\bullet} | \langle E^{\bullet}, e \rangle | \langle v, E^{\bullet} \rangle |$ $op^1 E^{\bullet} \mid op^2 E^{\bullet} e \mid op^2 v E^{\bullet}$ $E = E^{\bullet} \mid E \mid v \mid E \mid \langle E, e \rangle \mid \langle v, E \rangle \mid op^{1} \mid E \mid$ $op^2 E e \mid op^2 v E \mid dyn \tau E \mid stat \tau E$ $\Delta: op^1 \times \tau \longrightarrow \tau$ $\Delta(\text{fst}, \tau_0 \times \tau_1) = \tau_0$ $\Delta(\operatorname{snd}, \tau_0 \times \tau_1) = \tau_1$ $\Delta: op^2 \times \tau \times \tau \longrightarrow \tau$ $\Delta(op^2, Nat, Nat) = Nat$ $\Delta(op^2, Int, Int) = Int$ $\tau \leqslant : \tau$ $\frac{\tau_d' \leqslant : \tau_d \quad \tau_c \leqslant : \tau_c'}{\mathsf{Nat} \leqslant : \mathsf{Int}} \quad \frac{\tau_d' \leqslant : \tau_d \quad \tau_c \leqslant : \tau_c'}{\tau_d \Rightarrow \tau_c \leqslant : \tau_d' \Rightarrow \tau_c'} \quad \frac{\tau_0 \leqslant : \tau_0' \quad \tau_1 \leqslant : \tau_1'}{\tau_0 \times \tau_1 \leqslant : \tau_0' \times \tau_1'}$ $\frac{\tau \leqslant : \tau}{\tau \leqslant : \tau} \quad \frac{\tau \leqslant : \tau' \quad \tau' \leqslant : \tau''}{\tau \leqslant : \tau''}$ $\Gamma \vdash e$ $\frac{x \in \Gamma}{\Gamma \vdash x} \quad \frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x. e} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash i} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}$ $\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0 \ e_1} \quad \frac{\Gamma \vdash e}{\Gamma \vdash op^1 \ e} \quad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2 \ e_0 \ e_1} \quad \frac{\Gamma \vdash \text{Err}}{\Gamma \vdash op^2 \ e_0 \ e_1}$ $\Gamma \vdash e : \tau$

 $\Gamma$  + stat  $\tau$  e

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9461
                     \Gamma \vdash_{\!\!\scriptscriptstyle{\mathsf{E}}} e : \tau
9462
                                \frac{(x \colon \tau) \in \Gamma}{\Gamma \vdash_{\!\!\!\text{F}} x \colon \tau} \quad \frac{(x \colon \tau_d), \Gamma \vdash_{\!\!\!\text{F}} e \colon \tau_c}{\Gamma \vdash_{\!\!\!\text{F}} \lambda(x \colon \tau_d). e \colon \tau_d \Rightarrow \tau_c} \quad \frac{i \in \mathbb{N}}{\Gamma \vdash_{\!\!\!\text{F}} i \colon \text{Nat}}
9463
9464
9465
                               9466
9467
9468
9469
                                      \begin{array}{cccc} & \Gamma \vdash_{\!\!\!F} e_0 : \tau_0 \\ \Gamma \vdash_{\!\!\!F} e_0 : \tau_0 & \Gamma \vdash_{\!\!\!F} e_1 : \tau_1 & \Gamma \vdash_{\!\!\!F} e : \tau' \\ \Delta(op^1, \tau_0) = \tau & \Delta(op^2, \tau_0, \tau_1) = \tau & \tau' <: \tau \end{array}
9470
9471
9472
                                       9474
9475
                                                              9476
9477
9478
                                                                     \Gamma \vdash_{\mathsf{F}} v_0 : \tau_0' \quad \Gamma \vdash_{\mathsf{F}} v_1 : \tau_1'
9479
                                                        \Gamma \vdash_{\Gamma} \mathsf{mon} (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle : (\tau_0 \times \tau_1)
9480
9481
9482
                                                                               \Gamma \vdash_{\mathsf{E}} v_0 \quad \Gamma \vdash_{\mathsf{E}} v_1
9483
                                                        \Gamma \vdash_{\mathsf{F}} \mathsf{mon} \left( \tau_0 \times \overline{\tau_1} \right) \left\langle \overline{v_0, v_1} \right\rangle : \left( \tau_0 \times \overline{\tau_1} \right)
9484
9485
                                                                                       \Gamma \vdash_{\mathsf{F}} \lambda x. e
9486
                                                       \frac{\cdot}{\Gamma \vdash_{\mathsf{F}} \mathsf{mon} \left(\tau_d \Rightarrow \tau_c\right) \lambda x. \, e : \left(\tau_d \Rightarrow \tau_c\right)}
9487
9488
                      \frac{\Gamma \vdash_{\!\!\!\text{F}} \lambda(x \colon\! \tau_d').\, e \colon\! \tau_d' \!\Rightarrow\! \tau_c'}{\Gamma \vdash_{\!\!\!\text{F}} \min\left(\tau_d \!\Rightarrow\! \tau_c\right) \lambda(x \colon\! \tau_d').\, e \colon\! (\tau_d \!\Rightarrow\! \tau_c)} \quad \frac{\Gamma \vdash_{\!\!\!\text{F}} e \colon\! \tau'}{\Gamma \vdash_{\!\!\!\text{F}} \operatorname{chk} \tau \,\, e \colon\! \tau}
9489
9490
9491
                     \delta(op^1, v) = e
9492
                      \delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0
9493
                      \delta(\operatorname{snd}, \langle v_0, v_1 \rangle) = v_1
9494
9495
                     \delta(op^2, v, v) = e
9496
                       \delta(\text{sum}, i_0, i_1)
                                                                        = i_0 + i_1
9497
                       \delta(\text{quotient}, i_0, 0) = \text{BndryErr}
9498
                       \delta(\text{quotient}, i_0, i_1) = \lfloor i_0/i_1 \rfloor
9499
                             if i_1 \neq 0
9500
                      \mathcal{D}_{\mathsf{F}}: \tau \times v \longrightarrow e
9501
                       \mathcal{D}_{\mathsf{F}}(\tau, v) = \mathcal{X}(\tau, v)
9502
9503
                     S_{\mathsf{F}}: \tau \times v \longrightarrow e
9504
                      S_{\rm F}(\tau, v) = X(\tau, v)
9505
9506
9507
```

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```
X: \tau \times v \longrightarrow e
 X(\tau_d \Rightarrow \tau_c, \lambda x. e)
X(\tau_d \Rightarrow \tau_c, \lambda(x:\tau). e)
X(\tau_d \Rightarrow \tau_c, \lambda(x:\tau). e)
                                                                   = mon(\tau_d \Rightarrow \tau_c)(\lambda x. e)
                                                                    = mon(\tau_d \Rightarrow \tau_c)(\lambda(x:\tau).e)
 X(\tau_d \Rightarrow \tau_c, \text{mon}(\tau_d' \Rightarrow \tau_c') v') = \text{mon}(\tau_d \Rightarrow \tau_c) v'
 X(\tau_0 \times \tau_1, \langle v_0, v_1 \rangle)
                                                                     = mon(\tau_0 \times \tau_1) \langle v_0, v_1 \rangle
 X(\tau_0 \times \tau_1, \text{mon}(\tau_0' \times \tau_1') v')
                                                                     = mon(\tau_0 \times \tau_1) v'
 \mathcal{X}(\mathsf{Int},i)
                                                                     =i
 X(Nat, i)
                                                                     =i
       if i \in \mathbb{N}
                                                                     = BndryErr
 X(\tau, v)
       otherwise
e \triangleright_{S-1} e
                                                                     \rhd_{S-1} \mathcal{D}_{\mathsf{F}}(\tau, v)
 dyn τ υ
 chk τ v
                                                                     \triangleright_{S-1} X(\tau, v)
 (\text{mon}(\tau_d \Rightarrow \tau_c)(\lambda x. e)) v
                                                                     \triangleright_{S-1} \operatorname{dyn} \tau_c e'
       where e' = (\lambda x. e) (X(\tau_d, v))
 (\text{mon}(\tau_d \Rightarrow \tau_c)(\lambda(x:\tau).e)) v \triangleright_{S-1} \text{chk } \tau_c e'
       where e' = (\lambda(x : \tau), e) (X(\tau, v))
 fst (mon (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle)
                                                                     \triangleright_{S-1} X(\tau_0, v_0)
 \operatorname{snd}\left(\operatorname{mon}\left(\tau_0 \times \tau_1\right) \langle v_0, v_1 \rangle\right)
                                                                     \rhd_{S-1} \mathcal{X}(\tau_1, v_1)
 (\lambda(x:\tau).e)v
                                                                     \triangleright_{S-1} e[x \leftarrow v]
 op^1 v
                                                                     \triangleright_{S-1} \delta(op^1, v)
 op^2 v_0 v_1
                                                                     \triangleright_{\mathsf{S-1}} \delta(op^2, v_0, v_1)
e \rhd_{\mathsf{D-1}} e
 stat \tau v
                                                                      \triangleright_{\mathsf{D-1}} \mathcal{S}_{\mathsf{F}}(\tau,v)
                                                                     ⊳<sub>D-1</sub> TagErr
 v_0 v_1
       if v_0 \in \mathbb{Z} or v_0 = \langle v, v' \rangle
 (\text{mon}(\tau_d \Rightarrow \tau_c)(\lambda x. e)) v
                                                                      \triangleright_{D-1} (\lambda x. e) v
 (\text{mon}(\tau_d \Rightarrow \tau_c)(\lambda(x:\tau).e)) v \triangleright_{D-1} \text{stat } \tau_c e'
       where e' = \operatorname{chk} \tau_c ((\lambda(x : \tau). e) (X(\tau, v)))
 (\lambda x. e) v
                                                                     \triangleright_{\mathsf{H-D}} e[x \leftarrow v]
 fst (mon (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle)
                                                                      \triangleright_{\mathsf{D}\text{-}1} \mathcal{X}(\tau_0, v_0)
 \operatorname{snd}\left(\operatorname{mon}\left(\tau_0 \times \tau_1\right) \langle v_0, v_1 \rangle\right)
                                                                     \triangleright_{\mathsf{D}\text{-}1} \mathcal{X}(\tau_1, v_1)
 op^1 v
                                                                     ⊳<sub>H-D</sub> TagErr
       if \delta(op^1, v) is undefined
                                                                     \triangleright_{\mathsf{H-D}} \delta(op^1, v)
 op^1 v
 op^2 v_0 v_1
                                                                     ⊳<sub>H-D</sub> TagErr
       if \delta(op^2, v_0, v_1) is undefined
                                                                     \triangleright_{\mathsf{H-D}} \delta(op^2, v_0, v_1)
 op^2 v_0 v_1
e \rightarrow_{\mathsf{F-S}} e
                                  \rightarrow_{\mathsf{F-S}} E^{\bullet}[e']
 E^{\bullet}[e]
       if e \triangleright_{S-1} e'
 E[\operatorname{stat} \tau E^{\bullet}[e]] \rightarrow_{\mathsf{F-S}} E[\operatorname{stat} \tau E^{\bullet}[e']]
       if e \triangleright_{S-1} e'
 E[\operatorname{dyn} \tau E^{\bullet}[e]] \rightarrow_{\mathsf{F-S}} E[\operatorname{dyn} \tau E^{\bullet}[e']]
       if e \triangleright_{D-1} e'
 E[Err]
                                      \rightarrow_{F-S} Err
```

```
9571
               e \rightarrow_{\mathsf{F-D}} e

\begin{array}{ccc}
\hline
E^{\bullet}[e] & \rightarrow_{\text{F-D}} E^{\bullet}[e'] \\
\text{if } e \rhd_{\text{D-1}} e' & -\hline
\end{array}

9572
9573
9574
                 E[\operatorname{stat} \tau E^{\bullet}[e]] \rightarrow_{\mathsf{F-D}} E[\operatorname{stat} \tau E^{\bullet}[e']]
9575
                      if e \triangleright_{S-1} e'
9576
                 E[\operatorname{dyn} \tau E^{\bullet}[e]] \rightarrow_{\mathsf{F-D}} E[\operatorname{dyn} \tau E^{\bullet}[e']]
9577
                      if e \triangleright_{D-1} e'
9578
                                                     \rightarrow_{F-D} Err
                 E[Err]
9579
                e \rightarrow_{F-S}^* e reflexive, transitive closure of \rightarrow_{F-S}
9580
9581
                e \rightarrow_{F-D}^* e reflexive, transitive closure of \rightarrow_{F-D}
9582
9583
9584
9585
9586
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9588
9589
```

```
A.6.2 Forgetful Theorems
9681
                                                                                                                                                                                                        9736
9682
                                                                                                                                                                                                        9737
         Theorem 6.0: static HF-soundness
9683
                                                                                                                                                                                                        9738
           If \vdash e : \tau then \vdash_{\mathsf{F}} e : \tau and one of the following holds:
9684
                                                                                                                                                                                                        9739
           • e \to_{\mathsf{F-S}}^* v and \vdash_{\mathsf{F}} v : \tau
9685
           • e \rightarrow_{F-S}^{*-S} E[\operatorname{dyn} \tau' e'] and e' \rhd_{D-1} \operatorname{TagErr}
• e \rightarrow_{F-S}^{*} \operatorname{BndryErr}
                                                                                                                                                                                                        9741
9686
9687
                                                                                                                                                                                                        9742
           • e diverges
9688
                                                                                                                                                                                                        9743
         Proof:
           1. \vdash_{\mathsf{F}} e : \tau
9690
                                                                                                                                                                                                        9745
               by static subset
9691
           2. QED by static progress and static preservation.
9692
                                                                                                                                                                                                        9747
9693
                                                                                                                                                                                                        9748
         Theorem 6.1: dynamic HF-soundness
9694
                                                                                                                                                                                                        9749
           If \vdash e then \vdash<sub>F</sub> e and one of the following holds:
9695
                                                                                                                                                                                                        9750
           • e \to_{\mathsf{F-D}}^* v and \vdash_{\mathsf{F}} v
9696
                                                                                                                                                                                                        9751
           • e \rightarrow_{\text{F-D}}^* E[e'] and e' \triangleright_{D-1} \text{TagErr}
9697
                                                                                                                                                                                                        9752
           • e \rightarrow_{F-D}^* BndryErr
9698
                                                                                                                                                                                                        9753
           • e diverges
9699
                                                                                                                                                                                                        9754
         Proof:
9700
                                                                                                                                                                                                        9755
           1. ⊢<sub>F</sub> e
9701
                                                                                                                                                                                                        9756
               by dynamic subset
9702
                                                                                                                                                                                                        9757
           2. QED by dynamic progress and dynamic preservation.
9703
                                                                                                                                                                                                        9758
9704
                                                                                                                                                                                                        9759
         Corollary 6.2 : HF static soundness
9705
                                                                                                                                                                                                        9760
           If \vdash e : \tau and e is boundary-free, then one of the following
9706
                                                                                                                                                                                                        9761
           holds:
9707
                                                                                                                                                                                                        9762
           \bullet \ e \to_{\mathsf{F-S}}^* v \text{ and } \vdash_{\mathsf{F}} v : \tau
9708
                                                                                                                                                                                                        9763
           • e \rightarrow_{F-S}^* BndryErr
9709
                                                                                                                                                                                                        9764
           • e diverges
9710
                                                                                                                                                                                                        9765
         Proof:
9711
            Consequence of the proof for static HF-soundness
                                                                                                                                                                                                        9766
9712
                                                                                                                                                                                                        9767
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9735
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```

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9791
               A.6.3 Forgetful Lemmas
                                                                                                                                                                                    CASE X(\tau_0 \times \tau_1, \langle v_0, v_1 \rangle) = \text{mon}(\tau_0 \times \tau_1) \langle v_0, v_1 \rangle:
                                                                                                                                                                                                                                                                                                                                           9846
9792
                                                                                                                                                                                         IF \Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle:
                                                                                                                                                                                                                                                                                                                                           9847
               Lemma 6.3 : X(\cdot, \cdot) soundness
                                                                                                                                                                                             1. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} \left( \tau_0 \times \tau_1 \right) \langle v_0, v_1 \rangle
9793
                                                                                                                                                                                                                                                                                                                                           9848
                  If \Gamma \vdash_{\mathsf{F}} v \text{ or } \Gamma \vdash_{\mathsf{F}} v : \tau
9794
                                                                                                                                                                                                   by \Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle
                                                                                                                                                                                                                                                                                                                                           9849
                  and X(\tau', v) = v',
9795
                                                                                                                                                                                             2. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle : \tau_0 \times \tau_1
                  then \Gamma \vdash_{\mathsf{F}} v' and \Gamma \vdash_{\mathsf{F}} v' : \tau'
                                                                                                                                                                                                                                                                                                                                           9851
                                                                                                                                                                                                   by \Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle
9796
               Proof:
9797
                                                                                                                                                                                            3. OED
                                                                                                                                                                                                                                                                                                                                           9852
                    By case analysis of the definition of X(\cdot, \cdot).
9798
                                                                                                                                                                                         ELSE \Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle : \tau_0' \times \tau_1' :
                                                                                                                                                                                                                                                                                                                                           9853
                    CASE X(\tau_d \Rightarrow \tau_c, v) = \text{mon}(\tau_d \Rightarrow \tau_c) v:
9799
                                                                                                                                                                                             1. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} \left( \tau_0 \times \tau_1 \right) \langle v_0, v_1 \rangle
                                                                                                                                                                                                                                                                                                                                           9854
                         IF v = \lambda x. e
9800
                                                                                                                                                                                                   by \Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle : \tau_0' \times \tau_1'
                                                                                                                                                                                                                                                                                                                                           9855
                                  9801
                                                                                                                                                                                             2. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle : \tau_0 \times \tau_1
                                                                                                                                                                                                                                                                                                                                           9856
                             1. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) v
9802
                                                                                                                                                                                                   by \Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle : \tau_0' \times \tau_1'
                                                                                                                                                                                                                                                                                                                                           9857
                                    by \Gamma \vdash_{\mathsf{E}} v
9803
                                                                                                                                                                                             3. QED
                                                                                                                                                                                                                                                                                                                                           9858
                             2. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) v : \tau_d \Rightarrow \tau_c
                                                                                                                                                                                    CASE X(\tau_0 \times \tau_1, \text{mon}(\tau_0' \times \tau_1') \langle v_0, v_1 \rangle) = \text{mon}(\tau_0 \times \tau_1) \langle v_0, v_1 \rangle
9804
                                                                                                                                                                                                                                                                                                                                           9859
                                    by \Gamma \vdash_{\mathsf{F}} v
9805
                                                                                                                                                                                                                                                                                                                                           9860
                             3. QED
9806
                                                                                                                                                                                         IF \Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle:
                                                                                                                                                                                                                                                                                                                                           9861
                          ELSE v = \lambda(x : \tau_x). e
9807
                                                                                                                                                                                            1. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle
                                                                                                                                                                                                                                                                                                                                           9862
                                         \wedge \Gamma \vdash_{\mathsf{F}} v : \tau_d' \Rightarrow \tau_c' :
9808
                                                                                                                                                                                                   by \Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle
                                                                                                                                                                                                                                                                                                                                           9863
                              1. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) v
9809
                                                                                                                                                                                             2. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle : \tau_0 \times \tau_1
                                                                                                                                                                                                                                                                                                                                           9864
                                   by \Gamma \vdash_{\mathsf{F}} v : \tau_d' \Rightarrow \tau_c'
9810
                                                                                                                                                                                                   by \Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle
                                                                                                                                                                                                                                                                                                                                           9865
                             2. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) v : \tau_d \Rightarrow \tau_c
9811
                                                                                                                                                                                             3. QED
                                                                                                                                                                                                                                                                                                                                           9866
                                    by \Gamma \vdash_{\mathsf{F}} v : \tau_d' \Rightarrow \tau_c'
                                                                                                                                                                                         ELSE \Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle : \tau_0^{\prime\prime} \times \tau_1^{\prime\prime} :
9812
                                                                                                                                                                                                                                                                                                                                           9867
9813
                                                                                                                                                                                                                                                                                                                                           9868
                                                                                                                                                                                             1. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle
                     CASE X(\tau_d \Rightarrow \tau_c, \text{mon}(\tau_d' \Rightarrow \tau_c') v') = \text{mon}(\tau_d \Rightarrow \tau_c) v':
9814
                                                                                                                                                                                                   by \Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle : \tau_0^{\prime\prime} \times \tau_1^{\prime\prime}
                         IF \Gamma \vdash_{\mathsf{F}} \mathsf{mon} \left(\tau_d' \Rightarrow \tau_c'\right) \tilde{v}':
9815
                               IF v' = \lambda x. e'
                                                                                                                                                                                             2. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle : \tau_0 \times \tau_1
                                                                                                                                                                                                                                                                                                                                           9870
9816
                                                                                                                                                                                                   by \Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle : \tau_0^{\prime\prime} \times \tau_1^{\prime\prime}
                                                                                                                                                                                                                                                                                                                                           9871
                                       9817
                                                                                                                                                                                             3. QED
                                                                                                                                                                                                                                                                                                                                           9872
                                  1. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) v'
                                                                                                                                                                                    CASE X(Int, i) = i:
                                                                                                                                                                                                                                                                                                                                           9873
                                         9819
                                                                                                                                                                                       1. \Gamma \vdash_{\mathsf{E}} i
                                                                                                                                                                                                                                                                                                                                           9874
                                  2. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) v' : \tau_d \Rightarrow \tau_c
9820
                                                                                                                                                                                       2. \Gamma \vdash_{\mathsf{F}} i : \mathsf{Int}
                                                                                                                                                                                                                                                                                                                                           9875
                                         by \Gamma \vdash_{\mathsf{F}} v'
9821
                                                                                                                                                                                        3. QED
                                                                                                                                                                                                                                                                                                                                           9876
                                  3. QED
                                                                                                                                                                                    CASE X(Nat, i) = i:
                                                                                                                                                                                                                                                                                                                                           9877
                               ELSE \upsilon' = \lambda(x : \tau_x). e'
                                                                                                                                                                                       1. \Gamma \vdash_{\mathsf{F}} i
9823
                                                                                                                                                                                                                                                                                                                                           9878
                                              \wedge \Gamma \vdash_{\mathsf{F}} v' : \tau''_d \Rightarrow \tau''_c :
9824
                                                                                                                                                                                       2. \Gamma \vdash_{\mathsf{F}} i : \mathsf{Nat}
                                                                                                                                                                                                                                                                                                                                           9879
                                  1. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) v'
                                                                                                                                                                                             by i \in \mathbb{N}
9825
                                                                                                                                                                                                                                                                                                                                           9880
                                         by \Gamma \vdash_{\mathsf{F}} v' : \tau''_d \Rightarrow \tau''_c
9826
                                                                                                                                                                                        3. QED
                                                                                                                                                                                                                                                                                                                                           9881
                                  2. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_d \Longrightarrow \tau_c) v' : \tau_d \Longrightarrow \tau_c
                                                                                                                                                                              9827
                                                                                                                                                                                                                                                                                                                                           9882
                                         by \Gamma \vdash_{\mathsf{F}} v' : \tau''_d \Rightarrow \tau''_c
9828
                                                                                                                                                                                                                                                                                                                                           9883
                                                                                                                                                                               Corollary 6.4 : \mathcal{D}_{\mathsf{F}} soundness
                          ELSE \Gamma \vdash_{\mathsf{F}} \mathsf{mon} \left( \tau_d' \Rightarrow \tau_c' \right) v' : \tau_d' \Rightarrow \tau_c' :
                                                                                                                                                                               If \vdash_{\mathsf{F}} v then \vdash_{\mathsf{F}} \mathcal{D}_{\mathsf{F}}(\tau, v) : \tau
                                                                                                                                                                                                                                                                                                                                           9885
9830
                               IF v' = \lambda x. e'
                                                                                                                                                                               Proof:
9831
                                                                                                                                                                                                                                                                                                                                           9886
                                       \wedge \; \Gamma \vdash_{\!\!\! F} \!\!\! v' :
                                                                                                                                                                                    QED by X soundness
9832
                                                                                                                                                                                                                                                                                                                                           9887
                                  1. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) v'
                                         by \Gamma \vdash_{\mathsf{E}} v'
                                                                                                                                                                               Corollary 6.5 : S_F soundness
9834
                                                                                                                                                                                                                                                                                                                                           9889
                                  2. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) v' : \tau_d \Rightarrow \tau_c
                                                                                                                                                                               If \vdash_{\mathsf{F}} v : \tau \text{ then } \vdash_{\mathsf{F}} \mathcal{S}_{\mathsf{F}}(\tau, v)
9835
                                                                                                                                                                                                                                                                                                                                           9890
                                         by \Gamma \vdash_{\mathsf{F}} v'
                                                                                                                                                                               Proof:
9836
                                                                                                                                                                                                                                                                                                                                           9891
                                  3. QED
                                                                                                                                                                                    QED by X soundness
                                                                                                                                                                                                                                                                                                                                           9892
                               ELSE v' = \lambda(x : \tau_x). e'
9838
                                                                                                                                                                                                                                                                                                                                           9893
                                              \land \Gamma \vdash_{\mathsf{F}} v' : \tau''_d \Rightarrow \tau''_c :
                                                                                                                                                                               Corollary 6.6: HF static subset
9839
                                                                                                                                                                                                                                                                                                                                           9894
                                  1. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_d \Longrightarrow \tau_c) v'
                                                                                                                                                                               If \Gamma \vdash e : \tau then \Gamma \vdash_{\mathsf{F}} e : \tau.
9840
                                                                                                                                                                                                                                                                                                                                           9895
                                         by \Gamma \vdash_{\mathsf{F}} v' : \tau''_d \Rightarrow \tau''_c
                                                                                                                                                                               Proof:
9841
                                                                                                                                                                                                                                                                                                                                           9896
                                  2. \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) v' : \tau_d \Rightarrow \tau_c
                                                                                                                                                                                    Consequence of the proof for the higher-order static
9842
                                                                                                                                                                                                                                                                                                                                           9897
                                         by \Gamma \vdash_{\mathsf{F}} v' : \tau''_d \Rightarrow \tau''_c
                                                                                                                                                                                    subset lemma; both \vdash_{\mathsf{F}} and \vdash_{\mathsf{H}} have the same typing rules
9843
                                                                                                                                                                                                                                                                                                                                           9898
                                  3. QED
                                                                                                                                                                                    for surface-language expressions.
9844
                                                                                                                                                                                                                                                                                                                                           9899
                                                                                                                                                                                                                                                                                                                                           9900
9845
                                                                                                                                                                     90
```

```
9901
                                                                                                                                       CASE e = E^{\bullet}[op^1 v]:
                                                                                                                                                                                                                                                          9956
                                                                                                                                          1. \vdash_{\mathsf{F}} op^1 v : \tau'
9902
                                                                                                                                                                                                                                                          9957
           Corollary 6.7: HF dynamic subset
                                                                                                                                               by static hole typing
9903
                                                                                                                                                                                                                                                          9958
            If \Gamma \vdash e then \Gamma \vdash_{\mathsf{F}} e.
9904
                                                                                                                                                                                                                                                          9959
                                                                                                                                          2. \vdash_{\mathsf{F}} v : \tau_0 \times \tau_1
           Proof:
9905
                                                                                                                                               by inversion
               Consequence of the proof for the higher-order dynamic
                                                                                                                                          3. v = \langle v_0, v_1 \rangle
                                                                                                                                                                                                                                                          9961
9906
                subset lemma.
                                                                                                                                               \forall v = \text{mon}(\tau_0' \times \tau_1') \langle v_0, v_1 \rangle
9907
                                                                                                                                                                                                                                                          9962
9908
                                                                                                                                               by canonical forms
                                                                                                                                                                                                                                                          9963
           Lemma 6.8: HF static progress
9909
                                                                                                                                          4. IF v = \langle v_0, v_1 \rangle
                                                                                                                                                                                                                                                          9964
              If \vdash_{\mathsf{E}} e : \tau then one of the following holds:
                                                                                                                                                     \wedge op^1 = fst :
9910
                                                                                                                                                                                                                                                          9965
              • e is a value
9911
                                                                                                                                                  a. \delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0
                                                                                                                                                                                                                                                          9966
              • e \in Err
9912
                                                                                                                                                  b. e \rightarrow_{\mathsf{F-S}} E^{\bullet}[v_0]
                                                                                                                                                                                                                                                          9967
              • e \rightarrow_{F-S} e'
9913
                                                                                                                                                      by op^1 v \triangleright_{S-1} v_0
                                                                                                                                                                                                                                                          9968
              • e \rightarrow_{F-S} BndryErr
9914
                                                                                                                                                  c. QED
                                                                                                                                                                                                                                                          9969
              • e = E[dyn \tau' e'] and e' \rightarrow_{F-D} TagErr
9915
                                                                                                                                               IF v = \langle v_0, v_1 \rangle
                                                                                                                                                                                                                                                          9970
9916
                                                                                                                                                      \wedge op^1 = \text{snd}:
                                                                                                                                                                                                                                                          9971
               By the boundary factoring lemma, there are eight possi-
9917
                                                                                                                                                  a. \delta(\operatorname{snd}, \langle v_0, v_1 \rangle) = v_1
                                                                                                                                                                                                                                                          9972
               ble cases.
9918
                                                                                                                                                  b. e \rightarrow_{\mathsf{F-S}} E^{\bullet}[v_1]
                                                                                                                                                                                                                                                          9973
               CASE e is a value :
9919
                                                                                                                                                      by op^1 v \triangleright_{S-1} v_1
                                                                                                                                                                                                                                                          9974
                  1. OED
9920
                CASE e = E^{\bullet}[v_0 \ v_1]:
                                                                                                                                                  c. QED
                                                                                                                                                                                                                                                          9975
9921
                                                                                                                                               IF v = \text{mon}\left(\tau_0' \times \tau_1'\right) \langle v_0, v_1 \rangle
                                                                                                                                                                                                                                                          9976
                  1. \vdash_{\mathsf{F}} v_0 \ v_1 : \tau'
                                                                                                                                                     \wedge op^1 = fst
9922
                                                                                                                                                                                                                                                          9977
                       by static hole typing
                                                                                                                                                     \wedge \mathcal{X}(\tau_0', \upsilon_0) = \upsilon_0':
                                                                                                                                                                                                                                                          9978
9923
                  2. \vdash_{\mathsf{F}} v_0 : \tau_d \Longrightarrow \tau_c
                                                                                                                                                  a. e \rightarrow_{\mathsf{F-S}} E^{\bullet}[v_0']
                                                                                                                                                                                                                                                          9979
                       \wedge \vdash_{\mathsf{F}} v_1 : \tau_d
9925
                       by inversion
                                                                                                                                                      by fst v \triangleright_{S-1} v_0'
                                                                                                                                                                                                                                                          9980
                                                                                                                                                 b. QED
                                                                                                                                                                                                                                                          9981
                  3. v_0 = \lambda(x : \tau_d'). e'
9927
                                                                                                                                               IF v = \text{mon}(\tau_0' \times \tau_1') \langle v_0, v_1 \rangle
                                                                                                                                                                                                                                                          9982
                       \forall v_0 = \text{mon}(\tau_d' \Rightarrow \tau_c') \lambda x. e'
9928
                                                                                                                                                     \wedge op^1 = fst
                                                                                                                                                                                                                                                          9983
                       \forall v_0 = \text{mon}(\tau_d' \Rightarrow \tau_c') \lambda(x : \tau_x). e'
9929
                                                                                                                                                     \wedge \mathcal{X}(\tau_0', v_0) = \text{BndryErr}:
                                                                                                                                                                                                                                                          9984
                       by canonical forms
9930
                                                                                                                                                                                                                                                          9985
                  4. IF v_0 = \lambda(x : \tau'_d). e':
                                                                                                                                                  a. e \rightarrow_{F-S} BndryErr
9931
                          a. e \rightarrow_{\mathsf{F-S}} E^{\bullet}[e'[x \leftarrow v_1]]
                                                                                                                                                      by fst v \triangleright_{S-1} BndryErr
                                                                                                                                                                                                                                                          9986
                                                                                                                                                                                                                                                          9987
                              by v_0 v_1 \triangleright_{S-1} e'[x \leftarrow v_1]
                                                                                                                                                 b. QED
9933
                                                                                                                                               IF v = \text{mon}\left(\tau_0' \times \tau_1'\right) \langle v_0, v_1 \rangle
                                                                                                                                                                                                                                                          9988
                          b. QED
                                                                                                                                                     \wedge op^1 = snd
                                                                                                                                                                                                                                                          9989
9934
                       IF v_0 = \text{mon}(\tau_d' \Rightarrow \tau_c') \lambda x. e'
9935
                                                                                                                                                     \wedge \mathcal{X}(\tau_1', v_1) = v_1':
                                                                                                                                                                                                                                                          9990
                             \wedge \mathcal{X}(\tau_d', v_1) = v_1' :
9936
                          a. e \rightarrow_{F-S} E^{\bullet}[\mathsf{dyn} \ \tau'_c \ (e'[x \leftarrow v'_1])]
                                                                                                                                                  a. e \rightarrow_{\mathsf{F-S}} E^{\bullet}[v_1']
                                                                                                                                                                                                                                                          9991
9937
                                                                                                                                                      by snd v \triangleright_{S-1} v_1'
                                                                                                                                                                                                                                                          9992
                               by v_0 \ v_1 >_{S-1} dyn \ \tau'_c \ (e'[x \leftarrow v'_1])
9938
                                                                                                                                                                                                                                                          9993
                                                                                                                                                  b. QED
                                                                                                                                               ELSE v = mon(\tau_0' \times \tau_1') \langle v_0, v_1 \rangle
                       IF v_0 = \text{mon}(\tau_d' \Rightarrow \tau_c') \lambda x. e'
                                                                                                                                                                                                                                                          9995
                                                                                                                                                           \wedge op^1 = \text{snd}
9940
                             \wedge \mathcal{X}(\tau_d', v_1) = \text{BndryErr}:
                                                                                                                                                           \wedge X(\tau_1', v_1) = \text{BndryErr}:
9941
                                                                                                                                                                                                                                                          9996
                          a. e \rightarrow_{F-S} BndryErr
9942
                                                                                                                                                                                                                                                          9997
                               by v_0 v_1 \triangleright_{S-1} BndryErr
                                                                                                                                                  a. e \rightarrow_{F-S} BndryErr
                                                                                                                                                      by snd v \triangleright_{S-1} BndryErr
                                                                                                                                                                                                                                                          9998
                                                                                                                                                                                                                                                          9999
9944
                       IF v_0 = \text{mon}(\tau_d' \Rightarrow \tau_c') \lambda(x : \tau_x). e'
                                                                                                                                                  b. QED
9945
                                                                                                                                       CASE e = E^{\bullet}[op^2 v_0 v_1]:
                                                                                                                                                                                                                                                          10000
                             \wedge \mathcal{X}(\tau_x, v_1) = v_1':
9946
                         a. e \rightarrow_{F-S} E^{\bullet}[\text{stat } \tau'_c (\text{chk } \tau'_c e'[x \leftarrow v'_1])]
                                                                                                                                          1. \vdash_{\mathsf{F}} op^2 v_0 v_1 : \tau'
                                                                                                                                                                                                                                                          10001
                                                                                                                                               by static hole typing
                                                                                                                                                                                                                                                          10002
                               by v_0 \ v_1 >_{S-1} \text{ stat } \tau'_c \text{ (chk } \tau'_c \ e'[x \leftarrow v'_1])
9948
                                                                                                                                                                                                                                                          10003
                                                                                                                                          2. \vdash_{\mathsf{F}} v_0 : \tau_0
                                                                                                                                                                                                                                                          10004
                       ELSE v_0 = \text{mon}(\tau_d' \Rightarrow \tau_c') \lambda(x : \tau_x). e'
                                                                                                                                               \wedge \vdash_{\mathsf{F}} v_1 : \tau_1
9950
                                                                                                                                                                                                                                                          10005
                                   \wedge \mathcal{X}(\tau_x, v_1) = \text{BndryErr}:
                                                                                                                                               \wedge \Delta(op^2, \tau_0, \tau_1) = \tau^{\prime\prime}
9951
                                                                                                                                                                                                                                                          10006
                          a. e \rightarrow_{F-S} BndryErr
                                                                                                                                               by inversion
9952
                                                                                                                                          3. \delta(op^2, v_0, v_1) = e'
                                                                                                                                                                                                                                                          10007
                               by v_0 v_1 \triangleright_{S-1} BndryErr
9953
                                                                                                                                                                                                                                                          10008
                          b. QED
                                                                                                                                               by \Delta type soundness (2)
                                                                                                                                                                                                                                                          10009
```

```
If \vdash_{\mathsf{E}} e then one of the following holds:
                4. op^2 v_0 v_1 \triangleright_{S-1} e'
10011
                                                                                                                                                                                                                                     10066
                                                                                                                           • e is a value
10012
                     by (3)
                                                                                                                                                                                                                                     10067
                                                                                                                           • e \in Err
                5. QED by e \rightarrow_{F-S} E^{\bullet}[e']
10013
                                                                                                                                                                                                                                     10068
                                                                                                                           • e \rightarrow_{\mathsf{F-D}} e'
              CASE e = E[\text{dyn } \tau' \ e'] \text{ and } e' \text{ is boundary-free} :
10014
                                                                                                                                                                                                                                     10069
                                                                                                                           • e \rightarrow_{F-D} BndryErr
10015
                1. e' is a value
                                                                                                                                                                                                                                     10070
                                                                                                                          • e \rightarrow_{F-D} TagErr
                     \forall e' \in Err
                                                                                                                                                                                                                                     10071
10016
                                                                                                                         Proof:
                     \vee e' \rightarrow_{\mathsf{F-D}} e''
10017
                                                                                                                                                                                                                                     10072
                                                                                                                            By the boundary factoring lemma, there are seven cases.
                     \vee e' \rightarrow_{\mathsf{F-D}} \mathsf{BndryErr}
10018
                                                                                                                                                                                                                                     10073
                                                                                                                            CASE e is a value :
10019
                     \vee e' = E'[e''] and e'' \triangleright_{D-1} \mathsf{TagErr}
                                                                                                                                                                                                                                     10074
                                                                                                                               1. QED
                     by dynamic progress
10020
                                                                                                                                                                                                                                     10075
                                                                                                                            CASE e = E^{\bullet}[v_0 \ v_1]:
10021
                2. IF e' is a value :
                                                                                                                                                                                                                                     10076
                                                                                                                                IF v_0 = \lambda x. e':
                       a. QED e \rightarrow_{\mathsf{F-S}} E[\mathcal{D}_{\mathsf{F}}(\tau',e')]
10022
                                                                                                                                                                                                                                     10077
                                                                                                                                  1. e \rightarrow_{\mathsf{F-D}} E^{\bullet}[e'[x \leftarrow v_1]]
                     IF e' \in Err:
10023
                                                                                                                                                                                                                                     10078
                                                                                                                                       by v_0 \ v_1 \rhd_{\mathsf{D}\text{-}1} e'[x \leftarrow v_1]
                       a. QED e \rightarrow_{F-S} e'
10024
                                                                                                                                                                                                                                     10079
                                                                                                                                  2. QED
10025
                     IF e' \rightarrow_{F-D} e'':
                                                                                                                                                                                                                                     10080
                                                                                                                                IF v_0 = \text{mon}(\tau_d \Rightarrow \tau_c)(\lambda x. e'):
10026
                       a. QED e \rightarrow_{F-S} E[dyn \tau' e'']
                                                                                                                                                                                                                                     10081
                                                                                                                                  1. e \rightarrow_{\mathsf{F-D}} E^{\bullet}[e'[x \leftarrow v_1]]
10027
                     IF e' \rightarrow_{F-D} BndryErr:
                                                                                                                                                                                                                                     10082
                                                                                                                                       by v_0 v_1 \triangleright_{D-1} e'[x \leftarrow v_1]
10028
                       a. QED e \rightarrow_{F-S} E[dyn \tau' BndryErr]
                                                                                                                                                                                                                                     10083
                                                                                                                                  2. OED
                     ELSE e' = E'[e''] and e'' >_{D-1} TagErr:
10029
                                                                                                                                                                                                                                     10084
                                                                                                                                IF v_0 = \text{mon}(\tau_d \Rightarrow \tau_c)(\lambda(x:\tau_x). e')
10030
                       a. E' \in E^{\bullet}
                                                                                                                                                                                                                                     10085
                                                                                                                                      \wedge \mathcal{X}(\tau_x, v_1) = v_1':
10031
                            by e' is boundary-free
                                                                                                                                                                                                                                     10086
                                                                                                                                  1. e \rightarrow_{\mathsf{F-D}} E^{\bullet}[\mathsf{stat} \ \tau_c \ (\mathsf{chk} \ \tau_c \ e'[x \leftarrow v'_1])]
10032
                                                                                                                                                                                                                                     10087
                                                                                                                                       by v_0 \ v_1 \rhd_{D-1} stat \tau_c (chk \tau_c \ e'[x \leftarrow v'_1])
              CASE e = E[\text{stat } \tau' \ e'] \text{ and } e' \text{ is boundary-free} :
                                                                                                                                                                                                                                     10088
10033
                                                                                                                                  2. OED
                1. e' is a value
10034
                                                                                                                                                                                                                                     10089
                                                                                                                                IF v_0 = \text{mon}(\tau_d \Rightarrow \tau_c)(\lambda(x:\tau_x).e')
                     \forall e' \in Err
10035
                                                                                                                                                                                                                                     10090
                                                                                                                                      \wedge X(\tau_x, v_1) = \text{BndryErr}:
10036
                     \vee e' \rightarrow_{\mathsf{F-S}} e''
                                                                                                                                                                                                                                     10091
                                                                                                                                  1. e \rightarrow_{F-D} BndryErr
10037
                     \vee e' \rightarrow_{F-S} BndryErr
                                                                                                                                                                                                                                     10092
                                                                                                                                       by v_0 v_1 \triangleright_{D-1} BndryErr
                     \vee e' = E''[\mathsf{dyn} \ \tau'' \ E^{\bullet''}[e'']] \ \mathsf{and} \ e'' \rhd_{\mathsf{D-1}} \mathsf{TagErr}
10038
                                                                                                                                                                                                                                     10093
                                                                                                                                  2. QED
                     by static progress
10039
                                                                                                                                                                                                                                     10094
                                                                                                                                IF v_0 = \lambda(x:\tau_x). e':
10040
                2. IF e' is a value :
                                                                                                                                                                                                                                     10095
                                                                                                                                  1. Contradiction by \vdash_{\mathbf{E}} e
10041
                       a. QED e \rightarrow_{F-S} E[S_F(\tau', e')]
                                                                                                                                                                                                                                     10096
                                                                                                                                ELSE v_0 = i
                     IF e' \in Err:
                                                                                                                                                                                                                                     10097
                                                                                                                                           \vee v_0 = \langle v, v' \rangle
10043
                       a. QED e \rightarrow_{F-S} e'
                                                                                                                                                                                                                                     10098
                                                                                                                                           \vee v_0 = \text{mon } \tau_0 \times \tau_1 v':
10044
                     IF e' \rightarrow_{F-S} e'':
                                                                                                                                                                                                                                     10099
                                                                                                                                  1. e \rightarrow_{F-D} TagErr
10045
                       a. QED e \rightarrow_{F-S} E[\text{stat } \tau' e'']
                                                                                                                                                                                                                                     10100
                                                                                                                                       by (v_0 \ v_1) \triangleright_{D-1} \mathsf{TagErr}
                     IF e' \rightarrow_{F-S} BndryErr:
10046
                                                                                                                                                                                                                                     10101
                                                                                                                                  2. OED
10047
                       a. QED e \rightarrow_{F-S} E[\text{stat } \tau' \text{ BndryErr}]
                                                                                                                                                                                                                                     10102
                                                                                                                            CASE e = E^{\bullet}[op^1 v]:
                     ELSE e' = E''[\text{dyn } \tau'' E^{\bullet ''}[e'']] and e'' \rhd_{D-1} \text{TagErr}
10048
                                                                                                                                                                                                                                     10103
                                                                                                                                IF v = \text{mon } \tau_0 \times \tau_1 \langle v_0, v_1 \rangle
10049
                                                                                                                                                                                                                                     10104
                                                                                                                                      \wedge op^1 = fst
                       a. Contradiction by e' is boundary-free
                                                                                                                                                                                                                                     10105
10050
                                                                                                                                      \wedge X(\tau_0, v_0) = v_0':
              CASE e = E[Err]:
10051
                                                                                                                                                                                                                                     10106
                                                                                                                                  1. e \rightarrow_{\mathsf{F-D}} E^{\bullet}[v_{\mathsf{n}}']
                1. QED e \rightarrow_{F-S} Err
10052
                                                                                                                                                                                                                                     10107
                                                                                                                                       by op^1 v \triangleright_{D-1} v'_0
              CASE e = E^{\bullet}[\operatorname{chk} \tau' \upsilon]:
10053
                                                                                                                                                                                                                                     10108
                                                                                                                                  2. QED
                 IF X(\tau, v) = v':
10054
                                                                                                                                                                                                                                     10109
                                                                                                                                IF v = \text{mon } \tau_0 \times \tau_1 \langle v_0, v_1 \rangle
10055
                    1. e \rightarrow_{\mathsf{F-S}} E^{\bullet}[v']
                                                                                                                                                                                                                                     10110
                                                                                                                                      \wedge op^1 = fst
                        by (chk \tau v) \triangleright_{S-1} v'
10056
                                                                                                                                                                                                                                     10111
                                                                                                                                      \wedge X(\tau_0, v_0) = \text{BndryErr}:
10057
                    2. OED
                                                                                                                                                                                                                                     10112
                                                                                                                                  1. e \rightarrow_{F-D} BndryErr
                 ELSE \mathcal{X}(\tau, v) = \text{BndryErr}:
10058
                                                                                                                                                                                                                                     10113
                                                                                                                                       by op^1 v \triangleright_{D-1} BndryErr
10059
                    1. e \rightarrow_{F-S} BndryErr
                                                                                                                                                                                                                                     10114
                                                                                                                                  2. QED
10060
                        by (chk \tau v) \triangleright_{S-1} BndryErr
                                                                                                                                                                                                                                     10115
                                                                                                                                IF v = \text{mon } \tau_0 \times \tau_1 \langle v_0, v_1 \rangle
10061
                                                                                                                                                                                                                                     10116
                    2. QED
                                                                                                                                      \wedge op^1 = \text{snd}
10062
                                                                                                                                                                                                                                     10117
                                                                                                                                     \wedge \mathcal{X}(\tau_1, v_1) = v_1':
10063
                                                                                                                                                                                                                                     10118
          Lemma 6.9: HF dynamic progress
10064
                                                                                                                                                                                                                                     10119
10065
                                                                                                                                                                                                                                     10120
                                                                                                                  92
```

```
10121
                    1. e \rightarrow_{\mathsf{F-D}} E^{\bullet}[v_1']
                                                                                                                                      a. OED e \rightarrow_{\mathsf{F-S}} e'
                                                                                                                                                                                                                                      10176
                                                                                                                                    IF e' \rightarrow_{F-S} e'':
10122
                        by op^1 v \triangleright_{D-1} v'_1
                                                                                                                                                                                                                                      10177
10123
                    2. OED
                                                                                                                                      a. QED e \rightarrow_{F-S} E[\text{stat } \tau' e'']
                                                                                                                                                                                                                                      10178
                                                                                                                                    IF e' \rightarrow_{F-S} BndryErr:
                                                                                                                                                                                                                                      10179
10124
                 IF v = \text{mon } \tau_0 \times \tau_1 \langle v_0, v_1 \rangle
                                                                                                                                      a. QED e \rightarrow_{F-S} E[\text{stat } \tau' \text{ BndryErr}]
10125
                       \wedge op^1 = \text{snd}
                                                                                                                                                                                                                                      10180
                       \wedge \mathcal{X}(\tau_1, v_1) = \text{BndryErr}:
                                                                                                                                    ELSE e' = E''[\text{dyn } \tau'' E^{\bullet ''}[e'']] and e'' \triangleright_{D-1} \text{TagErr}
                                                                                                                                                                                                                                      10181
10126
                                                                                                                                                                                                                                      10182
10127
                    1. e \rightarrow_{F-D} BndryErr
                        by op^1 v \triangleright_{D-1} BndryErr
                                                                                                                                      a. Contradiction by e' is boundary-free
                                                                                                                                                                                                                                      10183
10128
10129
                    2. QED
                                                                                                                            CASE e = E[Err]:
                                                                                                                                                                                                                                      10184
10130
                 IF \delta(op^1, v) = e':
                                                                                                                               1. QED e \rightarrow_{F-D} Err
                                                                                                                                                                                                                                      10185
10131
                    1. (op^1 v) >_{D-1} e'
                                                                                                                                                                                                                                      10186
                    2. QED by e \rightarrow_{\mathsf{F-D}} E^{\bullet}[e']
10132
                                                                                                                         Lemma 6.10 : HF static preservation
                                                                                                                                                                                                                                      10187
10133
                 ELSE \delta(op^1, v) is undefined :
                                                                                                                                                                                                                                      10188
                                                                                                                         If \vdash_{\mathsf{F}} e : \tau and e \rightarrow_{\mathsf{F-S}} e' then \vdash_{\mathsf{F}} e' : \tau
                    1. e \rightarrow_{F-D} TagErr
10134
                                                                                                                                                                                                                                      10189
                                                                                                                         Proof:
10135
                        by (op^1 v) \triangleright_{D-1} TagErr
                                                                                                                                                                                                                                      10190
                                                                                                                            By the boundary factoring lemma there are eight cases.
10136
                    2. QED
                                                                                                                                                                                                                                      10191
                                                                                                                            CASE e is a value :
10137
              CASE e = E^{\bullet}[op^2 v_0 v_1]:
                                                                                                                                                                                                                                      10192
                                                                                                                               1. Contradiction by e \rightarrow_{F-S} e'
10138
                 IF \delta(op^2, v_0, v_1) = e'':
                                                                                                                                                                                                                                      10193
                                                                                                                             CASE e = E^{\bullet}[v_0 \ v_1]:
                    1. op^2 v_0 v_1 \triangleright_{D-1} e''
10139
                                                                                                                                                                                                                                      10194
                                                                                                                                IF v_0 = \lambda(x : \tau_x). e'
10140
                    2. OED
                                                                                                                                                                                                                                      10195
                                                                                                                                      \wedge e \rightarrow_{\mathsf{F-S}} e'[x \leftarrow v_1]:
10141
                 ELSE \delta(op^2, v_0, v_1) is undefined :
                                                                                                                                                                                                                                      10196
                                                                                                                                   1. \vdash_{\mathsf{E}} v_0 \ v_1 : \tau'
10142
                    1. e \rightarrow_{F-D} TagErr
                                                                                                                                                                                                                                      10197
                                                                                                                                       by static hole typing
                        by op^2 v_0 v_1 \triangleright_{D-1} TagErr
                                                                                                                                                                                                                                      10198
10143
                                                                                                                                   2. \vdash_{\mathsf{F}} v_0 : \tau_d \Rightarrow \tau_c
10144
                    2. OED
                                                                                                                                                                                                                                      10199
                                                                                                                                       \wedge \vdash_{\mathsf{F}} v_1 : \tau_d
              CASE e = E[dyn \tau' e'] and e' is boundary-free:
10145
                                                                                                                                       \wedge \tau_c \leqslant : \tau'
                                                                                                                                                                                                                                      10200
10146
                1. e' is a value
                                                                                                                                                                                                                                      10201
                                                                                                                                       by inversion
10147
                     \forall e' \in Err
                                                                                                                                                                                                                                      10202
                                                                                                                                  3. (x:\tau_x) \vdash_{\mathsf{E}} e' : \tau_c
                     \vee e' \rightarrow_{\mathsf{F-D}} e''
10148
                                                                                                                                                                                                                                      10203
                                                                                                                                       by inversion (2)
                     \vee e' \rightarrow_{\mathsf{F-D}} \mathsf{BndryErr}
10149
                                                                                                                                                                                                                                      10204
                                                                                                                                   4. \tau_d \leqslant : \tau_x
                     \vee e' = E^{\bullet}[e''] and e'' \rhd_{D-1} \mathsf{TagErr}
10150
                                                                                                                                                                                                                                      10205
                                                                                                                                       by canonical forms (2)
10151
                     by dynamic progress
                                                                                                                                                                                                                                      10206
                                                                                                                                   5. \vdash_{\mathsf{F}} v_1 : \tau_x
10152
                2. IF e' is a value :
                                                                                                                                                                                                                                      10207
                                                                                                                                       by (2, 4)
10153
                       a. QED e \rightarrow_{F-D} E[\mathcal{D}_F(\tau', e')]
                                                                                                                                                                                                                                      10208
                                                                                                                                   6. \vdash_{\mathsf{E}} e'[x \leftarrow v_1] : \tau_c
10154
                     IF e' \in Err:
                                                                                                                                                                                                                                      10209
                                                                                                                                       by substitution (3, 5)
10155
                       a. OED e \rightarrow_{\mathsf{F-D}} e'
                                                                                                                                                                                                                                      10210
                                                                                                                                   7. \vdash_{\mathsf{E}} e'[x \leftarrow v_1] : \tau'
                     IF e' \rightarrow_{F-D} e'':
10156
                                                                                                                                                                                                                                      10211
                                                                                                                                       by (2, 6)
10157
                       a. QED e \rightarrow_{F-S} E[dyn \tau' e'']
                                                                                                                                                                                                                                      10212
                                                                                                                                   8. QED by hole substitution
10158
                     IF e' \rightarrow_{F-D} BndryErr:
                                                                                                                                                                                                                                      10213
                                                                                                                                IF v_0 = \text{mon } \tau_d \Rightarrow \tau_c \lambda x. e'
                       a. QED e \rightarrow_{F-D} E[dyn \tau' BndryErr]
10159
                                                                                                                                                                                                                                      10214
                                                                                                                                      \land e \rightarrow_{\mathsf{F-S}} E^{\bullet}[\mathsf{dyn} \ \tau_c \ ((\lambda x. e') \ (\mathcal{X}(\tau_d, v_1))] :
                     ELSE e' = E[e''] and e'' \triangleright_{D-1} TagErr:
                                                                                                                                                                                                                                      10215
10160
                                                                                                                                   1. \vdash_{\mathsf{F}} v_0 \ v_1 : \tau'
                       a. E \in E^{\bullet}
10161
                                                                                                                                                                                                                                      10216
                                                                                                                                       by static hole typing
10162
                            by e' is boundary-free
                                                                                                                                                                                                                                      10217
                                                                                                                                   2. \vdash_{\mathsf{F}} v_0 : \tau_d' \Rightarrow \tau_c'
10163
                       b. oed
                                                                                                                                                                                                                                      10218
                                                                                                                                       \wedge \tau_c' \leqslant : \tau'
              CASE e = E[\text{stat } \tau' e'] and e' is boundary-free:
                                                                                                                                                                                                                                      10219
10164
                                                                                                                                       by inversion
10165
                1. e' is a value
                                                                                                                                                                                                                                      10220
                                                                                                                                   3. \tau_d \Rightarrow \tau_c \leqslant : \tau_d' \Rightarrow \tau_c'
                     \forall e' \in Err
10166
                                                                                                                                                                                                                                      10221
                                                                                                                                       by canonical forms (2)
                     \vee e' \rightarrow_{\mathsf{F-S}} e''
10167
                                                                                                                                                                                                                                      10222
                                                                                                                                   4. \vdash_{\mathsf{F}} \lambda x. e'
                     \vee e' \rightarrow_{\mathsf{F-S}} \mathsf{BndryErr}
10168
                                                                                                                                                                                                                                      10223
                                                                                                                                       by inversion (2)
10169
                     \vee e' = E''[\mathsf{dyn} \ \tau'' \ E^{\bullet}''[e'']] \ \mathsf{and} \ e'' \rhd_{\mathsf{D-1}} \mathsf{TagErr}
                                                                                                                                                                                                                                      10224
                                                                                                                                   5. \tau'_d \leqslant : \tau_d
10170
                     by static progress
                                                                                                                                                                                                                                      10225
                                                                                                                                       \wedge \tau_c \leqslant : \tau'_c
10171
                2. IF e' is a value :
                                                                                                                                                                                                                                      10226
                                                                                                                                       by (3)
10172
                       a. QED e \rightarrow_{F-S} E[S_F(\tau', e')]
                                                                                                                                                                                                                                      10227
                                                                                                                                   6. \vdash_{\mathsf{E}} X(\tau_d, v_1)
10173
                     IF e' \in Err:
                                                                                                                                                                                                                                      10228
                                                                                                                                       by X soundness
10174
                                                                                                                                                                                                                                      10229
10175
                                                                                                                                                                                                                                      10230
                                                                                                                  93
```

10231	$7 + (\lambda m c') Y(\tau, m)$	2   /2   2   \ 7   \ 7	10286
10231	7. $\vdash_{F} (\lambda x. e') X(\tau_d, v_1)$ by $(4, 6)$	2. $\vdash_{F} \langle v_0, v_1 \rangle : \tau_0 \times \tau_1 \\ \land \tau_1 \leqslant : \tau'$	10287
10232	8. $\vdash_{F} dyn \ \tau_c \ ((\lambda x. \ e') \ \mathcal{X}(\tau_d, v_1)) : \tau_c$	by inversion	10288
10234	by (7)	3. $\vdash_{F} v_1 : \tau_1$	10289
10235	9. $\vdash_{F} dyn \ \tau_{C} \left( (\lambda x. \ e') \ \mathcal{X}(\tau_d, v_1) \right) : \tau'$	by inversion	10290
10236	by $(2, 5, 8)$	4. $\vdash_{\mathbf{F}} v_1 : \tau'$	10291
10237	10. QED by hole substitution	by (2)	10292
10238	ELSE $v_0 = \text{mon } \tau_d \Rightarrow \tau_c (\lambda(x:\tau_x). e')$	5. QED by hole substitution	10293
10239	$\wedge e \rightarrow_{F-S} E^{\bullet}[stat \ \tau_{c} \ (chk \ \tau_{c} \ ((\lambda(x : \tau_{x}). \ e') \ X(\tau_{x}, v_{1})))]$	<b>IF</b> $v = \text{mon}(\tau_0 \times \tau_1) \langle v_0, v_1 \rangle$	10294
10240	$ \begin{array}{c} \mathcal{H} = \left\{ \begin{array}c} \mathcal{H} = \left\{ \begin{array}{c} \mathcal{H} = \left\{ \begin{array}{c} \mathcal{H} = \left\{ \begin{array}c} \mathcal{H} = \left\{ \begin{array}{c} \mathcal{H} = \left\{ \begin{array}c} \mathcal{H} = \mathcal{H} = \left\{ \begin{array}c} \mathcal{H} = \left\{ \begin{array}c} \mathcal{H} = \mathcal{H} = \left\{ \begin{array}c} \mathcal{H} = \mathcal{H} = \left\{ \begin{array}c} \mathcal{H} = \mathcal$	$\wedge op^1 = \text{fst}$	10295
10241	1. $\vdash_{F} v_0 \ v_1 : \tau'$	$\wedge e \to_{F-S} E^{\bullet}[X(\tau_0, v_0)]:$	10296
10242	by static hole typing	1. $\vdash_{F} fst  v : \tau'$	10297
10243	$2. \vdash_{F} v_0 : \tau_d' \Rightarrow \tau_c'$	by static hole typing	10298
10244	$\wedge \vdash_{F} v_1 : \tau'_d$	$2. \vdash_{F} v : \tau_0' \times \tau_1'$	10299
10245	$\wedge \tau'_c \leqslant : \tau'$	$\wedge \tau_0' <: \tau'$	10300
10246	by inversion	by inversion (1)	10301
10247	3. $\vdash_{F} \lambda(x : \tau_x). \ e' : \tau_x \Rightarrow \tau'_x$	3. $\tau_0 \leqslant \tau_0'$	10302
10248	by inversion	by canonical forms (2)	10303
10249	4. $\tau_d \Rightarrow \tau_c \leqslant : \tau_d' \Rightarrow \tau_c'$	4. $\vdash_{F} \langle v_0, v_1 \rangle$	10304
10250	by canonical forms (2)	$\vee \vdash_{F} \langle v_0, v_1 \rangle : \tau''$	10305
10251	5. $\tau_c \leqslant : \tau_c'$	by inversion (2)	10306
10252	by (4)	5. ⊢ <sub>F</sub> <i>v</i> <sub>0</sub>	10307
10253	6. $\vdash_{F} X(\tau_x, v_1) : \tau_x$	$\vee$ $\vdash_{F} v_0 : \tau'''$	10308
10254	by X soundness	by inversion (4)	10309
10255	7. $\vdash_{F} (\lambda(x : \tau_x). e') X(\tau_x, \upsilon_1) : \tau_x'$	6. $\vdash_{F} X(\tau_0, v_0) : \tau_0$	10310
10256	by (3, 6)	by X soundness (5)	10311
10257	8. $\vdash_{F} chk \ \tau_c \ ((\lambda(x : \tau_x). \ e') \ \mathcal{X}(\tau_x, v_1)) : \tau_c$	7. $\vdash_{F} X(\tau_0, v_0) : \tau'$	10312
10258	by (7)	by (2, 3, 6)	10313
10259	9. $\vdash_{F} stat \ \tau_c \ (chk \ \tau_c \ ((\lambda(x : \tau_x). \ e') \ \mathcal{X}(\tau_x, \upsilon_1))) : \tau_c$	8. QED by hole substitution	10314
10260	by (8)	<b>ELSE</b> $v = mon(\tau_0 \times \tau_1) \langle v_0, v_1 \rangle$	10315
10261	10. $\vdash_{F} stat \ \tau_c \ (chk \ \tau_c \ ((\lambda(x : \tau_x). \ e') \ \mathcal{X}(\tau_x, \upsilon_1))) : \tau'$	$\wedge op^1 = \text{snd}$	10316
10262	by (2, 5, 9)	$\wedge e \rightarrow_{F-S} E^{\bullet}[X(\tau_1, v_1)]:$	10317
10263	11. QED by hole substitution	1. $\vdash_{F} snd v : \tau'$	10318
10264	CASE $e = E^{\bullet}[op^1 v]$ :	by static hole typing	10319
10265	<b>IF</b> $v = \langle v_0, v_1 \rangle$	$2. \vdash_{F} \upsilon : \tau_0' \times \tau_1'$	10320
10266	$\wedge op^1 = fst$	$\wedge \tau_1' <: \tau'$	10321
10267	$\wedge e \rightarrow_{F-S} E^{\bullet}[v_0]:$	by inversion (1)	10322
10268	1. $\vdash_{F} fst \langle v_0, v_1 \rangle : \tau'$	3. $\tau_1 \leqslant : \tau_1'$	10323
10269	by static hole typing	by canonical forms (2)	10324
10270	2. $\vdash_{F} \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$	4. $\vdash_{F} \langle v_0, v_1 \rangle$	10325
10271	$\wedge \tau_0 \leqslant : \tau'$	$\lor \vdash_{F} \langle v_0, v_1 \rangle :  au''$	10326
10272	by inversion	by inversion (2)	10327
10273	3. $\vdash_{F} v_0 : \tau_0$	5. $\vdash_{F} v_1$	10328
10274	by inversion	$\vee \vdash_{F} v_1 : \tau'''$	10329
10275	4. $\vdash_{F} v_0 : \tau'$	by inversion (4)	10330
10276	by (2)	6. $\vdash_{F} X(\tau_1, v_1) : \tau_1$	10331
10277	5. QED by hole substitution	by $X$ soundness (5)	10332
10278	$\mathbf{IF} \ \ v = \langle v_0, v_1 \rangle$	7. $\vdash_{F} X(\tau_1, v_1) : \tau'$	10333
10279	$\wedge op^1 = \operatorname{snd}$	by (2, 3, 6)	10334
10280	$\wedge e \to_{F-S} E^{\bullet}[v_1]:$	8. QED by hole substitution	10335
10281	1. $\vdash_{F} snd \langle v_0, v_1 \rangle : \tau'$	<b>CASE</b> $e = E^{\bullet}[op^2 v_0 v_1]$	10336
10282	by static hole typing	$\wedge \delta(op^2, v_0, v_1) = v$	10337
10283		$\wedge e \rightarrow_{F-S} E^{\bullet}[v]:$	10338
10284			10339
10285	94		10340

```
1. \vdash_{\mathsf{F}} op^2 v_0 v_1 : \tau'
10341
                                                                                                                                      6. QED by hole substitution (5)
                                                                                                                                                                                                                                           10396
10342
                     by static hole typing
                                                                                                                               CASE e = E[Err]:
                                                                                                                                                                                                                                           10397
10343
                 2. \vdash_{\mathsf{F}} v_0 : \tau_0
                                                                                                                                  1. e \rightarrow_{F-S} Err
                                                                                                                                                                                                                                           10398
10344
                                                                                                                                  2. QED by \vdash_{\mathsf{E}} \mathsf{Err} : \tau
                     \wedge \vdash_{\mathsf{F}} v_1 : \tau_1
                                                                                                                                                                                                                                           10399
10345
                      \wedge \Delta(op^2, \tau_0, \tau_1) = \tau^{\prime\prime}
                                                                                                                                CASE e = E^{\bullet}[\operatorname{chk} \tau v]:
                                                                                                                                                                                                                                           10400
                     \wedge \tau'' \leqslant : \tau'
10346
                                                                                                                                  1. \vdash_{\mathsf{E}} \mathsf{chk} \; \tau \; v : \tau
                                                                                                                                                                                                                                           10401
10347
                     by inversion
                                                                                                                                       by static hole typing
                                                                                                                                                                                                                                           10402
                 3. \vdash_{\mathsf{F}} \upsilon : \tau^{\prime\prime}
10348
                                                                                                                                  2. \vdash_{\mathsf{E}} v : \tau'
                                                                                                                                                                                                                                           10403
10349
                     by \Delta type soundness (2)
                                                                                                                                       by inversion (1)
                 4. ⊨ υ: τ'
10350
                                                                                                                                  3. \vdash_{\mathsf{F}} \mathcal{X}(\tau, \upsilon') : \tau
                                                                                                                                                                                                                                           10405
10351
                     by (2, 3)
                                                                                                                                      by X soundness (2)
                                                                                                                                                                                                                                           10406
                 5. QED by hole substitution (4)
                                                                                                                                  4. QED by hole substitution (3)
10352
                                                                                                                                                                                                                                           10407
                                                                                                                           10353
              CASE e = E[dyn \tau' e'] and e' is boundary-free :
                                                                                                                                                                                                                                           10408
                  IF e' is a value :
10354
                                                                                                                           Lemma 6.11: HF dynamic preservation
                                                                                                                                                                                                                                           10409
10355
                     1. e \rightarrow_{\mathsf{F-S}} E[\mathcal{D}_{\mathsf{F}}(\tau', e')]
                                                                                                                                                                                                                                           10410
                                                                                                                            If \vdash_{\mathsf{F}} e and e \rightarrow_{\mathsf{F}\text{-}\mathsf{D}} e' then \vdash_{\mathsf{F}} e'
10356
                    2. \vdash dyn \tau' e' : \tau'
                                                                                                                                                                                                                                           10411
                                                                                                                            Proof:
10357
                         by boundary hole typing
                                                                                                                                                                                                                                           10412
                                                                                                                               By the boundary factoring lemma, there are seven cases.
10358
                                                                                                                                                                                                                                           10413
                    3. \ e e ′
                                                                                                                                CASE e is a value :
10359
                         by inversion (2)
                                                                                                                                                                                                                                           10414
                                                                                                                                  1. Contradiction by e \rightarrow_{F-D} e'
10360
                    4. \vdash_{\mathsf{E}} \mathcal{D}_{\mathsf{F}}(\tau',e'):\tau'
                                                                                                                                                                                                                                           10415
                                                                                                                                CASE e = E^{\bullet}[v_0 \ v_1]:
10361
                         by \mathcal{D}_{\mathsf{F}} soundness (3)
                                                                                                                                                                                                                                           10416
                                                                                                                                   IF v_0 = \lambda x. e'
                     5. QED by hole substitution (4)
10362
                                                                                                                                                                                                                                           10417
                                                                                                                                         \wedge e \rightarrow_{\mathsf{F-D}} E^{\bullet}[e'[x \leftarrow v_1]]:
                  ELSE e' \rightarrow_{F-D} e'':
                                                                                                                                                                                                                                           10418
10363
                                                                                                                                      1. \vdash_{\mathsf{E}} v_0 \ v_1
                     1. e \rightarrow_{F-S} E[dyn \tau' e'']
10364
                                                                                                                                                                                                                                           10419
                                                                                                                                          by dynamic hole typing
                    2. \vdash_{\mathsf{E}} \mathsf{dyn} \ \tau' \ e' : \tau'
10365
                                                                                                                                                                                                                                           10420
                                                                                                                                      2. ⊢ υ<sub>0</sub>
10366
                         by boundary hole typing
                                                                                                                                                                                                                                           10421
                                                                                                                                          \wedge \vdash_{\mathsf{F}} v_1
10367
                                                                                                                                                                                                                                           10422
                    3. ⊢ e′
                                                                                                                                          by inversion (1)
10368
                         by inversion (2)
                                                                                                                                                                                                                                           10423
                                                                                                                                     3. x \vdash_{\mathsf{F}} e'
10369
                     4. ⊦<sub>e</sub> e''
                                                                                                                                                                                                                                           10424
                                                                                                                                          by inversion (2)
10370
                         by dynamic preservation (3)
                                                                                                                                                                                                                                           10425
                                                                                                                                      4. \vdash_{\mathsf{F}} e'[x \leftarrow v_1]
10371
                     5. \vdash_{\mathsf{E}} \mathsf{dyn} \; \tau' \; e'' : \tau'
                                                                                                                                                                                                                                           10426
                                                                                                                                          by substitution (2, 3)
10372
                         by (4)
                                                                                                                                                                                                                                           10427
                                                                                                                                      5. QED hole substitution (4)
10373
                    6. QED by hole substitution (5)
                                                                                                                                                                                                                                           10428
                                                                                                                                   IF v_0 = \text{mon}(\tau_d \Rightarrow \tau_c) \lambda x. e'
10374
              CASE e = E[\text{stat } \tau' e'] and e' is boundary-free:
                                                                                                                                                                                                                                           10429
                                                                                                                                         \wedge e \rightarrow_{\mathsf{F-D}} E^{\bullet}[(\lambda x. e') v_1]:
10375
                  IF e' is a value :
                                                                                                                                                                                                                                           10430
                                                                                                                                      1. \vdash_{\mathsf{E}} v_0 \ v_1
10376
                     1. e \rightarrow_{\mathsf{F-S}} E[\mathcal{S}_{\mathsf{F}}(\tau', e')]
                                                                                                                                                                                                                                           10431
                                                                                                                                          by dynamic hole typing
10377

 1. ⊢ stat τ' e'

                                                                                                                                                                                                                                           10432
                                                                                                                                      2. ⊢ υ<sub>0</sub>
10378
                                                                                                                                                                                                                                           10433
                         by boundary hole typing
                                                                                                                                          \Lambda \vdash_{\mathsf{E}} v_1
                    3. \vdash_{\mathsf{F}} e' : \tau'
10379
                                                                                                                                                                                                                                           10434
                                                                                                                                          by inversion (1)
                         by inversion (2)
                                                                                                                                                                                                                                           10435
10380
                                                                                                                                      3. \vdash_{\mathsf{F}} (\lambda x. e') v_1
                     4. \vdash_{\mathsf{E}} \mathcal{S}_{\mathsf{F}}(\tau',e')
10381
                                                                                                                                                                                                                                           10436
                                                                                                                                          by (2)
10382
                         by S_{\rm F} soundness (3)
                                                                                                                                                                                                                                           10437
                                                                                                                                      4. QED hole substitution (5)
                    5. QED by hole substitution (4)
10383
                                                                                                                                                                                                                                           10438
                                                                                                                                   ELSE v_0 = \text{mon}(\tau_d \Rightarrow \tau_c) \lambda(x : \tau_x). e'
                  ELSE e' \rightarrow_{F-S} e'':
10384
                                                                                                                                                                                                                                           10439
                                                                                                                                             \land e \rightarrow_{\mathsf{F-D}} E^{\bullet}[\mathsf{stat}\ \tau_c\ (\mathsf{chk}\ \tau_c\ ((\lambda(x\!:\!\tau_x).\ e')\ \mathcal{X}(\tau_x,\upsilon_1)))]
                    1. e \rightarrow_{F-S} E[\text{stat } \tau' e'']
10385
                                                                                                                                                                                                                                           10440
                    2. \vdash_{\mathsf{F}} stat \tau' e'
10386
                                                                                                                                                                                                                                           10441
                                                                                                                                      1. \vdash_{\mathsf{F}} v_0 \ v_1
10387
                         by boundary hole typing
                                                                                                                                                                                                                                           10442
                                                                                                                                          by dynamic hole typing
10388
                    10443
                                                                                                                                      2. ⊢ υ<sub>0</sub>
10389
                         by inversion (2)
                                                                                                                                                                                                                                           10444
                                                                                                                                          \wedge \vdash_{\mathsf{F}} v_1
                     4. \vdash_{\scriptscriptstyle{\mathsf{E}}} e^{\prime\prime} : \tau^{\prime}
10390
                                                                                                                                                                                                                                           10445
                                                                                                                                          by inversion
10391
                         by static preservation (3)
                                                                                                                                                                                                                                           10446
                                                                                                                                      3. \vdash_{\mathsf{F}} \lambda(x:\tau_x). e':\tau_x \Longrightarrow \tau'_x
                    5. \vdash<sub>F</sub> stat \tau' e''
10392
                                                                                                                                                                                                                                           10447
                                                                                                                                          by inversion (2)
10393
                         by (4)
                                                                                                                                                                                                                                           10448
10394
                                                                                                                                                                                                                                           10449
10395
                                                                                                                                                                                                                                           10450
                                                                                                                     95
```

10451	$4. \vdash_{F} X(\tau_x, \upsilon_1) : \tau_x$	4. $\vdash_{F} \mathcal{X}(\tau_1, \upsilon_1)$	10506
10452	by X soundness (2)	by $X$ soundness (3)	10507
10453	5. $\vdash_{F} ((\lambda(x:\tau_x).e') X(\tau_x,v_1)) : \tau_x'$	5. QED by hole substitution	10508
10454	by (3, 4)	$\mathbf{CASE} \ \ e = E^{\bullet}[op^2 \ v_0 \ v_1]$	10509
10455	6. $\vdash_{F} chk \ \tau_c \ ((\lambda(x : \tau_x). \ e') \ \mathcal{X}(\tau_x, v_1)) : \tau_c$	$\wedge  \delta(op^2, v_0, v_1) = v$	10510
10456	by (5)	$\wedge e \to_{F-D} E^{\bullet}[v]:$	10511
10457	7. $\vdash_{F} stat \ \tau_c \ (chk \ \tau_c \ ((\lambda(x : \tau_x). \ e') \ \mathcal{X}(\tau_x, v_1)))$	1. $\vdash_{F} op^2 v_0 v_1$	10512
10458	by (6)	by dynamic hole typing	10513
10459	8. QED hole substitution	2. F <sub>F</sub> v <sub>0</sub>	10514
10460	<b>CASE</b> $e = E^{\bullet}[op^1 v]$ :	$\wedge \vdash_{\mathcal{E}} v_1$	10515 10516
10461 10462	$ \mathbf{IF}  v = \langle v_0, v_1 \rangle \\ \wedge op^1 = \mathbf{fst} $	by inversion (1)	10516
10463	*	3. $\vdash_{F} v$	10517
10464		by $\delta$ preservation (2) 4. QED by hole substitution (3)	10518
10465	by dynamic hole typing	<b>CASE</b> $e = E[\text{dyn } \tau' e']$ and $e'$ is boundary-free :	10519
10466	2. ⊧ <sub>F</sub> v	<b>IF</b> $e'$ is a value:	10521
10467	by inversion (1)	1. $e \rightarrow_{F-D} E[\mathcal{D}_{F}(\tau', e')]$	10522
10468	•	2. F <sub>ε</sub> dyn τ' e' : τ'	10523
10469	3. $\vdash_{F} v_0$ by inversion (2)	by boundary hole typing	10524
10470	4. QED by hole substitution	3. \	10525
10471	IF $v = \langle v_0, v_1 \rangle$	by inversion (2)	10526
10472	$\wedge op^1 = \text{snd}$	4. $\vdash_{F} \mathcal{D}_{F}(\tau',e') : \tau'$	10527
10473	$\wedge e \to_{F-D} E^{\bullet}[v_1]:$	by $\mathcal{D}_{F}$ soundness (3)	10528
10474	1. ⊢ <sub>F</sub> op¹ v	5. QED by hole substitution (4)	10529
10475	by dynamic hole typing	ELSE $e' \rightarrow_{F-D} e''$ :	10530
10476	2. ⊢ <sub>F</sub> v	1. $e \rightarrow_{\text{F-D}} E[\text{dyn } \tau' e'']$	10531
10477	by inversion (1)	2. $\vdash_{F} dyn \; \tau' \; e' : \tau'$	10532
10478	$3. \vdash_{F} v_1$	by boundary hole typing	10533
10479	by inversion (2)	3. ⊢ <sub>F</sub> e'	10534
10480	4. QED by hole substitution	$\wedge \tau' \leqslant : \tau''$	10535
10481	IF $v = mon(\tau_0 \times \tau_1) \langle v_0, v_1 \rangle$	by inversion (2)	10536
10482	$\wedge op^1 = fst$	4. ⊢ <sub>F</sub> e''	10537
10483	$\wedge e \rightarrow_{F-D} E^{\bullet}[\mathcal{X}(\tau_0, v_0)]:$	by dynamic preservation (3)	10538
10484	1. $\vdash_{F} op^1 v$	5. ⊦ <sub>F</sub> dyn τ' e'' : τ'	10539
10485	by dynamic hole typing	by (4)	10540
10486	2. $\vdash_{F} mon\left(\tau_0 \times \tau_1\right) \langle v_0, v_1 \rangle$	6. QED by hole substitution (5)	10541
10487	by inversion (1)	<b>CASE</b> $e = E[\text{stat } \tau' \ e']$ and $e'$ is boundary-free:	10542
10488	3. $\vdash_{F} v_0$	IF $e' \in v$ :	10543
10489	$ee$ $arphi_{ extsf{F}}$ $v_0: au_0'$	1. $e \rightarrow_{F-D} E[\mathcal{S}_{F}(\tau',e')]$	10544
10490	by inversion (2)	2. ⊢ <sub>F</sub> stat τ' e'	10545
10491	4. $\vdash_{F} \mathcal{X}(\tau_0, \nu_0)$	by boundary hole typing	10546
10492	by $X$ soundness (3)	3. $\vdash_{F} e' : \tau'$	10547
10493	5. QED by hole substitution	by inversion (2)	10548
10494	<b>ELSE</b> $v = mon(\tau_0 \times \tau_1) \langle v_0, v_1 \rangle$	$4. \vdash_{F} \mathcal{S}_{F}(\tau',e')$	10549
10495	$\wedge op^1 = \text{snd}$	by $S_{F}$ soundness (3)	10550
10496	$\wedge e \to_{F-D} E^{\bullet}[X(\tau_1, v_1)]:$	5. QED by hole substitution (5)	10551
10497	1. $\vdash_{F} op^1 v$	ELSE $e' \rightarrow_{F-S} e''$ :	10552
10498	by dynamic hole typing	1. $e \rightarrow_{F-D} E[stat \ \tau' \ e'']$	10553
10499	2. $\vdash_{F} mon (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle$	2. $\vdash_{F} stat \ \tau' \ e'$	10554
10500	by inversion (1)	by boundary hole typing	10555
10501 10502	3. ⊢ <sub>F</sub> v <sub>1</sub>	3. $\vdash_{F} e' : \tau'$	10556 10557
10502	$\forall \vdash_{F} v_1 : \tau'_1$ by inversion (2)	by inversion (2) $A = a'' : \pi'$	10557
10503	by inversion (2)	4. $\vdash_{F} e'' : \tau'$ by static preservation (3)	10558
10504		96	10559
10303		70	10300

```
5. \vdash<sub>F</sub> stat \tau' e''
                                                                                                                          a. QED e = E'[\text{stat } \tau E^{\bullet}[op^2 v_0 v_1]]
10561
                                                                                                                                                                                                                 10616
                                                                                                                 CASE e = E[\operatorname{chk} \tau v]:
10562
                      by (4)
                                                                                                                                                                                                                 10617
                                                                                                                    1. E = E^{\bullet}
10563
                  6. QED by hole substitution (5)
                                                                                                                                                                                                                 10618
            CASE e = E[Err]:
                                                                                                                        \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
10564
                                                                                                                                                                                                                 10619
10565
               1. e \rightarrow_{F-D} Err
                                                                                                                        \vee E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                                 10620
               2. QED ⊨ Err
                                                                                                                        by inner boundary
                                                                                                                                                                                                                 10621
10566
10567
                                                                                                                    2. IF E = E^{\bullet} :
                                                                                                                                                                                                                 10622
                                                                                                                          a. QED e = E^{\bullet}[\operatorname{chk} \tau \upsilon]
10568
                                                                                                                                                                                                                 10623
         Lemma 6.12: HF static boundary factoring
10569
                                                                                                                        IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
            If \vdash_{\mathsf{E}} e : \tau then one of the following holds:
                                                                                                                          a. Contradiction by \vdash_{\mathsf{E}} e : \tau
10570
                                                                                                                                                                                                                 10625
            • e is a value
10571
                                                                                                                        ELSE E = E'[\text{stat } \tau E^{\bullet}]:
                                                                                                                                                                                                                 10626
            • e = E^{\bullet}[v_0 \ v_1]
                                                                                                                          a. QED e = E'[\text{stat } \tau E^{\bullet}[\text{chk } \tau v]]
10572
            • e = E^{\bullet}[op^1 v]
                                                                                                                                                                                                                 10627
10573
                                                                                                                 CASE e = E[\mathsf{dyn} \ \tau \ v]:
                                                                                                                                                                                                                 10628
            • e = E^{\bullet}[op^2 v_0 v_1]
                                                                                                                   1. QED v is boundary-free
10574
            • e = E^{\bullet}[\operatorname{chk} \tau v]
                                                                                                                                                                                                                 10629
10575
                                                                                                                 CASE e = E[\operatorname{stat} \tau v]:
                                                                                                                                                                                                                 10630
            • e = E[dyn \tau e'] where e' is boundary-free
10576
                                                                                                                    1. QED v is boundary-free
                                                                                                                                                                                                                 10631
            • e = E[\text{stat } \tau e'] where e' is boundary-free
                                                                                                                 CASE e = E[Err]:
10577
                                                                                                                                                                                                                 10632
            • e = E[Err]
10578
         Proof:
                                                                                                                                                                                                                 10633
                                                                                                                    1. QED
            By the unique static evaluation contexts lemma, there are
10579
                                                                                                                                                                                                                 10634
10580
            eight cases.
                                                                                                                                                                                                                 10635
                                                                                                              Lemma 6.13: HF unique static evaluation contexts
10581
            CASE e is a value :
                                                                                                                                                                                                                 10636
                                                                                                               If \vdash_{\mathsf{E}} e : \tau then one of the following holds:
               1. QED
10582
                                                                                                                • e is a value
                                                                                                                                                                                                                 10637
            CASE e = E[v_0 \ v_1]:
10583
                                                                                                                \bullet e = E[v_0 \ v_1]
                                                                                                                                                                                                                 10638
               1. E=E^{\bullet}
10584
                                                                                                                \bullet \ e = E[op^1 \ v]
                                                                                                                                                                                                                 10639
                   \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
10585
                                                                                                                • e = E[op^2 v_0 v_1]
                                                                                                                                                                                                                 10640
10586
                   \vee E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                • e = E[\operatorname{chk} \tau v]
                                                                                                                                                                                                                 10641
10587
                   by inner boundary
                                                                                                                • e = E[\mathsf{dyn} \ \tau \ v]
                                                                                                                                                                                                                 10642
               2. IF E = E^{\bullet}:
                                                                                                                • e = E[\text{stat } \tau \ v]
                                                                                                                                                                                                                 10643
                     a. QED e = E^{\bullet}[v_0 \ v_1]
10589
                                                                                                                • e = E[Err]
                                                                                                                                                                                                                 10644
                   IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                              Proof:
10590
                                                                                                                                                                                                                 10645
                     a. QED e = E'[\mathsf{dyn} \ \tau \ E^{\bullet}[v_0 \ v_1]]
                                                                                                                 By induction on the structure of e.
10591
                                                                                                                                                                                                                 10646
                   ELSE E = E'[\text{stat } \tau E^{\bullet}]:
                                                                                                                 CASE e = x
                                                                                                                                                                                                                 10647
                     a. QED e = E'[\text{stat } \tau E^{\bullet}[v_0 \ v_1]]
                                                                                                                           \vee e = \lambda x. e'
10593
                                                                                                                                                                                                                 10648
            CASE e = E[op^1 v]:
                                                                                                                           \vee e = \operatorname{stat} \tau e':
10594
                                                                                                                                                                                                                 10649
                                                                                                                   1. Contradiction by \vdash_{\mathsf{F}} e : \tau
               1. E = E^{\bullet}
10595
                                                                                                                                                                                                                 10650
                                                                                                                 CASE e = i
                   \forall E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
10596
                                                                                                                                                                                                                 10651
                                                                                                                           \vee e = \lambda(x : \tau_d). e'
10597
                   \vee E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                                 10652
10598
                   by inner boundary
                                                                                                                           \vee e = \text{mon} (\tau_d \Rightarrow \tau_c) v:
                                                                                                                                                                                                                 10653
               2. IF E = E^{\bullet}:
                                                                                                                    1. QED e is a value
10599
                     a. QED e = E^{\bullet}[op^1 v]
                                                                                                                 CASE e = \langle e_0, e_1 \rangle:
                                                                                                                                                                                                                 10655
10600
                                                                                                                    IF e_0 \notin v:
                   IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
10601
                                                                                                                                                                                                                 10656
                     a. QED e = E'[\text{dyn } \tau E^{\bullet}[op^1 v]]
                                                                                                                       1. \vdash_{\mathsf{F}} e_0 : \tau_0
10602
                                                                                                                                                                                                                 10657
                   ELSE E = E'[\text{stat } \tau E^{\bullet}]:
                                                                                                                           by inversion
10603
                                                                                                                                                                                                                 10658
                     a. QED e = E'[\text{stat } \tau E^{\bullet}[op^1 v]]
                                                                                                                       2. e_0 = E_0[e'_0]
10604
                                                                                                                                                                                                                 10659
                                                                                                                           by the induction hypothesis (1)
             CASE e = E[op^2 v_0 v_1]:
10605
                                                                                                                                                                                                                 10660
               1. E = E^{\bullet}
                                                                                                                       3. E = \langle E_0, e_1 \rangle
10606
                                                                                                                                                                                                                 10661
                   \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                       4. QED e = E[e'_0]
10607
                                                                                                                                                                                                                 10662
                                                                                                                     IF e_0 \in v
                   \vee E = E'[\text{stat } \tau E^{\bullet}]
10608
                                                                                                                                                                                                                 10663
                   by inner boundary
                                                                                                                          \wedge e_1 \notin v:
10609
                                                                                                                                                                                                                 10664
               2. IF E = E^{\bullet}:
                                                                                                                       1. \vdash_{\mathsf{F}} e_1 : \tau_1
10610
                                                                                                                                                                                                                 10665
                     a. QED e = E^{\bullet}[op^2 v_0 v_1]
                                                                                                                           by inversion
10611
                                                                                                                                                                                                                 10666
                   IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                                       2. e_1 = E_1[e_1']
10612
                                                                                                                                                                                                                 10667
                     a. QED e = E'[\operatorname{dyn} \tau E^{\bullet}[op^2 v_0 v_1]]
                                                                                                                           by the induction hypothesis (1)
10613
                                                                                                                                                                                                                 10668
                   ELSE E = E'[\text{stat } \tau E^{\bullet}]:
                                                                                                                       3. E = \langle e_0, E_1 \rangle
10614
                                                                                                                                                                                                                 10669
10615
                                                                                                                                                                                                                 10670
                                                                                                        97
```

```
10671
                 4. QED e = E[e'_1]
                                                                                                             1. E = []
                                                                                                                                                                                                10726
              ELSE e_0 \in v
                                                                                                             2. QED e = E[op^2 e_0 e_1]
10672
                                                                                                                                                                                                10727
                                                                                                        CASE e = \operatorname{chk} \tau e_0:
10673
                       \land e_1 \in v:
                                                                                                                                                                                                10728
                 1. E = []
                                                                                                           IF e_0 \notin v:
10674
                                                                                                                                                                                                10729
10675
                 2. QED e is a value
                                                                                                             1. \vdash_{\mathsf{F}} e_0 : \tau_0
                                                                                                                                                                                                10730
                                                                                                                by inversion
                                                                                                                                                                                                10731
           CASE e = e_0 e_1:
10676
                                                                                                                                                                                                10732
10677
              IF e_0 \notin v:
                                                                                                             2. e_0 = E_0[e'_0]
                                                                                                                by the induction hypothesis (1)
                                                                                                                                                                                                10733
10678
                 1. \vdash_{\mathsf{F}} e_0 : \tau_0
10679
                    by inversion
                                                                                                             3. E = \operatorname{chk} \tau E_0
                                                                                                                                                                                                10734
                                                                                                             4. QED e = E[e'_0]
10680
                 2. e_0 = E_0[e'_0]
                                                                                                                                                                                                10735
                                                                                                           ELSE e_0 \in v:
10681
                    by the induction hypothesis (1)
                                                                                                                                                                                                10736
                 3. E = E_0 e_1
                                                                                                             1. E = []
10682
                                                                                                                                                                                                10737
                 4. QED e = E[e'_0]
10683
                                                                                                             2. QED e = E[\operatorname{chk} \tau \ e_0]
                                                                                                                                                                                                10738
10684
              IF e_0 \in v
                                                                                                        CASE e = \text{dyn } \tau e_0:
                                                                                                                                                                                                10739
10685
                                                                                                                                                                                                10740
                   \land e_1 \notin v:
                                                                                                           IF e_0 \notin v:
10686
                                                                                                                                                                                                10741
                 1. \vdash_{\mathsf{F}} e_1 : \tau_1
                                                                                                             1. \vdash_{\mathsf{F}} e_0
10687
                    by inversion
                                                                                                                by inversion
                                                                                                                                                                                                10742
10688
                 2. e_1 = E_1[e_1']
                                                                                                             2. e_0 = E_0[e'_0]
                                                                                                                                                                                                10743
                    by the induction hypothesis (1)
                                                                                                                by unique dynamic evaluation contexts (1)
10689
                                                                                                                                                                                                10744
10690
                 3. E = e_0 E_1
                                                                                                             3. E = \operatorname{dyn} \tau E_0
                                                                                                                                                                                                10745
10691
                 4. QED e = E[e'_1]
                                                                                                             4. QED e = E[e'_0]
                                                                                                                                                                                                10746
              ELSE e_0 \in v
                                                                                                           ELSE e_0 \in v:
10692
                                                                                                                                                                                                10747
                                                                                                             1. E = []
                                                                                                                                                                                                10748
10693
                       \land e_1 \in v:
                 1. E = []
10694
                                                                                                             2. QED e = E[dyn \tau e_0]
                                                                                                                                                                                                10749
                 2. QED e = E[e_0 \ e_1]
                                                                                                        CASE e = Err:
10695
                                                                                                                                                                                                10750
10696
            CASE e = op^1 e_0:
                                                                                                          1. E = []
                                                                                                                                                                                                10751
              IF e_0 \notin v:
                                                                                                          2. QED e = E[Err]
                                                                                                                                                                                                10752
10697
                 1. \vdash_{\mathsf{F}} e_0 : \tau_0
10698
                                                                                                                                                                                                10753
                    by inversion
10699
                                                                                                                                                                                                10754
                                                                                                    Lemma 6.14: HF inner boundary
10700
                 2. e_0 = E_0[e'_0]
                                                                                                                                                                                                10755
                                                                                                      For all contexts E, one of the following holds:
10701
                    by the induction hypothesis (1)
                                                                                                      • E = E^{\bullet}
                                                                                                                                                                                                10756
10702
                 3. E = op^1 E_0
                                                                                                      • E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                                                                                10757
                 4. QED e = E[e'_0]
10703
                                                                                                      • E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                10758
                                                                                                     Proof:
10704
              ELSE e_0 \in v:
                                                                                                                                                                                                10759
                                                                                                        By induction on the structure of E.
10705
                 1. E = []
                                                                                                                                                                                                10760
                                                                                                        CASE E = E^{\bullet}:
                2. QED e = E[op^1 e_0]
10706
                                                                                                                                                                                                10761
                                                                                                          1. QED
10707
            CASE e = op^2 e_0 e_1:
                                                                                                                                                                                                10762
10708
              IF e_0 \notin v:
                                                                                                        CASE E = E_0 e_1:
                                                                                                                                                                                                10763
                                                                                                          1. E_0 = E^{\bullet}
10709
                 1. \vdash_{\mathsf{F}} e_0 : \tau_0
                                                                                                                                                                                                10764
                                                                                                              \vee E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                                                                                10765
                    by inversion
10710
                                                                                                              \vee E_0 = E_0'[\operatorname{stat} \tau E^{\bullet}]
10711
                                                                                                                                                                                                10766
                 2. e_0 = E_0[e'_0]
                                                                                                              by the induction hypothesis
                    by the induction hypothesis (1)
10712
                                                                                                                                                                                                10767
                                                                                                          2. IF E_0 = E^{\bullet}:
                3. E = op^2 E_0 e_1
10713
                                                                                                                                                                                                10768
                 4. QED e = E[e'_0]
                                                                                                                a. QED E is boundary-free
10714
                                                                                                                                                                                                10769
                                                                                                              IF E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
10715
              IF e_0 \in v
                                                                                                                                                                                                10770
                                                                                                                a. E' = E'_0 e_1
10716
                   \wedge e_1 \notin v:
                                                                                                                                                                                                10771
                                                                                                                b. QED E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
10717
                 1. \vdash_{\mathsf{F}} e_1 : \tau_1
                                                                                                                                                                                                10772
                                                                                                              ELSE E_0 = E_0'[\text{stat } \tau E^{\bullet}]:
10718
                    by inversion
                                                                                                                                                                                                10773
                 2. e_1 = E_1[e_1']
                                                                                                                a. E' = E'_0 e_1
10719
                                                                                                                                                                                                10774
                                                                                                                b. QED E = E'[\text{stat } \tau E^{\bullet}]
10720
                    by the induction hypothesis (1)
                                                                                                                                                                                                10775
                                                                                                        CASE E = v_0 E_1:
10721
                 3. E = op^2 e_0 E_1
                                                                                                                                                                                                10776
                                                                                                         1. E_1 = E^{\bullet}
10722
                 4. QED e = E[e'_1]
                                                                                                                                                                                                10777
                                                                                                              \vee E_1 = E_1'[\mathsf{dyn} \ \tau \ E^{\bullet}]
10723
              ELSE e_0 \in v
                                                                                                                                                                                                10778
10724
                       \land e_1 \in v:
                                                                                                                                                                                                10779
10725
                                                                                                                                                                                                10780
                                                                                               98
```

```
10781
                     \vee E_1 = E_1'[\operatorname{stat} \tau E^{\bullet}]
                                                                                                                                    \vee E_0 = E_0'[\operatorname{stat} \tau E^{\bullet}]
                                                                                                                                                                                                                                       10836
10782
                     by the induction hypothesis
                                                                                                                                    by the induction hypothesis
                                                                                                                                                                                                                                       10837
                 2. IF E_1 = E^{\bullet}:
                                                                                                                               2. IF E_0 = E^{\bullet}:
10783
                                                                                                                                                                                                                                       10838
                       a. QED E is boundary-free
                                                                                                                                       a. QED E is boundary-free
10784
                                                                                                                                                                                                                                       10839
10785
                     IF E_1 = E_1'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                                                    IF E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                                                                                                                                                       10840
                       a. E' = v_0 E'_1
                                                                                                                                       a. E' = op^2 E'_0 e_1
                                                                                                                                                                                                                                       10841
10786
                                                                                                                                       b. QED E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                       b. QED E = E'[\text{dyn } \tau E^{\bullet}]
10787
                                                                                                                                                                                                                                       10842
                     ELSE E_1 = E_1'[\text{stat } \tau E^{\bullet}]:
                                                                                                                                    ELSE E_0 = E_0'[\text{stat } \tau E^{\bullet}]:
10788
                                                                                                                                                                                                                                       10843
10789
                       a. E' = v_0 E'_1
                                                                                                                                       a. E' = op^2 E'_0 e_1
                                                                                                                                                                                                                                       10844
                       b. QED E = E'[\text{stat } \tau E^{\bullet}]
10790
                                                                                                                                       b. QED E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                                                       10845
10791
              CASE E = \langle E_0, e_1 \rangle:
                                                                                                                             CASE E = op^2 v_0 E_1:
                                                                                                                                                                                                                                       10846
                1. E_0 = E^{\bullet}
10792
                                                                                                                                1. E_1 = E^{\bullet}
                                                                                                                                                                                                                                       10847
10793
                     \vee E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                                                                                                                       10848
                                                                                                                                    \vee E_1 = E_1'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                    \vee E_1 = E_1'[\operatorname{stat} \tau E^{\bullet}]
10794
                     \vee E_0 = E_0'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                                                       10849
10795
                     by the induction hypothesis
                                                                                                                                                                                                                                       10850
                                                                                                                                    by the induction hypothesis
10796
                 2. IF E_0 = E^{\bullet}:
                                                                                                                               2. IF E_1 = E^{\bullet}:
                                                                                                                                                                                                                                       10851
                       a. QED E is boundary-free
10797
                                                                                                                                                                                                                                       10852
                                                                                                                                       a. QED E is boundary-free
10798
                     IF E_0 = E_0'[\text{dyn } \tau \ E^{\bullet}]:
                                                                                                                                    IF E_1 = E_1'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                                                                                                                                                       10853
                       a. E' = \langle E'_0, e_1 \rangle
10799
                                                                                                                                                                                                                                       10854
                                                                                                                                       a. E' = op^2 v_0 E'_1
10800
                       b. QED E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                       b. QED E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                                                                                                                       10855
10801
                     ELSE E_0 = E_0'[\text{stat } \tau E^{\bullet}]:
                                                                                                                                    ELSE E_1 = E_1'[\text{stat } \tau E^{\bullet}]:
                                                                                                                                                                                                                                       10856
                       a. E' = \langle E'_0, e_1 \rangle
                                                                                                                                       a. E' = op^2 v_0 E'_1
10802
                                                                                                                                                                                                                                       10857
                       b. QED E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                                                       10858
10803
                                                                                                                                       b. QED E = E'[\text{stat } \tau E^{\bullet}]
              CASE E = \langle v_0, E_1 \rangle:
10804
                                                                                                                                                                                                                                       10859
                                                                                                                             CASE E = \operatorname{chk} \tau E_0:
                1. E_1 = E^{\bullet}
10805
                                                                                                                                1. E_0 = E^{\bullet}
                                                                                                                                                                                                                                       10860
10806
                     \vee E_1 = E_1'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                    \vee E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                                                                                                                       10861
                     \vee E_1 = E_1'[\operatorname{stat} \tau E^{\bullet}]
10807
                                                                                                                                    \vee E_0 = E_0'[\operatorname{stat} \tau E^{\bullet}]
                                                                                                                                                                                                                                       10862
10808
                     by the induction hypothesis
                                                                                                                                                                                                                                       10863
                                                                                                                                    by the induction hypothesis
                2. IF E_1 = E^{\bullet}:
10809
                                                                                                                                2. IF E_0 = E^{\bullet}:
                                                                                                                                                                                                                                       10864
10810
                       a. QED E is boundary-free
                                                                                                                                                                                                                                       10865
                                                                                                                                       a. QED E is boundary-free
                     IF E_1 = E_1'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
10811
                                                                                                                                    IF E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                                                                                                                                                       10866
10812
                       a. E' = \langle v_0, E'_1 \rangle
                                                                                                                                       a. E' = \operatorname{chk} \tau E'_0
                                                                                                                                                                                                                                       10867
10813
                       b. QED E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                                                                                                                                       10868
                                                                                                                                       b. QED E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
10814
                     ELSE E_1 = E_1'[\text{stat } \tau E^{\bullet}]:
                                                                                                                                    ELSE E_0 = E_0'[\text{stat } \tau E^{\bullet}]:
                                                                                                                                                                                                                                       10869
10815
                       a. E' = \langle v_0, E'_1 \rangle
                                                                                                                                                                                                                                       10870
                                                                                                                                       a. E' = \operatorname{chk} \tau E'_0
                       b. QED E = E'[\text{stat } \tau E^{\bullet}]
10816
                                                                                                                                       b. QED E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                                                       10871
10817
              CASE E = op^1 E_0:
                                                                                                                             CASE E = \text{dyn } \tau E_0:
                                                                                                                                                                                                                                       10872
                 1. E_0 = E^{\bullet}
10818
                                                                                                                                                                                                                                       10873
                                                                                                                                1. E_0 = E^{\bullet}
                     \vee E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]
10819
                                                                                                                                    \vee E_0 = E_0'[\operatorname{dyn} \tau' E^{\bullet}]
                                                                                                                                                                                                                                       10874
                     \vee E_0 = E_0'[\text{stat } \tau E^{\bullet}]
                                                                                                                                                                                                                                       10875
10820
                                                                                                                                    \vee E_0 = E_0'[\operatorname{stat} \tau' E^{\bullet}]
                     by the induction hypothesis
10821
                                                                                                                                    by the induction hypothesis
                                                                                                                                                                                                                                       10876
10822
                2. IF E_0 = E^{\bullet}:
                                                                                                                                                                                                                                       10877
                                                                                                                                2. IF E_0 = E^{\bullet}:
                       a. QED E is boundary-free
10823
                                                                                                                                                                                                                                       10878
                                                                                                                                      a. QED
                     IF E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
10824
                                                                                                                                    IF E_0 = E_0'[\text{dyn } \tau' E^{\bullet}]:
                                                                                                                                                                                                                                       10879
                       a. E' = op^1 E'_0
10825
                                                                                                                                                                                                                                       10880
                                                                                                                                       a. E' = \operatorname{dyn} \tau E'_0
                       b. QED E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
10826
                                                                                                                                       b. QED E = E'[\text{dyn } \tau' E^{\bullet}]
                                                                                                                                                                                                                                       10881
10827
                     ELSE E_0 = E_0'[\text{stat } \tau E^{\bullet}]:
                                                                                                                                    ELSE E_0 = E_0'[\operatorname{stat} \tau' E^{\bullet}]:
                                                                                                                                                                                                                                       10882
                       a. E' = op^1 E'_0
                                                                                                                                       a. E' = \operatorname{dyn} \tau E'_0
10828
                                                                                                                                                                                                                                       10883
10829
                       b. QED E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                                       b. QED E = E'[\text{stat } \tau' E^{\bullet}]
                                                                                                                                                                                                                                       10884
10830
              CASE E = op^2 E_0 e_1:
                                                                                                                                                                                                                                       10885
                                                                                                                             CASE E = \text{stat } \tau E_0:
10831
                1. E_0 = E^{\bullet}
                                                                                                                                                                                                                                       10886
                                                                                                                               1. E_0 = E^{\bullet}
                     \vee E_0 = E_0'[\mathsf{dyn} \ \tau \ E^{\bullet}]
10832
                                                                                                                                    \vee E_0 = E_0'[\mathsf{dyn} \ \tau' \ E^{\bullet}]
                                                                                                                                                                                                                                       10887
10833
                                                                                                                                                                                                                                       10888
10834
                                                                                                                                                                                                                                       10889
10835
                                                                                                                                                                                                                                       10890
```

```
IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
10891
                   \vee E_0 = E_0'[\operatorname{stat} \tau' E^{\bullet}]
                                                                                                                                                                                                               10946
                                                                                                                        a. QED e = E'[\operatorname{dyn} \tau E^{\bullet}[\operatorname{op}^2 v_0 v_1]]
10892
                   by the induction hypothesis
                                                                                                                                                                                                               10947
              2. IF E_0 = E^{\bullet}:
                                                                                                                      ELSE E = E'[\text{stat } \tau E^{\bullet}]:
10893
                                                                                                                                                                                                               10948
                                                                                                                        a. QED e = E'[\text{stat } \tau E^{\bullet}[op^2 v_0 v_1]]
                     a. QED
                                                                                                                                                                                                               10949
10894
                   IF E_0 = E_0'[\mathsf{dyn} \ \tau' \ E^{\bullet}]:
10895
                                                                                                                CASE e = E[\operatorname{chk} \tau' \upsilon]:
                                                                                                                                                                                                               10950
                     a. E' = \text{stat } \tau E'_0
                                                                                                                  1. E = E^{\bullet}
                                                                                                                                                                                                               10951
10896
                     b. QED E = E'[\text{dyn } \tau' E^{\bullet}]
                                                                                                                      \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
10897
                                                                                                                                                                                                               10952
                   ELSE E_0 = E_0'[\text{stat } \tau' E^{\bullet}]:
                                                                                                                      \vee E = E'[\text{stat } \tau E^{\bullet}]
10898
                                                                                                                                                                                                               10953
10899
                     a. E' = \operatorname{stat} \tau E'_0
                                                                                                                      by inner boundary
                                                                                                                                                                                                               10954
                     b. QED E = E'[\text{stat } \tau' E^{\bullet}]
                                                                                                                  2. IF E = E^{\bullet}:
10900
                                                                                                                                                                                                               10955
10901
                                                                                                                        a. Contradiction by \vdash_{\mathsf{E}} e
                                                                                                                                                                                                               10956
                                                                                                                      IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
10902
         Lemma 6.15: HF dynamic boundary factoring
                                                                                                                                                                                                               10957
10903
                                                                                                                        a. Contradiction by \vdash_{\mathsf{E}} e
                                                                                                                                                                                                               10958
           If \vdash_{\mathsf{E}} e then one of the following holds:
           • e is a value
                                                                                                                      ELSE E = E'[\text{stat } \tau E^{\bullet}]:
10904
                                                                                                                                                                                                               10959
           • e = E^{\bullet}[v_0 \ v_1]
10905
                                                                                                                        a. QED e = E'[\text{stat } \tau E^{\bullet}[\text{chk } \tau' v]]
                                                                                                                                                                                                               10960
           • e = E^{\bullet}[op^1 v]
10906
                                                                                                                CASE e = E[\mathsf{dyn} \ \tau \ v]:
                                                                                                                                                                                                               10961
10907
                                                                                                                  1. QED v is boundary-free
           • e = E^{\bullet}[op^2 v_0 v_1]
                                                                                                                                                                                                               10962
10908
                                                                                                                CASE e = E[\text{stat } \tau \ v]:
                                                                                                                                                                                                               10963
           • e = E[dyn \tau e'] where e' is boundary-free
                                                                                                                  1. QED v is boundary-free
10909
                                                                                                                                                                                                               10964
           • e = E[\text{stat } \tau e'] where e' is boundary-free
10910
           • e = E[Err]
                                                                                                                CASE e = E[Err]:
                                                                                                                                                                                                               10965
10911
         Proof:
                                                                                                                  1. QED
                                                                                                                                                                                                               10966
            By the unique dynamic evaluation contexts lemma, there
10912
                                                                                                                                                                                                               10967
            are eight cases.
                                                                                                                                                                                                               10968
10913
                                                                                                             Lemma 6.16: HF unique dynamic evaluation contexts
            CASE e is a value :
10914
                                                                                                                                                                                                               10969
                                                                                                             If \vdash_{\mathsf{F}} e then one of the following holds:
              1. QED
10915
                                                                                                               • e is a value
                                                                                                                                                                                                               10970
            CASE e = E[v_0 \ v_1]:
10916
                                                                                                               • e = E[v_0 \ v_1]
                                                                                                                                                                                                               10971
              1. E = E^{\bullet}
10917
                                                                                                               • e = E[op^1 v]
                                                                                                                                                                                                               10972
                   \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
10918
                                                                                                               • e = E[op^2 v_0 v_1]
                                                                                                                                                                                                               10973
                   \vee E = E'[\text{stat } \tau E^{\bullet}]
10919
                                                                                                               • e = E[\operatorname{chk} \tau v]
                                                                                                                                                                                                               10974
                  by inner boundary
10920
                                                                                                               • e = E[\mathsf{dyn} \ \tau \ v]
                                                                                                                                                                                                               10975
              2. IF E = E^{\bullet}:
10921
                                                                                                               • e = E[\operatorname{stat} \tau v]
                                                                                                                                                                                                               10976
                     a. QED e = E^{\bullet}[v_0 \ v_1]
                                                                                                               • e = E[Err]
                                                                                                                                                                                                               10977
                   IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                             Proof:
10923
                                                                                                                                                                                                               10978
                     a. QED e = E'[\text{dyn } \tau E^{\bullet}[v_0 \ v_1]]
                                                                                                                By induction on the structure of e.
10924
                                                                                                                                                                                                               10979
                   ELSE E = E'[\text{stat } \tau E^{\bullet}]:
                                                                                                                CASE e = x
10925
                                                                                                                                                                                                               10980
                     a. QED e = E'[\text{stat } \tau E^{\bullet}[v_0 \ v_1]]
                                                                                                                          \vee e = \lambda(x:\tau).e'
10926
                                                                                                                                                                                                               10981
                                                                                                                          \vee e = dyn \tau e':
            CASE e = E[op^1 v]:
10927
                                                                                                                                                                                                               10982
10928
              1. E=E^{\bullet}
                                                                                                                  1. Contradiction by \vdash_{E} e
                                                                                                                                                                                                               10983
                   \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                CASE e = i
10929
                                                                                                                                                                                                               10984
                   \vee E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                          \vee e = \lambda x. e'
10930
                                                                                                                                                                                                               10985
                                                                                                                          \vee e = \text{mon}(\tau_d \Rightarrow \tau_c)v:
                   by inner boundary
10931
                                                                                                                                                                                                               10986
              2. IF E = E^{\bullet}:
                                                                                                                  1. QED e is a value
10932
                                                                                                                                                                                                               10987
                     a. QED e = E^{\bullet}[op^1 v]
                                                                                                                CASE e = Err:
10933
                                                                                                                                                                                                               10988
                   IF E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]:
                                                                                                                  1. E = []
10934
                                                                                                                                                                                                               10989
                     a. QED e = E'[\text{dyn } \tau E^{\bullet}[op^1 v]]
                                                                                                                  2. QED e = E[Err]
10935
                                                                                                                                                                                                               10990
                   ELSE E = E'[\text{stat } \tau E^{\bullet}]:
                                                                                                                CASE e = \langle e_0, e_1 \rangle:
10936
                                                                                                                                                                                                               10991
                     a. QED e = E'[\text{stat } \tau E^{\bullet}[op^1 v]]
                                                                                                                   IF e_0 \notin v:
10937
                                                                                                                                                                                                               10992
            CASE e = E[op^2 v_0 v_1]:
                                                                                                                     1. \vdash_{\mathsf{F}} e_0
10938
                                                                                                                                                                                                               10993
              1. E = E^{\bullet}
                                                                                                                         by inversion
10939
                                                                                                                                                                                                               10994
                   \vee E = E'[\mathsf{dyn} \ \tau \ E^{\bullet}]
                                                                                                                     2. e_0 = E_0[e'_0]
10940
                                                                                                                                                                                                               10995
                   \vee E = E'[\text{stat } \tau E^{\bullet}]
                                                                                                                         by the induction hypothesis (1)
10941
                                                                                                                                                                                                               10996
                   by inner boundary
                                                                                                                     3. E = \langle E_0, e_1 \rangle
10942
                                                                                                                                                                                                               10997
              2. IF E = E^{\bullet}:
                                                                                                                     4. QED e = E[e'_0]
10943
                                                                                                                                                                                                               10998
                     a. QED e = E^{\bullet}[op^2 v_0 v_1]
10944
                                                                                                                                                                                                               10999
10945
                                                                                                                                                                                                               11000
                                                                                                      100
```

```
11001
                IF e_0 \in v
                                                                                                                                                                                                            11056
                                                                                                                    1. \vdash_{\mathsf{F}} e_1
                                                                                                                        by inversion
11002
                     \land e_1 \notin v:
                                                                                                                                                                                                            11057
                                                                                                                    2. e_1 = E_1[e_1']
11003
                  1. \vdash_{\mathsf{F}} e_1
                                                                                                                                                                                                            11058
                                                                                                                        by the induction hypothesis (1)
                      by inversion
11004
                                                                                                                                                                                                            11059
11005
                  2. e_1 = E_1[e_1']
                                                                                                                    3. E = op^2 e_0 E_1
                                                                                                                                                                                                            11060
                      by the induction hypothesis (1)
                                                                                                                    4. QED e = E[e'_1]
                                                                                                                                                                                                            11061
11006
11007
                  3. E = \langle e_0, E_1 \rangle
                                                                                                                  ELSE e_0 \in v
                                                                                                                                                                                                            11062
11008
                  4. QED e = E[e'_1]
                                                                                                                           \land e_1 \in v:
                                                                                                                                                                                                            11063
11009
                ELSE e_0 \in v
                                                                                                                    1. E = []
                                                                                                                    2. QED e = E[op^2 e_0 e_1]
11010
                         \land e_1 \in v:
                                                                                                                                                                                                            11065
11011
                  1. E = []
                                                                                                              CASE e = \operatorname{chk} \tau e_0:
                                                                                                                                                                                                            11066
                  2. QED e is a value
                                                                                                                  Contradiction by \vdash_{E} e
11012
                                                                                                                                                                                                            11067
11013
            CASE e = e_0 e_1:
                                                                                                              CASE e = \text{stat } \tau \ e_0:
                                                                                                                                                                                                            11068
11014
               IF e_0 \notin v:
                                                                                                                  IF e_0 \notin v:
                                                                                                                                                                                                            11069
11015
                                                                                                                                                                                                            11070
                  1. \vdash_{\mathsf{F}} e_0
                                                                                                                    1. \vdash<sub>F</sub> e<sub>0</sub>
11016
                      by inversion
                                                                                                                        by inversion
                                                                                                                                                                                                            11071
11017
                  2. e_0 = E_0[e_0']
                                                                                                                    2. e_0 = E_0[e'_0]
                                                                                                                                                                                                            11072
11018
                      by the induction hypothesis (1)
                                                                                                                        by unique static evaluation contexts (1)
                                                                                                                                                                                                            11073
11019
                  3. E = E_0 e_1
                                                                                                                   3. E = \operatorname{stat} \tau E_0
                                                                                                                                                                                                            11074
                  4. QED e = E[e'_0]
                                                                                                                    4. QED e = E[e'_0]
11020
                                                                                                                                                                                                            11075
11021
                IF e_0 \in v
                                                                                                                  ELSE e_0 \in v:
                                                                                                                                                                                                            11076
                                                                                                                    1. E = []
11022
                     \land e_1 \notin v:
                                                                                                                                                                                                            11077
                                                                                                                    2. QED e = E[\text{stat } \tau e_0]
                                                                                                                                                                                                            11078
                  1. \vdash_{\mathsf{F}} e_1
11023
11024
                      by inversion
                                                                                                                                                                                                            11079
11025
                  2. e_1 = E_1[e_1']
                                                                                                           Lemma 6.17: HF static hole typing
                                                                                                                                                                                                            11080
                      by the induction hypothesis (1)
11026
                                                                                                                                                                                                            11081
                                                                                                           If \vdash_{\mathsf{F}} E^{\bullet}[e] : \tau then the derivation contains a sub-term \vdash_{\mathsf{F}} e : \tau'
                  3. E = e_0 E_1
                                                                                                                                                                                                            11082
11027
11028
                  4. QED e = E[e'_1]
                                                                                                                                                                                                            11083
                                                                                                              By induction on the structure of E^{\bullet}.
                ELSE e_0 \in v
11029
                                                                                                                                                                                                            11084
                                                                                                              CASE E^{\bullet} = []:
11030
                         \land e_1 \in v:
                                                                                                                                                                                                            11085
                                                                                                                 1. QED E^{\bullet}[e] = e
                  1. E = []
11031
                                                                                                                                                                                                            11086
                                                                                                              CASE E^{\bullet} = E^{\bullet}_{0} e_{1}:
11032
                  2. QED e = E[e_0 \ e_1]
                                                                                                                                                                                                            11087
                                                                                                                1. E^{\bullet}[e] = E^{\bullet}_{0}[e] e_{1}
11033
             CASE e = op^1 e_0:
                                                                                                                                                                                                            11088
                                                                                                                 2. \vdash_{\mathsf{E}} E^{\bullet}_{0}[e] : \tau_{d} \Longrightarrow \tau_{c}
11034
               IF e_0 \notin v:
                                                                                                                                                                                                            11089
                                                                                                                    by inversion
11035
                                                                                                                                                                                                            11090
                  1. F e<sub>0</sub>
                                                                                                                 3. QED by the induction hypothesis (2)
11036
                      by inversion
                                                                                                                                                                                                            11091
                                                                                                              CASE E^{\bullet} = v_0 E^{\bullet}_1:
11037
                  2. e_0 = E_0[e'_0]
                                                                                                                                                                                                            11092
                                                                                                                 1. E^{\bullet}[e] = v_0 E^{\bullet}_{1}[e]
                      by the induction hypothesis (1)
11038
                                                                                                                                                                                                            11093
                                                                                                                2. \vdash_{\mathsf{F}} E^{\bullet}_{1}[e] : \tau_{d}
                  3. E = op^1 E_0
11039
                                                                                                                                                                                                            11094
                                                                                                                    by inversion
                  4. QED e = E[e'_0]
                                                                                                                                                                                                            11095
11040
                                                                                                                3. QED by the induction hypothesis (2)
                ELSE e_0 \in v:
11041
                                                                                                                                                                                                            11096
                                                                                                              CASE E^{\bullet} = \langle E^{\bullet}_{0}, e_{1} \rangle:
11042
                  1. E = []
                                                                                                                                                                                                            11097
                                                                                                                 1. E^{\bullet}[e] = \langle E^{\bullet}_{0}[e], e_{1} \rangle
                  2. QED e = E[op^1 e_0]
11043
                                                                                                                                                                                                            11098
                                                                                                                 2. \vdash_{\mathsf{E}} E^{\bullet}_{0}[e] : \tau_{0}
            CASE e = op^2 e_0 e_1:
11044
                                                                                                                                                                                                            11099
                                                                                                                     by inversion
11045
               IF e_0 \notin v:
                                                                                                                                                                                                            11100
                                                                                                                3. QED by the induction hypothesis (2)
11046
                  1. \vdash_{\mathsf{E}} e_0
                                                                                                              CASE E^{\bullet} = \langle v_0, E^{\bullet}_1 \rangle:
                                                                                                                                                                                                            11101
11047
                      by inversion
                                                                                                                                                                                                            11102
                                                                                                                 1. E^{\bullet}[e] = \langle v_0, E^{\bullet}_1[e] \rangle
11048
                  2. e_0 = E_0[e'_0]
                                                                                                                                                                                                            11103
                                                                                                                 2. \vdash_{\mathsf{F}} E^{\bullet}_{1}[e] : \tau_{1}
11049
                      by the induction hypothesis (1)
                                                                                                                                                                                                            11104
                                                                                                                    by inversion
                  3. E = op^2 E_0 e_1
11050
                                                                                                                                                                                                            11105
                                                                                                                 3. QED by the induction hypothesis (2)
11051
                  4. QED e = E[e'_0]
                                                                                                                                                                                                            11106
                                                                                                              CASE E^{\bullet} = op^1 E^{\bullet}_0:
11052
                IF e_0 \in v
                                                                                                                                                                                                            11107
                                                                                                                 1. E^{\bullet}[e] = op^1 E^{\bullet}_{0}[e]
11053
                     \land e_1 \notin v:
                                                                                                                                                                                                            11108
11054
                                                                                                                                                                                                            11109
11055
                                                                                                                                                                                                            11110
                                                                                                     101
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11111
                2. \vdash_{\mathsf{F}} E^{\bullet}{}_{0}[e] : \tau_{0}
                                                                                                                         2. \vdash_{\mathsf{F}} E^{\bullet}{}_{0}[e]
                                                                                                                                                                                                                          11166
11112
                    by inversion
                                                                                                                             by inversion
                                                                                                                                                                                                                          11167
                3. QED by the induction hypothesis (2)
11113
                                                                                                                        3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                          11168
             CASE E^{\bullet} = op^2 E^{\bullet}_0 e_1:
                                                                                                                      CASE E^{\bullet} = op^2 \upsilon_0 E^{\bullet}_1:
11114
                                                                                                                                                                                                                          11169
                1. E^{\bullet}[e] = op^2 E^{\bullet}_{0}[e] e_1
                                                                                                                         1. E^{\bullet}[e] = op^2 v_0 E^{\bullet}_1[e]
11115
                                                                                                                                                                                                                          11170
                2. \vdash_{\mathsf{F}} E^{\bullet}{}_0[e] : \tau_0
                                                                                                                         2. \vdash_{\mathsf{E}} E^{\bullet}_{1}[e]
                                                                                                                                                                                                                          11171
11116
11117
                    by inversion
                                                                                                                             by inversion
                                                                                                                                                                                                                          11172
                3. QED by the induction hypothesis (2)
11118
                                                                                                                         3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                          11173
11119
             CASE E^{\bullet} = op^2 v_0 E^{\bullet}_1:
                                                                                                                                                                                                                          11174
                1. E^{\bullet}[e] = op^2 v_0 E^{\bullet}_1[e]
11120
                                                                                                                                                                                                                          11175
                                                                                                                   Lemma 6.19: HF boundary hole typing
11121
                2. \vdash_{\mathsf{F}} E^{\bullet}_{1}[e] : \tau_{1}
                                                                                                                     • If \vdash_{\mathsf{E}} E[\mathsf{dyn} \ \tau \ e] : \tau' then the derivation contains a sub-term
                    by inversion
11122
                                                                                                                                                                                                                          11177
                                                                                                                         \vdash_{\mathsf{E}} \mathsf{dyn} \; \tau \; e : \tau
11123
                3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                          11178
                                                                                                                     • If \vdash_{\mathsf{E}} E[\mathsf{dyn} \ \tau \ e] then the derivation contains a sub-term
             CASE E^{\bullet} = \operatorname{chk} \tau E^{\bullet}_{0}:
11124
                                                                                                                         \vdash_{\mathsf{E}} \mathsf{dyn} \; \tau \; e : \tau
                                                                                                                                                                                                                          11179
11125
                1. E^{\bullet}[e] = \operatorname{chk} \tau E^{\bullet}_{0}[e]
                                                                                                                                                                                                                          11180
                                                                                                                     • If \vdash_{\mathsf{E}} E[\mathsf{stat} \ \tau \ e] : \tau' then the derivation contains a sub-term
11126
                2. \vdash_{\mathsf{F}} E^{\bullet}_{0}[e] : \tau_{0}
                                                                                                                                                                                                                          11181
11127
                    by inversion
                                                                                                                                                                                                                          11182
                                                                                                                     • If \vdash_{\mathsf{E}} E[\mathsf{stat} \ \tau \ e] then the derivation contains a sub-term
11128
                3. QED by the induction hypothesis (2)
                                                                                                                        \vdash_{\mathsf{F}} \mathsf{stat} \ \tau \ e
                                                                                                                                                                                                                          11183
11129
                                                                                                                                                                                                                          11184
                                                                                                                   Proof:
11130
                                                                                                                      By the following four lemmas: static dyn hole typing,
                                                                                                                                                                                                                          11185
          Lemma 6.18: HF dynamic hole typing
11131
                                                                                                                      dynamic dyn hole typing, static stat hole typing, and
                                                                                                                                                                                                                          11186
          If \vdash_{\mathsf{E}} E^{\bullet}[e] then the derivation contains a sub-term \vdash_{\mathsf{E}} e
11132
                                                                                                                      dynamic stat hole typing.
                                                                                                                                                                                                                          11187
                                                                                                                                                                                                                          11188
11133
             By induction on the structure of E^{\bullet}.
                                                                                                                                                                                                                          11189
11134
             CASE E^{\bullet} = []:
                                                                                                                   Lemma 6.20: HF static dyn hole typing
11135
                1. QED E^{\bullet}[e] = e
                                                                                                                   If \vdash_{\mathsf{E}} E[\mathsf{dyn} \ \tau \ e] : \tau' then the derivation contains a sub-term
                                                                                                                                                                                                                          11190
11136
                                                                                                                                                                                                                          11191
             CASE E^{\bullet} = E^{\bullet}_{0} e_{1}:
                                                                                                                     \vdash_{\mathsf{F}} \mathsf{dyn} \ \tau \ e : \tau.
11137
                                                                                                                                                                                                                          11192
                                                                                                                   Proof:
                1. E^{\bullet}[e] = E^{\bullet}_{0}[e] e_{1}
11138
                                                                                                                                                                                                                          11193
                                                                                                                      By induction on the structure of E.
                2. \vdash_{\mathsf{F}} E^{\bullet}{}_{0}[e]
11139
                                                                                                                      case E \in E^{\bullet}:
                                                                                                                                                                                                                          11194
                    by inversion
11140
                                                                                                                        1. \vdash<sub>E</sub> dyn \tau e : \tau "
                                                                                                                                                                                                                          11195
                3. QED by the induction hypothesis (2)
11141
                                                                                                                             by static hole typing
                                                                                                                                                                                                                          11196
             CASE E^{\bullet} = v_0 E^{\bullet}_1:
11142
                                                                                                                                                                                                                          11197
                                                                                                                         2. \vdash<sub>F</sub> dyn \tau e : \tau
                1. E^{\bullet}[e] = v_0 E^{\bullet}_{1}[e]
11143
                                                                                                                                                                                                                          11198
                                                                                                                             by inversion (1)
                2. \vdash_{\mathsf{F}} E^{\bullet}_{1}[e]
11144
                                                                                                                                                                                                                          11199
                                                                                                                         3. QED
                    by inversion
11145
                                                                                                                      CASE E = E_0 e_1:
                                                                                                                                                                                                                          11200
                3. QED by the induction hypothesis (2)
11146
                                                                                                                         1. E[dyn \tau e] = E_0[dyn \tau e] e_1
                                                                                                                                                                                                                          11201
             CASE E^{\bullet} = \langle E^{\bullet}_{0}, e_{1} \rangle:
11147
                                                                                                                        2. \vdash_{\mathsf{E}} E_0[\mathsf{dyn} \ \tau \ e] : \tau_0
                                                                                                                                                                                                                          11202
                1. E^{\bullet}[e] = \langle E^{\bullet}_{0}[e], e_{1} \rangle
11148
                                                                                                                             by inversion
                                                                                                                                                                                                                          11203
                2. \vdash_{\mathsf{E}} E^{\bullet}{}_{0}[e]
11149
                                                                                                                         3. QED by the induction hypothesis (2)
                    by inversion
                                                                                                                                                                                                                          11205
11150
                                                                                                                      CASE E = v_0 E_1:
                3. QED by the induction hypothesis (2)
11151
                                                                                                                         1. E[dyn \tau e] = v_0 E_1[dyn \tau e]
                                                                                                                                                                                                                          11206
             CASE E^{\bullet} = \langle v_0, E^{\bullet}_1 \rangle:
11152
                                                                                                                                                                                                                          11207
                                                                                                                         2. \vdash_{\mathsf{F}} E_1[\mathsf{dyn} \ \tau \ e] : \tau_1
                1. E^{\bullet}[e] = \langle v_0, E^{\bullet}_1[e] \rangle
11153
                                                                                                                             by inversion
                                                                                                                                                                                                                          11208
                2. \vdash_{\mathsf{E}} E^{\bullet}_{1}[e]
11154
                                                                                                                         3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                          11209
                    by inversion
11155
                                                                                                                                                                                                                          11210
                                                                                                                      CASE E = \langle E_0, e_1 \rangle:
                3. QED by the induction hypothesis (2)
11156
                                                                                                                         1. E[dyn \tau e] = \langle E_0[dyn \tau e], e_1 \rangle
                                                                                                                                                                                                                          11211
             CASE E^{\bullet} = op^1 E^{\bullet}_0:
11157
                                                                                                                        2. \vdash_{\mathsf{F}} E_0[\mathsf{dyn} \ \tau \ e] : \tau_0
                                                                                                                                                                                                                          11212
                1. E^{\bullet}[e] = op^1 E^{\bullet}_{0}[e]
11158
                                                                                                                             by inversion
                                                                                                                                                                                                                          11213
                2. \vdash_{\mathsf{E}} E^{\bullet}{}_{0}[e]
11159
                                                                                                                        3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                          11214
                    by inversion
11160
                                                                                                                      CASE E = \langle v_0, E_1 \rangle:
                                                                                                                                                                                                                          11215
                3. QED by the induction hypothesis (2)
                                                                                                                                                                                                                          11216
11161
                                                                                                                         1. E[dyn \tau e] = \langle v_0, E_1[dyn \tau e] \rangle
             CASE E^{\bullet} = op^2 E^{\bullet}_0 e_1:
11162
                                                                                                                        2. \vdash_{\mathsf{E}} E_1[\mathsf{dyn}\ \tau\ e] : \tau_1
                                                                                                                                                                                                                          11217
                1. E^{\bullet}[e] = op^2 E^{\bullet}_{0}[e] e_1
11163
                                                                                                                             by inversion
                                                                                                                                                                                                                          11218
11164
                                                                                                                                                                                                                          11219
11165
                                                                                                            102
                                                                                                                                                                                                                          11220
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11221	3. QED by the induction hypothesis (2)	2. $\vdash_{F} E_1[dyn\ \tau\ e]$	11276
11222	CASE $E = op^1 E_0$ :	by inversion	11277
11223	1. $E[dyn \ \tau \ e] = op^1 E_0[dyn \ \tau \ e]$	3. QED by the induction hypothesis (2)	11278
11224	2. $\vdash_{F} E_0[dyn \ \tau \ e] : \tau_0$	CASE $E = op^1 E_0$ :	11279
11225	by inversion	1. $E[dyn \ \tau \ e] = op^1 E_0[dyn \ \tau \ e]$	11280
11226	3. QED by the induction hypothesis (2)	2. $\vdash_{F} E_0[dyn\ \tau\ e]$	11281
11227	CASE $E = op^2 E_0 e_1$ :	by inversion	11282
11228	1. $E[dyn \ \tau \ e] = op^2 E_0[dyn \ \tau \ e] e_1$	3. QED by the induction hypothesis (2)	11283
11229	$2. \vdash_{F} E_0[dyn \ \tau \ e] : \tau_0$	CASE $E = op^2 E_0 e_1$ :	11284
11230	by inversion	1. $E[dyn \ \tau \ e] = op^2 E_0[dyn \ \tau \ e] e_1$	11285
11231	3. QED by the induction hypothesis (2)	2. $\vdash_{F} E_0[dyn\ \tau\ e]$	11286
11232	CASE $E = op^2 v_0 E_1$ :	by inversion	11287
11233	1. $E[dyn \ \tau \ e] = op^2 \ v_0 \ E_1[dyn \ \tau \ e]$	3. QED by the induction hypothesis (2)	11288
11234	2. $\vdash_{F} E_1[dyn\ \tau\ e]$ : $\tau_1$	CASE $E = op^2 v_0 E_1$ :	11289
11235	by inversion	1. $E[dyn \ \tau \ e] = op^2 \ v_0 \ E_1[dyn \ \tau \ e]$	11290
11236	3. QED by the induction hypothesis (2)	2. $\vdash_{F} E_1[dyn\ \tau\ e]$	11291
11237	CASE $E = \operatorname{chk} \tau'' E_0$ :	by inversion	11292
11238	1. $E[dyn \ \tau \ e] = chk \ \tau'' \ E_0[dyn \ \tau \ e]$	3. QED by the induction hypothesis (2)	11293
11239	2. $\vdash_{F} E_0[dyn\ \tau\ e]$ : $\tau_0$	<b>CASE</b> $E = \operatorname{chk} \tau'' E_0$ :	11294
11240	by inversion	1. Contradiction by $\vdash_{F} E[dyn\ \tau\ e]$	11295
11241	3. QED by the induction hypothesis (2)		11296
11242	CASE $E = \text{dyn } \tau_0 E_0$ :	CASE $E = \text{dyn } \tau E_0$ :	11297
11243	1. $E[dyn \ \tau \ e] = dyn \ \tau_0 \ E_0[dyn \ \tau \ e]$	1. Contradiction by $\vdash_{F} E[dyn \ \tau \ e]$	11298
11244	2. $\vdash_{F} E_0[dyn \ \tau \ e]$	<b>CASE</b> $E = \operatorname{stat} \tau_0 E_0$ :	11299
11245	by inversion		11300
11246	3. QED by dynamic dyn hole typing (2)	1. $E[\operatorname{dyn} \tau \ e] = \operatorname{stat} \tau_0 E_0[\operatorname{dyn} \tau \ e]$	11301
11247	<b>CASE</b> $E = \text{stat } \tau_0 E_0$ :	2. $\vdash_{F} E_0[dyn \ \tau \ e] : \tau_0$	11302
11248	1. Contradiction by $\vdash_{F} E[dyn \ \tau \ e] : \tau'$	by inversion	11303
11249		3. QED by static dyn hole typing (2)	11304
11250	Lemma 6.21: HF dynamic dyn hole typing	}	11305
11251	If $\vdash_{F} E[dyn \ \tau \ e]$ then the derivation contains a sub-term	<b>Lemma 6.22</b> : HF static stat hole typing	11306
11252	$\vdash_{F} dyn \ \tau \ e : \tau.$	If $\vdash_{F} E[stat\ \tau\ e] : \tau'$ then the derivation contains a sub-term	11307
11253	Proof:	$\vdash_{F} stat \ \tau \ e.$	11308
11254	By induction on the structure of <i>E</i> .	Proof:	11309
11255	Case $E \in E^{\bullet}$ :	By induction on the structure of $E$ .	11310
11256	1. Contradiction by $\vdash_{F} E[dyn\ \tau\ e]$	CASE $E \in E^{\bullet}$ :	11311
11257	<b>CASE</b> $E = E_0 e_1$ :	1. Contradiction by $\vdash_{F} E[stat \ \tau \ e] : \tau'$	11312
11258	1. $E[dyn \ \tau \ e] = E_0[dyn \ \tau \ e] \ e_1$	CASE $E = E_0 e_1$ :	11313
11259	2. $\vdash_{F} E_0[dyn\ \tau\ e]$	1. $E[\operatorname{stat} \tau \ e] = E_0[\operatorname{stat} \tau \ e] \ e_1$	11314
11260	by inversion	2. $\vdash_{F} E_0[stat\ \tau\ e] : \tau_0$	11315
11261	3. QED by the induction hypothesis (2)	by inversion	11316
11262	CASE $E = v_0 E_1$ :	3. QED by the induction hypothesis (2)	11317
11263	1. $E[dyn \ \tau \ e] = v_0 \ E_1[dyn \ \tau \ e]$	$\mathbf{CASE} \ E = v_0 \ E_1 :$	11318
11264	2. $\vdash_{F} E_1[dyn\ \tau\ e]$	1. $E[\operatorname{stat} \tau \ e] = v_0 E_1[\operatorname{stat} \tau \ e]$	11319
11265	by inversion	2. $\vdash_{F} E_1[stat\ \tau\ e] : \tau_1$	11320
11266	3. QED by the induction hypothesis (2)	by inversion	11321
11267	CASE $E = \langle E_0, e_1 \rangle$ :	3. QED by the induction hypothesis (2)	11322
11268	1. $E[\operatorname{dyn} \tau \ e] = \langle E_0[\operatorname{dyn} \tau \ e], e_1 \rangle$	CASE $E = \langle E_0, e_1 \rangle$ :	11323
11269	2. $\vdash_{F} E_0[dyn\ \tau\ e]$	1. $E[\operatorname{stat} \tau \ e] = \langle E_0[\operatorname{stat} \tau \ e], e_1 \rangle$	11324
11270	by inversion	2. $\vdash_{F} E_0[stat \ \tau \ e] : \tau_0$	11325
11271	3. QED by the induction hypothesis (2)	by inversion	11326
11272	<b>CASE</b> $E = \langle v_0, E_1 \rangle$ :	3. QED by the induction hypothesis (2)	11327
11273	1. $E[dyn \ \tau \ e] = \langle v_0, E_1[dyn \ \tau \ e] \rangle$	<b>CASE</b> $E = \langle v_0, E_1 \rangle$ :	11328
11274		1. $E[\operatorname{stat} \tau \ e] = \langle v_0, E_1[\operatorname{stat} \tau \ e] \rangle$	11329
11275	10:		11330

11331	2. $\vdash_{F} E_1[stat \ \tau \ e] : \tau_1$	1. $E[\operatorname{stat} \tau \ e] = \langle v_0, E_1[\operatorname{stat} \tau \ e] \rangle$	11386
11332	by inversion	2. $\vdash_{F} E_1[stat \ \tau \ e]$	11387
11333	3. QED by the induction hypothesis (2)	by inversion	11388
11334	CASE $E = op^1 E_0$ :	3. QED by the induction hypothesis (2)	11389
11335	1. $E[\operatorname{stat} \tau \ e] = op^1 E_0[\operatorname{stat} \tau \ e]$	CASE $E = op^1 E_0$ :	11390
11336	2. $\vdash_{F} E_0[stat \ t \ e] = t_0  E_0[stat \ t \ e]$	1. $E[\operatorname{stat} \tau e] = op^1 E_0[\operatorname{stat} \tau e]$	11391
11337	by inversion	2. $\vdash_{F} E_0[stat \ \tau \ e]$	11392
11338	3. QED by the induction hypothesis (2)	by inversion	11393
11339	CASE $E = op^2 E_0 e_1$ :	3. QED by the induction hypothesis (2)	11394
11340	1. $E[\text{stat } \tau \ e] = op^2 E_0[\text{stat } \tau \ e] e_1$	CASE $E = op^2 E_0 e_1$ :	11395
11341	2. $\vdash_{F} E_0[stat \ t \ e] = \tau_0$	1. $E[\operatorname{stat} \tau e] = op^2 E_0[\operatorname{stat} \tau e] e_1$	11396
11342	by inversion $\frac{2.7}{100} = \frac{100}{100}$	2. $\vdash_{F} E_0[stat \ \tau \ e]$	11397
11343	3. QED by the induction hypothesis (2)	by inversion	11398
11344	CASE $E = op^2 v_0 E_1$ :	3. QED by the induction hypothesis (2)	11399
11345	1. $E[\text{stat } \tau \ e] = op^2 \ v_0 \ E_1[\text{stat } \tau \ e]$	CASE $E = op^2 v_0 E_1$ :	11400
11346	2. $\vdash_{F} E_1[stat\ t\ e] = t_0 t_0 t_1[stat\ t\ e]$	1. $E[\operatorname{stat} \tau e] = op^2 v_0 E_1[\operatorname{stat} \tau e]$	11401
11347	by inversion $ \begin{array}{c} \mathbf{z} \cdot \mathbf{r}_{F} E_{I}[stat \ t \ e] \cdot t_{I} \\ \mathbf{b} \cdot \mathbf{r}_{F} = \mathbf{r}_{F}[stat \ e] \cdot t_{I} \\ \mathbf{e} \cdot \mathbf{r}_{F} = \mathbf{r}_{F}[stat \ e] \cdot t_{I} \\ \mathbf{e} \cdot \mathbf{r}_{F} = \mathbf{r}_{F}[stat \ e] \cdot t_{I} \\ \mathbf{e} \cdot \mathbf{r}_{F} = \mathbf{r}_{F}[stat \ e] \cdot t_{I} \\ \mathbf{e} \cdot \mathbf{r}_{F} = \mathbf{r}_{F}[stat \ e] \cdot t_{I} \\ \mathbf{e} \cdot \mathbf{r}_{F} = \mathbf{r}_{F}[stat \ e] \cdot t_{I} \\ \mathbf{e} \cdot \mathbf{r}_{F} = \mathbf{r}_{F}[stat \ e] \cdot t_{I} \\ \mathbf{e} \cdot \mathbf{r}_{F} = \mathbf{r}_{F}[stat \ e] \cdot t_{I} \\ \mathbf{e} \cdot \mathbf{r}_{F} = \mathbf{r}_{F}[stat \ e] \cdot t_{I} \\ \mathbf{e} \cdot \mathbf{r}_{F} = \mathbf{r}_{F}[stat \ e] \cdot t_{I} \\ \mathbf{e} \cdot \mathbf{r}_{F} = \mathbf{r}_{F}[stat \ e] \cdot t_{I} \\ \mathbf{e} \cdot \mathbf{r}_{F} = \mathbf{r}_{F}[stat \ e] \cdot t_{I} \\ \mathbf{e} \cdot \mathbf{r}_{F} = \mathbf{r}_{F}[stat \ e] \cdot t_{I} \\ \mathbf{e} \cdot \mathbf{r}_{F} = \mathbf{r}_{F}[stat \ e] \cdot t_{F} \\ \mathbf{e} \cdot \mathbf{r}_{F} = \mathbf{r}_{F}[stat \ e] \cdot t_{F} \\ \mathbf{e} \cdot \mathbf{r}_{F} = \mathbf{r}_{F}[e] \cdot t_{F} \\ \mathbf{e} \cdot \mathbf{r}_{F}[e] \cdot t_{F} \\ \mathbf{e} \cdot \mathbf{r}_{F}[e] \cdot t_{F}[e] \cdot t_{F} \\ \mathbf{e} \cdot \mathbf{r}_{F}[e] \cdot t_{F}[e] \cdot t_{F}[e] \cdot t_{F}[e] \\ \mathbf{e} \cdot \mathbf{r}_{F}[e] \cdot t_{F}[e] \cdot t_{F}[e] \\ \mathbf{e} \cdot \mathbf{r}_{F}[e] \cdot t_{F}[e] \cdot t_{F}[e] \cdot t_{F}[e] $	2. $\vdash_{\mathbf{F}} E_1[\operatorname{stat} \tau \ e]$	11402
11348	3. QED by the induction hypothesis (2)	by inversion	11403
11349	case $E = \operatorname{chk} \tau'' E_0$ :	3. QED by the induction hypothesis (2)	11404
11350	1. $E[\operatorname{stat} \tau \ e] = \operatorname{chk} \tau'' E_0[\operatorname{stat} \tau \ e]$	CASE $E = \operatorname{chk} \tau'' E_0$ :	11405
11351	2. $\vdash_{F} E_0[stat\ t\ e] = Cirk\ t\ E_0[stat\ t\ e]$	1. Contradiction by $\vdash_{F} E[stat \ \tau \ e]$	11406
11352	by inversion $\frac{2.7}{100} = \frac{100}{100}$	CASE $E = \text{dyn } \tau E_0$ :	11407
11353	3. QED by the induction hypothesis (2)	1. Contradiction by $\vdash_{F} E[stat \ \tau \ e]$	11408
11354	case $E = \text{dyn } \tau_0 E_0$ :	CASE $E = \operatorname{stat} \tau_0 E_0$ :	11409
11355	1. $E[\operatorname{stat} \tau \ e] = \operatorname{dyn} \tau_0 E_0[\operatorname{stat} \tau \ e]$	1. $E[\operatorname{stat} \tau e] = \operatorname{stat} \tau_0 E_0[\operatorname{stat} \tau e]$	11410
11356	2. $\vdash_{F} E_0[stat \ t \ e]$	2. $\vdash_{F} E_0[stat \ \tau \ e] = stat \ \iota_0 E_0[stat \ \tau \ e]$	11411
11357	by inversion by $\frac{1}{2}$	by inversion by inversion	11412
11358	3. QED by dynamic stat hole typing (2)	3. QED by static stat hole typing (2)	11413
11359	CASE $E = \text{stat } \tau_0 E_0$ :	$\Box$	11414
11360	1. Contradiction by $\vdash_{F} E[stat \ \tau \ e] : \tau'$	_	11415
11361	. Contradiction by $r_F L[stat \ i \ e] \cdot i$	<b>Lemma 6.24</b> : HF static boundary-free hole substitution	11416
11362		If $\vdash_{F} E^{\bullet}[e] : \tau$ and the derivation contains a sub-term $\vdash_{F} e : \tau'$	11417
11363	Lemma 6.23: HF dynamic stat hole typing	and $\vdash_{F} e' : \tau'$ then $\vdash_{F} E^{\bullet}[e'] : \tau$ .  Proof:	11418
11364	If $\vdash_{F} E[\text{stat } \tau \ e]$ then the derivation contains a sub-term $\vdash_{F}$	By induction on the structure of $E^{\bullet}$	11419
11365	stat τ e. Proof:	CASE $E^{\bullet} = []$ :	11420
11366	By induction on the structure of <i>E</i> .	1. $E^{\bullet}[e] = e$	11421
11367	CASE $E \in E^{\bullet}$ :		11422
11368	1. QED by dynamic hole typing	λ Ε [ε] = ε 2. ⊢ <sub>ε</sub> e : τ	11423
11369	CASE $E = E_0 e_1$ :	by (1)	11424
11370	1. $E[\operatorname{stat} \tau \ e] = E_0[\operatorname{stat} \tau \ e] \ e_1$	3. $\tau' = \tau$	11425
11371	2. $\vdash_{F} E_0[stat \ \tau \ e]$	$4. \vdash_{F} e' : \tau$	11426
11372	by inversion	5. QED by (1, 4)	11427
11373	3. QED by the induction hypothesis (2)	CASE $E^{\bullet} = E^{\bullet}_{0} e_{1}$ :	11428
11374	CASE $E = v_0 E_1$ :	1. $E^{\bullet}[e] = E^{\bullet}_{0}[e] e_{1}$	11429
11375	1. $E[\operatorname{stat} \tau e] = v_0 E_1[\operatorname{stat} \tau e]$	$\wedge E^{\bullet}[e'] = E^{\bullet}_{0}[e'] e_{1}$ $\wedge E^{\bullet}[e'] = E^{\bullet}_{0}[e'] e_{1}$	11430
11376	2. $\vdash_{F} E_1[stat \ \tau \ e]$	2. $\vdash_{F} E^{\bullet}_{0}[e] e_{1} : \tau$	11431
11377	by inversion	3. $\vdash_{F} E^{\bullet}_{0}[e] : \tau_{0}$	11432
11378	3. QED by the induction hypothesis (2)		11433
11379	CASE $E = \langle E_0, e_1 \rangle$ :	by inversion	11434
11380	1. $E[\operatorname{stat} \tau \ e] = \langle E_0[\operatorname{stat} \tau \ e], e_1 \rangle$	4. $\vdash_{F} E^{\bullet}_{0}[e'] : \tau_{0}$	11435
11381	2. $\vdash_{F} E_0[stat \ \tau \ e]$	by the induction hypothesis (3)	11436
11382	by inversion	5. $\vdash_{F} E^{\bullet}_{0}[e'] e_{1} : \tau$	11437
11383	3. QED by the induction hypothesis (2)	by (2, 3, 4)	11438
11384	CASE $E = \langle v_0, E_1 \rangle$ :	~ <sub>J</sub> ( <del>u</del> , ·, · <sub>j</sub>	11439
11385	10	4	11440
	19		

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11441
                    6. QED by (1, 5)
                                                                                                                                                             3. \vdash_{\mathsf{F}} E^{\bullet}_{0}[e] : \tau_{0}
                                                                                                                                                                                                                                                                                           11496
                 CASE E^{\bullet} = v_0 E^{\bullet}_1:
11442
                                                                                                                                                                  \land \vdash_{\mathsf{F}} e_1 : \tau_1
                                                                                                                                                                                                                                                                                           11497
                                                                                                                                                                  by inversion
11443
                    1. E^{\bullet}[e] = v_0 E^{\bullet}[e]
                                                                                                                                                                                                                                                                                           11498
                           \wedge E^{\bullet}[e'] = v_0 E^{\bullet}_1[e']
                                                                                                                                                             4. \vdash_{\mathsf{E}} E^{\bullet}_{0}[e'] : \tau_{0}
11444
                                                                                                                                                                                                                                                                                           11499
11445
                    2. \vdash_{\mathsf{F}} v_0 E^{\bullet}_1[e] : \tau
                                                                                                                                                                  by the induction hypothesis (3)
                                                                                                                                                                                                                                                                                           11500
                                                                                                                                                             5. \vdash_{\mathsf{F}} op^2 E^{\bullet}_{0}[e'] e_1 : \tau
                    3. \vdash_{\mathsf{F}} v_0 : \tau_0
                                                                                                                                                                                                                                                                                           11501
11446
                          \wedge \vdash_{\mathsf{F}} E^{\bullet}_{1}[e] : \tau_{1}
                                                                                                                                                                  by (2, 3, 4)
11447
                                                                                                                                                                                                                                                                                           11502
                                                                                                                                                            6. QED by (1, 5)
11448
                          by inversion
                                                                                                                                                                                                                                                                                           11503
11449
                    4. \vdash_{\mathsf{F}} E^{\bullet}_{1}[e'] : \tau_{1}
                                                                                                                                                         CASE E^{\bullet} = op^2 v_0 E^{\bullet}_1:
                          by the induction hypothesis (3)
                                                                                                                                                             1. E^{\bullet}[e] = op^2 v_0 E^{\bullet}_1[e]
11450
                                                                                                                                                                                                                                                                                           11505
11451
                    5. \vdash_{\mathsf{F}} v_0 E^{\bullet}_1[e'] : \tau
                                                                                                                                                                  \wedge E^{\bullet}[e'] = op^2 v_0 E^{\bullet}_{1}[e']
                                                                                                                                                                                                                                                                                           11506
                                                                                                                                                             2. \vdash_{\mathsf{E}} op^2 v_0 E^{\bullet}_1[e] : \tau
11452
                          by (2, 3, 4)
                                                                                                                                                                                                                                                                                           11507
11453
                    6. QED by (1, 5)
                                                                                                                                                             3. \vdash_{\mathsf{F}} v_0 : \tau_0
                                                                                                                                                                                                                                                                                           11508
                  CASE E^{\bullet} = op^1 E^{\bullet}_0:
                                                                                                                                                                  \wedge \vdash_{\mathsf{F}} E^{\bullet}_{1}[e] : \tau_{1}
11454
                                                                                                                                                                                                                                                                                           11509
11455
                    1. E^{\bullet}[e] = op^1 E^{\bullet}_{0}[e]
                                                                                                                                                                  by inversion
                                                                                                                                                                                                                                                                                           11510
11456
                           \wedge E^{\bullet}[e'] = op^1 E^{\bullet}_{0}[e']
                                                                                                                                                             4. \vdash_{\mathsf{F}} E^{\bullet}_{1}[e'] : \tau_{1}
                                                                                                                                                                                                                                                                                           11511
                    2. \vdash_{\mathsf{E}} op^1 E^{\bullet}_{0}[e] : \tau
                                                                                                                                                                  by the induction hypothesis (3)
11457
                                                                                                                                                                                                                                                                                           11512
11458
                    3. \vdash_{\mathsf{E}} E^{\bullet}_{0}[e] : \tau_{0}
                                                                                                                                                             5. \vdash_{\mathsf{E}} op^2 v_0 E^{\bullet}_1[e'] : \tau
                                                                                                                                                                                                                                                                                           11513
                          by inversion
                                                                                                                                                                  by (2, 3, 4)
11459
                                                                                                                                                                                                                                                                                           11514
11460
                    4. \vdash_{\mathsf{E}} E^{\bullet}_{0}[e'] : \tau_{0}
                                                                                                                                                            6. QED by (1, 5)
                                                                                                                                                                                                                                                                                           11515
11461
                          by the induction hypothesis (3)
                                                                                                                                                         CASE E^{\bullet} = \operatorname{chk} \tau^{\prime\prime} E^{\bullet}_{0}:
                                                                                                                                                                                                                                                                                           11516
                    5. \vdash_{\mathsf{F}} op^1 E^{\bullet}_{0}[e'] : \tau
                                                                                                                                                             1. E^{\bullet}[e] = \text{chk } \tau'' E^{\bullet}_{0}[e]
11462
                                                                                                                                                                                                                                                                                           11517
                                                                                                                                                                  \wedge E^{\bullet}[e'] = \operatorname{chk} \tau'' E^{\bullet}_{0}[e']
                          by (2, 3, 4)
11463
                                                                                                                                                                                                                                                                                           11518
                    6. QED by (1, 5)
                                                                                                                                                            2. \vdash_{\mathsf{F}} \mathsf{chk} \; \tau'' \; E^{\bullet}_{0}[e] : \tau
11464
                                                                                                                                                                                                                                                                                           11519
                 CASE E^{\bullet} = \langle E^{\bullet}_{0}, e_{1} \rangle:
11465
                                                                                                                                                            3. \vdash_{\mathsf{F}} E^{\bullet}_{0}[e] : \tau_{0}
                                                                                                                                                                                                                                                                                           11520
                    1. E^{\bullet}[e] = \langle E^{\bullet}_{0}[e], e_{1} \rangle
                                                                                                                                                                  by inversion
                                                                                                                                                                                                                                                                                           11521
11467
                           \wedge E^{\bullet}[e'] = \langle E^{\bullet}_{0}[e'], e_{1} \rangle
                                                                                                                                                             4. \vdash_{\mathsf{F}} E^{\bullet}_{0}[e'] : \tau_{0}
                                                                                                                                                                                                                                                                                           11522
11468
                    2. \vdash_{\mathsf{F}} \langle E^{\bullet}_{0}[e], e_{1} \rangle : \tau
                                                                                                                                                                  by the induction hypothesis (3)
                                                                                                                                                                                                                                                                                           11523
                    3. \vdash_{\mathsf{E}} E^{\bullet}_{0}[e] : \tau_{0}
                                                                                                                                                             5. \vdash_{\mathsf{E}} chk \tau'' E^{\bullet}_{0}[e'] : \tau
11469
                                                                                                                                                                                                                                                                                           11524
11470
                           \wedge \vdash_{\mathsf{F}} e_1 : \tau_1
                                                                                                                                                                  by (2, 3, 4)
                                                                                                                                                                                                                                                                                           11525
11471
                          by inversion
                                                                                                                                                             6. QED by (1, 5)
                                                                                                                                                                                                                                                                                           11526
11472
                    4. \vdash_{\mathsf{F}} E^{\bullet}{}_{0}[e'] : \tau_{0}
                                                                                                                                                                                                                                                                                           11527
11473
                          by the induction hypothesis (3)
                                                                                                                                                                                                                                                                                           11528
                                                                                                                                                    Lemma 6.25: HF dynamic hole substitution
11474
                    5. \vdash_{\mathsf{F}} \langle E^{\bullet}_{0}[e'], e_{1} \rangle : \tau
                                                                                                                                                                                                                                                                                           11529
                                                                                                                                                      If \vdash_{\mathsf{F}} E^{\bullet}[e] and \vdash_{\mathsf{F}} e' then \vdash_{\mathsf{F}} E^{\bullet}[e']
11475
                          by (2, 3, 4)
                                                                                                                                                                                                                                                                                           11530
                                                                                                                                                     Proof:
                    6. QED by (1, 5)
11476
                                                                                                                                                                                                                                                                                           11531
                                                                                                                                                         By induction on the structure of E^{\bullet}
11477
                  CASE E^{\bullet} = \langle v_0, E^{\bullet}_1 \rangle:
                                                                                                                                                                                                                                                                                           11532
                                                                                                                                                         CASE E^{\bullet} = []:
11478
                    1. E^{\bullet}[e] = \langle v_0, E^{\bullet}_1[e] \rangle
                                                                                                                                                                                                                                                                                           11533
                                                                                                                                                            1. QED E^{\bullet}[e'] = e'
                           \wedge E^{\bullet}[e'] = \langle v_0, E^{\bullet}_1[e'] \rangle
11479
                                                                                                                                                         CASE E^{\bullet} = \langle E^{\bullet}_{0}, e_{1} \rangle:
                    2. \vdash_{\mathsf{E}} \langle v_0, E^{\bullet}_1[e] \rangle : \tau
                                                                                                                                                                                                                                                                                           11535
11480
                                                                                                                                                             1. E^{\bullet}[e] = \langle E^{\bullet}_{0}[e], e_{1} \rangle
11481
                    3. \vdash_{\mathsf{F}} v_0 : \tau_0
                                                                                                                                                                                                                                                                                           11536
                                                                                                                                                                  \wedge E^{\bullet}[e'] = \langle E^{\bullet}_{0}[e'], e_{1} \rangle
11482
                           \wedge \vdash_{\mathsf{E}} E^{\bullet}_{1}[e] : \tau_{1}
                                                                                                                                                                                                                                                                                           11537
                                                                                                                                                             2. \vdash_{\mathsf{F}} \langle E^{\bullet}_{0}[e], e_{1} \rangle
11483
                          by inversion
                                                                                                                                                                                                                                                                                           11538
                                                                                                                                                            3. \vdash_{\mathsf{E}} E^{\bullet}_{0}[e]
11484
                    4. \vdash_{\mathsf{F}} E^{\bullet}_{1}[e'] : \tau_{1}
                                                                                                                                                                                                                                                                                           11539
                                                                                                                                                                  \wedge \vdash_{\mathsf{F}} e_1
                          by the induction hypothesis (3)
11485
                                                                                                                                                                                                                                                                                           11540
                                                                                                                                                                  by inversion
                    5. \vdash_{\mathsf{E}} \langle v_0, E^{\bullet}_1[e'] \rangle : \tau
11486
                                                                                                                                                                                                                                                                                           11541
                                                                                                                                                            4. \vdash_{\mathsf{F}} E^{\bullet}_{0}[e']
11487
                          by (2, 3, 4)
                                                                                                                                                                                                                                                                                           11542
                                                                                                                                                                  by the induction hypothesis (3)
                    6. QED by (1, 5)
11488
                                                                                                                                                                                                                                                                                           11543
                                                                                                                                                            5. \vdash_{\mathsf{F}} \langle E^{\bullet}_{0}[e'], e_{1} \rangle
11489
                 CASE E^{\bullet} = op^2 E^{\bullet}_0 e_1:
                                                                                                                                                                                                                                                                                           11544
                                                                                                                                                                 by (3, 4)
                    1. E^{\bullet}[e] = op^2 E^{\bullet}_{0}[e] e_1
11490
                                                                                                                                                                                                                                                                                           11545
                                                                                                                                                             6. QED by (1, 5)
11491
                          \wedge E^{\bullet}[e'] = op^2 E^{\bullet}_{0}[e'] e_1
                                                                                                                                                                                                                                                                                           11546
                                                                                                                                                         CASE E^{\bullet} = \langle v_0, E^{\bullet}_1 \rangle:
                    2. \vdash_{\mathsf{F}} op^2 E^{\bullet}_{0}[e] e_1 : \tau
11492
                                                                                                                                                                                                                                                                                           11547
                                                                                                                                                            1. E^{\bullet}[e] = \langle v_0, E^{\bullet}_1[e] \rangle
11493
                                                                                                                                                                                                                                                                                           11548
                                                                                                                                                                  \wedge E^{\bullet}[e'] = \langle v_0, E^{\bullet}_1[e'] \rangle
11494
                                                                                                                                                                                                                                                                                           11549
11495
                                                                                                                                                                                                                                                                                           11550
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5. \vdash_{\mathsf{F}} op^2 E^{\bullet}_0[e'] e_1
11551
                 2. \vdash_{\mathsf{F}} \langle v_0, E^{\bullet}_1[e] \rangle
                                                                                                                                                                                                                                               11606
                                                                                                                                        by (3, 4)
11552
                 3. \vdash_{\mathsf{F}} v_0
                                                                                                                                                                                                                                               11607
11553
                      \wedge \vdash_{\mathsf{F}} E^{\bullet}_{1}[e]
                                                                                                                                    6. QED by (1, 5)
                                                                                                                                                                                                                                               11608
                                                                                                                                  CASE E^{\bullet} = op^2 \upsilon_0 E^{\bullet}_1:
                      by inversion
11554
                                                                                                                                                                                                                                               11609
                                                                                                                                    1. E^{\bullet}[e] = op^2 v_0 E^{\bullet}_1[e]
11555
                 4. \vdash_{\mathsf{F}} E^{\bullet}_{1}[e']
                                                                                                                                                                                                                                               11610
                      by the induction hypothesis (3)
                                                                                                                                         \wedge E^{\bullet}[e'] = op^2 v_0 E^{\bullet}_1[e']
                                                                                                                                                                                                                                               11611
11556
                                                                                                                                    2. \vdash_{\mathsf{F}} op^2 v_0 E^{\bullet}_1[e]
11557
                 5. \vdash_{\mathsf{F}} \langle v_0, E^{\bullet}_1[e'] \rangle
                                                                                                                                                                                                                                               11612
11558
                      by (3, 4)
                                                                                                                                    3. \vdash_{\mathsf{F}} v_0
                                                                                                                                                                                                                                               11613
11559
                 6. QED by (1, 5)
                                                                                                                                         \wedge \vdash_{\mathsf{F}} E^{\bullet}_{1}[e]
               CASE E^{\bullet} = E^{\bullet}_{0} e_{1}:
11560
                                                                                                                                         by inversion
                                                                                                                                                                                                                                               11615
11561
                 1. E^{\bullet}[e] = E^{\bullet}_{0}[e] e_{1}
                                                                                                                                    4. \vdash_{\mathsf{F}} E^{\bullet}_{1}[e']
                                                                                                                                                                                                                                               11616
                      \wedge E^{\bullet}[e'] = E^{\bullet}_{0}[e'] e_{1}
                                                                                                                                        by the induction hypothesis (3)
11562
                                                                                                                                                                                                                                               11617
11563
                 2. \vdash_{\mathsf{F}} E^{\bullet}{}_{0}[e] e_{1}
                                                                                                                                    5. \vdash_{\mathsf{F}} op^2 v_0 E^{\bullet}_1[e']
                                                                                                                                                                                                                                               11618
11564
                 3. \vdash_{\mathsf{F}} E^{\bullet}{}_{0}[e]
                                                                                                                                        by (3, 4)
                                                                                                                                                                                                                                               11619
11565
                                                                                                                                    6. QED by (1, 5)
                                                                                                                                                                                                                                               11620
                      \land \vdash_{\mathsf{F}} e_1
11566
                      by inversion
                                                                                                                                  CASE E^{\bullet} = \operatorname{chk} \tau_0 E^{\bullet}_0:
                                                                                                                                                                                                                                               11621
11567
                 4. \vdash_{\mathsf{E}} E^{\bullet}_{0}[e']
                                                                                                                                    1. Contradiction by \vdash_{E} E^{\bullet}[e]
                                                                                                                                                                                                                                               11622
                      by the induction hypothesis (3)
11568
                                                                                                                                                                                                                                               11623
                 5. \vdash_{\mathsf{E}} E^{\bullet}_{0}[e'] e_{1}
11569
                                                                                                                             Lemma 6.26: HF hole substitution
                                                                                                                                                                                                                                               11624
11570
                      by (3, 4)
                                                                                                                                                                                                                                               11625
                                                                                                                                • If \vdash_{\mathsf{E}} E[e] and the derivation contains a sub-term \vdash_{\mathsf{E}} e : \tau'
11571
                 6. QED by (1, 5)
                                                                                                                                                                                                                                               11626
                                                                                                                                    and \vdash_{\mathsf{E}} e' : \tau' then \vdash_{\mathsf{E}} E[e'].
               CASE E^{\bullet} = v_0 E^{\bullet}_1:
11572
                                                                                                                                                                                                                                               11627
                                                                                                                                • If \vdash_{E} E[e] and the derivation contains a sub-term \vdash_{E} e and
                 1. E^{\bullet}[e] = v_0 E^{\bullet}_{1}[e]
                                                                                                                                                                                                                                               11628
11573
                                                                                                                                    \vdash_{\mathsf{F}} e' \text{ then } \vdash_{\mathsf{F}} E[e'].
                      \wedge E^{\bullet}[e'] = v_0 E^{\bullet}[e']
11574
                                                                                                                                                                                                                                               11629
                                                                                                                                • If \vdash_{\mathsf{E}} E[e] : \tau and the derivation contains a sub-term \vdash_{\mathsf{E}} e : \tau'
11575
                 2. \vdash_{\mathsf{F}} v_0 E^{\bullet}_1[e]
                                                                                                                                    and \vdash_{\mathsf{F}} e' : \tau' then \vdash_{\mathsf{F}} E[e'] : \tau.
                                                                                                                                                                                                                                               11630
11576
                 3. \vdash_{\mathsf{F}} v_0
                                                                                                                                                                                                                                               11631
                                                                                                                                • If \vdash_{\mathsf{E}} E[e] : \tau and the derivation contains a sub-term \vdash_{\mathsf{E}} e
                      \wedge \vdash_{\mathsf{F}} E^{\bullet}_{1}[e]
11577
                                                                                                                                                                                                                                               11632
                                                                                                                                    and \vdash_{\mathsf{E}} e' then \vdash_{\mathsf{E}} E[e'] : \tau.
11578
                      by inversion
                                                                                                                                                                                                                                               11633
                                                                                                                              Proof:
                 4. \vdash_{\mathsf{E}} E^{\bullet}_{1}[e']
11579
                                                                                                                                 By the following four lemmas: dynamic context static hole
                                                                                                                                                                                                                                               11634
11580
                      by the induction hypothesis (3)
                                                                                                                                                                                                                                               11635
                                                                                                                                  substitution, dynamic context dynamic hole substitution,
11581
                 5. \vdash_{\mathsf{F}} v_0 E^{\bullet}_1[e']
                                                                                                                                  static context static hole substitution, and static context
                                                                                                                                                                                                                                               11636
11582
                      by (3, 4)
                                                                                                                                                                                                                                               11637
                                                                                                                                  dynamic hole substitution.
11583
                 6. QED by (1, 5)
                                                                                                                                                                                                                                               11638
               CASE E^{\bullet} = op^1 E^{\bullet}_0:
11584
                                                                                                                                                                                                                                               11639
                                                                                                                              Lemma 6.27: HF dynamic context static hole substitution
                 1. E^{\bullet}[e] = op^1 E^{\bullet}_0[e]
                                                                                                                                                                                                                                               11640
11585
                                                                                                                              If \vdash_{\mathsf{F}} E[e] and contains \vdash_{\mathsf{F}} e : \tau', and furthermore \vdash_{\mathsf{F}} e' : \tau',
                      \wedge E^{\bullet}[e'] = op^1 E^{\bullet}_0[e']
11586
                                                                                                                                                                                                                                               11641
                                                                                                                                then \vdash_{\mathsf{F}} E[e']
11587
                 2. \vdash_{\mathsf{F}} op^1 E^{\bullet}_0[e]
                                                                                                                                                                                                                                               11642
                                                                                                                              Proof:
                 3. \vdash_{\mathsf{E}} E^{\bullet}_{0}[e]
11588
                                                                                                                                                                                                                                               11643
                                                                                                                                 By induction on the structure of E.
                      by inversion
                                                                                                                                  case E \in E^{\bullet}:
11589
                                                                                                                                                                                                                                               11644
                 4. \vdash_{\mathsf{F}} E^{\bullet}_{0}[e']
                                                                                                                                                                                                                                               11645
11590
                                                                                                                                    1. Contradiction by \vdash_{E} E[e]
                      by the induction hypothesis (3)
11591
                                                                                                                                  CASE E = E_0 e_1:
                                                                                                                                                                                                                                               11646
11592
                 5. \vdash_{\mathsf{F}} op^1 E^{\bullet}_0[e']
                                                                                                                                                                                                                                               11647
                                                                                                                                    1. E[e] = E_0[e] e_1
11593
                      by (4)
                                                                                                                                                                                                                                               11648
                                                                                                                                    2. \vdash_{\mathsf{F}} E_0[e]
11594
                 6. QED by (1, 5)
                                                                                                                                                                                                                                               11649
                                                                                                                                         by inversion
               CASE E^{\bullet} = op^2 E^{\bullet}_0 e_1:
11595
                                                                                                                                                                                                                                               11650
                                                                                                                                    3. QED by the induction hypothesis (2)
                 1. E^{\bullet}[e] = op^2 E^{\bullet}_{0}[e] e_1
11596
                                                                                                                                  CASE E = v_0 E_1:
                                                                                                                                                                                                                                               11651
                      \wedge E^{\bullet}[e'] = op^2 E^{\bullet}_{0}[e'] e_1
11597
                                                                                                                                                                                                                                               11652
                                                                                                                                    1. E[e] = v_0 E_1[e]
                 2. \vdash_{\mathsf{F}} op^2 E^{\bullet}_0[e] e_1
11598
                                                                                                                                    2. \vdash_{\mathsf{F}} E_1[e]
                                                                                                                                                                                                                                               11653
11599
                 3. \vdash_{\mathsf{F}} E^{\bullet}{}_{0}[e]
                                                                                                                                                                                                                                               11654
                                                                                                                                        by inversion
11600
                      \wedge \vdash_{\mathsf{E}} e_1
                                                                                                                                                                                                                                               11655
                                                                                                                                    3. QED by the induction hypothesis (2)
11601
                      by inversion
                                                                                                                                                                                                                                               11656
                                                                                                                                  CASE E = \langle E_0, e_1 \rangle:
11602
                 4. \vdash_{\mathsf{E}} E^{\bullet}_{0}[e']
                                                                                                                                    1. E[e] = \langle E_0[e], e_1 \rangle
                                                                                                                                                                                                                                               11657
                      by the induction hypothesis (3)
11603
                                                                                                                                                                                                                                               11658
11604
                                                                                                                                                                                                                                               11659
11605
                                                                                                                                                                                                                                               11660
                                                                                                                      106
```

```
11661
                                                                                                        1. E[e] = \langle E_0[e], e_1 \rangle
                                                                                                                                                                                           11716
              2. \vdash_{\mathsf{F}} E_0[e]
                                                                                                        2. \vdash_{\mathsf{F}} E_0[e]
11662
                 by inversion
                                                                                                                                                                                           11717
             3. QED by the induction hypothesis (2)
                                                                                                           by inversion
11663
                                                                                                                                                                                           11718
            CASE E = \langle v_0, E_1 \rangle:
                                                                                                        3. QED by the induction hypothesis (2)
11664
                                                                                                                                                                                           11719
11665
             1. E[e] = \langle v_0, E_1[e] \rangle
                                                                                                     CASE E = \langle v_0, E_1 \rangle:
                                                                                                                                                                                           11720
             2. \vdash_{\mathsf{E}} E_1[e]
                                                                                                        1. E[e] = \langle v_0, E_1[e] \rangle
                                                                                                                                                                                           11721
11666
                 by inversion
11667
                                                                                                        2. \vdash_{\mathsf{F}} E_1[e]
                                                                                                                                                                                           11722
             3. QED by the induction hypothesis (2)
                                                                                                           by inversion
11668
                                                                                                                                                                                           11723
11669
            CASE E = op^1 E_0:
                                                                                                       3. QED by the induction hypothesis (2)
             1. E[e] = op^1 E_0[e]
                                                                                                     CASE E = op^1 E_0:
11670
                                                                                                                                                                                           11725
11671
              2. \vdash_{\mathsf{F}} E_0[e]
                                                                                                        1. E[e] = op^1 E_0[e]
                                                                                                                                                                                           11726
                 by inversion
                                                                                                        2. \vdash_{\mathsf{E}} E_0[e]
11672
                                                                                                                                                                                           11727
             3. QED by the induction hypothesis (2)
11673
                                                                                                           by inversion
                                                                                                                                                                                           11728
            CASE E = op^2 E_0 e_1:
                                                                                                       3. QED by the induction hypothesis (2)
11674
                                                                                                                                                                                           11729
11675
             1. E[e] = op^2 E_0[e] e_1
                                                                                                     CASE E = op^2 E_0 e_1:
                                                                                                                                                                                           11730
11676
             2. \vdash_{\mathsf{F}} E_0[e]
                                                                                                        1. E[e] = op^2 E_0[e] e_1
                                                                                                                                                                                           11731
                                                                                                       2. \vdash_{\mathsf{F}} E_0[e]
11677
                 by inversion
                                                                                                                                                                                           11732
11678
             3. QED by the induction hypothesis (2)
                                                                                                           by inversion
                                                                                                                                                                                           11733
            CASE E = op^2 v_0 E_1:
                                                                                                       3. QED by the induction hypothesis (2)
11679
                                                                                                                                                                                           11734
             1. E[e] = op^2 v_0 E_1[e]
11680
                                                                                                     CASE E = op^2 v_0 E_1:
                                                                                                                                                                                           11735
11681
             2. \vdash_{\mathsf{E}} E_1[e]
                                                                                                        1. E[e] = op^2 v_0 E_1[e]
                                                                                                                                                                                           11736
                 by inversion
                                                                                                        2. \vdash_{\mathsf{F}} E_1[e]
11682
                                                                                                                                                                                           11737
             3. QED by the induction hypothesis (2)
                                                                                                           by inversion
                                                                                                                                                                                           11738
11683
            CASE E = \operatorname{chk} \tau^{\prime\prime} E_0:
                                                                                                       3. QED by the induction hypothesis (2)
11684
                                                                                                                                                                                           11739
             1. E[e] = \text{chk } \tau'' E_0[e]
                                                                                                     CASE E = \operatorname{chk} \tau'' E_0:
11685
                                                                                                                                                                                           11740
             2. \vdash_{\mathsf{E}} E_0[e]
                                                                                                        1. Contradiction by \vdash_{\mathsf{E}} E[e]
                                                                                                                                                                                           11741
                 by inversion
                                                                                                     CASE E = \text{dyn } \tau'' E_0:
11687
                                                                                                                                                                                           11742
             3. QED by the induction hypothesis (2)
                                                                                                        1. Contradiction by \vdash_{E} E[e]
11688
                                                                                                                                                                                           11743
            CASE E = \text{dyn } \tau'' E_0:
                                                                                                     CASE E = \text{stat } \tau_0 E_0:
11689
                                                                                                                                                                                           11744
11690
             1. Contradiction by \vdash_{E} E[e]
                                                                                                        1. E[e] = \text{stat } \tau_0 E_0[e]
                                                                                                                                                                                           11745
11691
            CASE E = \text{stat } \tau_0 E_0:
                                                                                                       2. \vdash_{\mathsf{F}} E_0[e] : \tau_0
                                                                                                                                                                                           11746
11692
             1. E[e] = \text{stat } \tau_0 E_0[e]
                                                                                                           by inversion
                                                                                                                                                                                           11747
                                                                                                        3. QED by static context dynamic hole substitution (2)
11693
             2. \vdash_{\mathsf{F}} E_0[e] : \tau_0
                                                                                                                                                                                           11748
11694
                 by inversion
                                                                                                                                                                                           11749
11695
             3. OED by static context static hole substitution (2)
                                                                                                                                                                                           11750
                                                                                                  Lemma 6.29: HF static context static hole substitution
11696
                                                                                                                                                                                           11751
                                                                                                   If \vdash_{\mathsf{F}} E[e] : \tau and contains \vdash_{\mathsf{F}} e : \tau', and furthermore \vdash_{\mathsf{F}} e' : \tau',
                                                                                                                                                                                           11752
                                                                                                    then \vdash_{\mathsf{F}} E[e'] : \tau
        Lemma 6.28: HF dynamic context dynamic hole substitution
11698
                                                                                                  Proof:
                                                                                                                                                                                           11753
         If \vdash_{\mathsf{E}} E[e] and contains \vdash_{\mathsf{E}} e, and furthermore \vdash_{\mathsf{E}} e', then \vdash_{\mathsf{E}} E[e']
                                                                                                     By induction on the structure of E.
11699
                                                                                                                                                                                           11754
                                                                                                                                                                                           11755
11700
                                                                                                     CASE E \in E^{\bullet}:
           By induction on the structure of E.
11701
                                                                                                       1. QED by static boundary-free hole substitution
                                                                                                                                                                                           11756
           case E \in E^{\bullet}:
11702
                                                                                                     CASE E = E_0 e_1:
                                                                                                                                                                                           11757
             1. QED by dynamic boundary-free hole substitution
11703
                                                                                                        1. E[e] = E_0[e] e_1
                                                                                                                                                                                           11758
            CASE E = E_0 e_1:
11704
                                                                                                        2. \vdash_{\mathsf{E}} E_0[e] : \tau_0
                                                                                                                                                                                           11759
             1. E[e] = E_0[e] e_1
11705
                                                                                                           by inversion
                                                                                                                                                                                           11760
             2. \vdash_{\mathsf{F}} E_0[e]
                                                                                                       3. QED by the induction hypothesis (2)
11706
                                                                                                                                                                                           11761
                 by inversion
11707
                                                                                                     CASE E = v_0 E_1:
                                                                                                                                                                                           11762
             3. QED by the induction hypothesis (2)
11708
                                                                                                        1. E[e] = v_0 E_1[e]
                                                                                                                                                                                           11763
            CASE E = v_0 E_1:
                                                                                                       2. \vdash_{\mathsf{F}} E_1[e] : \tau_1
11709
                                                                                                                                                                                           11764
             1. E[e] = v_0 E_1[e]
11710
                                                                                                           by inversion
                                                                                                                                                                                           11765
             2. \vdash_{\mathsf{E}} E_1[e]
11711
                                                                                                       3. QED by the induction hypothesis (2)
                                                                                                                                                                                           11766
                 by inversion
11712
                                                                                                     CASE E = \langle E_0, e_1 \rangle:
                                                                                                                                                                                           11767
             3. QED by the induction hypothesis (2)
11713
                                                                                                        1. E[e] = \langle E_0[e], e_1 \rangle
                                                                                                                                                                                           11768
           CASE E = \langle E_0, e_1 \rangle:
11714
                                                                                                                                                                                           11769
11715
                                                                                                                                                                                           11770
```

```
2. \vdash_{\mathsf{F}} E_0[e] : \tau_0
11771
                                                                                                        CASE E = \langle E_0, e_1 \rangle:
                                                                                                                                                                                               11826
11772
                  by inversion
                                                                                                          1. E[e] = \langle E_0[e], e_1 \rangle
                                                                                                                                                                                               11827
11773
              3. QED by the induction hypothesis (2)
                                                                                                          2. \vdash_{\mathsf{F}} E_0[e] : \tau_0
                                                                                                                                                                                               11828
            CASE E = \langle v_0, E_1 \rangle:
11774
                                                                                                              by inversion
                                                                                                                                                                                               11829
11775
              1. E[e] = \langle v_0, E_1[e] \rangle
                                                                                                          3. QED by the induction hypothesis (2)
                                                                                                                                                                                               11830
              2. \vdash_{\mathsf{F}} E_1[e] : \tau_1
                                                                                                        CASE E = \langle v_0, E_1 \rangle:
                                                                                                                                                                                               11831
11776
                                                                                                          1. E[e] = \langle v_0, E_1[e] \rangle
11777
                 by inversion
                                                                                                                                                                                               11832
11778
              3. QED by the induction hypothesis (2)
                                                                                                          2. \vdash_{\mathsf{F}} E_1[e] : \tau_1
                                                                                                                                                                                               11833
11779
            CASE E = op^1 E_0:
                                                                                                              by inversion
                                                                                                                                                                                               11834
              1. E[e] = op^1 E_0[e]
                                                                                                          3. QED by the induction hypothesis (2)
11780
                                                                                                                                                                                               11835
11781
              2. \vdash_{\mathsf{F}} E_0[e] : \tau_0
                                                                                                        CASE E = op^1 E_0:
                                                                                                                                                                                               11836
                                                                                                          1. E[e] = op^1 E_0[e]
                  by inversion
11782
                                                                                                                                                                                               11837
              3. QED by the induction hypothesis (2)
11783
                                                                                                          2. \vdash_{\mathsf{F}} E_0[e] : \tau_0
                                                                                                                                                                                               11838
            CASE E = op^2 E_0 e_1:
                                                                                                             by inversion
11784
                                                                                                                                                                                               11839
11785
              1. E[e] = op^2 E_0[e] e_1
                                                                                                          3. QED by the induction hypothesis (2)
                                                                                                                                                                                               11840
11786
              2. \vdash_{\mathsf{F}} E_0[e] : \tau_0
                                                                                                        CASE E = op^2 E_0 e_1:
                                                                                                                                                                                               11841
11787
                  by inversion
                                                                                                          1. E[e] = op^2 E_0[e] e_1
                                                                                                                                                                                               11842
11788
              3. QED by the induction hypothesis (2)
                                                                                                          2. \vdash_{E} E_{0}[e] : \tau_{0}
                                                                                                                                                                                               11843
            CASE E = op^2 v_0 E_1:
                                                                                                             by inversion
11789
                                                                                                                                                                                               11844
              1. E[e] = op^2 v_0 E_1[e]
11790
                                                                                                          3. QED by the induction hypothesis (2)
                                                                                                                                                                                               11845
11791
              2. \vdash_{\mathsf{F}} E_1[e] : \tau_1
                                                                                                        CASE E = op^2 v_0 E_1:
                                                                                                                                                                                               11846
                 by inversion
                                                                                                          1. E[e] = op^2 v_0 E_1[e]
11792
                                                                                                                                                                                               11847
              3. QED by the induction hypothesis (2)
                                                                                                          2. \vdash_{\mathsf{F}} E_1[e] : \tau_1
                                                                                                                                                                                               11848
11793
            CASE E = \operatorname{chk} \tau^{\prime\prime} E_0:
11794
                                                                                                              by inversion
                                                                                                                                                                                               11849
              1. E[e] = \text{chk } \tau'' E_0[e]
                                                                                                          3. QED by the induction hypothesis (2)
11795
                                                                                                                                                                                               11850
11796
              2. \vdash_{\mathsf{F}} E_0[e] : \tau_0
                                                                                                        CASE E = \operatorname{chk} \tau'' E_0:
                                                                                                                                                                                               11851
11797
                  by inversion
                                                                                                          1. Contradiction by \vdash_{\mathsf{E}} E[e] : \tau
                                                                                                                                                                                               11852
              3. QED by the induction hypothesis (2)
                                                                                                        CASE E = \text{dyn } \tau_0 E_0:
                                                                                                                                                                                               11853
            CASE E = \text{dyn } \tau_0 E_0:
                                                                                                          1. E[e] = \text{dyn } \tau_0 E_0[e]
11799
                                                                                                                                                                                               11854
                                                                                                          2. \vdash_{\mathsf{F}} E_0[e]
11800
              1. E[e] = \text{dyn } \tau_0 E_0[e]
                                                                                                                                                                                               11855
11801
              2. \vdash_{\mathsf{E}} E_0[e]
                                                                                                              by inversion
                                                                                                                                                                                               11856
11802
                  by inversion
                                                                                                          3. QED by dynamic stat hole typing (2)
                                                                                                                                                                                               11857
11803
              3. QED by static dyn hole typing (2)
                                                                                                        CASE E = \text{stat } \tau_0 E_0:
                                                                                                                                                                                               11858
                                                                                                                                                                                               11859
11804
            CASE E = \text{stat } \tau_0 E_0:
                                                                                                          1. Contradiction by \vdash_{\mathsf{F}} E[e] : \tau
11805
              1. Contradiction by \vdash_{E} E[e] : \tau
                                                                                                                                                                                               11860
11806
                                                                                                                                                                                               11861
                                                                                                    Lemma 6.31: \vdash_F static inversion
11807
                                                                                                                                                                                               11862
         Lemma 6.30: HF static context dynamic hole substitution
11808
                                                                                                                                                                                               11863
          If \vdash_{\mathsf{F}} E[e] : \tau and contains \vdash_{\mathsf{F}} e, and furthermore \vdash_{\mathsf{F}} e', then
          \vdash_{\mathsf{E}} E[e'] : \tau
11810
                                                                                                                                                                                               11865
         Proof:
11811
           By induction on the structure of E.
                                                                                                                                                                                               11866
11812
                                                                                                                                                                                               11867
            CASE E \in E^{\bullet}:
11813
              1. Contradiction by \vdash_{\mathsf{E}} E[e] : \tau
11814
            CASE E = E_0 e_1:
                                                                                                                                                                                               11869
11815
                                                                                                                                                                                               11870
              1. E[e] = E_0[e] e_1
11816
              2. \vdash_{\mathsf{E}} E_0[e] : \tau_0
                                                                                                                                                                                               11871
11817
                                                                                                                                                                                               11872
                  by inversion
              3. QED by the induction hypothesis (2)
11818
                                                                                                                                                                                               11873
11819
                                                                                                                                                                                               11874
            CASE E = v_0 E_1:
11820
              1. E[e] = v_0 E_1[e]
                                                                                                                                                                                               11875
11821
                                                                                                                                                                                               11876
              2. \vdash_{E} E_{1}[e] : \tau_{1}
11822
                  by inversion
                                                                                                                                                                                               11877
11823
                                                                                                                                                                                               11878
              3. QED by the induction hypothesis (2)
11824
                                                                                                                                                                                               11879
11825
                                                                                                                                                                                               11880
                                                                                               108
```

11935

11989

```
• If \Gamma \vdash_{\mathsf{E}} x : \tau then (x : \tau') \in \Gamma and \tau' \leqslant \tau
                                                                                                                                                                    • If \vdash_{\mathsf{F}} v : \tau_0 \times \tau_1 then either:
11881
                                                                                                                                                                                                                                                                                                                 11936
                 • If \Gamma \vdash_{\mathsf{F}} \lambda(x : \tau'_d). e' : \tau then (x : \tau'_d), \Gamma \vdash_{\mathsf{F}} e' : \tau'_c and \tau'_d \Rightarrow \tau'_c \leqslant \varepsilon
                                                                                                                                                                        -v=\langle v_0,v_1\rangle
11882
                                                                                                                                                                                                                                                                                                                 11937
                                                                                                                                                                        - or v = \text{mon } \tau_0' \times \tau_1' \langle v_0, v_1 \rangle
11883
                                                                                                                                                                                                                                                                                                                 11938
                 • If \Gamma \vdash_{\mathsf{F}} \langle e_0, e_1 \rangle : \tau_0 \times \tau_1 then \Gamma \vdash_{\mathsf{F}} e_0 : \tau_0' and \Gamma \vdash_{\mathsf{F}} e_1 : \tau_1' and
                                                                                                                                                                             \wedge \tau_0' \times \tau_1' \leqslant \tau_0 \times \tau_1
11884
                                                                                                                                                                                                                                                                                                                 11939
                      \tau_0' \leqslant : \tau_0 \text{ and } \tau_1' \leqslant : \tau_1
                                                                                                                                                                    • If \vdash_{\mathsf{E}} v : \tau_d \Rightarrow \tau_c then either:
11885
                                                                                                                                                                                                                                                                                                                 11940
                 • If \Gamma \vdash_{\mathsf{F}} e_0 \ e_1 : \tau_c then \Gamma \vdash_{\mathsf{F}} e_0 : \tau_d' \Rightarrow \tau_c' and \Gamma \vdash_{\mathsf{F}} e_1 : \tau_d' and
                                                                                                                                                                        -v = \lambda(x:\tau_x).e'
                                                                                                                                                                                                                                                                                                                 11941
11886
                                                                                                                                                                             \wedge \tau_d \leqslant : \tau_x
                      \tau_c' \leqslant : \tau_c
11887
                                                                                                                                                                                                                                                                                                                 11942
                 • If \Gamma \vdash_{\mathsf{F}} \mathsf{fst}\, e : \tau then \Gamma \vdash_{\mathsf{F}} e : \tau_0 \times \tau_1 and \Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0
                                                                                                                                                                        - or v = \text{mon}(\tau_d' \Rightarrow \tau_c')(\lambda x. e)
11888
                                                                                                                                                                                                                                                                                                                 11943
                      and \tau_0 \leqslant : \tau
                                                                                                                                                                             \wedge \tau_d' \Rightarrow \tau_c' \leqslant : \tau_d \Rightarrow \tau_c
11889
                                                                                                                                                                        - or v = \text{mon}(\tau_d' \Rightarrow \tau_c') \lambda(x : \tau_x). e
                 • If \Gamma \vdash_{\mathsf{F}} \mathsf{snd} \ e : \tau \mathsf{ then } \Gamma \vdash_{\mathsf{F}} e : \tau_0 \times \tau_1 \mathsf{ and } \Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1
11890
                                                                                                                                                                                                                                                                                                                 11945
                                                                                                                                                                              \wedge \tau_d' \Rightarrow \tau_c' \leqslant : \tau_d \Rightarrow \tau_c
                      and \tau_1 \leqslant : \tau
11891
                                                                                                                                                                                                                                                                                                                 11946
                 • If \Gamma \vdash_{\mathsf{F}} op^2 e_0 e_1 : \tau then \Gamma \vdash_{\mathsf{F}} e_0 : \tau_0 and \Gamma \vdash_{\mathsf{F}} e_1 : \tau_1 and
                                                                                                                                                                    • If \vdash_{\mathsf{E}} v: Int then v = i
11892
                                                                                                                                                                                                                                                                                                                 11947
                      \Delta(op^2, \tau_0, \tau_1) = \tau' and \tau' \leqslant \tau
                                                                                                                                                                   • If \vdash_{\mathsf{F}} v: Nat then v = i and v \in \mathbb{N}
11893
                                                                                                                                                                                                                                                                                                                 11948
                                                                                                                                                                 Proof:
                 • If \Gamma \vdash_{\mathsf{F}} \mathsf{mon} \ \tau_0' \times \tau_1' \langle v_0, v_1 \rangle : \tau_0 \times \tau_1 \ \mathsf{then} \ \mathsf{either}:
11894
                                                                                                                                                                                                                                                                                                                 11949
                      -\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle
                                                                                                                                                                      QED by definition of \vdash_{\mathsf{F}} \cdot : \tau
11895
                                                                                                                                                                                                                                                                                                                 11950
                      - or \Gamma \vdash_{\mathsf{E}} \langle v_0, v_1 \rangle : \tau
11896
                                                                                                                                                                                                                                                                                                                 11951
                 • If \Gamma \vdash_{\mathsf{F}} \mathsf{mon}\, \tau'_d \Rightarrow \tau'_c \lambda x. \, e : \tau_d \Rightarrow \tau_c \text{ then } \Gamma \vdash_{\mathsf{F}} \lambda x. \, e \text{ and}
                                                                                                                                                                 Lemma 6.34 : \Delta type soundness
11897
                                                                                                                                                                                                                                                                                                                 11952
                      \tau_d' \Rightarrow \tau_c' \leqslant : \tau_d \Rightarrow \tau_c
                                                                                                                                                                    If \vdash_{\mathsf{F}} v_0 : \tau_0 and \vdash_{\mathsf{F}} v_1 : \tau_1 and \Delta(op^2, \tau_0, \tau_1) = \tau then one of
11898
                                                                                                                                                                                                                                                                                                                 11953
                 • If \Gamma \vdash_{\mathsf{F}} \mathsf{mon} \, \tau'_d \Rightarrow \tau'_c \, \lambda(x : \tau''_d). e : \tau_d \Rightarrow \tau_c \; \mathsf{then} \; \Gamma \vdash_{\mathsf{F}} \lambda(x : \tau''_d)
                                                                                                                                                                    the following holds:
11899
                                                                                                                                                                                                                                                                                                                 11954
                      \tau_x). e: \tau_d'' \Rightarrow \tau_c'' and \tau_d' \Rightarrow \tau_c' \leqslant \tau_d \Rightarrow \tau_c
                                                                                                                                                                    • \delta(op^2, v_0, v_1) = v and \vdash_{\mathsf{F}} v : \tau, or
11900
                                                                                                                                                                                                                                                                                                                 11955
                 • If \Gamma \vdash_{\mathsf{F}} \mathsf{dyn} \ \tau' \ e' : \tau \ \mathsf{then} \ \Gamma \vdash_{\mathsf{F}} e' \ \mathsf{and} \ \tau' \leqslant \tau
                                                                                                                                                                    • \delta(op^2, v_0, v_1) = \text{BndryErr}
11901
                                                                                                                                                                                                                                                                                                                 11956
                 • If \Gamma \vdash_{\mathsf{F}} \mathsf{chk} \ \tau \ e' : \tau \ \mathsf{then} \ \Gamma \vdash_{\mathsf{F}} e' : \tau'
                                                                                                                                                                 Proof (sketch): Similar to the proof for the higher-order \Delta
                                                                                                                                                                                                                                                                                                                 11957
                                                                                                                                                                 type soundness lemma.
                                                                                                                                                                                                                                                                                                  11958
11903
                   QED by the definition of \Gamma \vdash_{\mathsf{F}} e : \tau
                                                                                                                                                                 Lemma 6.35 : \delta preservation
                                                                                                                                                                                                                                                                                                                 11959
11904
                                                                                                                                                                   • If \vdash_{\mathsf{E}} v and \delta(op^1, v) = v' then \vdash_{\mathsf{E}} v'
11905
                                                                                                                                                                                                                                                                                                                 11960
11906
              Lemma 6.32 : \vdash_{\scriptscriptstyle E} dynamic inversion
                                                                                                                                                                   • If \vdash_{\mathsf{F}} v_0 and \vdash_{\mathsf{F}} v_1 and \delta(op^2, v_0, v_1) = v' then \vdash_{\mathsf{F}} v'
                                                                                                                                                                                                                                                                                                                 11961
                 • If \Gamma \vdash_{\mathsf{F}} x then x \in \Gamma
                                                                                                                                                                 Proof:
11907
                                                                                                                                                                                                                                                                                                                 11962
                 • If \Gamma \vdash_{\mathsf{F}} \lambda x. e' then x, \Gamma \vdash_{\mathsf{F}} e'
                                                                                                                                                                     Similar to the proof for the higher-order \delta preservation
                                                                                                                                                                                                                                                                                                                 11963
                 • If \Gamma \vdash_{\mathsf{F}} \langle e_0, e_1 \rangle then \Gamma \vdash_{\mathsf{F}} e_0 and \Gamma \vdash_{\mathsf{F}} e_1
                                                                                                                                                                     lemma.
11909
                                                                                                                                                                                                                                                                                                                 11964
                 11910
                                                                                                                                                                                                                                                                                                                 11965
                 • If \Gamma \vdash_{\vdash} op^1 e_0 then \Gamma \vdash_{\vdash} e_0
11911
                                                                                                                                                                 Lemma 6.36: HF substitution
                                                                                                                                                                                                                                                                                                                 11966
                 • If \Gamma \vdash_{\mathsf{E}} op^2 e_0 e_1 then \Gamma \vdash_{\mathsf{E}} e_0 and \Gamma \vdash_{\mathsf{E}} e_1
                                                                                                                                                                    • If (x:\tau_x), \Gamma \vdash_{\mathsf{F}} e and \vdash_{\mathsf{F}} v:\tau_x then \Gamma \vdash_{\mathsf{F}} e[x \leftarrow v]
                                                                                                                                                                                                                                                                                                                 11967
                 • If \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_d \Rightarrow \tau_c) \lambda x. e then \Gamma \vdash_{\mathsf{F}} \lambda x. e
                                                                                                                                                                    • If x, \Gamma \vdash_{\mathsf{F}} e and \vdash_{\mathsf{F}} v then \Gamma \vdash_{\mathsf{F}} e[x \leftarrow v]
11913
                                                                                                                                                                                                                                                                                                                 11968
                 • If \Gamma \vdash_{\mathsf{F}} \mathsf{mon}(\tau_d \Rightarrow \tau_c) \lambda(x : \tau_x). e \text{ then } \Gamma \vdash_{\mathsf{F}} \lambda(x : \tau_x). e : \tau_x \Rightarrow
11914
                                                                                                                                                                    • If (x:\tau_x), \Gamma \vdash_{\mathsf{E}} e : \tau and \vdash_{\mathsf{E}} v : \tau_x then \Gamma \vdash_{\mathsf{E}} e[x \leftarrow v] : \tau
                                                                                                                                                                                                                                                                                                                 11969
                                                                                                                                                                    • If x, \Gamma \vdash_{\scriptscriptstyle{E}} e : \tau and \vdash_{\scriptscriptstyle{E}} v then \Gamma \vdash_{\scriptscriptstyle{E}} e[x \leftarrow v] : \tau
11915
                                                                                                                                                                                                                                                                                                                 11970
                 • If \Gamma \vdash_{\mathsf{F}} \mathsf{mon} (\tau_0 \times \tau_1) \langle v_0, v_1 \rangle then either:
11916
                                                                                                                                                                                                                                                                                                                 11971
                      -\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle
                                                                                                                                                                     Similar to the proof for the higher-order substitution
11917
                                                                                                                                                                                                                                                                                                                 11972
11918
                      - or \Gamma \vdash_{\mathsf{E}} \langle v_0, v_1 \rangle : \tau'
                                                                                                                                                                     lemma.
                                                                                                                                                                                                                                                                                                                 11973
                                                                                                                                                                 • If \Gamma \vdash_{\mathsf{F}} \mathsf{stat} \ \tau' \ e' \ \mathsf{then} \ \Gamma \vdash_{\mathsf{F}} e' : \tau'
11919
                                                                                                                                                                                                                                                                                                                 11974
11920
                                                                                                                                                                 Lemma 6.37: weakening
                                                                                                                                                                                                                                                                                                                 11975
                   QED by the definition of \Gamma \vdash_{\mathsf{E}} e
                                                                                                                                                                  • If \Gamma \vdash_{\scriptscriptstyle{E}} e then x, \Gamma \vdash_{\scriptscriptstyle{E}} e
11921
                                                                                                                                                                                                                                                                                                                 11976
                                                                                                                                                                   • If \Gamma \vdash_{\mathsf{F}} e : \tau then (x : \tau'), \Gamma \vdash_{\mathsf{F}} e : \tau
11922
                                                                                                                                                                                                                                                                                                                 11977
                                                                                                                                                                 Proof:
11923
              Lemma 6.33: HF canonical forms
                                                                                                                                                                                                                                                                                                                 11978
                                                                                                                                                                     QED because e is closed under \Gamma
11924
                                                                                                                                                                                                                                                                                                                 11979
                                                                                                                                                                                                                                                                                                                 11980
11926
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11932
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11933
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```

11991	A.7 Embeddings Summary	12046
11992	The paragraphs in this section summarize the five embed-	12047
11993	dings with four slogans. Each slogan pertains to one aspect	12048
11994	of the embedding:	12049
11995	What kinds of checks does the embedding perform	12050
11996	when a value reaches a type boundary?	12051
11997	2. When, if ever, does the embedding wrap a value in a	12052
11998	monitor?	12053
11999	3. If an ill-typed value reaches a type boundary, when	12054
12000	does the embedding signal an error?	12055
12001	4. How do types affect behavior?	12056
12002	These embeddings are ordered on a speculative scale from	12057
12003	"most guarantees" to "least guarantees".	12058
12004		12059
12005	Higher-Order embedding	12060
12006	1. recursively check read-only values;	12061
12007 12008	2. monitor functional and mutable values;	12062 12063
12009	3. detect boundary errors as early as possible;	12064
12010	4. types globally constrain behavior.	12065
12011	Co-Natural embedding	12066
12012	1. tag-check all values;	12067
12013	2. monitor all data structures and functions;	12068
12014	3. detect boundary errors as late as possible;	12069
12015	4. types globally constrain behavior	12070
12016	Foundful ambadding	12071
12017	Forgetful embedding	12072
12018	1. tag-check all values;	12073
12019	2. apply at most one monitor to each value;	12074
12020	<ul><li>3. detect boundary errors as late as possible;</li><li>4. types (of values) locally constrain behavior.</li></ul>	12075
12021	4. types (of values) locally constrain behavior.	12076
12022	First-Order embedding	12077
12023	1. tag-check all values;	12078
12024	2. never allocate a monitor;	12079
12025	3. detect boundary errors as late as possible;	12080
12026 12027	4. types (of contexts) locally constrain behavior.	12081 12082
12027	Erasure embedding	12082
12029	1. never check values;	12084
12030	2. never allocate a monitor;	12085
12031	3. never detect a type boundary error;	12086
12032	4. types do not affect behavior	12087
12033	71	12088
12034		12089
12035		12090
12036		12091
12037		12092
12038		12093
12039		12094
12040		12095
12041		12096
12042		12097
12043		12098

## A.8 Simulation Lemmas A.8.1 Definitions Combined Language $= x \mid v \mid \langle e, e \rangle \mid e \mid op^1 \mid op^2 \mid e \mid op^2 \mid e \mid$ dyn $\tau$ e | stat $\tau$ e | Err | chk K e | dyn e | stat e $v = i \mid \langle v, v \rangle \mid \lambda x. e \mid \lambda(x:\tau). e \mid \text{mon}(\tau \Rightarrow \tau) v$ $E^{\bullet} = [] \mid E^{\bullet} e \mid v E^{\bullet} \mid \langle E^{\bullet}, e \rangle \mid \langle v, E^{\bullet} \rangle \mid$ $op^1 E^{\bullet} \mid op^2 E^{\bullet} e \mid op^2 v E^{\bullet} \mid chk K E^{\bullet}$ $E = E^{\bullet} \mid E \mid v \mid E \mid \langle E, e \rangle \mid \langle v, E \rangle \mid op^{1} \mid E \mid$ $op^2 E e \mid op^2 v E \mid dyn \tau E \mid stat \tau E$ $chk K E \mid dyn E \mid stat E$ $e_1 \lesssim_{\mathsf{E}} e$ $\frac{}{\mathsf{Err}\,\,{}_{1}\!\!\lesssim_{\mathsf{E}}\!\mathsf{Err}}\quad\frac{e^{1}\,\,{}_{1}\!\!\lesssim_{\mathsf{E}}\,e^{\mathsf{E}}}{\mathsf{chk}\,K\,e^{1}\,\,{}_{1}\!\!\lesssim_{\mathsf{F}}\,e^{\mathsf{E}}}$ $\frac{e^1 \underset{1 \lesssim_E}{\cdot} e^E}{\operatorname{dyn} e^1 \underset{1 \lesssim_E}{\cdot} e^E} \quad \frac{e^1 \underset{1 \lesssim_E}{\cdot} e^E}{\operatorname{stat} e^1 \underset{1 \lesssim_E}{\cdot} e^E} \quad \frac{1}{x \underset{1 \lesssim_E}{\cdot} x} \quad \frac{1}{i \underset{1 \lesssim_E}{\cdot} i}$ $\frac{e^1 \underset{1 \lesssim_E}{\sim} e^E}{\lambda x. e^1 \underset{1 \lesssim_E}{\sim} \lambda x. e^E} \quad \frac{e^1 \underset{1 \lesssim_E}{\sim} e^E}{\lambda (x \colon \tau). e^1 \underset{1 \lesssim_E}{\sim} \lambda (x \colon \tau). e^E}$ $\frac{e_0^1 \underset{1}{\sim} e_0^E e_0^E e_1^1 \underset{1}{\sim} E_E^E e_1^E}{e_0^1 e_1^1 \underset{1}{\sim} E_E^E e_1^E} \quad \frac{e_0^1 \underset{1}{\sim} E_E^E e_0^E e_1^1 \underset{1}{\sim} E_E^E e_1^E}{\langle e_0^1, e_1^1 \rangle \underset{1}{\sim} E_E \langle e_0^E, e_1^E \rangle}$ $\frac{e_0^1 \lesssim_E e_0^E}{op^1 e_0^1 \lesssim_E op^1 e_0^E} \frac{e_0^1 \lesssim_E e_0^E}{op^2 e_0^1 e_1^1 \lesssim_E op^2 e_0^E} \frac{e_1^1 \lesssim_E op^2 e_1^E}{op^2 e_0^1 e_1^1 \lesssim_E op^2 e_0^E e_1^E}$ $\frac{e_0^1 \lesssim_E e_0^E}{\operatorname{dyn} \tau e_0^1 \lesssim_E \operatorname{dyn} \tau e_0^E} \quad \frac{e_0^1 \lesssim_E e_0^E}{\operatorname{stat} \tau e_0^1 \lesssim_E \operatorname{stat} \tau e_0^E}$

<i>E</i> 1≲ <sub>E</sub> <i>E</i>		
	$E^1 \lesssim_{E} E^E$	
$chk K E^1 \lesssim_E E^{\overline{E}}$	$\overline{dynE^1}_{1} \lesssim_{E} E^{\overline{E}}$	stat $E^1 \lesssim_{E} E^{E}$
	$\frac{1 \lesssim_{E} e^{E}}{e^{E}}  \frac{v_0 \ 1 \lesssim_{E} \tau}{v_0 \ E^1}$	
	$\frac{1 \lesssim_{E} e^{E}}{\langle z^{E}, e^{E} \rangle}  \frac{v_0 \ 1 \lesssim_{E} v_0}{\langle v_0, E^1 \rangle}$	
	$\frac{E^1 \lesssim_E E^E}{op^2 E^1 e^1}$	-7 0 -
$v_0 \lesssim_{E} v_1  E^1$	· -	., 0=
$op^2 v_0 E^1 \lesssim_{E} op$	$\frac{\partial^2 v_1 E^{E}}{\partial g}  \overline{dyn} \ \tau E$	<sup>1</sup> $_{1}\lesssim_{E} dyn \ \tau \ E^{E}$
	$E^1 \lesssim_{E} E^E$	
sta	t $\tau E^1 \lesssim_{E} stat \ \tau E$	Ē.

#### A.8.2 Theorems

Theorem 8.0 : Err approximation

If  $e \in e_S$  and  $\vdash e : \tau$  then the following statements hold:

- if  $e \rightarrow_{\text{E-S}}^* \text{Err then } e \rightarrow_{\text{1-S}}^* \text{Err}$  if  $e \rightarrow_{\text{1-S}}^* \text{Err then } e \rightarrow_{\text{H-S}}^* \text{Err}$

QED by 1–E *approximation* and H–1 *approximation*.

# A.8.3 Lemmas

**Lemma 8.1**: 1–E approximation

If  $e \in e_{S}$  and  $\vdash e : \tau$  and  $e \to_{\mathsf{F-S}}^{*}$  Err then  $\vdash_{1} e \leadsto e'' : \lfloor \tau \rfloor$  and  $e^{\prime\prime} \rightarrow_{\text{1-S}}^* \text{Err}$ Proof:

- e" 1≤E e by 1–E reflexivity
- QED by 1-E simulation

**Lemma 8.2** : reflexivity

If  $\vdash e : \tau$  and  $\vdash_1 e \leadsto e'' : \lfloor \tau \rfloor$  then  $e'' \underset{1 \leq E}{\leq} e$ .

- e and e'' are identical up to chk expressions by definition of ↔
- QED by definition of  $1 \lesssim_E$

Lemma 8.3: E-1 simulation

If  $e_0^1 \lesssim_E e_0^E$  and  $e_0^E \to_{E-S} e_1^E$  and  $e_0^1 \notin Err$  then:

- $e_0^1 \to_{1-S} \dots \to_{1-S} e_n^1$   $\forall i \in \{1..n-1\}. e_i^1 \lesssim_E e_0^E$
- $e_n^1 \lesssim_{\mathsf{E}} e_1^{\mathsf{E}}$

Proof:

```
CASE e_{0}^{E} = E^{E}[e_{0'}^{E}]

\wedge e_{0'}^{E} \rhd_{E-S} e_{1'}^{E}

\wedge e_{1}^{E} = E^{E}[e_{1'}^{E}]:

1. e_{0}^{1} = E^{1}[e_{0'}^{1}]

\wedge E_{1}^{1} \lesssim_{E} E_{E}^{E}
                                                                                                                                             2. E^{1}[E_{0}^{1}[v_{0}]] \underset{1}{\lesssim}_{E} E^{E}[e^{E}]
12211
                                                                                                                                                                                                                                                               12266
                                                                                                                                                  by E^1[E_0^1[\text{stat } v_0]] \underset{1 \leq E}{\leq} E^{E}[e^{E}]
12212
                                                                                                                                                                                                                                                               12267
12213
                                                                                                                                             3. QED
                                                                                                                                                                                                                                                               12268
                                                                                                                                      12214
                                                                                                                                                                                                                                                               12269
12215
                                                                                                                                                                                                                                                               12270
                                                                                                                                      Lemma 8.6: 1–E step
                                                                                                                                                                                                                                                               12271
                       \land e_{0'}^1 \lesssim_{\mathbb{E}} e_{0'}^{\mathbb{E}}
by 1–E context factoring
                                                                                                                                       If E^1[e^1] \underset{1 \leq E}{\sim} E^E[e^{\overline{E}}] and e^1 \notin \{\text{chk } \tau e, \text{dyn } e, \text{stat } e\} and e^E \triangleright_{E-S} e_1^E \text{ then } E^1[e^1] \rightarrow_{1-S} E^1[e_1^1] \text{ and } e_1^1 \underset{1 \leq E}{\sim} e_1^E
12216
12217
                                                                                                                                                                                                                                                               12272
12218
                                                                                                                                                                                                                                                               12273
                  2. E^{1}[e_{0'}^{1}] \rightarrow_{1-S} \ldots \rightarrow_{1-S} E^{1}[e_{n-1'}^{1}]
                                                                                                                                                                                                                                                               12274
                        \land \forall i \in \{1..n-1\} . E^{1}[e_{i'}^{1}] \underset{1 \leq E}{\leq} E^{E}[e_{0'}^{E}]
                                                                                                                                          By boundary factoring and static hole typing, the inner
12220
                                                                                                                                                                                                                                                               12275
                       \wedge e_{n-1}^1 \neq \operatorname{chk} \tau e
\wedge e_{n-1}^1 \neq \operatorname{dyn} e
                                                                                                                                          expression e^1 is either typed or untyped.
                                                                                                                                                                                                                                                               12276
                                                                                                                                          CASE \vdash_1 e^1:
12222
                                                                                                                                                                                                                                                               12277
                                                                                                                                              QED by case analysis on e^{E} \triangleright_{E-S} e_{1}^{E}; either e^{1} steps in
                       \wedge e_{n-1}^{1} \neq \text{stat } e
                                                                                                                                                                                                                                                               12278
                                                                                                                                              the same manner, or e^1 steps to a boundary error due
                       by repeated uses of 1-E stutter (1)
12224
                                                                                                                                                                                                                                                               12279
                                                                                                                                              to the application of a typed function to an invalid
                  3. IF e_{n-1}^1 \in \text{Err}:
12225
                                                                                                                                                                                                                                                               12280
                       a. QED e_0^1 \to_{1-S}^* Err

ELSE e_{n-1}^1 \notin Err:

a. E^1[e_{n-1}^1] \to_{1-S} E^1[e_n^1]

\land e_n^1 \lesssim_E e_1^E
                                                                                                                                              argument
12226
                                                                                                                                                                                                                                                               12281
                                                                                                                                          CASE \vdash_1 e^1 : K :
12227
                                                                                                                                                                                                                                                               12282
                                                                                                                                              QED by case analysis on e^{E} \triangleright_{E-S} e_{1}^{E}; note that e_{1}^{E} \notin
12228
                                                                                                                                                                                                                                                               12283
                                                                                                                                              TagErr since e^1 is well-typed.
12229
                                                                                                                                      12284
                               by 1-E step (1, 2)
                                                                                                                                                                                                                                                               12285
                          b. QED 1-E context congruence
                                                                                                                                      Lemma 8.7: 1–E context congruence
12231
                                                                                                                                                                                                                                                               12286
                                                                                                                                       If e^1 \lesssim_E e^E and E^1 \lesssim_E E^E then E^1[e^1] \lesssim_E E^E[e^E]
                                                                                                                                                                                                                                                               12287
           Lemma 8.4 : 1–E context factoring
12233
                                                                                                                                                                                                                                                               12288
              If e_0^1 \lesssim_{\mathsf{E}} e_0^{\mathsf{E}} and e_0^{\mathsf{E}} = E_0^{\mathsf{E}}[e_{1'}^{\mathsf{E}}] then e_0^1 = E_0^1[e_{0'}^1] and E_0^1 \lesssim_{\mathsf{E}} E_0^{\mathsf{E}}
                                                                                                                                          QED by definition of 1 \lesssim E
12234
                                                                                                                                                                                                                                                               12289
              and e_{0'}^1 \lesssim_{\mathsf{E}} e_{0'}^{\mathsf{E}}
12235
                                                                                                                                                                                                                                                               12290
                                                                                                                                      Lemma 8.8: reflexivity
                QED by structural induction on the derivation of e_0^1 \lesssim_E
                                                                                                                                       If \vdash e : \tau then e \underset{1 \leq E}{\sim} e.
12237
                                                                                                                                                                                                                                                               12292
                e_0^{\mathsf{E}}
                                                                                                                                                                                                                                                               12293
                                                                                                                                          By structural induction on the derivation of \vdash e : \tau.
12239
                                                                                                                                                                                                                                                               12294
                                                                                                                                          TODO
12240
           Lemma 8.5 : 1-E stutter
                                                                                                                                                                                                                                                               12295
                                                                                                                                      12241
              If E^{1}[e^{1}] >_{1} \leq_{E} E^{E}[e^{E}] and e^{1} = E_{0}^{1}[Err]
                                                                                                                                                                                                                                                               12296
                                                                                                                                      Lemma 8.9: 1-H simulation
12242
                          \vee e^{1} = E_{0}^{1}[\operatorname{chk} K v_{0}]
                                                                                                                                                                                                                                                               12297
                                                                                                                                       yolo
12243
                          \vee e^1 = E_0^1[\mathsf{dyn}\ v_0]
                                                                                                                                                                                                                                                               12298
                                                                                                                                      Proof:
12244
                          \vee e^1 = E_0^1[\text{stat } v_0]
                                                                                                                                                                                                                                                               12299
12245
                                                                                                                                          nooo
              then E^{1}[e^{1}] \rightarrow_{1-S} E^{1}[e^{1}] and E^{1}[e^{1}] \underset{1}{\sim}_{E} E^{E}[e^{E}].
12246
                                                                                                                                                                                                                                                               12301
12247
               CASE e^1 = E_0^1[Err]:
12248
                  1. QED E^1[e^1] \rightarrow_{1-S} Err
                                                                                                                                                                                                                                                               12303
                CASE e^1 = E_0^1[\text{chk } K v_0]:
                                                                                                                                                                                                                                                               12304
12250
                   IF \mathcal{X}(K, v_0) = \text{BndryErr}:
                                                                                                                                                                                                                                                               12305
12251
                       1. QED E^1[E_0^1[\operatorname{chk} K v_0]] \rightarrow_{1-S} \operatorname{BndryErr}
12252
                   ELSE X(K, v_0) = v_0:
                                                                                                                                                                                                                                                               12307
                       1. E^{1}[E_{0}^{1}[\operatorname{chk} K v_{0}]] \rightarrow_{1-S} E^{1}[E_{0}^{1}[v_{0}]]
12254
                      2. E^{1}[E_{0}^{1}[v_{0}]] \underset{1}{\lesssim}_{\mathsf{E}} E^{\mathsf{E}}[e^{\mathsf{E}}]
                                                                                                                                                                                                                                                               12309
                                                                                                                                                                                                                                                               12310
                           by E^1[E_0^1[\operatorname{chk} K v_0]] \underset{1}{\lesssim_E} E^E[e^E]
12256
                                                                                                                                                                                                                                                               12311
                      3. QED
12257
                                                                                                                                                                                                                                                               12312
                CASE e^1 = E_0^1[\text{dyn } v_0]:
12258
                                                                                                                                                                                                                                                               12313
                  1. E^{1}[E_{0}^{1}[\mathsf{dyn}\ v_{0}]] \rightarrow_{1-S} E^{1}[E_{0}^{1}[v_{0}]]
12259
                                                                                                                                                                                                                                                               12314
                  2. E^{1}[E_{0}^{1}[v_{0}]] \underset{1 \leq E}{\leq} E^{E}[e^{E}]
12260
                                                                                                                                                                                                                                                               12315
                       by E^1[E_0^1[\mathsf{dyn}\ v_0]] \lesssim_{\mathsf{E}} E^{\mathsf{E}}[e^{\mathsf{E}}]
12261
                                                                                                                                                                                                                                                               12316
                  3. QED
12262
                                                                                                                                                                                                                                                               12317
                CASE e^1 = E_0^1[\text{stat } v_0]:
12263
                                                                                                                                                                                                                                                               12318
                  1. E^{1}[E_{0}^{1}[\text{stat }v_{0}]] \rightarrow_{1-S} E^{1}[E_{0}^{1}[v_{0}]]
12264
                                                                                                                                                                                                                                                               12319
12265
                                                                                                                                                                                                                                                               12320
                                                                                                                              112
```