A Spectrum of Soundness and Performance

Supplementary Material

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Contents

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A Models

This section contains full definitions of the languages and full proofs of our claims about each language.

Aside from the common notions in section 1.1, the definition and proofs of each model are independent and self-contained.

A.1 Preliminaries

Definition 1.0 : $\rightarrow^*$ *divergence*

Given a reduction relation $\rightarrow^*$, an expression e diverges if for all e' such that $e \rightarrow^* e'$ there exists an e'' such that $e' \rightarrow e''$.

Convention 1.1 : *variable convention*

All λ-bound variables in an expression are distinct from one another, and from any free variables in the expression.

Assumption 1.2 : $\vdash$ *permutation*

For all typing judgments and properties $\vdash$:
- If $x, x', \Gamma \vdash e$ then $x', x, \Gamma \vdash e$
- If $(x:\tau), (x':\tau'), \Gamma \vdash e$ then $(x':\tau'), (x:\tau), \Gamma \vdash e$

Definition 1.3 : $\vdash$ *boundary-free*

An expression e is *boundary free* if e does not contain a subterm of the form (dyn τ' e'), nor a subterm of the form (stat τ' e').

Notes:

- The upcoming models use a common surface syntax and typing system, but to keep each model self-contained we reprint this system in each definition.
- The proofs are written in a structured style, typically as a list of basic steps where each step is justified by an assumption, a lemma, or a previous step. Lemma names are *italicized* and hyperlinked to the actual lemma.

A.2 (H) Higher-Order Embedding

A.2.1 Higher-Order Definitions

$\boxed{\text{Language H}}$

$$
\begin{array}{lcl}
e & = & x \mid v \mid \langle e, e\rangle \mid e\ e \mid op^1\ e \mid op^2\ e\ e \mid \\
 & & \mathsf{dyn}\ \tau\ e \mid \mathsf{stat}\ \tau\ e \mid \mathsf{Err} \\
v & = & i \mid \langle v, v\rangle \mid \lambda x.\, e \mid \lambda(x\!:\!\tau).\, e \mid \\
 & & \mathsf{mon}\,(\tau \Rightarrow \tau)\, v \\
\tau & = & \mathsf{Nat} \mid \mathsf{Int} \mid \tau \times \tau \mid \tau \Rightarrow \tau \\
\Gamma & = & \cdot \mid x, \Gamma \mid (x\!:\!\tau), \Gamma \\
\mathsf{Err} & = & \mathsf{BndryErr} \mid \mathsf{TagErr} \\
r & = & v \mid \mathsf{Err} \\
E^{\bullet} & = & [\,] \mid E^{\bullet}\ e \mid v\ E^{\bullet} \mid \langle E^{\bullet}, e\rangle \mid \langle v, E^{\bullet}\rangle \mid \\
 & & op^1\ E^{\bullet} \mid op^2\ E^{\bullet}\ e \mid op^2\ v\ E^{\bullet} \\
E & = & E^{\bullet} \mid E\ e \mid v\ E \mid \langle E, e\rangle \mid \langle v, E\rangle \mid op^1\ E \mid \\
 & & op^2\ E\ e \mid op^2\ v\ E \mid \mathsf{dyn}\ \tau\ E \mid \mathsf{stat}\ \tau\ E
\end{array}
$$

$\boxed{\Delta : op^1 \times \tau \longrightarrow \tau}$

$$
\begin{array}{l}
\Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0 \\
\Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1
\end{array}
$$

$\boxed{\Delta : op^2 \times \tau \times \tau \longrightarrow \tau}$

$$
\begin{array}{l}
\Delta(op^2, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat} \\
\Delta(op^2, \mathsf{Int}, \mathsf{Int}) = \mathsf{Int}
\end{array}
$$

$\boxed{\tau \leqslant: \tau}$

$$
\frac{}{\mathsf{Nat} \leqslant: \mathsf{Int}} \qquad
\frac{\tau_d' \leqslant: \tau_d \quad \tau_c \leqslant: \tau_c'}{\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'} \qquad
\frac{\tau_0 \leqslant: \tau_0' \quad \tau_1 \leqslant: \tau_1'}{\tau_0 \times \tau_1 \leqslant: \tau_0' \times \tau_1'}
$$

$$
\frac{}{\tau \leqslant: \tau} \qquad
\frac{\tau \leqslant: \tau' \quad \tau' \leqslant: \tau''}{\tau \leqslant: \tau''}
$$

$\boxed{\Gamma \vdash e}$

$$
\frac{x \in \Gamma}{\Gamma \vdash x} \qquad
\frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x.\, e} \qquad
\frac{}{\Gamma \vdash i} \qquad
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1\rangle}
$$

$$
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0\ e_1} \qquad
\frac{\Gamma \vdash e}{\Gamma \vdash op^1\ e} \qquad
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2\ e_0\ e_1} \qquad
\frac{}{\Gamma \vdash \mathsf{Err}}
$$

$$
\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat}\ \tau\ e}
$$

$\boxed{\Gamma \vdash e : \tau}$

$$
\frac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash x : \tau} \qquad
\frac{(x\!:\!\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c} \qquad
\frac{i \in \mathbb{N}}{\Gamma \vdash i : \mathsf{Nat}}
$$

$$
\frac{}{\Gamma \vdash i : \mathsf{Int}} \qquad
\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1\rangle : \tau_0 \times \tau_1} \qquad
\frac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash e_1 : \tau_d}{\Gamma \vdash e_0\ e_1 : \tau_c}
$$

$$
\frac{\Gamma \vdash e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash op^1\ e_0 : \tau} \qquad
\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash op^2\ e_0\ e_1 : \tau} \qquad
\frac{\Gamma \vdash e : \tau' \quad \tau' \leqslant: \tau}{\Gamma \vdash e : \tau} \qquad
\frac{}{\Gamma \vdash \mathsf{Err} : \tau}
$$

$$
\frac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau}
$$

$\boxed{\Gamma \vdash_{\mathsf{H}} e}$

$$
\frac{x \in \Gamma}{\Gamma \vdash_{\mathsf{H}} x} \qquad
\frac{x, \Gamma \vdash_{\mathsf{H}} e}{\Gamma \vdash_{\mathsf{H}} \lambda x.\, e} \qquad
\frac{}{\Gamma \vdash_{\mathsf{H}} i} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} e_0 \quad \Gamma \vdash_{\mathsf{H}} e_1}{\Gamma \vdash_{\mathsf{H}} \langle e_0, e_1\rangle}
$$

$$
\frac{\Gamma \vdash_{\mathsf{H}} e_0 \quad \Gamma \vdash_{\mathsf{H}} e_1}{\Gamma \vdash_{\mathsf{H}} e_0\ e_1} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} e}{\Gamma \vdash_{\mathsf{H}} op^1\ e} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} e_0 \quad \Gamma \vdash_{\mathsf{H}} e_1}{\Gamma \vdash_{\mathsf{H}} op^2\ e_0\ e_1} \qquad
\frac{}{\Gamma \vdash_{\mathsf{H}} \mathsf{Err}}
$$

$$
\frac{\Gamma \vdash_{\mathsf{H}} e : \tau}{\Gamma \vdash_{\mathsf{H}} \mathsf{stat}\ \tau\ e} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} v : \tau_d \Rightarrow \tau_c}{\Gamma \vdash_{\mathsf{H}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v}
$$

$\boxed{\Gamma \vdash_{\mathsf{H}} e : \tau}$

$$
\frac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash_{\mathsf{H}} x : \tau} \qquad
\frac{(x\!:\!\tau_d), \Gamma \vdash_{\mathsf{H}} e : \tau_c}{\Gamma \vdash_{\mathsf{H}} \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c} \qquad
\frac{i \in \mathbb{N}}{\Gamma \vdash_{\mathsf{H}} i : \mathsf{Nat}}
$$

$$
\frac{}{\Gamma \vdash_{\mathsf{H}} i : \mathsf{Int}} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} e_0 : \tau_0 \quad \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1}{\Gamma \vdash_{\mathsf{H}} \langle e_0, e_1\rangle : \tau_0 \times \tau_1} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash_{\mathsf{H}} e_1 : \tau_d}{\Gamma \vdash_{\mathsf{H}} e_0\ e_1 : \tau_c}
$$

$$
\frac{\Gamma \vdash_{\mathsf{H}} e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash_{\mathsf{H}} op^1\ e_0 : \tau} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} e_0 : \tau_0 \quad \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash_{\mathsf{H}} op^2\ e_0\ e_1 : \tau} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} e : \tau' \quad \tau' <: \tau}{\Gamma \vdash_{\mathsf{H}} e : \tau}
$$

$$
\frac{}{\Gamma \vdash_{\mathsf{H}} \mathsf{Err} : \tau} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} e}{\Gamma \vdash_{\mathsf{H}} \mathsf{dyn}\ \tau\ e : \tau}
$$

$$
\frac{\Gamma \vdash_{\mathsf{H}} v}{\Gamma \vdash_{\mathsf{H}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v : (\tau_d \Rightarrow \tau_c)}
$$

$\boxed{\delta(op^1, v) = e}$

$$
\begin{array}{l}
\delta(\mathsf{fst}, \langle v_0, v_1\rangle) = v_0 \\
\delta(\mathsf{snd}, \langle v_0, v_1\rangle) = v_1
\end{array}
$$

$\boxed{\delta(op^2, v, v) = e}$

$$\begin{array}{ll}
\delta(\mathsf{sum}, i_0, i_1) & = i_0 + i_1 \\
\delta(\mathsf{quotient}, i_0, 0) & = \mathsf{BndryErr} \\
\delta(\mathsf{quotient}, i_0, i_1) & = \lfloor i_0 / i_1 \rfloor \\
\quad \text{if } i_1 \neq 0 &
\end{array}$$

$\boxed{\mathcal{D}_{\mathsf{H}} : \tau \times v \longrightarrow e}$

$$\begin{array}{ll}
\mathcal{D}_{\mathsf{H}}(\tau_d \Rightarrow \tau_c, v) & = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v \\
\quad \text{if } v = \lambda x.\, e \text{ or } v = \mathsf{mon}\, \tau'\, v' & \\
\mathcal{D}_{\mathsf{H}}(\tau_0 \times \tau_1, \langle v_0, v_1 \rangle) & = \langle \mathsf{dyn}\, \tau_0\, v_0, \mathsf{dyn}\, \tau_1\, v_1 \rangle \\
\mathcal{D}_{\mathsf{H}}(\mathsf{Int}, i) & = i \\
\mathcal{D}_{\mathsf{H}}(\mathsf{Nat}, i) & = i \\
\quad \text{if } i \in \mathbb{N} & \\
\mathcal{D}_{\mathsf{H}}(\tau, v) & = \mathsf{BndryErr} \\
\quad \text{otherwise} &
\end{array}$$

$\boxed{\mathcal{S}_{\mathsf{H}} : \tau \times v \longrightarrow e}$

$$\begin{array}{ll}
\mathcal{S}_{\mathsf{H}}(\tau_d \Rightarrow \tau_c, v) & = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v \\
\mathcal{S}_{\mathsf{H}}(\tau_0 \times \tau_1, \langle v_0, v_1 \rangle) & = \langle \mathsf{stat}\, \tau_0\, v_0, \mathsf{stat}\, \tau_1\, v_1 \rangle \\
\mathcal{S}_{\mathsf{H}}(\mathsf{Int}, v) & = v \\
\mathcal{S}_{\mathsf{H}}(\mathsf{Nat}, v) & = v
\end{array}$$

$\boxed{e \rhd_{\mathsf{H\text{-}S}} e}$

$$\begin{array}{lll}
\mathsf{dyn}\, \tau\, v & \rhd_{\mathsf{H\text{-}S}} & \mathcal{D}_{\mathsf{H}}(\tau, v) \\
(\mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v_f)\, v & \rhd_{\mathsf{H\text{-}S}} & \mathsf{dyn}\, \tau_c\, (v_f\, (\mathsf{stat}\, \tau_d\, v)) \\
(\lambda(x\!:\!\tau).\, e)\, v & \rhd_{\mathsf{H\text{-}S}} & e[x \leftarrow v] \\
op^1\, v & \rhd_{\mathsf{H\text{-}S}} & \delta(op^1, v) \\
op^2\, v_0\, v_1 & \rhd_{\mathsf{H\text{-}S}} & \delta(op^2, v_0, v_1)
\end{array}$$

$\boxed{e \rhd_{\mathsf{H\text{-}D}} e}$

$$\begin{array}{lll}
\mathsf{stat}\, \tau\, v & \rhd_{\mathsf{H\text{-}D}} & \mathcal{S}_{\mathsf{H}}(\tau, v) \\
v_0\, v_1 & \rhd_{\mathsf{H\text{-}D}} & \mathsf{TagErr} \\
\quad \text{if } v_0 \in \mathbb{Z} \text{ or } v_0 = \langle v, v' \rangle & & \\
(\mathsf{mon}\, \tau_d \Rightarrow \tau_c\, v_f)\, v & \rhd_{\mathsf{H\text{-}D}} & \mathsf{stat}\, \tau_c\, (v_f\, (\mathsf{dyn}\, \tau_d\, v)) \\
(\lambda x.\, e)\, v & \rhd_{\mathsf{H\text{-}D}} & e[x \leftarrow v] \\
op^1\, v & \rhd_{\mathsf{H\text{-}D}} & \mathsf{TagErr} \\
\quad \text{if } \delta(op^1, v) \text{ is undefined} & & \\
op^1\, v & \rhd_{\mathsf{H\text{-}D}} & \delta(op^1, v) \\
op^2\, v_0\, v_1 & \rhd_{\mathsf{H\text{-}D}} & \mathsf{TagErr} \\
\quad \text{if } \delta(op^2, v_0, v_1) \text{ is undefined} & & \\
op^2\, v_0\, v_1 & \rhd_{\mathsf{H\text{-}D}} & \delta(op^2, v_0, v_1)
\end{array}$$

$\boxed{e \rightarrow_{\mathsf{H\text{-}S}} e}$

$$\begin{array}{lll}
E^{\bullet}[e] & \rightarrow_{\mathsf{H\text{-}S}} & E^{\bullet}[e'] \\
\quad \text{if } e \rhd_{\mathsf{H\text{-}S}} e' & & \\
E[\mathsf{stat}\, \tau\, E^{\bullet}[e]] & \rightarrow_{\mathsf{H\text{-}S}} & E[\mathsf{stat}\, \tau\, E^{\bullet}[e']] \\
\quad \text{if } e \rhd_{\mathsf{H\text{-}S}} e' & & \\
E[\mathsf{dyn}\, \tau\, E^{\bullet}[e]] & \rightarrow_{\mathsf{H\text{-}S}} & E[\mathsf{dyn}\, \tau\, E^{\bullet}[e']] \\
\quad \text{if } e \rhd_{\mathsf{H\text{-}D}} e' & & \\
E[\mathsf{Err}] & \rightarrow_{\mathsf{H\text{-}S}} & \mathsf{Err}
\end{array}$$

$\boxed{e \rightarrow_{\mathsf{H\text{-}D}} e}$

$$\begin{array}{lll}
E^{\bullet}[e] & \rightarrow_{\mathsf{H\text{-}D}} & E^{\bullet}[e'] \\
\quad \text{if } e \rhd_{\mathsf{H\text{-}D}} e' & & \\
E[\mathsf{stat}\, \tau\, E^{\bullet}[e]] & \rightarrow_{\mathsf{H\text{-}D}} & E[\mathsf{stat}\, \tau\, E^{\bullet}[e']] \\
\quad \text{if } e \rhd_{\mathsf{H\text{-}S}} e' & & \\
E[\mathsf{dyn}\, \tau\, E^{\bullet}[e]] & \rightarrow_{\mathsf{H\text{-}D}} & E[\mathsf{dyn}\, \tau\, E^{\bullet}[e']] \\
\quad \text{if } e \rhd_{\mathsf{H\text{-}D}} e' & & \\
E[\mathsf{Err}] & \rightarrow_{\mathsf{H\text{-}D}} & \mathsf{Err}
\end{array}$$

$\boxed{e \rightarrow^{*}_{\mathsf{H\text{-}S}} e}$ reflexive, transitive closure of $\rightarrow_{\mathsf{H\text{-}S}}$

$\boxed{e \rightarrow^{*}_{\mathsf{H\text{-}D}} e}$ reflexive, transitive closure of $\rightarrow_{\mathsf{H\text{-}D}}$

A.2.2 Higher-Order Theorems

Theorem 2.0 : *static* H-*soundness*

If $\vdash e : \tau$ then $\vdash_{\mathsf{H}} e : \tau$ and one of the following holds:
- $e \rightarrow^{*}_{\mathsf{H\text{-}S}} v$ and $\vdash_{\mathsf{H}} v : \tau$
- $e \rightarrow^{*}_{\mathsf{H\text{-}S}} E[\mathsf{dyn}\ \tau'\ E^{\bullet}[e']]$ and $e' \rhd_{\mathsf{H\text{-}D}} \mathsf{TagErr}$
- $e \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
- e diverges

Proof:

1. $\vdash_{\mathsf{H}} e : \tau$
   by *static subset*
2. QED by *static progress* and *static preservation*.

□

Theorem 2.1 : *dynamic* H-*soundness*

If $\vdash e$ then $\vdash_{\mathsf{H}} e$ and one of the following holds:
- $e \rightarrow^{*}_{\mathsf{H\text{-}D}} v$ and $\vdash_{\mathsf{H}} v$
- $e \rightarrow^{*}_{\mathsf{H\text{-}D}} E[e']$ and $e' \rhd_{\mathsf{H\text{-}D}} \mathsf{TagErr}$
- $e \rightarrow^{*}_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
- e diverges

Proof:

1. $\vdash_{\mathsf{H}} e$
   by *dynamic subset*
2. QED by *dynamic progress* and *dynamic preservation*.

□

Corollary 2.2 : H *static soundness*

If $\vdash e : \tau$ and e is boundary-free, then one of the following holds:
- $e \rightarrow^{*}_{\mathsf{H}} v$ and $\vdash_{\mathsf{H}} v : \tau$
- $e \rightarrow^{*}_{\mathsf{H}} \mathsf{BndryErr}$
- e diverges

Proof:

Consequence of the proof for *static* H-*soundness*

□

Corollary 2.3 : H *compilation*

If $\vdash e : \tau$
and $\mathcal{D}'_{\mathsf{H}}$ extends $\mathcal{D}_{\mathsf{H}}$ with a rule to monitor a typed function:
$\mathcal{D}'_{\mathsf{H}}(\tau_d \Rightarrow \tau_c, \lambda(x\!:\!\tau).\,e) = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda(x\!:\!\tau).\,e)$
and $\rhd_{\mathsf{H\text{-}D}}'$ extends $\rhd_{\mathsf{H\text{-}D}}$ with a rule to apply a typed function:
$(\lambda(x\!:\!\tau).\,e)\ v\ \ \rhd_{\mathsf{H\text{-}D}}'\ \ e[x \leftarrow v]$
and $e \rightarrow_{\mathsf{H\text{-}D}}' e$ is defined as:

$$
\begin{array}{lll}
E[e] & \rightarrow_{\mathsf{H\text{-}D}}' & E[e'] \\
\quad \text{if } e \rhd_{\mathsf{H\text{-}D}}' e' & & \\
E[\mathsf{stat}\ \tau\ v] & \rightarrow_{\mathsf{H\text{-}D}}' & E[\mathcal{D}'_{\mathsf{H}}(\tau, v)] \\
E[\mathsf{dyn}\ \tau\ v] & \rightarrow_{\mathsf{H\text{-}D}}' & E[\mathcal{D}'_{\mathsf{H}}(\tau, v)] \\
E[\mathsf{Err}] & \rightarrow_{\mathsf{H\text{-}D}}' & \mathsf{Err}
\end{array}
$$

and $\rightarrow^{*}_{\mathsf{H\text{-}D}}{}'$ is the reflexive transitive closure of $\rightarrow_{\mathsf{H\text{-}D}}'$

then one of the following holds:
- $e \rightarrow^{*}_{\mathsf{H\text{-}D}}{}'\ v$ and $\vdash_{\mathsf{H}} v : \tau$
- $e \rightarrow^{*}_{\mathsf{H\text{-}D}}{}'\ \mathsf{TagErr}$
- $e \rightarrow^{*}_{\mathsf{H\text{-}D}}{}'\ \mathsf{BndryErr}$
- e diverges

Proof:

By *static* H-*soundness* and the fact that $\mathcal{S}_{\mathsf{H}}$ and $\rhd_{\mathsf{H\text{-}S}}$ are subsets of $\mathcal{D}'_{\mathsf{H}}$ and $\rhd_{\mathsf{H\text{-}D}}'$, respectively.

□

A.2.3 Higher-Order Lemmas

Lemma 2.4 : *$\mathcal{D}_{\mathsf{H}}$ soundness*

If $\Vdash_{\mathsf{H}} v$ then $\Vdash_{\mathsf{H}} \mathcal{D}_{\mathsf{H}}(\tau, v) : \tau$

Proof:

CASE $\mathcal{D}_{\mathsf{H}}(\tau_d \Rightarrow \tau_c, v) = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v$:

1. $\Vdash_{\mathsf{H}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v : \tau_d \Rightarrow \tau_c$
   by $\Vdash_{\mathsf{H}} v$
2. QED

CASE $v = \langle v_0, v_1 \rangle$
$\wedge\ \mathcal{D}_{\mathsf{H}}(\tau_0 \times \tau_1, v) = \langle \mathsf{dyn}\ \tau_0\ v_0, \mathsf{dyn}\ \tau_1\ v_1 \rangle$:

1. $\Vdash_{\mathsf{H}} v_0$
   $\wedge \Vdash_{\mathsf{H}} v_1$
   by *inversion*
2. $\Vdash_{\mathsf{H}} \mathsf{dyn}\ \tau_0\ v_0 : \tau_0$
   $\wedge \Vdash_{\mathsf{H}} \mathsf{dyn}\ \tau_1\ v_1 : \tau_1$
   by (1)
3. QED (2)

CASE $v = i$
$\wedge\ \mathcal{D}_{\mathsf{H}}(\mathsf{Int}, v) = v$:

1. QED

CASE $v \in \mathbb{N}$
$\wedge\ \mathcal{D}_{\mathsf{H}}(\mathsf{Nat}, v) = v$:

1. QED

CASE $\mathcal{D}_{\mathsf{H}}(\tau, v) = \mathsf{BndryErr}$:

1. QED

□

Lemma 2.5 : *$\mathcal{S}_{\mathsf{H}}$ soundness*

If $\Vdash_{\mathsf{H}} v : \tau$ then $\Vdash_{\mathsf{H}} \mathcal{S}_{\mathsf{H}}(\tau, v)$

Proof:

CASE $\Vdash_{\mathsf{H}} v : \tau_d \Rightarrow \tau_c$
$\wedge\ \mathcal{S}_{\mathsf{H}}(\tau_d \Rightarrow \tau_c, v) = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v$:

1. QED

CASE $\Vdash_{\mathsf{H}} v : \tau_0 \times \tau_1$
$\wedge\ \mathcal{S}_{\mathsf{H}}(\tau_0 \times \tau_1, v) = \langle \mathsf{stat}\ \tau_0\ v_0, \mathsf{stat}\ \tau_1\ v_1 \rangle$:

1. $v = \langle v_0, v_1 \rangle$
   by *canonical forms*
2. $\Vdash_{\mathsf{H}} v_0 : \tau_0$
   $\wedge \Vdash_{\mathsf{H}} v_1 : \tau_1$
   by *inversion* (1)
3. $\Vdash_{\mathsf{H}} \mathsf{stat}\ \tau_0\ v_0 : \tau_0$
   by the induction hypothesis (2)
4. $\Vdash_{\mathsf{H}} \mathsf{stat}\ \tau_1\ v_1 : \tau_1$
   by the induction hypothesis (2)
5. QED

CASE $\Vdash_{\mathsf{H}} v : \mathsf{Int}$
$\wedge\ \mathcal{S}_{\mathsf{H}}(\mathsf{Int}, v) = v$:

1. QED

CASE $\Vdash_{\mathsf{H}} v : \mathsf{Nat}$
$\wedge\ \mathcal{S}_{\mathsf{H}}(\mathsf{Nat}, v) = v$:

1. QED

□

Lemma 2.6 : H *static subset*

If $\Gamma \vdash e : \tau$ then $\Gamma \Vdash_{\mathsf{H}} e : \tau$.

Proof:

By structural induction on the derivation of $\Gamma \vdash e : \tau$.

CASE $\boxed{\dfrac{(x{:}\tau) \in \Gamma}{\Gamma \vdash x : \tau}}$:

1. $\Gamma \Vdash_{\mathsf{H}} x : \tau$
   by $(x{:}\tau) \in \Gamma$
2. QED

CASE $\boxed{\dfrac{(x{:}\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x{:}\tau_d).\, e : \tau_d \Rightarrow \tau_c}}$:

1. $(x{:}\tau_d), \Gamma \Vdash_{\mathsf{H}} e : \tau_c$
   by the induction hypothesis
2. $\Gamma \Vdash_{\mathsf{H}} \lambda(x{:}\tau_d).\, e : \tau_d \Rightarrow \tau_c$
3. QED

CASE $\boxed{\dfrac{i \in \mathbb{N}}{\Gamma \vdash i : \mathsf{Nat}}}$:

1. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash i : \mathsf{Int}}}$:

1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1}}$:

1. $\Gamma \Vdash_{\mathsf{H}} e_0 : \tau_0$
   $\wedge\ \Gamma \Vdash_{\mathsf{H}} e_1 : \tau_1$
   by the induction hypothesis
2. $\Gamma \Vdash_{\mathsf{H}} \langle e_0, e_1 \rangle : \tau_0 \times \tau_1$
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash e_1 : \tau_d}{\Gamma \vdash e_0\ e_1 : \tau_c}}$:

1. $\Gamma \Vdash_{\mathsf{H}} e_0 : \tau_d \Rightarrow \tau_c$
   $\wedge\ \Gamma \Vdash_{\mathsf{H}} e_1 : \tau_d$
   by the induction hypothesis
2. $\Gamma \Vdash_{\mathsf{H}} e_0\ e_1 : \tau_c$
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash op^1\ e_0 : \tau}}$:

1. $\Gamma \Vdash_{\mathsf{H}} e_0 : \tau_0$
   by the induction hypothesis
2. $\Gamma \Vdash_{\mathsf{H}} op^1\ e_0 : \tau$
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash op^2\ e_0\ e_1 : \tau}}$:

1. $\Gamma \Vdash_{\mathsf{H}} e_0 : \tau_0$
   $\wedge\ \Gamma \Vdash_{\mathsf{H}} e_1 : \tau_1$
   by the induction hypothesis
2. $\Gamma \Vdash_{\mathsf{H}} op^2\ e_0\ e_1 : \tau$
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau' \quad \tau' <: \tau}{\Gamma \vdash e : \tau}}$:

1. $\Gamma \vdash_{\mathsf{H}} e : \tau'$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{H}} e : \tau$
3. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash \mathsf{Err} : \tau}}$:

1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau}}$:

1. $\Gamma \vdash_{\mathsf{H}} e$
   by *dynamic subset*
2. $\Gamma \vdash_{\mathsf{H}} \mathsf{dyn}\ \tau\ e : \tau$
   by (1)
3. QED

□

Lemma 2.7 : H *dynamic subset*

If $\Gamma \vdash e$ then $\Gamma \vdash_{\mathsf{H}} e$.

Proof:

By structural induction on the derivation of $\Gamma \vdash e$.

CASE $\boxed{\dfrac{x \in \Gamma}{\Gamma \vdash x}}$:

1. $\Gamma \vdash_{\mathsf{H}} x$
   by $x \in \Gamma$
2. QED

CASE $\boxed{\dfrac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x.\, e}}$:

1. $x, \Gamma \vdash_{\mathsf{H}} e$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{H}} \lambda x.\, e$
   by (1)
3. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash i}}$:

1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}}$:

1. $\Gamma \vdash_{\mathsf{H}} e_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{H}} \langle e_0, e_1 \rangle$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0\ e_1}}$:

1. $\Gamma \vdash_{\mathsf{H}} e_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{H}} e_0\ e_1$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e}{\Gamma \vdash op^1\ e}}$:

1. $\Gamma \vdash_{\mathsf{H}} e$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{H}} op^1\ e$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2\ e_0\ e_1}}$:

1. $\Gamma \vdash_{\mathsf{H}} e_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{H}} op^2\ e_0\ e_1$
   by (1)
3. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash \mathsf{Err}}}$:

1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat}\ \tau\ e}}$:

1. $\Gamma \vdash_{\mathsf{H}} e : \tau$
   by *static subset*
2. $\Gamma \vdash_{\mathsf{H}} \mathsf{stat}\ \tau\ e$
   by (1)
3. QED

□

Lemma 2.8 : H *static progress*

If $\vdash_{\mathsf{H}} e : \tau$ then one of the following holds:

- e is a value
- $e \in \mathsf{Err}$
- $e \longrightarrow_{\mathsf{H\text{-}S}} e'$
- $e \longrightarrow_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
- $e = E[\mathsf{dyn}\ \tau'\ E^{\bullet}[e']]$ and $e' \rhd_{\mathsf{H\text{-}D}} \mathsf{TagErr}$

Proof:

By the *boundary factoring* lemma, there are seven possible cases.

CASE e is a value :

1. QED

CASE $e = E^{\bullet}[v_0\ v_1]$:

1. $\vdash_{\mathsf{H}} v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\mathsf{H}} v_0 : \tau_d \Rightarrow \tau_c$
   $\wedge \vdash_{\mathsf{H}} v_1 : \tau_d$
   by *inversion*

3. $v_0 = \lambda(x{:}\tau'_d).\, e'$
 $\vee\ v_0 = \mathsf{mon}\,(\tau'_d \Rightarrow \tau'_c)\, v_f$
 by *canonical forms*
4. **IF** $v_0 = \lambda(x{:}\tau'_d).\, e'$:
 a. $e \rightarrow_{\text{H-S}} E^{\bullet}[e'[x \leftarrow v_1]]$
  by $v_0\ v_1 \rhd_{\text{H-S}} e'[x \leftarrow v_1]$
 b. QED
 ELSE $v_0 = \mathsf{mon}\,(\tau'_d \Rightarrow \tau'_c)\, v_f$:
 a. $e \rightarrow_{\text{H-S}} E^{\bullet}[\mathsf{dyn}\ \tau'_c\ (v_f\ (\mathsf{stat}\ \tau'_d\ v_1))]$
  by $v_0\ v_1 \rhd_{\text{H-S}} \mathsf{dyn}\ \tau'_c\ (v_f\ (\mathsf{stat}\ \tau'_d\ v_1))$
 b. QED

CASE $e = E^{\bullet}[op^1\ v]$:
1. $\vdash_{\text{H}} op^1\ v : \tau'$
 by *static hole typing*
2. $\vdash_{\text{H}} v : \tau_0 \times \tau_1$
 by *inversion*
3. $v = \langle v_0, v_1 \rangle$
 by *canonical forms*
4. **IF** $op^1 = \mathsf{fst}$:
 a. $\delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0$
 b. $e \rightarrow_{\text{H-S}} E^{\bullet}[v_0]$
  by $op^1\ v \rhd_{\text{H-S}} v_0$
 c. QED
 ELSE $op^1 = \mathsf{snd}$:
 a. $\delta(\mathsf{snd}, \langle v_0, v_1 \rangle) = v_1$
 b. $e \rightarrow_{\text{H-S}} E^{\bullet}[v_1]$
  by $op^1\ v \rhd_{\text{H-S}} v_1$
 c. QED

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:
1. $\vdash_{\text{H}} op^2\ v_0\ v_1 : \tau'$
 by *static hole typing*
2. $\vdash_{\text{H}} v_0 : \tau_0$
 $\wedge \vdash_{\text{H}} v_1 : \tau_1$
 $\wedge\ \Delta(op^2, \tau_0, \tau_1) = \tau''$
 by *inversion*
3. $\delta(op^2, v_0, v_1) = e'$
 by *Δ type soundness* (2)
4. $op^2\ v_0\ v_1 \rhd_{\text{H-S}} e'$
 by (3)
5. QED by $e \rightarrow_{\text{H-S}} E^{\bullet}[e']$

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :
1. e' is a value
 $\vee\ e' \in \mathsf{Err}$
 $\vee\ e' \rightarrow_{\text{H-D}} e''$
 $\vee\ e' \rightarrow_{\text{H-D}} \mathsf{BndryErr}$
 $\vee\ e' = E'[e'']$ and $e'' \rhd_{\text{H-D}} \mathsf{TagErr}$
 by *dynamic progress*
2. **IF** e' is a value :
 a. QED $e \rightarrow_{\text{H-S}} E[\mathcal{D}_{\text{H}}(\tau', e')]$
 IF $e' \in \mathsf{Err}$:
 a. QED $e \rightarrow_{\text{H-S}} e'$
 IF $e' \rightarrow_{\text{H-D}} e''$:
 a. QED $e \rightarrow_{\text{H-S}} E[\mathsf{dyn}\ \tau'\ e'']$
 IF $e' \rightarrow_{\text{H-D}} \mathsf{BndryErr}$:
 a. QED $e \rightarrow_{\text{H-S}} E[\mathsf{dyn}\ \tau'\ \mathsf{BndryErr}]$
 ELSE $e' = E'[e'']$ and $e'' \rhd_{\text{H-D}} \mathsf{TagErr}$:
 a. $E' \in E^{\bullet}$
  by e' is boundary-free
 b. QED

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :
1. e' is a value
 $\vee\ e' \in \mathsf{Err}$
 $\vee\ e' \rightarrow_{\text{H-S}} e''$
 $\vee\ e' \rightarrow_{\text{H-S}} \mathsf{BndryErr}$
 $\vee\ e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{H-D}} \mathsf{TagErr}$
 by *static progress*
2. **IF** e' is a value :
 a. QED $e \rightarrow_{\text{H-S}} E[\mathcal{S}_{\text{H}}(\tau', e')]$
 IF $e' \in \mathsf{Err}$:
 a. QED $e \rightarrow_{\text{H-S}} e'$
 IF $e' \rightarrow_{\text{H-S}} e''$:
 a. QED $e \rightarrow_{\text{H-S}} E[\mathsf{stat}\ \tau'\ e'']$
 IF $e' \rightarrow_{\text{H-S}} \mathsf{BndryErr}$:
 a. QED $e \rightarrow_{\text{H-S}} E[\mathsf{stat}\ \tau'\ \mathsf{BndryErr}]$
 ELSE $e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{H-D}} \mathsf{TagErr}$
 :
 a. Contradiction by e' is boundary-free

CASE $e = E[\mathsf{Err}]$:
1. QED $e \rightarrow_{\text{H-S}} \mathsf{Err}$

□

Lemma 2.9 : H *dynamic progress*

If $\vdash_{\text{H}} e$ then one of the following holds:
- e is a value
- $e \in \mathsf{Err}$
- $e \rightarrow_{\text{H-D}} e'$
- $e \rightarrow_{\text{H-D}} \mathsf{BndryErr}$
- $e = E[e']$ and $e' \rhd_{\text{H-D}} \mathsf{TagErr}$

Proof:
By the *boundary factoring* lemma, there are seven cases.

CASE e is a value :
1. QED

CASE $e = E^{\bullet}[v_0\ v_1]$:
 IF $v_0 = \lambda x.\, e'$:
 1. $e \rightarrow_{\text{H-D}} E^{\bullet}[e'[x \leftarrow v_1]]$
  by $v_0\ v_1 \rhd_{\text{H-D}} e'[x \leftarrow v_1]$
 2. QED
 IF $v_0 = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v_f$:
 1. $e \rightarrow_{\text{H-D}} E^{\bullet}[\mathsf{stat}\ \tau_c\ (v_f\ (\mathsf{dyn}\ \tau_d\ v_1))]$
  by $v_0\ v_1 \rhd_{\text{H-D}} \mathsf{stat}\ \tau_c\ (v_f\ (\mathsf{dyn}\ \tau_d\ v_1))$
 2. QED
 ELSE $v_0 = i$
  $\vee\ v_0 = \langle v, v' \rangle$:
 1. $e \rightarrow_{\text{H-D}} \mathsf{TagErr}$
  by $(v_0\ v_1) \rhd_{\text{H-D}} \mathsf{TagErr}$
 2. QED

CASE $e = E^{\bullet}[op^1\ v]$:
 IF $\delta(op^1, v) = e'$:

1. $(op^1\ v) \rhd_{\text{H-D}} e'$
2. QED

ELSE $\delta(op^1, v)$ is undefined :

1. $e \rightarrow_{\text{H-D}} \mathsf{TagErr}$
   by $(op^1\ v) \rhd_{\text{H-D}} \mathsf{TagErr}$
2. QED

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:

IF $\delta(op^2, v_0, v_1) = e''$:

1. $op^2\ v_0\ v_1 \rhd_{\text{H-D}} e''$
2. QED

ELSE $\delta(op^2, v_0, v_1)$ is undefined :

1. $e \rightarrow_{\text{H-D}} \mathsf{TagErr}$
   by $op^2\ v_0\ v_1 \rhd_{\text{H-D}} \mathsf{TagErr}$
2. QED

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :

1. e' is a value
   $\vee\ e' \in \mathsf{Err}$
   $\vee\ e' \rightarrow_{\text{H-D}} e''$
   $\vee\ e' \rightarrow_{\text{H-D}} \mathsf{BndryErr}$
   $\vee\ e' = E[e'']$ and $e'' \rhd_{\text{H-D}} \mathsf{TagErr}$
   by *dynamic progress*
2. **IF** e' is a value :
   a. QED $e \rightarrow_{\text{H-D}} E[\mathcal{D}_{\text{H}}(\tau', e')]$

   IF $e' \in \mathsf{Err}$:
   a. QED $e \rightarrow_{\text{H-D}} e'$

   IF $e' \rightarrow_{\text{H-D}} e''$:
   a. QED $e \rightarrow_{\text{H-S}} E[\mathsf{dyn}\ \tau'\ e'']$

   IF $e' \rightarrow_{\text{H-D}} \mathsf{BndryErr}$:
   a. QED $e \rightarrow_{\text{H-D}} E[\mathsf{dyn}\ \tau'\ \mathsf{BndryErr}]$

   ELSE $e' = E[e'']$ and $e'' \rhd_{\text{H-D}} \mathsf{TagErr}$:
   a. $E \in E^{\bullet}$
      by e' is boundary-free
   b. QED

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :

1. e' is a value
   $\vee\ e' \in \mathsf{Err}$
   $\vee\ e' \rightarrow_{\text{H-S}} e''$
   $\vee\ e' \rightarrow_{\text{H-S}} \mathsf{BndryErr}$
   $\vee\ e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{H-D}} \mathsf{TagErr}$
   by *static progress*
2. **IF** e' is a value :
   a. QED $e \rightarrow_{\text{H-S}} E[\mathcal{S}_{\text{H}}(\tau', e')]$

   IF $e' \in \mathsf{Err}$:
   a. QED $e \rightarrow_{\text{H-S}} e'$

   IF $e' \rightarrow_{\text{H-S}} e''$:
   a. QED $e \rightarrow_{\text{H-S}} E[\mathsf{stat}\ \tau'\ e'']$

   IF $e' \rightarrow_{\text{H-S}} \mathsf{BndryErr}$:
   a. QED $e \rightarrow_{\text{H-S}} E[\mathsf{stat}\ \tau'\ \mathsf{BndryErr}]$

   ELSE $e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{H-D}} \mathsf{TagErr}$
   :
   a. Contradiction by e' is boundary-free

CASE $e = E[\mathsf{Err}]$:

1. QED $e \rightarrow_{\text{H-D}} \mathsf{Err}$

□

Lemma 2.10 : H *static preservation*

If $\vdash_{\text{H}} e : \tau$ and $e \rightarrow_{\text{H-S}} e'$ then $\vdash_{\text{H}} e' : \tau$

Proof:

By the *boundary factoring* lemma there are seven cases.

CASE e is a value :

1. Contradiction by $e \rightarrow_{\text{H-S}} e'$

CASE $e = E^{\bullet}[v_0\ v_1]$:

IF $v_0 = \lambda(x\!:\!\tau_x).\ e'$
$\quad\wedge\ e \rightarrow_{\text{H-S}} E^{\bullet}[e'[x \leftarrow v_1]]$:

1. $\vdash_{\text{H}} v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\text{H}} v_0 : \tau_d \Rightarrow \tau_c$
   $\wedge\ \vdash_{\text{H}} v_1 : \tau_d$
   $\wedge\ \tau_c \leqslant: \tau'$
   by *inversion*
3. $\tau_d \leqslant: \tau_x$
   by *canonical forms* (2)
4. $(x\!:\!\tau_x) \vdash_{\text{H}} e' : \tau_c$
   by *inversion* (2)
5. $\vdash_{\text{H}} v_1 : \tau_x$
   by (2, 3)
6. $\vdash_{\text{H}} e'[x \leftarrow v_1] : \tau_c$
   by *substitution* (4, 5)
7. $\vdash_{\text{H}} e'[x \leftarrow v_1] : \tau'$
   by (2, 6)
8. QED by *hole substitution* (7)

ELSE $v_0 = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v_f$
$\quad\wedge\ e \rightarrow_{\text{H-S}} E^{\bullet}[\mathsf{dyn}\ \tau_c\ (v_f\ (\mathsf{stat}\ \tau_d\ v_1))]$:

1. $\vdash_{\text{H}} v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\text{H}} v_0 : \tau_d' \Rightarrow \tau_c'$
   $\wedge\ \vdash_{\text{H}} v_1 : \tau_d'$
   $\wedge\ \tau_c' \leqslant: \tau'$
   by *inversion*
3. $\vdash_{\text{H}} v_f$
   by *inversion* (2)
4. $\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'$
   by *canonical forms* (2)
5. $\tau_d' \leqslant: \tau_d$
   $\wedge\ \tau_c \leqslant: \tau_c'$
   by (4)
6. $\vdash_{\text{H}} v_1 : \tau_d$
   by (2, 5)
7. $\vdash_{\text{H}} \mathsf{stat}\ \tau_d\ v_1$
   by (6)
8. $\vdash_{\text{H}} v_f\ (\mathsf{stat}\ \tau_d\ v_1)$
   by (3, 7)
9. $\vdash_{\text{H}} \mathsf{dyn}\ \tau_c\ v_f\ (\mathsf{stat}\ \tau_d\ v_1) : \tau_c$
   by (8)
10. $\vdash_{\text{H}} \mathsf{dyn}\ \tau_c\ v_f\ (\mathsf{stat}\ \tau_d\ v_1) : \tau'$
    by (2, 5, 9)
11. QED by *hole substitution* (10)

CASE $e = E^{\bullet}[op^1\ v]$:

IF $v = \langle v_0, v_1 \rangle$
$\land\ op^1 = \mathsf{fst}$
$\land\ e \rightarrow_{\text{H-S}} E^{\bullet}[v_0]$:

1. $\vdash_{\text{H}} \mathsf{fst}\ \langle v_0, v_1 \rangle : \tau'$
   by *static hole typing*
2. $\vdash_{\text{H}} \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$
   $\land\ \tau_0 \leqslant: \tau'$
   by *inversion* (1)
3. $\vdash_{\text{H}} v_0 : \tau_0$
   by *inversion* (2)
4. $\vdash_{\text{H}} v_0 : \tau'$
   by (2, 3)
5. QED by *hole substitution* (4)

ELSE $v = \langle v_0, v_1 \rangle$
$\land\ op^1 = \mathsf{snd}$
$\land\ e \rightarrow_{\text{H-S}} E^{\bullet}[v_1]$:

1. $\vdash_{\text{H}} \mathsf{snd}\ \langle v_0, v_1 \rangle : \tau'$
   by *static hole typing*
2. $\vdash_{\text{H}} \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$
   $\land\ \tau_1 \leqslant: \tau'$
   by *inversion* (1)
3. $\vdash_{\text{H}} v_1 : \tau_1$
   by *inversion* (2)
4. $\vdash_{\text{H}} v_1 : \tau'$
   by (2, 3)
5. QED by *hole substitution* (4)

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:

1. $e \rightarrow_{\text{H-S}} E^{\bullet}[\delta(op^2, v_0, v_1)]$
   by $e \rightarrow_{\text{H-S}} e'$
2. $\vdash_{\text{H}} op^2\ v_0\ v_1 : \tau'$
   by *static hole typing*
3. $\vdash_{\text{H}} v_0 : \tau_0$
   $\land \vdash_{\text{H}} v_1 : \tau_1$
   $\land\ \Delta(op^2, \tau_0, \tau_1) = \tau''$
   $\land\ \tau'' \leqslant: \tau'$
   by *inversion* (1)
4. $\vdash_{\text{H}} \delta(op^2, v_0, v_1) : \tau''$
   by Δ *type soundness* (2)
5. $\vdash_{\text{H}} \delta(op^2, v_0, v_1) : \tau'$
   by (2, 3)
6. QED by *hole substitution* (4)

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :

IF e' is a value :

1. $e \rightarrow_{\text{H-S}} E[\mathcal{D}_{\text{H}}(\tau', e')]$
2. $\vdash_{\text{H}} \mathsf{dyn}\ \tau'\ e' : \tau'$
   by *boundary hole typing*
3. $\vdash_{\text{H}} e'$
   by *inversion* (2)
4. $\vdash_{\text{H}} \mathcal{D}_{\text{H}}(\tau', e') : \tau'$
   by $\mathcal{D}_{\text{H}}$ *soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\text{H-D}} e''$:

1. $e \rightarrow_{\text{H-S}} E[\mathsf{dyn}\ \tau'\ e'']$
2. $\vdash_{\text{H}} \mathsf{dyn}\ \tau'\ e' : \tau'$
   by *boundary hole typing*
3. $\vdash_{\text{H}} e'$
   by *inversion* (2)
4. $\vdash_{\text{H}} e''$
   by *dynamic preservation* (3)
5. $\vdash_{\text{H}} \mathsf{dyn}\ \tau'\ e'' : \tau'$
   by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :

IF e' is a value :

1. $e \rightarrow_{\text{H-S}} E[\mathcal{S}_{\text{H}}(\tau', e')]$
2. $\vdash_{\text{H}} \mathsf{stat}\ \tau'\ e'$
   by *boundary hole typing*
3. $\vdash_{\text{H}} e' : \tau'$
   by *inversion* (2)
4. $\vdash_{\text{H}} \mathcal{S}_{\text{H}}(\tau', e')$
   by $\mathcal{S}_{\text{H}}$ *soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\text{H-S}} e''$:

1. $e \rightarrow_{\text{H-S}} E[\mathsf{stat}\ \tau'\ e'']$
2. $\vdash_{\text{H}} \mathsf{stat}\ \tau'\ e'$
   by *boundary hole typing*
3. $\vdash_{\text{H}} e' : \tau'$
   by *inversion* (2)
4. $\vdash_{\text{H}} e'' : \tau'$
   by *static preservation* (3)
5. $\vdash_{\text{H}} \mathsf{stat}\ \tau'\ e''$
   by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{Err}]$:

1. $e \rightarrow_{\text{H-S}} \mathsf{Err}$
2. QED by $\vdash_{\text{H}} \mathsf{Err} : \tau$

□

Lemma 2.11 : H *dynamic preservation*

If $\vdash_{\text{H}} e$ and $e \rightarrow_{\text{H-D}} e'$ then $\vdash_{\text{H}} e'$

Proof:

By the *boundary factoring* lemma, there are seven cases.

CASE e is a value :

1. Contradiction by $e \rightarrow_{\text{H-D}} e'$

CASE $e = E^{\bullet}[v_0\ v_1]$:

IF $v_0 = \lambda x.\, e'$
$\land\ e \rightarrow_{\text{H-D}} E^{\bullet}[e'[x \leftarrow v_1]]$:

1. $\vdash_{\text{H}} v_0\ v_1$
   by *dynamic hole typing*
2. $\vdash_{\text{H}} v_0$
   $\land \vdash_{\text{H}} v_1$
   by *inversion* (1)
3. $x \vdash_{\text{H}} e'$
   by *inversion* (2)
4. $\vdash_{\text{H}} e'[x \leftarrow v_1]$
   by *substitution* (2, 3)
5. QED *hole substitution* (4)

ELSE $v_0 = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v_f$
$\quad\wedge\ e \rightarrow_{\text{H-D}} E^\bullet[\mathsf{stat}\ \tau_c\ (v_f\ (\mathsf{dyn}\ \tau_d\ v_1))]$:

1. $\vdash_{\mathrm{H}} v_0\ v_1$
   by *dynamic hole typing*
2. $\vdash_{\mathrm{H}} v_0$
   $\wedge \vdash_{\mathrm{H}} v_1$
   by *inversion* (1)
3. $\vdash_{\mathrm{H}} v_f : \tau_d \Rightarrow \tau_c$
   by *inversion* (2)
4. $\vdash_{\mathrm{H}} \mathsf{dyn}\ \tau_d\ v_1 : \tau_d$
   by (2)
5. $\vdash_{\mathrm{H}} v_f\ (\mathsf{dyn}\ \tau_d\ v_1) : \tau_c$
   by (3, 4)
6. $\vdash_{\mathrm{H}} \mathsf{stat}\ \tau_c\ v_f\ (\mathsf{dyn}\ \tau_d\ v_1)$
   by (5)
7. QED by *hole substitution*

CASE $e = E^\bullet[op^1\ v]$:

IF $v = \langle v_0, v_1\rangle$
$\wedge\ op^1 = \mathsf{fst}$
$\wedge\ e \rightarrow_{\text{H-D}} E^\bullet[v_0]$:

1. $\vdash_{\mathrm{H}} op^1\ v$
   by *dynamic hole typing*
2. $\vdash_{\mathrm{H}} v$
   by *inversion* (1)
3. $\vdash_{\mathrm{H}} v_0$
   by *inversion* (2)
4. QED by *hole substitution*

ELSE $v = \langle v_0, v_1\rangle$
$\wedge\ op^1 = \mathsf{snd}$
$\wedge\ e \rightarrow_{\text{H-D}} E^\bullet[v_1]$:

1. $\vdash_{\mathrm{H}} op^1\ v$
   by *dynamic hole typing*
2. $\vdash_{\mathrm{H}} v$
   by *inversion* (1)
3. $\vdash_{\mathrm{H}} v_1$
   by *inversion* (2)
4. QED by *hole substitution*

CASE $e = E^\bullet[op^2\ v_0\ v_1]$:

1. $e \rightarrow_{\text{H-D}} E^\bullet[\delta(op^2, v_0, v_1)]$
2. $\vdash_{\mathrm{H}} op^2\ v_0\ v_1$
   by *dynamic hole typing*
3. $\vdash_{\mathrm{H}} v_0$
   $\wedge \vdash_{\mathrm{H}} v_1$
   by *inversion* (1)
4. $\vdash_{\mathrm{H}} \delta(op^2, v_0, v_1)$
   by *δ preservation* (2)
5. QED by *hole substitution* (3)

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :

IF e' is a value :

1. $e \rightarrow_{\text{H-D}} E[\mathcal{D}_{\mathrm{H}}(\tau', e')]$
2. $\vdash_{\mathrm{H}} \mathsf{dyn}\ \tau'\ e' : \tau'$
   by *boundary hole typing*
3. $\vdash_{\mathrm{H}} e'$
   by *inversion* (2)
4. $\vdash_{\mathrm{H}} \mathcal{D}_{\mathrm{H}}(\tau', e') : \tau'$
   by *$\mathcal{D}_{\mathrm{H}}$ soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\text{H-D}} e''$:

1. $e \rightarrow_{\text{H-D}} E[\mathsf{dyn}\ \tau'\ e'']$
2. $\vdash_{\mathrm{H}} \mathsf{dyn}\ \tau'\ e' : \tau'$
   by *boundary hole typing*
3. $\vdash_{\mathrm{H}} e'$
   $\wedge\ \tau' \leqslant: \tau''$
   by *inversion* (2)
4. $\vdash_{\mathrm{H}} e''$
   by *dynamic preservation* (3)
5. $\vdash_{\mathrm{H}} \mathsf{dyn}\ \tau'\ e'' : \tau'$
   by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :

IF $e' \in v$:

1. $e \rightarrow_{\text{H-D}} E[\mathcal{S}_{\mathrm{H}}(\tau', e')]$
2. $\vdash_{\mathrm{H}} \mathsf{stat}\ \tau'\ e'$
   by *boundary hole typing*
3. $\vdash_{\mathrm{H}} e' : \tau'$
   by *inversion* (2)
4. $\vdash_{\mathrm{H}} \mathcal{S}_{\mathrm{H}}(\tau', e')$
   by *$\mathcal{S}_{\mathrm{H}}$ soundness* (3)
5. QED by *hole substitution* (5)

ELSE $e' \rightarrow_{\text{H-S}} e''$:

1. $e \rightarrow_{\text{H-D}} E[\mathsf{stat}\ \tau'\ e'']$
2. $\vdash_{\mathrm{H}} \mathsf{stat}\ \tau'\ e'$
   by *boundary hole typing*
3. $\vdash_{\mathrm{H}} e' : \tau'$
   by *inversion* (2)
4. $\vdash_{\mathrm{H}} e'' : \tau'$
   by *static preservation* (3)
5. $\vdash_{\mathrm{H}} \mathsf{stat}\ \tau'\ e''$
   by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{Err}]$:

1. $e \rightarrow_{\text{H-D}} \mathsf{Err}$
2. QED $\vdash_{\mathrm{H}} \mathsf{Err}$

□

Lemma 2.12 : H *static boundary factoring*
If $\vdash_{\mathrm{H}} e : \tau$ then one of the following holds:
- e is a value
- $e = E^\bullet[v_0\ v_1]$
- $e = E^\bullet[op^1\ v]$
- $e = E^\bullet[op^2\ v_0\ v_1]$
- $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{Err}]$

Proof:
By the *unique static evaluation contexts* lemma, there are seven cases.

CASE e is a value :

1. QED

case $e = E[v_0\ v_1]$:
1. $E = E^\bullet$
   $\lor\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\lor\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
   by *inner boundary*
2. **if** $E = E^\bullet$:
   a. qed $e = E^\bullet[v_0\ v_1]$
   if $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. qed $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[v_0\ v_1]]$
   else $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. qed $e = E'[\mathsf{stat}\ \tau\ E^\bullet[v_0\ v_1]]$

case $e = E[op^1\ v]$:
1. $E = E^\bullet$
   $\lor\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\lor\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
   by *inner boundary*
2. **if** $E = E^\bullet$:
   a. qed $e = E^\bullet[op^1\ v]$
   if $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. qed $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^1\ v]]$
   else $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. qed $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^1\ v]]$

case $e = E[op^2\ v_0\ v_1]$:
1. $E = E^\bullet$
   $\lor\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\lor\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
   by *inner boundary*
2. **if** $E = E^\bullet$:
   a. qed $e = E^\bullet[op^2\ v_0\ v_1]$
   if $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. qed $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$
   else $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. qed $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$

case $e = E[\mathsf{dyn}\ \tau\ v]$:
1. qed v is boundary-free

case $e = E[\mathsf{stat}\ \tau\ v]$:
1. qed v is boundary-free

case $e = E[\mathsf{Err}]$:
1. qed

□

Lemma 2.13 : H *unique static evaluation contexts*

If $\vdash_{\mathsf{H}} e : \tau$ then one of the following holds:
- e is a value
- $e = E[v_0\ v_1]$
- $e = E[op^1\ v]$
- $e = E[op^2\ v_0\ v_1]$
- $e = E[\mathsf{dyn}\ \tau\ v]$
- $e = E[\mathsf{stat}\ \tau\ v]$
- $e = E[\mathsf{Err}]$

Proof:

By induction on the structure of e.

case $e = x$
   $\lor\ e = \lambda x.\, e'$
   $\lor\ e = \mathsf{stat}\ \tau\ e'$:
1. Contradiction by $\vdash_{\mathsf{H}} e : \tau$

case $e = i$
   $\lor\ e = \lambda(x\!:\!\tau_d).\, e'$
   $\lor\ e = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v$:
1. qed e is a value

case $e = \langle e_0, e_1 \rangle$:

if $e_0 \notin v$:
1. $\vdash_{\mathsf{H}} e_0 : \tau_0$
   by *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis (1)
3. $E = \langle E_0, e_1 \rangle$
4. qed $e = E[e_0']$

if $e_0 \in v$
   $\land\ e_1 \notin v$:
1. $\vdash_{\mathsf{H}} e_1 : \tau_1$
   by *inversion*
2. $e_1 = E_1[e_1']$
   by the induction hypothesis (1)
3. $E = \langle e_0, E_1 \rangle$
4. qed $e = E[e_1']$

else $e_0 \in v$
   $\land\ e_1 \in v$:
1. $E = [\,]$
2. qed e is a value

case $e = e_0\ e_1$:

if $e_0 \notin v$:
1. $\vdash_{\mathsf{H}} e_0 : \tau_0$
   by *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis (1)
3. $E = E_0\ e_1$
4. qed $e = E[e_0']$

if $e_0 \in v$
   $\land\ e_1 \notin v$:
1. $\vdash_{\mathsf{H}} e_1 : \tau_1$
   by *inversion*
2. $e_1 = E_1[e_1']$
   by the induction hypothesis (1)
3. $E = e_0\ E_1$
4. qed $e = E[e_1']$

else $e_0 \in v$
   $\land\ e_1 \in v$:
1. $E = [\,]$
2. qed $e = E[e_0\ e_1]$

case $e = op^1\ e_0$:

if $e_0 \notin v$:
1. $\vdash_{\mathsf{H}} e_0 : \tau_0$
   by *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis (1)
3. $E = op^1\ E_0$
4. qed $e = E[e_0']$

else $e_0 \in v$:

1. $E = [\,]$
2. QED $e = E[op^1\ e_0]$

CASE $e = op^2\ e_0\ e_1$:

IF $e_0 \notin v$:

1. $\vdash_H e_0 : \tau_0$
   by *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis (1)
3. $E = op^2\ E_0\ e_1$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\ e_1 \notin v$:

1. $\vdash_H e_1 : \tau_1$
   by *inversion*
2. $e_1 = E_1[e_1']$
   by the induction hypothesis (1)
3. $E = op^2\ e_0\ E_1$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\ e_1 \in v$:

1. $E = [\,]$
2. QED $e = E[op^2\ e_0\ e_1]$

CASE $e = \mathsf{dyn}\ \tau\ e_0$:

IF $e_0 \notin v$:

1. $\vdash_H e_0$
   by *inversion*
2. $e_0 = E_0[e_0']$
   by *unique dynamic evaluation contexts* (1)
3. $E = \mathsf{dyn}\ \tau\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:

1. $E = [\,]$
2. QED $e = E[\mathsf{dyn}\ \tau\ e_0]$

CASE $e = \mathsf{stat}\ \tau\ e_0$:

Contradiction by $\vdash_H e : \tau$

CASE $e = \mathsf{Err}$:

1. $E = [\,]$
2. QED $e = E[\mathsf{Err}]$

☐

Lemma 2.14 : H *inner boundary*

For all contexts E, one of the following holds:

- $E = E^\bullet$
- $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
- $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

Proof:

By induction on the structure of E.

CASE $E = E^\bullet$:

1. QED

CASE $E = E_0\ e_1$:

1. $E_0 = E^\bullet$
   $\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\vee\ E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
   by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
   a. QED E is boundary-free
   IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. $E' = E_0'\ e_1$
   b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. $E' = E_0'\ e_1$
   b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = v_0\ E_1$:

1. $E_1 = E^\bullet$
   $\vee\ E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\vee\ E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$
   by the induction hypothesis
2. **IF** $E_1 = E^\bullet$:
   a. QED E is boundary-free
   IF $E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. $E' = v_0\ E_1'$
   b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   ELSE $E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. $E' = v_0\ E_1'$
   b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = \langle E_0, e_1\rangle$:

1. $E_0 = E^\bullet$
   $\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\vee\ E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
   by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
   a. QED E is boundary-free
   IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. $E' = \langle E_0', e_1\rangle$
   b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. $E' = \langle E_0', e_1\rangle$
   b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = \langle v_0, E_1\rangle$:

1. $E_1 = E^\bullet$
   $\vee\ E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\vee\ E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$
   by the induction hypothesis
2. **IF** $E_1 = E^\bullet$:
   a. QED E is boundary-free
   IF $E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. $E' = \langle v_0, E_1'\rangle$
   b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   ELSE $E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. $E' = \langle v_0, E_1'\rangle$
   b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = op^1\ E_0$:

1. $E_0 = E^\bullet$
   $\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\vee\ E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
   by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
   a. QED E is boundary-free
   IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:

a. $E' = op^1\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = op^1\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
CASE $E = op^2\ E_0\ e_1$:
1. $E_0 = E^\bullet$
$\lor\ E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\lor\ E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = op^2\ E_0'\ e_1$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = op^2\ E_0'\ e_1$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
CASE $E = op^2\ v_0\ E_1$:
1. $E_1 = E^\bullet$
$\lor\ E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\lor\ E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_1 = E^\bullet$:
a. QED E is boundary-free
IF $E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = op^2\ v_0\ E_1'$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = op^2\ v_0\ E_1'$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
CASE $E = \mathsf{dyn}\ \tau\ E_0$:
1. $E_0 = E^\bullet$
$\lor\ E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$
$\lor\ E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED
IF $E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau'\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau'\ E^\bullet]$
CASE $E = \mathsf{stat}\ \tau\ E_0$:
1. $E_0 = E^\bullet$
$\lor\ E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$
$\lor\ E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED
IF $E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{stat}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau'\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{stat}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau'\ E^\bullet]$

□

Lemma 2.15 : H *dynamic boundary factoring*

If $\vdash_{\mathrm{H}} e$ then one of the following holds:

- e is a value
- $e = E^\bullet[v_0\ v_1]$
- $e = E^\bullet[op^1\ v]$
- $e = E^\bullet[op^2\ v_0\ v_1]$
- $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{Err}]$

Proof:
By the *unique dynamic evaluation contexts* lemma, there are seven cases.
CASE e is a value :
1. QED
CASE $e = E[v_0\ v_1]$:
1. $E = E^\bullet$
$\lor\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\lor\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. QED $e = E^\bullet[v_0\ v_1]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[v_0\ v_1]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[v_0\ v_1]]$
CASE $e = E[op^1\ v]$:
1. $E = E^\bullet$
$\lor\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\lor\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. QED $e = E^\bullet[op^1\ v]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^1\ v]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^1\ v]]$
CASE $e = E[op^2\ v_0\ v_1]$:
1. $E = E^\bullet$
$\lor\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\lor\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. QED $e = E^\bullet[op^2\ v_0\ v_1]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$
CASE $e = E[\mathsf{dyn}\ \tau\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{stat}\ \tau\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{Err}]$:
1. QED

□

Lemma 2.16 : H *unique dynamic evaluation contexts*

If $\vdash_{\mathrm{H}} e$ then one of the following holds:
- e is a value
- $e = E[v_0\ v_1]$
- $e = E[op^1\ v]$
- $e = E[op^2\ v_0\ v_1]$
- $e = E[\mathsf{dyn}\ \tau\ v]$
- $e = E[\mathsf{stat}\ \tau\ v]$
- $e = E[\mathsf{Err}]$

Proof:

By induction on the structure of e.

CASE $e = x$
$\lor\ e = \lambda(x\!:\!\tau).\, e'$
$\lor\ e = \mathsf{dyn}\ \tau\ e'$:
1. Contradiction by $\vdash_{\mathrm{H}} e$

CASE $e = i$
$\lor\ e = \lambda x.\, e'$
$\lor\ e = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v$:
1. QED e is a value

CASE $e = \mathsf{Err}$:
1. $E = [\,]$
2. QED $e = E[\mathsf{Err}]$

CASE $e = \langle e_0, e_1 \rangle$:

IF $e_0 \notin v$:
1. $\vdash_{\mathrm{H}} e_0$
   by *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis (1)
3. $E = \langle E_0, e_1 \rangle$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\land\ e_1 \notin v$:
1. $\vdash_{\mathrm{H}} e_1$
   by *inversion*
2. $e_1 = E_1[e_1']$
   by the induction hypothesis (1)
3. $E = \langle e_0, E_1 \rangle$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\land\ e_1 \in v$:
1. $E = [\,]$
2. QED e is a value

CASE $e = e_0\ e_1$:

IF $e_0 \notin v$:
1. $\vdash_{\mathrm{H}} e_0$
   by *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis (1)
3. $E = E_0\ e_1$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\land\ e_1 \notin v$:
1. $\vdash_{\mathrm{H}} e_1$
   by *inversion*
2. $e_1 = E_1[e_1']$
   by the induction hypothesis (1)
3. $E = e_0\ E_1$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\land\ e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[e_0\ e_1]$

CASE $e = op^1\ e_0$:

IF $e_0 \notin v$:
1. $\vdash_{\mathrm{H}} e_0$
   by *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis (1)
3. $E = op^1\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[op^1\ e_0]$

CASE $e = op^2\ e_0\ e_1$:

IF $e_0 \notin v$:
1. $\vdash_{\mathrm{H}} e_0$
   by *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis (1)
3. $E = op^2\ E_0\ e_1$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\land\ e_1 \notin v$:
1. $\vdash_{\mathrm{H}} e_1$
   by *inversion*
2. $e_1 = E_1[e_1']$
   by the induction hypothesis (1)
3. $E = op^2\ e_0\ E_1$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\land\ e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[op^2\ e_0\ e_1]$

CASE $e = \mathsf{stat}\ \tau\ e_0$:

IF $e_0 \notin v$:
1. $\vdash_{\mathrm{H}} e_0$
   by *inversion*
2. $e_0 = E_0[e_0']$
   by *unique static evaluation contexts* (1)
3. $E = \mathsf{stat}\ \tau\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$

2. QED $e = E[\text{stat}\ \tau\ e_0]$

□

Lemma 2.17 : H *static hole typing*

If $\vdash_H E^\bullet[e] : \tau$ then the derivation contains a sub-term $\vdash_H e : \tau'$

Proof:

By induction on the structure of $E^\bullet$.

CASE $E^\bullet = [\,]$:

1. QED $E^\bullet[e] = e$

CASE $E^\bullet = E^\bullet{}_0\ e_1$:

1. $E^\bullet[e] = E^\bullet{}_0[e]\ e_1$
2. $\vdash_H E^\bullet{}_0[e] : \tau_d \Rightarrow \tau_c$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = v_0\ E^\bullet{}_1$:

1. $E^\bullet[e] = v_0\ E^\bullet{}_1[e]$
2. $\vdash_H E^\bullet{}_1[e] : \tau_d$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = \langle E^\bullet{}_0, e_1 \rangle$:

1. $E^\bullet[e] = \langle E^\bullet{}_0[e], e_1 \rangle$
2. $\vdash_H E^\bullet{}_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = \langle v_0, E^\bullet{}_1 \rangle$:

1. $E^\bullet[e] = \langle v_0, E^\bullet{}_1[e] \rangle$
2. $\vdash_H E^\bullet{}_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = op^1\ E^\bullet{}_0$:

1. $E^\bullet[e] = op^1\ E^\bullet{}_0[e]$
2. $\vdash_H E^\bullet{}_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = op^2\ E^\bullet{}_0\ e_1$:

1. $E^\bullet[e] = op^2\ E^\bullet{}_0[e]\ e_1$
2. $\vdash_H E^\bullet{}_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = op^2\ v_0\ E^\bullet{}_1$:

1. $E^\bullet[e] = op^2\ v_0\ E^\bullet{}_1[e]$
2. $\vdash_H E^\bullet{}_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

□

Lemma 2.18 : H *dynamic hole typing*

If $\vdash_H E^\bullet[e]$ then the derivation contains a sub-term $\vdash_H e$

Proof:

By induction on the structure of $E^\bullet$.

CASE $E^\bullet = [\,]$:

1. QED $E^\bullet[e] = e$

CASE $E^\bullet = E^\bullet{}_0\ e_1$:

1. $E^\bullet[e] = E^\bullet{}_0[e]\ e_1$
2. $\vdash_H E^\bullet{}_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = v_0\ E^\bullet{}_1$:

1. $E^\bullet[e] = v_0\ E^\bullet{}_1[e]$
2. $\vdash_H E^\bullet{}_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = \langle E^\bullet{}_0, e_1 \rangle$:

1. $E^\bullet[e] = \langle E^\bullet{}_0[e], e_1 \rangle$
2. $\vdash_H E^\bullet{}_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = \langle v_0, E^\bullet{}_1 \rangle$:

1. $E^\bullet[e] = \langle v_0, E^\bullet{}_1[e] \rangle$
2. $\vdash_H E^\bullet{}_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = op^1\ E^\bullet{}_0$:

1. $E^\bullet[e] = op^1\ E^\bullet{}_0[e]$
2. $\vdash_H E^\bullet{}_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = op^2\ E^\bullet{}_0\ e_1$:

1. $E^\bullet[e] = op^2\ E^\bullet{}_0[e]\ e_1$
2. $\vdash_H E^\bullet{}_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = op^2\ v_0\ E^\bullet{}_1$:

1. $E^\bullet[e] = op^2\ v_0\ E^\bullet{}_1[e]$
2. $\vdash_H E^\bullet{}_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

□

Lemma 2.19 : H *boundary hole typing*

- If $\vdash_H E[\text{dyn}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_H \text{dyn}\ \tau\ e : \tau$
- If $\vdash_H E[\text{dyn}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_H \text{dyn}\ \tau\ e : \tau$
- If $\vdash_H E[\text{stat}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_H \text{stat}\ \tau\ e$
- If $\vdash_H E[\text{stat}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_H \text{stat}\ \tau\ e$

Proof:

By the following four lemmas: *static* dyn *hole typing*, *dynamic* dyn *hole typing*, *static* stat *hole typing*, and *dynamic* stat *hole typing*.

□

Lemma 2.20 : H *static* dyn *hole typing*

If $\vdash_H E[\text{dyn}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_H \text{dyn}\ \tau\ e : \tau$.

Proof:

By induction on the structure of E.

CASE $E \in E^\bullet$:

1. $\vdash_H$ dyn τ e : τ''
   by *static hole typing*
2. $\vdash_H$ dyn τ e : τ
   by *inversion* (1)
3. QED

CASE $E = E_0\ e_1$:

1. E[dyn τ e] = E_0[dyn τ e] e_1
2. $\vdash_H$ E_0[dyn τ e] : τ_0
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:

1. E[dyn τ e] = v_0 E_1[dyn τ e]
2. $\vdash_H$ E_1[dyn τ e] : τ_1
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:

1. E[dyn τ e] = $\langle E_0$[dyn τ e]$, e_1\rangle$
2. $\vdash_H$ E_0[dyn τ e] : τ_0
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:

1. E[dyn τ e] = $\langle v_0, E_1$[dyn τ e]$\rangle$
2. $\vdash_H$ E_1[dyn τ e] : τ_1
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:

1. E[dyn τ e] = op^1 E_0[dyn τ e]
2. $\vdash_H$ E_0[dyn τ e] : τ_0
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:

1. E[dyn τ e] = op^2 E_0[dyn τ e] e_1
2. $\vdash_H$ E_0[dyn τ e] : τ_0
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:

1. E[dyn τ e] = op^2 v_0 E_1[dyn τ e]
2. $\vdash_H$ E_1[dyn τ e] : τ_1
   by *inversion*
3. QED by the induction hypothesis (2)

CASE E = dyn τ_0 E_0 :

1. E[dyn τ e] = dyn τ_0 E_0[dyn τ e]
2. $\vdash_H$ E_0[dyn τ e]
   by *inversion*
3. QED by *dynamic* dyn *hole typing* (2)

CASE E = stat τ_0 E_0 :

1. Contradiction by $\vdash_H$ E[dyn τ e] : τ'

□

Lemma 2.21 : H *dynamic* dyn *hole typing*

If $\vdash_H$ E[dyn τ e] then the derivation contains a sub-term $\vdash_H$ dyn τ e : τ.

Proof:

By induction on the structure of E.

CASE $E \in E^\bullet$:

1. Contradiction by $\vdash_H$ E[dyn τ e]

CASE $E = E_0\ e_1$:

1. E[dyn τ e] = E_0[dyn τ e] e_1
2. $\vdash_H$ E_0[dyn τ e]
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:

1. E[dyn τ e] = v_0 E_1[dyn τ e]
2. $\vdash_H$ E_1[dyn τ e]
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:

1. E[dyn τ e] = $\langle E_0$[dyn τ e]$, e_1\rangle$
2. $\vdash_H$ E_0[dyn τ e]
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:

1. E[dyn τ e] = $\langle v_0, E_1$[dyn τ e]$\rangle$
2. $\vdash_H$ E_1[dyn τ e]
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:

1. E[dyn τ e] = op^1 E_0[dyn τ e]
2. $\vdash_H$ E_0[dyn τ e]
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:

1. E[dyn τ e] = op^2 E_0[dyn τ e] e_1
2. $\vdash_H$ E_0[dyn τ e]
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:

1. E[dyn τ e] = op^2 v_0 E_1[dyn τ e]
2. $\vdash_H$ E_1[dyn τ e]
   by *inversion*
3. QED by the induction hypothesis (2)

CASE E = dyn τ E_0 :

1. Contradiction by $\vdash_H$ E[dyn τ e]

CASE E = stat τ_0 E_0 :

1. E[dyn τ e] = stat τ_0 E_0[dyn τ e]
2. $\vdash_H$ E_0[dyn τ e] : τ_0
   by *inversion*
3. QED by *static* dyn *hole typing* (2)

□

Lemma 2.22 : H *static* stat *hole typing*

If $\vdash_H$ E[stat τ e] : τ' then the derivation contains a sub-term $\vdash_H$ stat τ e.

Proof:

By induction on the structure of E.

CASE $E \in E^\bullet$:

1. Contradiction by $\vdash_H$ E[stat τ e] : τ'

case $E = E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by *inversion*
3. **qed** by the induction hypothesis (2)

case $E = v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{H}} E_1[\mathsf{stat}\ \tau\ e] : \tau_1$
   by *inversion*
3. **qed** by the induction hypothesis (2)

case $E = \langle E_0, e_1 \rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle E_0[\mathsf{stat}\ \tau\ e], e_1 \rangle$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by *inversion*
3. **qed** by the induction hypothesis (2)

case $E = \langle v_0, E_1 \rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle v_0, E_1[\mathsf{stat}\ \tau\ e] \rangle$
2. $\vdash_{\mathsf{H}} E_1[\mathsf{stat}\ \tau\ e] : \tau_1$
   by *inversion*
3. **qed** by the induction hypothesis (2)

case $E = op^1\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = op^1\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by *inversion*
3. **qed** by the induction hypothesis (2)

case $E = op^2\ E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by *inversion*
3. **qed** by the induction hypothesis (2)

case $E = op^2\ v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{H}} E_1[\mathsf{stat}\ \tau\ e] : \tau_1$
   by *inversion*
3. **qed** by the induction hypothesis (2)

case $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{dyn}\ \tau_0\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. **qed** by *dynamic* stat *hole typing* (2)

case $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_{\mathsf{H}} E[\mathsf{stat}\ \tau\ e] : \tau'$

□

Lemma 2.23 : H *dynamic* stat *hole typing*

If $\vdash_{\mathsf{H}}\ E[\mathsf{stat}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau\ e$.

Proof:

By induction on the structure of E.

case $E \in E^{\bullet}$:
1. **qed** by *dynamic hole typing*

case $E = E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. **qed** by the induction hypothesis (2)

case $E = v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{H}} E_1[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. **qed** by the induction hypothesis (2)

case $E = \langle E_0, e_1 \rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle E_0[\mathsf{stat}\ \tau\ e], e_1 \rangle$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. **qed** by the induction hypothesis (2)

case $E = \langle v_0, E_1 \rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle v_0, E_1[\mathsf{stat}\ \tau\ e] \rangle$
2. $\vdash_{\mathsf{H}} E_1[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. **qed** by the induction hypothesis (2)

case $E = op^1\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = op^1\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. **qed** by the induction hypothesis (2)

case $E = op^2\ E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. **qed** by the induction hypothesis (2)

case $E = op^2\ v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{H}} E_1[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. **qed** by the induction hypothesis (2)

case $E = \mathsf{dyn}\ \tau\ E_0$:
1. Contradiction by $\vdash_{\mathsf{H}} E[\mathsf{stat}\ \tau\ e]$

case $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{stat}\ \tau_0\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by *inversion*
3. **qed** by *static* stat *hole typing* (2)

□

Lemma 2.24 : H *static boundary-free hole substitution*

If $\vdash_{\mathsf{H}} E^{\bullet}[e] : \tau$ and the derivation contains a sub-term $\vdash_{\mathsf{H}} e : \tau'$ and $\vdash_{\mathsf{H}} e' : \tau'$ then $\vdash_{\mathsf{H}} E^{\bullet}[e'] : \tau$.

Proof:

By induction on the structure of $E^{\bullet}$

case $E^{\bullet} = [\,]$:
1. $E^{\bullet}[e] = e$
   $\wedge\ E^{\bullet}[e'] = e'$
2. $\vdash_{\mathsf{H}} e : \tau$
   by (1)
3. $\tau' = \tau$
4. $\vdash_{\mathsf{H}} e' : \tau$

5. QED by (1, 4)

CASE $E^\bullet = E^\bullet{}_0\ e_1$:

1. $E^\bullet[e] = E^\bullet{}_0[e]\ e_1$
   $\wedge\ E^\bullet[e'] = E^\bullet{}_0[e']\ e_1$
2. $\vdash_{\mathsf{H}} E^\bullet{}_0[e]\ e_1 : \tau$
3. $\vdash_{\mathsf{H}} E^\bullet{}_0[e] : \tau_0$
   $\wedge \vdash_{\mathsf{H}} e_1 : \tau_1$
   by *inversion*
4. $\vdash_{\mathsf{H}} E^\bullet{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{H}} E^\bullet{}_0[e']\ e_1 : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^\bullet = v_0\ E^\bullet{}_1$:

1. $E^\bullet[e] = v_0\ E^\bullet{}_1[e]$
   $\wedge\ E^\bullet[e'] = v_0\ E^\bullet{}_1[e']$
2. $\vdash_{\mathsf{H}} v_0\ E^\bullet{}_1[e] : \tau$
3. $\vdash_{\mathsf{H}} v_0 : \tau_0$
   $\wedge \vdash_{\mathsf{H}} E^\bullet{}_1[e] : \tau_1$
   by *inversion*
4. $\vdash_{\mathsf{H}} E^\bullet{}_1[e'] : \tau_1$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{H}} v_0\ E^\bullet{}_1[e'] : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^\bullet = op^1\ E^\bullet{}_0$:

1. $E^\bullet[e] = op^1\ E^\bullet{}_0[e]$
   $\wedge\ E^\bullet[e'] = op^1\ E^\bullet{}_0[e']$
2. $\vdash_{\mathsf{H}} op^1\ E^\bullet{}_0[e] : \tau$
3. $\vdash_{\mathsf{H}} E^\bullet{}_0[e] : \tau_0$
   by *inversion*
4. $\vdash_{\mathsf{H}} E^\bullet{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{H}} op^1\ E^\bullet{}_0[e'] : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^\bullet = \langle E^\bullet{}_0, e_1\rangle$:

1. $E^\bullet[e] = \langle E^\bullet{}_0[e], e_1\rangle$
   $\wedge\ E^\bullet[e'] = \langle E^\bullet{}_0[e'], e_1\rangle$
2. $\vdash_{\mathsf{H}} \langle E^\bullet{}_0[e], e_1\rangle : \tau$
3. $\vdash_{\mathsf{H}} E^\bullet{}_0[e] : \tau_0$
   $\wedge \vdash_{\mathsf{H}} e_1 : \tau_1$
   by *inversion*
4. $\vdash_{\mathsf{H}} E^\bullet{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{H}} \langle E^\bullet{}_0[e'], e_1\rangle : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^\bullet = \langle v_0, E^\bullet{}_1\rangle$:

1. $E^\bullet[e] = \langle v_0, E^\bullet{}_1[e]\rangle$
   $\wedge\ E^\bullet[e'] = \langle v_0, E^\bullet{}_1[e']\rangle$
2. $\vdash_{\mathsf{H}} \langle v_0, E^\bullet{}_1[e]\rangle : \tau$
3. $\vdash_{\mathsf{H}} v_0 : \tau_0$
   $\wedge \vdash_{\mathsf{H}} E^\bullet{}_1[e] : \tau_1$
   by *inversion*
4. $\vdash_{\mathsf{H}} E^\bullet{}_1[e'] : \tau_1$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{H}} \langle v_0, E^\bullet{}_1[e']\rangle : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^\bullet = op^2\ E^\bullet{}_0\ e_1$:

1. $E^\bullet[e] = op^2\ E^\bullet{}_0[e]\ e_1$
   $\wedge\ E^\bullet[e'] = op^2\ E^\bullet{}_0[e']\ e_1$
2. $\vdash_{\mathsf{H}} op^2\ E^\bullet{}_0[e]\ e_1 : \tau$
3. $\vdash_{\mathsf{H}} E^\bullet{}_0[e] : \tau_0$
   $\wedge \vdash_{\mathsf{H}} e_1 : \tau_1$
   by *inversion*
4. $\vdash_{\mathsf{H}} E^\bullet{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{H}} op^2\ E^\bullet{}_0[e']\ e_1 : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^\bullet = op^2\ v_0\ E^\bullet{}_1$:

1. $E^\bullet[e] = op^2\ v_0\ E^\bullet{}_1[e]$
   $\wedge\ E^\bullet[e'] = op^2\ v_0\ E^\bullet{}_1[e']$
2. $\vdash_{\mathsf{H}} op^2\ v_0\ E^\bullet{}_1[e] : \tau$
3. $\vdash_{\mathsf{H}} v_0 : \tau_0$
   $\wedge \vdash_{\mathsf{H}} E^\bullet{}_1[e] : \tau_1$
   by *inversion*
4. $\vdash_{\mathsf{H}} E^\bullet{}_1[e'] : \tau_1$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{H}} op^2\ v_0\ E^\bullet{}_1[e'] : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

$\square$

Lemma 2.25 : H *dynamic hole substitution*

If $\vdash_{\mathsf{H}} E^\bullet[e]$ and $\vdash_{\mathsf{H}} e'$ then $\vdash_{\mathsf{H}} E^\bullet[e']$

Proof:

By induction on the structure of $E^\bullet$

CASE $E^\bullet = [\,]$:

1. QED $E^\bullet[e'] = e'$

CASE $E^\bullet = \langle E^\bullet{}_0, e_1\rangle$:

1. $E^\bullet[e] = \langle E^\bullet{}_0[e], e_1\rangle$
   $\wedge\ E^\bullet[e'] = \langle E^\bullet{}_0[e'], e_1\rangle$
2. $\vdash_{\mathsf{H}} \langle E^\bullet{}_0[e], e_1\rangle$
3. $\vdash_{\mathsf{H}} E^\bullet{}_0[e]$
   $\wedge \vdash_{\mathsf{H}} e_1$
   by *inversion*
4. $\vdash_{\mathsf{H}} E^\bullet{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{H}} \langle E^\bullet{}_0[e'], e_1\rangle$
   by (3, 4)
6. QED by (1, 5)

CASE $E^\bullet = \langle v_0, E^\bullet{}_1\rangle$:

1. $E^{\bullet}[e] = \langle v_0, E^{\bullet}{}_1[e] \rangle$
   $\wedge\, E^{\bullet}[e'] = \langle v_0, E^{\bullet}{}_1[e'] \rangle$
2. $\vdash_{\mathrm{H}} \langle v_0, E^{\bullet}{}_1[e] \rangle$
3. $\vdash_{\mathrm{H}} v_0$
   $\wedge \vdash_{\mathrm{H}} E^{\bullet}{}_1[e]$
   by *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_1[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} \langle v_0, E^{\bullet}{}_1[e'] \rangle$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = E^{\bullet}{}_0\ e_1$:
1. $E^{\bullet}[e] = E^{\bullet}{}_0[e]\ e_1$
   $\wedge\, E^{\bullet}[e'] = E^{\bullet}{}_0[e']\ e_1$
2. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e]\ e_1$
3. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e]$
   $\wedge \vdash_{\mathrm{H}} e_1$
   by *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e']\ e_1$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = v_0\ E^{\bullet}{}_1$:
1. $E^{\bullet}[e] = v_0\ E^{\bullet}{}_1[e]$
   $\wedge\, E^{\bullet}[e'] = v_0\ E^{\bullet}{}_1[e']$
2. $\vdash_{\mathrm{H}} v_0\ E^{\bullet}{}_1[e]$
3. $\vdash_{\mathrm{H}} v_0$
   $\wedge \vdash_{\mathrm{H}} E^{\bullet}{}_1[e]$
   by *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_1[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} v_0\ E^{\bullet}{}_1[e']$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^1\ E^{\bullet}{}_0$:
1. $E^{\bullet}[e] = op^1\ E^{\bullet}{}_0[e]$
   $\wedge\, E^{\bullet}[e'] = op^1\ E^{\bullet}{}_0[e']$
2. $\vdash_{\mathrm{H}} op^1\ E^{\bullet}{}_0[e]$
3. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e]$
   by *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} op^1\ E^{\bullet}{}_0[e']$
   by (4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ E^{\bullet}{}_0\ e_1$:
1. $E^{\bullet}[e] = op^2\ E^{\bullet}{}_0[e]\ e_1$
   $\wedge\, E^{\bullet}[e'] = op^2\ E^{\bullet}{}_0[e']\ e_1$
2. $\vdash_{\mathrm{H}} op^2\ E^{\bullet}{}_0[e]\ e_1$
3. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e]$
   $\wedge \vdash_{\mathrm{H}} e_1$
   by *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} op^2\ E^{\bullet}{}_0[e']\ e_1$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ v_0\ E^{\bullet}{}_1$:
1. $E^{\bullet}[e] = op^2\ v_0\ E^{\bullet}{}_1[e]$
   $\wedge\, E^{\bullet}[e'] = op^2\ v_0\ E^{\bullet}{}_1[e']$
2. $\vdash_{\mathrm{H}} op^2\ v_0\ E^{\bullet}{}_1[e]$
3. $\vdash_{\mathrm{H}} v_0$
   $\wedge \vdash_{\mathrm{H}} E^{\bullet}{}_1[e]$
   by *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_1[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} op^2\ v_0\ E^{\bullet}{}_1[e']$
   by (3, 4)
6. QED by (1, 5)

□

Lemma 2.26 : H *hole substitution*

- If $\vdash_{\mathrm{H}} E[e]$ and the derivation contains a sub-term $\vdash_{\mathrm{H}} e : \tau'$ and $\vdash_{\mathrm{H}} e' : \tau'$ then $\vdash_{\mathrm{H}} E[e']$.
- If $\vdash_{\mathrm{H}} E[e]$ and the derivation contains a sub-term $\vdash_{\mathrm{H}} e$ and $\vdash_{\mathrm{H}} e'$ then $\vdash_{\mathrm{H}} E[e']$.
- If $\vdash_{\mathrm{H}} E[e] : \tau$ and the derivation contains a sub-term $\vdash_{\mathrm{H}} e : \tau'$ and $\vdash_{\mathrm{H}} e' : \tau'$ then $\vdash_{\mathrm{H}} E[e'] : \tau$.
- If $\vdash_{\mathrm{H}} E[e] : \tau$ and the derivation contains a sub-term $\vdash_{\mathrm{H}} e$ and $\vdash_{\mathrm{H}} e'$ then $\vdash_{\mathrm{H}} E[e'] : \tau$.

Proof:

By the following four lemmas: *dynamic context static hole substitution*, *dynamic context dynamic hole substitution*, *static context static hole substitution*, and *static context dynamic hole substitution.*

□

Lemma 2.27 : H *dynamic context static hole substitution*

If $\vdash_{\mathrm{H}} E[e]$ and contains $\vdash_{\mathrm{H}} e : \tau'$, and furthermore $\vdash_{\mathrm{H}} e' : \tau'$, then $\vdash_{\mathrm{H}} E[e']$

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:
1. Contradiction by $\vdash_{\mathrm{H}} E[e]$

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_{\mathrm{H}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_{\mathrm{H}} E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[e] = \langle E_0[e], e_1 \rangle$

2. $\vdash_H E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_H E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_H E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_H E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_H E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau''\ E_0$:
1. Contradiction by $\vdash_H E[e]$

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{stat}\ \tau_0\ E_0[e]$
2. $\vdash_H E_0[e] : \tau_0$
   by *inversion*
3. QED by *static context static hole substitution* (2)

□

Lemma 2.28 : H *dynamic context dynamic hole substitution*

If $\vdash_H E[e]$ and contains $\vdash_H e$, and furthermore $\vdash_H e'$, then $\vdash_H E[e']$

Proof:

By induction on the structure of E.

CASE $E \in E^\bullet$:
1. QED by *dynamic boundary-free hole substitution*

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_H E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_H E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_H E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_H E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_H E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_H E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_H E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau''\ E_0$:
1. Contradiction by $\vdash_H E[e]$

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{stat}\ \tau_0\ E_0[e]$
2. $\vdash_H E_0[e] : \tau_0$
   by *inversion*
3. QED by *static context dynamic hole substitution* (2)

□

Lemma 2.29 : H *static context static hole substitution*

If $\vdash_H E[e] : \tau$ and contains $\vdash_H e : \tau'$, and furthermore $\vdash_H e' : \tau'$, then $\vdash_H E[e'] : \tau$

Proof:

By induction on the structure of E.

CASE $E \in E^\bullet$:
1. QED by *static boundary-free hole substitution*

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_H E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_H E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_H E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$

2. $\vdash_{\mathsf{H}} E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_{\mathsf{H}} E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_{\mathsf{H}} E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_{\mathsf{H}} E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{dyn}\ \tau_0\ E_0[e]$
2. $\vdash_{\mathsf{H}} E_0[e]$
   by *inversion*
3. QED by *static* dyn *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_{\mathsf{H}} E[e] : \tau$

□

Lemma 2.30 : H *static context dynamic hole substitution*

If $\vdash_{\mathsf{H}} E[e] : \tau$ and contains $\vdash_{\mathsf{H}} e$, and furthermore $\vdash_{\mathsf{H}} e'$, then $\vdash_{\mathsf{H}} E[e'] : \tau$

Proof:

By induction on the structure of E.

CASE $E \in E^\bullet$:
1. Contradiction by $\vdash_{\mathsf{H}} E[e] : \tau$

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_{\mathsf{H}} E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_{\mathsf{H}} E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_{\mathsf{H}} E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_{\mathsf{H}} E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_{\mathsf{H}} E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_{\mathsf{H}} E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_{\mathsf{H}} E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{dyn}\ \tau_0\ E_0[e]$
2. $\vdash_{\mathsf{H}} E_0[e]$
   by *inversion*
3. QED by *dynamic* stat *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_{\mathsf{H}} E[e] : \tau$

□

Lemma 2.31 : $\vdash_H$ *static inversion*

- If $\Gamma \vdash_{\mathsf{H}} x : \tau$ then $(x : \tau') \in \Gamma$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{H}} \lambda(x : \tau'_d).\, e' : \tau$ then $(x : \tau'_d), \Gamma \vdash_{\mathsf{H}} e' : \tau'_c$ and $\tau'_d \Rightarrow \tau'_c \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{H}} \langle e_0, e_1 \rangle : \tau$ then $\Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$ and $\Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$ and $\tau_0 \times \tau_1 \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{H}} e_0\ e_1 : \tau_c$ then $\Gamma \vdash_{\mathsf{H}} e_0 : \tau'_d \Rightarrow \tau'_c$ and $\Gamma \vdash_{\mathsf{H}} e_1 : \tau'_d$ and $\tau'_c \leqslant: \tau_c$
- If $\Gamma \vdash_{\mathsf{H}} \mathsf{fst}\, e : \tau$ then $\Gamma \vdash_{\mathsf{H}} e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0$ and $\tau_0 \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{H}} \mathsf{snd}\, e : \tau$ then $\Gamma \vdash_{\mathsf{H}} e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1$ and $\tau_1 \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{H}} op^2\ e_0\ e_1 : \tau$ then $\Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$ and $\Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau'$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{H}} \mathsf{mon}\ \tau_d \Rightarrow \tau_c\ v' : \tau$ then $\Gamma \vdash_{\mathsf{H}} v'$ and $\tau_d \Rightarrow \tau_c \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{H}} \mathsf{dyn}\ \tau'\ e' : \tau$ then $\Gamma \vdash_{\mathsf{H}} e'$ and $\tau' \leqslant: \tau$

Proof:

QED by the definition of $\Gamma \vdash_{\mathsf{H}} e : \tau$

□

Lemma 2.32 : $\vdash_H$ *dynamic inversion*

- If $\Gamma \vdash_{\mathsf{H}} x$ then $x \in \Gamma$
- If $\Gamma \vdash_{\mathsf{H}} \lambda x.\, e'$ then $x, \Gamma \vdash_{\mathsf{H}} e'$
- If $\Gamma \vdash_{\mathsf{H}} \langle e_0, e_1 \rangle$ then $\Gamma \vdash_{\mathsf{H}} e_0$ and $\Gamma \vdash_{\mathsf{H}} e_1$
- If $\Gamma \vdash_{\mathsf{H}} e_0\ e_1$ then $\Gamma \vdash_{\mathsf{H}} e_0$ and $\Gamma \vdash_{\mathsf{H}} e_1$
- If $\Gamma \vdash_{\mathsf{H}} op^1\ e_0$ then $\Gamma \vdash_{\mathsf{H}} e_0$
- If $\Gamma \vdash_{\mathsf{H}} op^2\ e_0\ e_1$ then $\Gamma \vdash_{\mathsf{H}} e_0$ and $\Gamma \vdash_{\mathsf{H}} e_1$
- If $\Gamma \vdash_{\mathsf{H}} \mathsf{mon}\ \tau_d \Rightarrow \tau_c\ v'$ then $\Gamma \vdash_{\mathsf{H}} v' : \tau_d \Rightarrow \tau_c$
- If $\Gamma \vdash_{\mathsf{H}} \mathsf{stat}\ \tau'\ e'$ then $\Gamma \vdash_{\mathsf{H}} e' : \tau'$

Proof:

QED by the definition of $\Gamma \vdash_{\mathsf{H}} e$

□

Lemma 2.33 : H *canonical forms*

- If $\vdash_{\mathsf{H}} v : \tau_0 \times \tau_1$ then $v = \langle v_0, v_1 \rangle$
- If $\vdash_{\mathsf{H}} v : \tau_d \Rightarrow \tau_c$ then either:
  - $v = \lambda(x\!:\!\tau_x).\, e'$ $\wedge\ \tau_d \leqslant: \tau_x$
  - or $v = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\, v'$ $\wedge\ \tau_d' \Rightarrow \tau_c' \leqslant: \tau_d \Rightarrow \tau_c$
- If $\vdash_{\mathsf{H}} v : \mathsf{Int}$ then $v \in i$
- If $\vdash_{\mathsf{H}} v : \mathsf{Nat}$ then $v \in \mathbb{N}$

Proof:

QED by definition of $\vdash_{\mathsf{H}} e : \tau$

□

Lemma 2.34 : Δ *type soundness*

If $\vdash_{\mathsf{H}} v_0 : \tau_0$ and $\vdash_{\mathsf{H}} v_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau$ then $\vdash_{\mathsf{H}} \delta(op^2, v_0, v_1) : \tau$.

Proof:

By case analysis on the definition of Δ.

CASE $\Delta(\mathsf{sum}, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat}$:
1. $v_0 = i_0, i_0 \in \mathbb{N}$
   $\wedge\ v_1 = i_1, i_1 \in \mathbb{N}$
   by *canonical forms*
2. $\delta(\mathsf{sum}, i_0, i_1) = i_0 + i_1 \in \mathbb{N}$
3. QED

CASE $\Delta(\mathsf{sum}, \mathsf{Int}, \mathsf{Int}) = \mathsf{Int}$:
1. $v_0 = i_0$
   $\wedge\ v_1 = i_1$
   by *canonical forms*
2. $\delta(\mathsf{sum}, i_0, i_1) = i_0 + i_1 \in i$
3. QED

CASE $\Delta(\mathsf{quotient}, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat}$:
1. $v_0 = i_0, i_0 \in \mathbb{N}$
   $\wedge\ v_1 = i_1, i_1 \in \mathbb{N}$
   by *canonical forms*
2. **IF** $i_1 = 0$:
   a. $\delta(\mathsf{quotient}, i_0, i_1) = \mathsf{BndryErr}$
   b. QED by $\vdash_{\mathsf{H}} \mathsf{BndryErr} : \tau$
   ELSE $i_1 \neq 0$:
   a. $\delta(\mathsf{quotient}, i_0, i_1) = \lfloor i_0 / i_1 \rfloor \in \mathbb{N}$
   b. QED

CASE $\Delta(\mathsf{quotient}, \mathsf{Int}, \mathsf{Int}) = \mathsf{Int}$:
1. $v_0 = i_0$
   $\wedge\ v_1 = i_1$
   by *canonical forms*
2. **IF** $i_1 = 0$:
   a. $\delta(\mathsf{quotient}, i_0, i_1) = \mathsf{BndryErr}$
   b. QED by $\vdash_{\mathsf{H}} \mathsf{BndryErr} : \tau$
   ELSE $i_1 \neq 0$:
   a. $\delta(\mathsf{quotient}, i_0, i_1) = \lfloor i_0 / i_1 \rfloor \in i$
   b. QED

□

Lemma 2.35 : δ *preservation*

- If $\vdash_{\mathsf{H}} v$ and $\delta(op^1, v) = e$ then $\vdash_{\mathsf{H}} e$
- If $\vdash_{\mathsf{H}} v_0$ and $\vdash_{\mathsf{H}} v_1$ and $\delta(op^2, v_0, v_1) = e$ then $\vdash_{\mathsf{H}} e$

Proof:

CASE $\delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0$:
1. $\vdash_{\mathsf{H}} v_0$
   by *inversion*
2. QED

CASE $\delta(\mathsf{snd}, \langle v_0, v_1 \rangle) = v_1$:
1. $\vdash_{\mathsf{H}} v_1$
   by *inversion*
2. QED

CASE $\delta(\mathsf{sum}, v_0, v_1) = v_0 + v_1$:
1. QED

CASE $\delta(\mathsf{quotient}, v_0, v_1) = \lfloor v_0 / v_1 \rfloor$:
1. QED

CASE $\delta(op^2, v_0, v_1) = \mathsf{BndryErr}$:
1. QED

□

Lemma 2.36 : H *substitution*

- If $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e$ and $\vdash_{\mathsf{H}} v : \tau_x$ then $\Gamma \vdash_{\mathsf{H}} e[x \leftarrow v]$
- If $x, \Gamma \vdash_{\mathsf{H}} e$ and $\vdash_{\mathsf{H}} v$ then $\Gamma \vdash_{\mathsf{H}} e[x \leftarrow v]$
- If $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e : \tau$ and $\vdash_{\mathsf{H}} v : \tau_x$ then $\Gamma \vdash_{\mathsf{H}} e[x \leftarrow v] : \tau$
- If $x, \Gamma \vdash_{\mathsf{H}} e : \tau$ and $\vdash_{\mathsf{H}} v$ then $\Gamma \vdash_{\mathsf{H}} e[x \leftarrow v] : \tau$

Proof:

By the following four lemmas: *dynamic context static value substitution*, *dynamic context dynamic value substitution*, *static context static value substitution*, and *static context dynamic value substitution.*

□

Lemma 2.37 : H *dynamic-static substitution*

If $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e$ and $\vdash_{\mathsf{H}} v : \tau_x$ then $\Gamma \vdash_{\mathsf{H}} e[x \leftarrow v]$

Proof:

By induction on the structure of e.

CASE $e = x$:
1. Contradiction by $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} x$

CASE $e = x'$:
1. QED by $(x'[x \leftarrow v]) = x'$

CASE $e = i$:
1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x'.\, e'$:
1. $e[x \leftarrow v] = \lambda x'.\, (e'[x \leftarrow v])$
2. $x', (x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e'$
   by *inversion*
3. $x', \Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} \lambda x'.\, (e'[x \leftarrow v])$
   by (3)
5. QED

CASE $e = \lambda(x'\!:\!\tau').\, e'$:
1. Contradiction by $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e$

CASE $e = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'$:
1. $e[x \leftarrow v] = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'[x \leftarrow v]$

2. $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} v' : \tau_d \Rightarrow \tau_c$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} v'[x \leftarrow v] : \tau_d \Rightarrow \tau_c$
   by *static context static value substitution* (2)
4. $\Gamma \vdash_{\mathsf{H}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \langle e_0, e_1 \rangle$:
1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e_0$
   $\wedge\ (x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e_1$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
   by (3)
5. QED

CASE $e = e_0\ e_1$:
1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e_0$
   $\wedge\ (x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e_1$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^1\ e_0$:
1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e_0$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} op^1\ e_0[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:
1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e_0$
   $\wedge\ (x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e_1$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \mathsf{stat}\ \tau'\ e'$:
1. $e[x \leftarrow v] = \mathsf{stat}\ \tau'\ e'[x \leftarrow v]$
2. $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e' : \tau'$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v] : \tau'$
   by *static context static value substitution* (2)
4. $\Gamma \vdash_{\mathsf{H}} \mathsf{stat}\ \tau'\ e'[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \mathsf{Err}$:
1. QED $\mathsf{Err} = \mathsf{Err}[x \leftarrow v]$

□

Lemma 2.38 : H *dynamic-dynamic substitution*

If $x, \Gamma \vdash_{\mathsf{H}} e$ and $\vdash_{\mathsf{H}} v$ then $\Gamma \vdash_{\mathsf{H}} e[x \leftarrow v]$

Proof:

By induction on the structure of e

CASE $e = x$:
1. $e[x \leftarrow v] = v$
2. $\Gamma \vdash_{\mathsf{H}} v$
   by *weakening*
3. QED

CASE $e = x'$:
1. QED by $(x'[x \leftarrow v]) = x'$

CASE $e = i$:
1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x'.\, e'$:
1. $e[x \leftarrow v] = \lambda x'.\, (e'[x \leftarrow v])$
2. $x', x, \Gamma \vdash_{\mathsf{H}} e'$
   by *inversion*
3. $x', \Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} \lambda x'.\, (e'[x \leftarrow v])$
   by (3)
5. QED

CASE $e = \lambda(x'\!:\!\tau').\, e'$:
1. Contradiction by $x, \Gamma \vdash_{\mathsf{H}} e$

CASE $e = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'$:
1. $e[x \leftarrow v] = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} v' : \tau_d \Rightarrow \tau_c$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} v'[x \leftarrow v] : \tau_d \Rightarrow \tau_c$
   by *static context dynamic value substitution* (2)
4. $\Gamma \vdash_{\mathsf{H}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \langle e_0, e_1 \rangle$:
1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $x, \Gamma \vdash_{\mathsf{H}} e_0$
   $\wedge\ x, \Gamma \vdash_{\mathsf{H}} e_1$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
   by (3)
5. QED

CASE $e = e_0\ e_1$:

1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} e_0$
   $\wedge\ x, \Gamma \vdash_{\mathsf{H}} e_1$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^1\ e_0$:

1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} e_0$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} op^1\ e_0[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:

1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} e_0$
   $\wedge\ x, \Gamma \vdash_{\mathsf{H}} e_1$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \mathsf{stat}\ \tau'\ e'$:

1. $e[x \leftarrow v] = \mathsf{stat}\ \tau'\ e'[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} e' : \tau'$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v] : \tau'$
   by *static context static value substitution* (2)
4. $\Gamma \vdash_{\mathsf{H}} \mathsf{stat}\ \tau'\ e'[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \mathsf{Err}$:

1. QED $\mathsf{Err} = \mathsf{Err}[x \leftarrow v]$

□

Lemma 2.39 : H *static-static substitution*

If $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} e : \tau$ and $\vdash_{\mathsf{H}} v : \tau_x$ then $\Gamma \vdash_{\mathsf{H}} e[x \leftarrow v] : \tau$

Proof:

By induction on the structure of e.

CASE $e = x$:

1. $e[x \leftarrow v] = v$
2. $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} x : \tau$
3. $\tau_x \leqslant: \tau$
   by *inversion*
4. $\vdash_{\mathsf{H}} v : \tau$
   by (3)
5. $\Gamma \vdash_{\mathsf{H}} v : \tau$
   by *weakening*
6. QED

CASE $e = x'$:

1. QED by $(x'[x \leftarrow v]) = x'$

CASE $e = i$:

1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x'.\ e'$:

1. Contradiction by $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} e : \tau$

CASE $e = \lambda(x' : \tau').\ e'$:

1. $e[x \leftarrow v] = \lambda(x' : \tau').\ (e'[x \leftarrow v])$
2. $(x' : \tau'), (x : \tau_x), \Gamma \vdash_{\mathsf{H}} e' : \tau'_c$
   $\wedge\ \tau' \Rightarrow \tau'_c \leqslant: \tau$
3. $(x' : \tau'), \Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v] : \tau'_c$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} \lambda(x' : \tau').\ e' : \tau$
5. QED

CASE $e = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'$:

1. $e[x \leftarrow v] = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'[x \leftarrow v]$
2. $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} v'$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} v'[x \leftarrow v]$
   by *dynamic context static value substitution* (2)
4. $\Gamma \vdash_{\mathsf{H}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = \langle e_0, e_1 \rangle$:

1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   $\wedge\ (x : \tau_x), \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v] : \tau_1$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle : \tau$
   by (3)
5. QED

CASE $e = e_0\ e_1$:

1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   $\wedge\ (x : \tau_x), \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v] : \tau_1$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]\ e_1[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = op^1\ e_0$:

1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0$
   by the induction hypothesis (2)

4. $\Gamma \vdash_{\mathsf{H}} op^1\ e_0[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:

1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   $\wedge (x : \tau_x), \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v] : \tau_1$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = \mathsf{dyn}\ \tau'\ e'$:

1. $e[x \leftarrow v] = \mathsf{dyn}\ \tau'\ e'[x \leftarrow v]$
2. $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} e'$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v]$
   by *dynamic context static value substitution* (2)
4. $\Gamma \vdash_{\mathsf{H}} \mathsf{dyn}\ \tau'\ e'[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = \mathsf{Err}$:

1. QED by $\mathsf{Err} = \mathsf{Err}[x \leftarrow v]$

□

Lemma 2.40 : H *static-dynamic substitution*

If $x, \Gamma \vdash_{\mathsf{H}} e : \tau$ and $\vdash_{\mathsf{H}} v$ then $\Gamma \vdash_{\mathsf{H}} e[x \leftarrow v] : \tau$

Proof:

By induction on the structure of e.

CASE $e = x$:

1. Contradiction by $x, \Gamma \vdash_{\mathsf{H}} x : \tau$

CASE $e = x'$:

1. QED by $(x'[x \leftarrow v]) = x'$

CASE $e = i$:

1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x'.\ e'$:

1. Contradiction by $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} e : \tau$

CASE $e = \lambda(x' : \tau').\ e'$:

1. $e[x \leftarrow v] = \lambda(x' : \tau').\ (e'[x \leftarrow v])$
2. $(x' : \tau'), x, \Gamma \vdash_{\mathsf{H}} e' : \tau'_c$
   $\wedge\ \tau' \Rightarrow \tau'_c \leqslant: \tau$
3. $(x' : \tau'), \Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v] : \tau'_c$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} \lambda(x' : \tau').\ e' : \tau$
5. QED

CASE $e = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\ v'$:

1. $e[x \leftarrow v] = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\ v'[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} v'$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} v'[x \leftarrow v]$
   by *dynamic context dynamic value substitution* (2)
4. $\Gamma \vdash_{\mathsf{H}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\ v'[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = \langle e_0, e_1 \rangle$:

1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $x, \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   $\wedge\ x, \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v] : \tau_1$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle : \tau$
   by (3)
5. QED

CASE $e = e_0\ e_1$:

1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   $\wedge\ x, \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v] : \tau_1$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]\ e_1[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = op^1\ e_0$:

1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} op^1\ e_0[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:

1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   $\wedge\ x, \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v] : \tau_1$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = \mathsf{dyn}\ \tau'\ e'$:

1. $e[x \leftarrow v] = \mathsf{dyn}\ \tau'\ e'[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} e'$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v]$
   by *dynamic context dynamic value substitution* (2)
4. $\Gamma \vdash_{\mathsf{H}} \mathsf{dyn}\ \tau'\ e'[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = \mathsf{Err}$:

1. QED by $\mathsf{Err} = \mathsf{Err}[x \leftarrow v]$

□

Lemma 2.41 : *weakening*

- If $\Gamma \vdash_{\mathsf{H}} e$ then $x, \Gamma \vdash_{\mathsf{H}} e$
- If $\Gamma \vdash_{\mathsf{H}} e : \tau$ then $(x : \tau'), \Gamma \vdash_{\mathsf{H}} e : \tau$

Proof:

- e is closed under Γ
  by $\Gamma \vdash_{\mathsf{H}} e$
  $\vee\, \Gamma \vdash_{\mathsf{H}} e : \tau$
- QED x is unused in the derivation

□

A.3 (E) Erasure Embedding

A.3.1 Erasure Definitions

$\boxed{\text{Language E}}$

$$
\begin{array}{lcl}
e & = & x \mid v \mid \langle e, e\rangle \mid e\ e \mid op^1\ e \mid op^2\ e\ e \mid \\
 & & \mathsf{dyn}\ \tau\ e \mid \mathsf{stat}\ \tau\ e \mid \mathsf{Err} \\
v & = & i \mid \langle v, v\rangle \mid \lambda x.\, e \mid \lambda(x\!:\!\tau).\, e \\
\tau & = & \mathsf{Nat} \mid \mathsf{Int} \mid \tau\times\tau \mid \tau \Rightarrow \tau \\
\Gamma & = & \cdot \mid x, \Gamma \mid (x\!:\!\tau), \Gamma \\
\mathsf{Err} & = & \mathsf{BndryErr} \mid \mathsf{TagErr} \\
r & = & v \mid \mathsf{Err} \\
E^\bullet & = & [\,] \mid E^\bullet\ e \mid v\ E^\bullet \mid \langle E^\bullet, e\rangle \mid \langle v, E^\bullet\rangle \mid \\
 & & op^1\ E^\bullet \mid op^2\ E^\bullet\ e \mid op^2\ v\ E^\bullet \\
E & = & E^\bullet \mid E\ e \mid v\ E \mid \langle E, e\rangle \mid \langle v, E\rangle \mid op^1\ E \mid \\
 & & op^2\ E\ e \mid op^2\ v\ E \mid \mathsf{dyn}\ \tau\ E \mid \mathsf{stat}\ \tau\ E
\end{array}
$$

$\boxed{\Delta : op^1 \times \tau \longrightarrow \tau}$

$$\Delta(\mathsf{fst}, \tau_0\times\tau_1) = \tau_0$$
$$\Delta(\mathsf{snd}, \tau_0\times\tau_1) = \tau_1$$

$\boxed{\Delta : op^2 \times \tau \times \tau \longrightarrow \tau}$

$$\Delta(op^2, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat}$$
$$\Delta(op^2, \mathsf{Int}, \mathsf{Int}) = \mathsf{Int}$$

$\boxed{\tau \leqslant: \tau}$

$$\frac{}{\mathsf{Nat} \leqslant: \mathsf{Int}} \qquad \frac{\tau_d' \leqslant: \tau_d \quad \tau_c \leqslant: \tau_c'}{\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'} \qquad \frac{\tau_0 \leqslant: \tau_0' \quad \tau_1 \leqslant: \tau_1'}{\tau_0\times\tau_1 \leqslant: \tau_0'\times\tau_1'}$$

$$\frac{}{\tau \leqslant: \tau} \qquad \frac{\tau \leqslant: \tau' \quad \tau' \leqslant: \tau''}{\tau \leqslant: \tau''}$$

$\boxed{\Gamma \vdash e}$

$$\frac{x \in \Gamma}{\Gamma \vdash x} \qquad \frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x.\, e} \qquad \frac{}{\Gamma \vdash i} \qquad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1\rangle}$$

$$\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0\ e_1} \qquad \frac{\Gamma \vdash e}{\Gamma \vdash op^1\ e} \qquad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2\ e_0\ e_1} \qquad \frac{}{\Gamma \vdash \mathsf{Err}}$$

$$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat}\ \tau\ e}$$

$\boxed{\Gamma \vdash e : \tau}$

$$\frac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash x : \tau} \qquad \frac{(x\!:\!\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c} \qquad \frac{i \in \mathbb{N}}{\Gamma \vdash i : \mathsf{Nat}}$$

$$\frac{}{\Gamma \vdash i : \mathsf{Int}} \qquad \frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1\rangle : \tau_0\times\tau_1} \qquad \frac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash e_1 : \tau_d}{\Gamma \vdash e_0\ e_1 : \tau_c}$$

$$\frac{\Gamma \vdash e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash op^1\ e_0 : \tau} \qquad \frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash op^2\ e_0\ e_1 : \tau} \qquad \frac{\Gamma \vdash e : \tau' \quad \tau' \leqslant: \tau}{\Gamma \vdash e : \tau} \qquad \frac{}{\Gamma \vdash \mathsf{Err} : \tau}$$

$$\frac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau}$$

$\boxed{\Gamma \vdash_{\mathsf{E}} e}$

$$\frac{x \in \Gamma}{\Gamma \vdash_{\mathsf{E}} x} \qquad \frac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash_{\mathsf{E}} x} \qquad \frac{x, \Gamma \vdash_{\mathsf{E}} e}{\Gamma \vdash_{\mathsf{E}} \lambda x.\, e} \qquad \frac{(x\!:\!\tau), \Gamma \vdash_{\mathsf{E}} e}{\Gamma \vdash_{\mathsf{E}} \lambda(x\!:\!\tau).\, e} \qquad \frac{}{\Gamma \vdash_{\mathsf{E}} i}$$

$$\frac{\Gamma \vdash_{\mathsf{E}} e_0 \quad \Gamma \vdash_{\mathsf{E}} e_1}{\Gamma \vdash_{\mathsf{E}} \langle e_0, e_1\rangle} \qquad \frac{\Gamma \vdash_{\mathsf{E}} e_0 \quad \Gamma \vdash_{\mathsf{E}} e_1}{\Gamma \vdash_{\mathsf{E}} e_0\ e_1} \qquad \frac{\Gamma \vdash_{\mathsf{E}} e}{\Gamma \vdash_{\mathsf{E}} op^1\ e}$$

$$\frac{\Gamma \vdash_{\mathsf{E}} e_0 \quad \Gamma \vdash_{\mathsf{E}} e_1}{\Gamma \vdash_{\mathsf{E}} op^2\ e_0\ e_1} \qquad \frac{}{\Gamma \vdash_{\mathsf{E}} \mathsf{Err}} \qquad \frac{\Gamma \vdash_{\mathsf{E}} e}{\Gamma \vdash_{\mathsf{E}} \mathsf{dyn}\ \tau\ e} \qquad \frac{\Gamma \vdash_{\mathsf{E}} e}{\Gamma \vdash_{\mathsf{E}} \mathsf{stat}\ \tau\ e}$$

$\boxed{\delta(op^1, v) = e}$

$$\delta(\mathsf{fst}, \langle v_0, v_1\rangle) = v_0$$
$$\delta(\mathsf{snd}, \langle v_0, v_1\rangle) = v_1$$

$\boxed{\delta(op^2, v, v) = e}$

$$\delta(\mathsf{sum}, i_0, i_1) = i_0 + i_1$$
$$\delta(\mathsf{quotient}, i_0, 0) = \mathsf{BndryErr}$$
$$\delta(\mathsf{quotient}, i_0, i_1) = \lfloor i_0 / i_1 \rfloor \quad \text{if } i_1 \neq 0$$

$\boxed{\mathcal{D}_{\mathsf{E}} : \tau \times v \longrightarrow e}$

$$\mathcal{D}_{\mathsf{E}}(\tau, v) = v$$

$\boxed{\mathcal{S}_{\mathsf{E}} : \tau \times v \longrightarrow e}$

$$\mathcal{S}_{\mathsf{E}}(\tau, v) = v$$

$\boxed{e \rhd_{\text{E-S}} e}$

$$
\begin{array}{lll}
\text{dyn}\ \tau\ v & \rhd_{\text{E-S}} & \mathcal{D}_{\text{E}}(\tau, v) \\
\text{stat}\ \tau\ v & \rhd_{\text{E-S}} & \mathcal{S}_{\text{E}}(\tau, v) \\
v_0\ v_1 & \rhd_{\text{E-S}} & \text{TagErr} \\
\quad \text{if } v_0 \in \mathbb{Z} \text{ or } v_0 = \langle v, v' \rangle & & \\
(\lambda(x\!:\!\tau).\, e)\ v & \rhd_{\text{E-S}} & e[x \leftarrow v] \\
(\lambda x.\, e)\ v & \rhd_{\text{E-S}} & e[x \leftarrow v] \\
op^1\ v & \rhd_{\text{E-S}} & \text{TagErr} \\
\quad \text{if } \delta(op^1, v) \text{ is undefined} & & \\
op^1\ v & \rhd_{\text{E-S}} & \delta(op^1, v) \\
op^2\ v_0\ v_1 & \rhd_{\text{E-S}} & \text{TagErr} \\
\quad \text{if } \delta(op^2, v_0, v_1) \text{ is undefined} & & \\
op^2\ v_0\ v_1 & \rhd_{\text{E-S}} & \delta(op^2, v_0, v_1)
\end{array}
$$

$\boxed{e \rhd_{\text{E-D}} e}$

$$
\begin{array}{lll}
\text{stat}\ \tau\ v & \rhd_{\text{E-D}} & \mathcal{S}_{\text{E}}(\tau, v) \\
\text{dyn}\ \tau\ v & \rhd_{\text{E-D}} & \mathcal{D}_{\text{E}}(\tau, v) \\
v_0\ v_1 & \rhd_{\text{E-D}} & \text{TagErr} \\
\quad \text{if } v_0 \in \mathbb{Z} \text{ or } v_0 = \langle v, v' \rangle & & \\
(\lambda(x\!:\!\tau).\, e)\ v & \rhd_{\text{E-D}} & e[x \leftarrow v] \\
(\lambda x.\, e)\ v & \rhd_{\text{E-D}} & e[x \leftarrow v] \\
op^1\ v & \rhd_{\text{E-D}} & \text{TagErr} \\
\quad \text{if } \delta(op^1, v) \text{ is undefined} & & \\
op^1\ v & \rhd_{\text{E-D}} & \delta(op^1, v) \\
op^2\ v_0\ v_1 & \rhd_{\text{E-D}} & \text{TagErr} \\
\quad \text{if } \delta(op^2, v_0, v_1) \text{ is undefined} & & \\
op^2\ v_0\ v_1 & \rhd_{\text{E-D}} & \delta(op^2, v_0, v_1)
\end{array}
$$

$\boxed{e \rightarrow_{\text{E-S}} e}$

$$
\begin{array}{lll}
E[e] & \rightarrow_{\text{E-S}} & E[e'] \\
\quad \text{if } e \rhd_{\text{E-S}} e' & & \\
E[\text{Err}] & \rightarrow_{\text{E-S}} & \text{Err}
\end{array}
$$

$\boxed{e \rightarrow_{\text{E-D}} e}$

$$
\begin{array}{lll}
E[e] & \rightarrow_{\text{E-D}} & E[e'] \\
\quad \text{if } e \rhd_{\text{E-D}} e' & & \\
E[\text{Err}] & \rightarrow_{\text{E-D}} & \text{Err}
\end{array}
$$

$\boxed{e \rightarrow^*_{\text{E-S}} e}$ reflexive, transitive closure of $\rightarrow_{\text{E-S}}$

$\boxed{e \rightarrow^*_{\text{E-D}} e}$ reflexive, transitive closure of $\rightarrow_{\text{E-D}}$

A.3.2 Erasure Theorems

Theorem 3.0 : *static* E*-soundness*

> If $\vdash e : \tau$ then $\vdash_{\mathsf{E}} e$ and one of the following holds:
> - $e \rightarrow^{*}_{\mathsf{E\text{-}S}} v$ and $\vdash_{\mathsf{E}} v$
> - $e \rightarrow^{*}_{\mathsf{E\text{-}S}} \mathsf{TagErr}$
> - $e \rightarrow^{*}_{\mathsf{E\text{-}S}} \mathsf{BndryErr}$
> - e diverges

Proof :

1. $\vdash_{\mathsf{E}} e$
   by *static subset*
2. QED by *progress* and *preservation*

□

Theorem 3.1 : *dynamic* E*-soundness*

> If $\vdash e$ then $\vdash_{\mathsf{E}} e$ and one of the following holds:
> - $e \rightarrow^{*}_{\mathsf{E\text{-}D}} v$ and $\vdash_{\mathsf{E}} v$
> - $e \rightarrow^{*}_{\mathsf{E\text{-}D}} \mathsf{TagErr}$
> - $e \rightarrow^{*}_{\mathsf{E\text{-}D}} \mathsf{BndryErr}$
> - e diverges

Proof :

1. $\rightarrow^{*}_{\mathsf{E\text{-}D}} = \rightarrow^{*}_{\mathsf{E\text{-}S}}$
   by definition
2. QED by *static* E *soundness*

□

Remark 3.2 : E*-compilation*

> The $\rightarrow^{*}_{\mathsf{E\text{-}S}}$ and $\rightarrow^{*}_{\mathsf{E\text{-}D}}$ relations are identical. In practice, uses of $\rightarrow^{*}_{\mathsf{E\text{-}S}}$ may be replaced with $\rightarrow^{*}_{\mathsf{E\text{-}D}}$.

Theorem 3.3 : *boundary-free* E*-soundness*

> If $\vdash e : \tau$ and e is boundary-free then one of the following holds:
> - $e \rightarrow^{*}_{\mathsf{E\text{-}S}} v$ and $\vdash v : \tau$
> - $e \rightarrow^{*}_{\mathsf{E\text{-}S}} \mathsf{BndryErr}$
> - e diverges

Proof :

QED by *boundary-free progress* and *broundary-free preservation.*

□

A.3.3 Erasure Lemmas

Lemma 3.4 : *$\mathcal{D}_{\mathsf{E}}$ soundness*

If $\vdash_{\mathsf{E}} v$ then $\vdash_{\mathsf{E}} \mathcal{D}_{\mathsf{E}}(\tau, v)$.

Proof:

CASE $\mathcal{D}_{\mathsf{E}}(\tau, v) = v$:

1. QED

□

Lemma 3.5 : *$\mathcal{S}_{\mathsf{E}}$ soundness*

If $\vdash_{\mathsf{E}} v$ then $\vdash_{\mathsf{E}} \mathcal{S}_{\mathsf{E}}(\tau, v)$.

Proof:

CASE $\mathcal{S}_{\mathsf{E}}(\tau, v) = v$:

1. QED

□

Lemma 3.6 : *static subset*

If $\Gamma \vdash e : \tau$ then $\Gamma \vdash_{\mathsf{E}} e$.

Proof:

By structural induction on the typing relation.

CASE $\boxed{\dfrac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash x : \tau}}$:

1. $(x\!:\!\tau) \in \Gamma$
2. $\Gamma \vdash_{\mathsf{E}} x$
   by (1)
3. QED

CASE $\boxed{\dfrac{(x\!:\!\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c}}$:

1. $(x\!:\!\tau_d), \Gamma \vdash_{\mathsf{E}} e$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} \lambda(x\!:\!\tau_d).\, e$
   by (1)
3. QED

CASE $\boxed{\dfrac{i \in \mathbb{N}}{\Gamma \vdash i : \mathsf{Nat}}}$:

1. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash_{\mathsf{E}} i : \mathsf{Int}}}$:

1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1}}$:

1. $\Gamma \vdash_{\mathsf{E}} e_0$
   $\wedge\, \Gamma \vdash_{\mathsf{E}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} \langle e_0, e_1 \rangle$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash e_1 : \tau_d}{\Gamma \vdash e_0\ e_1 : \tau_c}}$:

1. $\Gamma \vdash_{\mathsf{E}} e_0$
   $\wedge\, \Gamma \vdash_{\mathsf{E}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} e_0\ e_1$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash op^1\ e_0 : \tau}}$:

1. $\Gamma \vdash_{\mathsf{E}} e_0$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} op^1\ e_0$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash op^2\ e_0\ e_1 : \tau}}$:

1. $\Gamma \vdash_{\mathsf{E}} e_0$
   $\wedge\, \Gamma \vdash_{\mathsf{E}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} op^2\ e_0\ e_1$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau' \quad \tau' <: \tau}{\Gamma \vdash e : \tau}}$:

1. $\Gamma \vdash_{\mathsf{E}} e$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash \mathsf{Err} : \tau}}$:

1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau}}$:

1. $\Gamma \vdash_{\mathsf{E}} e$
   by *dynamic subset*
2. $\Gamma \vdash_{\mathsf{E}} \mathsf{dyn}\ \tau\ e$
   by (1)
3. QED

□

Lemma 3.7 : *dynamic subset*

If $\Gamma \vdash e$ then $\Gamma \vdash_{\mathsf{E}} e$.

Proof:

By structural induction on the $\vdash$ relation.

CASE $\boxed{\dfrac{x \in \Gamma}{\Gamma \vdash x}}$:

1. $x \in \Gamma$
2. $\Gamma \vdash_{\mathsf{E}} x$
   by (1)
3. QED

CASE $\boxed{\dfrac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x.\, e}}$:

1. $x, \Gamma \vdash_{\mathsf{E}} e$
   by the induction hypothesis

2. $\Gamma \vdash_{\mathsf{E}} \lambda x.\, e$
   by (1)
3. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash i}}$:

1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}}$:

1. $\Gamma \vdash_{\mathsf{E}} e_0$
   $\wedge\, \Gamma \vdash_{\mathsf{E}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} \langle e_0, e_1 \rangle$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0\ e_1}}$:

1. $\Gamma \vdash_{\mathsf{E}} e_0$
   $\wedge\, \Gamma \vdash_{\mathsf{E}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} e_0\ e_1$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e}{\Gamma \vdash op^1\ e}}$:

1. $\Gamma \vdash_{\mathsf{E}} e_0$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} op^1\ e_0$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2\ e_0\ e_1}}$:

1. $\Gamma \vdash_{\mathsf{E}} e_0$
   $\wedge\, \Gamma \vdash_{\mathsf{E}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} op^2\ e_0\ e_1$
   by (1)
3. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash \mathsf{Err}}}$:

1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat}\ \tau\ e}}$:

1. $\Gamma \vdash_{\mathsf{E}} e$
   by *static subset*
2. $\Gamma \vdash_{\mathsf{E}} \mathsf{stat}\ \tau\ e$
   by (1)
3. QED

□

Lemma 3.8 : E *progress*

If $\vdash_{\mathsf{E}} e$ then one of the following holds:

- e is a value
- $e \in \mathsf{Err}$
- $e \rightarrow_{\mathsf{E\text{-}S}} e'$
- $e \rightarrow_{\mathsf{E\text{-}S}} \mathsf{TagErr}$
- $e \rightarrow_{\mathsf{E\text{-}S}} \mathsf{BndryErr}$

Proof:

By the *unique evaluation contexts* lemma, there are seven possible cases.

CASE e is a value :

1. QED

CASE $e = E[v_0\ v_1]$:

IF $v_0 = \lambda x.\, e'$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} E[e'[x \leftarrow v_1]]$
   by $v_0\ v_1 \rhd_{\mathsf{E\text{-}S}} e'[x \leftarrow v_1]$
2. QED

IF $v_0 = \lambda(x : \tau).\, e'$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} E[e'[x \leftarrow v_1]]$
   by $v_0\ v_1 \rhd_{\mathsf{E\text{-}S}} e'[x \leftarrow v_1]$
2. QED

ELSE $v_0 = i$
$\vee\, v_0 = \langle v, v' \rangle$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} \mathsf{TagErr}$
   by $v_0\ v_1 \rhd_{\mathsf{E\text{-}S}} \mathsf{TagErr}$
2. QED

CASE $e = E[op^1\ v]$:

IF $\delta(op^1, v) = e''$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} E[e'']$
   by $(op^1\ v) \rhd_{\mathsf{E\text{-}S}} e''$
2. QED

ELSE $\delta(op^1, v)$ is undefined :

1. $e \rightarrow_{\mathsf{E\text{-}S}} \mathsf{TagErr}$
   by $(op^1\ v) \rhd_{\mathsf{E\text{-}S}} \mathsf{TagErr}$
2. QED

CASE $e = E[op^2\ v_0\ v_1]$:

IF $\delta(op^2, v_0, v_1) = e''$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} E[e'']$
   by $(op^2\ v_0\ v_1) \rhd_{\mathsf{E\text{-}S}} e''$
2. QED

IF $\delta(op^2, v_0, v_1) = \mathsf{BndryErr}$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} \mathsf{BndryErr}$
   by $(op^2\ v_0\ v_1) \rhd_{\mathsf{E\text{-}S}} \mathsf{BndryErr}$
2. QED

ELSE $\delta(op^2, v_0, v_1)$ is undefined :

1. $e \rightarrow_{\mathsf{E\text{-}S}} \mathsf{TagErr}$
   by $(op^2\ v_0\ v_1) \rhd_{\mathsf{E\text{-}S}} \mathsf{TagErr}$
2. QED

CASE $e = E[\mathsf{dyn}\ \tau\ v]$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} E[\mathcal{D}_{\mathsf{E}}(\tau, v)]$
2. QED

CASE $e = E[\mathsf{stat}\ \tau\ v]$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} E[\mathcal{S}_{\mathsf{E}}(\tau, v)]$
2. QED

CASE $eE[\mathsf{Err}]$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} \mathsf{Err}$
2. QED

□

Lemma 3.9 : E *preservation*

If $\vdash_{\mathsf{E}} e$ and $e \rightarrow_{\mathsf{E\text{-}S}} e'$ then $\vdash_{\mathsf{E}} e'$.

Proof:

By *unique evaluation contexts* there are seven cases.

CASE e is a value :

1. Contradiction by $e \rightarrow_{\mathsf{E\text{-}S}} e'$

CASE $e = E[v_0\ v_1]$:

1. $v_0 = \lambda x.\, e'$ or $v_0 = \lambda(x\!:\!\tau).\, e'$
   $\wedge\ E[v_0\ v_1] \rightarrow_{\mathsf{E\text{-}S}} E[e'[x \leftarrow v_1]]$
2. $\vdash_{\mathsf{E}} v_0\ v_1$
   by *hole typing*
3. $\vdash_{\mathsf{E}} v_0$
   $\wedge \vdash_{\mathsf{E}} v_1$
   by *inversion* (2)
4. $x \vdash_{\mathsf{E}} e'$
   by *inversion* (3)
5. $\vdash_{\mathsf{E}} e'[x \leftarrow v_1]$
   by *substitution* (3, 4)
6. QED by *hole substitution* (5)

CASE $e = E[op^1\ v]$:

1. $E[op^1\ v] \rightarrow_{\mathsf{E\text{-}S}} E[v']$
   $\wedge\ \delta(op^1, v) = e''$
2. $\vdash_{\mathsf{E}} op^1\ v$
   by *hole typing*
3. $\vdash_{\mathsf{E}} v$
   by *inversion* (2)
4. $\vdash_{\mathsf{E}} e''$
   by δ *preservation* (1,3)
5. QED by *hole substitution* (4)

CASE $e = E[op^2\ v_0\ v_1]$:

1. $E[op^2\ v_0\ v_1] \rightarrow_{\mathsf{E\text{-}S}} E[v']$
   $\wedge\ \delta(op^2, v_0, v_1) = e''$
2. $\vdash_{\mathsf{E}} op^2\ v_0\ v_1$
   by *hole typing*
3. $\vdash_{\mathsf{E}} v_0$
   $\wedge \vdash_{\mathsf{E}} v_1$
   by *inversion* (2)
4. $\vdash_{\mathsf{E}} e''$
   by δ *preservation* (3)
5. QED by *hole substitution* (4)

CASE $e = E[\mathsf{dyn}\ \tau\ v]$:

1. $E[\mathsf{dyn}\ \tau\ v] \rightarrow_{\mathsf{E\text{-}S}} E[\mathcal{D}_{\mathsf{E}}(\tau, v)]$
2. $\vdash_{\mathsf{E}} \mathsf{dyn}\ \tau\ v$
   by *hole typing*
3. $\vdash_{\mathsf{E}} v$
   by *inversion* (2)
4. $\vdash_{\mathsf{E}} \mathcal{D}_{\mathsf{E}}(\tau, v)$
   by $\mathcal{D}_{\mathsf{E}}$ *soundness* (3)
5. QED by *hole substitution* (4)

CASE $e = E[\mathsf{stat}\ \tau\ v]$:

1. $E[\mathsf{stat}\ \tau\ v] \rightarrow_{\mathsf{E\text{-}S}} \mathcal{S}_{\mathsf{E}}(\tau, v)$
2. $\vdash_{\mathsf{E}} \mathsf{stat}\ \tau\ v$
   by *hole typing*
3. $\vdash_{\mathsf{E}} v$
   by *inversion* (2)
4. $\vdash_{\mathsf{E}} \mathcal{S}_{\mathsf{E}}(\tau, v)$
   by $\mathcal{S}_{\mathsf{E}}$ *soundness* (3)
5. QED by *hole substitution* (4)

CASE $e = E[\mathsf{Err}]$:

1. $E[\mathsf{Err}] \rightarrow_{\mathsf{E\text{-}S}} \mathsf{Err}$
2. QED

□

Lemma 3.10 : E *boundary-free progress*

If $\vdash e : \tau$ and e is boundary-free, then one of the following holds:

- e is a value
- $e \in \mathsf{Err}$
- $e \rightarrow_{\mathsf{E\text{-}S}} e'$
- $e \rightarrow_{\mathsf{E\text{-}S}} \mathsf{BndryErr}$

Proof:

By the *unique static evaluation contexts* lemma, there are five cases:

CASE $e = v$:

1. QED

CASE $e = E[v_0\ v_1]$:

IF $v_0 = \lambda(x\!:\!\tau').\, e'$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} E[e'[x \leftarrow v_1]]$
   by $v_0\ v_1 \rhd_{\mathsf{E\text{-}S}} e'[x \leftarrow v_1]$
2. QED

ELSE $v_0 = \lambda x.\, e'$
$\vee\ v_0 = i$
$\vee\ v_0 = \langle v, v' \rangle$:

1. Contradiction by $\vdash e : \tau$

CASE $e = E[op^1\ v]$:

IF $\delta(op^1, v) = e''$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} E[e'']$
   by $(op^1\ v) \rhd_{\mathsf{E\text{-}S}} e''$
2. QED

ELSE $\delta(op^1, v)$ is undefined :

1. Contradiction by $\vdash e : \tau$

CASE $e = E[op^2\ v_0\ v_1]$:

IF $\delta(op^2, v_0, v_1) = e''$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} E[e'']$
   by $(op^2\ v_0\ v_1) \rhd_{\mathsf{E\text{-}S}} e''$
2. QED

IF $\delta(op^2, v_0, v_1) = \mathsf{BndryErr}$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} \mathsf{BndryErr}$
   by $(op^2\ v_0\ v_1) \rhd_{\mathsf{E\text{-}S}} \mathsf{BndryErr}$
2. QED

ELSE $\delta(op^2, v_0, v_1)$ is undefined :

1. Contradiction by $\vdash e : \tau$

CASE $e = E[\mathsf{Err}]$:

1. $E[\mathsf{Err}] \rightarrow_{\mathsf{E\text{-}S}} \mathsf{Err}$
2. QED

□

Lemma 3.11 : E *boundary-free preservation*

If $\vdash e : \tau$ and e is boundary-free and $e \rightarrow_{\text{E-S}} e'$ then $\vdash e' : \tau$ and e' is boundary-free.

Proof:

By the *unique static evaluation contexts* lemma, there are five cases.

CASE e is a value :
1. Contradiction by $e \rightarrow_{\text{E-S}} e'$

CASE $e = E[v_0\ v_1]$:

IF $v_0 = \lambda(x\!:\!\tau_d).\ e'$:
1. $E[v_0\ v_1] \rightarrow_{\text{E-S}} E[e'[x \leftarrow v_1]]$
2. $\vdash v_0\ v_1 : \tau_c$
3. $\vdash v_0 : \tau_d \Rightarrow \tau_c$
$\wedge \vdash v_1 : \tau_d$
by (2)
4. $(x\!:\!\tau_d) \vdash e' : \tau_c$
by (3)
5. $\vdash e'[x \leftarrow v_1] : \tau_c$
by *substitution* (3, 4)
6. $e'[x \leftarrow v_1]$ is boundary-free
by e' and v_1 are boundary-free
7. QED

ELSE :
1. Contradiction by $\vdash e : \tau$

CASE $e = E[op^1\ v]$:
1. $E[op^1\ v] \rightarrow_{\text{E-S}} E[v']$
$\wedge\ \delta(op^1, v) = e''$
2. $\vdash op^1\ v : \tau'$
3. $\vdash v : \tau_0$
4. $\vdash e'' : \tau'$
by δ *preservation* (3)
5. QED

CASE $e = E[op^2\ v_0\ v_1]$:
1. $E[op^2\ v_0\ v_1] \rightarrow_{\text{E-S}} E[v']$
$\wedge\ \delta(op^2, v_0, v_1) = e''$
2. $\vdash op^2\ v_0\ v_1 : \tau'$
3. $\vdash v_0 : \tau_0$
$\wedge \vdash v_1 : \tau_1$
4. $\vdash e'' : \tau'$
by δ *preservation* (3)
5. QED

CASE $e = E[\text{Err}]$:
1. $E[\text{Err}] \rightarrow_{\text{E-S}} \text{Err}$
2. QED by $\vdash \text{Err} : \tau$

□

Lemma 3.12 : E *unique evaluation contexts*

If $\vdash_{\text{E}} e$ then one of the following holds:
- e is a value
- $e = E[v_0\ v_1]$
- $e = E[op^1\ v]$
- $e = E[op^2\ v_0\ v_1]$
- $e = E[\text{dyn}\ \tau\ v]$
- $e = E[\text{stat}\ \tau\ v]$
- $e = E[\text{Err}]$

Proof:

By induction on the structure of e.

CASE $e = x$:
1. Contradiction by $\vdash_{\text{E}} e$

CASE $e = i$
$\vee\ e = \lambda x.\ e'$
$\vee\ e = \lambda(x\!:\!\tau_d).\ e'$:
1. QED

CASE $e = \langle e_0, e_1 \rangle$:

IF $e_0 \notin v$:
1. $\vdash_{\text{E}} e_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis
3. $E = \langle E_0, e_1 \rangle$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\ e_1 \notin v$:
1. $\vdash_{\text{E}} e_1$
by *inversion*
2. $e_1 = E_1[e_1']$
by the induction hypothesis
3. $E = \langle e_0, E_1 \rangle$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\ e_1 \in v$:
1. QED

CASE $e = e_0\ e_1$:

IF $e_0 \notin v$:
1. $\vdash_{\text{E}} e_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis
3. $E = E_0\ e_1$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\ e_1 \notin v$:
1. $\vdash_{\text{E}} e_1$
by *inversion*
2. $e_1 = E_1[e_1']$
by the induction hypothesis
3. $E = e_0\ E_1$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\ e_1 \in v$:
1. $E = [\,]$

2. QED

CASE $e = op^1\ e_0$:

IF $e_0 \notin v$:

1. $\vdash_{\mathsf{E}} e_0$
   by *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis
3. $E = op^1\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:

1. $E = [\,]$
2. QED

CASE $e = op^2\ e_0\ e_1$:

IF $e_0 \notin v$:

1. $\vdash_{\mathsf{E}} e_0$
   by *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis
3. $E = op^2\ E_0\ e_1$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\ e_1 \notin v$:

1. $\vdash_{\mathsf{E}} e_1$
   by *inversion*
2. $e_1 = E_1[e_1']$
   by the induction hypothesis
3. $E = op^2\ e_0\ E_1$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\ e_1 \in v$:

1. $E = [\,]$
2. QED

CASE $e = \mathsf{dyn}\ \tau\ e_0$:

IF $e_0 \notin v$:

1. $\vdash_{\mathsf{E}} e_0$
   by *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis
3. $E = \mathsf{dyn}\ \tau\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:

1. $E = [\,]$
2. QED

CASE $e = \mathsf{stat}\ \tau\ e_0$:

IF $e_0 \notin v$:

1. $\vdash_{\mathsf{E}} e_0$
   by *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis
3. $E = \mathsf{stat}\ \tau\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:

1. $E = [\,]$
2. QED

CASE $e = \mathsf{Err}$:

1. $E = [\,]$
2. QED

□

Lemma 3.13 : E *hole typing*

If $\vdash_{\mathsf{E}} E[e]$ then the derivation contains a sub-term $\vdash_{\mathsf{E}} e$

Proof:

By induction on the structure of E.

CASE $E = [\,]$:

1. QED $E[e] = e$

CASE $E = E_0\ e_1$:

1. $E[e] = E_0[e]\ e_1$
2. $\vdash_{\mathsf{E}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:

1. $E[e] = v_0\ E_1[e]$
2. $\vdash_{\mathsf{E}} E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:

1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_{\mathsf{E}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:

1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_{\mathsf{E}} E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:

1. $E[e] = op^1\ E_0[e]$
2. $\vdash_{\mathsf{E}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:

1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_{\mathsf{E}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:

1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_{\mathsf{E}} E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau\ E_0$:

1. $E[e] = \mathsf{dyn}\ \tau\ E_0[e]$
2. $\vdash_{\mathsf{E}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{stat}\ \tau\ E_0$:

1. $E[e] = \mathsf{stat}\ \tau\ E_0[e]$

2. $\vdash_{\mathsf{E}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

$\square$

Lemma 3.14 : E *hole substitution*

If $\vdash_{\mathsf{E}} E[e]$ and $\vdash_{\mathsf{E}} e'$ then $\vdash_{\mathsf{E}} E[e']$

Proof:

By induction on the structure of E.

CASE $E = [\,]$:

1. QED $E[e'] = e'$

CASE $E = \langle E_0, e_1 \rangle$:

1. $E[e] = \langle E_0[e], e_1 \rangle$
   $\wedge\ E[e'] = \langle E_0[e'], e_1 \rangle$
2. $\vdash_{\mathsf{E}} \langle E_0[e], e_1 \rangle$
3. $\vdash_{\mathsf{E}} E_0[e]$
   $\wedge \vdash_{\mathsf{E}} e_1$
   by *inversion*
4. $\vdash_{\mathsf{E}} E_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{E}} \langle E_0[e'], e_1 \rangle$
   by (3, 4)
6. QED by (1, 5)

CASE $E = \langle v_0, E_1 \rangle$:

1. $E[e] = \langle v_0, E_1[e] \rangle$
   $\wedge\ E[e'] = \langle v_0, E_1[e'] \rangle$
2. $\vdash_{\mathsf{E}} \langle v_0, E_1[e] \rangle$
3. $\vdash_{\mathsf{E}} v_0$
   $\wedge \vdash_{\mathsf{E}} E_1[e]$
   by *inversion*
4. $\vdash_{\mathsf{E}} E_1[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{E}} \langle v_0, E_1[e'] \rangle$
   by (3, 4)
6. QED by (1, 5)

CASE $E = E_0\ e_1$:

1. $E[e] = E_0[e]\ e_1$
   $\wedge\ E[e'] = E_0[e']\ e_1$
2. $\vdash_{\mathsf{E}} E_0[e]\ e_1$
3. $\vdash_{\mathsf{E}} E_0[e]$
   $\wedge \vdash_{\mathsf{E}} e_1$
   by *inversion*
4. $\vdash_{\mathsf{E}} E_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{E}} E_0[e']\ e_1$
   by (3, 4)
6. QED by (1, 5)

CASE $E = v_0\ E_1$:

1. $E[e] = v_0\ E_1[e]$
   $\wedge\ E[e'] = v_0\ E_1[e']$
2. $\vdash_{\mathsf{E}} v_0\ E_1[e]$
3. $\vdash_{\mathsf{E}} v_0$
   $\wedge \vdash_{\mathsf{E}} E_1[e]$
   by *inversion*
4. $\vdash_{\mathsf{E}} E_1[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{E}} v_0\ E_1[e']$
   by (3, 4)
6. QED by (1, 5)

CASE $E = op^1\ E_0$:

1. $E[e] = op^1\ E_0[e]$
   $\wedge\ E[e'] = op^1\ E_0[e']$
2. $\vdash_{\mathsf{E}} op^1\ E_0[e]$
3. $\vdash_{\mathsf{E}} E_0[e]$
   by *inversion*
4. $\vdash_{\mathsf{E}} E_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{E}} op^1\ E_0[e']$
   by (3, 4)
6. QED by (1, 5)

CASE $E = op^2\ E_0\ e_1$:

1. $E[e] = op^2\ E_0[e]\ e_1$
   $\wedge\ E[e'] = op^2\ E_0[e']\ e_1$
2. $\vdash_{\mathsf{E}} op^2\ E_0[e]\ e_1$
3. $\vdash_{\mathsf{E}} E_0[e]$
   $\wedge \vdash_{\mathsf{E}} e_1$
   by *inversion*
4. $\vdash_{\mathsf{E}} E_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{E}} op^2\ E_0[e']\ e_1$
   by (3, 4)
6. QED by (1, 5)

CASE $E = op^2\ v_0\ E_1$:

1. $E[e] = op^2\ v_0\ E_1[e]$
   $\wedge\ E[e'] = op^2\ v_0\ E_1[e']$
2. $\vdash_{\mathsf{E}} op^2\ v_0\ E_1[e]$
3. $\vdash_{\mathsf{E}} v_0$
   $\wedge \vdash_{\mathsf{E}} E_1[e]$
   by *inversion*
4. $\vdash_{\mathsf{E}} E_1[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{E}} op^2\ v_0\ E_1[e']$
   by (3, 4)
6. QED by (1, 5)

CASE $E = \mathsf{dyn}\ \tau\ E_0$:

1. $E[e] = \mathsf{dyn}\ \tau\ E_0[e]$
   $\wedge\ E[e'] = \mathsf{dyn}\ \tau\ E_0[e']$
2. $\vdash_{\mathsf{E}} \mathsf{dyn}\ \tau\ E_0[e]$
3. $\vdash_{\mathsf{E}} E_0[e]$
   by *inversion*
4. $\vdash_{\mathsf{E}} E_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{E}} \mathsf{dyn}\ \tau\ E_0[e']$
   by (3, 4)
6. QED by (1, 5)

CASE $E = \mathsf{stat}\ \tau\ E_0$:

1. $E[e] = \mathsf{stat}\ \tau\ E_0[e]$
   $\wedge\ E[e'] = \mathsf{stat}\ \tau\ E_0[e']$

2. $\vdash_E \text{stat}\ \tau\ E_0[e]$
3. $\vdash_E E_0[e]$
   by *inversion*
4. $\vdash_E E_0[e']$
   by the induction hypothesis (3)
5. $\vdash_E \text{stat}\ \tau\ E_0[e']$
   by (3, 4)
6. QED by (1, 5)

□

Lemma 3.15 : $\vdash_E$ *inversion*

- If $\Gamma \vdash_E e_0\ e_1$ then $\Gamma \vdash_E e_0$ and $\Gamma \vdash_E e_1$
- If $\Gamma \vdash_E \lambda x.\ e$ then $x, \Gamma \vdash_E e$
- If $\Gamma \vdash_E \lambda(x\!:\!\tau).\ e$ then $(x\!:\!\tau), \Gamma \vdash_E e$
- If $\Gamma \vdash_E op^1\ e$ then $\Gamma \vdash_E e$
- If $\Gamma \vdash_E op^2\ e_0\ e_1$ then $\Gamma \vdash_E e_0$ and $\Gamma \vdash_E e_1$
- If $\Gamma \vdash_E \text{dyn}\ \tau\ e$ then $\Gamma \vdash_E e$
- If $\Gamma \vdash_E \text{stat}\ \tau\ e$ then $\Gamma \vdash_E e$

Proof:

QED by the definition of $\vdash_E e$.

□

Lemma 3.16 : E *substitution*

If $x, \Gamma \vdash_E e$ or $(x\!:\!\tau), \Gamma \vdash_E e$, and $\vdash_E v$ then $\Gamma \vdash_E e[x \leftarrow v]$

Proof:

By induction on the structure of e.

CASE $e = x$:
1. $e[x \leftarrow v] = v$
2. $\Gamma \vdash_E v$
   by *weakening*
3. QED

CASE $e = x'$:
1. QED by $x'[x \leftarrow v] = x'$

CASE $e = i$:
1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x.\ e'$:
1. QED by $(\lambda x.\ e')[x \leftarrow v] = \lambda x.\ e'$

CASE $e = \lambda(x\!:\!\tau').\ e'$:
1. QED by $(\lambda(x\!:\!\tau').\ e')[x \leftarrow v] = \lambda(x\!:\!\tau').\ e'$

CASE $e = \lambda x'.\ e'$:
1. $e[x \leftarrow v] = \lambda x'.\ (e'[x \leftarrow v])$
2. $x', x, \Gamma \vdash_E e'$
   by *inversion*
3. $x', \Gamma \vdash_E e'[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_E \lambda x'.\ e'[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \lambda(x'\!:\!\tau').\ e'$:
1. $e[x \leftarrow v] = \lambda(x'\!:\!\tau').\ (e'[x \leftarrow v])$
2. $(x'\!:\!\tau'), x, \Gamma \vdash_E e'$
   by *inversion*
3. $(x'\!:\!\tau'), \Gamma \vdash_E e'[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_E \lambda(x'\!:\!\tau').\ (e'[x \leftarrow v])$
   by (3)
5. QED

CASE $e = \langle e_0, e_1 \rangle$:
1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $x, \Gamma \vdash_E e_0$
   $\wedge\ x, \Gamma \vdash_E e_1$
   by *inversion*
3. $\Gamma \vdash_E e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_E e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_E \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
   by (3)
5. QED

CASE $e = e_0\ e_1$:
1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_E e_0$
   $\wedge\ x, \Gamma \vdash_E e_1$
   by *inversion*
3. $\Gamma \vdash_E e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_E e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_E e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^1\ e_0$:
1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $x, \Gamma \vdash_E e_0$
   by *inversion*
3. $\Gamma \vdash_E e_0[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_E op^1\ e_0[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:
1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_E e_0$
   $\wedge\ x, \Gamma \vdash_E e_1$
   by *inversion*
3. $\Gamma \vdash_E e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_E e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_E op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \text{dyn}\ \tau'\ e'$:
1. $e[x \leftarrow v] = \text{dyn}\ \tau'\ e'[x \leftarrow v]$
2. $x, \Gamma \vdash_E e'$
   by *inversion*
3. $\Gamma \vdash_E e'[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_E \text{dyn}\ \tau'\ (e'[x \leftarrow v])$
   by (3)
5. QED

case $e = \mathsf{stat}\ \tau'\ e'$:
1. $e[x \leftarrow v] = \mathsf{stat}\ \tau'\ e'[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{E}} e'$
   by *inversion*
3. $\Gamma \vdash_{\mathsf{E}} e'[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{E}} \mathsf{stat}\ \tau'\ (e'[x \leftarrow v])$
   by (3)
5. qed

case $e = \mathsf{Err}$:
1. qed by $\mathsf{Err}[x \leftarrow v] = \mathsf{Err}$

□

Lemma 3.17 : *δ preservation*
- If $\vdash_{\mathsf{E}} v$ and $\delta(op^1, v) = e'$ then $\vdash_{\mathsf{E}} e'$
- If $\vdash_{\mathsf{E}} v_0$ and $\vdash_{\mathsf{E}} v_1$ and $\delta(op^2, v_0, v_1) = e'$ then $\vdash_{\mathsf{E}} v'$

Proof:

case $\delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0$:
1. $\vdash_{\mathsf{E}} v_0$
   by *inversion*
2. qed

case $\delta(\mathsf{snd}, \langle v_0, v_1 \rangle) = v_1$:
1. $\vdash_{\mathsf{E}} v_1$
   by *inversion*
2. qed

case $\delta(\mathsf{sum}, v_0, v_1) = v_0 + v_1$:
1. qed

case $\delta(\mathsf{quotient}, v_0, v_1) = \lfloor v_0 / v_1 \rfloor$:
1. qed

case $\delta(\mathsf{quotient}, v_0, v_1) = \mathsf{BndryErr}$:
1. qed

□

Lemma 3.18 : *weakening*
- If $\Gamma \vdash_{\mathsf{E}} e$ then $x, \Gamma \vdash_{\mathsf{E}} e$
- If $\Gamma \vdash_{\mathsf{E}} e$ then $(x\!:\!\tau), \Gamma \vdash_{\mathsf{E}} e$

Proof:

qed because e is closed under Γ

□

Lemma 3.19 : *unique static evaluation contexts*

If $\vdash e : \tau$ then one of the following holds:
- e is a value
- $e = E[v_0\ v_1]$
- $e = E[op^1\ v]$
- $e = E[op^2\ v_0\ v_1]$
- $e = E[\mathsf{Err}]$

Proof:

By induction on the structure of e.

case $e = x$:
1. Contradiction by $\vdash e : \tau$

case $e = i$
  $\vee\ e = \lambda(x\!:\!\tau_d).\ e'$:
1. qed e is a value

case $e = \mathsf{stat}\ \tau\ e'$:
1. Contradiction by $\vdash_1 e : K$

case $e = \langle e_0, e_1 \rangle$:

if $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
   by the induction hypothesis
2. $E = \langle E_0, e_1 \rangle$
3. qed by $e = E[e_0']$

if $e_0 \in v$
  $\wedge\ e_1 \notin v$:
1. $e_1 = E_1[e_1']$
   by the induction hypothesis
2. $E = \langle e_0, E_1 \rangle$
3. qed by $e = E[e_1']$

else $e_0 \in v$
  $\wedge\ e_1 \in v$:
1. $E = [\,]$
2. qed $e = E[\langle e_0, e_1 \rangle]$

case $e = e_0\ e_1$:

if $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
   by the induction hypothesis
2. $E = E_0\ e_1$
3. qed by $e = E[e_0']$

if $e_0 \in v$
  $\wedge\ e_1 \notin v$:
1. $e_1 = E_1[e_1']$
   by the induction hypothesis
2. $E = e_0\ E_1$
3. qed by $e = E[e_1']$

else $e_0 \in v$
  $\wedge\ e_1 \in v$:
1. $E = [\,]$
2. qed $e = E[e_0\ e_1]$

case $e = op^1\ e_0$:

if $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
   by the induction hypothesis
2. $E = op^1\ E_0$
3. qed $e = E[e_0']$

else $e_0 \in v$:
1. $E = [\,]$
2. qed $e = E[op^1\ e_0]$

case $e = op^2\ e_0\ e_1$:

if $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
   by the induction hypothesis
2. $E = op^2\ E_0\ e_1$
3. qed $e = E[e_0']$

if $e_0 \in v$
  $\wedge\ e_1 \notin v$:
1. $e_1 = E_1[e_1']$
   by the induction hypothesis
2. $E = op^2\ e_0\ E_1$
3. qed $e = E[e_1']$

ELSE $e_0 \in v$
$\land\, e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[op^2\ e_0\ e_1]$

CASE $e = \mathsf{dyn}\ \tau\ e_0$:
1. Contradiction by e is boundary-free

CASE $e = \mathsf{stat}\ \tau\ e_0$:
1. Contradiction by $\vdash e : \tau$

CASE $e = \mathsf{Err}$:
1. $E = [\,]$
2. QED $e = E[\mathsf{Err}]$

□

Lemma 3.20 : $\vdash$ *static inversion*

- If $\Gamma \vdash x : \tau$ then $(x{:}\tau') \in \Gamma$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash \lambda(x{:}\tau'_d).\, e' : \tau$ then $(x{:}\tau'_d), \Gamma \vdash e' : \tau'_c$ and $\tau'_d \Rightarrow \tau'_c \leqslant: \tau$
- If $\Gamma \vdash \langle e_0, e_1 \rangle : \tau$ then $\Gamma \vdash e_0 : \tau_0$ and $\Gamma \vdash e_1 : \tau_1$ and $\tau_0 \times \tau_1 \leqslant: \tau$
- If $\Gamma \vdash e_0\ e_1 : \tau_c$ then $\Gamma \vdash e_0 : \tau'_d \Rightarrow \tau'_c$ and $\Gamma \vdash e_1 : \tau'_d$ and $\tau'_c \leqslant: \tau_c$
- If $\Gamma \vdash \mathsf{fst}\, e : \tau$ then $\Gamma \vdash e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0$ and $\tau_0 \leqslant: \tau$
- If $\Gamma \vdash \mathsf{snd}\, e : \tau$ then $\Gamma \vdash e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1$ and $\tau_1 \leqslant: \tau$
- If $\Gamma \vdash op^2\ e_0\ e_1 : \tau$ then $\Gamma \vdash e_0 : \tau_0$ and $\Gamma \vdash e_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau'$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash \mathsf{dyn}\ \tau'\ e' : \tau$ then $\Gamma \vdash e'$ and $\tau' \leqslant: \tau$

Proof:

QED by the definition of $\Gamma \vdash e : \tau$

□

Lemma 3.21 : *canonical forms*

- If $\vdash v : \tau_0 \times \tau_1$ then $v = \langle v_0, v_1 \rangle$
- If $\vdash v : \tau_d \Rightarrow \tau_c$ then $v = \lambda(x{:}\tau_x).\, e'$
  $\land\, \tau_d \leqslant: \tau_x$
- If $\vdash v : \mathsf{Int}$ then $v = i$
- If $\vdash v : \mathsf{Nat}$ then $v = i$ and $v \in \mathbb{N}$

Proof:

QED by definition of $\vdash e : \tau$

□

Lemma 3.22 : *substitution*

If $(x{:}\tau_x), \Gamma \vdash e : \tau$, and e is boundary-free and $\vdash v : \tau_x$ then $\Gamma \vdash e[x \leftarrow v] : \tau$

Proof:

By induction on the structure of e.

CASE $e = x$:
1. $e[x \leftarrow v] = v$
2. $\tau_x = \tau$
3. $\Gamma \vdash v : \tau$
   by *weakening*
4. QED

CASE $e = x'$:
1. QED by $x'[x \leftarrow v] = x'$

CASE $e = i$:
1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x.\, e'$:
1. Contradiction by $(x{:}\tau_x), \Gamma \vdash e : \tau$

CASE $e = \lambda(x{:}\tau').\, e'$:
1. QED by $(\lambda(x{:}\tau').\, e')[x \leftarrow v] = \lambda(x{:}\tau').\, e'$

CASE $e = \lambda(x'{:}\tau').\, e'$:
1. $e[x \leftarrow v] = \lambda(x'{:}\tau').\, (e'[x \leftarrow v])$
2. $(x'{:}\tau'), x, \Gamma \vdash e'$
   by *static inversion forms*
3. $(x'{:}\tau'), \Gamma \vdash e'[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash \lambda(x'{:}\tau').\, (e'[x \leftarrow v])$
   by (3)
5. QED

CASE $e = \langle e_0, e_1 \rangle$:
1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $x, \Gamma \vdash e_0$
   $\land\, x, \Gamma \vdash e_1$
   by *static inversion forms*
3. $\Gamma \vdash e_0[x \leftarrow v]$
   $\land\, \Gamma \vdash e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
   by (3)
5. QED

CASE $e = e_0\ e_1$:
1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash e_0$
   $\land\, x, \Gamma \vdash e_1$
   by *static inversion forms*
3. $\Gamma \vdash e_0[x \leftarrow v]$
   $\land\, \Gamma \vdash e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^1\ e_0$:
1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $x, \Gamma \vdash e_0$
   by *static inversion forms*
3. $\Gamma \vdash e_0[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash op^1\ e_0[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:
1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash e_0$
   $\land\, x, \Gamma \vdash e_1$
   by *static inversion forms*
3. $\Gamma \vdash e_0[x \leftarrow v]$
   $\land\, \Gamma \vdash e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)

5. QED

CASE $e = \mathsf{dyn}\ \tau'\ e'$:

1. Contradiction by e is boundary-free

CASE $e = \mathsf{stat}\ \tau'\ e'$:

1. Contradiction by e is boundary-free

CASE $e = \mathsf{Err}$:

1. QED $\mathsf{Err}[x \leftarrow v] = \mathsf{Err}$

□

Lemma 3.23 : *δ preservation*

- If $\vdash v$ and $\delta(op^1, v) = v'$ then $\vdash e'$
- If $\vdash v_0$ and $\vdash v_1$ and $\delta(op^2, v_0, v_1) = e'$ then $\vdash v'$

Proof:

CASE $\delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0$:

1. $\vdash v_0$
   by *static inversion forms*
2. QED

CASE $\delta(\mathsf{snd}, \langle v_0, v_1 \rangle) = v_1$:

1. $\vdash v_1$
   by *static inversion forms*
2. QED

CASE $\delta(\mathsf{sum}, v_0, v_1) = v_0 + v_1$:

1. QED

CASE $\delta(\mathsf{quotient}, v_0, v_1) = \lfloor v_0 / v_1 \rfloor$:

1. QED

CASE $\delta(\mathsf{quotient}, v_0, v_1) = \mathsf{BndryErr}$:

1. QED

□

Lemma 3.24 : *weakening*

- If $\Gamma \vdash e$ then $x, \Gamma \vdash e$
- If $\Gamma \vdash e$ then $(x : \tau), \Gamma \vdash e$

Proof:

QED because e is closed under Γ

□

A.4 (1) First-Order Embedding

A.4.1 First-Order Definitions

$\boxed{\text{Language 1}}$

$$
\begin{array}{lcl}
e & = & x \mid v \mid e\ e \mid \langle e, e\rangle \mid op^1\ e \mid op^2\ e\ e \mid \\
 & & \mathsf{dyn}\ \tau\ e \mid \mathsf{stat}\ \tau\ e \mid \mathsf{Err} \mid \mathsf{chk}\ K\ e \mid \mathsf{dyn}\ e \mid \mathsf{stat}\ e \\
v & = & i \mid \langle v, v\rangle \mid \lambda x.\,e \mid \lambda(x\!:\!\tau).\,e \\
\tau & = & \mathsf{Nat} \mid \mathsf{Int} \mid \tau\times\tau \mid \tau \Rightarrow \tau \\
K & = & \mathsf{Nat} \mid \mathsf{Int} \mid \mathsf{Pair} \mid \mathsf{Fun} \mid \mathsf{Any} \\
\Gamma & = & \cdot \mid x, \Gamma \mid (x\!:\!\tau), \Gamma \\
\mathsf{Err} & = & \mathsf{BndryErr} \mid \mathsf{TagErr} \\
r & = & v \mid \mathsf{Err} \\
E^\bullet & = & [\,] \mid E^\bullet\ e \mid v\ E^\bullet \mid \langle E^\bullet, e\rangle \mid \langle v, E^\bullet\rangle \mid \\
 & & op^1\ E^\bullet \mid op^2\ E^\bullet\ e \mid op^2\ v\ E^\bullet \mid \mathsf{chk}\ K\ E^\bullet \\
E & = & E^\bullet \mid E\ e \mid v\ E \mid \langle E, e\rangle \mid \langle v, E\rangle \mid op^1\ E \mid \\
 & & op^2\ E\ e \mid op^2\ v\ E \mid \mathsf{dyn}\ \tau\ E \mid \mathsf{stat}\ \tau\ E \mid \\
 & & \mathsf{chk}\ K\ E \mid \mathsf{dyn}\ E \mid \mathsf{stat}\ E
\end{array}
$$

$\boxed{\Delta : op^1 \times \tau \longrightarrow \tau}$

$$
\begin{array}{l}
\Delta(\mathsf{fst}, \tau_0\times\tau_1) = \tau_0 \\
\Delta(\mathsf{snd}, \tau_0\times\tau_1) = \tau_1
\end{array}
$$

$\boxed{\Delta : op^2 \times \tau \times \tau \longrightarrow \tau}$

$$
\begin{array}{l}
\Delta(op^2, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat} \\
\Delta(op^2, \mathsf{Int}, \mathsf{Int}) = \mathsf{Int}
\end{array}
$$

$\boxed{\tau \leqslant: \tau}$

$$
\frac{}{\mathsf{Nat} \leqslant: \mathsf{Int}}
\qquad
\frac{\tau_d' \leqslant: \tau_d \quad \tau_c \leqslant: \tau_c'}{\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'}
\qquad
\frac{\tau_0 \leqslant: \tau_0' \quad \tau_1 \leqslant: \tau_1'}{\tau_0\times\tau_1 \leqslant: \tau_0'\times\tau_1'}
$$

$$
\frac{}{\tau \leqslant: \tau}
\qquad
\frac{\tau \leqslant: \tau' \quad \tau' \leqslant: \tau''}{\tau \leqslant: \tau''}
$$

$\boxed{\Gamma \vdash e}$

$$
\frac{x \in \Gamma}{\Gamma \vdash x}
\qquad
\frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x.\,e}
\qquad
\frac{}{\Gamma \vdash i}
\qquad
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1\rangle}
$$

$$
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0\ e_1}
\qquad
\frac{\Gamma \vdash e}{\Gamma \vdash op^1\ e}
\qquad
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2\ e_0\ e_1}
\qquad
\frac{}{\Gamma \vdash \mathsf{Err}}
$$

$$
\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat}\ \tau\ e}
$$

$\boxed{\Gamma \vdash e : \tau}$

$$
\frac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash x : \tau}
\qquad
\frac{(x\!:\!\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x\!:\!\tau_d).\,e : \tau_d \Rightarrow \tau_c}
\qquad
\frac{i \in \mathbb{N}}{\Gamma \vdash i : \mathsf{Nat}}
$$

$$
\frac{}{\Gamma \vdash i : \mathsf{Int}}
\qquad
\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1\rangle : \tau_0\times\tau_1}
\qquad
\frac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash e_1 : \tau_d}{\Gamma \vdash e_0\ e_1 : \tau_c}
$$

$$
\frac{\Gamma \vdash e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash op^1\ e_0 : \tau}
\qquad
\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash op^2\ e_0\ e_1 : \tau}
\qquad
\frac{\Gamma \vdash e : \tau' \quad \tau' \leqslant: \tau}{\Gamma \vdash e : \tau}
\qquad
\frac{}{\Gamma \vdash \mathsf{Err} : \tau}
$$

$$
\frac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau}
$$

$\boxed{K \leqslant: K}$

$$
\frac{}{K \leqslant: \mathsf{Any}}
\qquad
\frac{}{\mathsf{Nat} \leqslant: \mathsf{Int}}
\qquad
\frac{}{K \leqslant: K}
\qquad
\frac{K \leqslant: K' \quad K' \leqslant: K''}{K \leqslant: K''}
$$

$\boxed{\lfloor \tau \rfloor = K}$

$$
\begin{array}{lcl}
\lfloor \mathsf{Nat} \rfloor & = & \mathsf{Nat} \\
\lfloor \mathsf{Int} \rfloor & = & \mathsf{Int} \\
\lfloor \tau_0\times\tau_1 \rfloor & = & \mathsf{Pair} \\
\lfloor \tau_d \Rightarrow \tau_c \rfloor & = & \mathsf{Fun}
\end{array}
$$

$\boxed{\Gamma \vdash e \rightsquigarrow e}$

$$
\frac{}{\Gamma \vdash i \rightsquigarrow i}
\qquad
\frac{\Gamma \vdash e_0 \rightsquigarrow e_0' \quad \Gamma \vdash e_1 \rightsquigarrow e_1'}{\Gamma \vdash \langle e_0, e_1\rangle \rightsquigarrow \langle e_0', e_1'\rangle}
\qquad
\frac{x, \Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \lambda x.\,e \rightsquigarrow \lambda x.\,e'}
$$

$$
\frac{}{\Gamma \vdash x \rightsquigarrow x}
\qquad
\frac{\Gamma \vdash e_0 \rightsquigarrow e_0' \quad \Gamma \vdash e_1 \rightsquigarrow e_1'}{\Gamma \vdash e_0\ e_1 \rightsquigarrow e_0'\ e_1'}
\qquad
\frac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash op^1\ e \rightsquigarrow op^1\ e'}
$$

$$
\frac{\Gamma \vdash e_0 \rightsquigarrow e_0' \quad \Gamma \vdash e_1 \rightsquigarrow e_1'}{\Gamma \vdash op^2\ e_0\ e_1 \rightsquigarrow op^2\ e_0'\ e_1'}
\qquad
\frac{}{\Gamma \vdash \mathsf{Err} \rightsquigarrow \mathsf{Err}}
$$

$$
\frac{\Gamma \vdash e : \tau \rightsquigarrow e'}{\Gamma \vdash \mathsf{stat}\ \tau\ e \rightsquigarrow \mathsf{stat}\ \tau\ e'}
$$

$\boxed{\Gamma \vdash e : \tau \rightsquigarrow e}$

$$\frac{}{\Gamma \vdash i : \mathsf{Nat} \rightsquigarrow i} \qquad \frac{}{\Gamma \vdash i : \mathsf{Int} \rightsquigarrow i}$$

$$\frac{\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e_0' \quad \Gamma \vdash e_1 : \tau_1 \rightsquigarrow e_1'}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 {\times} \tau_1 \rightsquigarrow \langle e_0', e_1' \rangle}$$

$$\frac{(x\!:\!\tau_d), \Gamma \vdash e : \tau_c \rightsquigarrow e'}{\Gamma \vdash \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c \rightsquigarrow \lambda(x\!:\!\tau_d).\, e'} \qquad \frac{}{\Gamma \vdash x : \tau \rightsquigarrow x}$$

$$\frac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \rightsquigarrow e_0' \quad \Gamma \vdash e_1 : \tau_d \rightsquigarrow e_1' \quad \lfloor \tau_c \rfloor = K}{\Gamma \vdash e_0\ e_1 : \tau_c \rightsquigarrow \mathsf{chk}\ K\ (e_0'\ e_1')}$$

$$\frac{\Gamma \vdash e : \tau_0 {\times} \tau_1 \rightsquigarrow e' \quad \lfloor \tau_0 \rfloor = K}{\Gamma \vdash \mathsf{fst}\ e : \tau_0 \rightsquigarrow \mathsf{chk}\ K\ (\mathsf{fst}\ e')}$$

$$\frac{\Gamma \vdash e : \tau_0 {\times} \tau_1 \rightsquigarrow e' \quad \lfloor \tau_1 \rfloor = K}{\Gamma \vdash \mathsf{snd}\ e : \tau_1 \rightsquigarrow \mathsf{chk}\ K\ (\mathsf{snd}\ e')}$$

$$\frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e_0' \\ \Gamma \vdash e_1 : \tau_1 \rightsquigarrow e_1'\end{array}}{\Gamma \vdash op^2\ e_0\ e_1 : \tau \rightsquigarrow op^2\ e_0'\ e_1'} \qquad \frac{\begin{array}{c}\Gamma \vdash e : \tau' \rightsquigarrow e' \\ \tau' \leqslant: \tau\end{array}}{\Gamma \vdash e : \tau \rightsquigarrow e'}$$

$$\frac{}{\Gamma \vdash \mathsf{Err} : \tau \rightsquigarrow \mathsf{Err}} \qquad \frac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau \rightsquigarrow \mathsf{dyn}\ \tau\ e'}$$

$\boxed{\Gamma \vdash_1 e}$

$$\frac{}{\Gamma \vdash_1 i} \qquad \frac{\Gamma \vdash_1 e_0 \quad \Gamma \vdash_1 e_1}{\Gamma \vdash_1 \langle e_0, e_1 \rangle} \qquad \frac{x, \Gamma \vdash_1 e}{\Gamma \vdash_1 \lambda x.\, e} \qquad \frac{(x\!:\!\tau), \Gamma \vdash_1 e : \mathsf{Any}}{\Gamma \vdash_1 \lambda(x\!:\!\tau).\, e}$$

$$\frac{x \in \Gamma}{\Gamma \vdash_1 x} \qquad \frac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash_1 x} \qquad \frac{\Gamma \vdash_1 e_0 \quad \Gamma \vdash_1 e_1}{\Gamma \vdash_1 e_0\ e_1} \qquad \frac{\Gamma \vdash_1 e}{\Gamma \vdash_1 op^1\ e}$$

$$\frac{\Gamma \vdash_1 e_0 \quad \Gamma \vdash_1 e_1}{\Gamma \vdash_1 op^2\ e_0\ e_1} \qquad \frac{}{\Gamma \vdash_1 \mathsf{Err}} \qquad \frac{\Gamma \vdash_1 e : \lfloor \tau \rfloor}{\Gamma \vdash_1 \mathsf{stat}\ \tau\ e} \qquad \frac{\Gamma \vdash_1 e : \mathsf{Any}}{\Gamma \vdash_1 \mathsf{stat}\ e}$$

$\boxed{\Gamma \vdash_1 e : K}$

$$\frac{i \in \mathbb{N}}{\Gamma \vdash_1 i : \mathsf{Nat}} \qquad \frac{}{\Gamma \vdash_1 i : \mathsf{Int}} \qquad \frac{\begin{array}{c}\Gamma \vdash_1 e_0 : \mathsf{Any} \\ \Gamma \vdash_1 e_1 : \mathsf{Any}\end{array}}{\Gamma \vdash_1 \langle e_0, e_1 \rangle : \mathsf{Pair}}$$

$$\frac{x, \Gamma \vdash_1 e}{\Gamma \vdash_1 \lambda x.\, e : \mathsf{Fun}} \qquad \frac{(x\!:\!\tau), \Gamma \vdash_1 e : \mathsf{Any}}{\Gamma \vdash_1 \lambda(x\!:\!\tau).\, e : \mathsf{Fun}} \qquad \frac{x \in \Gamma}{\Gamma \vdash_1 x : \mathsf{Any}}$$

$$\frac{\begin{array}{c}(x\!:\!\tau) \in \Gamma \\ \lfloor \tau \rfloor = K\end{array}}{\Gamma \vdash_1 x : K} \qquad \frac{\begin{array}{c}\Gamma \vdash_1 e_0 : \mathsf{Fun} \\ \Gamma \vdash_1 e_1 : \mathsf{Any}\end{array}}{\Gamma \vdash_1 e_0\ e_1 : \mathsf{Any}} \qquad \frac{\Gamma \vdash_1 e : \mathsf{Pair}}{\Gamma \vdash_1 \mathsf{fst}\ e : \mathsf{Any}}$$

$$\frac{\Gamma \vdash_1 e : \mathsf{Pair}}{\Gamma \vdash_1 \mathsf{snd}\ e : \mathsf{Any}} \qquad \frac{\begin{array}{c}\Gamma \vdash_1 e_0 : K_0 \quad \Gamma \vdash_1 e_1 : K_1 \\ \Delta(op^2, K_0, K_1) = K\end{array}}{\Gamma \vdash_1 op^2\ e_0\ e_1 : K}$$

$$\frac{\Gamma \vdash_1 e : K' \quad K' \leqslant: K}{\Gamma \vdash_1 e : K} \qquad \frac{}{\Gamma \vdash_1 \mathsf{Err} : K} \qquad \frac{\Gamma \vdash_1 e \quad \lfloor \tau \rfloor = K}{\Gamma \vdash_1 \mathsf{dyn}\ \tau\ e : K}$$

$$\frac{\Gamma \vdash_1 e}{\Gamma \vdash_1 \mathsf{dyn}\ e : \mathsf{Any}} \qquad \frac{\Gamma \vdash_1 e : \mathsf{Any}}{\Gamma \vdash_1 \mathsf{chk}\ K\ e : K}$$

$\boxed{\delta(op^1, v) = e}$

$$\begin{array}{ll}\delta(\mathsf{fst}, \langle v_0, v_1 \rangle) & = v_0 \\ \delta(\mathsf{snd}, \langle v_0, v_1 \rangle) & = v_1\end{array}$$

$\boxed{\delta(op^2, v, v) = e}$

$$\begin{array}{ll}\delta(\mathsf{sum}, i_0, i_1) & = i_0 + i_1 \\ \delta(\mathsf{quotient}, i_0, 0) & = \mathsf{BndryErr} \\ \delta(\mathsf{quotient}, i_0, i_1) & = \lfloor i_0 / i_1 \rfloor \\ \quad \text{if } i_1 \neq 0 & \end{array}$$

$\boxed{\mathcal{D}_1 : \tau {\times} v \longrightarrow v}$

$$\mathcal{D}_1(\tau, v) = \mathcal{X}(\lfloor \tau \rfloor, v)$$

$\boxed{\mathcal{S}_1 : \tau {\times} v \longrightarrow v}$

$$\mathcal{S}_1(\tau, v) = v$$

$\boxed{\mathcal{X} : K {\times} v \longrightarrow v}$

$$\begin{array}{ll}\mathcal{X}(\mathsf{Fun}, \lambda x.\, e) & = \lambda x.\, e \\ \mathcal{X}(\mathsf{Fun}, \lambda(x\!:\!\tau).\, e) & = \lambda(x\!:\!\tau).\, e \\ \mathcal{X}(\mathsf{Pair}, \langle v_0, v_1 \rangle) & = \langle v_0, v_1 \rangle \\ \mathcal{X}(\mathsf{Int}, i) & = i \\ \mathcal{X}(\mathsf{Nat}, i) & = i \\ \quad \text{if } i \in \mathbb{N} & \\ \mathcal{X}(K, v) & = \mathsf{BndryErr} \\ \quad \text{otherwise} & \end{array}$$

$\boxed{e \rhd_{\text{1-S}} e}$

$\mathsf{dyn}\ v \quad \rhd_{\text{1-S}}\ v$

$\mathsf{dyn}\ \tau\ v \quad \rhd_{\text{1-S}}\ \mathcal{D}(\tau, v)$

$\mathsf{chk}\ K\ v \quad \rhd_{\text{1-S}}\ \mathcal{X}(K, v)$

$(\lambda(x\!:\!\tau).\, e)\ v \quad \rhd_{\text{1-S}}\ \mathsf{BndryErr}$
  if $\mathcal{X}(\lfloor\tau\rfloor, v) = \mathsf{BndryErr}$

$(\lambda(x\!:\!\tau).\, e)\ v \quad \rhd_{\text{1-S}}\ e[x \leftarrow \mathcal{X}(\lfloor\tau\rfloor, v)]$

$(\lambda x.\, e)\ v \quad \rhd_{\text{1-S}}\ \mathsf{dyn}\ (e[x \leftarrow v])$

$op^1\ v \quad \rhd_{\text{1-S}}\ \delta(op^1, v)$

$op^2\ v_0\ v_1 \quad \rhd_{\text{1-S}}\ \delta(op^2, v_0, v_1)$

$\boxed{e \rhd_{\text{1-D}} e}$

$\mathsf{stat}\ v \quad \rhd_{\text{1-D}}\ v$

$\mathsf{stat}\ \tau\ v \quad \rhd_{\text{1-D}}\ \mathcal{S}(\tau, v)$

$v_0\ v_1 \quad \rhd_{\text{1-D}}\ \mathsf{TagErr}$
  if $v_0 \in \mathbb{Z}$ or $v_0 = \langle v, v' \rangle$

$(\lambda(x\!:\!\tau).\, e)\ v \quad \rhd_{\text{1-D}}\ \mathsf{BndryErr}$
  if $\mathcal{X}(\lfloor\tau\rfloor, v) = \mathsf{BndryErr}$

$(\lambda(x\!:\!\tau).\, e)\ v \quad \rhd_{\text{1-D}}\ \mathsf{stat}\ (e[x \leftarrow \mathcal{X}(\lfloor\tau\rfloor, v)])$

$(\lambda x.\, e)\ v \quad \rhd_{\text{1-D}}\ e[x \leftarrow v]$

$op^1\ v \quad \rhd_{\text{1-D}}\ \mathsf{TagErr}$
  if $\delta(op^1, v)$ is undefined

$op^1\ v \quad \rhd_{\text{1-D}}\ \delta(op^1, v)$

$op^2\ v_0\ v_1 \quad \rhd_{\text{1-D}}\ \mathsf{TagErr}$
  if $\delta(op^2, v_0, v_1)$ is undefined

$op^2\ v_0\ v_1 \quad \rhd_{\text{1-D}}\ \delta(op^2, v_0, v_1)$

$\boxed{e \rightarrow_{\text{1-S}} e}$

$E^{\bullet}[e] \quad \rightarrow_{\text{1-S}}\ E^{\bullet}[e']$
  if $e \rhd_{\text{1-S}} e'$

$E[\mathsf{stat}\ \tau\ E^{\bullet}[e]] \quad \rightarrow_{\text{1-S}}\ E[\mathsf{stat}\ \tau\ E^{\bullet}[e']]$
  if $e \rhd_{\text{1-S}} e'$

$E[\mathsf{dyn}\ \tau\ E^{\bullet}[e]] \quad \rightarrow_{\text{1-S}}\ E[\mathsf{dyn}\ \tau\ E^{\bullet}[e']]$
  if $e \rhd_{\text{1-D}} e'$

$E[\mathsf{Err}] \quad \rightarrow_{\text{1-S}}\ \mathsf{Err}$

$\boxed{e \rightarrow_{\text{1-D}} e}$

$E^{\bullet}[e] \quad \rightarrow_{\text{1-D}}\ E^{\bullet}[e']$
  if $e \rhd_{\text{1-D}} e'$

$E[\mathsf{stat}\ \tau\ E^{\bullet}[e]] \quad \rightarrow_{\text{1-D}}\ E[\mathsf{stat}\ \tau\ E^{\bullet}[e']]$
  if $e \rhd_{\text{1-S}} e'$

$E[\mathsf{dyn}\ \tau\ E^{\bullet}[e]] \quad \rightarrow_{\text{1-D}}\ E[\mathsf{dyn}\ \tau\ E^{\bullet}[e']]$
  if $e \rhd_{\text{1-D}} e'$

$E[\mathsf{Err}] \quad \rightarrow_{\text{1-D}}\ \mathsf{Err}$

$\boxed{e \rightarrow^*_{\text{1-S}} e}$ reflexive, transitive closure of $\rightarrow_{\text{1-S}}$

$\boxed{e \rightarrow^*_{\text{1-D}} e}$ reflexive, transitive closure of $\rightarrow_{\text{1-D}}$

Definition 4.0 : 1 *boundary-free*

An expression e is *boundary free* if e does not contain a subterm of the form:

- $(\mathsf{dyn}\ \tau'\ e')$,
- $(\mathsf{stat}\ \tau'\ e')$,
- $(\mathsf{dyn}\ e')$, or
- $(\mathsf{stat}\ e')$.

A.4.2 First-Order Theorems

Theorem 4.1 : *static* 1*-soundness*

If $\vdash e : \tau$ then $\vdash e : \tau \rightsquigarrow e''$ and $\vdash_1 e'' : \lfloor \tau \rfloor$ and one of the following holds:
- $e'' \rightarrow^*_{\text{1-S}} v$ and $\vdash_1 v : \lfloor \tau \rfloor$
- $e'' \rightarrow^*_{\text{1-S}} E[\mathsf{dyn}\ \tau'\ E^\bullet[e']]$ and $e' \rhd_{\text{1-D}} \mathsf{TagErr}$
- $e'' \rightarrow^*_{\text{1-S}} E[\mathsf{dyn}\ E^\bullet[e']]$ and $e' \rhd_{\text{1-D}} \mathsf{TagErr}$
- $e'' \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$
- e'' diverges

Proof:
1. $\vdash_1 e : \tau \rightsquigarrow e''$
   $\wedge \vdash_1 e'' : \lfloor \tau \rfloor$
   by $\rightsquigarrow$ *static soundness*
2. QED by *static progress* and *static preservation*

□

Theorem 4.2 : *dynamic* 1*-soundness*

If $\vdash e$ then $\vdash e \rightsquigarrow e''$ and $\vdash_1 e''$ and one of the following holds:
- $e'' \rightarrow^*_{\text{1-D}} v$ and $\vdash_1 v$
- $e'' \rightarrow^*_{\text{1-D}} E[e']$ and $e' \rhd_{\text{1-D}} \mathsf{TagErr}$
- $e'' \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
- e'' diverges

Proof:
1. $\vdash_1 e \rightsquigarrow e''$
   $\wedge \vdash_1 e''$
   by $\rightsquigarrow$ *dynamic soundness*
2. QED by *dynamic progress* and *dynamic preservation*

□

Theorem 4.3 : *boundary-free* 1*-soundness*

If $\vdash e : \tau$ and e is boundary-free then one of the following holds:
- $e \rightarrow^*_{\text{1-S}} v$ and $\vdash v : \tau$
- $e \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$
- e diverges

Proof:

QED by *progress* and *preservation*

□

Theorem 4.4 : H/1 *base type equivalence*

If $\vdash e : \tau$ and all boundary terms in e are of the following four forms:
- $\mathsf{dyn}\ \mathsf{Int}\ e'$
- $\mathsf{stat}\ \mathsf{Int}\ e'$
- $\mathsf{stat}\ \mathsf{Nat}\ e'$
- $\mathsf{dyn}\ \mathsf{Nat}\ e'$

and $\vdash e : \tau \rightsquigarrow e''$, then $e \rightarrow^*_{\text{H-S}} v$ if and only if $e'' \rightarrow^*_{\text{1-S}} v$.

Proof:
1. $\mathcal{D}_{\text{H}}(\mathsf{Int}, v) = \mathcal{D}_1(\mathsf{Int}, v)$
   by by definition
2. $\mathcal{D}_{\text{H}}(\mathsf{Nat}, v) = \mathcal{D}_1(\mathsf{Nat}, v)$
   by by definition
3. $\mathcal{S}_{\text{H}}(\mathsf{Int}, v) = \mathcal{S}_1(\mathsf{Int}, v)$
   by by definition
4. $\mathcal{S}_{\text{H}}(\mathsf{Nat}, v) = \mathcal{S}_1(\mathsf{Nat}, v)$
   by by definition
5. QED

□

Corollary 4.5 : 1 *compilation*

If $\vdash e : \tau$
and $\vdash e : \tau \rightsquigarrow e''$
and $\vdash_1 e'' : \lfloor \tau \rfloor$
and $\rhd_{\text{1-D}}'$ is similar to $\rhd_{\text{1-D}}$ but without the no-op boundaries, as follows:

$$
\begin{array}{lll}
\mathsf{chk}\ K\ v & \rhd_{\text{1-D}}' & \mathcal{X}(K, v) \\
v_0\ v_1 & \rhd_{\text{1-D}}' & \mathsf{TagErr} \\
\quad \text{if } v_0 \in \mathbb{Z} \text{ or } v_0 = \langle v, v' \rangle & & \\
(\lambda(x\!:\!\tau).\, e)\ v & \rhd_{\text{1-D}}' & \mathsf{BndryErr} \\
\quad \text{if } \mathcal{X}(\lfloor \tau \rfloor, v) = \mathsf{BndryErr} & & \\
(\lambda(x\!:\!\tau).\, e)\ v & \rhd_{\text{1-D}}' & e[x \leftarrow \mathcal{X}(\lfloor \tau \rfloor, v)] \\
(\lambda x.\, e)\ v & \rhd_{\text{1-D}}' & e[x \leftarrow v] \\
op^1\ v & \rhd_{\text{1-D}}' & \mathsf{TagErr} \\
\quad \text{if } \delta(op^1, v) \text{ is undefined} & & \\
op^1\ v & \rhd_{\text{1-D}}' & \delta(op^1, v) \\
op^2\ v_0\ v_1 & \rhd_{\text{1-D}}' & \mathsf{TagErr} \\
\quad \text{if } \delta(op^2, v_0, v_1) \text{ is undefined} & & \\
op^2\ v_0\ v_1 & \rhd_{\text{1-D}}' & \delta(op^2, v_0, v_1)
\end{array}
$$

and $e \rightarrow_{\text{1-D}}' e$ is defined as:

$$
\begin{array}{lll}
E[e] & \rightarrow_{\text{1-D}}' & E[e'] \\
\quad \text{if } e \rhd_{\text{1-D}}' e' & & \\
E[\mathsf{stat}\ \tau\ v] & \rightarrow_{\text{1-D}}' & E[\mathcal{D}_1(\tau, v)] \\
E[\mathsf{dyn}\ \tau\ v] & \rightarrow_{\text{1-D}}' & E[\mathcal{D}_1(\tau, v)] \\
E[\mathsf{Err}] & \rightarrow_{\text{1-D}}' & \mathsf{Err}
\end{array}
$$

and $\rightarrow^*_{\text{1-D}}{}'$ is the reflexive transitive closure of $\rightarrow_{\text{1-D}}'$

then one of the following holds:
- $e'' \rightarrow^*_{\text{1-D}}{}'\ v$ and $\vdash_1 v : \lfloor \tau \rfloor$
- $e'' \rightarrow^*_{\text{1-D}}{}'\ \mathsf{TagErr}$
- $e'' \rightarrow^*_{\text{1-D}}{}'\ \mathsf{BndryErr}$
- e diverges

Proof (sketch): By *static* 1*-soundness* and the fact that $\rhd_{\text{1-S}}$ is a subset of $\rhd_{\text{1-D}}'$ (modulo the dyn e and stat e boundaries).□

A.4.3 First-Order Lemmas

Lemma 4.6 : *$\mathcal{D}_1$ soundness*

If $\vdash_1 v$ then $\vdash_1 \mathcal{D}_1(\tau, v) : \lfloor \tau \rfloor$.

Proof:

- $\mathcal{D}_1(\tau, v) = \mathcal{X}(\lfloor \tau \rfloor, v)$
- QED by *check soundness*

☐

Lemma 4.7 : *$\mathcal{S}_1$ soundness*

If $\vdash_1 v : \tau$ then $\vdash_1 \mathcal{S}_1(\tau, v)$.

Proof:

- $\mathcal{S}_1(\tau, v) = \mathcal{X}(\lfloor \tau \rfloor, v)$
- QED *check soundness*

☐

Lemma 4.8 : *$\rightsquigarrow$ static soundness*

If $\Gamma \vdash e : \tau$ then $\Gamma \vdash e : \tau \rightsquigarrow e'$ and $\Gamma \vdash_1 e' : \lfloor \tau \rfloor$.

Proof:

By induction on the structure of $\Gamma \vdash e : \tau$.

CASE $\boxed{\dfrac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash x : \tau}}$:

1. $\Gamma \vdash x \rightsquigarrow x$
2. $\Gamma \vdash_1 x : \lfloor \tau \rfloor$
   by $(x\!:\!\tau) \in \Gamma$
3. QED

CASE $\boxed{\dfrac{(x\!:\!\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c}}$:

1. $\Gamma \vdash e : \tau_c \rightsquigarrow e'$
   $\wedge\ (x\!:\!\tau_d), \Gamma \vdash e' : \lfloor \tau_c \rfloor$
   by the induction hypothesis
2. $(x\!:\!\tau_d), \Gamma \vdash e' : \mathsf{Any}$
   by $\lfloor \tau_c \rfloor <: \mathsf{Any}$
3. $\lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c \rightsquigarrow \lambda(x\!:\!\tau_d).\, e'$
4. $\Gamma \vdash_1 \lambda(x\!:\!\tau_d).\, e' : \mathsf{Fun}$
   by (2)
5. QED (3, 4)

CASE $\boxed{\dfrac{i \in \mathbb{N}}{\Gamma \vdash i : \mathsf{Nat}}}$:

1. $\Gamma \vdash i : \mathsf{Nat} \rightsquigarrow i$
2. QED by $\Gamma \vdash_1 i : \mathsf{Nat}$

CASE $\boxed{\dfrac{}{\Gamma \vdash i : \mathsf{Int}}}$:

1. $\Gamma \vdash i : \mathsf{Int} \rightsquigarrow i$
2. QED by $\Gamma \vdash_1 i : \mathsf{Int}$

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \qquad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1}}$:

1. $\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e_0'$
   $\wedge\ \Gamma \vdash e_0' : \lfloor \tau_0 \rfloor$
   by the induction hypothesis
2. $\Gamma \vdash e_1 : \tau_1 \rightsquigarrow e_1'$
   $\wedge\ \Gamma \vdash e_1' : \lfloor \tau_1 \rfloor$
   by the induction hypothesis
3. $\Gamma \vdash_1 e_0 : \mathsf{Any}$
   by $\lfloor \tau_0 \rfloor <: \mathsf{Any}$
4. $\Gamma \vdash_1 e_1 : \mathsf{Any}$
   by $\lfloor \tau_1 \rfloor <: \mathsf{Any}$
5. $\Gamma \vdash \langle e_0, e_1 \rangle : \tau \rightsquigarrow \langle e_0', e_1' \rangle$
   by (1, 2)
6. $\Gamma \vdash_1 \langle e_0', e_1' \rangle : \mathsf{Pair}$
   by (3, 4)
7. QED by (5, 6)

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \qquad \Gamma \vdash e_1 : \tau_d}{\Gamma \vdash e_0\ e_1 : \tau_c}}$:

1. $\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \rightsquigarrow e_0'$
   $\wedge\ \Gamma \vdash_1 e_0' : \lfloor \tau_d \Rightarrow \tau_c \rfloor$
   by the induction hypothesis
2. $\Gamma \vdash e_1 : \tau_d \rightsquigarrow e_1'$
   $\wedge\ \Gamma \vdash_1 e_1' : \lfloor \tau_c \rfloor$
   by the induction hypothesis
3. $\Gamma \vdash_1 e_0' : \mathsf{Fun}$
   by $\lfloor \tau_d \Rightarrow \tau_c \rfloor = \mathsf{Fun}$
4. $\Gamma \vdash_1 e_1' : \mathsf{Any}$
   by $\lfloor \tau_c \rfloor <: \mathsf{Any}$
5. $\Gamma \vdash e_0\ e_1 : \tau_c \rightsquigarrow \mathsf{chk}\ \lfloor \tau_c \rfloor\ (e_0'\ e_1')$
   by (1, 2)
6. $\Gamma \vdash_1 \mathsf{chk}\ \lfloor \tau_c \rfloor\ (e_0'\ e_1') : \lfloor \tau_c \rfloor$
   by (3, 4)
7. QED by (5, 6)

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \qquad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash op^1\ e_0 : \tau}}$:

IF $op^1 = \mathsf{fst}$:

1. $\Delta(\mathsf{fst}, \tau_0) = \tau$
2. $\tau_0 = \tau \times \tau'$
   by Δ *inversion*
3. $\Gamma \vdash e_0 : \tau \times \tau' \rightsquigarrow e_0'$
   $\wedge\ \Gamma \vdash_1 e_0' : \lfloor \tau \times \tau' \rfloor$
   by the induction hypothesis
4. $\Gamma \vdash_1 e_0' : \mathsf{Pair}$
   by $\lfloor \tau \times \tau' \rfloor = \mathsf{Pair}$
5. $\Gamma \vdash \mathsf{fst}\ e_0 : \tau \rightsquigarrow \mathsf{chk}\ \lfloor \tau \rfloor\ (\mathsf{fst}\ e_0')$
   by (2)
6. $\Gamma \vdash_1 \mathsf{chk}\ \lfloor \tau \rfloor\ (\mathsf{fst}\ e_0') : \lfloor \tau \rfloor$
   by (3)
7. QED by 4,5

ELSE $op^1 = \mathsf{snd}$:

1. $\Delta(\mathsf{snd}, \tau_0) = \tau$
2. $\tau_0 = \tau' \times \tau$
   by Δ *inversion*
3. $\Gamma \vdash e_0 : \tau' \times \tau \rightsquigarrow e_0'$
   $\wedge\ \Gamma \vdash_1 e_0' : \lfloor \tau' \times \tau \rfloor$
   by the induction hypothesis

4. $\Gamma \vdash_1 e_0' : \mathsf{Pair}$
   by $\lfloor \tau' \times \tau \rfloor = \mathsf{Pair}$
5. $\Gamma \vdash \mathsf{snd}\ e_0 : \tau \rightsquigarrow \mathsf{chk}\ \lfloor \tau \rfloor\ (\mathsf{snd}\ e_0')$
   by (2)
6. $\Gamma \vdash_1 \mathsf{chk}\ \lfloor \tau \rfloor\ (\mathsf{snd}\ e_0') : \lfloor \tau \rfloor$
   by (3)
7. QED by 4,5

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash op^2\ e_0\ e_1 : \tau}}$:

1. $\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e_0'$
   $\wedge\, \Gamma \vdash_1 e_0' : \lfloor \tau_0 \rfloor$
   by the induction hypothesis
2. $\Gamma \vdash e_1 : \tau_1 \rightsquigarrow e_1'$
   $\wedge\, \Gamma \vdash_1 e_1' : \lfloor \tau_1 \rfloor$
   by the induction hypothesis
3. $\Delta(op^2, \lfloor \tau_0 \rfloor, \lfloor \tau_1 \rfloor) = \lfloor \tau \rfloor$
   by Δ *tag preservation*
4. $\Gamma \vdash op^2\ e_0\ e_1 : \tau \rightsquigarrow op^2\ e_0'\ e_1'$
   by (1, 2)
5. $\Gamma \vdash_1 op^2\ e_0'\ e_1' : \lfloor \tau \rfloor$
   by (1, 2, 3)
6. QED by (5, 6)

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau' \quad \tau' <: \tau}{\Gamma \vdash e : \tau}}$:

1. $\Gamma \vdash e : \tau' \rightsquigarrow e'$
   $\wedge\, \Gamma \vdash_1 e' : \lfloor \tau' \rfloor$
   by the induction hypothesis
2. $\lfloor \tau' \rfloor \leqslant: \lfloor \tau \rfloor$
   by *subtyping preservation*
3. $\Gamma \vdash_1 e' : \lfloor \tau \rfloor$
   by (2)
4. QED by (1, 3)

CASE $\boxed{\dfrac{}{\Gamma \vdash \mathsf{Err} : \tau}}$:

1. $\Gamma \vdash \mathsf{Err} : \tau \rightsquigarrow \mathsf{Err}$
2. $\Gamma \vdash_1 \mathsf{Err} : \tau$
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau}}$:

1. $\Gamma \vdash e \rightsquigarrow e'$
   $\wedge\, \Gamma \vdash_1 e'$
   by $\rightsquigarrow$ *dynamic soundness*
2. $\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau \rightsquigarrow \mathsf{dyn}\ \tau\ e'$
   by (1)
3. $\Gamma \vdash_1 \mathsf{dyn}\ \tau\ e' : \lfloor \tau \rfloor$
   by (1)
4. QED by (2, 3)

□

Lemma 4.9 : $\rightsquigarrow$ *dynamic soundness*

If $\Gamma \vdash e$ then $\Gamma \vdash e \rightsquigarrow e'$ and $\Gamma \vdash_1 e'$

Proof:

By induction on the structure of $\Gamma \vdash e$.

CASE $\boxed{\dfrac{x \in \Gamma}{\Gamma \vdash x}}$:

1. $\Gamma \vdash x \rightsquigarrow x$
2. $\Gamma \vdash_1 x$
   by $x \in \Gamma$
3. QED

CASE $\boxed{\dfrac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x.\, e}}$:

1. $x, \Gamma \vdash e \rightsquigarrow e'$
   $\wedge\, x, \Gamma \vdash_1 e'$
   by the induction hypothesis
2. $\Gamma \vdash \lambda x.\, e \rightsquigarrow \lambda x.\, e'$
   by (1)
3. $\Gamma \vdash_1 \lambda x.\, e'$
   by (1)
4. QED by (2, 3)

CASE $\boxed{\dfrac{}{\Gamma \vdash i}}$:

1. $\Gamma \vdash i \rightsquigarrow i$
2. $\Gamma \vdash_1 i$
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}}$:

1. $\Gamma \vdash e_0 \rightsquigarrow e_0'$
   $\wedge\, \Gamma \vdash_1 e_0'$
   by the induction hypothesis
2. $\Gamma \vdash e_1 \rightsquigarrow e_1'$
   $\wedge\, \Gamma \vdash_1 e_1'$
   by the induction hypothesis
3. $\Gamma \vdash \langle e_0, e_1 \rangle \rightsquigarrow \langle e_0', e_1' \rangle$
   by (1, 2)
4. $\Gamma \vdash_1 \langle e_0', e_1' \rangle$
   by (1, 2)
5. QED by (3, 4)

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0\ e_1}}$:

1. $\Gamma \vdash e_0 \rightsquigarrow e_0'$
   $\wedge\, \Gamma \vdash_1 e_0'$
   by the induction hypothesis
2. $\Gamma \vdash e_1 \rightsquigarrow e_1'$
   $\wedge\, \Gamma \vdash_1 e_1'$
   by the induction hypothesis
3. $\Gamma \vdash e_0\ e_1 \rightsquigarrow e_0'\ e_1'$
   by (1, 2)
4. $\Gamma \vdash_1 e_0'\ e_1'$
   by (1, 2)
5. QED by (3, 4)

CASE $\boxed{\dfrac{\Gamma \vdash e}{\Gamma \vdash op^1\ e}}$:

1. $\Gamma \vdash e \rightsquigarrow e'$
   $\wedge\ \Gamma \vdash_1 e'$
   by the induction hypothesis
2. $\Gamma \vdash op^1\ e \rightsquigarrow op^1\ e'$
   by (1)
3. $\Gamma \vdash_1 op^1\ e'$
   by (1)
4. QED by (2, 3)

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2\ e_0\ e_1}}$:

1. $\Gamma \vdash e_0 \rightsquigarrow e_0'$
   $\wedge\ \Gamma \vdash_1 e_0'$
   by the induction hypothesis
2. $\Gamma \vdash e_1 \rightsquigarrow e_1'$
   $\wedge\ \Gamma \vdash_1 e_1'$
   by the induction hypothesis
3. $\Gamma \vdash op^2\ e_0\ e_1 \rightsquigarrow op^2\ e_0'\ e_1'$
   by (1, 2)
4. $\Gamma \vdash_1 op^2\ e_0'\ e_1'$
   by (1, 2)
5. QED by 3,4

CASE $\boxed{\dfrac{}{\Gamma \vdash \mathsf{Err}}}$:

1. $\Gamma \vdash \mathsf{Err} \rightsquigarrow \mathsf{Err}$
2. $\Gamma \vdash_1 \mathsf{Err}$
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat}\ \tau\ e}}$:

1. $\Gamma \vdash e : \tau \rightsquigarrow e'$
   $\wedge\ \Gamma \vdash_1 e' : \lfloor \tau \rfloor$
   by $\rightsquigarrow$ *static soundness*
2. $\Gamma \vdash \mathsf{stat}\ \tau\ e \rightsquigarrow \mathsf{stat}\ \tau\ e'$
   by (1)
3. $\Gamma \vdash_1 \mathsf{stat}\ \tau\ e$
   by (1)
4. QED by (2,3)

□

Lemma 4.10 : 1 *static progress*

If $\vdash_1 e : K$ then one of the following holds:

- e is a value
- $e \in \mathsf{Err}$
- $e \rightarrow_{1\text{-S}} e'$
- $e \rightarrow_{1\text{-S}} \mathsf{BndryErr}$
- $e = E[\mathsf{dyn}\ \tau'\ E^\bullet[e']]$ and $e' \rightarrow_{1\text{-D}} \mathsf{TagErr}$
- $e = E[\mathsf{dyn}\ E^\bullet[e']]$ and $e' \rightarrow_{1\text{-D}} \mathsf{TagErr}$

Proof:

By the *boundary factoring* lemma, there are ten cases.

CASE e is a value :

1. QED

CASE $e = E^\bullet[v_0\ v_1]$:

1. $\vdash_1 v_0\ v_1 : K'$
   by *static hole typing*
2. $\vdash_1 v_0 : \mathsf{Fun}$
   by *inversion* (1)
3. $v_0 = \lambda x.\, e'$
   $\vee\ v_0 = \lambda(x\!:\!\tau_d).\, e'$
   by *canonical forms* (2)
4. **IF** $v_0 = \lambda x.\, e'$:
   a. $e \rightarrow_{1\text{-S}} E^\bullet[\mathsf{dyn}\ (e'[x \leftarrow v_1])]$
      by $(\lambda x.\, e')\ v_1 \rhd_{1\text{-S}} (\mathsf{dyn}\ (e'[x \leftarrow v_1]))$
   b. QED

   IF $v_0 = \lambda(x\!:\!\tau_d).\, e'$
   $\wedge\ \mathcal{X}(\lfloor \tau_d \rfloor, v_1) = v_1$:
   a. $e \rightarrow_{1\text{-S}} E^\bullet[e'[x \leftarrow v_1]]$
      by $(\lambda(x\!:\!\tau_d).\, e')\ v_1 \rhd_{1\text{-S}} e'[x \leftarrow v_1]$
   b. QED

   ELSE $v_0 = \lambda(x\!:\!\tau_d).\, e'$
   $\wedge\ \mathcal{X}(\lfloor \tau_d \rfloor, v_1) = \mathsf{BndryErr}$:
   a. $e \rightarrow_{1\text{-S}} E^\bullet[\mathsf{BndryErr}]$
      by $(\lambda(x\!:\!\tau_d).\, e')\ v_1 \rhd_{1\text{-S}} \mathsf{BndryErr}$
   b. QED

CASE $e = E^\bullet[op^1\ v]$:

1. $op^1 = \mathsf{fst}$
   $\vee\ op^1 = \mathsf{snd}$
2. $\vdash_1 op^1\ v : K'$
   by *static hole typing*
3. $\vdash_1 v : \mathsf{Pair}$
   by *inversion* (2)
4. $v = \langle v_0, v_1 \rangle$
   by *canonical forms* (3)
5. $\delta(op^1, v) = v_i$ where $i \in \{0, 1\}$
   by (1, 3)
6. $e \rightarrow_{1\text{-S}} E^\bullet[v_i]$
   by $(op^1\ v) \rhd_{1\text{-S}} v_i$
7. QED

CASE $e = E^\bullet[op^2\ v_0\ v_1]$:

1. $\vdash_1 op^2\ v_0\ v_1 : K'$
   by *static hole typing*
2. $\vdash_1 v_0 : K_0$
   $\wedge \vdash_1 v_1 : K_1$
   $\wedge\ \Delta(op^2, K_0, K_1) = K_2$
   by *inversion* (1)
3. $\delta(op^2, v_0, v_1) = e''$
   by Δ *tag soundness*
4. QED by $e \rightarrow_{1\text{-S}} E^\bullet[e'']$

CASE $e = E^\bullet[\mathsf{chk}\ K\ v_0]$:

1. $e \rightarrow_{1\text{-S}} E^\bullet[\mathcal{X}(K, v)]$
2. QED

CASE $e = E[\mathsf{dyn}\ e']$ where e' is boundary-free :

1. e' is a value
   $\vee\ e' \in \mathsf{Err}$
   $\vee\ e' \rightarrow_{1\text{-D}} e''$

$\lor\ e' \rightarrow_{\text{1-D}} \mathsf{BndryErr}$
$\lor\ e' = E'[e''] \text{ and } e'' \rhd_{\text{1-D}} \mathsf{TagErr}$
by *dynamic progress*
2. IF e' is a value :
 a. QED $e \rightarrow_{\text{1-S}} E[v]$
 IF $e' \in \mathsf{Err}$:
 a. QED $e \rightarrow_{\text{1-S}} e'$
 IF $e' \rightarrow_{\text{1-D}} e''$:
 a. QED $e \rightarrow_{\text{1-S}} E[\mathsf{dyn}\ e'']$
 IF $e' \rightarrow_{\text{1-D}} \mathsf{BndryErr}$:
 a. QED $e \rightarrow_{\text{1-S}} E[\mathsf{dyn}\ \mathsf{BndryErr}]$
 ELSE $e' = E'[e''] \text{ and } e'' \rhd_{\text{1-D}} \mathsf{TagErr}$:
 a. $E' \in E^{\bullet}$
 by e' is boundary-free
 b. QED

CASE $e = E[\mathsf{stat}\ e']$ where e' is boundary-free :
1. e' is a value
$\lor\ e' \in \mathsf{Err}$
$\lor\ e' \rightarrow_{\text{1-S}} e''$
$\lor\ e' \rightarrow_{\text{1-S}} \mathsf{BndryErr}$
$\lor\ e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']] \text{ and } e'' \rhd_{\text{1-D}} \mathsf{TagErr}$
$\lor\ e' = E''[\mathsf{dyn}\ E^{\bullet\prime\prime}[e'']] \text{ and } e'' \rhd_{\text{1-D}} \mathsf{TagErr}$
by *static progress*
2. IF e' is a value :
 a. QED $e \rightarrow_{\text{1-S}} E[e']$
 IF $e' \in \mathsf{Err}$:
 a. QED $e \rightarrow_{\text{1-S}} e'$
 IF $e' \rightarrow_{\text{1-S}} e''$:
 a. QED $e \rightarrow_{\text{1-S}} E[\mathsf{stat}\ e'']$
 IF $e' \rightarrow_{\text{1-S}} \mathsf{BndryErr}$:
 a. QED $e \rightarrow_{\text{1-S}} E[\mathsf{stat}\ \mathsf{BndryErr}]$
 IF $e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']] \text{ and } e'' \rhd_{\text{1-D}} \mathsf{TagErr}$:
 a. Contradiction by e' is boundary-free
 ELSE $e' = E''[\mathsf{dyn}\ E^{\bullet\prime\prime}[e'']] \text{ and } e'' \rhd_{\text{1-D}} \mathsf{TagErr}$:
 a. Contradiction by e' is boundary-free

CASE $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free :
1. e' is a value
$\lor\ e' \in \mathsf{Err}$
$\lor\ e' \rightarrow_{\text{1-D}} e''$
$\lor\ e' \rightarrow_{\text{1-D}} \mathsf{BndryErr}$
$\lor\ e' = E'[e''] \text{ and } e'' \rhd_{\text{1-D}} \mathsf{TagErr}$
by *dynamic progress*
2. IF e' is a value :
 a. QED $e \rightarrow_{\text{1-S}} E[\mathcal{D}_1(\tau', e')]$
 IF $e' \in \mathsf{Err}$:
 a. QED $e \rightarrow_{\text{1-S}} e'$
 IF $e' \rightarrow_{\text{1-D}} e''$:
 a. QED $e \rightarrow_{\text{1-S}} E[\mathsf{dyn}\ \tau'\ e'']$
 IF $e' \rightarrow_{\text{1-D}} \mathsf{BndryErr}$:
 a. QED $e \rightarrow_{\text{1-S}} E[\mathsf{dyn}\ \tau'\ \mathsf{BndryErr}]$
 ELSE $e' = E'[e''] \text{ and } e'' \rhd_{\text{1-D}} \mathsf{TagErr}$:
 a. $E' \in E^{\bullet}$
 by e' is boundary-free
 b. QED

CASE $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free :
1. e' is a value
$\lor\ e' \in \mathsf{Err}$
$\lor\ e' \rightarrow_{\text{1-S}} e''$
$\lor\ e' \rightarrow_{\text{1-S}} \mathsf{BndryErr}$
$\lor\ e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']] \text{ and } e'' \rhd_{\text{1-D}} \mathsf{TagErr}$
$\lor\ e' = E''[\mathsf{dyn}\ E^{\bullet\prime\prime}[e'']] \text{ and } e'' \rhd_{\text{1-D}} \mathsf{TagErr}$
by *static progress*
2. IF e' is a value :
 a. QED $e \rightarrow_{\text{1-S}} E[\mathcal{S}_1(\tau', e')]$
 IF $e' \in \mathsf{Err}$:
 a. QED $e \rightarrow_{\text{1-S}} e'$
 IF $e' \rightarrow_{\text{1-S}} e''$:
 a. QED $e \rightarrow_{\text{1-S}} E[\mathsf{stat}\ \tau'\ e'']$
 IF $e' \rightarrow_{\text{1-S}} \mathsf{BndryErr}$:
 a. QED $e \rightarrow_{\text{1-S}} E[\mathsf{stat}\ \tau'\ \mathsf{BndryErr}]$
 IF $e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']] \text{ and } e'' \rhd_{\text{1-D}} \mathsf{TagErr}$:
 a. Contradiction by e' is boundary-free
 ELSE $e' = E''[\mathsf{dyn}\ E^{\bullet\prime\prime}[e'']] \text{ and } e'' \rhd_{\text{1-D}} \mathsf{TagErr}$:
 a. Contradiction by e' is boundary-free

CASE $e = E[\mathsf{Err}]$:
1. QED $e \rightarrow_{\text{1-S}} \mathsf{Err}$

☐

Lemma 4.11 : 1 *dynamic progress*

If $\vdash_1 e : K$ then one of the following holds:
- e is a value
- $e \in \mathsf{Err}$
- $e \rightarrow_{\text{1-D}} e'$
- $e \rightarrow_{\text{1-D}} \mathsf{BndryErr}$
- $e = E[e'] \text{ and } e' \rightarrow_{\text{1-D}} \mathsf{TagErr}$

Proof:

By the *boundary factoring* lemma, there are nine cases.

CASE $e = v$:
1. QED

CASE $e = E^{\bullet}[v_0\ v_1]$:
 IF $v_0 = \lambda x.\, e_0$:
 1. $e \rightarrow_{\text{1-D}} E^{\bullet}[e_0[x \leftarrow v_1]]$
 by $(\lambda x.\, e_0)\ v_1 \rhd_{\text{1-D}} e_0[x \leftarrow v_1]$
 2. QED
 IF $v_0 = \lambda(x\!:\!\tau_d).\, e_0$
 $\land\ \mathcal{X}(\lfloor \tau_d \rfloor, v_1) = v_1$:
 1. $e \rightarrow_{\text{1-D}} E^{\bullet}[\mathsf{stat}\ (e_0[x \leftarrow v_1])]$
 by $(\lambda(x\!:\!\tau_d).\, e_0)\ v_1 \rhd_{\text{1-D}} (\mathsf{stat}\ e_0[x \leftarrow v_1])$
 2. QED
 IF $v_0 = \lambda(x\!:\!\tau_d).\, e_0$
 $\land\ \mathcal{X}(\lfloor \tau_d \rfloor, v_1) = \mathsf{BndryErr}$:
 1. $e \rightarrow_{\text{1-D}} E^{\bullet}[\mathsf{BndryErr}]$
 by $(\lambda(x\!:\!\tau_d).\, e_0)\ v_1 \rhd_{\text{1-D}} \mathsf{BndryErr}$
 2. QED

ELSE $v_0 = i$
$\vee\; v_0 = \langle v, v' \rangle$:
1. $e \rightarrow_{\text{1-D}} E^{\bullet}[\mathsf{TagErr}]$
by $v_0\; v_1 \rhd_{\text{1-D}} \mathsf{TagErr}$
2. QED

CASE $e = E^{\bullet}[op^1\; v]$:
IF $\delta(op^1, v) = v'$:
1. $e \rightarrow_{\text{1-D}} E^{\bullet}[v']$
by $(op^1\; v) \rhd_{\text{1-D}} v'$
2. QED

ELSE $\delta(op^1, v)$ is undefined :
1. $e \rightarrow_{\text{1-D}} \mathsf{TagErr}$
by $(op^1\; v) \rhd_{\text{1-D}} \mathsf{TagErr}$
2. QED

CASE $e = E^{\bullet}[op^2\; v_0\; v_1]$:
IF $\delta(op^2, v_0, v_1) = e''$:
1. QED by $e \rightarrow_{\text{1-D}} E[e'']$

ELSE $\delta(op^2, v_0, v_1)$ is undefined :
1. $e \rightarrow_{\text{1-D}} E^{\bullet}[\mathsf{TagErr}]$
by $(op^2\; v_0\; v_1) \rhd_{\text{1-D}} \mathsf{TagErr}$
2. QED

CASE $e = E^{\bullet}[\mathsf{chk}\; K\; v_0]$:
1. Contradiction by $\vdash_1 e$

CASE $e = E[\mathsf{dyn}\; v]$ where e' is boundary-free :
1. e' is a value
$\vee\; e' \in \mathsf{Err}$
$\vee\; e' \rightarrow_{\text{1-D}} e''$
$\vee\; e' \rightarrow_{\text{1-D}} \mathsf{BndryErr}$
$\vee\; e' = E'[e'']$ and $e'' \rhd_{\text{1-D}} \mathsf{TagErr}$
by *dynamic progress*
2. **IF** e' is a value :
a. QED $e \rightarrow_{\text{1-S}} E[v]$
IF $e' \in \mathsf{Err}$:
a. QED $e \rightarrow_{\text{1-S}} e'$
IF $e' \rightarrow_{\text{1-D}} e''$:
a. QED $e \rightarrow_{\text{1-S}} E[\mathsf{dyn}\; e'']$
IF $e' \rightarrow_{\text{1-D}} \mathsf{BndryErr}$:
a. QED $e \rightarrow_{\text{1-S}} E[\mathsf{dyn}\; \mathsf{BndryErr}]$
ELSE $e' = E'[e'']$ and $e'' \rhd_{\text{1-D}} \mathsf{TagErr}$:
a. $E' \in E^{\bullet}$
by e' is boundary-free
b. QED

CASE $e = E[\mathsf{stat}\; e']$ where e' is boundary-free :
1. e' is a value
$\vee\; e' \in \mathsf{Err}$
$\vee\; e' \rightarrow_{\text{1-S}} e''$
$\vee\; e' \rightarrow_{\text{1-S}} \mathsf{BndryErr}$
$\vee\; e' = E''[\mathsf{dyn}\; \tau''\; E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{1-D}} \mathsf{TagErr}$
$\vee\; e' = E''[\mathsf{dyn}\; E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{1-D}} \mathsf{TagErr}$
by *static progress*
2. **IF** e' is a value :
a. QED $e \rightarrow_{\text{1-S}} E[e']$
IF $e' \in \mathsf{Err}$:
a. QED $e \rightarrow_{\text{1-S}} e'$
IF $e' \rightarrow_{\text{1-S}} e''$:
a. QED $e \rightarrow_{\text{1-S}} E[\mathsf{stat}\; e'']$
IF $e' \rightarrow_{\text{1-S}} \mathsf{BndryErr}$:
a. QED $e \rightarrow_{\text{1-S}} E[\mathsf{stat}\; \mathsf{BndryErr}]$
IF $e' = E''[\mathsf{dyn}\; \tau''\; E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{1-D}} \mathsf{TagErr}$:
a. Contradiction by e' is boundary-free
ELSE $e' = E''[\mathsf{dyn}\; E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{1-D}} \mathsf{TagErr}$:
a. Contradiction by e' is boundary-free

CASE $e = E[\mathsf{dyn}\; \tau\; e']$ where e' is boundary-free :
1. e' is a value
$\vee\; e' \in \mathsf{Err}$
$\vee\; e' \rightarrow_{\text{1-D}} e''$
$\vee\; e' \rightarrow_{\text{1-D}} \mathsf{BndryErr}$
$\vee\; e' = E[e'']$ and $e'' \rhd_{\text{1-D}} \mathsf{TagErr}$
by *dynamic progress*
2. **IF** e' is a value :
a. QED $e \rightarrow_{\text{1-D}} E[\mathcal{D}_1(\tau', e')]$
IF $e' \in \mathsf{Err}$:
a. QED $e \rightarrow_{\text{1-D}} e'$
IF $e' \rightarrow_{\text{1-D}} e''$:
a. QED $e \rightarrow_{\text{1-S}} E[\mathsf{dyn}\; \tau'\; e'']$
IF $e' \rightarrow_{\text{1-D}} \mathsf{BndryErr}$:
a. QED $e \rightarrow_{\text{1-D}} E[\mathsf{dyn}\; \tau'\; \mathsf{BndryErr}]$
ELSE $e' = E[e'']$ and $e'' \rhd_{\text{1-D}} \mathsf{TagErr}$:
a. $E \in E^{\bullet}$
by e' is boundary-free
b. QED

CASE $e = E[\mathsf{stat}\; \tau\; e']$ where e' is boundary-free :
1. e' is a value
$\vee\; e' \in \mathsf{Err}$
$\vee\; e' \rightarrow_{\text{1-S}} e''$
$\vee\; e' \rightarrow_{\text{1-S}} \mathsf{BndryErr}$
$\vee\; e' = E''[\mathsf{dyn}\; \tau''\; E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{1-D}} \mathsf{TagErr}$
$\vee\; e' = E''[\mathsf{dyn}\; \tau''\; E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{1-D}} \mathsf{TagErr}$
by *static progress*
2. **IF** e' is a value :
a. QED $e \rightarrow_{\text{1-S}} E[\mathcal{S}_1(\tau', e')]$
IF $e' \in \mathsf{Err}$:
a. QED $e \rightarrow_{\text{1-S}} e'$
IF $e' \rightarrow_{\text{1-S}} e''$:
a. QED $e \rightarrow_{\text{1-S}} E[\mathsf{stat}\; \tau'\; e'']$
IF $e' \rightarrow_{\text{1-S}} \mathsf{BndryErr}$:
a. QED $e \rightarrow_{\text{1-S}} E[\mathsf{stat}\; \tau'\; \mathsf{BndryErr}]$
IF $e' = E''[\mathsf{dyn}\; \tau''\; E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{1-D}} \mathsf{TagErr}$:
a. Contradiction by e' is boundary-free
ELSE $e' = E''[\mathsf{dyn}\; E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{1-D}} \mathsf{TagErr}$:
a. Contradiction by e' is boundary-free

CASE $e = E[\mathsf{Err}]$:
1. $e \rightarrow_{\text{1-D}} \mathsf{Err}$
2. QED

□

Lemma 4.12 : 1 *static preservation*

If $\vdash_1 e : K$ and $e \rightarrow_{1\text{-S}} e'$ then $\vdash_1 e' : K$

Proof:

By the *boundary factoring* lemma, there are ten cases to consider.

CASE e is a value :
1. Contradiction by $e \rightarrow_{1\text{-S}} e'$

CASE $e = E^{\bullet}[v_0\ v_1]$:

IF $v_0 = \lambda x.\, e'$
$\wedge\ e \rightarrow_{1\text{-S}} E^{\bullet}[\mathsf{dyn}\ e'[x \leftarrow v_1]]$:
1. $\vdash_1 v_0\ v_1 : \mathsf{Any}$
   by *static hole typing*
2. $\vdash_1 v_0 : \mathsf{Fun}$
   $\wedge \vdash_1 v_1 : \mathsf{Any}$
   by *inversion*
3. $x \vdash_1 e'$
   by *inversion* (2)
4. $\vdash_1 v_1$
   by *static value inversion* (2)
5. $\vdash_1 e'[x \leftarrow v_1]$
   by *substitution* (3, 4)
6. $\vdash_1 \mathsf{dyn}\ (e'[x \leftarrow v_1]) : \mathsf{Any}$
   by (5)
7. QED by *hole substitution*

IF $v_0 = \lambda(x\!:\!\tau).\, e'$
$\wedge\ \mathcal{X}(\lfloor\tau\rfloor, v_1) = \mathsf{BndryErr}$
$\wedge\ e \rightarrow_{1\text{-D}} E^{\bullet}[\mathsf{BndryErr}]$:
1. $\vdash_1 v_0\ v_1 : \mathsf{Any}$
   by *static hole typing*
2. $\vdash_1 \mathsf{BndryErr} : \mathsf{Any}$
3. QED by *hole substitution* (2)

ELSE $v_0 = \lambda(x\!:\!\tau).\, e'$
$\wedge\ e \rightarrow_{1\text{-S}} E^{\bullet}[e'[x \leftarrow \mathcal{X}(\lfloor\tau\rfloor, v_1)]]$:
1. $\vdash_1 v_0\ v_1 : \mathsf{Any}$
   by *static hole typing*
2. $\vdash_1 v_0 : \mathsf{Fun}$
   $\wedge \vdash_1 v_1 : \mathsf{Any}$
   by *inversion* (1)
3. $(x\!:\!\tau) \vdash_1 e' : \mathsf{Any}$
   by *inversion* (2)
4. $\vdash_1 \mathcal{X}(\lfloor\tau\rfloor, v_1) : \lfloor\tau\rfloor$
   by *check soundness* (2)
5. $\vdash_1 e[x \leftarrow \mathcal{X}(\lfloor\tau\rfloor, v_1)] : \mathsf{Any}$
   by *substitution* (3, 4)
6. QED by *hole substitution*

CASE $e = E^{\bullet}[op^1\ v]$
$\wedge\ \delta(op^1, v) = v'$
$\wedge\ e \rightarrow_{1\text{-S}} E^{\bullet}[v']$:
1. $\vdash_1 op^1\ v : \mathsf{Any}$
   by *static hole typing*
2. $\vdash_1 v : \mathsf{Pair}$
   by *inversion*
3. $v = \langle v_0, v_1\rangle$
   by *canonical forms*
4. $\vdash_1 v_0 : \mathsf{Any}$
   $\wedge \vdash_1 v_1 : \mathsf{Any}$
   by *inversion* (2, 3)
5. $v' = v_0$
   $\vee\ v' = v_1$
   by $\delta(\mathsf{fst}, v) = v_0$
   $\wedge\ \delta(\mathsf{snd}, v) = v_1$
6. QED by *hole substitution* (5)

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$
$\wedge\ \delta(op^2, v_0, v_1) = e''$
$\wedge\ e \rightarrow_{1\text{-S}} E^{\bullet}[e'']$:
1. $\vdash_1 op^2\ v_0\ v_1 : K'$
   by *static hole typing*
2. $\vdash_1 v_0 : K_0$
   $\wedge \vdash_1 v_1 : K_1$
   $\wedge\ \Delta(op^2, K_0, K_1) = K''$
   $\wedge\ K'' <: K'$
   by *inversion* (1)
3. $\vdash_1 e'' : K''$
   by Δ *tag soundness* (3)
4. $\vdash_1 e'' : K'$
   by (2, 3)
5. QED by *hole substitution* (4)

CASE $e = E^{\bullet}[\mathsf{chk}\ K_0\ v_0]$:
1. $E^{\bullet}[\mathsf{chk}\ K_0\ v_0] \rightarrow_{1\text{-S}} E^{\bullet}[\mathcal{X}(K_0, v_0)]$
2. $\vdash_1 \mathsf{chk}\ K_0\ v : K''$
   by *static hole typing*
3. $K_0 \leqslant: K''$
   by *inversion*
4. $\vdash_1 \mathcal{X}(K_0, v_0) : K_0$
   by *check soundness*
5. QED by (3, 4, *hole substitution*)

CASE $e = E[\mathsf{dyn}\ e']$ where e' is boundary-free :

IF e' is a value :
1. $e \rightarrow_{1\text{-S}} E[e']$
2. $\vdash_1 \mathsf{dyn}\ e' : \mathsf{Any}$
   by *boundary hole typing*
3. $\vdash_1 e'$
   by *inversion* (2)
4. $\vdash_1 e' : \mathsf{Any}$
   by *dynamic value inversion* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{1\text{-D}} e''$:
1. $e \rightarrow_{1\text{-S}} E[\mathsf{dyn}\ e'']$
2. $\vdash_1 \mathsf{dyn}\ e' : \mathsf{Any}$
   by *boundary hole typing*
3. $\vdash_1 e'$
   by *inversion* (2)
4. $\vdash_1 e''$
   by *dynamic preservation* (3)

5. $\vdash_1$ dyn e'' : Any
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\text{stat } e']$ where e' is boundary-free :
IF e' is a value :
1. $e \rightarrow_{1\text{-S}} E[e']$
2. $\vdash_1$ stat e'
by *boundary hole typing*
3. $\vdash_1 e'$: Any
by *inversion* (2)
4. $\vdash_1 e'$
by *static value inversion* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{1\text{-S}} e''$:
1. $e \rightarrow_{1\text{-S}} E[\text{stat } e'']$
2. $\vdash_1$ stat e'
by *boundary hole typing*
3. $\vdash_1 e'$: Any
by *inversion* (2)
4. $\vdash_1 e''$: Any
by *static preservation* (3)
5. $\vdash_1$ stat e''
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\text{dyn } \tau\ e']$ where e' is boundary-free :
IF e' is a value :
1. $e \rightarrow_{1\text{-S}} E[\mathcal{D}_1(\tau', e')]$
2. $\vdash_1$ dyn $\tau'\ e' : \lfloor \tau' \rfloor$
by *boundary hole typing*
3. $\vdash_1 e'$
by *inversion* (2)
4. $\vdash_1 \mathcal{D}_1(\tau', e') : \lfloor \tau' \rfloor$
by $\mathcal{D}_1$ *soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{1\text{-D}} e''$:
1. $e \rightarrow_{1\text{-S}} E[\text{dyn } \tau'\ e'']$
2. $\vdash_1$ dyn $\tau'\ e' : \lfloor \tau' \rfloor$
by *boundary hole typing*
3. $\vdash_1 e'$
by *inversion* (2)
4. $\vdash_1 e''$
by *dynamic preservation* (3)
5. $\vdash_1$ dyn $\tau'\ e'' : \lfloor \tau' \rfloor$
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\text{stat } \tau\ e']$ where e' is boundary-free :
IF e' is a value :
1. $e \rightarrow_{1\text{-S}} E[\mathcal{S}_1(\tau', e')]$
2. $\vdash_1$ stat $\tau'\ e'$
by *boundary hole typing*
3. $\vdash_1 e' : \lfloor \tau' \rfloor$
by *inversion* (2)
4. $\vdash_1 \mathcal{S}_1(\tau', e')$
by $\mathcal{S}_1$ *soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{1\text{-S}} e''$:
1. $e \rightarrow_{1\text{-S}} E[\text{stat } \tau'\ e'']$
2. $\vdash_1$ stat $\tau'\ e'$
by *boundary hole typing*
3. $\vdash_1 e' : \lfloor \tau' \rfloor$
by *inversion* (2)
4. $\vdash_1 e'' : \lfloor \tau' \rfloor$
by *static preservation* (3)
5. $\vdash_1$ stat $\tau'\ e''$
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\text{Err}]$:
1. $e \rightarrow_{1\text{-S}}$ Err
2. QED $\vdash_1$ Err : K

□

Lemma 4.13 : 1 *dynamic preservation*

If $\vdash_1 e$ and $e \rightarrow_{1\text{-D}} e'$ then $\vdash_1 e'$

Proof:

By *boundary factoring* there are nine cases.

CASE e is a value :
1. Contradiction by $e \rightarrow_{1\text{-D}} e'$

CASE $e = E^{\bullet}[v_0\ v_1]$:
IF $v_0 = \lambda x.\, e'$
$\wedge\ e \rightarrow_{1\text{-D}} E^{\bullet}[e'[x \leftarrow v_1]]$:
1. $\vdash_1 v_0\ v_1$
by *dynamic hole typing*
2. $\vdash_1 v_0$
$\wedge \vdash_1 v_1$
by *inversion* (1)
3. $x \vdash_1 e'$
by *inversion* (2)
4. $\vdash_1 e'[x \leftarrow v_1]$
by *substitution* (2, 3)
5. QED by *hole substitution*

IF $v_0 = \lambda(x{:}\tau).\, e'$
$\wedge\ \mathcal{X}(\lfloor \tau \rfloor, v_1) = \text{BndryErr}$
$\wedge\ e \rightarrow_{1\text{-D}} E^{\bullet}[\text{BndryErr}]$:
1. $\vdash_1 v_0\ v_1$
by *dynamic hole typing*
2. $\vdash_1$ BndryErr
3. QED by *hole substitution* (2)

ELSE $v_0 = \lambda(x{:}\tau).\, e'$
$\wedge\ e \rightarrow_{1\text{-D}} E^{\bullet}[\text{stat } (e'[x \leftarrow \mathcal{X}(\lfloor \tau \rfloor, v_1)])]$:
1. $\vdash_1 v_0\ v_1$
by *dynamic hole typing*
2. $\vdash_1 v_0$
$\wedge \vdash_1 v_1$
by *inversion* (1)
3. $(x{:}\tau) \vdash_1 e$: Any
by *inversion* (2)
4. $\vdash_1 \mathcal{X}(\lfloor \tau \rfloor, v_1) : \lfloor \tau \rfloor$
by *check soundness* (2)

5. $\vdash_1 e[x \leftarrow \mathcal{X}(\lfloor\tau\rfloor, v_1)] : \mathsf{Any}$
   by *substitution* (3, 4)
6. $\vdash_1 \mathsf{stat}\ (e[x \leftarrow \mathcal{X}(\lfloor\tau\rfloor, v_1)])$
   by (5)
7. QED by *hole substitution* (6)

CASE $e = E^{\bullet}[op^1\ v]$
$\wedge\ \delta(op^1, v) = v'$
$\wedge\ e \rightarrow_{1\text{-D}} E^{\bullet}[v']$:

1. $\vdash_1 op^1\ v$
   by *dynamic hole typing*
2. $\vdash_1 v$
   by *inversion* (1)
3. $\vdash_1 v'$
   by *δ preservation* (2)
4. QED by *hole substitution* (3)

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$
$\wedge\ \delta(op^2, v_0, v_1) = e''$
$\wedge\ e \rightarrow_{1\text{-D}} E^{\bullet}[e'']$:

1. $\vdash_1 op^2\ v_0\ v_1$
   by *dynamic hole typing*
2. $\vdash_1 v_0$
   $\wedge \vdash_1 v_1$
   by *inversion* (1)
3. $\vdash_1 e''$
   by *δ preservation* (2)
4. QED by *hole substitution* (3)

CASE $e = E[\mathsf{dyn}\ e']$ where e' is boundary-free :

IF e' is a value :

1. $e \rightarrow_{1\text{-D}} E[e']$
2. $\vdash_1 \mathsf{dyn}\ e' : \mathsf{Any}$
   by *boundary hole typing*
3. $\vdash_1 e'$
   by *inversion* (2)
4. $\vdash_1 e' : \mathsf{Any}$
   by *$\mathcal{D}_1$ soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{1\text{-D}} e''$:

1. $e \rightarrow_{1\text{-D}} E[\mathsf{dyn}\ e'']$
2. $\vdash_1 \mathsf{dyn}\ e' : \mathsf{Any}$
   by *boundary hole typing*
3. $\vdash_1 e'$
   by *inversion* (2)
4. $\vdash_1 e''$
   by *dynamic preservation* (3)
5. $\vdash_1 \mathsf{dyn}\ e'' : \mathsf{Any}$
   by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{stat}\ e']$ where e' is boundary-free :

IF $e' \in v$:

1. $e \rightarrow_{1\text{-D}} E[e']$
2. $\vdash_1 \mathsf{stat}\ e'$
   by *boundary hole typing*
3. $\vdash_1 e' : \mathsf{Any}$
   by *inversion* (2)
4. $\vdash_1 e'$
   by *static value inversion* (3)
5. QED by *hole substitution* (5)

ELSE $e' \rightarrow_{1\text{-S}} e''$:

1. $e \rightarrow_{1\text{-D}} E[\mathsf{stat}\ e'']$
2. $\vdash_1 \mathsf{stat}\ e'$
   by *boundary hole typing*
3. $\vdash_1 e' : \mathsf{Any}$
   by *inversion* (2)
4. $\vdash_1 e'' : \mathsf{Any}$
   by *static preservation* (3)
5. $\vdash_1 \mathsf{stat}\ e''$
   by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free :

IF e' is a value :

1. $e \rightarrow_{1\text{-D}} E[\mathcal{D}_1(\tau', e')]$
2. $\vdash_1 \mathsf{dyn}\ \tau'\ e' : \lfloor\tau'\rfloor$
   by *boundary hole typing*
3. $\vdash_1 e'$
   by *inversion* (2)
4. $\vdash_1 \mathcal{D}_1(\tau', e') : \lfloor\tau'\rfloor$
   by *$\mathcal{D}_1$ soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{1\text{-D}} e''$:

1. $e \rightarrow_{1\text{-D}} E[\mathsf{dyn}\ \tau'\ e'']$
2. $\vdash_1 \mathsf{dyn}\ \tau'\ e' : \lfloor\tau'\rfloor$
   by *boundary hole typing*
3. $\vdash_1 e'$
   $\wedge\ \tau' \leqslant: \tau''$
   by *inversion* (2)
4. $\vdash_1 e''$
   by *dynamic preservation* (3)
5. $\vdash_1 \mathsf{dyn}\ \tau'\ e'' : \lfloor\tau'\rfloor$
   by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free :

IF $e' \in v$:

1. $e \rightarrow_{1\text{-D}} E[\mathcal{S}_1(\tau', e')]$
2. $\vdash_1 \mathsf{stat}\ \tau'\ e'$
   by *boundary hole typing*
3. $\vdash_1 e' : \lfloor\tau'\rfloor$
   by *inversion* (2)
4. $\vdash_1 \mathcal{S}_1(\tau', e')$
   by *$\mathcal{S}_1$ soundness* (3)
5. QED by *hole substitution* (5)

ELSE $e' \rightarrow_{1\text{-S}} e''$:

1. $e \rightarrow_{1\text{-D}} E[\mathsf{stat}\ \tau'\ e'']$
2. $\vdash_1 \mathsf{stat}\ \tau'\ e'$
   by *boundary hole typing*
3. $\vdash_1 e' : \lfloor\tau'\rfloor$
   by *inversion* (2)
4. $\vdash_1 e'' : \lfloor\tau'\rfloor$
   by *static preservation* (3)

5. $\vdash_1$ stat τ' e''
   by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{Err}]$:
1. $e \rightarrow_{\text{1-D}} \mathsf{Err}$
2. QED $\vdash_1 \mathsf{Err}$

□

Lemma 4.14 : *boundary-free progress*

If $\vdash e : \tau$ and e is boundary-free, then one of the following holds:
- e is a value
- $e \rightarrow_{\text{1-S}} e'$
- $e \rightarrow_{\text{1-S}} \mathsf{BndryErr}$

Proof:

By the L *unique static evaluation contexts* lemma, there are five cases.

CASE $e = v$:
1. QED

CASE $e = E^{\bullet}[v_0\ v_1]$:

IF $v_0 = \lambda(x\!:\!\tau').\,e'$:
1. $e \rightarrow_{\text{1-S}} E^{\bullet}[e'[x \leftarrow v_1]]$
   by $v_0\ v_1 \rhd_{\text{1-S}} e'[x \leftarrow v_1]$
2. QED

ELSE $v_0 = \lambda x.\,e'$
$\lor\, v_0 = i$
$\lor\, v_0 = \langle v, v' \rangle$:
1. Contradiction by $\vdash e : \tau$

CASE $e = E^{\bullet}[op^1\ v]$:

IF $\delta(op^1, v) = e''$:
1. $e \rightarrow_{\text{1-S}} E^{\bullet}[e'']$
   by $(op^1\ v) \rhd_{\text{1-S}} e''$
2. QED

ELSE $\delta(op^1, v)$ is undefined :
1. Contradiction by $\vdash e : \tau$

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:

IF $\delta(op^2, v_0, v_1) = e''$:
1. $e \rightarrow_{\text{1-S}} E^{\bullet}[e'']$
   by $(op^2\ v_0\ v_1) \rhd_{\text{1-S}} e''$
2. QED

IF $\delta(op^2, v_0, v_1) = \mathsf{BndryErr}$:
1. $e \rightarrow_{\text{1-S}} \mathsf{BndryErr}$
   by $(op^2\ v_0\ v_1) \rhd_{\text{1-S}} \mathsf{BndryErr}$
2. QED

ELSE $\delta(op^2, v_0, v_1)$ is undefined :
1. Contradiction by $\vdash e : \tau$

CASE $e = E^{\bullet}[\mathsf{Err}]$:
1. $E^{\bullet}[\mathsf{Err}] \rightarrow_{\text{1-S}} \mathsf{Err}$
2. QED

□

Lemma 4.15 : 1 *boundary-free preservation*

If $\vdash e : \tau$ and e is boundary-free and $e \rightarrow_{\text{1-S}} e'$ then $\vdash e' : \tau$ and e' is boundary-free.

Proof:

By the L *unique static evaluation contexts* lemma, there are five cases.

CASE e is a value :
1. Contradiction by $e \rightarrow_{\text{1-S}} e'$

CASE $e = E^{\bullet}[v_0\ v_1]$:

IF $v_0 = \lambda(x\!:\!\tau_d).\,e'$:
1. $E^{\bullet}[v_0\ v_1] \rightarrow_{\text{1-S}} E^{\bullet}[e'[x \leftarrow v_1]]$
2. $\vdash v_0\ v_1 : \tau_c$
3. $\vdash v_0 : \tau_d \Rightarrow \tau_c$
   $\land \vdash v_1 : \tau_d$
   by (2)
4. $(x\!:\!\tau_d) \vdash e' : \tau_c$
   by (3)
5. $\vdash e'[x \leftarrow v_1] : \tau_c$
   by *substitution* (3, 4)
6. $e'[x \leftarrow v_1]$ is boundary-free
   by e' and v_1 are boundary-free
7. QED

ELSE :
1. Contradiction by $\vdash e : \tau$

CASE $e = E^{\bullet}[op^1\ v]$:
1. $E^{\bullet}[op^1\ v] \rightarrow_{\text{1-S}} E^{\bullet}[v']$
   $\land\, \delta(op^1, v) = e''$
2. $\vdash op^1\ v : \tau'$
3. $\vdash v : \tau_0$
4. $\vdash e'' : \tau'$
   by δ *preservation* (3)
5. QED

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:
1. $E^{\bullet}[op^2\ v_0\ v_1] \rightarrow_{\text{1-S}} E^{\bullet}[v']$
   $\land\, \delta(op^2, v_0, v_1) = e''$
2. $\vdash op^2\ v_0\ v_1 : \tau'$
3. $\vdash v_0 : \tau_0$
   $\land \vdash v_1 : \tau_1$
4. $\vdash e'' : \tau'$
   by δ *preservation* (3)
5. QED

CASE $e = E^{\bullet}[\mathsf{Err}]$:
1. $E^{\bullet}[\mathsf{Err}] \rightarrow_{\text{1-S}} \mathsf{Err}$
2. QED by $\vdash \mathsf{Err} : \tau$

□

Lemma 4.16 : $\mathcal{X}$ *soundness*

For all K and v, $\vdash_1 \mathcal{X}(K, v) : K$.

Proof:

CASE $\vdash_1 v : K$:
1. $\mathcal{X}(K, v) = v$
2. QED

CASE $\nvdash_1 v : K$:
1. $\mathcal{X}(K, v) = \mathsf{BndryErr}$
2. QED

□

Lemma 4.17 : 1 *static boundary factoring*

If $\vdash_1 e : K$ then one of the following holds:
- e is a value
- $e = E^\bullet[v_0\ v_1]$
- $e = E^\bullet[op^1\ v]$
- $e = E^\bullet[op^2\ v_0\ v_1]$
- $e = E^\bullet[\mathsf{chk}\ K\ v]$
- $e = E[\mathsf{dyn}\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ e']$ where e' is boundary-free
- $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{Err}]$

Proof:

By the *unique evaluation contexts* lemma, there are ten cases.

CASE e is a value :
1. QED

CASE $e = E[v_0\ v_1]$:
1. $E = E^\bullet$
   $\vee\ E = E'[\mathsf{dyn}\ E^\bullet]$
   $\vee\ E = E'[\mathsf{stat}\ E^\bullet]$
   $\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
   by *inner boundary*
2. **IF** $E = E^\bullet$:
   a. QED $e = E^\bullet[v_0\ v_1]$
   IF $E = E'[\mathsf{dyn}\ E^\bullet]$:
   a. QED $e = E'[\mathsf{dyn}\ E^\bullet[v_0\ v_1]]$
   IF $E = E'[\mathsf{stat}\ E^\bullet]$:
   a. QED $e = E'[\mathsf{stat}\ E^\bullet[v_0\ v_1]]$
   IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[v_0\ v_1]]$
   ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[v_0\ v_1]]$

CASE $e = E[op^1\ v]$:
1. $E = E^\bullet$
   $\vee\ E = E'[\mathsf{dyn}\ E^\bullet]$
   $\vee\ E = E'[\mathsf{stat}\ E^\bullet]$
   $\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
   by *inner boundary*
2. **IF** $E = E^\bullet$:
   a. QED $e = E^\bullet[op^1\ v]$
   IF $E = E'[\mathsf{dyn}\ E^\bullet]$:
   a. QED $e = E'[\mathsf{dyn}\ E^\bullet[op^1\ v]]$
   IF $E = E'[\mathsf{stat}\ E^\bullet]$:
   a. QED $e = E'[\mathsf{stat}\ E^\bullet[op^1\ v]]$
   IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^1\ v]]$
   ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^1\ v]]$

CASE $e = E[op^2\ v_0\ v_1]$:
1. $E = E^\bullet$
   $\vee\ E = E'[\mathsf{dyn}\ E^\bullet]$
   $\vee\ E = E'[\mathsf{stat}\ E^\bullet]$
   $\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
   by *inner boundary*
2. **IF** $E = E^\bullet$:
   a. QED $e = E^\bullet[op^2\ v_0\ v_1]$
   IF $E = E'[\mathsf{dyn}\ E^\bullet]$:
   a. QED $e = E'[\mathsf{dyn}\ E^\bullet[op^2\ v_0\ v_1]]$
   IF $E = E'[\mathsf{stat}\ E^\bullet]$:
   a. QED $e = E'[\mathsf{stat}\ E^\bullet[op^2\ v_0\ v_1]]$
   IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$
   ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$

CASE $e = E[\mathsf{dyn}\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{stat}\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{dyn}\ \tau\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{stat}\ \tau\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{Err}]$:
1. QED

□

Lemma 4.18 : 1 *unique static evaluation contexts*

If $\vdash_1 e : K$ then one of the following holds:
- e is a value
- $e = E[v_0\ v_1]$
- $e = E[op^1\ v]$
- $e = E[op^2\ v_0\ v_1]$
- $e = E[\mathsf{chk}\ K\ v]$
- $e = E[\mathsf{dyn}\ v]$
- $e = E[\mathsf{stat}\ v]$
- $e = E[\mathsf{dyn}\ \tau\ v]$
- $e = E[\mathsf{stat}\ \tau\ v]$
- $e = E[\mathsf{Err}]$

Proof:

By induction on the structure of e.

CASE $e = x$:
1. Contradiction by $\vdash_1 e : K$

CASE $e = i$
   $\vee\ e = \lambda x.\, e'$
   $\vee\ e = \lambda(x\!:\!\tau_d).\, e'$:
1. QED e is a value

CASE $e = \langle e_0, e_1\rangle$:
IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
   by the induction hypothesis
2. $E = \langle E_0, e_1\rangle$
3. QED by $e = E[e_0']$

IF $e_0 \in v$
   $\wedge\ e_1 \notin v$:
1. $e_1 = E_1[e_1']$
   by the induction hypothesis

2. $E = \langle e_0, E_1 \rangle$
3. QED by $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[\langle e_0, e_1 \rangle]$

CASE $e = e_0\ e_1$:
IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
by the induction hypothesis
2. $E = E_0\ e_1$
3. QED by $e = E[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:
1. $e_1 = E_1[e_1']$
by the induction hypothesis
2. $E = e_0\ E_1$
3. QED by $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[e_0\ e_1]$

CASE $e = op^1\ e_0$:
1. **IF** $e_0 \notin v$:
a. $e_0 = E_0[e_0']$
by the induction hypothesis
b. $E = op^1\ E_0$
c. QED $e = E[e_0']$
2. **ELSE** $e_0 \in v$:
a. $E = [\,]$
b. QED $e = E[op^1\ e_0]$

CASE $e = op^2\ e_0\ e_1$:
IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
by the induction hypothesis
2. $E = op^2\ E_0\ e_1$
3. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:
1. $e_1 = E_1[e_1']$
by the induction hypothesis
2. $E = op^2\ e_0\ E_1$
3. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[op^2\ e_0\ e_1]$

CASE $e = \mathsf{chk}\ K\ e_0$:
IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
by the induction hypothesis
2. $E = \mathsf{chk}\ K\ E_0$
3. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[\mathsf{chk}\ K\ e_0]$

CASE $e = \mathsf{dyn}\ e_0$:
IF $e_0 \notin v$:
1. $\vdash_1 e_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by *unique evaluation contexts* (1)
3. $E = \mathsf{dyn}\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[\mathsf{dyn}\ e_0]$

CASE $e = \mathsf{stat}\ e_0$:
1. Contradiction by $\vdash_1 e : K$

CASE $e = \mathsf{dyn}\ \tau\ e_0$:
IF $e_0 \notin v$:
1. $\vdash_1 e_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by *unique evaluation contexts* (1)
3. $E = \mathsf{dyn}\ \tau\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[\mathsf{dyn}\ \tau\ e_0]$

CASE $e = \mathsf{stat}\ K'\ e_0$:
1. Contradiction by $\vdash_1 e : K$

CASE $e = \mathsf{Err}$:
1. $E = [\,]$
2. QED $e = E[\mathsf{Err}]$

□

Lemma 4.19 : 1 *inner boundary*

For all contexts E, one of the following holds:
- $E = E^\bullet$
- $E = E'[\mathsf{dyn}\ v]$
- $E = E'[\mathsf{stat}\ v]$
- $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
- $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

Proof:

By induction on the structure of E.

CASE $E = E^\bullet$:
1. QED

CASE $E = E_0\ e_1$:
1. $E_0 = E^\bullet$
$\vee\, E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$
$\vee\, E_0 = E_0'[\mathsf{stat}\ E^\bullet]$
$\vee\, E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\, E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED E is boundary-free

IF $E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$:
a. $E' = E_0'\ e_1$

b. QED $E = E'[\text{dyn } E^\bullet]$
IF $E_0 = E_0'[\text{stat } E^\bullet]$:
a. $E' = E_0' \; e_1$
b. QED $E = E'[\text{stat } E^\bullet]$
IF $E_0 = E_0'[\text{dyn } \tau \; E^\bullet]$:
a. $E' = E_0' \; e_1$
b. QED $E = E'[\text{dyn } \tau \; E^\bullet]$
ELSE $E_0 = E_0'[\text{stat } \tau \; E^\bullet]$:
a. $E' = E_0' \; e_1$
b. QED $E = E'[\text{stat } \tau \; E^\bullet]$
CASE $E = v_0 \; E_1$:
1. $E_1 = E^\bullet$
$\vee \; E_1 = E_1'[\text{dyn } E^\bullet]$
$\vee \; E_1 = E_1'[\text{stat } E^\bullet]$
$\vee \; E_1 = E_1'[\text{dyn } \tau \; E^\bullet]$
$\vee \; E_1 = E_1'[\text{stat } \tau \; E^\bullet]$
by the induction hypothesis
2. IF $E_1 = E^\bullet$:
a. QED E is boundary-free
IF $E_1 = E_1'[\text{dyn } E^\bullet]$:
a. $E' = v_0 \; E_1'$
b. QED $E = E'[\text{dyn } E^\bullet]$
IF $E_1 = E_1'[\text{stat } E^\bullet]$:
a. $E' = v_0 \; E_1'$
b. QED $E = E'[\text{stat } E^\bullet]$
IF $E_1 = E_1'[\text{dyn } \tau \; E^\bullet]$:
a. $E' = v_0 \; E_1'$
b. QED $E = E'[\text{dyn } \tau \; E^\bullet]$
ELSE $E_1 = E_1'[\text{stat } \tau \; E^\bullet]$:
a. $E' = v_0 \; E_1'$
b. QED $E = E'[\text{stat } \tau \; E^\bullet]$
CASE $E = \langle E_0, e_1 \rangle$:
1. $E_0 = E^\bullet$
$\vee \; E_0 = E_0'[\text{dyn } E^\bullet]$
$\vee \; E_0 = E_0'[\text{stat } E^\bullet]$
$\vee \; E_0 = E_0'[\text{dyn } \tau \; E^\bullet]$
$\vee \; E_0 = E_0'[\text{stat } \tau \; E^\bullet]$
by the induction hypothesis
2. IF $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E_0'[\text{dyn } E^\bullet]$:
a. $E' = \langle E_0', e_1 \rangle$
b. QED $E = E'[\text{dyn } E^\bullet]$
IF $E_0 = E_0'[\text{stat } E^\bullet]$:
a. $E' = \langle E_0', e_1 \rangle$
b. QED $E = E'[\text{stat } E^\bullet]$
IF $E_0 = E_0'[\text{dyn } \tau \; E^\bullet]$:
a. $E' = \langle E_0', e_1 \rangle$
b. QED $E = E'[\text{dyn } \tau \; E^\bullet]$
ELSE $E_0 = E_0'[\text{stat } \tau \; E^\bullet]$:
a. $E' = \langle E_0', e_1 \rangle$
b. QED $E = E'[\text{stat } \tau \; E^\bullet]$
CASE $E = \langle v_0, E_1 \rangle$:
1. $E_1 = E^\bullet$
$\vee \; E_0 = E_0'[\text{dyn } E^\bullet]$
$\vee \; E_0 = E_0'[\text{stat } E^\bullet]$
$\vee \; E_0 = E_0'[\text{dyn } \tau \; E^\bullet]$
$\vee \; E_0 = E_0'[\text{stat } \tau \; E^\bullet]$
by the induction hypothesis
2. IF $E_1 = E^\bullet$:
a. QED E is boundary-free
IF $E_1 = E_1'[\text{dyn } E^\bullet]$:
a. $E' = \langle v_0, E_1' \rangle$
b. QED $E = E'[\text{dyn } E^\bullet]$
IF $E_1 = E_1'[\text{stat } E^\bullet]$:
a. $E' = \langle v_0, E_1' \rangle$
b. QED $E = E'[\text{stat } E^\bullet]$
IF $E_1 = E_1'[\text{dyn } \tau \; E^\bullet]$:
a. $E' = \langle v_0, E_1' \rangle$
b. QED $E = E'[\text{dyn } \tau \; E^\bullet]$
ELSE $E_1 = E_1'[\text{stat } \tau \; E^\bullet]$:
a. $E' = \langle v_0, E_1' \rangle$
b. QED $E = E'[\text{stat } \tau \; E^\bullet]$
CASE $E = op^1 \; E_0$:
1. $E_0 = E^\bullet$
$\vee \; E_0 = E_0'[\text{dyn } E^\bullet]$
$\vee \; E_0 = E_0'[\text{stat } E^\bullet]$
$\vee \; E_0 = E_0'[\text{dyn } \tau \; E^\bullet]$
$\vee \; E_0 = E_0'[\text{stat } \tau \; E^\bullet]$
by the induction hypothesis
2. IF $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E_0'[\text{dyn } E^\bullet]$:
a. $E' = op^1 \; E_0'$
b. QED $E = E'[\text{dyn } E^\bullet]$
IF $E_0 = E_0'[\text{stat } E^\bullet]$:
a. $E' = op^1 \; E_0'$
b. QED $E = E'[\text{stat } E^\bullet]$
IF $E_0 = E_0'[\text{dyn } \tau \; E^\bullet]$:
a. $E' = op^1 \; E_0'$
b. QED $E = E'[\text{dyn } \tau \; E^\bullet]$
ELSE $E_0 = E_0'[\text{stat } \tau \; E^\bullet]$:
a. $E' = op^1 \; E_0'$
b. QED $E = E'[\text{stat } \tau \; E^\bullet]$
CASE $E = op^2 \; E_0 \; e_1$:
1. $E_0 = E^\bullet$
$\vee \; E_0 = E_0'[\text{dyn } E^\bullet]$
$\vee \; E_0 = E_0'[\text{stat } E^\bullet]$
$\vee \; E_0 = E_0'[\text{dyn } \tau \; E^\bullet]$
$\vee \; E_0 = E_0'[\text{stat } \tau \; E^\bullet]$
by the induction hypothesis
2. IF $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E_0'[\text{dyn } E^\bullet]$:
a. $E' = op^2 \; E_0' \; e_1$
b. QED $E = E'[\text{dyn } E^\bullet]$

IF $E_0 = E_0'[\mathsf{stat}\ E^\bullet]$:
a. $E' = op^2\ E_0'\ e_1$
b. QED $E = E'[\mathsf{stat}\ E^\bullet]$
IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = op^2\ E_0'\ e_1$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = op^2\ E_0'\ e_1$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
CASE $E = op^2\ v_0\ E_1$:
1. $E_1 = E^\bullet$
$\vee\ E_1 = E_1'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E_1 = E_1'[\mathsf{stat}\ E^\bullet]$
$\vee\ E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_1 = E^\bullet$:
a. QED E is boundary-free
IF $E_1 = E_1'[\mathsf{dyn}\ E^\bullet]$:
a. $E' = op^2\ v_0\ E_1'$
b. QED $E = E'[\mathsf{dyn}\ E^\bullet]$
IF $E_1 = E_1'[\mathsf{stat}\ E^\bullet]$:
a. $E' = op^2\ v_0\ E_1'$
b. QED $E = E'[\mathsf{stat}\ E^\bullet]$
IF $E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = op^2\ v_0\ E_1'$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = op^2\ v_0\ E_1'$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
CASE $E = \mathsf{dyn}\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED
IF $E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ E^\bullet]$
IF $E_0 = E_0'[\mathsf{stat}\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ E_0'$
b. QED $E = E'[\mathsf{stat}\ E^\bullet]$
IF $E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau'\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau'\ E^\bullet]$
CASE $E = \mathsf{stat}\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED
IF $E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$:
a. $E' = \mathsf{stat}\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ E^\bullet]$
IF $E_0 = E_0'[\mathsf{stat}\ E^\bullet]$:
a. $E' = \mathsf{stat}\ E_0'$
b. QED $E = E'[\mathsf{stat}\ E^\bullet]$
IF $E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{stat}\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau'\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{stat}\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau'\ E^\bullet]$
CASE $E = \mathsf{dyn}\ \tau\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED
IF $E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ E^\bullet]$
IF $E_0 = E_0'[\mathsf{stat}\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{stat}\ E^\bullet]$
IF $E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau'\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau'\ E^\bullet]$
CASE $E = \mathsf{stat}\ \tau\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED
IF $E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$:
a. $E' = \mathsf{stat}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ E^\bullet]$
IF $E_0 = E_0'[\mathsf{stat}\ E^\bullet]$:

a. $E' = \mathsf{stat}\ \tau\ E'_0$
b. QED $E = E'[\mathsf{stat}\ E^\bullet]$
IF $E_0 = E'_0[\mathsf{dyn}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{stat}\ \tau\ E'_0$
b. QED $E = E'[\mathsf{dyn}\ \tau'\ E^\bullet]$
ELSE $E_0 = E'_0[\mathsf{stat}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{stat}\ \tau\ E'_0$
b. QED $E = E'[\mathsf{stat}\ \tau'\ E^\bullet]$
CASE $E = \mathsf{chk}\ K_0\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E'_0[\mathsf{dyn}\ E^\bullet]$
$\vee\ E_0 = E'_0[\mathsf{stat}\ E^\bullet]$
$\vee\ E_0 = E'_0[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E_0 = E'_0[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E'_0[\mathsf{dyn}\ E^\bullet]$:
a. $E' = \mathsf{chk}\ K_0\ E'_0$
b. QED $E = E'[\mathsf{dyn}\ E^\bullet]$
IF $E_0 = E'_0[\mathsf{stat}\ E^\bullet]$:
a. $E' = \mathsf{chk}\ K_0\ E'_0$
b. QED $E = E'[\mathsf{stat}\ E^\bullet]$
IF $E_0 = E'_0[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = \mathsf{chk}\ K_0\ E'_0$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_0 = E'_0[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = \mathsf{chk}\ K_0\ E'_0$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

□

Lemma 4.20 : 1 *dynamic boundary factoring*

If $\vdash_1 e$ then one of the following holds:
- e is a value
- $e = E^\bullet[v_0\ v_1]$
- $e = E^\bullet[op^1\ v]$
- $e = E^\bullet[op^2\ v_0\ v_1]$
- $e = E[\mathsf{dyn}\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ e']$ where e' is boundary-free
- $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{Err}]$

Proof:
By the *unique evaluation contexts* lemma, there are ten cases.
CASE e is a value :
1. QED
CASE $e = E[v_0\ v_1]$:
1. $E = E^\bullet$
$\vee\ E = E'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E = E'[\mathsf{stat}\ E^\bullet]$
$\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. QED $e = E^\bullet[v_0\ v_1]$
IF $E = E'[\mathsf{dyn}\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ E^\bullet[v_0\ v_1]]$
IF $E = E'[\mathsf{stat}\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ E^\bullet[v_0\ v_1]]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[v_0\ v_1]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[v_0\ v_1]]$
CASE $e = E[op^1\ v]$:
1. $E = E^\bullet$
$\vee\ E = E'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E = E'[\mathsf{stat}\ E^\bullet]$
$\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. QED $e = E^\bullet[op^1\ v]$
IF $E = E'[\mathsf{dyn}\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ E^\bullet[op^1\ v]]$
IF $E = E'[\mathsf{stat}\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ E^\bullet[op^1\ v]]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^1\ v]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^1\ v]]$
CASE $e = E[op^2\ v_0\ v_1]$:
1. $E = E^\bullet$
$\vee\ E = E'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E = E'[\mathsf{stat}\ E^\bullet]$
$\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. QED $e = E^\bullet[op^2\ v_0\ v_1]$
IF $E = E'[\mathsf{dyn}\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ E^\bullet[op^2\ v_0\ v_1]]$
IF $E = E'[\mathsf{stat}\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ E^\bullet[op^2\ v_0\ v_1]]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$
CASE $e = E[\mathsf{chk}\ K'\ v]$:
1. $E = E^\bullet$
$\vee\ E = E'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E = E'[\mathsf{stat}\ E^\bullet]$
$\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. QED $e = E^\bullet[\mathsf{chk}\ K'\ v]$
IF $E = E'[\mathsf{dyn}\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ E^\bullet[\mathsf{chk}\ K'\ v]]$

IF $E = E'[\mathsf{stat}\ E^{\bullet}]$:
a. QED $e = E'[\mathsf{stat}\ E^{\bullet}[\mathsf{chk}\ K'\ v]]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^{\bullet}[\mathsf{chk}\ K'\ v]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^{\bullet}[\mathsf{chk}\ K'\ v]]$
CASE $e = E[\mathsf{dyn}\ v]$:
1. QED v is boundary-free
CASE $e = E[\mathsf{stat}\ v]$:
1. QED v is boundary-free
CASE $e = E[\mathsf{dyn}\ \tau\ v]$:
1. QED v is boundary-free
CASE $e = E[\mathsf{stat}\ \tau\ v]$:
1. QED v is boundary-free
CASE $e = E[\mathsf{Err}]$:
1. QED

□

Lemma 4.21 : 1 *unique dynamic evaluation contexts*
If $\vdash_1 e$ then one of the following holds:
- e is a value
- $e = E[v_0\ v_1]$
- $e = E[op^1\ v]$
- $e = E[op^2\ v_0\ v_1]$
- $e = E[\mathsf{chk}\ K\ v]$
- $e = E[\mathsf{dyn}\ v]$
- $e = E[\mathsf{stat}\ v]$
- $e = E[\mathsf{dyn}\ \tau\ v]$
- $e = E[\mathsf{stat}\ \tau\ v]$
- $e = E[\mathsf{Err}]$

Proof:
By induction on the structure of e.
CASE $e = x$:
1. Contradiction by $\vdash_1 e$
CASE $e = i$
$\lor\ e = \lambda x.\, e'$
$\lor\ e = \lambda(x\!:\!\tau_d).\, e'$:
1. QED e is a value
CASE $e = \langle e_0, e_1\rangle$:
IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
by the induction hypothesis
2. $E = \langle E_0, e_1\rangle$
3. QED $e = E[e_0']$
IF $e_0 \in v$
$\land\ e_1 \notin v$:
1. $e_1 = E_1[e_1']$
by the induction hypothesis
2. $E = \langle e_0, E_1\rangle$
3. QED $e = E[e_1']$
ELSE $e_0 \in v$
$\land\ e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[\langle e_0, e_1\rangle]$
CASE $e = e_0\ e_1$:
IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
by the induction hypothesis
2. $E = E_0\ e_1$
3. QED $e = E[e_0']$
IF $e_0 \in v$
$\land\ e_1 \notin v$:
1. $e_1 = E_1[e_1']$
by the induction hypothesis
2. $E = e_0\ E_1$
3. QED $e = E[e_1']$
ELSE $e_0 \in v$
$\land\ e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[e_0\ e_1]$
CASE $e = op^1\ e_0$:
IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
by the induction hypothesis
2. $E = op^1\ E_0$
3. QED $e = E[e_0']$
ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[op^1\ e_0]$
CASE $e = op^2\ e_0\ e_1$:
IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
by the induction hypothesis
2. $E = op^2\ E_0\ e_1$
3. QED $e = E[e_0']$
IF $e_0 \in v$
$\land\ e_1 \notin v$:
1. $e_1 = E_1[e_1']$
by the induction hypothesis
2. $E = op^2\ e_0\ E_1$
3. QED $e = E[e_1']$
ELSE $e_0 \in v$
$\land\ e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[op^2\ e_0\ e_1]$
CASE $e = \mathsf{chk}\ K\ e'$:
1. Contradiction by $\vdash_1 e$
CASE $e = \mathsf{dyn}\ e_0$:
1. Contradiction by $\vdash_1 e$
CASE $e = \mathsf{stat}\ e_0$:
IF $e_0 \notin v$:
1. $\vdash_1 e_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by *unique evaluation contexts* (1)
3. $E = \mathsf{stat}\ E_0$
4. QED $e = E[e_0']$
ELSE $e_0 \in v$:
1. $E = [\,]$

2. QED $e = E[\text{stat } e_0]$

CASE $e = \text{dyn } e_0$:

Contradiction by $\vdash_1 e$

CASE $e = \text{stat } K_0\ e_0$:

IF $e_0 \notin v$:

1. $\vdash_1 e_0$
   by *inversion*
2. $e_0 = E_0[e_0']$
   by *unique evaluation contexts* (1)
3. $E = \text{stat } \tau\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:

1. $E = [\,]$
2. QED $e = E[\text{stat } \tau\ e_0]$

□

Lemma 4.22 : 1 *static hole typing*

If $\vdash_1 E^\bullet[e] : K$ then the typing derivation contains a sub-term $\vdash_1 e : K'$ for some K'.

Proof:

By induction on the structure of $E^\bullet$.

CASE $E^\bullet = [\,]$:

1. QED $E^\bullet[e] = e$

CASE $E^\bullet = E^\bullet{}_0\ e_1$:

1. $E^\bullet[e] = E^\bullet{}_0[e]\ e_1$
2. $\vdash_1 E^\bullet{}_0[e] : \text{Fun}$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = v_0\ E^\bullet{}_1$:

1. $E^\bullet[e] = v_0\ E^\bullet{}_1[e]$
2. $\vdash_1 E^\bullet{}_1[e] : \text{Any}$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = \langle E^\bullet{}_0, e_1\rangle$:

1. $E^\bullet[e] = \langle E^\bullet{}_0[e], e_1\rangle$
2. $\vdash_1 E^\bullet{}_0[e] : \text{Any}$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = \langle v_0, E^\bullet{}_1\rangle$:

1. $E^\bullet[e] = \langle v_0, E^\bullet{}_1[e]\rangle$
2. $\vdash_1 E^\bullet{}_1[e] : \text{Any}$
   by *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = op^1\ E^\bullet{}_0$:

1. $E^\bullet[e] = op^1\ E^\bullet{}_0[e]$
2. $\vdash_1 E^\bullet{}_0[e] : \text{Pair}$
   by *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = op^2\ E^\bullet{}_0\ e_1$:

1. $E^\bullet[e] = op^2\ E^\bullet{}_0[e]\ e_1$
2. $\vdash_1 E^\bullet{}_0[e] : K_0$
   by *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = op^2\ v_0\ E^\bullet{}_1$:

1. $E^\bullet[e] = op^2\ v_0\ E^\bullet{}_1[e]$
2. $\vdash_1 E^\bullet{}_1[e] : K_1$
   by *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = \text{chk } K\ E^\bullet{}_0$:

1. $E^\bullet[e] = \text{chk } K\ E^\bullet{}_0[e]$
2. $\vdash_1 E^\bullet{}_0[e] : \text{Any}$
   by *inversion*
3. QED the induction hypothesis (2)

□

Lemma 4.23 : 1 *dynamic hole typing*

If $\vdash_1 E^\bullet[e]$ then the derivation contains a sub-term $\vdash_1 e$

Proof:

By induction on the structure of $E^\bullet$.

CASE $E^\bullet = [\,]$:

1. QED $E^\bullet[e] = e$

CASE $E^\bullet = E^\bullet{}_0\ e_1$:

1. $E^\bullet[e] = E^\bullet{}_0[e]\ e_1$
2. $\vdash_1 E^\bullet{}_0[e]$
   by *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = v_0\ E^\bullet{}_1$:

1. $E^\bullet[e] = v_0\ E^\bullet{}_1[e]$
2. $\vdash_1 E^\bullet{}_1[e]$
   by *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = \langle E^\bullet{}_0, e_1\rangle$:

1. $E^\bullet[e] = \langle E^\bullet{}_0[e], e_1\rangle$
2. $\vdash_1 E^\bullet{}_0[e]$
   by *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = \langle v_0, E^\bullet{}_1\rangle$:

1. $E^\bullet[e] = \langle v_0, E^\bullet{}_1[e]\rangle$
2. $\vdash_1 E^\bullet{}_1[e]$
   by *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = op^1\ E^\bullet{}_0$:

1. $E^\bullet[e] = op^1\ E^\bullet{}_0[e]$
2. $\vdash_1 E^\bullet{}_0[e]$
   by *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = op^2\ E^\bullet{}_0\ e_1$:

1. $E^\bullet[e] = op^2\ E^\bullet{}_0[e]\ e_1$
2. $\vdash_1 E^\bullet{}_0[e]$
   by *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = op^2\ v_0\ E^\bullet{}_1$:

1. $E^\bullet[e] = op^2\ v_0\ E^\bullet{}_1[e]$
2. $\vdash_1 E^\bullet{}_1[e]$
   by *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = \text{chk } K\ E^\bullet{}_0$:

1. Contradiction by $\vdash_1 E^\bullet[e]$

□

Lemma 4.24 : 1 *boundary hole typing*

- If $\vdash_1 E[\text{dyn } e]$ then the derivation contains a sub-term $\vdash_1 \text{dyn } e : \text{Any}$
- If $\vdash_1 E[\text{dyn } e] : K'$ then the derivation contains a sub-term $\vdash_1 \text{dyn } e : \text{Any}$
- If $\vdash_1 E[\text{stat } e]$ then the derivation contains a sub-term $\vdash_1 \text{stat } e$
- If $\vdash_1 E[\text{stat } e] : K'$ then the derivation contains a sub-term $\vdash_1 \text{stat } e$
- If $\vdash_1 E[\text{dyn } \tau\ e]$ then the derivation contains a sub-term $\vdash_1 \text{dyn } \tau\ e : \lfloor\tau\rfloor$
- If $\vdash_1 E[\text{dyn } \tau\ e] : K'$ then the derivation contains a sub-term $\vdash_1 \text{dyn } \tau\ e : \lfloor\tau\rfloor$
- If $\vdash_1 E[\text{stat } \tau\ e]$ then the derivation contains a sub-term $\vdash_1 \text{stat } \tau\ e$
- If $\vdash_1 E[\text{stat } \tau\ e] : K'$ then the derivation contains a sub-term $\vdash_1 \text{stat } \tau\ e$

Proof:

By the following four lemmas: *static* dyn *hole typing*, *dynamic* dyn *hole typing*, *static* stat *hole typing*, and *dynamic* stat *hole typing*.

□

Lemma 4.25 : 1 *static* dyn *hole typing*

If $\vdash_1 E[\text{dyn } \tau\ e] : K'$ then the derivation contains a sub-term $\vdash_1 \text{dyn } \tau\ e : \lfloor\tau\rfloor$.

Proof:

By induction on the structure of E.

case $E \in E^\bullet$:
1. $\vdash_1 \text{dyn } \tau\ e : K''$
   by *static hole typing*
2. $\vdash_1 \text{dyn } \tau\ e : \lfloor\tau\rfloor$
   by *inversion* (1)
3. qed

case $E = E_0\ e_1$:
1. $E[\text{dyn } \tau\ e] = E_0[\text{dyn } \tau\ e]\ e_1$
2. $\vdash_1 E_0[\text{dyn } \tau\ e] : K_0$
   by *inversion*
3. qed by the induction hypothesis (2)

case $E = v_0\ E_1$:
1. $E[\text{dyn } \tau\ e] = v_0\ E_1[\text{dyn } \tau\ e]$
2. $\vdash_1 E_1[\text{dyn } \tau\ e] : K_1$
   by *inversion*
3. qed by the induction hypothesis (2)

case $E = \langle E_0, e_1\rangle$:
1. $E[\text{dyn } \tau\ e] = \langle E_0[\text{dyn } \tau\ e], e_1\rangle$
2. $\vdash_1 E_0[\text{dyn } \tau\ e] : K_0$
   by *inversion*
3. qed by the induction hypothesis (2)

case $E = \langle v_0, E_1\rangle$:
1. $E[\text{dyn } \tau\ e] = \langle v_0, E_1[\text{dyn } \tau\ e]\rangle$
2. $\vdash_1 E_1[\text{dyn } \tau\ e] : K_1$
   by *inversion*
3. qed by the induction hypothesis (2)

case $E = op^1\ E_0$:
1. $E[\text{dyn } \tau\ e] = op^1\ E_0[\text{dyn } \tau\ e]$
2. $\vdash_1 E_0[\text{dyn } \tau\ e] : K_0$
   by *inversion*
3. qed by the induction hypothesis (2)

case $E = op^2\ E_0\ e_1$:
1. $E[\text{dyn } \tau\ e] = op^2\ E_0[\text{dyn } \tau\ e]\ e_1$
2. $\vdash_1 E_0[\text{dyn } \tau\ e] : K_0$
   by *inversion*
3. qed by the induction hypothesis (2)

case $E = op^2\ v_0\ E_1$:
1. $E[\text{dyn } \tau\ e] = op^2\ v_0\ E_1[\text{dyn } \tau\ e]$
2. $\vdash_1 E_1[\text{dyn } \tau\ e] : K_1$
   by *inversion*
3. qed by the induction hypothesis (2)

case $E = \text{dyn } E_0$:
1. $E[\text{dyn } \tau\ e] = \text{dyn } E_0[\text{dyn } \tau\ e]$
2. $\vdash_1 E_0[\text{dyn } \tau\ e]$
   by *inversion*
3. qed by *dynamic* dyn *hole typing* (2)

case $E = \text{stat } E_0$:
1. Contradiction by $\vdash_1 E[\text{dyn } \tau\ e] : \tau'$

case $E = \text{dyn } \tau_0\ E_0$:
1. $E[\text{dyn } \tau\ e] = \text{dyn } \tau_0\ E_0[\text{dyn } \tau\ e]$
2. $\vdash_1 E_0[\text{dyn } \tau\ e]$
   by *inversion*
3. qed by *dynamic* dyn *hole typing* (2)

case $E = \text{stat } \tau_0\ E_0$:
1. Contradiction by $\vdash_1 E[\text{dyn } \tau\ e] : \tau'$

case $E = \text{chk } K_0\ E_0$:
1. $E[\text{dyn } \tau\ e] = \text{chk } K_0\ E_0[\text{dyn } \tau\ e]$
2. $\vdash_1 E_0[\text{dyn } \tau\ e] : \text{Any}$
   by *inversion*
3. qed by the induction hypothesis (2)

□

Lemma 4.26 : 1 *dynamic* dyn *hole typing*

If $\vdash_1 E[\text{dyn } \tau\ e]$ then the derivation contains a sub-term $\vdash_1 \text{dyn } \tau\ e : \lfloor\tau\rfloor$.

Proof:

By induction on the structure of E.

case $E \in E^\bullet$:
1. Contradiction by $\vdash_1 E[\text{dyn } \tau\ e]$

case $E = E_0\ e_1$:
1. $E[\text{dyn } \tau\ e] = E_0[\text{dyn } \tau\ e]\ e_1$
2. $\vdash_1 E_0[\text{dyn } \tau\ e]$
   by *inversion*
3. qed by the induction hypothesis (2)

case $E = v_0\ E_1$:
1. $E[\text{dyn } \tau\ e] = v_0\ E_1[\text{dyn } \tau\ e]$
2. $\vdash_1 E_1[\text{dyn } \tau\ e]$
   by *inversion*
3. qed by the induction hypothesis (2)

case $E = \langle E_0, e_1\rangle$:

1. $E[\mathsf{dyn}\ \tau\ e] = \langle E_0[\mathsf{dyn}\ \tau\ e], e_1\rangle$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1\rangle$:
1. $E[\mathsf{dyn}\ \tau\ e] = \langle v_0, E_1[\mathsf{dyn}\ \tau\ e]\rangle$
2. $\vdash_1 E_1[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[\mathsf{dyn}\ \tau\ e] = op^1\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ E_0[\mathsf{dyn}\ \tau\ e]\ e_1$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_1 E_1[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{dyn}\ \tau\ e]$

CASE $E = \mathsf{stat}\ E_0$:
1. $E[\mathsf{dyn}\ \tau\ e] = \mathsf{stat}\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e] : \lfloor \tau_0 \rfloor$
   by *inversion*
3. QED by *static* dyn *hole typing* (2)

CASE $E = \mathsf{dyn}\ \tau\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{dyn}\ \tau\ e]$

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[\mathsf{dyn}\ \tau\ e] = \mathsf{stat}\ \tau_0\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e] : \lfloor \tau_0 \rfloor$
   by *inversion*
3. QED by *static* dyn *hole typing* (2)

CASE $E = \mathsf{chk}\ K_0\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{dyn}\ \tau\ e]$

□

Lemma 4.27 : 1 *static* stat *hole typing*

If $\vdash_1 E[\mathsf{stat}\ \tau\ e] : K'$ then the derivation contains a sub-term $\vdash_1$ stat τ e.

Proof:

By induction on the structure of E.

CASE $E \in E^\bullet$:
1. Contradiction by $\vdash_1 E[\mathsf{stat}\ \tau\ e] : \tau'$

CASE $E = E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e] : K_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_1[\mathsf{stat}\ \tau\ e] : K_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1\rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle E_0[\mathsf{stat}\ \tau\ e], e_1\rangle$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e] : K_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1\rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle v_0, E_1[\mathsf{stat}\ \tau\ e]\rangle$
2. $\vdash_1 E_1[\mathsf{stat}\ \tau\ e] : K_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = op^1\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e] : K_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e] : K_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_1[\mathsf{stat}\ \tau\ e] : K_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{dyn}\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by *dynamic* stat *hole typing* (2)

CASE $E = \mathsf{stat}\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{stat}\ \tau\ e] : \tau'$

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{dyn}\ \tau_0\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by *dynamic* stat *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{stat}\ \tau\ e] : \tau'$

CASE $E = \mathsf{chk}\ K_0\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{chk}\ K_0\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e] : \mathsf{Any}$
   by *inversion*
3. QED by the induction hypothesis (2)

□

Lemma 4.28 : 1 *dynamic* stat *hole typing*

If $\vdash_1$ $E[\mathsf{stat}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_1$ stat τ e.

Proof:

By induction on the structure of E.

case $E \in E^\bullet$:
1. qed by *dynamic hole typing*

case $E = E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. qed by the induction hypothesis (2)

case $E = v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_1[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. qed by the induction hypothesis (2)

case $E = \langle E_0, e_1 \rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle E_0[\mathsf{stat}\ \tau\ e], e_1 \rangle$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. qed by the induction hypothesis (2)

case $E = \langle v_0, E_1 \rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle v_0, E_1[\mathsf{stat}\ \tau\ e] \rangle$
2. $\vdash_1 E_1[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. qed by the induction hypothesis (2)

case $E = op^1\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = op^1\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. qed by the induction hypothesis (2)

case $E = op^2\ E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. qed by the induction hypothesis (2)

case $E = op^2\ v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_1[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. qed by the induction hypothesis (2)

case $E = \mathsf{dyn}\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{stat}\ \tau\ e]$

case $E = \mathsf{stat}\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{stat}\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e] : \lfloor \tau_0 \rfloor$
   by *inversion*
3. qed by *static* stat *hole typing* (2)

case $E = \mathsf{dyn}\ \tau\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{stat}\ \tau\ e]$

case $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{stat}\ \tau_0\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e] : \lfloor \tau_0 \rfloor$
   by *inversion*
3. qed by *static* stat *hole typing* (2)

case $E = \mathsf{chk}\ K_0\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{stat}\ \tau\ e]$

□

Lemma 4.29 : 1 *static hole substitution*

If $\vdash_1 E^\bullet[e] : K$ and the derivation contains a sub-term $\vdash_1 e : K'$ and $\vdash_1 e' : K'$, then $\vdash_1 E^\bullet[e'] : K$

Proof:

By induction on the structure of $E^\bullet$.

case $E^\bullet = [\,]$:
1. $E^\bullet[e] = e$
   $\wedge\ E^\bullet[e'] = e'$
2. $\vdash_1 e : K$
   by (1)
3. $K' = K$
4. qed

case $E^\bullet = \langle E^\bullet{}_0, e_1 \rangle$:
1. $E^\bullet[e] = \langle E^\bullet{}_0[e], e_1 \rangle$
   $\wedge\ E^\bullet[e'] = \langle E^\bullet{}_0[e'], e_1 \rangle$
2. $\vdash_1 \langle E^\bullet{}_0[e], e_1 \rangle : K$
3. $\vdash_1 E^\bullet{}_0[e] : K_0$
   $\wedge \vdash_1 e_1 : K_1$
   by *inversion*
4. $\vdash_1 E^\bullet{}_0[e'] : K_0$
   by the induction hypothesis (3)
5. $\vdash_1 \langle E^\bullet{}_0[e'], e_1 \rangle : K$
   by (2, 3, 4)
6. qed by (1, 5)

case $E^\bullet = \langle v_0, E^\bullet{}_1 \rangle$:
1. $E^\bullet[e] = \langle v_0, E^\bullet{}_1[e] \rangle$
   $\wedge\ E^\bullet[e'] = \langle v_0, E^\bullet{}_1[e'] \rangle$
2. $\vdash_1 \langle v_0, E^\bullet{}_1[e] \rangle : K$
3. $\vdash_1 v_0 : K_0$
   $\wedge \vdash_1 E^\bullet{}_1[e] : K_1$
   by *inversion*
4. $\vdash_1 E^\bullet{}_1[e'] : K_1$
   by the induction hypothesis (3)
5. $\vdash_1 \langle v_0, E^\bullet{}_1[e'] \rangle : K$
   by (2, 3, 4)
6. qed by (1, 5)

case $E^\bullet = E^\bullet{}_0\ e_1$:
1. $E^\bullet[e] = E^\bullet{}_0[e]\ e_1$
   $\wedge\ E^\bullet[e'] = E^\bullet{}_0[e']\ e_1$
2. $\vdash_1 E^\bullet{}_0[e]\ e_1 : K$
3. $\vdash_1 E^\bullet{}_0[e] : K_0$
   $\wedge \vdash_1 e_1 : K_1$
   by *inversion*
4. $\vdash_1 E^\bullet{}_0[e'] : K_0$
   by the induction hypothesis (3)
5. $\vdash_1 E^\bullet{}_0[e']\ e_1 : K$
   by (2, 3, 4)
6. qed by (1, 5)

case $E^\bullet = v_0\ E^\bullet{}_1$:
1. $E^\bullet[e] = v_0\ E^\bullet{}_1[e]$
   $\wedge\ E^\bullet[e'] = v_0\ E^\bullet{}_1[e']$
2. $\vdash_1 v_0\ E^\bullet{}_1[e] : K$

3. $\vdash v_0 : K_0$
 $\wedge \vdash E^{\bullet}{}_1[e] : K_1$
 by *inversion*
4. $\vdash E^{\bullet}{}_1[e'] : K_1$
 by the induction hypothesis (3)
5. $\vdash v_0\ E^{\bullet}{}_1[e'] : K$
 by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^1\ E^{\bullet}{}_0$:

1. $E^{\bullet}[e] = op^1\ E^{\bullet}{}_0[e]$
 $\wedge E^{\bullet}[e'] = op^1\ E^{\bullet}{}_0[e']$
2. $\vdash op^1\ E^{\bullet}{}_0[e] : K$
3. $\vdash E^{\bullet}{}_0[e] : K_0$
 by *inversion*
4. $\vdash E^{\bullet}{}_0[e'] : K_0$
 by the induction hypothesis (3)
5. $\vdash op^1\ E^{\bullet}{}_0[e'] : K$
 by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ E^{\bullet}{}_0\ e_1$:

1. $E^{\bullet}[e] = op^2\ E^{\bullet}{}_0[e]\ e_1$
 $\wedge E^{\bullet}[e'] = op^2\ E^{\bullet}{}_0[e']\ e_1$
2. $\vdash op^2\ E^{\bullet}{}_0[e]\ e_1 : K$
3. $\vdash E^{\bullet}{}_0[e] : K_0$
 $\wedge \vdash e_1 : K_1$
 by *inversion*
4. $\vdash E^{\bullet}{}_0[e'] : K_0$
 by the induction hypothesis (3)
5. $\vdash op^2\ E^{\bullet}{}_0[e']\ e_1 : K$
 by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ v_0\ E^{\bullet}{}_1$:

1. $E^{\bullet}[e] = op^2\ v_0\ E^{\bullet}{}_1[e]$
 $\wedge E^{\bullet}[e'] = op^2\ v_0\ E^{\bullet}{}_1[e']$
2. $\vdash op^2\ v_0\ E^{\bullet}{}_1[e] : K$
3. $\vdash v_0 : K_0$
 $\wedge \vdash E^{\bullet}{}_1[e] : K_1$
 by *inversion*
4. $\vdash E^{\bullet}{}_1[e'] : K_1$
 by the induction hypothesis (3)
5. $\vdash op^2\ v_0\ E^{\bullet}{}_1[e'] : K$
 by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \mathrm{chk}\ K_c\ E^{\bullet}{}_0$:

1. $E^{\bullet}[e] = \mathrm{chk}\ K_c\ E^{\bullet}{}_0[e]$
 $\wedge E^{\bullet}[e'] = \mathrm{chk}\ K_c\ E^{\bullet}{}_0[e']$
2. $\vdash \mathrm{chk}\ K_c\ E^{\bullet}{}_0[e] : K$
3. $\vdash E^{\bullet}{}_0[e] : K_0$
 by *inversion*
4. $\vdash E^{\bullet}{}_0[e'] : K_0$
 by the induction hypothesis (3)
5. $\vdash \mathrm{chk}\ K_c\ E^{\bullet}{}_0[e'] : K$
 by (2, 3, 4)
6. QED by (1, 5)

□

Lemma 4.30 : 1 *dynamic hole substitution*

If $\vdash E^{\bullet}[e]$ and $\vdash e'$ then $\vdash E^{\bullet}[e']$

Proof:

By induction on the structure of $E^{\bullet}$.

CASE $E^{\bullet} = [\,]$:

1. QED $E^{\bullet}[e'] = e'$

CASE $E^{\bullet} = \langle E^{\bullet}{}_0, e_1\rangle$:

1. $E^{\bullet}[e] = \langle E^{\bullet}{}_0[e], e_1\rangle$
 $\wedge E^{\bullet}[e'] = \langle E^{\bullet}{}_0[e'], e_1\rangle$
2. $\vdash \langle E^{\bullet}{}_0[e], e_1\rangle$
3. $\vdash E^{\bullet}{}_0[e]$
 $\wedge \vdash e_1$
 by *inversion*
4. $\vdash E^{\bullet}{}_0[e']$
 by the induction hypothesis (3)
5. $\vdash \langle E^{\bullet}{}_0[e'], e_1\rangle$
 by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \langle v_0, E^{\bullet}{}_1\rangle$:

1. $E^{\bullet}[e] = \langle v_0, E^{\bullet}{}_1[e]\rangle$
 $\wedge E^{\bullet}[e'] = \langle v_0, E^{\bullet}{}_1[e']\rangle$
2. $\vdash \langle v_0, E^{\bullet}{}_1[e]\rangle$
3. $\vdash v_0$
 $\wedge \vdash E^{\bullet}{}_1[e]$
 by *inversion*
4. $\vdash E^{\bullet}{}_1[e']$
 by the induction hypothesis (3)
5. $\vdash \langle v_0, E^{\bullet}{}_1[e']\rangle$
 by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = E^{\bullet}{}_0\ e_1$:

1. $E^{\bullet}[e] = E^{\bullet}{}_0[e]\ e_1$
 $\wedge E^{\bullet}[e'] = E^{\bullet}{}_0[e']\ e_1$
2. $\vdash E^{\bullet}{}_0[e]\ e_1$
3. $\vdash E^{\bullet}{}_0[e]$
 $\wedge \vdash e_1$
 by *inversion*
4. $\vdash E^{\bullet}{}_0[e']$
 by the induction hypothesis (3)
5. $\vdash E^{\bullet}{}_0[e']\ e_1$
 by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = v_0\ E^{\bullet}{}_1$:

1. $E^{\bullet}[e] = v_0\ E^{\bullet}{}_1[e]$
 $\wedge E^{\bullet}[e'] = v_0\ E^{\bullet}{}_1[e']$
2. $\vdash v_0\ E^{\bullet}{}_1[e]$
3. $\vdash v_0$
 $\wedge \vdash E^{\bullet}{}_1[e]$
 by *inversion*
4. $\vdash E^{\bullet}{}_1[e']$
 by the induction hypothesis (3)

5. $\vdash_1 v_0\ E^{\bullet}{}_1[e']$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^1\ E^{\bullet}{}_0$:

1. $E^{\bullet}[e] = op^1\ E^{\bullet}{}_0[e]$
   $\wedge\ E^{\bullet}[e'] = op^1\ E^{\bullet}{}_0[e']$
2. $\vdash_1 op^1\ E^{\bullet}{}_0[e]$
3. $\vdash_1 E^{\bullet}{}_0[e]$
   by *inversion*
4. $\vdash_1 E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_1 op^1\ E^{\bullet}{}_0[e']$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ E^{\bullet}{}_0\ e_1$:

1. $E^{\bullet}[e] = op^2\ E^{\bullet}{}_0[e]\ e_1$
   $\wedge\ E^{\bullet}[e'] = op^2\ E^{\bullet}{}_0[e']\ e_1$
2. $\vdash_1 op^2\ E^{\bullet}{}_0[e]\ e_1$
3. $\vdash_1 E^{\bullet}{}_0[e]$
   $\wedge \vdash_1 e_1$
   by *inversion*
4. $\vdash_1 E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_1 op^2\ E^{\bullet}{}_0[e']\ e_1$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ v_0\ E^{\bullet}{}_1$:

1. $E^{\bullet}[e] = op^2\ v_0\ E^{\bullet}{}_1[e]$
   $\wedge\ E^{\bullet}[e'] = op^2\ v_0\ E^{\bullet}{}_1[e']$
2. $\vdash_1 op^2\ v_0\ E^{\bullet}{}_1[e]$
3. $\vdash_1 v_0$
   $\wedge \vdash_1 E^{\bullet}{}_1[e]$
   by *inversion*
4. $\vdash_1 E^{\bullet}{}_1[e']$
   by the induction hypothesis (3)
5. $\vdash_1 op^2\ v_0\ E^{\bullet}{}_1[e']$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \mathsf{chk}\ K_c\ E^{\bullet}{}_0$:

1. Contradiction by $\vdash_1 E^{\bullet}[e]$

□

Lemma 4.31 : 1 *hole substitution*

- If $\vdash_1 E[e] : K$ and the derivation contains a sub-term $\vdash_1 e : K'$ and $\vdash_1 e' : K'$ then $\vdash_1 E[e'] : K$.
- If $\vdash_1 E[e] : K$ and the derivation contains a sub-term $\vdash_1 e$ and $\vdash_1 e'$ then $\vdash_1 E[e'] : K$.
- If $\vdash_1 E[e]$ and the derivation contains a sub-term $\vdash_1 e : K'$ and $\vdash_1 e' : K'$ then $\vdash_1 E[e']$.
- If $\vdash_1 E[e]$ and the derivation contains a sub-term $\vdash_1 e$ and $\vdash_1 e'$ then $\vdash_1 E[e']$.

Proof:

By the following four lemmas: *dynamic context static hole substitution*, *dynamic context dynamic hole substitution*, *static context static hole substitution*, and *static context dynamic hole substitution.*

□

Lemma 4.32 : 1 *dynamic context static hole substitution*

If $\vdash_1 E[e]$ and contains $\vdash_1 e : K'$, and furthermore $\vdash_1 e' : K'$, then $\vdash_1 E[e']$

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:

1. Contradiction by $\vdash_1 E[e]$

CASE $E = E_0\ e_1$:

1. $E[e] = E_0[e]\ e_1$
2. $\vdash_1 E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:

1. $E[e] = v_0\ E_1[e]$
2. $\vdash_1 E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:

1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_1 E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:

1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_1 E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:

1. $E[e] = op^1\ E_0[e]$
2. $\vdash_1 E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:

1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_1 E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:

1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_1 E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ E_0$:

1. Contradiction by $\vdash_1 E[e]$

CASE $E = \mathsf{stat}\ E_0$:

1. $E[e] = \mathsf{stat}\ E_0[e]$
2. $\vdash_1 E_0[e] : \mathsf{Any}$
   by *inversion*
3. QED by *static context static hole substitution* (2)

CASE $E = \mathsf{dyn}\ \tau''\ E_0$:

1. Contradiction by $\vdash_1 E[e]$

case $E = \mathsf{stat}\ \tau_0\ E_0$:

1. $E[e] = \mathsf{stat}\ \tau_0\ E_0[e]$
2. $\vdash_1 E_0[e] : \lfloor \tau_0 \rfloor$
   by *inversion*
3. QED by *static context static hole substitution* (2)

case $E = \mathsf{chk}\ K_0\ E_0$:

1. Contradiction by $\vdash_1 E[e]$

□

Lemma 4.33 : 1 *dynamic context dynamic hole substitution*

If $\vdash_1 E[e]$ and contains $\vdash_1 e$, and furthermore $\vdash_1 e'$, then $\vdash_1 E[e']$

Proof:

By induction on the structure of E.

case $E \in E^\bullet$:

1. QED by *dynamic boundary-free hole substitution*

case $E = E_0\ e_1$:

1. $E[e] = E_0[e]\ e_1$
2. $\vdash_1 E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

case $E = v_0\ E_1$:

1. $E[e] = v_0\ E_1[e]$
2. $\vdash_1 E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

case $E = \langle E_0, e_1 \rangle$:

1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_1 E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

case $E = \langle v_0, E_1 \rangle$:

1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_1 E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

case $E = op^1\ E_0$:

1. $E[e] = op^1\ E_0[e]$
2. $\vdash_1 E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

case $E = op^2\ E_0\ e_1$:

1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_1 E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

case $E = op^2\ v_0\ E_1$:

1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_1 E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

case $E = \mathsf{dyn}\ E_0$:

1. Contradiction by $\vdash_1 E[e]$

case $E = \mathsf{stat}\ E_0$:

1. $E[e] = \mathsf{stat}\ E_0[e]$
2. $\vdash_1 E_0[e] : \mathsf{Any}$
   by *inversion*
3. QED by *static context dynamic hole substitution* (2)

case $E = \mathsf{dyn}\ \tau''\ E_0$:

1. Contradiction by $\vdash_1 E[e]$

case $E = \mathsf{stat}\ \tau_0\ E_0$:

1. $E[e] = \mathsf{stat}\ \tau_0\ E_0[e]$
2. $\vdash_1 E_0[e] : \lfloor \tau_0 \rfloor$
   by *inversion*
3. QED by *static context dynamic hole substitution* (2)

case $E = \mathsf{chk}\ K_0\ E_0$:

1. Contradiction by $\vdash_1 E[e]$

□

Lemma 4.34 : 1 *static context static hole substitution*

If $\vdash_1 E[e] : K$ and contains $\vdash_1 e : K'$, and furthermore $\vdash_1 e' : K'$, then $\vdash_1 E[e'] : K$

Proof:

By induction on the structure of E.

case $E \in E^\bullet$:

1. QED by *static boundary-free hole substitution*

case $E = E_0\ e_1$:

1. $E[e] = E_0[e]\ e_1$
2. $\vdash_1 E_0[e] : K_0$
   by *inversion*
3. QED by the induction hypothesis (2)

case $E = v_0\ E_1$:

1. $E[e] = v_0\ E_1[e]$
2. $\vdash_1 E_1[e] : K_1$
   by *inversion*
3. QED by the induction hypothesis (2)

case $E = \langle E_0, e_1 \rangle$:

1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_1 E_0[e] : K_0$
   by *inversion*
3. QED by the induction hypothesis (2)

case $E = \langle v_0, E_1 \rangle$:

1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_1 E_1[e] : K_1$
   by *inversion*
3. QED by the induction hypothesis (2)

case $E = op^1\ E_0$:

1. $E[e] = op^1\ E_0[e]$
2. $\vdash_1 E_0[e] : K_0$
   by *inversion*
3. QED by the induction hypothesis (2)

case $E = op^2\ E_0\ e_1$:

1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_1 E_0[e] : K_0$
   by *inversion*
3. QED by the induction hypothesis (2)

case $E = op^2\ v_0\ E_1$:

1. $E[e] = op^2\ v_0\ E_1[e]$

2. $\vdash_1 E_1[e] : K_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ E_0$:
1. $E[e] = \mathsf{dyn}\ E_0[e]$
2. $\vdash_1 E_0[e]$
   by *inversion*
3. QED by *static* dyn *hole typing* (2)

CASE $E = \mathsf{stat}\ E_0$:
1. Contradiction by $\vdash_1 E[e] : K$

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{dyn}\ \tau_0\ E_0[e]$
2. $\vdash_1 E_0[e]$
   by *inversion*
3. QED by *static* dyn *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_1 E[e] : K$

CASE $E = \mathsf{chk}\ K_0\ E_0$:
1. $E[e] = \mathsf{chk}\ K_0\ E_0[e]$
2. $\vdash_1 E_0[e] : \mathsf{Any}$
   by *inversion*
3. QED by the induction hypothesis (2)

□

Lemma 4.35 : 1 *static context dynamic hole substitution*

If $\vdash_1 E[e] : K$ and contains $\vdash_1 e$, and furthermore $\vdash_1 e'$, then $\vdash_1 E[e'] : K$

Proof:

By induction on the structure of E.

CASE $E \in E^\bullet$:
1. Contradiction by $\vdash_1 E[e] : K$

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_1 E_0[e] : K_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_1 E_1[e] : K_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1\rangle$:
1. $E[e] = \langle E_0[e], e_1\rangle$
2. $\vdash_1 E_0[e] : K_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1\rangle$:
1. $E[e] = \langle v_0, E_1[e]\rangle$
2. $\vdash_1 E_1[e] : K_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_1 E_0[e] : K_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_1 E_0[e] : K_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_1 E_1[e] : K_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ E_0$:
1. $E[e] = \mathsf{dyn}\ E_0[e]$
2. $\vdash_1 E_0[e]$
   by *inversion*
3. QED by *dynamic* stat *hole typing* (2)

CASE $E = \mathsf{stat}\ E_0$:
1. Contradiction by $\vdash_1 E[e] : K$

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{dyn}\ \tau_0\ E_0[e]$
2. $\vdash_1 E_0[e]$
   by *inversion*
3. QED by *dynamic* stat *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_1 E[e] : K$

CASE $E = \mathsf{chk}\ K_0\ E_0$:
1. $E[e] = \mathsf{chk}\ K_0\ E_0[e]$
2. $\vdash_1 E_0[e] : \mathsf{Any}$
   by *inversion*
3. QED by the induction hypothesis (2)

□

Lemma 4.36 : 1 *static inversion*

- If $\vdash_1 \langle e_0, e_1\rangle : K$ then $\vdash_1 e_0 : \mathsf{Any}$ and $\vdash_1 e_1 : \mathsf{Any}$
- If $\vdash_1 \lambda x.\, e : K$ then $x \vdash_1 e$
- If $\vdash_1 \lambda(x\!:\!\tau).\, e : K$ then $(x\!:\!\tau) \vdash_1 e : \mathsf{Any}$
- If $\vdash_1 e_0\ e_1 : K$ then $K = \mathsf{Any}$ and $\vdash_1 e_0 : \mathsf{Fun}$ and $\vdash_1 e_1 : \mathsf{Any}$
- If $\vdash_1 op^1\ e_0 : K$ then $K = \mathsf{Any}$ and $\vdash_1 e_0 : \mathsf{Pair}$
- If $\vdash_1 op^2\ e_0\ e_1 : K$ then $\vdash_1 e_0 : K_0$ and $\vdash_1 e_1 : K_1$ and $\Delta(op^2, K_0, K_1) = K'$ and $K' <: K$
- If $\vdash_1 \mathsf{dyn}\ \tau\ e : K$ then $\vdash_1 e$ and $\lfloor\tau\rfloor \leqslant: K$
- If $\vdash_1 \mathsf{chk}\ K'\ e_0 : K$ then $\vdash_1 e_0 : \mathsf{Any}$ and $K' \leqslant: K$

Proof:

QED by the definition of $\Gamma \vdash_1 e : \tau$

□

Lemma 4.37 : 1 *dynamic inversion*

- If $\vdash_1 \langle e_0, e_1\rangle$ then $\vdash_1 e_0$ and $\vdash_1 e_1$
- If $\vdash_1 \lambda x.\, e$ then $x \vdash_1 e$
- If $\vdash_1 \lambda(x\!:\!\tau).\, e$ then $(x\!:\!\tau) \vdash_1 e : \mathsf{Any}$
- If $\vdash_1 e_0\ e_1$ then $\vdash_1 e_0$ and $\vdash_1 e_1$
- If $\vdash_1 op^1\ e_0$ then $\vdash_1 e_0$
- If $\vdash_1 op^2\ e_0\ e_1$ then $\vdash_1 e_0$ and $\vdash_1 e_1$
- If $\vdash_1 \mathsf{stat}\ \tau\ e$ then $\vdash_1 e : \lfloor\tau\rfloor$
- If $\vdash_1 \mathsf{stat}\ e$ then $\vdash_1 e : \mathsf{Any}$

Proof:

QED by the definition of $\vdash_1 e$.

□

Lemma 4.38 : 1 *canonical forms*

- If $\vdash_1 v : \mathsf{Pair}$ then $v = \langle v_0, v_1\rangle$
- If $\vdash_1 v : \mathsf{Fun}$ then $v = \lambda x.\, e'$ or $v = \lambda(x\!:\!\tau_d).\, e'$
- If $\vdash_1 v : \mathsf{Int}$ then $v = i$
- If $\vdash_1 v : \mathsf{Nat}$ then $v \in \mathbb{N}$

Proof:

QED by definition of $\vdash_1 \cdot : K$

□

Lemma 4.39 : Δ *tag soundness*

If $\vdash_1 v_0 : K_0$ and $\vdash_1 v_1 : K_1$ and $\Delta(op^2, K_0, K_1) = K$ then $\vdash_1 \delta(op^2, v_0, v_1) : K$.

Proof:

By case analysis on Δ.

CASE $\Delta(\mathsf{sum}, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat}$:

1. $v_0 = i_0, i_0 \in \mathbb{N}$
   $\wedge\ v_1 = i_1, i_1 \in \mathbb{N}$
   by *canonical forms*
2. $\delta(\mathsf{sum}, i_0, i_1) = i_0 + i_1 \in \mathbb{N}$
3. QED

CASE $\Delta(\mathsf{sum}, \mathsf{Int}, \mathsf{Int}) = \mathsf{Int}$:

1. $v_0 = i_0$
   $\wedge\ v_1 = i_1$
   by *canonical forms*
2. $\delta(\mathsf{sum}, i_0, i_1) = i_0 + i_1 \in i$
3. QED

CASE $\Delta(\mathsf{quotient}, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat}$:

1. $v_0 = i_0, i_0 \in \mathbb{N}$
   $\wedge\ v_1 = i_1, i_1 \in \mathbb{N}$
   by *canonical forms*
2. IF $i_1 = 0$:
   a. $\delta(\mathsf{quotient}, i_0, i_1) = \mathsf{BndryErr}$
   b. QED by $\vdash_1 \mathsf{BndryErr} : K$
   ELSE $i_1 \neq 0$:
   a. $\delta(\mathsf{quotient}, i_0, i_1) = \lfloor i_0 / i_1\rfloor \in \mathbb{N}$
   b. QED

CASE $\Delta(\mathsf{quotient}, \mathsf{Int}, \mathsf{Int}) = \mathsf{Int}$:

1. $v_0 = i_0$
   $\wedge\ v_1 = i_1$
   by *canonical forms*
2. IF $i_1 = 0$:
   a. $\delta(\mathsf{quotient}, i_0, i_1) = \mathsf{BndryErr}$
   b. QED by $\vdash_1 \mathsf{BndryErr} : K$
   ELSE $i_1 \neq 0$:
   a. $\delta(\mathsf{quotient}, i_0, i_1) = \lfloor i_0 / i_1\rfloor \in i$
   b. QED

□

Lemma 4.40 : δ *preservation*

- If $\vdash_1 v$ and $\delta(op^1, v) = e$ then $\vdash_1 e$
- If $\vdash_1 v_0$ and $\vdash_1 v_1$ and $\delta(op^2, v_0, v_1) = e$ then $\vdash_1 e$

Proof:

CASE $\delta(\mathsf{fst}, \langle v_0, v_1\rangle) = v_0$:

1. $\vdash_1 v_0$
   by *inversion*
2. QED

CASE $\delta(\mathsf{snd}, \langle v_0, v_1\rangle) = v_1$:

1. $\vdash_1 v_1$
   by *inversion*
2. QED

CASE $\delta(\mathsf{sum}, v_0, v_1) = v_0 + v_1$:

1. QED

CASE $\delta(\mathsf{quotient}, v_0, v_1) = \lfloor v_0 / v_1\rfloor$:

1. QED

CASE $\delta(op^2, v_0, v_1) = \mathsf{BndryErr}$:

1. QED

□

Lemma 4.41 : Δ *preservation*

If $\Delta(op^2, \tau_0, \tau_1) = \tau$ then $\Delta(op^2, \lfloor\tau_0\rfloor, \lfloor\tau_1\rfloor) = \lfloor\tau\rfloor$.

Proof:

By case analysis on the definition of Δ

CASE $\Delta(op^2, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat}$:

1. QED by $\lfloor\mathsf{Nat}\rfloor = \mathsf{Nat}$

CASE $\Delta(op^2, \mathsf{Int}, \mathsf{Int}) = \mathsf{Int}$:

1. QED by $\lfloor\mathsf{Int}\rfloor = \mathsf{Int}$

□

Lemma 4.42 : Δ *inversion*

- If $\Delta(\mathsf{fst}, \tau) = \tau'$ then $\tau = \tau_0 \times \tau_1$ and $\tau' = \tau_0$
- If $\Delta(\mathsf{snd}, \tau) = \tau'$ then $\tau = \tau_0 \times \tau_1$ and $\tau' = \tau_1$

Proof:

QED by the definition of Δ

□

Lemma 4.43 : <: *preservation*

If $\tau <: \tau'$ then $\lfloor\tau\rfloor \leqslant: \lfloor\tau'\rfloor$

Proof:

By case analysis on the last rule used to show $\tau <: \tau'$.

CASE $\mathsf{Nat} <: \mathsf{Int}$:

1. QED $\lfloor\mathsf{Nat}\rfloor <: \lfloor\mathsf{Int}\rfloor$

CASE $\tau_d \Rightarrow \tau_c <: \tau_d' \Rightarrow \tau_c'$:

1. $\lfloor\tau_d \Rightarrow \tau_c\rfloor = \mathsf{Fun}$
   $\wedge\ \lfloor\tau_d' \Rightarrow \tau_c'\rfloor = \mathsf{Fun}$
2. QED

CASE $\tau_0 \times \tau_1 <: \tau_0' \times \tau_1'$:

1. $\lfloor\tau_0 \times \tau_1\rfloor = \mathsf{Pair}$
   $\wedge\ \lfloor\tau_0' \times \tau_1'\rfloor = \mathsf{Pair}$
2. QED

□

Lemma 4.44 : 1 *static value inversion*

If $\vdash_1 v : \mathsf{Any}$ then $\vdash_1 v$

Proof:

By induction on the structure of v.

CASE $v = i$:

1. QED by $\vdash_1 v$

CASE $v = \langle v_0, v_1 \rangle$:

1. $\vdash_1 v_0 : \mathsf{Any}$ $\wedge \vdash_1 v_1 : \mathsf{Any}$ by *inversion*
2. $\vdash_1 v_0$ $\wedge \vdash_1 v_1$ by the induction hypothesis
3. QED by (2)

CASE $v = \lambda x.\, e$:

1. $x \vdash_1 e$ by *inversion*
2. QED

CASE $v = \lambda(x\!:\!\tau).\, e$:

1. $(x\!:\!\tau) \vdash_1 e : \mathsf{Any}$ by *inversion*
2. QED

□

Lemma 4.45 : 1 *dynamic value inversion*

If $\vdash_1 v$ then $\vdash_1 v : \mathsf{Any}$

Proof:

By induction on the structure of v.

CASE $v = i$:

1. $\vdash_1 v : \mathsf{Int}$
2. QED by $\mathsf{Int} <: \mathsf{Any}$

CASE $v = \langle v_0, v_1 \rangle$:

1. $\vdash_1 v_0$ $\wedge \vdash_1 v_1$ by *inversion*
2. $\vdash_1 v_0 : \mathsf{Any}$ $\wedge \vdash_1 v_1 : \mathsf{Any}$ by the induction hypothesis
3. $\vdash_1 \langle v_0, v_1 \rangle : \mathsf{Pair}$ by (2)
4. QED by $\mathsf{Pair} <: \mathsf{Any}$

CASE $v = \lambda x.\, e$:

1. $x \vdash_1 e$ by *inversion*
2. $\vdash_1 \lambda x.\, e : \mathsf{Fun}$ by (1)
3. QED by $\mathsf{Fun} <: \mathsf{Any}$

CASE $v = \lambda(x\!:\!\tau).\, e$:

1. $(x\!:\!\tau) \vdash_1 e : \mathsf{Any}$ by *inversion*
2. $\vdash_1 \lambda(x\!:\!\tau).\, e : \mathsf{Fun}$ by (1)
3. QED by $\mathsf{Fun} <: \mathsf{Any}$

□

Lemma 4.46 : 1 *substitution*

- If $(x\!:\!\tau), \Gamma \vdash_1 e$ and $\vdash_1 v : \lfloor \tau \rfloor$ then $\Gamma \vdash_1 e[x \leftarrow v]$
- If $x, \Gamma \vdash_1 e$ and $\vdash_1 v$ then $\Gamma \vdash_1 e[x \leftarrow v]$
- If $(x\!:\!\tau_x), \Gamma \vdash_1 e : K$ and $\vdash_1 v : \lfloor \tau_x \rfloor$ then $\Gamma \vdash_1 e[x \leftarrow v] : K$
- If $x, \Gamma \vdash_1 e : K$ and $\vdash_1 v$ then $\Gamma \vdash_1 e[x \leftarrow v] : K$

Proof:

By the following four lemmas: *dynamic context static value substitution*, *dynamic context dynamic value substitution*, *static context static value substitution*, and *static context dynamic value substitution.*

□

Lemma 4.47 : 1 *dynamic-static substitution*

If $(x\!:\!\tau), \Gamma \vdash_1 e$ and $\vdash_1 v : \lfloor \tau \rfloor$ then $\Gamma \vdash_1 e[x \leftarrow v]$

Proof:

By induction on the structure of e.

CASE $e = x$:

1. $e[x \leftarrow v] = v$
2. $\vdash_1 v : \mathsf{Any}$ by $\lfloor \tau \rfloor <: \mathsf{Any}$
3. $\vdash_1 v$ by *static value inversion* (2)
4. $\Gamma \vdash_1 v$ by *weakening* (3)
5. QED

CASE $e = x'$:

1. QED by $x'[x \leftarrow v] = x'$

CASE $e = i$:

1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x.\, e'$:

1. QED by $(\lambda x.\, e')[x \leftarrow v] = \lambda x.\, e'$

CASE $e = \lambda(x\!:\!\tau').\, e'$:

1. QED by $(\lambda(x\!:\!\tau').\, e')[x \leftarrow v] = \lambda(x\!:\!\tau').\, e'$

CASE $e = \lambda x'.\, e'$:

1. $e[x \leftarrow v] = \lambda x'.\, (e'[x \leftarrow v])$
2. $x', (x\!:\!\tau), \Gamma \vdash_1 e'$ by *inversion*
3. $x', \Gamma \vdash_1 e'[x \leftarrow v]$ by *dynamic context static value substitution*
4. $\Gamma \vdash_1 \lambda x'.\, e'[x \leftarrow v]$ by (3)
5. QED

CASE $e = \lambda(x'\!:\!\tau').\, e'$:

1. $e[x \leftarrow v] = \lambda(x'\!:\!\tau').\, (e'[x \leftarrow v])$
2. $(x'\!:\!\tau'), (x\!:\!\tau), \Gamma \vdash_1 e' : \mathsf{Any}$ by *inversion*
3. $(x'\!:\!\tau'), \Gamma \vdash_1 e'[x \leftarrow v] : \mathsf{Any}$ by *static context static value substitution*
4. $\Gamma \vdash_1 \lambda(x'\!:\!\tau').\, (e'[x \leftarrow v])$
5. QED

CASE $e = \langle e_0, e_1 \rangle$:

1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $(x\!:\!\tau), \Gamma \vdash_1 e_0$ $\wedge (x\!:\!\tau), \Gamma \vdash_1 e_1$ by *inversion*

3. $\Gamma \vdash_1 e_0[x \leftarrow v]$
   $\land\ \Gamma \vdash_1 e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
   by (3)
5. QED

CASE $e = e_0\ e_1$:

1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $(x : \tau), \Gamma \vdash_1 e_0$
   $\land\ (x : \tau), \Gamma \vdash_1 e_1$
   by *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v]$
   $\land\ \Gamma \vdash_1 e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^1\ e_0$:

1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $(x : \tau), \Gamma \vdash_1 e_0$
   by *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 op^1\ e_0[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:

1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $(x : \tau), \Gamma \vdash_1 e_0$
   $\land\ (x : \tau), \Gamma \vdash_1 e_1$
   by *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v]$
   $\land\ \Gamma \vdash_1 e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \mathsf{dyn}\ \tau'\ e'$
$\lor\ e = \mathsf{dyn}\ e'$
$\lor\ e = \mathsf{chk}\ K'\ e'$:

1. Contradiction by $(x : \tau), \Gamma \vdash_1 e$

CASE $e = \mathsf{stat}\ \tau'\ e'$:

1. $e[x \leftarrow v] = \mathsf{stat}\ \tau'\ e'[x \leftarrow v]$
2. $(x : \tau), \Gamma \vdash_1 e' : \lfloor \tau' \rfloor$
   by *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v] : \lfloor \tau' \rfloor$
   by *static context static value substitution* (2)
4. $\Gamma \vdash_1 \mathsf{stat}\ \tau'\ (e'[x \leftarrow v])$
   by (3)
5. QED

CASE $e = \mathsf{stat}\ e'$:

1. $e[x \leftarrow v] = \mathsf{stat}\ e'[x \leftarrow v]$
2. $(x : \tau), \Gamma \vdash_1 e' : \mathsf{Any}$
   by *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v] : \mathsf{Any}$
   by *static context static value substitution* (2)
4. $\Gamma \vdash_1 \mathsf{stat}\ (e'[x \leftarrow v])$
   by (3)
5. QED

□

Lemma 4.48 : 1 *dynamic-dynamic substitution*

If $x, \Gamma \vdash_1 e$ and $\vdash_1 v$ then $\Gamma \vdash_1 e[x \leftarrow v]$

Proof:

By induction on the structure of e.

CASE $e = x$:

1. $e[x \leftarrow v] = v$
2. $\Gamma \vdash_1 v$
   by *weakening* (3)
3. QED

CASE $e = x'$:

1. QED by $x'[x \leftarrow v] = x'$

CASE $e = i$:

1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x.\ e'$:

1. QED by $(\lambda x.\ e')[x \leftarrow v] = \lambda x.\ e'$

CASE $e = \lambda(x : \tau').\ e'$:

1. QED by $(\lambda(x : \tau').\ e')[x \leftarrow v] = \lambda(x : \tau').\ e'$

CASE $e = \lambda x'.\ e'$:

1. $e[x \leftarrow v] = \lambda x'.\ (e'[x \leftarrow v])$
2. $x', x, \Gamma \vdash_1 e'$
   by *inversion*
3. $x', \Gamma \vdash_1 e'[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 \lambda x'.\ e'[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \lambda(x' : \tau').\ e'$:

1. $e[x \leftarrow v] = \lambda(x' : \tau').\ (e'[x \leftarrow v])$
2. $(x' : \tau'), x, \Gamma \vdash_1 e' : \mathsf{Any}$
   by *inversion*
3. $(x' : \tau'), \Gamma \vdash_1 e'[x \leftarrow v] : \mathsf{Any}$
   by *static context dynamic value substitution*
4. $\Gamma \vdash_1 \lambda(x' : \tau').\ (e'[x \leftarrow v])$
5. QED

CASE $e = \langle e_0, e_1 \rangle$:

1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $x, \Gamma \vdash_1 e_0$
   $\land\ x, \Gamma \vdash_1 e_1$
   by *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v]$
   $\land\ \Gamma \vdash_1 e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
   by (3)
5. QED

CASE $e = e_0\ e_1$:

1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$

2. $x, \Gamma \vdash_1 e_0$
   $\wedge\, x, \Gamma \vdash_1 e_1$
   by *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v]$
   $\wedge\, \Gamma \vdash_1 e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^1\ e_0$:

1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e_0$
   by *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 op^1\ e_0[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:

1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e_0$
   $\wedge\, x, \Gamma \vdash_1 e_1$
   by *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v]$
   $\wedge\, \Gamma \vdash_1 e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \mathsf{dyn}\ \tau'\ e'$
$\vee\ e = \mathsf{dyn}\ e'$
$\vee\ e = \mathsf{chk}\ K'\ e'$:

1. Contradiction by $\Gamma \vdash_1 e$

CASE $e = \mathsf{stat}\ \tau'\ e'$:

1. $e[x \leftarrow v] = \mathsf{stat}\ \tau'\ e'[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e' : \lfloor \tau' \rfloor$
   by *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v] : \lfloor \tau' \rfloor$
   by *static context dynamic value substitution* (2)
4. $\Gamma \vdash_1 \mathsf{stat}\ \tau'\ (e'[x \leftarrow v])$
   by (3)
5. QED

CASE $e = \mathsf{stat}\ e'$:

1. $e[x \leftarrow v] = \mathsf{stat}\ e'[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e' : \mathsf{Any}$
   by *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v] : \mathsf{Any}$
   by *static context dynamic value substitution* (2)
4. $\Gamma \vdash_1 \mathsf{stat}\ (e'[x \leftarrow v])$
   by (3)
5. QED

☐

Lemma 4.49 : 1 *static-static substitution*

If $(x : \tau), \Gamma \vdash_1 e : K$ and $\vdash_1 v : \lfloor \tau \rfloor$ then $\Gamma \vdash_1 e[x \leftarrow v] : K$

Proof:

By induction on the structure of e.

CASE $e = x$:

1. $\lfloor \tau \rfloor \leqslant: K$
   by $(x : \tau), \Gamma \vdash_1 x : K$
2. $e[x \leftarrow v] = v$
3. $\vdash_1 v : K$
   by (1)
4. $\Gamma \vdash_1 v : K$
   by *weakening* (3)
5. QED

CASE $e = x'$:

1. QED by $(x'[x \leftarrow v]) = x'$

CASE $e = i$:

1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x.\, e'$:

1. QED by $(\lambda x.\, e')[x \leftarrow v] = \lambda x.\, e'$

CASE $e = \lambda x'.\, e'$:

1. $e[x \leftarrow v] = \lambda x'.\, (e'[x \leftarrow v])$
2. $x', (x : \tau), \Gamma \vdash_1 e'$
   by *inversion*
3. $x', \Gamma \vdash_1 e'[x \leftarrow v]$
   by *dynamic context static value substitution*
4. $\Gamma \vdash_1 \lambda x'.\, e'[x \leftarrow v] : K$
   by (3)
5. QED

CASE $e = \lambda(x' : \tau').\, e'$:

1. $e[x \leftarrow v] = \lambda(x' : \tau').\, (e'[x \leftarrow v])$
2. $(x' : \tau'), (x : \tau), \Gamma \vdash_1 e' : \mathsf{Any}$
   by *inversion*
3. $(x' : \tau'), \Gamma \vdash_1 e'[x \leftarrow v] : \mathsf{Any}$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 \lambda(x' : \tau').\, (e'[x \leftarrow v]) : K$
5. QED

CASE $e = \langle e_0, e_1 \rangle$:

1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $(x : \tau), \Gamma \vdash_1 e_0 : \mathsf{Any}$
   $\wedge\, (x : \tau), \Gamma \vdash_1 e_1 : \mathsf{Any}$
   by *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v] : \mathsf{Any}$
   $\wedge\, \Gamma \vdash_1 e_1[x \leftarrow v] : \mathsf{Any}$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle : K$
   by (3)
5. QED

CASE $e = e_0\ e_1$:

1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $(x : \tau), \Gamma \vdash_1 e_0 : K_0$
   $\wedge\, (x : \tau), \Gamma \vdash_1 e_1 : K_1$
   by *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$
   $\wedge\, \Gamma \vdash_1 e_1[x \leftarrow v] : K_1$
   by the induction hypothesis (2)

4. $\Gamma \vdash_1 e_0[x \leftarrow v]\ e_1[x \leftarrow v] : K$
by (3)
5. QED

CASE $e = op^1\ e_0$:
1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $(x : \tau), \Gamma \vdash_1 e_0 : K_0$
by *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$
by the induction hypothesis (2)
4. $\Gamma \vdash_1 op^1\ e_0[x \leftarrow v] : K$
by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:
1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $(x : \tau), \Gamma \vdash_1 e_0 : K_0$
$\wedge\ (x : \tau), \Gamma \vdash_1 e_1 : K_1$
by *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$
$\wedge\ \Gamma \vdash_1 e_1[x \leftarrow v] : K_1$
by the induction hypothesis (2)
4. $\Gamma \vdash_1 op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v] : K$
by (3)
5. QED

CASE $e = \mathsf{dyn}\ \tau'\ e'$:
1. $e[x \leftarrow v] = \mathsf{dyn}\ \tau'\ e'[x \leftarrow v]$
2. $(x : \tau), \Gamma \vdash_1 e'$
by *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v]$
by *dynamic context static value substitution* (2)
4. $\Gamma \vdash_1 \mathsf{dyn}\ \tau'\ (e'[x \leftarrow v]) : K$
by (3)
5. QED

CASE $e = \mathsf{dyn}\ e'$:
1. $e[x \leftarrow v] = \mathsf{dyn}\ e'[x \leftarrow v]$
2. $(x : \tau), \Gamma \vdash_1 e'$
by *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v]$
by *dynamic context static value substitution* (2)
4. $\Gamma \vdash_1 \mathsf{dyn}\ (e'[x \leftarrow v]) : K$
by (3)
5. QED

CASE $e = \mathsf{chk}\ K'\ e'$:
1. $e[x \leftarrow v] = \mathsf{chk}\ K'\ (e'[x \leftarrow v])$
2. $(x : \tau), \Gamma \vdash_1 e' : \mathsf{Any}$
by *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v] : \mathsf{Any}$
by the induction hypothesis (2)
4. $\Gamma \vdash_1 \mathsf{chk}\ K'\ (e'[x \leftarrow v]) : K$
by (3)
5. QED

CASE $e = \mathsf{stat}\ \tau'\ e'$
$\vee\ e = \mathsf{stat}\ e'$:
1. Contradiction by $\Gamma \vdash_1 e : K$

□

Lemma 4.50 : 1 *static-dynamic substitution*
If $x, \Gamma \vdash_1 e : K$ and $\vdash_1 v$ then $\Gamma \vdash_1 e[x \leftarrow v] : K$

Proof:
By induction on the structure of e.

CASE $e = x$:
1. $K = \mathsf{Any}$
by $x, \Gamma \vdash_1 x : K$
2. $e[x \leftarrow v] = v$
3. $\vdash_1 v : K$
by *dynamic value inversion*
4. $\vdash_1 v : \mathsf{Any}$
by $K \leqslant: \mathsf{Any}$
5. $\Gamma \vdash_1 v : \mathsf{Any}$
by *weakening* (3)
6. QED

CASE $e = x'$:
1. QED by $x'[x \leftarrow v] = x'$

CASE $e = i$:
1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x.\, e'$:
1. QED by $(\lambda x.\, e')[x \leftarrow v] = \lambda x.\, e'$

CASE $e = \lambda(x : \tau').\, e'$:
1. QED by $(\lambda(x : \tau').\, e')[x \leftarrow v] = \lambda(x : \tau').\, e'$

CASE $e = \lambda x'.\, e'$:
1. $e[x \leftarrow v] = \lambda x'.\, (e'[x \leftarrow v])$
2. $x', x, \Gamma \vdash_1 e'$
by *inversion*
3. $x', \Gamma \vdash_1 e'[x \leftarrow v]$
by *dynamic context dynamic value substitution*
4. $\Gamma \vdash_1 \lambda x'.\, e'[x \leftarrow v] : K$
by (3)
5. QED

CASE $e = \lambda(x' : \tau').\, e'$:
1. $e[x \leftarrow v] = \lambda(x' : \tau').\, (e'[x \leftarrow v])$
2. $(x' : \tau'), x, \Gamma \vdash_1 e' : \mathsf{Any}$
by *inversion*
3. $(x' : \tau'), \Gamma \vdash_1 e'[x \leftarrow v] : \mathsf{Any}$
by *static context dynamic value substitution*
4. $\Gamma \vdash_1 \lambda(x' : \tau').\, (e'[x \leftarrow v]) : K$
5. QED

CASE $e = \langle e_0, e_1 \rangle$:
1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $x, \Gamma \vdash_1 e_0 : \mathsf{Any}$
$\wedge\ x, \Gamma \vdash_1 e_1 : \mathsf{Any}$
by *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v] : \mathsf{Any}$
$\wedge\ \Gamma \vdash_1 e_1[x \leftarrow v] : \mathsf{Any}$
by the induction hypothesis (2)
4. $\Gamma \vdash_1 \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle : K$
by (3)
5. QED

CASE $e = e_0\ e_1$:

1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e_0 : K_0$
   $\wedge\ x, \Gamma \vdash_1 e_1 : K_1$
   by *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$
   $\wedge\ \Gamma \vdash_1 e_1[x \leftarrow v] : K_1$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 e_0[x \leftarrow v]\ e_1[x \leftarrow v] : K$
   by (3)
5. QED

CASE $e = op^1\ e_0$:
1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e_0 : K_0$
   by *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 op^1\ e_0[x \leftarrow v] : K$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:
1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e_0 : K_0$
   $\wedge\ x, \Gamma \vdash_1 e_1 : K_1$
   by *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$
   $\wedge\ \Gamma \vdash_1 e_1[x \leftarrow v] : K_1$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v] : K$
   by (3)
5. QED

CASE $e = \mathsf{dyn}\ \tau'\ e'$:
1. $e[x \leftarrow v] = \mathsf{dyn}\ \tau'\ e'[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e'$
   by *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v]$
   by *dynamic context dynamic value substitution* (2)
4. $\Gamma \vdash_1 \mathsf{dyn}\ \tau'\ (e'[x \leftarrow v]) : K$
   by (3)
5. QED

CASE $e = \mathsf{dyn}\ e'$:
1. $e[x \leftarrow v] = \mathsf{dyn}\ e'[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e'$
   by *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v]$
   by *dynamic context dynamic value substitution* (2)
4. $\Gamma \vdash_1 \mathsf{dyn}\ (e'[x \leftarrow v]) : K$
   by (3)
5. QED

CASE $e = \mathsf{chk}\ K'\ e'$:
1. $e[x \leftarrow v] = \mathsf{chk}\ K'\ (e'[x \leftarrow v])$
2. $x, \Gamma \vdash_1 e' : \mathsf{Any}$
   by *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v] : \mathsf{Any}$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 \mathsf{chk}\ K'\ (e'[x \leftarrow v]) : K$
   by (3)
5. QED

CASE $e = \mathsf{stat}\ \tau'\ e'$
$\vee\ e = \mathsf{stat}\ e'$:
1. Contradiction by $\Gamma \vdash_1 e : K$

□

Lemma 4.51 : *weakening*
- If $\Gamma \vdash_1 e$ then $x, \Gamma \vdash_1 e$
- If $\Gamma \vdash_1 e : \tau$ then $(x{:}\tau'), \Gamma \vdash_1 e : \tau$

Proof:
- e is closed under Γ
  by $\Gamma \vdash_1 e$
  $\vee\ \Gamma \vdash_1 e : \tau$ QED

□

Lemma 4.52 : *unique static evaluation contexts*
If $\vdash e : \tau$ and e is boundary-free then one of the following holds:
- e is a value
- $e = E^\bullet[v_0\ v_1]$
- $e = E^\bullet[op^1\ v]$
- $e = E^\bullet[op^2\ v_0\ v_1]$
- $e = E^\bullet[\mathsf{Err}]$

Proof:
By induction on the structure of e.

CASE $e = x$:
1. Contradiction by $\vdash e : \tau$

CASE $e = i$
$\vee\ e = \lambda(x{:}\tau_d).\ e'$:
1. QED e is a value

CASE $e = \langle e_0, e_1 \rangle$:

IF $e_0 \notin v$:
1. $e_0 = E^\bullet{}_0[e_0']$
   by the induction hypothesis
2. $E^\bullet = \langle E^\bullet{}_0, e_1 \rangle$
3. QED by $e = E^\bullet[e_0']$

IF $e_0 \in v$
$\wedge\ e_1 \notin v$:
1. $e_1 = E^\bullet{}_1[e_1']$
   by the induction hypothesis
2. $E^\bullet = \langle e_0, E^\bullet{}_1 \rangle$
3. QED by $e = E^\bullet[e_1']$

ELSE $e_0 \in v$
$\wedge\ e_1 \in v$:
1. $E^\bullet = [\,]$
2. QED $e = E^\bullet[\langle e_0, e_1 \rangle]$

CASE $e = e_0\ e_1$:

IF $e_0 \notin v$:
1. $e_0 = E^\bullet{}_0[e_0']$
   by the induction hypothesis
2. $E^\bullet = E^\bullet{}_0\ e_1$
3. QED by $e = E^\bullet[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:
1. $e_1 = E^{\bullet}{}_1[e_1']$
   by the induction hypothesis
2. $E^{\bullet} = e_0\ E^{\bullet}{}_1$
3. QED by $e = E^{\bullet}[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E^{\bullet} = [\,]$
2. QED $e = E^{\bullet}[e_0\ e_1]$

CASE $e = op^1\ e_0$:

IF $e_0 \notin v$:
1. $e_0 = E^{\bullet}{}_0[e_0']$
   by the induction hypothesis
2. $E^{\bullet} = op^1\ E^{\bullet}{}_0$
3. QED $e = E^{\bullet}[e_0']$

ELSE $e_0 \in v$:
1. $E^{\bullet} = [\,]$
2. QED $e = E^{\bullet}[op^1\ e_0]$

CASE $e = op^2\ e_0\ e_1$:

IF $e_0 \notin v$:
1. $e_0 = E^{\bullet}{}_0[e_0']$
   by the induction hypothesis
2. $E^{\bullet} = op^2\ E^{\bullet}{}_0\ e_1$
3. QED $e = E^{\bullet}[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:
1. $e_1 = E^{\bullet}{}_1[e_1']$
   by the induction hypothesis
2. $E^{\bullet} = op^2\ e_0\ E^{\bullet}{}_1$
3. QED $e = E^{\bullet}[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E^{\bullet} = [\,]$
2. QED $e = E^{\bullet}[op^2\ e_0\ e_1]$

CASE $e = \mathsf{chk}\ K'\ e'$:
1. Contradiction by $\vdash e : \tau$

CASE $e = \mathsf{dyn}\ e_0$:
1. Contradiction by $\vdash e : \tau$

CASE $e = \mathsf{stat}\ e'$:
1. Contradiction by $\vdash e : \tau$

CASE $e = \mathsf{dyn}\ \tau\ e_0$:
1. QED e is boundary-free

CASE $e = \mathsf{stat}\ \tau\ e'$:
1. Contradiction by $\vdash e : \tau$

CASE $e = \mathsf{Err}$:
1. $E^{\bullet} = [\,]$
2. QED

□

Lemma 4.53 : $\vdash$ *static inversion*

- If $\Gamma \vdash x : \tau$ then $(x\!:\!\tau') \in \Gamma$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash \lambda(x\!:\!\tau_d').\,e' : \tau$ then $(x\!:\!\tau_d'), \Gamma \vdash e' : \tau_c'$ and $\tau_d' \Rightarrow \tau_c' \leqslant: \tau$
- If $\Gamma \vdash \langle e_0, e_1\rangle : \tau$ then $\Gamma \vdash e_0 : \tau_0$ and $\Gamma \vdash e_1 : \tau_1$ and $\tau_0 \times \tau_1 \leqslant: \tau$
- If $\Gamma \vdash e_0\ e_1 : \tau_c$ then $\Gamma \vdash e_0 : \tau_d' \Rightarrow \tau_c'$ and $\Gamma \vdash e_1 : \tau_d'$ and $\tau_c' \leqslant: \tau_c$
- If $\Gamma \vdash \mathsf{fst}\ e : \tau$ then $\Gamma \vdash e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0$ and $\tau_0 \leqslant: \tau$
- If $\Gamma \vdash \mathsf{snd}\ e : \tau$ then $\Gamma \vdash e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1$ and $\tau_1 \leqslant: \tau$
- If $\Gamma \vdash op^2\ e_0\ e_1 : \tau$ then $\Gamma \vdash e_0 : \tau_0$ and $\Gamma \vdash e_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau'$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash \mathsf{dyn}\ \tau'\ e' : \tau$ then $\Gamma \vdash e'$ and $\tau' \leqslant: \tau$

Proof:
QED by the definition of $\Gamma \vdash e : \tau$
□

Lemma 4.54 : *canonical forms*

- If $\vdash v : \tau_0 \times \tau_1$ then $v = \langle v_0, v_1\rangle$
- If $\vdash v : \tau_d \Rightarrow \tau_c$ then $v = \lambda(x\!:\!\tau_x).\,e'$
  $\wedge\, \tau_d \leqslant: \tau_x$
- If $\vdash v : \mathsf{Int}$ then $v = i$
- If $\vdash v : \mathsf{Nat}$ then $v = i$ and $v \in \mathbb{N}$

Proof:
QED by definition of $\vdash e : \tau$
□

Lemma 4.55 : *substitution*

If $(x\!:\!\tau_x), \Gamma \vdash e : \tau$, and e is boundary-free and $\vdash v : \tau_x$ then $\Gamma \vdash e[x \leftarrow v] : \tau$

Proof:
By induction on the structure of e.

CASE $e = x$:
1. $e[x \leftarrow v] = v$
2. $\tau_x = \tau$
3. $\Gamma \vdash v : \tau$
   by *weakening*
4. QED

CASE $e = x'$:
1. QED by $x'[x \leftarrow v] = x'$

CASE $e = i$:
1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x.\,e'$:
1. Contradiction by $(x\!:\!\tau_x), \Gamma \vdash e : \tau$

CASE $e = \lambda(x\!:\!\tau').\,e'$:
1. QED by $(\lambda(x\!:\!\tau').\,e')[x \leftarrow v] = \lambda(x\!:\!\tau').\,e'$

CASE $e = \lambda(x'\!:\!\tau').\,e'$:
1. $e[x \leftarrow v] = \lambda(x'\!:\!\tau').\,(e'[x \leftarrow v])$
2. $(x'\!:\!\tau'), x, \Gamma \vdash e'$
   by *static inversion forms*
3. $(x'\!:\!\tau'), \Gamma \vdash e'[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash \lambda(x'\!:\!\tau').\,(e'[x \leftarrow v])$
   by (3)
5. QED

CASE $e = \langle e_0, e_1 \rangle$:

1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $x, \Gamma \vdash e_0$
   $\wedge\, x, \Gamma \vdash e_1$
   by *static inversion forms*
3. $\Gamma \vdash e_0[x \leftarrow v]$
   $\wedge\, \Gamma \vdash e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
   by (3)
5. QED

CASE $e = e_0\ e_1$:

1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash e_0$
   $\wedge\, x, \Gamma \vdash e_1$
   by *static inversion forms*
3. $\Gamma \vdash e_0[x \leftarrow v]$
   $\wedge\, \Gamma \vdash e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^1\ e_0$:

1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $x, \Gamma \vdash e_0$
   by *static inversion forms*
3. $\Gamma \vdash e_0[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash op^1\ e_0[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:

1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash e_0$
   $\wedge\, x, \Gamma \vdash e_1$
   by *static inversion forms*
3. $\Gamma \vdash e_0[x \leftarrow v]$
   $\wedge\, \Gamma \vdash e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \mathsf{chk}\ K'\ e'$:

1. Contradiction by $\vdash e : \tau$

CASE $e = \mathsf{dyn}\ e'$:

1. Contradiction by $\vdash e : \tau$

CASE $e = \mathsf{stat}\ e'$:

1. Contradiction by $\vdash e : \tau$

CASE $e = \mathsf{dyn}\ \tau'\ e'$:

1. Contradiction by e is boundary-free

CASE $e = \mathsf{stat}\ \tau'\ e'$:

1. Contradiction by $\vdash e : \tau$

CASE $e = \mathsf{Err}$:

1. QED $\mathsf{Err}[x \leftarrow v] = \mathsf{Err}$

□

Lemma 4.56 : *δ preservation*

- If $\vdash v$ and $\delta(op^1, v) = v'$ then $\vdash e'$
- If $\vdash v_0$ and $\vdash v_1$ and $\delta(op^2, v_0, v_1) = e'$ then $\vdash v'$

Proof:

CASE $\delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0$:

1. $\vdash v_0$
   by *static inversion forms*
2. QED

CASE $\delta(\mathsf{snd}, \langle v_0, v_1 \rangle) = v_1$:

1. $\vdash v_1$
   by *static inversion forms*
2. QED

CASE $\delta(\mathsf{sum}, v_0, v_1) = v_0 + v_1$:

1. QED

CASE $\delta(\mathsf{quotient}, v_0, v_1) = \lfloor v_0 / v_1 \rfloor$:

1. QED

CASE $\delta(\mathsf{quotient}, v_0, v_1) = \mathsf{BndryErr}$:

1. QED

□

Lemma 4.57 : *weakening*

- If $\Gamma \vdash e$ then $x, \Gamma \vdash e$
- If $\Gamma \vdash e$ then $(x : \tau), \Gamma \vdash e$

Proof:

QED because e is closed under Γ

□

A.5 (HC) Co-Natural Embedding

A.5.1 Co-Natural Definitions

$\boxed{\text{Language HC}}$

$$
\begin{array}{lcl}
e & = & x \mid v \mid \langle e, e\rangle \mid e\ e \mid op^1\ e \mid op^2\ e\ e \mid \\
  &   & \mathsf{dyn}\ \tau\ e \mid \mathsf{stat}\ \tau\ e \mid \mathsf{Err} \\
v & = & i \mid \langle v, v\rangle \mid \lambda x.\, e \mid \lambda(x{:}\tau).\, e \\
  &   & \mathsf{mon}\,(\tau \Rightarrow \tau)\, v \mid \mathsf{mon}\,(\tau \times \tau)\, v \\
\tau & = & \mathsf{Nat} \mid \mathsf{Int} \mid \tau \times \tau \mid \tau \Rightarrow \tau \\
\Gamma & = & \cdot \mid x, \Gamma \mid (x{:}\tau), \Gamma \\
\mathsf{Err} & = & \mathsf{BndryErr} \mid \mathsf{TagErr} \\
r & = & v \mid \mathsf{Err} \\
E^\bullet & = & [\,] \mid E^\bullet\ e \mid v\ E^\bullet \mid \langle E^\bullet, e\rangle \mid \langle v, E^\bullet\rangle \mid \\
  &   & op^1\ E^\bullet \mid op^2\ E^\bullet\ e \mid op^2\ v\ E^\bullet \\
E & = & E^\bullet \mid E\ e \mid v\ E \mid \langle E, e\rangle \mid \langle v, E\rangle \mid op^1\ E \mid \\
  &   & op^2\ E\ e \mid op^2\ v\ E \mid \mathsf{dyn}\ \tau\ E \mid \mathsf{stat}\ \tau\ E
\end{array}
$$

$\boxed{\Delta : op^1 \times \tau \longrightarrow \tau}$

$$
\begin{array}{l}
\Delta(\mathsf{fst}, \tau_0 \times \tau_1) \ = \tau_0 \\
\Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1
\end{array}
$$

$\boxed{\Delta : op^2 \times \tau \times \tau \longrightarrow \tau}$

$$
\begin{array}{l}
\Delta(op^2, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat} \\
\Delta(op^2, \mathsf{Int}, \mathsf{Int}) \ \ = \mathsf{Int}
\end{array}
$$

$\boxed{\tau \leqslant: \tau}$

$$
\frac{}{\mathsf{Nat} \leqslant: \mathsf{Int}} \qquad
\frac{\tau_d' \leqslant: \tau_d \quad \tau_c \leqslant: \tau_c'}{\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'} \qquad
\frac{\tau_0 \leqslant: \tau_0' \quad \tau_1 \leqslant: \tau_1'}{\tau_0 \times \tau_1 \leqslant: \tau_0' \times \tau_1'}
$$

$$
\frac{}{\tau \leqslant: \tau} \qquad
\frac{\tau \leqslant: \tau' \quad \tau' \leqslant: \tau''}{\tau \leqslant: \tau''}
$$

$\boxed{\Gamma \vdash e}$

$$
\frac{x \in \Gamma}{\Gamma \vdash x} \qquad
\frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x.\, e} \qquad
\frac{}{\Gamma \vdash i} \qquad
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1\rangle}
$$

$$
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0\ e_1} \qquad
\frac{\Gamma \vdash e}{\Gamma \vdash op^1\ e} \qquad
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2\ e_0\ e_1} \qquad
\frac{}{\Gamma \vdash \mathsf{Err}}
$$

$$
\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat}\ \tau\ e}
$$

$\boxed{\Gamma \vdash e : \tau}$

$$
\frac{(x{:}\tau) \in \Gamma}{\Gamma \vdash x : \tau} \qquad
\frac{(x{:}\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x{:}\tau_d).\, e : \tau_d \Rightarrow \tau_c} \qquad
\frac{i \in \mathbb{N}}{\Gamma \vdash i : \mathsf{Nat}}
$$

$$
\frac{}{\Gamma \vdash i : \mathsf{Int}} \qquad
\frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_0 \\ \Gamma \vdash e_1 : \tau_1\end{array}}{\Gamma \vdash \langle e_0, e_1\rangle : \tau_0 \times \tau_1} \qquad
\frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \\ \Gamma \vdash e_1 : \tau_d\end{array}}{\Gamma \vdash e_0\ e_1 : \tau_c}
$$

$$
\frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_0 \\ \Delta(op^1, \tau_0) = \tau\end{array}}{\Gamma \vdash op^1\ e_0 : \tau} \qquad
\frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_0 \\ \Gamma \vdash e_1 : \tau_1 \\ \Delta(op^2, \tau_0, \tau_1) = \tau\end{array}}{\Gamma \vdash op^2\ e_0\ e_1 : \tau} \qquad
\frac{\begin{array}{c}\Gamma \vdash e : \tau' \\ \tau' \leqslant: \tau\end{array}}{\Gamma \vdash e : \tau} \qquad
\frac{}{\Gamma \vdash \mathsf{Err} : \tau}
$$

$$
\frac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau}
$$

$\boxed{\Gamma \vdash_{\mathsf{C}} e}$

$$
\frac{x \in \Gamma}{\Gamma \vdash_{\mathsf{C}} x} \qquad
\frac{x, \Gamma \vdash_{\mathsf{C}} e}{\Gamma \vdash_{\mathsf{C}} \lambda x.\, e} \qquad
\frac{}{\Gamma \vdash_{\mathsf{C}} i} \qquad
\frac{\Gamma \vdash_{\mathsf{C}} e_0 \quad \Gamma \vdash_{\mathsf{C}} e_1}{\Gamma \vdash_{\mathsf{C}} \langle e_0, e_1\rangle}
$$

$$
\frac{\Gamma \vdash_{\mathsf{C}} e_0 \quad \Gamma \vdash_{\mathsf{C}} e_1}{\Gamma \vdash_{\mathsf{C}} e_0\ e_1} \qquad
\frac{\Gamma \vdash_{\mathsf{C}} e}{\Gamma \vdash_{\mathsf{C}} op^1\ e} \qquad
\frac{\Gamma \vdash_{\mathsf{C}} e_0 \quad \Gamma \vdash_{\mathsf{C}} e_1}{\Gamma \vdash_{\mathsf{C}} op^2\ e_0\ e_1} \qquad
\frac{}{\Gamma \vdash_{\mathsf{C}} \mathsf{Err}}
$$

$$
\frac{\Gamma \vdash_{\mathsf{C}} e : \tau}{\Gamma \vdash_{\mathsf{C}} \mathsf{stat}\ \tau\ e} \qquad
\frac{\Gamma \vdash_{\mathsf{C}} v : \tau_0 \times \tau_1}{\Gamma \vdash_{\mathsf{C}} \mathsf{mon}\,(\tau_0 \times \tau_1)\, v} \qquad
\frac{\Gamma \vdash_{\mathsf{C}} v : \tau_d \Rightarrow \tau_c}{\Gamma \vdash_{\mathsf{C}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v}
$$

$\boxed{\Gamma \vdash_{\mathsf{C}} e : \tau}$

$$
\frac{(x{:}\tau) \in \Gamma}{\Gamma \vdash_{\mathsf{C}} x : \tau} \qquad
\frac{(x{:}\tau_d), \Gamma \vdash_{\mathsf{C}} e : \tau_c}{\Gamma \vdash_{\mathsf{C}} \lambda(x{:}\tau_d).\, e : \tau_d \Rightarrow \tau_c} \qquad
\frac{i \in \mathbb{N}}{\Gamma \vdash_{\mathsf{C}} i : \mathsf{Nat}}
$$

$$
\frac{}{\Gamma \vdash_{\mathsf{C}} i : \mathsf{Int}} \qquad
\frac{\begin{array}{c}\Gamma \vdash_{\mathsf{C}} e_0 : \tau_0 \\ \Gamma \vdash_{\mathsf{C}} e_1 : \tau_1\end{array}}{\Gamma \vdash_{\mathsf{C}} \langle e_0, e_1\rangle : \tau_0 \times \tau_1} \qquad
\frac{\begin{array}{c}\Gamma \vdash_{\mathsf{C}} e_0 : \tau_d \Rightarrow \tau_c \\ \Gamma \vdash_{\mathsf{C}} e_1 : \tau_d\end{array}}{\Gamma \vdash_{\mathsf{C}} e_0\ e_1 : \tau_c}
$$

$$
\frac{\begin{array}{c}\Gamma \vdash_{\mathsf{C}} e_0 : \tau_0 \\ \Delta(op^1, \tau_0) = \tau\end{array}}{\Gamma \vdash_{\mathsf{C}} op^1\ e_0 : \tau} \qquad
\frac{\begin{array}{c}\Gamma \vdash_{\mathsf{C}} e_0 : \tau_0 \\ \Gamma \vdash_{\mathsf{C}} e_1 : \tau_1 \\ \Delta(op^2, \tau_0, \tau_1) = \tau\end{array}}{\Gamma \vdash_{\mathsf{C}} op^2\ e_0\ e_1 : \tau} \qquad
\frac{\begin{array}{c}\Gamma \vdash_{\mathsf{C}} e : \tau' \\ \tau' <: \tau\end{array}}{\Gamma \vdash_{\mathsf{C}} e : \tau}
$$

$$
\frac{}{\Gamma \vdash_{\mathsf{C}} \mathsf{Err} : \tau} \qquad
\frac{\Gamma \vdash_{\mathsf{C}} e}{\Gamma \vdash_{\mathsf{C}} \mathsf{dyn}\ \tau\ e : \tau}
$$

$$
\frac{\Gamma \vdash_{\mathsf{C}} v}{\Gamma \vdash_{\mathsf{C}} \mathsf{mon}\,(\tau_0 \times \tau_1)\, v : (\tau_0 \times \tau_1)}
$$

$$
\frac{\Gamma \vdash_{\mathsf{C}} v}{\Gamma \vdash_{\mathsf{C}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v : (\tau_d \Rightarrow \tau_c)}
$$

$\boxed{\delta(op^1, v) = e}$

$\delta(\mathsf{fst}, \langle v_0, v_1\rangle) = v_0$

$\delta(\mathsf{snd}, \langle v_0, v_1\rangle) = v_1$

$\boxed{\delta(op^2, v, v) = e}$

$\delta(\mathsf{sum}, i_0, i_1) = i_0 + i_1$

$\delta(\mathsf{quotient}, i_0, 0) = \mathsf{BndryErr}$

$\delta(\mathsf{quotient}, i_0, i_1) = \lfloor i_0/i_1 \rfloor$
  if $i_1 \neq 0$

$\boxed{\mathcal{D}_\mathsf{C} : \tau \times v \longrightarrow e}$

$\mathcal{D}_\mathsf{C}(\tau_d \Rightarrow \tau_c, v) = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v$
  if $v = \lambda x.\, e$ or $v = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\, v'$

$\mathcal{D}_\mathsf{C}(\tau_0 \times \tau_1, v) = \mathsf{mon}\,(\tau_0 \times \tau_1)\, v$
  if $v = \langle v_0, v_1\rangle$ or $v = \mathsf{mon}\,(\tau_0' \times \tau_1')\, v'$

$\mathcal{D}_\mathsf{C}(\mathsf{Int}, i) = i$

$\mathcal{D}_\mathsf{C}(\mathsf{Nat}, i) = i$
  if $i \in \mathbb{N}$

$\mathcal{D}_\mathsf{C}(\tau, v) = \mathsf{BndryErr}$
  otherwise

$\boxed{\mathcal{S}_\mathsf{C} : \tau \times v \longrightarrow e}$

$\mathcal{S}_\mathsf{C}(\tau_d \Rightarrow \tau_c, v) = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v$

$\mathcal{S}_\mathsf{C}(\tau_0 \times \tau_1, v) = \mathsf{mon}\,(\tau_0 \times \tau_1)\, v$

$\mathcal{S}_\mathsf{C}(\tau, v) = v$
  otherwise

$\boxed{e \rhd_{\mathsf{S\text{-}C}} e}$

$\mathsf{dyn}\ \tau\ v \rhd_{\mathsf{S\text{-}C}} \mathcal{D}_\mathsf{C}(\tau, v)$

$(\mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v_f)\ v \rhd_{\mathsf{S\text{-}C}} \mathsf{dyn}\ \tau_c\ (v_f\ e')$
  where $e' = \mathsf{stat}\ \tau_d\ v$

$(\lambda(x{:}\tau).\, e)\ v \rhd_{\mathsf{S\text{-}C}} e[x \leftarrow v]$

$\mathsf{fst}\,(\mathsf{mon}\,(\tau_0 \times \tau_1)\, v) \rhd_{\mathsf{S\text{-}C}} \mathsf{dyn}\ \tau_0\ (\mathsf{fst}\, v)$

$\mathsf{snd}\,(\mathsf{mon}\,(\tau_0 \times \tau_1)\, v) \rhd_{\mathsf{S\text{-}C}} \mathsf{dyn}\ \tau_1\ (\mathsf{snd}\, v)$

$op^1\ v \rhd_{\mathsf{S\text{-}C}} \delta(op^1, v)$

$op^2\ v_0\ v_1 \rhd_{\mathsf{S\text{-}C}} \delta(op^2, v_0, v_1)$

$\boxed{e \rhd_{\mathsf{D\text{-}C}} e}$

$\mathsf{stat}\ \tau\ v \rhd_{\mathsf{D\text{-}C}} \mathcal{S}_\mathsf{C}(\tau, v)$

$v_0\ v_1 \rhd_{\mathsf{D\text{-}C}} \mathsf{TagErr}$
  if $v_0 \in \mathbb{Z}$ or $v_0 = \langle v, v'\rangle$

$(\mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v_f)\ v \rhd_{\mathsf{D\text{-}C}} \mathsf{stat}\ \tau_c\ (v_f\ e')$
  where $e' = \mathsf{dyn}\ \tau_d\ v$

$(\lambda x.\, e)\ v \rhd_{\mathsf{D\text{-}C}} e[x \leftarrow v]$

$\mathsf{fst}\,(\mathsf{mon}\,(\tau_0 \times \tau_1)\, v) \rhd_{\mathsf{D\text{-}C}} \mathsf{stat}\ \tau_0\ (\mathsf{fst}\, v)$

$\mathsf{snd}\,(\mathsf{mon}\,(\tau_0 \times \tau_1)\, v) \rhd_{\mathsf{D\text{-}C}} \mathsf{stat}\ \tau_1\ (\mathsf{snd}\, v)$

$op^1\ v \rhd_{\mathsf{D\text{-}C}} \mathsf{TagErr}$
  if $\delta(op^1, v)$ is undefined

$op^1\ v \rhd_{\mathsf{D\text{-}C}} \delta(op^1, v)$

$op^2\ v_0\ v_1 \rhd_{\mathsf{D\text{-}C}} \mathsf{TagErr}$
  if $\delta(op^2, v_0, v_1)$ is undefined

$op^2\ v_0\ v_1 \rhd_{\mathsf{D\text{-}C}} \delta(op^2, v_0, v_1)$

$\boxed{e \rightarrow_{\mathsf{C\text{-}S}} e}$

$E^\bullet[e] \rightarrow_{\mathsf{C\text{-}S}} E^\bullet[e']$
  if $e \rhd_{\mathsf{S\text{-}C}} e'$

$E[\mathsf{stat}\ \tau\ E^\bullet[e]] \rightarrow_{\mathsf{C\text{-}S}} E[\mathsf{stat}\ \tau\ E^\bullet[e']]$
  if $e \rhd_{\mathsf{S\text{-}C}} e'$

$E[\mathsf{dyn}\ \tau\ E^\bullet[e]] \rightarrow_{\mathsf{C\text{-}S}} E[\mathsf{dyn}\ \tau\ E^\bullet[e']]$
  if $e \rhd_{\mathsf{D\text{-}C}} e'$

$E[\mathsf{Err}] \rightarrow_{\mathsf{C\text{-}S}} \mathsf{Err}$

$\boxed{e \rightarrow_{\mathsf{C\text{-}D}} e}$

$E^\bullet[e] \rightarrow_{\mathsf{C\text{-}D}} E^\bullet[e']$
  if $e \rhd_{\mathsf{D\text{-}C}} e'$

$E[\mathsf{stat}\ \tau\ E^\bullet[e]] \rightarrow_{\mathsf{C\text{-}D}} E[\mathsf{stat}\ \tau\ E^\bullet[e']]$
  if $e \rhd_{\mathsf{S\text{-}C}} e'$

$E[\mathsf{dyn}\ \tau\ E^\bullet[e]] \rightarrow_{\mathsf{C\text{-}D}} E[\mathsf{dyn}\ \tau\ E^\bullet[e']]$
  if $e \rhd_{\mathsf{D\text{-}C}} e'$

$E[\mathsf{Err}] \rightarrow_{\mathsf{C\text{-}D}} \mathsf{Err}$

$\boxed{e \rightarrow^*_{\mathsf{C\text{-}S}} e}$ reflexive, transitive closure of $\rightarrow_{\mathsf{C\text{-}S}}$

$\boxed{e \rightarrow^*_{\mathsf{C\text{-}D}} e}$ reflexive, transitive closure of $\rightarrow_{\mathsf{C\text{-}D}}$

A.5.2 Co-Natural Theorems

Theorem 5.0 : *static* HC *soundness*

If $\vdash e : \tau$ then $\vdash_C e : \tau$ and one of the following holds:

- $e \rightarrow^*_{\text{C-S}} v$ and $\vdash_C v : \tau$
- $e \rightarrow^*_{\text{C-S}} E[\text{dyn}\ \tau'\ e']$ and $e' \rhd_{\text{D-C}} \text{TagErr}$
- $e \rightarrow^*_{\text{C-S}} \text{BndryErr}$
- e diverges

Proof:

1. $\vdash_C e : \tau$
   by *static subset*
2. QED by *static progress* and *static preservation.*

□

Theorem 5.1 : *dynamic* HC-*soundness*

If $\vdash e$ then $\vdash_C e$ and one of the following holds:

- $e \rightarrow^*_{\text{C-D}} v$ and $\vdash_C v$
- $e \rightarrow^*_{\text{C-D}} E[e']$ and $e' \rhd_{\text{D-C}} \text{TagErr}$
- $e \rightarrow^*_{\text{C-D}} \text{BndryErr}$
- e diverges

Proof:

1. $\vdash_C e$
   by *dynamic subset*
2. QED by *dynamic progress* and *dynamic preservation.*

□

Corollary 5.2 : HC *static soundness*

If $\vdash e : \tau$ and e is boundary-free, then one of the following holds:

- $e \rightarrow^*_{\text{C-S}} v$ and $\vdash_C v : \tau$
- $e \rightarrow^*_{\text{C-S}} \text{BndryErr}$
- e diverges

Proof:

Consequence of the proof for *static* HC-*soundness*

□

A.5.3 Co-Natural Lemmas

Lemma 5.3 : $\mathcal{D}_{\mathrm{C}}$ *soundness*

If $\vdash_{\mathrm{C}} v$ then $\vdash_{\mathrm{C}} \mathcal{D}_{\mathrm{C}}(\tau, v) : \tau$

Proof:

CASE $\mathcal{D}_{\mathrm{C}}(\tau_d \Rightarrow \tau_c, v) = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v$:

1. $\vdash_{\mathrm{C}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v : \tau_d \Rightarrow \tau_c$
   by $\vdash_{\mathrm{C}} v$
2. QED

CASE $\mathcal{D}_{\mathrm{C}}(\tau_0 \times \tau_1, v) = \mathsf{mon}\,(\tau_0 \times \tau_1)\,v$:

1. $\vdash_{\mathrm{C}} \mathsf{mon}\,(\tau_0 \times \tau_1)\,v : \tau_0 \times \tau_1$
   by $\vdash_{\mathrm{C}} v$
2. QED

CASE $v = i$
$\wedge\ \mathcal{D}_{\mathrm{C}}(\mathsf{Int}, v) = v$:

1. QED

CASE $v \in \mathbb{N}$
$\wedge\ \mathcal{D}_{\mathrm{C}}(\mathsf{Nat}, v) = v$:

1. QED

CASE $\mathcal{D}_{\mathrm{C}}(\tau, v) = \mathsf{BndryErr}$:

1. QED

□

Lemma 5.4 : $\mathcal{S}_{\mathrm{C}}$ *soundness*

If $\vdash_{\mathrm{C}} v : \tau$ then $\vdash_{\mathrm{C}} \mathcal{S}_{\mathrm{C}}(\tau, v)$

Proof:

CASE $\vdash_{\mathrm{C}} v : \tau_d \Rightarrow \tau_c$
$\wedge\ \mathcal{S}_{\mathrm{C}}(\tau_d \Rightarrow \tau_c, v) = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v$:

1. QED

CASE $\vdash_{\mathrm{C}} v : \tau_0 \times \tau_1$
$\wedge\ \mathcal{S}_{\mathrm{C}}(\tau_0 \times \tau_1, v) = \mathsf{mon}\,(\tau_0 \times \tau_1)\,v$:

1. QED

CASE $\vdash_{\mathrm{C}} v : \mathsf{Int}$
$\wedge\ \mathcal{S}_{\mathrm{C}}(\mathsf{Int}, v) = v$:

1. QED

CASE $\vdash_{\mathrm{C}} v : \mathsf{Nat}$
$\wedge\ \mathcal{S}_{\mathrm{C}}(\mathsf{Nat}, v) = v$:

1. QED

□

Corollary 5.5 : HC *static subset*

If $\Gamma \vdash e : \tau$ then $\Gamma \vdash_{\mathrm{C}} e : \tau$.

Proof:

Consequence of the proof for the higher-order *static subset* lemma; both $\vdash_{\mathrm{C}}$ and $\vdash_{\mathrm{H}}$ have the same typing rules for surface-language expressions.

□

Corollary 5.6 : HC *dynamic subset*

If $\Gamma \vdash e$ then $\Gamma \vdash_{\mathrm{C}} e$.

Proof:

Consequence of the proof for the higher-order *dynamic subset* lemma.

□

Lemma 5.7 : HC *static progress*

If $\vdash_{\mathrm{C}} e : \tau$ then one of the following holds:

- e is a value
- $e \in \mathsf{Err}$
- $e \rightarrow_{\mathsf{C\text{-}S}} e'$
- $e \rightarrow_{\mathsf{C\text{-}S}} \mathsf{BndryErr}$
- $e = E[\mathsf{dyn}\ \tau'\ e']$ and $e' \rightarrow_{\mathsf{C\text{-}D}} \mathsf{TagErr}$

Proof:

By the *boundary factoring* lemma, there are seven possible cases.

CASE e is a value :

1. QED

CASE $e = E^{\bullet}[v_0\ v_1]$:

1. $\vdash_{\mathrm{C}} v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\mathrm{C}} v_0 : \tau_d \Rightarrow \tau_c$
   $\wedge \vdash_{\mathrm{C}} v_1 : \tau_d$
   by *inversion*
3. $v_0 = \lambda(x\!:\!\tau_d').\,e'$
   $\vee\ v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,v_f$
   by *canonical forms*
4. IF $v_0 = \lambda(x\!:\!\tau_d').\,e'$:
   a. $e \rightarrow_{\mathsf{C\text{-}S}} E^{\bullet}[e'[x \leftarrow v_1]]$
      by $v_0\ v_1 \rhd_{\mathsf{S\text{-}C}} e'[x \leftarrow v_1]$
   b. QED

   ELSE $v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,v_f$:
   a. $e \rightarrow_{\mathsf{C\text{-}S}} E^{\bullet}[\mathsf{dyn}\ \tau_c'\ (v_f\ (\mathsf{stat}\ \tau_d'\ v_1))]$
      by $v_0\ v_1 \rhd_{\mathsf{S\text{-}C}} \mathsf{dyn}\ \tau_c'\ (v_f\ (\mathsf{stat}\ \tau_d'\ v_1))$
   b. QED

CASE $e = E^{\bullet}[op^1\ v]$:

1. $\vdash_{\mathrm{C}} op^1\ v : \tau'$
   by *static hole typing*
2. $\vdash_{\mathrm{C}} v : \tau_0 \times \tau_1$
   by *inversion*
3. $v = \langle v_0, v_1 \rangle$
   $\vee\ v = \mathsf{mon}\,(\tau_0 \times \tau_1)\,v'$
   by *canonical forms*
4. IF $v = \langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \mathsf{fst}$:
   a. $\delta(op^1, \langle v_0, v_1 \rangle) = v_0$
      by definition
   b. $e \rightarrow_{\mathsf{C\text{-}S}} E^{\bullet}[v_0]$
      by $op^1\ v \rhd_{\mathsf{S\text{-}C}} v_0$
   c. QED

   IF $v = \langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \mathsf{snd}$:
   a. $\delta(op^1, \langle v_0, v_1 \rangle) = v_1$
      by definition
   b. $e \rightarrow_{\mathsf{C\text{-}S}} E^{\bullet}[v_1]$
      by $op^1\ v \rhd_{\mathsf{S\text{-}C}} v_1$
   c. QED

   IF $v = \mathsf{mon}\,(\tau_0 \times \tau_1)\,v'$
   $\wedge\ op^1 = \mathsf{fst}$:
   a. $e \rightarrow_{\mathsf{C\text{-}S}} E^{\bullet}[\mathsf{dyn}\ \tau_0\ (op^1\ v')]$
      by definition

b. QED

ELSE $v = \mathsf{mon}\,(\tau_0{\times}\tau_1)\,v'$
$\wedge\ op^1 = \mathsf{snd}$:

a. $e \rightarrow_{\text{C-S}} E^{\bullet}[\mathsf{dyn}\ \tau_1\ (op^1\ v')]$
by definition

b. QED

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:

1. $\vdash_{\mathrm{C}} op^2\ v_0\ v_1 : \tau'$
by *static hole typing*
2. $\vdash_{\mathrm{C}} v_0 : \tau_0$
$\wedge \vdash_{\mathrm{C}} v_1 : \tau_1$
$\wedge\ \Delta(op^2, \tau_0, \tau_1) = \tau''$
by *inversion*
3. $\delta(op^2, v_0, v_1) = e'$
by Δ *type soundness* (2)
4. $op^2\ v_0\ v_1 \rhd_{\text{S-C}} e'$
by (3)
5. QED by $e \rightarrow_{\text{C-S}} E^{\bullet}[e']$

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :

1. e' is a value
$\vee\ e' \in \mathsf{Err}$
$\vee\ e' \rightarrow_{\text{C-D}} e''$
$\vee\ e' \rightarrow_{\text{C-D}} \mathsf{BndryErr}$
$\vee\ e' = E'[e'']$ and $e'' \rhd_{\text{D-C}} \mathsf{TagErr}$
by *dynamic progress*
2. **IF** e' is a value :
a. QED $e \rightarrow_{\text{C-S}} E[\mathcal{D}_{\mathrm{C}}(\tau', e')]$
IF $e' \in \mathsf{Err}$:
a. QED $e \rightarrow_{\text{C-S}} e'$
IF $e' \rightarrow_{\text{C-D}} e''$:
a. QED $e \rightarrow_{\text{C-S}} E[\mathsf{dyn}\ \tau'\ e'']$
IF $e' \rightarrow_{\text{C-D}} \mathsf{BndryErr}$:
a. QED $e \rightarrow_{\text{C-S}} E[\mathsf{dyn}\ \tau'\ \mathsf{BndryErr}]$
ELSE $e' = E'[e'']$ and $e'' \rhd_{\text{D-C}} \mathsf{TagErr}$:
a. $E' \in E^{\bullet}$
by e' is boundary-free
b. QED

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :

1. e' is a value
$\vee\ e' \in \mathsf{Err}$
$\vee\ e' \rightarrow_{\text{C-S}} e''$
$\vee\ e' \rightarrow_{\text{C-S}} \mathsf{BndryErr}$
$\vee\ e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{D-C}} \mathsf{TagErr}$
by *static progress*
2. **IF** e' is a value :
a. QED $e \rightarrow_{\text{C-S}} E[\mathcal{S}_{\mathrm{C}}(\tau', e')]$
IF $e' \in \mathsf{Err}$:
a. QED $e \rightarrow_{\text{C-S}} e'$
IF $e' \rightarrow_{\text{C-S}} e''$:
a. QED $e \rightarrow_{\text{C-S}} E[\mathsf{stat}\ \tau'\ e'']$
IF $e' \rightarrow_{\text{C-S}} \mathsf{BndryErr}$:
a. QED $e \rightarrow_{\text{C-S}} E[\mathsf{stat}\ \tau'\ \mathsf{BndryErr}]$
ELSE $e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{D-C}} \mathsf{TagErr}$
:
a. Contradiction by e' is boundary-free

CASE $e = E[\mathsf{Err}]$:

1. QED $e \rightarrow_{\text{C-S}} \mathsf{Err}$

□

Lemma 5.8 : HC *dynamic progress*

If $\vdash_{\mathrm{C}} e$ then one of the following holds:

- e is a value
- $e \in \mathsf{Err}$
- $e \rightarrow_{\text{C-D}} e'$
- $e \rightarrow_{\text{C-D}} \mathsf{BndryErr}$
- $e \rightarrow_{\text{C-D}} \mathsf{TagErr}$

Proof:

By the *boundary factoring* lemma, there are seven cases.

CASE e is a value :

1. QED

CASE $e = E^{\bullet}[v_0\ v_1]$:

IF $v_0 = \lambda x.\, e'$:
1. $e \rightarrow_{\text{C-D}} E^{\bullet}[e'[x \leftarrow v_1]]$
by $v_0\ v_1 \rhd_{\text{D-C}} e'[x \leftarrow v_1]$
2. QED

IF $v_0 = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v_f$:
1. $e \rightarrow_{\text{C-D}} E^{\bullet}[\mathsf{stat}\ \tau_c\ (v_f\ (\mathsf{dyn}\ \tau_d\ v_1))]$
by $v_0\ v_1 \rhd_{\text{D-C}} \mathsf{stat}\ \tau_c\ (v_f\ (\mathsf{dyn}\ \tau_d\ v_1))$
2. QED

ELSE $v_0 = i$
$\vee\ v_0 = \langle v, v' \rangle$:
1. $e \rightarrow_{\text{C-D}} \mathsf{TagErr}$
by $(v_0\ v_1) \rhd_{\text{D-C}} \mathsf{TagErr}$
2. QED

CASE $e = E^{\bullet}[op^1\ v]$:

IF $v = \mathsf{mon}\,(\tau_0{\times}\tau_1)\,v'$
$\wedge\ op^1 = \mathsf{fst}$:
1. $e \rightarrow_{\text{C-D}} E^{\bullet}[\mathsf{stat}\ \tau_0\ op^1\ v']$
by $op^1\ v \rhd_{\text{D-C}} \mathsf{stat}\ \tau_0\ op^1\ v'$
2. QED

IF $v = \mathsf{mon}\,(\tau_0{\times}\tau_1)\,v'$
$\wedge\ op^1 = \mathsf{snd}$:
1. $e \rightarrow_{\text{C-D}} E^{\bullet}[\mathsf{stat}\ \tau_1\ op^1\ v']$
by $op^1\ v \rhd_{\text{D-C}} \mathsf{stat}\ \tau_1\ op^1\ v'$
2. QED

IF $\delta(op^1, v) = e'$:
1. $(op^1\ v) \rhd_{\text{D-C}} e'$
2. QED

ELSE $\delta(op^1, v)$ is undefined :
1. $e \rightarrow_{\text{C-D}} \mathsf{TagErr}$
by $(op^1\ v) \rhd_{\text{D-C}} \mathsf{TagErr}$
2. QED

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:

IF $\delta(op^2, v_0, v_1) = e''$:
1. $op^2\ v_0\ v_1 \rhd_{\text{D-C}} e''$
2. QED

ELSE $\delta(op^2, v_0, v_1)$ is undefined :

1. $e \rightarrow_{\text{C-D}} \mathsf{TagErr}$
   by $op^2\ v_0\ v_1 \rhd_{\text{D-C}} \mathsf{TagErr}$
2. QED

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :

1. e' is a value
   $\vee\ e' \in \mathsf{Err}$
   $\vee\ e' \rightarrow_{\text{C-D}} e''$
   $\vee\ e' \rightarrow_{\text{C-D}} \mathsf{BndryErr}$
   $\vee\ e' = E[e'']$ and $e'' \rhd_{\text{D-C}} \mathsf{TagErr}$
   by *dynamic progress*
2. **IF** e' is a value :
   a. QED $e \rightarrow_{\text{C-D}} E[\mathcal{D}_{\text{C}}(\tau', e')]$
   IF $e' \in \mathsf{Err}$:
   a. QED $e \rightarrow_{\text{C-D}} e'$
   IF $e' \rightarrow_{\text{C-D}} e''$:
   a. QED $e \rightarrow_{\text{C-S}} E[\mathsf{dyn}\ \tau'\ e'']$
   IF $e' \rightarrow_{\text{C-D}} \mathsf{BndryErr}$:
   a. QED $e \rightarrow_{\text{C-D}} E[\mathsf{dyn}\ \tau'\ \mathsf{BndryErr}]$
   ELSE $e' = E[e'']$ and $e'' \rhd_{\text{D-C}} \mathsf{TagErr}$:
   a. $E \in E^{\bullet}$
      by e' is boundary-free
   b. QED

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :

1. e' is a value
   $\vee\ e' \in \mathsf{Err}$
   $\vee\ e' \rightarrow_{\text{C-S}} e''$
   $\vee\ e' \rightarrow_{\text{C-S}} \mathsf{BndryErr}$
   $\vee\ e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{D-C}} \mathsf{TagErr}$
   by *static progress*
2. **IF** e' is a value :
   a. QED $e \rightarrow_{\text{C-S}} E[\mathcal{S}_{\text{C}}(\tau', e')]$
   IF $e' \in \mathsf{Err}$:
   a. QED $e \rightarrow_{\text{C-S}} e'$
   IF $e' \rightarrow_{\text{C-S}} e''$:
   a. QED $e \rightarrow_{\text{C-S}} E[\mathsf{stat}\ \tau'\ e'']$
   IF $e' \rightarrow_{\text{C-S}} \mathsf{BndryErr}$:
   a. QED $e \rightarrow_{\text{C-S}} E[\mathsf{stat}\ \tau'\ \mathsf{BndryErr}]$
   ELSE $e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{D-C}} \mathsf{TagErr}$
   :
   a. Contradiction by e' is boundary-free

CASE $e = E[\mathsf{Err}]$:

1. QED $e \rightarrow_{\text{C-D}} \mathsf{Err}$

$\square$

Lemma 5.9 : HC *static preservation*

If $\vdash_{\text{C}} e : \tau$ and $e \rightarrow_{\text{C-S}} e'$ then $\vdash_{\text{C}} e' : \tau$

Proof:

By the *boundary factoring* lemma there are seven cases.

CASE e is a value :

1. Contradiction by $e \rightarrow_{\text{C-S}} e'$

CASE $e = E^{\bullet}[v_0\ v_1]$:

IF $v_0 = \lambda(x\!:\!\tau_x).\ e'$
$\wedge\ e \rightarrow_{\text{C-S}} E^{\bullet}[e'[x \leftarrow v_1]]$:

1. $\vdash_{\text{C}} v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\text{C}} v_0 : \tau_d \Rightarrow \tau_c$
   $\wedge \vdash_{\text{C}} v_1 : \tau_d$
   $\wedge\ \tau_c \leqslant: \tau'$
   by *inversion*
3. $\tau_d \leqslant: \tau_x$
   by *canonical forms* (2)
4. $(x\!:\!\tau_x) \vdash_{\text{C}} e' : \tau_c$
   by *inversion* (2)
5. $\vdash_{\text{C}} v_1 : \tau_x$
   by (2, 3)
6. $\vdash_{\text{C}} e'[x \leftarrow v_1] : \tau_c$
   by *substitution* (4, 5)
7. $\vdash_{\text{C}} e'[x \leftarrow v_1] : \tau'$
   by (2, 6)
8. QED by *hole substitution* (7)

ELSE $v_0 = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v_f$
$\wedge\ e \rightarrow_{\text{C-S}} E^{\bullet}[\mathsf{dyn}\ \tau_c\ (v_f\ (\mathsf{stat}\ \tau_d\ v_1))]$:

1. $\vdash_{\text{C}} v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\text{C}} v_0 : \tau_d' \Rightarrow \tau_c'$
   $\wedge \vdash_{\text{C}} v_1 : \tau_d'$
   $\wedge\ \tau_c' \leqslant: \tau'$
   by *inversion*
3. $\vdash_{\text{C}} v_f$
   by *inversion* (2)
4. $\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'$
   by *canonical forms* (2)
5. $\tau_d' \leqslant: \tau_d$
   $\wedge\ \tau_c \leqslant: \tau_c'$
   by (4)
6. $\vdash_{\text{C}} v_1 : \tau_d$
   by (2, 5)
7. $\vdash_{\text{C}} \mathsf{stat}\ \tau_d\ v_1$
   by (6)
8. $\vdash_{\text{C}} v_f\ (\mathsf{stat}\ \tau_d\ v_1)$
   by (3, 7)
9. $\vdash_{\text{C}} \mathsf{dyn}\ \tau_c\ (v_f\ (\mathsf{stat}\ \tau_d\ v_1)) : \tau_c$
   by (8)
10. $\vdash_{\text{C}} \mathsf{dyn}\ \tau_c\ (v_f\ (\mathsf{stat}\ \tau_d\ v_1)) : \tau'$
    by (2, 5, 9)
11. QED by *hole substitution* (10)

CASE $e = E^{\bullet}[op^1\ v]$:

IF $v = \mathsf{mon}\,(\tau_0 \times \tau_1)\, v'$
$\wedge\ op^1 = \mathsf{fst}$
$\wedge\ e \rightarrow_{\text{C-S}} E^{\bullet}[\mathsf{dyn}\ \tau_0\ (\mathsf{fst}\ v')]$:

1. $\vdash_{\text{C}} \mathsf{fst}\ v : \tau'$
   by *static hole typing*
2. $\vdash_{\text{C}} v : \tau_0' \times \tau_1'$
   $\wedge\ \tau_0' <: \tau'$
   by *inversion*
3. $\vdash_{\text{C}} v'$
   by *inversion* (2)

4. $\tau_0{\times}\tau_1 \leqslant: \tau_0'{\times}\tau_1'$
by *canonical forms* (2)
5. $\tau_0 \leqslant: \tau_0'$
6. $\vdash_C \mathsf{fst}\, v'$
by (3)
7. $\vdash_C \mathsf{dyn}\ \tau_0\ (\mathsf{fst}\, v') : \tau_0$
by (6)
8. $\vdash_C \mathsf{dyn}\ \tau_0\ (\mathsf{fst}\, v') : \tau'$
by (2, 5, 7)
9. QED by *hole substitution*

IF $v = \mathsf{mon}\,(\tau_0{\times}\tau_1)\,\langle v_0, v_1\rangle$
$\land\ op^1 = \mathsf{snd}$
$\land\ e \rightarrow_{\text{C-S}} E^\bullet[\mathsf{dyn}\ \tau_0\ (\mathsf{snd}\, v')]$:
1. $\vdash_C \mathsf{snd}\, v : \tau'$
by *static hole typing*
2. $\vdash_C v : \tau_0'{\times}\tau_1'$
$\land\ \tau_1' <: \tau'$
by *inversion*
3. $\vdash_C v'$
by *inversion* (2)
4. $\tau_0{\times}\tau_1 \leqslant: \tau_0'{\times}\tau_1'$
by *canonical forms* (2)
5. $\tau_1 \leqslant: \tau_1'$
6. $\vdash_C \mathsf{snd}\, v'$
by (3)
7. $\vdash_C \mathsf{dyn}\ \tau_0\ (\mathsf{snd}\, v') : \tau_0$
by (5)
8. $\vdash_C \mathsf{dyn}\ \tau_0\ (\mathsf{snd}\, v') : \tau'$
by (2, 5, 7)
9. QED by *hole substitution*

IF $v = \langle v_0, v_1\rangle$
$\land\ op^1 = \mathsf{fst}$
$\land\ e \rightarrow_{\text{C-S}} E^\bullet[v_0]$:
1. $\vdash_C \mathsf{fst}\,\langle v_0, v_1\rangle : \tau'$
by *static hole typing*
2. $\vdash_C \langle v_0, v_1\rangle : \tau_0{\times}\tau_1$
$\land\ \tau_0 \leqslant: \tau'$
by *inversion* (1)
3. $\vdash_C v_0 : \tau_0$
by *inversion* (2)
4. $\vdash_C v_0 : \tau'$
by (2, 3)
5. QED by *hole substitution* (4)

ELSE $v = \langle v_0, v_1\rangle$
$\land\ op^1 = \mathsf{snd}$
$\land\ e \rightarrow_{\text{C-S}} E^\bullet[v_1]$:
1. $\vdash_C \mathsf{snd}\,\langle v_0, v_1\rangle : \tau'$
by *static hole typing*
2. $\vdash_C \langle v_0, v_1\rangle : \tau_0{\times}\tau_1$
$\land\ \tau_1 \leqslant: \tau'$
by *inversion* (1)
3. $\vdash_C v_1 : \tau_1$
by *inversion* (2)
4. $\vdash_C v_1 : \tau'$
by (2, 3)
5. QED by *hole substitution* (4)

CASE $e = E^\bullet[op^2\ v_0\ v_1]$:
1. $e \rightarrow_{\text{C-S}} E^\bullet[\delta(op^2, v_0, v_1)]$
by $e \rightarrow_{\text{C-S}} e'$
2. $\vdash_C op^2\ v_0\ v_1 : \tau'$
by *static hole typing*
3. $\vdash_C v_0 : \tau_0$
$\land \vdash_C v_1 : \tau_1$
$\land\ \Delta(op^2, \tau_0, \tau_1) = \tau''$
$\land\ \tau'' \leqslant: \tau'$
by *inversion* (1)
4. $\vdash_C \delta(op^2, v_0, v_1) : \tau''$
by Δ *type soundness* (2)
5. $\vdash_C \delta(op^2, v_0, v_1) : \tau'$
by (2, 3)
6. QED by *hole substitution* (4)

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :

IF e' is a value :
1. $e \rightarrow_{\text{C-S}} E[\mathcal{D}_C(\tau', e')]$
2. $\vdash_C \mathsf{dyn}\ \tau'\ e' : \tau'$
by *boundary hole typing*
3. $\vdash_C e'$
by *inversion* (2)
4. $\vdash_C \mathcal{D}_C(\tau', e') : \tau'$
by $\mathcal{D}_C$ *soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\text{C-D}} e''$:
1. $e \rightarrow_{\text{C-S}} E[\mathsf{dyn}\ \tau'\ e'']$
2. $\vdash_C \mathsf{dyn}\ \tau'\ e' : \tau'$
by *boundary hole typing*
3. $\vdash_C e'$
by *inversion* (2)
4. $\vdash_C e''$
by *dynamic preservation* (3)
5. $\vdash_C \mathsf{dyn}\ \tau'\ e'' : \tau'$
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :

IF e' is a value :
1. $e \rightarrow_{\text{C-S}} E[\mathcal{S}_C(\tau', e')]$
2. $\vdash_C \mathsf{stat}\ \tau'\ e'$
by *boundary hole typing*
3. $\vdash_C e' : \tau'$
by *inversion* (2)
4. $\vdash_C \mathcal{S}_C(\tau', e')$
by $\mathcal{S}_C$ *soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\text{C-S}} e''$:
1. $e \rightarrow_{\text{C-S}} E[\mathsf{stat}\ \tau'\ e'']$
2. $\vdash_C \mathsf{stat}\ \tau'\ e'$
by *boundary hole typing*

3. $\vdash_C e' : \tau'$
   by *inversion* (2)
4. $\vdash_C e'' : \tau'$
   by *static preservation* (3)
5. $\vdash_C \mathsf{stat}\ \tau'\ e''$
   by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{Err}]$:

1. $e \rightarrow_{\text{C-S}} \mathsf{Err}$
2. QED by $\vdash_C \mathsf{Err} : \tau$

☐

Lemma 5.10 : HC *dynamic preservation*

If $\vdash_C e$ and $e \rightarrow_{\text{C-D}} e'$ then $\vdash_C e'$

Proof:

By the *boundary factoring* lemma, there are seven cases.

CASE e is a value :

1. Contradiction by $e \rightarrow_{\text{C-D}} e'$

CASE $e = E^\bullet[v_0\ v_1]$:

IF $v_0 = \lambda x.\, e'$
$\wedge\ e \rightarrow_{\text{C-D}} E^\bullet[e'[x \leftarrow v_1]]$:

1. $\vdash_C v_0\ v_1$
   by *dynamic hole typing*
2. $\vdash_C v_0$
   $\wedge \vdash_C v_1$
   by *inversion* (1)
3. $x \vdash_C e'$
   by *inversion* (2)
4. $\vdash_C e'[x \leftarrow v_1]$
   by *substitution* (2, 3)
5. QED *hole substitution* (4)

ELSE $v_0 = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v_f$
$\wedge\ e \rightarrow_{\text{C-D}} E^\bullet[\mathsf{stat}\ \tau_c\ (v_f\ (\mathsf{dyn}\ \tau_d\ v_1))]$:

1. $\vdash_C v_0\ v_1$
   by *dynamic hole typing*
2. $\vdash_C v_0$
   $\wedge \vdash_C v_1$
   by *inversion* (1)
3. $\vdash_C v_f : \tau_d \Rightarrow \tau_c$
   by *inversion* (2)
4. $\vdash_C \mathsf{dyn}\ \tau_d\ v_1 : \tau_d$
   by (2)
5. $\vdash_C v_f\ (\mathsf{dyn}\ \tau_d\ v_1) : \tau_c$
   by (3, 4)
6. $\vdash_C \mathsf{stat}\ \tau_c\ (v_f\ (\mathsf{dyn}\ \tau_d\ v_1))$
   by (5)
7. QED by *hole substitution*

CASE $e = E^\bullet[op^1\ v]$:

IF $v = \mathsf{mon}\,(\tau_0 \times \tau_1)\, v'$
$\wedge\ op^1 = \mathsf{fst}$
$\wedge\ e \rightarrow_{\text{C-D}} E[\mathsf{stat}\ \tau_0\ (\mathsf{fst}\ v')]$:

1. $\vdash_C op^1\ v$
   by *dynamic hole typing*
2. $\vdash_C \mathsf{mon}\,(\tau_0 \times \tau_1)\, v'$
   by *inversion* (1)
3. $\vdash_C v' : \tau_0 \times \tau_1$
   by *inversion* (2)
4. $\vdash_C \mathsf{fst}\ v' : \tau_0$
   by (3)
5. $\vdash_C \mathsf{stat}\ \tau_0\ (\mathsf{fst}\ v')$
   by (4)
6. QED by *hole substitution*

IF $v = \mathsf{mon}\,(\tau_0 \times \tau_1)\, v'$
$\wedge\ op^1 = \mathsf{snd}$
$\wedge\ e \rightarrow_{\text{C-D}} E[\mathsf{stat}\ \tau_0\ (\mathsf{snd}\ v')]$:

1. $\vdash_C op^1\ v$
   by *dynamic hole typing*
2. $\vdash_C \mathsf{mon}\,(\tau_0 \times \tau_1)\, v'$
   by *inversion* (1)
3. $\vdash_C v' : \tau_0 \times \tau_1$
   by *inversion* (2)
4. $\vdash_C \mathsf{snd}\ v' : \tau_1$
   by (3)
5. $\vdash_C \mathsf{stat}\ \tau_1\ (\mathsf{snd}\ v')$
   by (4)
6. QED by *hole substitution*

IF $v = \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{fst}$
$\wedge\ e \rightarrow_{\text{C-D}} E^\bullet[v_0]$:

1. $\vdash_C op^1\ v$
   by *dynamic hole typing*
2. $\vdash_C v$
   by *inversion* (1)
3. $\vdash_C v_0$
   by *inversion* (2)
4. QED by *hole substitution*

ELSE $v = \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{snd}$
$\wedge\ e \rightarrow_{\text{C-D}} E^\bullet[v_1]$:

1. $\vdash_C op^1\ v$
   by *dynamic hole typing*
2. $\vdash_C v$
   by *inversion* (1)
3. $\vdash_C v_1$
   by *inversion* (2)
4. QED by *hole substitution*

CASE $e = E^\bullet[op^2\ v_0\ v_1]$:

1. $e \rightarrow_{\text{C-D}} E^\bullet[\delta(op^2, v_0, v_1)]$
2. $\vdash_C op^2\ v_0\ v_1$
   by *dynamic hole typing*
3. $\vdash_C v_0$
   $\wedge \vdash_C v_1$
   by *inversion* (1)
4. $\vdash_C \delta(op^2, v_0, v_1)$
   by *δ preservation* (2)
5. QED by *hole substitution* (3)

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :

IF e' is a value :
1. $e \rightarrow_{\text{C-D}} E[\mathcal{D}_{\text{C}}(\tau', e')]$
2. $\vdash_{\text{C}} \mathsf{dyn}\ \tau'\ e' : \tau'$ by *boundary hole typing*
3. $\vdash_{\text{C}} e'$ by *inversion* (2)
4. $\vdash_{\text{C}} \mathcal{D}_{\text{C}}(\tau', e') : \tau'$ by $\mathcal{D}_{\text{C}}$ *soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\text{C-D}} e''$:
1. $e \rightarrow_{\text{C-D}} E[\mathsf{dyn}\ \tau'\ e'']$
2. $\vdash_{\text{C}} \mathsf{dyn}\ \tau'\ e' : \tau'$ by *boundary hole typing*
3. $\vdash_{\text{C}} e'$ $\wedge\ \tau' \leqslant: \tau''$ by *inversion* (2)
4. $\vdash_{\text{C}} e''$ by *dynamic preservation* (3)
5. $\vdash_{\text{C}} \mathsf{dyn}\ \tau'\ e'' : \tau'$ by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :

IF $e' \in v$:
1. $e \rightarrow_{\text{C-D}} E[\mathcal{S}_{\text{C}}(\tau', e')]$
2. $\vdash_{\text{C}} \mathsf{stat}\ \tau'\ e'$ by *boundary hole typing*
3. $\vdash_{\text{C}} e' : \tau'$ by *inversion* (2)
4. $\vdash_{\text{C}} \mathcal{S}_{\text{C}}(\tau', e')$ by $\mathcal{S}_{\text{C}}$ *soundness* (3)
5. QED by *hole substitution* (5)

ELSE $e' \rightarrow_{\text{C-S}} e''$:
1. $e \rightarrow_{\text{C-D}} E[\mathsf{stat}\ \tau'\ e'']$
2. $\vdash_{\text{C}} \mathsf{stat}\ \tau'\ e'$ by *boundary hole typing*
3. $\vdash_{\text{C}} e' : \tau'$ by *inversion* (2)
4. $\vdash_{\text{C}} e'' : \tau'$ by *static preservation* (3)
5. $\vdash_{\text{C}} \mathsf{stat}\ \tau'\ e''$ by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{Err}]$:
1. $e \rightarrow_{\text{C-D}} \mathsf{Err}$
2. QED $\vdash_{\text{C}} \mathsf{Err}$

□

Lemma 5.11 : HC *static boundary factoring*

If $\vdash_{\text{C}} e : \tau$ then one of the following holds:
- e is a value
- $e = E^{\bullet}[v_0\ v_1]$
- $e = E^{\bullet}[op^1\ v]$
- $e = E^{\bullet}[op^2\ v_0\ v_1]$
- $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{Err}]$

Proof:
By the *boundary factoring* lemma for the higher-order embedding. (The only difference is the meaning of *e is a value*.)
□

Lemma 5.12 : HC *dynamic boundary factoring*

If $\vdash_{\text{C}} e$ then one of the following holds:
- e is a value
- $e = E^{\bullet}[v_0\ v_1]$
- $e = E^{\bullet}[op^1\ v]$
- $e = E^{\bullet}[op^2\ v_0\ v_1]$
- $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{Err}]$

Proof:
By the *boundary factoring* lemma for the higher-order embedding.
□

Lemma 5.13 : HC *static hole typing*

If $\vdash_{\text{C}} E^{\bullet}[e] : \tau$ then the derivation contains a sub-term $\vdash_{\text{C}} e : \tau'$

Proof (sketch): Similar to the *static hole typing* lemma for the higher-order embedding. □

Lemma 5.14 : HC *dynamic hole typing*

If $\vdash_{\text{C}} E^{\bullet}[e]$ then the derivation contains a sub-term $\vdash_{\text{C}} e$

Proof (sketch): Similar to the *static hole typing* lemma for the higher-order embedding. □

Lemma 5.15 : HC *boundary hole typing*
- If $\vdash_{\text{C}} E[\mathsf{dyn}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_{\text{C}} \mathsf{dyn}\ \tau\ e : \tau$
- If $\vdash_{\text{C}} E[\mathsf{dyn}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_{\text{C}} \mathsf{dyn}\ \tau\ e : \tau$
- If $\vdash_{\text{C}} E[\mathsf{stat}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_{\text{C}} \mathsf{stat}\ \tau\ e$
- If $\vdash_{\text{C}} E[\mathsf{stat}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_{\text{C}} \mathsf{stat}\ \tau\ e$

Proof (sketch): Similar to the proof for the higher-order *boundary hole typing* lemma. □

Lemma 5.16 : HC *hole substitution*
- If $\vdash_{\text{C}} E[e]$ and the derivation contains a sub-term $\vdash_{\text{C}} e : \tau'$ and $\vdash_{\text{C}} e' : \tau'$ then $\vdash_{\text{C}} E[e']$.
- If $\vdash_{\text{C}} E[e]$ and the derivation contains a sub-term $\vdash_{\text{C}} e$ and $\vdash_{\text{C}} e'$ then $\vdash_{\text{C}} E[e']$.
- If $\vdash_{\text{C}} E[e] : \tau$ and the derivation contains a sub-term $\vdash_{\text{C}} e : \tau'$ and $\vdash_{\text{C}} e' : \tau'$ then $\vdash_{\text{C}} E[e'] : \tau$.
- If $\vdash_{\text{C}} E[e] : \tau$ and the derivation contains a sub-term $\vdash_{\text{C}} e$ and $\vdash_{\text{C}} e'$ then $\vdash_{\text{C}} E[e'] : \tau$.

Proof (sketch): Similar to the proof of the higher-order *hole substitution* lemma, just replacing $\vdash_{\mathsf{H}}$ with $\vdash_{\mathsf{C}}$. □

Lemma 5.17 : $\vdash_{\mathsf{C}}$ *static inversion*

- If $\Gamma \vdash_{\mathsf{C}} x : \tau$ then $(x{:}\tau') \in \Gamma$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{C}} \lambda(x : \tau'_d).\, e' : \tau$ then $(x : \tau'_d), \Gamma \vdash_{\mathsf{C}} e' : \tau'_c$ and $\tau'_d \Rightarrow \tau'_c \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{C}} \langle e_0, e_1 \rangle : \tau_0 {\times} \tau_1$ then $\Gamma \vdash_{\mathsf{C}} e_0 : \tau'_0$ and $\Gamma \vdash_{\mathsf{C}} e_1 : \tau'_1$ and $\tau'_0 \leqslant: \tau_0$ and $\tau'_1 \leqslant: \tau_1$
- If $\Gamma \vdash_{\mathsf{C}} e_0\ e_1 : \tau_c$ then $\Gamma \vdash_{\mathsf{C}} e_0 : \tau'_d \Rightarrow \tau'_c$ and $\Gamma \vdash_{\mathsf{C}} e_1 : \tau'_d$ and $\tau'_c \leqslant: \tau_c$
- If $\Gamma \vdash_{\mathsf{C}} \mathsf{fst}\, e : \tau$ then $\Gamma \vdash_{\mathsf{C}} e : \tau_0 {\times} \tau_1$ and $\Delta(\mathsf{fst}, \tau_0 {\times} \tau_1) = \tau_0$ and $\tau_0 \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{C}} \mathsf{snd}\, e : \tau$ then $\Gamma \vdash_{\mathsf{C}} e : \tau_0 {\times} \tau_1$ and $\Delta(\mathsf{snd}, \tau_0 {\times} \tau_1) = \tau_1$ and $\tau_1 \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{C}} op^2\ e_0\ e_1 : \tau$ then $\Gamma \vdash_{\mathsf{C}} e_0 : \tau_0$ and $\Gamma \vdash_{\mathsf{C}} e_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau'$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{C}} \mathsf{mon}\ \tau'_0 {\times} \tau'_1\ v' : \tau_0 {\times} \tau_1$ then $\Gamma \vdash_{\mathsf{C}} v'$ and $\tau'_0 {\times} \tau'_1 \leqslant: \tau_0 {\times} \tau_1$
- If $\Gamma \vdash_{\mathsf{C}} \mathsf{mon}\ \tau'_d \Rightarrow \tau'_c\ v' : \tau_d \Rightarrow \tau_c$ then $\Gamma \vdash_{\mathsf{C}} v'$ and $\tau'_d \Rightarrow \tau'_c \leqslant: \tau_d \Rightarrow \tau_c$
- If $\Gamma \vdash_{\mathsf{C}} \mathsf{dyn}\ \tau'\ e' : \tau$ then $\Gamma \vdash_{\mathsf{C}} e'$ and $\tau' \leqslant: \tau$

Proof:

QED by the definition of $\Gamma \vdash_{\mathsf{C}} e : \tau$

□

Lemma 5.18 : $\vdash_{\mathsf{C}}$ *dynamic inversion*

- If $\Gamma \vdash_{\mathsf{C}} x$ then $x \in \Gamma$
- If $\Gamma \vdash_{\mathsf{C}} \lambda x.\, e'$ then $x, \Gamma \vdash_{\mathsf{C}} e'$
- If $\Gamma \vdash_{\mathsf{C}} \langle e_0, e_1 \rangle$ then $\Gamma \vdash_{\mathsf{C}} e_0$ and $\Gamma \vdash_{\mathsf{C}} e_1$
- If $\Gamma \vdash_{\mathsf{C}} e_0\ e_1$ then $\Gamma \vdash_{\mathsf{C}} e_0$ and $\Gamma \vdash_{\mathsf{C}} e_1$
- If $\Gamma \vdash_{\mathsf{C}} op^1\ e_0$ then $\Gamma \vdash_{\mathsf{C}} e_0$
- If $\Gamma \vdash_{\mathsf{C}} op^2\ e_0\ e_1$ then $\Gamma \vdash_{\mathsf{C}} e_0$ and $\Gamma \vdash_{\mathsf{C}} e_1$
- If $\Gamma \vdash_{\mathsf{C}} \mathsf{mon}\ \tau_d \Rightarrow \tau_c\ v'$ then $\Gamma \vdash_{\mathsf{C}} v' : \tau_d \Rightarrow \tau_c$
- If $\Gamma \vdash_{\mathsf{C}} \mathsf{mon}\ \tau_0 {\times} \tau_1\ v'$ then $\Gamma \vdash_{\mathsf{C}} v' : \tau_0 {\times} \tau_1$
- If $\Gamma \vdash_{\mathsf{C}} \mathsf{stat}\ \tau'\ e'$ then $\Gamma \vdash_{\mathsf{C}} e' : \tau'$

Proof:

QED by the definition of $\Gamma \vdash_{\mathsf{C}} e$

□

Lemma 5.19 : HC *canonical forms*

- If $\vdash_{\mathsf{C}} v : \tau_0 {\times} \tau_1$ then either:
  - $v = \langle v_0, v_1 \rangle$
  - or $v = \mathsf{mon}\,(\tau'_0 {\times} \tau'_1)\, v'$
    $\wedge\ \tau'_0 {\times} \tau'_1 \leqslant: \tau_0 {\times} \tau_1$
- If $\vdash_{\mathsf{C}} v : \tau_d \Rightarrow \tau_c$ then either:
  - $v = \lambda(x{:}\tau_x).\, e'$
    $\wedge\ \tau_d \leqslant: \tau_x$
  - or $v = \mathsf{mon}\,(\tau'_d \Rightarrow \tau'_c)\, v'$
    $\wedge\ \tau'_d \Rightarrow \tau'_c \leqslant: \tau_d \Rightarrow \tau_c$
- If $\vdash_{\mathsf{C}} v : \mathsf{Int}$ then $v = i$
- If $\vdash_{\mathsf{C}} v : \mathsf{Nat}$ then $v = i$ and $v \in \mathbb{N}$

Proof:

QED by definition of $\vdash_{\mathsf{C}} e : \tau$

□

Lemma 5.20 : Δ *type soundness*

If $\vdash_{\mathsf{C}} v_0 : \tau_0$ and $\vdash_{\mathsf{C}} v_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau$ then one of the following holds:

- $\delta(op^2, v_0, v_1) = v$ and $\vdash_{\mathsf{C}} v : \tau$, or
- $\delta(op^2, v_0, v_1) = \mathsf{BndryErr}$

Proof (sketch): Similar to the proof for the higher-order Δ *type soundness* lemma. □

Lemma 5.21 : δ *preservation*

- If $\vdash_{\mathsf{C}} v$ and $\delta(op^1, v) = v'$ then $\vdash_{\mathsf{C}} v'$
- If $\vdash_{\mathsf{C}} v_0$ and $\vdash_{\mathsf{C}} v_1$ and $\delta(op^2, v_0, v_1) = v'$ then $\vdash_{\mathsf{C}} v'$

Proof (sketch): Similar to the proof for the higher-order δ *preservation* lemma. □

Lemma 5.22 : HC *substitution*

- If $(x{:}\tau_x), \Gamma \vdash_{\mathsf{C}} e$ and $\vdash_{\mathsf{C}} v : \tau_x$ then $\Gamma \vdash_{\mathsf{C}} e[x{\leftarrow}v]$
- If $x, \Gamma \vdash_{\mathsf{C}} e$ and $\vdash_{\mathsf{C}} v$ then $\Gamma \vdash_{\mathsf{C}} e[x{\leftarrow}v]$
- If $(x{:}\tau_x), \Gamma \vdash_{\mathsf{C}} e : \tau$ and $\vdash_{\mathsf{C}} v : \tau_x$ then $\Gamma \vdash_{\mathsf{C}} e[x{\leftarrow}v] : \tau$
- If $x, \Gamma \vdash_{\mathsf{C}} e : \tau$ and $\vdash_{\mathsf{C}} v$ then $\Gamma \vdash_{\mathsf{C}} e[x{\leftarrow}v] : \tau$

Proof (sketch): Similar to the proof for the higher-order *substitution* lemma. □

Lemma 5.23 : *weakening*

- If $\Gamma \vdash_{\mathsf{C}} e$ then $x, \Gamma \vdash_{\mathsf{C}} e$
- If $\Gamma \vdash_{\mathsf{C}} e : \tau$ then $(x{:}\tau'), \Gamma \vdash_{\mathsf{C}} e : \tau$

Proof:

QED because e is closed under Γ

□

A.6 (HF) Forgetful Embedding

A.6.1 Forgetful Definitions

$\boxed{\text{Language HF}}$

$$
\begin{array}{lcl}
e & = & x \mid v \mid \langle e, e\rangle \mid e\; e \mid op^1\; e \mid op^2\; e\; e \mid \\
 & & \mathsf{dyn}\; \tau\; e \mid \mathsf{stat}\; \tau\; e \mid \mathsf{Err} \mid \mathsf{chk}\; \tau\; e \\
v & = & i \mid \langle v, v\rangle \mid \lambda x.\, e \mid \lambda(x{:}\tau).\, e \mid \\
 & & \mathsf{mon}\,(\tau \Rightarrow \tau)\,(\lambda x.\, e) \mid \mathsf{mon}\,(\tau \Rightarrow \tau)\,(\lambda(x{:}\tau).\, e) \mid \\
 & & \mathsf{mon}\,(\tau \times \tau)\,\langle v, v\rangle \\
\tau & = & \mathsf{Nat} \mid \mathsf{Int} \mid \tau \times \tau \mid \tau \Rightarrow \tau \\
\Gamma & = & \cdot \mid x, \Gamma \mid (x{:}\tau), \Gamma \\
\mathsf{Err} & = & \mathsf{BndryErr} \mid \mathsf{TagErr} \\
r & = & v \mid \mathsf{Err} \\
E^{\bullet} & = & [\,] \mid E^{\bullet}\; e \mid v\; E^{\bullet} \mid \langle E^{\bullet}, e\rangle \mid \langle v, E^{\bullet}\rangle \mid \\
 & & op^1\; E^{\bullet} \mid op^2\; E^{\bullet}\; e \mid op^2\; v\; E^{\bullet} \\
E & = & E^{\bullet} \mid E\; e \mid v\; E \mid \langle E, e\rangle \mid \langle v, E\rangle \mid op^1\; E \mid \\
 & & op^2\; E\; e \mid op^2\; v\; E \mid \mathsf{dyn}\; \tau\; E \mid \mathsf{stat}\; \tau\; E
\end{array}
$$

$\boxed{\Delta : op^1 \times \tau \longrightarrow \tau}$

$$
\begin{array}{l}
\Delta(\mathsf{fst}, \tau_0 \times \tau_1) \;\; = \tau_0 \\
\Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1
\end{array}
$$

$\boxed{\Delta : op^2 \times \tau \times \tau \longrightarrow \tau}$

$$
\begin{array}{l}
\Delta(op^2, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat} \\
\Delta(op^2, \mathsf{Int}, \mathsf{Int}) \;\;\; = \mathsf{Int}
\end{array}
$$

$\boxed{\tau \leqslant: \tau}$

$$
\frac{}{\mathsf{Nat} \leqslant: \mathsf{Int}} \qquad
\frac{\tau_d' \leqslant: \tau_d \quad \tau_c \leqslant: \tau_c'}{\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'} \qquad
\frac{\tau_0 \leqslant: \tau_0' \quad \tau_1 \leqslant: \tau_1'}{\tau_0 \times \tau_1 \leqslant: \tau_0' \times \tau_1'}
$$

$$
\frac{}{\tau \leqslant: \tau} \qquad
\frac{\tau \leqslant: \tau' \quad \tau' \leqslant: \tau''}{\tau \leqslant: \tau''}
$$

$\boxed{\Gamma \vdash e}$

$$
\frac{x \in \Gamma}{\Gamma \vdash x} \qquad
\frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x.\, e} \qquad
\frac{}{\Gamma \vdash i} \qquad
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1\rangle}
$$

$$
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0\; e_1} \qquad
\frac{\Gamma \vdash e}{\Gamma \vdash op^1\; e} \qquad
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2\; e_0\; e_1} \qquad
\frac{}{\Gamma \vdash \mathsf{Err}}
$$

$$
\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat}\; \tau\; e}
$$

$\boxed{\Gamma \vdash e : \tau}$

$$
\frac{(x{:}\tau) \in \Gamma}{\Gamma \vdash x : \tau} \qquad
\frac{(x{:}\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x{:}\tau_d).\, e : \tau_d \Rightarrow \tau_c} \qquad
\frac{i \in \mathbb{N}}{\Gamma \vdash i : \mathsf{Nat}}
$$

$$
\frac{}{\Gamma \vdash i : \mathsf{Int}} \qquad
\frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_0 \\ \Gamma \vdash e_1 : \tau_1\end{array}}{\Gamma \vdash \langle e_0, e_1\rangle : \tau_0 \times \tau_1} \qquad
\frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \\ \Gamma \vdash e_1 : \tau_d\end{array}}{\Gamma \vdash e_0\; e_1 : \tau_c}
$$

$$
\frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_0 \\ \Delta(op^1, \tau_0) = \tau\end{array}}{\Gamma \vdash op^1\; e_0 : \tau} \qquad
\frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_0 \\ \Gamma \vdash e_1 : \tau_1 \\ \Delta(op^2, \tau_0, \tau_1) = \tau\end{array}}{\Gamma \vdash op^2\; e_0\; e_1 : \tau} \qquad
\frac{\begin{array}{c}\Gamma \vdash e : \tau' \\ \tau' \leqslant: \tau\end{array}}{\Gamma \vdash e : \tau} \qquad
\frac{}{\Gamma \vdash \mathsf{Err} : \tau}
$$

$$
\frac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn}\; \tau\; e : \tau}
$$

$\boxed{\Gamma \vdash_{\mathsf{F}} e}$

$$
\frac{x \in \Gamma}{\Gamma \vdash_{\mathsf{F}} x} \qquad
\frac{x, \Gamma \vdash_{\mathsf{F}} e}{\Gamma \vdash_{\mathsf{F}} \lambda x.\, e} \qquad
\frac{}{\Gamma \vdash_{\mathsf{F}} i} \qquad
\frac{\Gamma \vdash_{\mathsf{F}} e_0 \quad \Gamma \vdash_{\mathsf{F}} e_1}{\Gamma \vdash_{\mathsf{F}} \langle e_0, e_1\rangle}
$$

$$
\frac{\Gamma \vdash_{\mathsf{F}} e_0 \quad \Gamma \vdash_{\mathsf{F}} e_1}{\Gamma \vdash_{\mathsf{F}} e_0\; e_1} \qquad
\frac{\Gamma \vdash_{\mathsf{F}} e}{\Gamma \vdash_{\mathsf{F}} op^1\; e} \qquad
\frac{\Gamma \vdash_{\mathsf{F}} e_0 \quad \Gamma \vdash_{\mathsf{F}} e_1}{\Gamma \vdash_{\mathsf{F}} op^2\; e_0\; e_1} \qquad
\frac{}{\Gamma \vdash_{\mathsf{F}} \mathsf{Err}}
$$

$$
\frac{\Gamma \vdash_{\mathsf{F}} e : \tau}{\Gamma \vdash_{\mathsf{F}} \mathsf{stat}\; \tau\; e} \qquad
\frac{\Gamma \vdash_{\mathsf{F}} v_0 : \tau_0' \quad \Gamma \vdash_{\mathsf{F}} v_1 : \tau_1'}{\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1\rangle}
$$

$$
\frac{\Gamma \vdash_{\mathsf{F}} v_0 \quad \Gamma \vdash_{\mathsf{F}} v_1}{\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1\rangle} \qquad
\frac{\Gamma \vdash_{\mathsf{F}} \lambda x.\, e}{\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,\lambda x.\, e}
$$

$$
\frac{\Gamma \vdash_{\mathsf{F}} \lambda(x{:}\tau_d').\, e : \tau_d' \Rightarrow \tau_c'}{\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,\lambda(x{:}\tau_d').\, e}
$$

$\boxed{\Gamma \vdash_{\mathsf{F}} e : \tau}$

$$\frac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash_{\mathsf{F}} x : \tau} \qquad \frac{(x\!:\!\tau_d), \Gamma \vdash_{\mathsf{F}} e : \tau_c}{\Gamma \vdash_{\mathsf{F}} \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c} \qquad \frac{i \in \mathbb{N}}{\Gamma \vdash_{\mathsf{F}} i : \mathsf{Nat}}$$

$$\frac{}{\Gamma \vdash_{\mathsf{F}} i : \mathsf{Int}} \qquad \frac{\begin{array}{c}\Gamma \vdash_{\mathsf{F}} e_0 : \tau_0 \\ \Gamma \vdash_{\mathsf{F}} e_1 : \tau_1\end{array}}{\Gamma \vdash_{\mathsf{F}} \langle e_0, e_1 \rangle : \tau_0 \times \tau_1} \qquad \frac{\begin{array}{c}\Gamma \vdash_{\mathsf{F}} e_0 : \tau_d \Rightarrow \tau_c \\ \Gamma \vdash_{\mathsf{F}} e_1 : \tau_d\end{array}}{\Gamma \vdash_{\mathsf{F}} e_0\ e_1 : \tau_c}$$

$$\frac{\begin{array}{c}\Gamma \vdash_{\mathsf{F}} e_0 : \tau_0 \\ \Delta(op^1, \tau_0) = \tau\end{array}}{\Gamma \vdash_{\mathsf{F}} op^1\ e_0 : \tau} \qquad \frac{\begin{array}{c}\Gamma \vdash_{\mathsf{F}} e_0 : \tau_0 \\ \Gamma \vdash_{\mathsf{F}} e_1 : \tau_1 \\ \Delta(op^2, \tau_0, \tau_1) = \tau\end{array}}{\Gamma \vdash_{\mathsf{F}} op^2\ e_0\ e_1 : \tau} \qquad \frac{\begin{array}{c}\Gamma \vdash_{\mathsf{F}} e : \tau' \\ \tau' <: \tau\end{array}}{\Gamma \vdash_{\mathsf{F}} e : \tau}$$

$$\frac{}{\Gamma \vdash_{\mathsf{F}} \mathsf{Err} : \tau} \qquad \frac{\Gamma \vdash_{\mathsf{F}} e}{\Gamma \vdash_{\mathsf{F}} \mathsf{dyn}\ \tau\ e : \tau}$$

$$\frac{\Gamma \vdash_{\mathsf{F}} v_0 : \tau_0' \quad \Gamma \vdash_{\mathsf{F}} v_1 : \tau_1'}{\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\, \langle v_0, v_1 \rangle : (\tau_0 \times \tau_1)}$$

$$\frac{\Gamma \vdash_{\mathsf{F}} v_0 \quad \Gamma \vdash_{\mathsf{F}} v_1}{\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\, \langle v_0, v_1 \rangle : (\tau_0 \times \tau_1)}$$

$$\frac{\Gamma \vdash_{\mathsf{F}} \lambda x.\, e}{\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, \lambda x.\, e : (\tau_d \Rightarrow \tau_c)}$$

$$\frac{\Gamma \vdash_{\mathsf{F}} \lambda(x\!:\!\tau_d').\, e : \tau_d' \Rightarrow \tau_c'}{\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, \lambda(x\!:\!\tau_d').\, e : (\tau_d \Rightarrow \tau_c)} \qquad \frac{\Gamma \vdash_{\mathsf{F}} e : \tau'}{\Gamma \vdash_{\mathsf{F}} \mathsf{chk}\ \tau\ e : \tau}$$

$\boxed{\delta(op^1, v) = e}$

$$\begin{array}{ll}
\delta(\mathsf{fst}, \langle v_0, v_1 \rangle) & = v_0 \\
\delta(\mathsf{snd}, \langle v_0, v_1 \rangle) & = v_1
\end{array}$$

$\boxed{\delta(op^2, v, v) = e}$

$$\begin{array}{ll}
\delta(\mathsf{sum}, i_0, i_1) & = i_0 + i_1 \\
\delta(\mathsf{quotient}, i_0, 0) & = \mathsf{BndryErr} \\
\delta(\mathsf{quotient}, i_0, i_1) & = \lfloor i_0 / i_1 \rfloor \\
\quad \text{if } i_1 \neq 0 &
\end{array}$$

$\boxed{\mathcal{D}_{\mathsf{F}} : \tau \times v \longrightarrow e}$

$$\mathcal{D}_{\mathsf{F}}(\tau, v) = \mathcal{X}(\tau, v)$$

$\boxed{\mathcal{S}_{\mathsf{F}} : \tau \times v \longrightarrow e}$

$$\mathcal{S}_{\mathsf{F}}(\tau, v) = \mathcal{X}(\tau, v)$$

$\boxed{\mathcal{X} : \tau \times v \longrightarrow e}$

$$\begin{array}{ll}
\mathcal{X}(\tau_d \Rightarrow \tau_c, \lambda x.\, e) & = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda x.\, e) \\
\mathcal{X}(\tau_d \Rightarrow \tau_c, \lambda(x\!:\!\tau).\, e) & = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda(x\!:\!\tau).\, e) \\
\mathcal{X}(\tau_d \Rightarrow \tau_c, \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\, v') & = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v' \\
\mathcal{X}(\tau_0 \times \tau_1, \langle v_0, v_1 \rangle) & = \mathsf{mon}\,(\tau_0 \times \tau_1)\, \langle v_0, v_1 \rangle \\
\mathcal{X}(\tau_0 \times \tau_1, \mathsf{mon}\,(\tau_0' \times \tau_1')\, v') & = \mathsf{mon}\,(\tau_0 \times \tau_1)\, v' \\
\mathcal{X}(\mathsf{Int}, i) & = i \\
\mathcal{X}(\mathsf{Nat}, i) & = i \\
\quad \text{if } i \in \mathbb{N} & \\
\mathcal{X}(\tau, v) & = \mathsf{BndryErr} \\
\quad \text{otherwise} &
\end{array}$$

$\boxed{e \rhd_{\mathsf{S\text{-}1}} e}$

$$\begin{array}{ll}
\mathsf{dyn}\ \tau\ v & \rhd_{\mathsf{S\text{-}1}} \mathcal{D}_{\mathsf{F}}(\tau, v) \\
\mathsf{chk}\ \tau\ v & \rhd_{\mathsf{S\text{-}1}} \mathcal{X}(\tau, v) \\
(\mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda x.\, e))\ v & \rhd_{\mathsf{S\text{-}1}} \mathsf{dyn}\ \tau_c\ e' \\
\quad \text{where } e' = (\lambda x.\, e)\ (\mathcal{X}(\tau_d, v)) & \\
(\mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda(x\!:\!\tau).\, e))\ v & \rhd_{\mathsf{S\text{-}1}} \mathsf{chk}\ \tau_c\ e' \\
\quad \text{where } e' = (\lambda(x\!:\!\tau).\, e)\ (\mathcal{X}(\tau, v)) & \\
\mathsf{fst}\,(\mathsf{mon}\,(\tau_0 \times \tau_1)\, \langle v_0, v_1 \rangle) & \rhd_{\mathsf{S\text{-}1}} \mathcal{X}(\tau_0, v_0) \\
\mathsf{snd}\,(\mathsf{mon}\,(\tau_0 \times \tau_1)\, \langle v_0, v_1 \rangle) & \rhd_{\mathsf{S\text{-}1}} \mathcal{X}(\tau_1, v_1) \\
(\lambda(x\!:\!\tau).\, e)\ v & \rhd_{\mathsf{S\text{-}1}} e[x \leftarrow v] \\
op^1\ v & \rhd_{\mathsf{S\text{-}1}} \delta(op^1, v) \\
op^2\ v_0\ v_1 & \rhd_{\mathsf{S\text{-}1}} \delta(op^2, v_0, v_1)
\end{array}$$

$\boxed{e \rhd_{\mathsf{D\text{-}1}} e}$

$$\begin{array}{ll}
\mathsf{stat}\ \tau\ v & \rhd_{\mathsf{D\text{-}1}} \mathcal{S}_{\mathsf{F}}(\tau, v) \\
v_0\ v_1 & \rhd_{\mathsf{D\text{-}1}} \mathsf{TagErr} \\
\quad \text{if } v_0 \in \mathbb{Z} \text{ or } v_0 = \langle v, v' \rangle & \\
(\mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda x.\, e))\ v & \rhd_{\mathsf{D\text{-}1}} (\lambda x.\, e)\ v \\
(\mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda(x\!:\!\tau).\, e))\ v & \rhd_{\mathsf{D\text{-}1}} \mathsf{stat}\ \tau_c\ e' \\
\quad \text{where } e' = \mathsf{chk}\ \tau_c\ ((\lambda(x\!:\!\tau).\, e)\ (\mathcal{X}(\tau, v))) & \\
(\lambda x.\, e)\ v & \rhd_{\mathsf{H\text{-}D}} e[x \leftarrow v] \\
\mathsf{fst}\,(\mathsf{mon}\,(\tau_0 \times \tau_1)\, \langle v_0, v_1 \rangle) & \rhd_{\mathsf{D\text{-}1}} \mathcal{X}(\tau_0, v_0) \\
\mathsf{snd}\,(\mathsf{mon}\,(\tau_0 \times \tau_1)\, \langle v_0, v_1 \rangle) & \rhd_{\mathsf{D\text{-}1}} \mathcal{X}(\tau_1, v_1) \\
op^1\ v & \rhd_{\mathsf{H\text{-}D}} \mathsf{TagErr} \\
\quad \text{if } \delta(op^1, v) \text{ is undefined} & \\
op^1\ v & \rhd_{\mathsf{H\text{-}D}} \delta(op^1, v) \\
op^2\ v_0\ v_1 & \rhd_{\mathsf{H\text{-}D}} \mathsf{TagErr} \\
\quad \text{if } \delta(op^2, v_0, v_1) \text{ is undefined} & \\
op^2\ v_0\ v_1 & \rhd_{\mathsf{H\text{-}D}} \delta(op^2, v_0, v_1)
\end{array}$$

$\boxed{e \rightarrow_{\mathsf{F\text{-}S}} e}$

$$\begin{array}{lll}
E^{\bullet}[e] & \rightarrow_{\mathsf{F\text{-}S}} & E^{\bullet}[e'] \\
\quad \text{if } e \rhd_{\mathsf{S\text{-}1}} e' & & \\
E[\mathsf{stat}\ \tau\ E^{\bullet}[e]] & \rightarrow_{\mathsf{F\text{-}S}} & E[\mathsf{stat}\ \tau\ E^{\bullet}[e']] \\
\quad \text{if } e \rhd_{\mathsf{S\text{-}1}} e' & & \\
E[\mathsf{dyn}\ \tau\ E^{\bullet}[e]] & \rightarrow_{\mathsf{F\text{-}S}} & E[\mathsf{dyn}\ \tau\ E^{\bullet}[e']] \\
\quad \text{if } e \rhd_{\mathsf{D\text{-}1}} e' & & \\
E[\mathsf{Err}] & \rightarrow_{\mathsf{F\text{-}S}} & \mathsf{Err}
\end{array}$$

$\boxed{e \rightarrow_{\mathsf{F\text{-}D}} e}$

$$
\begin{array}{lll}
E^{\bullet}[e] & \rightarrow_{\mathsf{F\text{-}D}} & E^{\bullet}[e'] \\
\quad \text{if } e \rhd_{\mathsf{D\text{-}1}} e' & & \\
E[\mathsf{stat}\ \tau\ E^{\bullet}[e]] & \rightarrow_{\mathsf{F\text{-}D}} & E[\mathsf{stat}\ \tau\ E^{\bullet}[e']] \\
\quad \text{if } e \rhd_{\mathsf{S\text{-}1}} e' & & \\
E[\mathsf{dyn}\ \tau\ E^{\bullet}[e]] & \rightarrow_{\mathsf{F\text{-}D}} & E[\mathsf{dyn}\ \tau\ E^{\bullet}[e']] \\
\quad \text{if } e \rhd_{\mathsf{D\text{-}1}} e' & & \\
E[\mathsf{Err}] & \rightarrow_{\mathsf{F\text{-}D}} & \mathsf{Err}
\end{array}
$$

$\boxed{e \rightarrow^{*}_{\mathsf{F\text{-}S}} e}$ reflexive, transitive closure of $\rightarrow_{\mathsf{F\text{-}S}}$

$\boxed{e \rightarrow^{*}_{\mathsf{F\text{-}D}} e}$ reflexive, transitive closure of $\rightarrow_{\mathsf{F\text{-}D}}$

A.6.2 Forgetful Theorems

Theorem 6.0 : *static* HF-*soundness*

If $\vdash e : \tau$ then $\vdash_{\mathsf{F}} e : \tau$ and one of the following holds:

- $e \rightarrow^{*}_{\text{F-S}} v$ and $\vdash_{\mathsf{F}} v : \tau$
- $e \rightarrow^{*}_{\text{F-S}} E[\mathsf{dyn}\ \tau'\ e']$ and $e' \rhd_{\text{D-1}}$ TagErr
- $e \rightarrow^{*}_{\text{F-S}}$ BndryErr
- e diverges

Proof:

1. $\vdash_{\mathsf{F}} e : \tau$
   by *static subset*
2. QED by *static progress* and *static preservation*.

□

Theorem 6.1 : *dynamic* HF-*soundness*

If $\vdash e$ then $\vdash_{\mathsf{F}} e$ and one of the following holds:

- $e \rightarrow^{*}_{\text{F-D}} v$ and $\vdash_{\mathsf{F}} v$
- $e \rightarrow^{*}_{\text{F-D}} E[e']$ and $e' \rhd_{\text{D-1}}$ TagErr
- $e \rightarrow^{*}_{\text{F-D}}$ BndryErr
- e diverges

Proof:

1. $\vdash_{\mathsf{F}} e$
   by *dynamic subset*
2. QED by *dynamic progress* and *dynamic preservation*.

□

Corollary 6.2 : HF *static soundness*

If $\vdash e : \tau$ and e is boundary-free, then one of the following holds:

- $e \rightarrow^{*}_{\text{F-S}} v$ and $\vdash_{\mathsf{F}} v : \tau$
- $e \rightarrow^{*}_{\text{F-S}}$ BndryErr
- e diverges

Proof:

Consequence of the proof for *static* HF-*soundness*

□

A.6.3 Forgetful Lemmas

Lemma 6.3 : $\mathcal{X}(\cdot,\cdot)$ *soundness*

If $\Gamma \vdash_{\mathsf{F}} v$ or $\Gamma \vdash_{\mathsf{F}} v : \tau$
and $\mathcal{X}(\tau', v) = v'$,
then $\Gamma \vdash_{\mathsf{F}} v'$ and $\Gamma \vdash_{\mathsf{F}} v' : \tau'$

Proof:

By case analysis of the definition of $\mathcal{X}(\cdot,\cdot)$.

CASE $\mathcal{X}(\tau_d \Rightarrow \tau_c, v) = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v$:

IF $v = \lambda x.\, e$
$\wedge\ \Gamma \vdash_{\mathsf{F}} v$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v$
   by $\Gamma \vdash_{\mathsf{F}} v$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v : \tau_d \Rightarrow \tau_c$
   by $\Gamma \vdash_{\mathsf{F}} v$
3. QED

ELSE $v = \lambda(x\,{:}\,\tau_x).\, e$
$\wedge\ \Gamma \vdash_{\mathsf{F}} v : \tau_d' \Rightarrow \tau_c'$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v$
   by $\Gamma \vdash_{\mathsf{F}} v : \tau_d' \Rightarrow \tau_c'$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v : \tau_d \Rightarrow \tau_c$
   by $\Gamma \vdash_{\mathsf{F}} v : \tau_d' \Rightarrow \tau_c'$
3. QED

CASE $\mathcal{X}(\tau_d \Rightarrow \tau_c, \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,v') = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v'$:

IF $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,v'$:

IF $v' = \lambda x.\, e'$
$\wedge\ \Gamma \vdash_{\mathsf{F}} v'$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v'$
   by $\Gamma \vdash_{\mathsf{F}} v'$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v' : \tau_d \Rightarrow \tau_c$
   by $\Gamma \vdash_{\mathsf{F}} v'$
3. QED

ELSE $v' = \lambda(x\,{:}\,\tau_x).\, e'$
$\wedge\ \Gamma \vdash_{\mathsf{F}} v' : \tau_d'' \Rightarrow \tau_c''$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v'$
   by $\Gamma \vdash_{\mathsf{F}} v' : \tau_d'' \Rightarrow \tau_c''$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v' : \tau_d \Rightarrow \tau_c$
   by $\Gamma \vdash_{\mathsf{F}} v' : \tau_d'' \Rightarrow \tau_c''$
3. QED

ELSE $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,v' : \tau_d' \Rightarrow \tau_c'$:

IF $v' = \lambda x.\, e'$
$\wedge\ \Gamma \vdash_{\mathsf{F}} v'$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v'$
   by $\Gamma \vdash_{\mathsf{F}} v'$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v' : \tau_d \Rightarrow \tau_c$
   by $\Gamma \vdash_{\mathsf{F}} v'$
3. QED

ELSE $v' = \lambda(x\,{:}\,\tau_x).\, e'$
$\wedge\ \Gamma \vdash_{\mathsf{F}} v' : \tau_d'' \Rightarrow \tau_c''$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v'$
   by $\Gamma \vdash_{\mathsf{F}} v' : \tau_d'' \Rightarrow \tau_c''$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v' : \tau_d \Rightarrow \tau_c$
   by $\Gamma \vdash_{\mathsf{F}} v' : \tau_d'' \Rightarrow \tau_c''$
3. QED

CASE $\mathcal{X}(\tau_0{\times}\tau_1, \langle v_0, v_1\rangle) = \mathsf{mon}\,(\tau_0{\times}\tau_1)\,\langle v_0, v_1\rangle$:

IF $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0{\times}\tau_1)\,\langle v_0, v_1\rangle$
   by $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0{\times}\tau_1)\,\langle v_0, v_1\rangle : \tau_0{\times}\tau_1$
   by $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle$
3. QED

ELSE $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle : \tau_0'{\times}\tau_1'$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0{\times}\tau_1)\,\langle v_0, v_1\rangle$
   by $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle : \tau_0'{\times}\tau_1'$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0{\times}\tau_1)\,\langle v_0, v_1\rangle : \tau_0{\times}\tau_1$
   by $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle : \tau_0'{\times}\tau_1'$
3. QED

CASE $\mathcal{X}(\tau_0{\times}\tau_1, \mathsf{mon}\,(\tau_0'{\times}\tau_1')\,\langle v_0, v_1\rangle) = \mathsf{mon}\,(\tau_0{\times}\tau_1)\,\langle v_0, v_1\rangle$:

IF $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0{\times}\tau_1)\,\langle v_0, v_1\rangle$
   by $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0{\times}\tau_1)\,\langle v_0, v_1\rangle : \tau_0{\times}\tau_1$
   by $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle$
3. QED

ELSE $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle : \tau_0''{\times}\tau_1''$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0{\times}\tau_1)\,\langle v_0, v_1\rangle$
   by $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle : \tau_0''{\times}\tau_1''$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0{\times}\tau_1)\,\langle v_0, v_1\rangle : \tau_0{\times}\tau_1$
   by $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle : \tau_0''{\times}\tau_1''$
3. QED

CASE $\mathcal{X}(\mathsf{Int}, i) = i$:

1. $\Gamma \vdash_{\mathsf{F}} i$
2. $\Gamma \vdash_{\mathsf{F}} i : \mathsf{Int}$
3. QED

CASE $\mathcal{X}(\mathsf{Nat}, i) = i$:

1. $\Gamma \vdash_{\mathsf{F}} i$
2. $\Gamma \vdash_{\mathsf{F}} i : \mathsf{Nat}$
   by $i \in \mathbb{N}$
3. QED

□

Corollary 6.4 : $\mathcal{D}_{\mathsf{F}}$ *soundness*

If $\vdash_{\mathsf{F}} v$ then $\vdash_{\mathsf{F}} \mathcal{D}_{\mathsf{F}}(\tau, v) : \tau$

Proof:

QED by $\mathcal{X}$ *soundness*

□

Corollary 6.5 : $\mathcal{S}_{\mathsf{F}}$ *soundness*

If $\vdash_{\mathsf{F}} v : \tau$ then $\vdash_{\mathsf{F}} \mathcal{S}_{\mathsf{F}}(\tau, v)$

Proof:

QED by $\mathcal{X}$ *soundness*

□

Corollary 6.6 : HF *static subset*

If $\Gamma \vdash e : \tau$ then $\Gamma \vdash_{\mathsf{F}} e : \tau$.

Proof:

Consequence of the proof for the higher-order *static subset* lemma; both $\vdash_{\mathsf{F}}$ and $\vdash_{\mathsf{H}}$ have the same typing rules for surface-language expressions.

□

Corollary 6.7 : HF *dynamic subset*

If $\Gamma \vdash e$ then $\Gamma \vdash_{\mathsf{F}} e$.

Proof:

Consequence of the proof for the higher-order *dynamic subset* lemma.

□

Lemma 6.8 : HF *static progress*

If $\vdash_{\mathsf{F}} e : \tau$ then one of the following holds:

- e is a value
- $e \in \mathsf{Err}$
- $e \rightarrow_{\mathsf{F\text{-}S}} e'$
- $e \rightarrow_{\mathsf{F\text{-}S}} \mathsf{BndryErr}$
- $e = E[\mathsf{dyn}\ \tau'\ e']$ and $e' \rightarrow_{\mathsf{F\text{-}D}} \mathsf{TagErr}$

Proof:

By the *boundary factoring* lemma, there are eight possible cases.

CASE e is a value :

1. QED

CASE $e = E^{\bullet}[v_0\ v_1]$:

1. $\vdash_{\mathsf{F}} v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\mathsf{F}} v_0 : \tau_d \Rightarrow \tau_c$
   $\wedge \vdash_{\mathsf{F}} v_1 : \tau_d$
   by *inversion*
3. $v_0 = \lambda(x\!:\!\tau_d').\, e'$
   $\vee\ v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,\lambda x.\, e'$
   $\vee\ v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,\lambda(x\!:\!\tau_x).\, e'$
   by *canonical forms*
4. **IF** $v_0 = \lambda(x\!:\!\tau_d').\, e'$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[e'[x \leftarrow v_1]]$
      by $v_0\ v_1 \rhd_{\mathsf{S\text{-}1}} e'[x \leftarrow v_1]$
   b. QED

   IF $v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,\lambda x.\, e'$
   $\wedge\ \mathcal{X}(\tau_d', v_1) = v_1'$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[\mathsf{dyn}\ \tau_c'\ (e'[x \leftarrow v_1'])]$
      by $v_0\ v_1 \rhd_{\mathsf{S\text{-}1}} \mathsf{dyn}\ \tau_c'\ (e'[x \leftarrow v_1'])$
   b. QED

   IF $v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,\lambda x.\, e'$
   $\wedge\ \mathcal{X}(\tau_d', v_1) = \mathsf{BndryErr}$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} \mathsf{BndryErr}$
      by $v_0\ v_1 \rhd_{\mathsf{S\text{-}1}} \mathsf{BndryErr}$
   b. QED

   IF $v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,\lambda(x\!:\!\tau_x).\, e'$
   $\wedge\ \mathcal{X}(\tau_x, v_1) = v_1'$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[\mathsf{stat}\ \tau_c'\ (\mathsf{chk}\ \tau_c'\ e'[x \leftarrow v_1'])]$
      by $v_0\ v_1 \rhd_{\mathsf{S\text{-}1}} \mathsf{stat}\ \tau_c'\ (\mathsf{chk}\ \tau_c'\ e'[x \leftarrow v_1'])$
   b. QED

   ELSE $v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,\lambda(x\!:\!\tau_x).\, e'$
   $\wedge\ \mathcal{X}(\tau_x, v_1) = \mathsf{BndryErr}$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} \mathsf{BndryErr}$
      by $v_0\ v_1 \rhd_{\mathsf{S\text{-}1}} \mathsf{BndryErr}$
   b. QED

CASE $e = E^{\bullet}[op^1\ v]$:

1. $\vdash_{\mathsf{F}} op^1\ v : \tau'$
   by *static hole typing*
2. $\vdash_{\mathsf{F}} v : \tau_0 \times \tau_1$
   by *inversion*
3. $v = \langle v_0, v_1 \rangle$
   $\vee\ v = \mathsf{mon}\,(\tau_0' \times \tau_1')\,\langle v_0, v_1 \rangle$
   by *canonical forms*
4. **IF** $v = \langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \mathsf{fst}$:
   a. $\delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0$
   b. $e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[v_0]$
      by $op^1\ v \rhd_{\mathsf{S\text{-}1}} v_0$
   c. QED

   IF $v = \langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \mathsf{snd}$:
   a. $\delta(\mathsf{snd}, \langle v_0, v_1 \rangle) = v_1$
   b. $e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[v_1]$
      by $op^1\ v \rhd_{\mathsf{S\text{-}1}} v_1$
   c. QED

   IF $v = \mathsf{mon}\,(\tau_0' \times \tau_1')\,\langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \mathsf{fst}$
   $\wedge\ \mathcal{X}(\tau_0', v_0) = v_0'$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[v_0']$
      by $\mathsf{fst}\ v \rhd_{\mathsf{S\text{-}1}} v_0'$
   b. QED

   IF $v = \mathsf{mon}\,(\tau_0' \times \tau_1')\,\langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \mathsf{fst}$
   $\wedge\ \mathcal{X}(\tau_0', v_0) = \mathsf{BndryErr}$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} \mathsf{BndryErr}$
      by $\mathsf{fst}\ v \rhd_{\mathsf{S\text{-}1}} \mathsf{BndryErr}$
   b. QED

   IF $v = \mathsf{mon}\,(\tau_0' \times \tau_1')\,\langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \mathsf{snd}$
   $\wedge\ \mathcal{X}(\tau_1', v_1) = v_1'$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[v_1']$
      by $\mathsf{snd}\ v \rhd_{\mathsf{S\text{-}1}} v_1'$
   b. QED

   ELSE $v = \mathsf{mon}\,(\tau_0' \times \tau_1')\,\langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \mathsf{snd}$
   $\wedge\ \mathcal{X}(\tau_1', v_1) = \mathsf{BndryErr}$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} \mathsf{BndryErr}$
      by $\mathsf{snd}\ v \rhd_{\mathsf{S\text{-}1}} \mathsf{BndryErr}$
   b. QED

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:

1. $\vdash_{\mathsf{F}} op^2\ v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\mathsf{F}} v_0 : \tau_0$
   $\wedge \vdash_{\mathsf{F}} v_1 : \tau_1$
   $\wedge\ \Delta(op^2, \tau_0, \tau_1) = \tau''$
   by *inversion*
3. $\delta(op^2, v_0, v_1) = e'$
   by Δ *type soundness* (2)

4. $op^2\ v_0\ v_1 \rhd_{\text{S-1}} e'$
   by (3)
5. QED by $e \rightarrow_{\text{F-S}} E^\bullet[e']$

CASE $e = E[\text{dyn}\ \tau'\ e']$ and e' is boundary-free :

1. e' is a value
   $\vee\ e' \in \text{Err}$
   $\vee\ e' \rightarrow_{\text{F-D}} e''$
   $\vee\ e' \rightarrow_{\text{F-D}} \text{BndryErr}$
   $\vee\ e' = E'[e'']$ and $e'' \rhd_{\text{D-1}} \text{TagErr}$
   by *dynamic progress*
2. **IF** e' is a value :
   a. QED $e \rightarrow_{\text{F-S}} E[\mathcal{D}_{\text{F}}(\tau', e')]$
   IF $e' \in \text{Err}$:
   a. QED $e \rightarrow_{\text{F-S}} e'$
   IF $e' \rightarrow_{\text{F-D}} e''$:
   a. QED $e \rightarrow_{\text{F-S}} E[\text{dyn}\ \tau'\ e'']$
   IF $e' \rightarrow_{\text{F-D}} \text{BndryErr}$:
   a. QED $e \rightarrow_{\text{F-S}} E[\text{dyn}\ \tau'\ \text{BndryErr}]$
   ELSE $e' = E'[e'']$ and $e'' \rhd_{\text{D-1}} \text{TagErr}$:
   a. $E' \in E^\bullet$
      by e' is boundary-free
   b. QED

CASE $e = E[\text{stat}\ \tau'\ e']$ and e' is boundary-free :

1. e' is a value
   $\vee\ e' \in \text{Err}$
   $\vee\ e' \rightarrow_{\text{F-S}} e''$
   $\vee\ e' \rightarrow_{\text{F-S}} \text{BndryErr}$
   $\vee\ e' = E''[\text{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{D-1}} \text{TagErr}$
   by *static progress*
2. **IF** e' is a value :
   a. QED $e \rightarrow_{\text{F-S}} E[\mathcal{S}_{\text{F}}(\tau', e')]$
   IF $e' \in \text{Err}$:
   a. QED $e \rightarrow_{\text{F-S}} e'$
   IF $e' \rightarrow_{\text{F-S}} e''$:
   a. QED $e \rightarrow_{\text{F-S}} E[\text{stat}\ \tau'\ e'']$
   IF $e' \rightarrow_{\text{F-S}} \text{BndryErr}$:
   a. QED $e \rightarrow_{\text{F-S}} E[\text{stat}\ \tau'\ \text{BndryErr}]$
   ELSE $e' = E''[\text{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{D-1}} \text{TagErr}$
   :
   a. Contradiction by e' is boundary-free

CASE $e = E[\text{Err}]$:

1. QED $e \rightarrow_{\text{F-S}} \text{Err}$

CASE $e = E^\bullet[\text{chk}\ \tau'\ v]$:

IF $\mathcal{X}(\tau, v) = v'$:
1. $e \rightarrow_{\text{F-S}} E^\bullet[v']$
   by $(\text{chk}\ \tau\ v) \rhd_{\text{S-1}} v'$
2. QED

ELSE $\mathcal{X}(\tau, v) = \text{BndryErr}$:
1. $e \rightarrow_{\text{F-S}} \text{BndryErr}$
   by $(\text{chk}\ \tau\ v) \rhd_{\text{S-1}} \text{BndryErr}$
2. QED

□

Lemma 6.9 : HF *dynamic progress*

If $\vdash_{\text{F}} e$ then one of the following holds:

- e is a value
- $e \in \text{Err}$
- $e \rightarrow_{\text{F-D}} e'$
- $e \rightarrow_{\text{F-D}} \text{BndryErr}$
- $e \rightarrow_{\text{F-D}} \text{TagErr}$

Proof:

By the *boundary factoring* lemma, there are seven cases.

CASE e is a value :

1. QED

CASE $e = E^\bullet[v_0\ v_1]$:

IF $v_0 = \lambda x.\, e'$:
1. $e \rightarrow_{\text{F-D}} E^\bullet[e'[x \leftarrow v_1]]$
   by $v_0\ v_1 \rhd_{\text{D-1}} e'[x \leftarrow v_1]$
2. QED

IF $v_0 = \text{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda x.\, e')$:
1. $e \rightarrow_{\text{F-D}} E^\bullet[e'[x \leftarrow v_1]]$
   by $v_0\ v_1 \rhd_{\text{D-1}} e'[x \leftarrow v_1]$
2. QED

IF $v_0 = \text{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda(x\!:\!\tau_x).\, e')$
$\wedge\ \mathcal{X}(\tau_x, v_1) = v_1'$:
1. $e \rightarrow_{\text{F-D}} E^\bullet[\text{stat}\ \tau_c\ (\text{chk}\ \tau_c\ e'[x \leftarrow v_1'])]$
   by $v_0\ v_1 \rhd_{\text{D-1}} \text{stat}\ \tau_c\ (\text{chk}\ \tau_c\ e'[x \leftarrow v_1'])$
2. QED

IF $v_0 = \text{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda(x\!:\!\tau_x).\, e')$
$\wedge\ \mathcal{X}(\tau_x, v_1) = \text{BndryErr}$:
1. $e \rightarrow_{\text{F-D}} \text{BndryErr}$
   by $v_0\ v_1 \rhd_{\text{D-1}} \text{BndryErr}$
2. QED

IF $v_0 = \lambda(x\!:\!\tau_x).\, e'$:
1. Contradiction by $\vdash_{\text{F}} e$

ELSE $v_0 = i$
$\vee\ v_0 = \langle v, v' \rangle$
$\vee\ v_0 = \text{mon}\ \tau_0\!\times\!\tau_1\ v'$:
1. $e \rightarrow_{\text{F-D}} \text{TagErr}$
   by $(v_0\ v_1) \rhd_{\text{D-1}} \text{TagErr}$
2. QED

CASE $e = E^\bullet[op^1\ v]$:

IF $v = \text{mon}\ \tau_0\!\times\!\tau_1\ \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \text{fst}$
$\wedge\ \mathcal{X}(\tau_0, v_0) = v_0'$:
1. $e \rightarrow_{\text{F-D}} E^\bullet[v_0']$
   by $op^1\ v \rhd_{\text{D-1}} v_0'$
2. QED

IF $v = \text{mon}\ \tau_0\!\times\!\tau_1\ \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \text{fst}$
$\wedge\ \mathcal{X}(\tau_0, v_0) = \text{BndryErr}$:
1. $e \rightarrow_{\text{F-D}} \text{BndryErr}$
   by $op^1\ v \rhd_{\text{D-1}} \text{BndryErr}$
2. QED

IF $v = \text{mon}\ \tau_0\!\times\!\tau_1\ \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \text{snd}$
$\wedge\ \mathcal{X}(\tau_1, v_1) = v_1'$:

1. $e \rightarrow_{\text{F-D}} E^\bullet[v_1']$
   by $op^1\ v \rhd_{\text{D-1}} v_1'$
2. QED

IF $v = \text{mon}\ \tau_0{\times}\tau_1\ \langle v_0, v_1\rangle$
$\wedge\ op^1 = \text{snd}$
$\wedge\ \mathcal{X}(\tau_1, v_1) = \text{BndryErr}$:
1. $e \rightarrow_{\text{F-D}} \text{BndryErr}$
   by $op^1\ v \rhd_{\text{D-1}} \text{BndryErr}$
2. QED

IF $\delta(op^1, v) = e'$:
1. $(op^1\ v) \rhd_{\text{D-1}} e'$
2. QED by $e \rightarrow_{\text{F-D}} E^\bullet[e']$

ELSE $\delta(op^1, v)$ is undefined :
1. $e \rightarrow_{\text{F-D}} \text{TagErr}$
   by $(op^1\ v) \rhd_{\text{D-1}} \text{TagErr}$
2. QED

CASE $e = E^\bullet[op^2\ v_0\ v_1]$:

IF $\delta(op^2, v_0, v_1) = e''$:
1. $op^2\ v_0\ v_1 \rhd_{\text{D-1}} e''$
2. QED

ELSE $\delta(op^2, v_0, v_1)$ is undefined :
1. $e \rightarrow_{\text{F-D}} \text{TagErr}$
   by $op^2\ v_0\ v_1 \rhd_{\text{D-1}} \text{TagErr}$
2. QED

CASE $e = E[\text{dyn}\ \tau'\ e']$ and e' is boundary-free :
1. e' is a value
   $\vee\ e' \in \text{Err}$
   $\vee\ e' \rightarrow_{\text{F-D}} e''$
   $\vee\ e' \rightarrow_{\text{F-D}} \text{BndryErr}$
   $\vee\ e' = E^\bullet[e'']$ and $e'' \rhd_{\text{D-1}} \text{TagErr}$
   by *dynamic progress*
2. IF e' is a value :
   a. QED $e \rightarrow_{\text{F-D}} E[\mathcal{D}_{\text{F}}(\tau', e')]$
   IF $e' \in \text{Err}$:
   a. QED $e \rightarrow_{\text{F-D}} e'$
   IF $e' \rightarrow_{\text{F-D}} e''$:
   a. QED $e \rightarrow_{\text{F-S}} E[\text{dyn}\ \tau'\ e'']$
   IF $e' \rightarrow_{\text{F-D}} \text{BndryErr}$:
   a. QED $e \rightarrow_{\text{F-D}} E[\text{dyn}\ \tau'\ \text{BndryErr}]$
   ELSE $e' = E[e'']$ and $e'' \rhd_{\text{D-1}} \text{TagErr}$:
   a. $E \in E^\bullet$
      by e' is boundary-free
   b. QED

CASE $e = E[\text{stat}\ \tau'\ e']$ and e' is boundary-free :
1. e' is a value
   $\vee\ e' \in \text{Err}$
   $\vee\ e' \rightarrow_{\text{F-S}} e''$
   $\vee\ e' \rightarrow_{\text{F-S}} \text{BndryErr}$
   $\vee\ e' = E''[\text{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{D-1}} \text{TagErr}$
   by *static progress*
2. IF e' is a value :
   a. QED $e \rightarrow_{\text{F-S}} E[\mathcal{S}_{\text{F}}(\tau', e')]$
   IF $e' \in \text{Err}$:
   a. QED $e \rightarrow_{\text{F-S}} e'$
   IF $e' \rightarrow_{\text{F-S}} e''$:
   a. QED $e \rightarrow_{\text{F-S}} E[\text{stat}\ \tau'\ e'']$
   IF $e' \rightarrow_{\text{F-S}} \text{BndryErr}$:
   a. QED $e \rightarrow_{\text{F-S}} E[\text{stat}\ \tau'\ \text{BndryErr}]$
   ELSE $e' = E''[\text{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{D-1}} \text{TagErr}$
   :
   a. Contradiction by e' is boundary-free

CASE $e = E[\text{Err}]$:
1. QED $e \rightarrow_{\text{F-D}} \text{Err}$

□

Lemma 6.10 : HF *static preservation*

If $\vdash_{\text{F}} e : \tau$ and $e \rightarrow_{\text{F-S}} e'$ then $\vdash_{\text{F}} e' : \tau$

Proof:

By the *boundary factoring* lemma there are eight cases.

CASE e is a value :
1. Contradiction by $e \rightarrow_{\text{F-S}} e'$

CASE $e = E^\bullet[v_0\ v_1]$:

IF $v_0 = \lambda(x\!:\!\tau_x).\ e'$
$\wedge\ e \rightarrow_{\text{F-S}} e'[x \leftarrow v_1]$:
1. $\vdash_{\text{F}} v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\text{F}} v_0 : \tau_d \Rightarrow \tau_c$
   $\wedge \vdash_{\text{F}} v_1 : \tau_d$
   $\wedge\ \tau_c \leqslant: \tau'$
   by *inversion*
3. $(x\!:\!\tau_x) \vdash_{\text{F}} e' : \tau_c$
   by *inversion* (2)
4. $\tau_d \leqslant: \tau_x$
   by *canonical forms* (2)
5. $\vdash_{\text{F}} v_1 : \tau_x$
   by (2, 4)
6. $\vdash_{\text{F}} e'[x \leftarrow v_1] : \tau_c$
   by *substitution* (3, 5)
7. $\vdash_{\text{F}} e'[x \leftarrow v_1] : \tau'$
   by (2, 6)
8. QED by *hole substitution*

IF $v_0 = \text{mon}\ \tau_d \Rightarrow \tau_c\ \lambda x.\ e'$
$\wedge\ e \rightarrow_{\text{F-S}} E^\bullet[\text{dyn}\ \tau_c\ ((\lambda x.\ e')\ (\mathcal{X}(\tau_d, v_1)))]$:
1. $\vdash_{\text{F}} v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\text{F}} v_0 : \tau_d' \Rightarrow \tau_c'$
   $\wedge\ \tau_c' \leqslant: \tau'$
   by *inversion*
3. $\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'$
   by *canonical forms* (2)
4. $\vdash_{\text{F}} \lambda x.\ e'$
   by *inversion* (2)
5. $\tau_d' \leqslant: \tau_d$
   $\wedge\ \tau_c \leqslant: \tau_c'$
   by (3)
6. $\vdash_{\text{F}} \mathcal{X}(\tau_d, v_1)$
   by $\mathcal{X}$ *soundness*

7. $\vdash_{\mathsf{F}} (\lambda x.\, e')\, \mathcal{X}(\tau_d, v_1)$
by (4, 6)
8. $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau_c\ ((\lambda x.\, e')\, \mathcal{X}(\tau_d, v_1)) : \tau_c$
by (7)
9. $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau_c\ ((\lambda x.\, e')\, \mathcal{X}(\tau_d, v_1)) : \tau'$
by (2, 5, 8)
10. QED by *hole substitution*

ELSE $v_0 = \mathsf{mon}\ \tau_d \Rightarrow \tau_c\ (\lambda(x{:}\tau_x).\, e')$
$\wedge e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[\mathsf{stat}\ \tau_c\ (\mathsf{chk}\ \tau_c\ ((\lambda(x{:}\tau_x).\, e')\, \mathcal{X}(\tau_x, v_1)))]$
:

1. $\vdash_{\mathsf{F}} v_0\ v_1 : \tau'$
by *static hole typing*
2. $\vdash_{\mathsf{F}} v_0 : \tau_d' \Rightarrow \tau_c'$
$\wedge \vdash_{\mathsf{F}} v_1 : \tau_d'$
$\wedge \tau_c' \leqslant: \tau'$
by *inversion*
3. $\vdash_{\mathsf{F}} \lambda(x{:}\tau_x).\, e' : \tau_x \Rightarrow \tau_x'$
by *inversion*
4. $\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'$
by *canonical forms* (2)
5. $\tau_c \leqslant: \tau_c'$
by (4)
6. $\vdash_{\mathsf{F}} \mathcal{X}(\tau_x, v_1) : \tau_x$
by $\mathcal{X}$ *soundness*
7. $\vdash_{\mathsf{F}} (\lambda(x{:}\tau_x).\, e')\, \mathcal{X}(\tau_x, v_1) : \tau_x'$
by (3, 6)
8. $\vdash_{\mathsf{F}} \mathsf{chk}\ \tau_c\ ((\lambda(x{:}\tau_x).\, e')\, \mathcal{X}(\tau_x, v_1)) : \tau_c$
by (7)
9. $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau_c\ (\mathsf{chk}\ \tau_c\ ((\lambda(x{:}\tau_x).\, e')\, \mathcal{X}(\tau_x, v_1))) : \tau_c$
by (8)
10. $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau_c\ (\mathsf{chk}\ \tau_c\ ((\lambda(x{:}\tau_x).\, e')\, \mathcal{X}(\tau_x, v_1))) : \tau'$
by (2, 5, 9)
11. QED by *hole substitution*

CASE $e = E^{\bullet}[op^1\ v]$:

IF $v = \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{fst}$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[v_0]$:

1. $\vdash_{\mathsf{F}} \mathsf{fst}\ \langle v_0, v_1 \rangle : \tau'$
by *static hole typing*
2. $\vdash_{\mathsf{F}} \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$
$\wedge\ \tau_0 \leqslant: \tau'$
by *inversion*
3. $\vdash_{\mathsf{F}} v_0 : \tau_0$
by *inversion*
4. $\vdash_{\mathsf{F}} v_0 : \tau'$
by (2)
5. QED by *hole substitution*

IF $v = \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{snd}$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[v_1]$:

1. $\vdash_{\mathsf{F}} \mathsf{snd}\ \langle v_0, v_1 \rangle : \tau'$
by *static hole typing*
2. $\vdash_{\mathsf{F}} \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$
$\wedge\ \tau_1 \leqslant: \tau'$
by *inversion*
3. $\vdash_{\mathsf{F}} v_1 : \tau_1$
by *inversion*
4. $\vdash_{\mathsf{F}} v_1 : \tau'$
by (2)
5. QED by *hole substitution*

IF $v = \mathsf{mon}\, (\tau_0 \times \tau_1)\, \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{fst}$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[\mathcal{X}(\tau_0, v_0)]$:

1. $\vdash_{\mathsf{F}} \mathsf{fst}\ v : \tau'$
by *static hole typing*
2. $\vdash_{\mathsf{F}} v : \tau_0' \times \tau_1'$
$\wedge\ \tau_0' <: \tau'$
by *inversion* (1)
3. $\tau_0 \leqslant: \tau_0'$
by *canonical forms* (2)
4. $\vdash_{\mathsf{F}} \langle v_0, v_1 \rangle$
$\vee \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle : \tau''$
by *inversion* (2)
5. $\vdash_{\mathsf{F}} v_0$
$\vee \vdash_{\mathsf{F}} v_0 : \tau'''$
by *inversion* (4)
6. $\vdash_{\mathsf{F}} \mathcal{X}(\tau_0, v_0) : \tau_0$
by $\mathcal{X}$ *soundness* (5)
7. $\vdash_{\mathsf{F}} \mathcal{X}(\tau_0, v_0) : \tau'$
by (2, 3, 6)
8. QED by *hole substitution*

ELSE $v = \mathsf{mon}\, (\tau_0 \times \tau_1)\, \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{snd}$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[\mathcal{X}(\tau_1, v_1)]$:

1. $\vdash_{\mathsf{F}} \mathsf{snd}\ v : \tau'$
by *static hole typing*
2. $\vdash_{\mathsf{F}} v : \tau_0' \times \tau_1'$
$\wedge\ \tau_1' <: \tau'$
by *inversion* (1)
3. $\tau_1 \leqslant: \tau_1'$
by *canonical forms* (2)
4. $\vdash_{\mathsf{F}} \langle v_0, v_1 \rangle$
$\vee \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle : \tau''$
by *inversion* (2)
5. $\vdash_{\mathsf{F}} v_1$
$\vee \vdash_{\mathsf{F}} v_1 : \tau'''$
by *inversion* (4)
6. $\vdash_{\mathsf{F}} \mathcal{X}(\tau_1, v_1) : \tau_1$
by $\mathcal{X}$ *soundness* (5)
7. $\vdash_{\mathsf{F}} \mathcal{X}(\tau_1, v_1) : \tau'$
by (2, 3, 6)
8. QED by *hole substitution*

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$
$\wedge\ \delta(op^2, v_0, v_1) = v$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[v]$:

1. $\vdash_{\mathsf{F}} op^2\ v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\mathsf{F}} v_0 : \tau_0$
   $\wedge \vdash_{\mathsf{F}} v_1 : \tau_1$
   $\wedge\ \Delta(op^2, \tau_0, \tau_1) = \tau''$
   $\wedge\ \tau'' \leqslant: \tau'$
   by *inversion*
3. $\vdash_{\mathsf{F}} v : \tau''$
   by Δ *type soundness* (2)
4. $\vdash_{\mathsf{F}} v : \tau'$
   by (2, 3)
5. QED by *hole substitution* (4)

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :

IF e' is a value :

1. $e \rightarrow_{\mathsf{F\text{-}S}} E[\mathcal{D}_{\mathsf{F}}(\tau', e')]$
2. $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau'\ e' : \tau'$
   by *boundary hole typing*
3. $\vdash_{\mathsf{F}} e'$
   by *inversion* (2)
4. $\vdash_{\mathsf{F}} \mathcal{D}_{\mathsf{F}}(\tau', e') : \tau'$
   by $\mathcal{D}_{\mathsf{F}}$ *soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\mathsf{F\text{-}D}} e''$:

1. $e \rightarrow_{\mathsf{F\text{-}S}} E[\mathsf{dyn}\ \tau'\ e'']$
2. $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau'\ e' : \tau'$
   by *boundary hole typing*
3. $\vdash_{\mathsf{F}} e'$
   by *inversion* (2)
4. $\vdash_{\mathsf{F}} e''$
   by *dynamic preservation* (3)
5. $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau'\ e'' : \tau'$
   by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :

IF e' is a value :

1. $e \rightarrow_{\mathsf{F\text{-}S}} E[\mathcal{S}_{\mathsf{F}}(\tau', e')]$
2. $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau'\ e'$
   by *boundary hole typing*
3. $\vdash_{\mathsf{F}} e' : \tau'$
   by *inversion* (2)
4. $\vdash_{\mathsf{F}} \mathcal{S}_{\mathsf{F}}(\tau', e')$
   by $\mathcal{S}_{\mathsf{F}}$ *soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\mathsf{F\text{-}S}} e''$:

1. $e \rightarrow_{\mathsf{F\text{-}S}} E[\mathsf{stat}\ \tau'\ e'']$
2. $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau'\ e'$
   by *boundary hole typing*
3. $\vdash_{\mathsf{F}} e' : \tau'$
   by *inversion* (2)
4. $\vdash_{\mathsf{F}} e'' : \tau'$
   by *static preservation* (3)
5. $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau'\ e''$
   by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{Err}]$:

1. $e \rightarrow_{\mathsf{F\text{-}S}} \mathsf{Err}$
2. QED by $\vdash_{\mathsf{F}} \mathsf{Err} : \tau$

CASE $e = E^{\bullet}[\mathsf{chk}\ \tau\ v]$:

1. $\vdash_{\mathsf{F}} \mathsf{chk}\ \tau\ v : \tau$
   by *static hole typing*
2. $\vdash_{\mathsf{F}} v : \tau'$
   by *inversion* (1)
3. $\vdash_{\mathsf{F}} \mathcal{X}(\tau, v') : \tau$
   by $\mathcal{X}$ *soundness* (2)
4. QED by *hole substitution* (3)

□

Lemma 6.11 : HF *dynamic preservation*

If $\vdash_{\mathsf{F}} e$ and $e \rightarrow_{\mathsf{F\text{-}D}} e'$ then $\vdash_{\mathsf{F}} e'$

Proof:

By the *boundary factoring* lemma, there are seven cases.

CASE e is a value :

1. Contradiction by $e \rightarrow_{\mathsf{F\text{-}D}} e'$

CASE $e = E^{\bullet}[v_0\ v_1]$:

IF $v_0 = \lambda x.\ e'$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}D}} E^{\bullet}[e'[x \leftarrow v_1]]$:

1. $\vdash_{\mathsf{F}} v_0\ v_1$
   by *dynamic hole typing*
2. $\vdash_{\mathsf{F}} v_0$
   $\wedge \vdash_{\mathsf{F}} v_1$
   by *inversion* (1)
3. $x \vdash_{\mathsf{F}} e'$
   by *inversion* (2)
4. $\vdash_{\mathsf{F}} e'[x \leftarrow v_1]$
   by *substitution* (2, 3)
5. QED *hole substitution* (4)

IF $v_0 = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,\lambda x.\ e'$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}D}} E^{\bullet}[(\lambda x.\ e')\ v_1]$:

1. $\vdash_{\mathsf{F}} v_0\ v_1$
   by *dynamic hole typing*
2. $\vdash_{\mathsf{F}} v_0$
   $\wedge \vdash_{\mathsf{F}} v_1$
   by *inversion* (1)
3. $\vdash_{\mathsf{F}} (\lambda x.\ e')\ v_1$
   by (2)
4. QED *hole substitution* (5)

ELSE $v_0 = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,\lambda(x\!:\!\tau_x).\ e'$
$\wedge e \rightarrow_{\mathsf{F\text{-}D}} E^{\bullet}[\mathsf{stat}\ \tau_c\ (\mathsf{chk}\ \tau_c\ ((\lambda(x\!:\!\tau_x).\ e')\ \mathcal{X}(\tau_x, v_1)))]$
:

1. $\vdash_{\mathsf{F}} v_0\ v_1$
   by *dynamic hole typing*
2. $\vdash_{\mathsf{F}} v_0$
   $\wedge \vdash_{\mathsf{F}} v_1$
   by *inversion*
3. $\vdash_{\mathsf{F}} \lambda(x\!:\!\tau_x).\ e' : \tau_x \Rightarrow \tau'_x$
   by *inversion* (2)

4. $\vdash_{\mathsf{F}} \mathcal{X}(\tau_x, v_1) : \tau_x$
by *$\mathcal{X}$ soundness* (2)
5. $\vdash_{\mathsf{F}} ((\lambda(x\!:\!\tau_x).\, e')\ \mathcal{X}(\tau_x, v_1)) : \tau_x'$
by (3, 4)
6. $\vdash_{\mathsf{F}} \mathsf{chk}\ \tau_c\ ((\lambda(x\!:\!\tau_x).\, e')\ \mathcal{X}(\tau_x, v_1)) : \tau_c$
by (5)
7. $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau_c\ (\mathsf{chk}\ \tau_c\ ((\lambda(x\!:\!\tau_x).\, e')\ \mathcal{X}(\tau_x, v_1)))$
by (6)
8. QED *hole substitution*

CASE $e = E^{\bullet}[op^1\ v]$:

IF $v = \langle v_0, v_1\rangle$
$\wedge\ op^1 = \mathsf{fst}$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}D}} E^{\bullet}[v_0]$:
1. $\vdash_{\mathsf{F}} op^1\ v$
by *dynamic hole typing*
2. $\vdash_{\mathsf{F}} v$
by *inversion* (1)
3. $\vdash_{\mathsf{F}} v_0$
by *inversion* (2)
4. QED by *hole substitution*

IF $v = \langle v_0, v_1\rangle$
$\wedge\ op^1 = \mathsf{snd}$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}D}} E^{\bullet}[v_1]$:
1. $\vdash_{\mathsf{F}} op^1\ v$
by *dynamic hole typing*
2. $\vdash_{\mathsf{F}} v$
by *inversion* (1)
3. $\vdash_{\mathsf{F}} v_1$
by *inversion* (2)
4. QED by *hole substitution*

IF $v = \mathsf{mon}\,(\tau_0\!\times\!\tau_1)\,\langle v_0, v_1\rangle$
$\wedge\ op^1 = \mathsf{fst}$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}D}} E^{\bullet}[\mathcal{X}(\tau_0, v_0)]$:
1. $\vdash_{\mathsf{F}} op^1\ v$
by *dynamic hole typing*
2. $\vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0\!\times\!\tau_1)\,\langle v_0, v_1\rangle$
by *inversion* (1)
3. $\vdash_{\mathsf{F}} v_0$
$\vee \vdash_{\mathsf{F}} v_0 : \tau_0'$
by *inversion* (2)
4. $\vdash_{\mathsf{F}} \mathcal{X}(\tau_0, v_0)$
by *$\mathcal{X}$ soundness* (3)
5. QED by *hole substitution*

ELSE $v = \mathsf{mon}\,(\tau_0\!\times\!\tau_1)\,\langle v_0, v_1\rangle$
$\wedge\ op^1 = \mathsf{snd}$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}D}} E^{\bullet}[\mathcal{X}(\tau_1, v_1)]$:
1. $\vdash_{\mathsf{F}} op^1\ v$
by *dynamic hole typing*
2. $\vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0\!\times\!\tau_1)\,\langle v_0, v_1\rangle$
by *inversion* (1)
3. $\vdash_{\mathsf{F}} v_1$
$\vee \vdash_{\mathsf{F}} v_1 : \tau_1'$
by *inversion* (2)
4. $\vdash_{\mathsf{F}} \mathcal{X}(\tau_1, v_1)$
by *$\mathcal{X}$ soundness* (3)
5. QED by *hole substitution*

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$
$\wedge\ \delta(op^2, v_0, v_1) = v$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}D}} E^{\bullet}[v]$:
1. $\vdash_{\mathsf{F}} op^2\ v_0\ v_1$
by *dynamic hole typing*
2. $\vdash_{\mathsf{F}} v_0$
$\wedge \vdash_{\mathsf{F}} v_1$
by *inversion* (1)
3. $\vdash_{\mathsf{F}} v$
by *δ preservation* (2)
4. QED by *hole substitution* (3)

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :

IF e' is a value :
1. $e \rightarrow_{\mathsf{F\text{-}D}} E[\mathcal{D}_{\mathsf{F}}(\tau', e')]$
2. $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau'\ e' : \tau'$
by *boundary hole typing*
3. $\vdash_{\mathsf{F}} e'$
by *inversion* (2)
4. $\vdash_{\mathsf{F}} \mathcal{D}_{\mathsf{F}}(\tau', e') : \tau'$
by *$\mathcal{D}_{\mathsf{F}}$ soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\mathsf{F\text{-}D}} e''$:
1. $e \rightarrow_{\mathsf{F\text{-}D}} E[\mathsf{dyn}\ \tau'\ e'']$
2. $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau'\ e' : \tau'$
by *boundary hole typing*
3. $\vdash_{\mathsf{F}} e'$
$\wedge\ \tau' \leqslant: \tau''$
by *inversion* (2)
4. $\vdash_{\mathsf{F}} e''$
by *dynamic preservation* (3)
5. $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau'\ e'' : \tau'$
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :

IF $e' \in v$:
1. $e \rightarrow_{\mathsf{F\text{-}D}} E[\mathcal{S}_{\mathsf{F}}(\tau', e')]$
2. $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau'\ e'$
by *boundary hole typing*
3. $\vdash_{\mathsf{F}} e' : \tau'$
by *inversion* (2)
4. $\vdash_{\mathsf{F}} \mathcal{S}_{\mathsf{F}}(\tau', e')$
by *$\mathcal{S}_{\mathsf{F}}$ soundness* (3)
5. QED by *hole substitution* (5)

ELSE $e' \rightarrow_{\mathsf{F\text{-}S}} e''$:
1. $e \rightarrow_{\mathsf{F\text{-}D}} E[\mathsf{stat}\ \tau'\ e'']$
2. $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau'\ e'$
by *boundary hole typing*
3. $\vdash_{\mathsf{F}} e' : \tau'$
by *inversion* (2)
4. $\vdash_{\mathsf{F}} e'' : \tau'$
by *static preservation* (3)

5. $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau'\ e''$
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{Err}]$:
1. $e \rightarrow_{\mathsf{F\text{-}D}} \mathsf{Err}$
2. QED $\vdash_{\mathsf{F}} \mathsf{Err}$

□

Lemma 6.12 : HF *static boundary factoring*

If $\vdash_{\mathsf{F}} e : \tau$ then one of the following holds:
- e is a value
- $e = E^{\bullet}[v_0\ v_1]$
- $e = E^{\bullet}[op^1\ v]$
- $e = E^{\bullet}[op^2\ v_0\ v_1]$
- $e = E^{\bullet}[\mathsf{chk}\ \tau\ v]$
- $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{Err}]$

Proof:

By the *unique static evaluation contexts* lemma, there are eight cases.

CASE e is a value :
1. QED

CASE $e = E[v_0\ v_1]$:
1. $E = E^{\bullet}$
$\vee\ E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$
$\vee\ E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$
by *inner boundary*
2. **IF** $E = E^{\bullet}$:
a. QED $e = E^{\bullet}[v_0\ v_1]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^{\bullet}[v_0\ v_1]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^{\bullet}[v_0\ v_1]]$

CASE $e = E[op^1\ v]$:
1. $E = E^{\bullet}$
$\vee\ E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$
$\vee\ E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$
by *inner boundary*
2. **IF** $E = E^{\bullet}$:
a. QED $e = E^{\bullet}[op^1\ v]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^{\bullet}[op^1\ v]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^{\bullet}[op^1\ v]]$

CASE $e = E[op^2\ v_0\ v_1]$:
1. $E = E^{\bullet}$
$\vee\ E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$
$\vee\ E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$
by *inner boundary*
2. **IF** $E = E^{\bullet}$:
a. QED $e = E^{\bullet}[op^2\ v_0\ v_1]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^{\bullet}[op^2\ v_0\ v_1]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^{\bullet}[op^2\ v_0\ v_1]]$

CASE $e = E[\mathsf{chk}\ \tau\ v]$:
1. $E = E^{\bullet}$
$\vee\ E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$
$\vee\ E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$
by *inner boundary*
2. **IF** $E = E^{\bullet}$:
a. QED $e = E^{\bullet}[\mathsf{chk}\ \tau\ v]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$:
a. Contradiction by $\vdash_{\mathsf{F}} e : \tau$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^{\bullet}[\mathsf{chk}\ \tau\ v]]$

CASE $e = E[\mathsf{dyn}\ \tau\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{stat}\ \tau\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{Err}]$:
1. QED

□

Lemma 6.13 : HF *unique static evaluation contexts*

If $\vdash_{\mathsf{F}} e : \tau$ then one of the following holds:
- e is a value
- $e = E[v_0\ v_1]$
- $e = E[op^1\ v]$
- $e = E[op^2\ v_0\ v_1]$
- $e = E[\mathsf{chk}\ \tau\ v]$
- $e = E[\mathsf{dyn}\ \tau\ v]$
- $e = E[\mathsf{stat}\ \tau\ v]$
- $e = E[\mathsf{Err}]$

Proof:

By induction on the structure of e.

CASE $e = x$
$\vee\ e = \lambda x.\, e'$
$\vee\ e = \mathsf{stat}\ \tau\ e'$:
1. Contradiction by $\vdash_{\mathsf{F}} e : \tau$

CASE $e = i$
$\vee\ e = \lambda(x\!:\!\tau_d).\, e'$
$\vee\ e = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v$:
1. QED e is a value

CASE $e = \langle e_0, e_1\rangle$:
IF $e_0 \notin v$:
1. $\vdash_{\mathsf{F}} e_0 : \tau_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis (1)
3. $E = \langle E_0, e_1\rangle$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\ e_1 \notin v$:
1. $\vdash_{\mathsf{F}} e_1 : \tau_1$
by *inversion*
2. $e_1 = E_1[e_1']$
by the induction hypothesis (1)
3. $E = \langle e_0, E_1\rangle$

4. QED $e = E[e_1']$
ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E = [\,]$
2. QED e is a value

CASE $e = e_0\ e_1$:
IF $e_0 \notin v$:
1. $\vdash_{\mathsf{F}} e_0 : \tau_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis (1)
3. $E = E_0\ e_1$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:
1. $\vdash_{\mathsf{F}} e_1 : \tau_1$
by *inversion*
2. $e_1 = E_1[e_1']$
by the induction hypothesis (1)
3. $E = e_0\ E_1$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[e_0\ e_1]$

CASE $e = op^1\ e_0$:
IF $e_0 \notin v$:
1. $\vdash_{\mathsf{F}} e_0 : \tau_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis (1)
3. $E = op^1\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[op^1\ e_0]$

CASE $e = op^2\ e_0\ e_1$:
IF $e_0 \notin v$:
1. $\vdash_{\mathsf{F}} e_0 : \tau_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis (1)
3. $E = op^2\ E_0\ e_1$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:
1. $\vdash_{\mathsf{F}} e_1 : \tau_1$
by *inversion*
2. $e_1 = E_1[e_1']$
by the induction hypothesis (1)
3. $E = op^2\ e_0\ E_1$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[op^2\ e_0\ e_1]$

CASE $e = \mathsf{chk}\ \tau\ e_0$:
IF $e_0 \notin v$:
1. $\vdash_{\mathsf{F}} e_0 : \tau_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis (1)
3. $E = \mathsf{chk}\ \tau\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[\mathsf{chk}\ \tau\ e_0]$

CASE $e = \mathsf{dyn}\ \tau\ e_0$:
IF $e_0 \notin v$:
1. $\vdash_{\mathsf{F}} e_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by *unique dynamic evaluation contexts* (1)
3. $E = \mathsf{dyn}\ \tau\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[\mathsf{dyn}\ \tau\ e_0]$

CASE $e = \mathsf{Err}$:
1. $E = [\,]$
2. QED $e = E[\mathsf{Err}]$

☐

Lemma 6.14 : HF *inner boundary*

For all contexts E, one of the following holds:
- $E = E^{\bullet}$
- $E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$
- $E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$

Proof:
By induction on the structure of E.

CASE $E = E^{\bullet}$:
1. QED

CASE $E = E_0\ e_1$:
1. $E_0 = E^{\bullet}$
$\vee\, E_0 = E_0'[\mathsf{dyn}\ \tau\ E^{\bullet}]$
$\vee\, E_0 = E_0'[\mathsf{stat}\ \tau\ E^{\bullet}]$
by the induction hypothesis
2. **IF** $E_0 = E^{\bullet}$:
a. QED E is boundary-free
IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^{\bullet}]$:
a. $E' = E_0'\ e_1$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^{\bullet}]$:
a. $E' = E_0'\ e_1$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$

CASE $E = v_0\ E_1$:
1. $E_1 = E^{\bullet}$
$\vee\, E_1 = E_1'[\mathsf{dyn}\ \tau\ E^{\bullet}]$

$\vee\ E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_1 = E^\bullet$:
a. QED E is boundary-free
IF $E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = v_0\ E_1'$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = v_0\ E_1'$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
CASE $E = \langle E_0, e_1\rangle$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = \langle E_0', e_1\rangle$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = \langle E_0', e_1\rangle$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
CASE $E = \langle v_0, E_1\rangle$:
1. $E_1 = E^\bullet$
$\vee\ E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_1 = E^\bullet$:
a. QED E is boundary-free
IF $E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = \langle v_0, E_1'\rangle$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = \langle v_0, E_1'\rangle$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
CASE $E = op^1\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = op^1\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = op^1\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
CASE $E = op^2\ E_0\ e_1$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = op^2\ E_0'\ e_1$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = op^2\ E_0'\ e_1$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
CASE $E = op^2\ v_0\ E_1$:
1. $E_1 = E^\bullet$
$\vee\ E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_1 = E^\bullet$:
a. QED E is boundary-free
IF $E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = op^2\ v_0\ E_1'$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = op^2\ v_0\ E_1'$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
CASE $E = \mathsf{chk}\ \tau\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = \mathsf{chk}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = \mathsf{chk}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
CASE $E = \mathsf{dyn}\ \tau\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED
IF $E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau'\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau'\ E^\bullet]$
CASE $E = \mathsf{stat}\ \tau\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$

$\lor\ E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED
IF $E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{stat}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau'\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{stat}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau'\ E^\bullet]$
□

Lemma 6.15 : HF *dynamic boundary factoring*
If $\vdash_{\mathsf{F}} e$ then one of the following holds:
- e is a value
- $e = E^\bullet[v_0\ v_1]$
- $e = E^\bullet[op^1\ v]$
- $e = E^\bullet[op^2\ v_0\ v_1]$
- $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{Err}]$

Proof:
By the *unique dynamic evaluation contexts* lemma, there are eight cases.
CASE e is a value :
1. QED
CASE $e = E[v_0\ v_1]$:
1. $E = E^\bullet$
$\lor\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\lor\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. QED $e = E^\bullet[v_0\ v_1]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[v_0\ v_1]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[v_0\ v_1]]$
CASE $e = E[op^1\ v]$:
1. $E = E^\bullet$
$\lor\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\lor\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. QED $e = E^\bullet[op^1\ v]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^1\ v]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^1\ v]]$
CASE $e = E[op^2\ v_0\ v_1]$:
1. $E = E^\bullet$
$\lor\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\lor\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. QED $e = E^\bullet[op^2\ v_0\ v_1]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$
CASE $e = E[\mathsf{chk}\ \tau'\ v]$:
1. $E = E^\bullet$
$\lor\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\lor\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. Contradiction by $\vdash_{\mathsf{F}} e$
IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. Contradiction by $\vdash_{\mathsf{F}} e$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[\mathsf{chk}\ \tau'\ v]]$
CASE $e = E[\mathsf{dyn}\ \tau\ v]$:
1. QED v is boundary-free
CASE $e = E[\mathsf{stat}\ \tau\ v]$:
1. QED v is boundary-free
CASE $e = E[\mathsf{Err}]$:
1. QED
□

Lemma 6.16 : HF *unique dynamic evaluation contexts*
If $\vdash_{\mathsf{F}} e$ then one of the following holds:
- e is a value
- $e = E[v_0\ v_1]$
- $e = E[op^1\ v]$
- $e = E[op^2\ v_0\ v_1]$
- $e = E[\mathsf{chk}\ \tau\ v]$
- $e = E[\mathsf{dyn}\ \tau\ v]$
- $e = E[\mathsf{stat}\ \tau\ v]$
- $e = E[\mathsf{Err}]$

Proof:
By induction on the structure of e.
CASE $e = x$
$\lor\ e = \lambda(x\!:\!\tau).\,e'$
$\lor\ e = \mathsf{dyn}\ \tau\ e'$:
1. Contradiction by $\vdash_{\mathsf{F}} e$
CASE $e = i$
$\lor\ e = \lambda x.\,e'$
$\lor\ e = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v$:
1. QED e is a value
CASE $e = \mathsf{Err}$:
1. $E = [\,]$
2. QED $e = E[\mathsf{Err}]$
CASE $e = \langle e_0, e_1\rangle$:
IF $e_0 \notin v$:
1. $\vdash_{\mathsf{F}} e_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis (1)
3. $E = \langle E_0, e_1\rangle$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:
1. $\vdash_F e_1$
by *inversion*
2. $e_1 = E_1[e_1']$
by the induction hypothesis (1)
3. $E = \langle e_0, E_1 \rangle$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E = [\,]$
2. QED e is a value

CASE $e = e_0\ e_1$:
IF $e_0 \notin v$:
1. $\vdash_F e_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis (1)
3. $E = E_0\ e_1$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:
1. $\vdash_F e_1$
by *inversion*
2. $e_1 = E_1[e_1']$
by the induction hypothesis (1)
3. $E = e_0\ E_1$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[e_0\ e_1]$

CASE $e = op^1\ e_0$:
IF $e_0 \notin v$:
1. $\vdash_F e_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis (1)
3. $E = op^1\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[op^1\ e_0]$

CASE $e = op^2\ e_0\ e_1$:
IF $e_0 \notin v$:
1. $\vdash_F e_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis (1)
3. $E = op^2\ E_0\ e_1$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:
1. $\vdash_F e_1$
by *inversion*
2. $e_1 = E_1[e_1']$
by the induction hypothesis (1)
3. $E = op^2\ e_0\ E_1$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[op^2\ e_0\ e_1]$

CASE $e = \mathsf{chk}\ \tau\ e_0$:
Contradiction by $\vdash_F e$

CASE $e = \mathsf{stat}\ \tau\ e_0$:
IF $e_0 \notin v$:
1. $\vdash_F e_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by *unique static evaluation contexts* (1)
3. $E = \mathsf{stat}\ \tau\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[\mathsf{stat}\ \tau\ e_0]$

□

Lemma 6.17 : HF *static hole typing*
If $\vdash_F E^\bullet[e] : \tau$ then the derivation contains a sub-term $\vdash_F e : \tau'$
Proof:
By induction on the structure of $E^\bullet$.

CASE $E^\bullet = [\,]$:
1. QED $E^\bullet[e] = e$

CASE $E^\bullet = E^\bullet{}_0\ e_1$:
1. $E^\bullet[e] = E^\bullet{}_0[e]\ e_1$
2. $\vdash_F E^\bullet{}_0[e] : \tau_d \Rightarrow \tau_c$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = v_0\ E^\bullet{}_1$:
1. $E^\bullet[e] = v_0\ E^\bullet{}_1[e]$
2. $\vdash_F E^\bullet{}_1[e] : \tau_d$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = \langle E^\bullet{}_0, e_1 \rangle$:
1. $E^\bullet[e] = \langle E^\bullet{}_0[e], e_1 \rangle$
2. $\vdash_F E^\bullet{}_0[e] : \tau_0$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = \langle v_0, E^\bullet{}_1 \rangle$:
1. $E^\bullet[e] = \langle v_0, E^\bullet{}_1[e] \rangle$
2. $\vdash_F E^\bullet{}_1[e] : \tau_1$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = op^1\ E^\bullet{}_0$:
1. $E^\bullet[e] = op^1\ E^\bullet{}_0[e]$

2. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = op^2\ E^{\bullet}{}_0\ e_1$:
1. $E^{\bullet}[e] = op^2\ E^{\bullet}{}_0[e]\ e_1$
2. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = op^2\ v_0\ E^{\bullet}{}_1$:
1. $E^{\bullet}[e] = op^2\ v_0\ E^{\bullet}{}_1[e]$
2. $\vdash_{\mathsf{F}} E^{\bullet}{}_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = \mathsf{chk}\ \tau\ E^{\bullet}{}_0$:
1. $E^{\bullet}[e] = \mathsf{chk}\ \tau\ E^{\bullet}{}_0[e]$
2. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

□

Lemma 6.18 : HF *dynamic hole typing*

If $\vdash_{\mathsf{F}} E^{\bullet}[e]$ then the derivation contains a sub-term $\vdash_{\mathsf{F}} e$

Proof:

By induction on the structure of $E^{\bullet}$.

CASE $E^{\bullet} = [\,]$:
1. QED $E^{\bullet}[e] = e$

CASE $E^{\bullet} = E^{\bullet}{}_0\ e_1$:
1. $E^{\bullet}[e] = E^{\bullet}{}_0[e]\ e_1$
2. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = v_0\ E^{\bullet}{}_1$:
1. $E^{\bullet}[e] = v_0\ E^{\bullet}{}_1[e]$
2. $\vdash_{\mathsf{F}} E^{\bullet}{}_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = \langle E^{\bullet}{}_0, e_1\rangle$:
1. $E^{\bullet}[e] = \langle E^{\bullet}{}_0[e], e_1\rangle$
2. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = \langle v_0, E^{\bullet}{}_1\rangle$:
1. $E^{\bullet}[e] = \langle v_0, E^{\bullet}{}_1[e]\rangle$
2. $\vdash_{\mathsf{F}} E^{\bullet}{}_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = op^1\ E^{\bullet}{}_0$:
1. $E^{\bullet}[e] = op^1\ E^{\bullet}{}_0[e]$
2. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = op^2\ E^{\bullet}{}_0\ e_1$:
1. $E^{\bullet}[e] = op^2\ E^{\bullet}{}_0[e]\ e_1$
2. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = op^2\ v_0\ E^{\bullet}{}_1$:
1. $E^{\bullet}[e] = op^2\ v_0\ E^{\bullet}{}_1[e]$
2. $\vdash_{\mathsf{F}} E^{\bullet}{}_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

□

Lemma 6.19 : HF *boundary hole typing*

- If $\vdash_{\mathsf{F}} E[\mathsf{dyn}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau\ e : \tau$
- If $\vdash_{\mathsf{F}} E[\mathsf{dyn}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau\ e : \tau$
- If $\vdash_{\mathsf{F}} E[\mathsf{stat}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau\ e$
- If $\vdash_{\mathsf{F}} E[\mathsf{stat}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau\ e$

Proof:

By the following four lemmas: *static* dyn *hole typing*, *dynamic* dyn *hole typing*, *static* stat *hole typing*, and *dynamic* stat *hole typing*.

□

Lemma 6.20 : HF *static* dyn *hole typing*

If $\vdash_{\mathsf{F}} E[\mathsf{dyn}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau\ e : \tau$.

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:
1. $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau\ e : \tau''$
   by *static hole typing*
2. $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau\ e : \tau$
   by *inversion* (1)
3. QED

CASE $E = E_0\ e_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = E_0[\mathsf{dyn}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = v_0\ E_1[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{dyn}\ \tau\ e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1\rangle$:
1. $E[\mathsf{dyn}\ \tau\ e] = \langle E_0[\mathsf{dyn}\ \tau\ e], e_1\rangle$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1\rangle$:
1. $E[\mathsf{dyn}\ \tau\ e] = \langle v_0, E_1[\mathsf{dyn}\ \tau\ e]\rangle$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{dyn}\ \tau\ e] : \tau_1$
   by *inversion*

3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:

1. $E[\mathsf{dyn}\ \tau\ e] = op^1\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:

1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ E_0[\mathsf{dyn}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:

1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{dyn}\ \tau\ e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{chk}\ \tau''\ E_0$:

1. $E[\mathsf{dyn}\ \tau\ e] = \mathsf{chk}\ \tau''\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:

1. $E[\mathsf{dyn}\ \tau\ e] = \mathsf{dyn}\ \tau_0\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by *dynamic* dyn *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:

1. Contradiction by $\vdash_{\mathsf{F}} E[\mathsf{dyn}\ \tau\ e] : \tau'$

□

Lemma 6.21 : HF *dynamic* dyn *hole typing*

> If $\vdash_{\mathsf{F}}\ E[\mathsf{dyn}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau\ e : \tau$.

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:

1. Contradiction by $\vdash_{\mathsf{F}} E[\mathsf{dyn}\ \tau\ e]$

CASE $E = E_0\ e_1$:

1. $E[\mathsf{dyn}\ \tau\ e] = E_0[\mathsf{dyn}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:

1. $E[\mathsf{dyn}\ \tau\ e] = v_0\ E_1[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:

1. $E[\mathsf{dyn}\ \tau\ e] = \langle E_0[\mathsf{dyn}\ \tau\ e], e_1 \rangle$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:

1. $E[\mathsf{dyn}\ \tau\ e] = \langle v_0, E_1[\mathsf{dyn}\ \tau\ e] \rangle$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:

1. $E[\mathsf{dyn}\ \tau\ e] = op^1\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:

1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ E_0[\mathsf{dyn}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:

1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{chk}\ \tau''\ E_0$:

1. Contradiction by $\vdash_{\mathsf{F}} E[\mathsf{dyn}\ \tau\ e]$

□

CASE $E = \mathsf{dyn}\ \tau\ E_0$:

1. Contradiction by $\vdash_{\mathsf{F}} E[\mathsf{dyn}\ \tau\ e]$

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:

1. $E[\mathsf{dyn}\ \tau\ e] = \mathsf{stat}\ \tau_0\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by *static* dyn *hole typing* (2)

}

Lemma 6.22 : HF *static* stat *hole typing*

> If $\vdash_{\mathsf{F}} E[\mathsf{stat}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau\ e$.

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:

1. Contradiction by $\vdash_{\mathsf{F}} E[\mathsf{stat}\ \tau\ e] : \tau'$

CASE $E = E_0\ e_1$:

1. $E[\mathsf{stat}\ \tau\ e] = E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:

1. $E[\mathsf{stat}\ \tau\ e] = v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{stat}\ \tau\ e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:

1. $E[\mathsf{stat}\ \tau\ e] = \langle E_0[\mathsf{stat}\ \tau\ e], e_1 \rangle$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:

1. $E[\mathsf{stat}\ \tau\ e] = \langle v_0, E_1[\mathsf{stat}\ \tau\ e] \rangle$

2. $\vdash_{\mathsf{F}} E_1[\mathsf{stat}\ \tau\ e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = op^1\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{stat}\ \tau\ e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{chk}\ \tau''\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{chk}\ \tau''\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{dyn}\ \tau_0\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by *dynamic* stat *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_{\mathsf{F}} E[\mathsf{stat}\ \tau\ e] : \tau'$

□

Lemma 6.23 : HF *dynamic* stat *hole typing*

If $\vdash_{\mathsf{F}} E[\mathsf{stat}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_{\mathsf{F}}$ $\mathsf{stat}\ \tau\ e$.

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:
1. QED by *dynamic hole typing*

CASE $E = E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1\rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle E_0[\mathsf{stat}\ \tau\ e], e_1\rangle$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1\rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle v_0, E_1[\mathsf{stat}\ \tau\ e]\rangle$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = op^1\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{chk}\ \tau''\ E_0$:
1. Contradiction by $\vdash_{\mathsf{F}} E[\mathsf{stat}\ \tau\ e]$

CASE $E = \mathsf{dyn}\ \tau\ E_0$:
1. Contradiction by $\vdash_{\mathsf{F}} E[\mathsf{stat}\ \tau\ e]$

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{stat}\ \tau_0\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by *static* stat *hole typing* (2)

□

Lemma 6.24 : HF *static boundary-free hole substitution*

If $\vdash_{\mathsf{F}} E^{\bullet}[e] : \tau$ and the derivation contains a sub-term $\vdash_{\mathsf{F}} e : \tau'$ and $\vdash_{\mathsf{F}} e' : \tau'$ then $\vdash_{\mathsf{F}} E^{\bullet}[e'] : \tau$.

Proof:

By induction on the structure of $E^{\bullet}$

CASE $E^{\bullet} = [\,]$:
1. $E^{\bullet}[e] = e$
   $\wedge\ E^{\bullet}[e'] = e'$
2. $\vdash_{\mathsf{F}} e : \tau$
   by (1)
3. $\tau' = \tau$
4. $\vdash_{\mathsf{F}} e' : \tau$
5. QED by (1, 4)

CASE $E^{\bullet} = E^{\bullet}{}_0\ e_1$:
1. $E^{\bullet}[e] = E^{\bullet}{}_0[e]\ e_1$
   $\wedge\ E^{\bullet}[e'] = E^{\bullet}{}_0[e']\ e_1$
2. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e]\ e_1 : \tau$
3. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e] : \tau_0$
   $\wedge \vdash_{\mathsf{F}} e_1 : \tau_1$
   by *inversion*
4. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e']\ e_1 : \tau$
   by (2, 3, 4)

6. QED by (1, 5)

CASE $E^{\bullet} = v_0\ E^{\bullet}{}_1$:

1. $E^{\bullet}[e] = v_0\ E^{\bullet}{}_1[e]$
   $\wedge\ E^{\bullet}[e'] = v_0\ E^{\bullet}{}_1[e']$
2. $\vdash_{F} v_0\ E^{\bullet}{}_1[e] : \tau$
3. $\vdash_{F} v_0 : \tau_0$
   $\wedge \vdash_{F} E^{\bullet}{}_1[e] : \tau_1$
   by *inversion*
4. $\vdash_{F} E^{\bullet}{}_1[e'] : \tau_1$
   by the induction hypothesis (3)
5. $\vdash_{F} v_0\ E^{\bullet}{}_1[e'] : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^1\ E^{\bullet}{}_0$:

1. $E^{\bullet}[e] = op^1\ E^{\bullet}{}_0[e]$
   $\wedge\ E^{\bullet}[e'] = op^1\ E^{\bullet}{}_0[e']$
2. $\vdash_{F} op^1\ E^{\bullet}{}_0[e] : \tau$
3. $\vdash_{F} E^{\bullet}{}_0[e] : \tau_0$
   by *inversion*
4. $\vdash_{F} E^{\bullet}{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{F} op^1\ E^{\bullet}{}_0[e'] : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \langle E^{\bullet}{}_0, e_1\rangle$:

1. $E^{\bullet}[e] = \langle E^{\bullet}{}_0[e], e_1\rangle$
   $\wedge\ E^{\bullet}[e'] = \langle E^{\bullet}{}_0[e'], e_1\rangle$
2. $\vdash_{F} \langle E^{\bullet}{}_0[e], e_1\rangle : \tau$
3. $\vdash_{F} E^{\bullet}{}_0[e] : \tau_0$
   $\wedge \vdash_{F} e_1 : \tau_1$
   by *inversion*
4. $\vdash_{F} E^{\bullet}{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{F} \langle E^{\bullet}{}_0[e'], e_1\rangle : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \langle v_0, E^{\bullet}{}_1\rangle$:

1. $E^{\bullet}[e] = \langle v_0, E^{\bullet}{}_1[e]\rangle$
   $\wedge\ E^{\bullet}[e'] = \langle v_0, E^{\bullet}{}_1[e']\rangle$
2. $\vdash_{F} \langle v_0, E^{\bullet}{}_1[e]\rangle : \tau$
3. $\vdash_{F} v_0 : \tau_0$
   $\wedge \vdash_{F} E^{\bullet}{}_1[e] : \tau_1$
   by *inversion*
4. $\vdash_{F} E^{\bullet}{}_1[e'] : \tau_1$
   by the induction hypothesis (3)
5. $\vdash_{F} \langle v_0, E^{\bullet}{}_1[e']\rangle : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ E^{\bullet}{}_0\ e_1$:

1. $E^{\bullet}[e] = op^2\ E^{\bullet}{}_0[e]\ e_1$
   $\wedge\ E^{\bullet}[e'] = op^2\ E^{\bullet}{}_0[e']\ e_1$
2. $\vdash_{F} op^2\ E^{\bullet}{}_0[e]\ e_1 : \tau$
3. $\vdash_{F} E^{\bullet}{}_0[e] : \tau_0$
   $\wedge \vdash_{F} e_1 : \tau_1$
   by *inversion*
4. $\vdash_{F} E^{\bullet}{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{F} op^2\ E^{\bullet}{}_0[e']\ e_1 : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ v_0\ E^{\bullet}{}_1$:

1. $E^{\bullet}[e] = op^2\ v_0\ E^{\bullet}{}_1[e]$
   $\wedge\ E^{\bullet}[e'] = op^2\ v_0\ E^{\bullet}{}_1[e']$
2. $\vdash_{F} op^2\ v_0\ E^{\bullet}{}_1[e] : \tau$
3. $\vdash_{F} v_0 : \tau_0$
   $\wedge \vdash_{F} E^{\bullet}{}_1[e] : \tau_1$
   by *inversion*
4. $\vdash_{F} E^{\bullet}{}_1[e'] : \tau_1$
   by the induction hypothesis (3)
5. $\vdash_{F} op^2\ v_0\ E^{\bullet}{}_1[e'] : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \mathsf{chk}\ \tau''\ E^{\bullet}{}_0$:

1. $E^{\bullet}[e] = \mathsf{chk}\ \tau''\ E^{\bullet}{}_0[e]$
   $\wedge\ E^{\bullet}[e'] = \mathsf{chk}\ \tau''\ E^{\bullet}{}_0[e']$
2. $\vdash_{F} \mathsf{chk}\ \tau''\ E^{\bullet}{}_0[e] : \tau$
3. $\vdash_{F} E^{\bullet}{}_0[e] : \tau_0$
   by *inversion*
4. $\vdash_{F} E^{\bullet}{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{F} \mathsf{chk}\ \tau''\ E^{\bullet}{}_0[e'] : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

□

Lemma 6.25 : HF *dynamic hole substitution*

If $\vdash_{F} E^{\bullet}[e]$ and $\vdash_{F} e'$ then $\vdash_{F} E^{\bullet}[e']$

Proof:

By induction on the structure of $E^{\bullet}$

CASE $E^{\bullet} = [\,]$:

1. QED $E^{\bullet}[e'] = e'$

CASE $E^{\bullet} = \langle E^{\bullet}{}_0, e_1\rangle$:

1. $E^{\bullet}[e] = \langle E^{\bullet}{}_0[e], e_1\rangle$
   $\wedge\ E^{\bullet}[e'] = \langle E^{\bullet}{}_0[e'], e_1\rangle$
2. $\vdash_{F} \langle E^{\bullet}{}_0[e], e_1\rangle$
3. $\vdash_{F} E^{\bullet}{}_0[e]$
   $\wedge \vdash_{F} e_1$
   by *inversion*
4. $\vdash_{F} E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{F} \langle E^{\bullet}{}_0[e'], e_1\rangle$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \langle v_0, E^{\bullet}{}_1\rangle$:

1. $E^{\bullet}[e] = \langle v_0, E^{\bullet}{}_1[e]\rangle$
   $\wedge\ E^{\bullet}[e'] = \langle v_0, E^{\bullet}{}_1[e']\rangle$

2. $\vdash_{\mathsf{F}} \langle v_0, E^{\bullet}{}_1[e] \rangle$
3. $\vdash_{\mathsf{F}} v_0$
   $\wedge \vdash_{\mathsf{F}} E^{\bullet}{}_1[e]$
   by *inversion*
4. $\vdash_{\mathsf{F}} E^{\bullet}{}_1[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{F}} \langle v_0, E^{\bullet}{}_1[e'] \rangle$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = E^{\bullet}{}_0 \; e_1$:
1. $E^{\bullet}[e] = E^{\bullet}{}_0[e] \; e_1$
   $\wedge E^{\bullet}[e'] = E^{\bullet}{}_0[e'] \; e_1$
2. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e] \; e_1$
3. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e]$
   $\wedge \vdash_{\mathsf{F}} e_1$
   by *inversion*
4. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e'] \; e_1$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = v_0 \; E^{\bullet}{}_1$:
1. $E^{\bullet}[e] = v_0 \; E^{\bullet}{}_1[e]$
   $\wedge E^{\bullet}[e'] = v_0 \; E^{\bullet}{}_1[e']$
2. $\vdash_{\mathsf{F}} v_0 \; E^{\bullet}{}_1[e]$
3. $\vdash_{\mathsf{F}} v_0$
   $\wedge \vdash_{\mathsf{F}} E^{\bullet}{}_1[e]$
   by *inversion*
4. $\vdash_{\mathsf{F}} E^{\bullet}{}_1[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{F}} v_0 \; E^{\bullet}{}_1[e']$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^1 \; E^{\bullet}{}_0$:
1. $E^{\bullet}[e] = op^1 \; E^{\bullet}{}_0[e]$
   $\wedge E^{\bullet}[e'] = op^1 \; E^{\bullet}{}_0[e']$
2. $\vdash_{\mathsf{F}} op^1 \; E^{\bullet}{}_0[e]$
3. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e]$
   by *inversion*
4. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{F}} op^1 \; E^{\bullet}{}_0[e']$
   by (4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2 \; E^{\bullet}{}_0 \; e_1$:
1. $E^{\bullet}[e] = op^2 \; E^{\bullet}{}_0[e] \; e_1$
   $\wedge E^{\bullet}[e'] = op^2 \; E^{\bullet}{}_0[e'] \; e_1$
2. $\vdash_{\mathsf{F}} op^2 \; E^{\bullet}{}_0[e] \; e_1$
3. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e]$
   $\wedge \vdash_{\mathsf{F}} e_1$
   by *inversion*
4. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{F}} op^2 \; E^{\bullet}{}_0[e'] \; e_1$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2 \; v_0 \; E^{\bullet}{}_1$:
1. $E^{\bullet}[e] = op^2 \; v_0 \; E^{\bullet}{}_1[e]$
   $\wedge E^{\bullet}[e'] = op^2 \; v_0 \; E^{\bullet}{}_1[e']$
2. $\vdash_{\mathsf{F}} op^2 \; v_0 \; E^{\bullet}{}_1[e]$
3. $\vdash_{\mathsf{F}} v_0$
   $\wedge \vdash_{\mathsf{F}} E^{\bullet}{}_1[e]$
   by *inversion*
4. $\vdash_{\mathsf{F}} E^{\bullet}{}_1[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{F}} op^2 \; v_0 \; E^{\bullet}{}_1[e']$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \mathsf{chk} \; \tau_0 \; E^{\bullet}{}_0$:
1. Contradiction by $\vdash_{\mathsf{F}} E^{\bullet}[e]$

□

Lemma 6.26 : HF *hole substitution*

- If $\vdash_{\mathsf{F}} E[e]$ and the derivation contains a sub-term $\vdash_{\mathsf{F}} e : \tau'$ and $\vdash_{\mathsf{F}} e' : \tau'$ then $\vdash_{\mathsf{F}} E[e']$.
- If $\vdash_{\mathsf{F}} E[e]$ and the derivation contains a sub-term $\vdash_{\mathsf{F}} e$ and $\vdash_{\mathsf{F}} e'$ then $\vdash_{\mathsf{F}} E[e']$.
- If $\vdash_{\mathsf{F}} E[e] : \tau$ and the derivation contains a sub-term $\vdash_{\mathsf{F}} e : \tau'$ and $\vdash_{\mathsf{F}} e' : \tau'$ then $\vdash_{\mathsf{F}} E[e'] : \tau$.
- If $\vdash_{\mathsf{F}} E[e] : \tau$ and the derivation contains a sub-term $\vdash_{\mathsf{F}} e$ and $\vdash_{\mathsf{F}} e'$ then $\vdash_{\mathsf{F}} E[e'] : \tau$.

Proof:

By the following four lemmas: *dynamic context static hole substitution*, *dynamic context dynamic hole substitution*, *static context static hole substitution*, and *static context dynamic hole substitution.*

□

Lemma 6.27 : HF *dynamic context static hole substitution*

If $\vdash_{\mathsf{F}} E[e]$ and contains $\vdash_{\mathsf{F}} e : \tau'$, and furthermore $\vdash_{\mathsf{F}} e' : \tau'$, then $\vdash_{\mathsf{F}} E[e']$

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:
1. Contradiction by $\vdash_{\mathsf{F}} E[e]$

CASE $E = E_0 \; e_1$:
1. $E[e] = E_0[e] \; e_1$
2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0 \; E_1$:
1. $E[e] = v_0 \; E_1[e]$
2. $\vdash_{\mathsf{F}} E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[e] = \langle E_0[e], e_1 \rangle$

2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_{\mathsf{F}} E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_{\mathsf{F}} E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{chk}\ \tau''\ E_0$:
1. $E[e] = \mathsf{chk}\ \tau''\ E_0[e]$
2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau''\ E_0$:
1. Contradiction by $\vdash_{\mathsf{F}} E[e]$

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{stat}\ \tau_0\ E_0[e]$
2. $\vdash_{\mathsf{F}} E_0[e] : \tau_0$
   by *inversion*
3. QED by *static context static hole substitution* (2)

□

Lemma 6.28 : HF *dynamic context dynamic hole substitution*

If $\vdash_{\mathsf{F}} E[e]$ and contains $\vdash_{\mathsf{F}} e$, and furthermore $\vdash_{\mathsf{F}} e'$, then $\vdash_{\mathsf{F}} E[e']$

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:
1. QED by *dynamic boundary-free hole substitution*

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_{\mathsf{F}} E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_{\mathsf{F}} E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_{\mathsf{F}} E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{chk}\ \tau''\ E_0$:
1. Contradiction by $\vdash_{\mathsf{F}} E[e]$

CASE $E = \mathsf{dyn}\ \tau''\ E_0$:
1. Contradiction by $\vdash_{\mathsf{F}} E[e]$

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{stat}\ \tau_0\ E_0[e]$
2. $\vdash_{\mathsf{F}} E_0[e] : \tau_0$
   by *inversion*
3. QED by *static context dynamic hole substitution* (2)

□

Lemma 6.29 : HF *static context static hole substitution*

If $\vdash_{\mathsf{F}} E[e] : \tau$ and contains $\vdash_{\mathsf{F}} e : \tau'$, and furthermore $\vdash_{\mathsf{F}} e' : \tau'$, then $\vdash_{\mathsf{F}} E[e'] : \tau$

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:
1. QED by *static boundary-free hole substitution*

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_{\mathsf{F}} E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[e] = \langle E_0[e], e_1 \rangle$

2. $\vdash_{\mathsf{F}} E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_{\mathsf{F}} E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_{\mathsf{F}} E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_{\mathsf{F}} E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{chk}\ \tau''\ E_0$:
1. $E[e] = \mathsf{chk}\ \tau''\ E_0[e]$
2. $\vdash_{\mathsf{F}} E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{dyn}\ \tau_0\ E_0[e]$
2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by *static* dyn *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_{\mathsf{F}} E[e] : \tau$

□

Lemma 6.30 : HF *static context dynamic hole substitution*

If $\vdash_{\mathsf{F}} E[e] : \tau$ and contains $\vdash_{\mathsf{F}} e$, and furthermore $\vdash_{\mathsf{F}} e'$, then $\vdash_{\mathsf{F}} E[e'] : \tau$

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:
1. Contradiction by $\vdash_{\mathsf{F}} E[e] : \tau$

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_{\mathsf{F}} E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_{\mathsf{F}} E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_{\mathsf{F}} E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_{\mathsf{F}} E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_{\mathsf{F}} E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{chk}\ \tau''\ E_0$:
1. Contradiction by $\vdash_{\mathsf{F}} E[e] : \tau$

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{dyn}\ \tau_0\ E_0[e]$
2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by *dynamic* stat *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_{\mathsf{F}} E[e] : \tau$

□

Lemma 6.31 : $\vdash_{\mathsf{F}}$ *static inversion*

- If $\Gamma \vdash_{\mathsf{F}} x : \tau$ then $(x\!:\!\tau') \in \Gamma$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{F}} \lambda(x\!:\!\tau'_d).\, e' : \tau$ then $(x\!:\!\tau'_d), \Gamma \vdash_{\mathsf{F}} e' : \tau'_c$ and $\tau'_d \Rightarrow \tau'_c \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{F}} \langle e_0, e_1 \rangle : \tau_0 \times \tau_1$ then $\Gamma \vdash_{\mathsf{F}} e_0 : \tau'_0$ and $\Gamma \vdash_{\mathsf{F}} e_1 : \tau'_1$ and $\tau'_0 \leqslant: \tau_0$ and $\tau'_1 \leqslant: \tau_1$
- If $\Gamma \vdash_{\mathsf{F}} e_0\ e_1 : \tau_c$ then $\Gamma \vdash_{\mathsf{F}} e_0 : \tau'_d \Rightarrow \tau'_c$ and $\Gamma \vdash_{\mathsf{F}} e_1 : \tau'_d$ and $\tau'_c \leqslant: \tau_c$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{fst}\ e : \tau$ then $\Gamma \vdash_{\mathsf{F}} e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0$ and $\tau_0 \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{snd}\ e : \tau$ then $\Gamma \vdash_{\mathsf{F}} e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1$ and $\tau_1 \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{F}} op^2\ e_0\ e_1 : \tau$ then $\Gamma \vdash_{\mathsf{F}} e_0 : \tau_0$ and $\Gamma \vdash_{\mathsf{F}} e_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau'$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\ \tau'_0 \times \tau'_1\ \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$ then either:
  - $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle$
  - or $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle : \tau$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\ \tau'_d \Rightarrow \tau'_c\ \lambda x.\, e : \tau_d \Rightarrow \tau_c$ then $\Gamma \vdash_{\mathsf{F}} \lambda x.\, e$ and $\tau'_d \Rightarrow \tau'_c \leqslant: \tau_d \Rightarrow \tau_c$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\ \tau'_d \Rightarrow \tau'_c\ \lambda(x : \tau''_d).\, e : \tau_d \Rightarrow \tau_c$ then $\Gamma \vdash_{\mathsf{F}} \lambda(x : \tau_x).\, e : \tau''_d \Rightarrow \tau''_c$ and $\tau'_d \Rightarrow \tau'_c \leqslant: \tau_d \Rightarrow \tau_c$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{dyn}\ \tau'\ e' : \tau$ then $\Gamma \vdash_{\mathsf{F}} e'$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{chk}\ \tau\ e' : \tau$ then $\Gamma \vdash_{\mathsf{F}} e' : \tau'$

Proof:

QED by the definition of $\Gamma \vdash_{\mathsf{F}} e : \tau$

□

Lemma 6.32 : $\vdash_{\mathsf{F}}$ *dynamic inversion*

- If $\Gamma \vdash_{\mathsf{F}} x$ then $x \in \Gamma$
- If $\Gamma \vdash_{\mathsf{F}} \lambda x.\, e'$ then $x, \Gamma \vdash_{\mathsf{F}} e'$
- If $\Gamma \vdash_{\mathsf{F}} \langle e_0, e_1 \rangle$ then $\Gamma \vdash_{\mathsf{F}} e_0$ and $\Gamma \vdash_{\mathsf{F}} e_1$
- If $\Gamma \vdash_{\mathsf{F}} e_0\ e_1$ then $\Gamma \vdash_{\mathsf{F}} e_0$ and $\Gamma \vdash_{\mathsf{F}} e_1$
- If $\Gamma \vdash_{\mathsf{F}} op^1\ e_0$ then $\Gamma \vdash_{\mathsf{F}} e_0$
- If $\Gamma \vdash_{\mathsf{F}} op^2\ e_0\ e_1$ then $\Gamma \vdash_{\mathsf{F}} e_0$ and $\Gamma \vdash_{\mathsf{F}} e_1$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,\lambda x.\, e$ then $\Gamma \vdash_{\mathsf{F}} \lambda x.\, e$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,\lambda(x\!:\!\tau_x).\, e$ then $\Gamma \vdash_{\mathsf{F}} \lambda(x\!:\!\tau_x).\, e : \tau_x \Rightarrow \tau'_x$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1 \rangle$ then either:
  - $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle$
  - or $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle : \tau'$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{stat}\ \tau'\ e'$ then $\Gamma \vdash_{\mathsf{F}} e' : \tau'$

Proof:

QED by the definition of $\Gamma \vdash_{\mathsf{F}} e$

□

Lemma 6.33 : HF *canonical forms*

- If $\vdash_{\mathsf{F}} v : \tau_0 \times \tau_1$ then either:
  - $v = \langle v_0, v_1 \rangle$
  - or $v = \mathsf{mon}\ \tau'_0 \times \tau'_1\ \langle v_0, v_1 \rangle$ $\wedge\ \tau'_0 \times \tau'_1 \leqslant: \tau_0 \times \tau_1$
- If $\vdash_{\mathsf{F}} v : \tau_d \Rightarrow \tau_c$ then either:
  - $v = \lambda(x\!:\!\tau_x).\, e'$ $\wedge\ \tau_d \leqslant: \tau_x$
  - or $v = \mathsf{mon}\,(\tau'_d \Rightarrow \tau'_c)\,(\lambda x.\, e$ $\wedge\ \tau'_d \Rightarrow \tau'_c \leqslant: \tau_d \Rightarrow \tau_c$
  - or $v = \mathsf{mon}\,(\tau'_d \Rightarrow \tau'_c)\,\lambda(x\!:\!\tau_x).\, e$ $\wedge\ \tau'_d \Rightarrow \tau'_c \leqslant: \tau_d \Rightarrow \tau_c$
- If $\vdash_{\mathsf{F}} v : \mathsf{Int}$ then $v = i$
- If $\vdash_{\mathsf{F}} v : \mathsf{Nat}$ then $v = i$ and $v \in \mathbb{N}$

Proof:

QED by definition of $\vdash_{\mathsf{F}} \cdot : \tau$

□

Lemma 6.34 : Δ *type soundness*

If $\vdash_{\mathsf{F}} v_0 : \tau_0$ and $\vdash_{\mathsf{F}} v_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau$ then one of the following holds:

- $\delta(op^2, v_0, v_1) = v$ and $\vdash_{\mathsf{F}} v : \tau$, or
- $\delta(op^2, v_0, v_1) = \mathsf{BndryErr}$

Proof (sketch): Similar to the proof for the higher-order Δ *type soundness* lemma. □

Lemma 6.35 : δ *preservation*

- If $\vdash_{\mathsf{F}} v$ and $\delta(op^1, v) = v'$ then $\vdash_{\mathsf{F}} v'$
- If $\vdash_{\mathsf{F}} v_0$ and $\vdash_{\mathsf{F}} v_1$ and $\delta(op^2, v_0, v_1) = v'$ then $\vdash_{\mathsf{F}} v'$

Proof:

Similar to the proof for the higher-order δ *preservation* lemma.

□

Lemma 6.36 : HF *substitution*

- If $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{F}} e$ and $\vdash_{\mathsf{F}} v : \tau_x$ then $\Gamma \vdash_{\mathsf{F}} e[x \leftarrow v]$
- If $x, \Gamma \vdash_{\mathsf{F}} e$ and $\vdash_{\mathsf{F}} v$ then $\Gamma \vdash_{\mathsf{F}} e[x \leftarrow v]$
- If $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{F}} e : \tau$ and $\vdash_{\mathsf{F}} v : \tau_x$ then $\Gamma \vdash_{\mathsf{F}} e[x \leftarrow v] : \tau$
- If $x, \Gamma \vdash_{\mathsf{F}} e : \tau$ and $\vdash_{\mathsf{F}} v$ then $\Gamma \vdash_{\mathsf{F}} e[x \leftarrow v] : \tau$

Proof:

Similar to the proof for the higher-order *substitution* lemma.

□

Lemma 6.37 : *weakening*

- If $\Gamma \vdash_{\mathsf{F}} e$ then $x, \Gamma \vdash_{\mathsf{F}} e$
- If $\Gamma \vdash_{\mathsf{F}} e : \tau$ then $(x\!:\!\tau'), \Gamma \vdash_{\mathsf{F}} e : \tau$

Proof:

QED because e is closed under Γ

□

A.7 Embeddings Summary

The paragraphs in this section summarize the five embeddings with four slogans. Each slogan pertains to one aspect of the embedding:

1. What kinds of checks does the embedding perform when a value reaches a type boundary?
2. When, if ever, does the embedding wrap a value in a monitor?
3. If an ill-typed value reaches a type boundary, when does the embedding signal an error?
4. How do types affect behavior?

These embeddings are ordered on a speculative scale from "most guarantees" to "least guarantees".

Higher-Order embedding

1. recursively check read-only values;
2. monitor functional and mutable values;
3. detect boundary errors as early as possible;
4. types globally constrain behavior.

Co-Natural embedding

1. tag-check all values;
2. monitor all data structures and functions;
3. detect boundary errors as late as possible;
4. types globally constrain behavior

Forgetful embedding

1. tag-check all values;
2. apply at most one monitor to each value;
3. detect boundary errors as late as possible;
4. types (of values) locally constrain behavior.

First-Order embedding

1. tag-check all values;
2. never allocate a monitor;
3. detect boundary errors as late as possible;
4. types (of contexts) locally constrain behavior.

Erasure embedding

1. never check values;
2. never allocate a monitor;
3. never detect a type boundary error;
4. types do not affect behavior

A.8 Simulation Lemmas

A.8.1 Definitions

$\boxed{\text{Combined Language}}$

$$\begin{array}{rcl} e & = & x \mid v \mid \langle e, e\rangle \mid e\ e \mid op^1\ e \mid op^2\ e\ e \mid \\ & & \mathsf{dyn}\ \tau\ e \mid \mathsf{stat}\ \tau\ e \mid \mathsf{Err} \mid \mathsf{chk}\ K\ e \mid \mathsf{dyn}\ e \mid \mathsf{stat}\ e \\ v & = & i \mid \langle v, v\rangle \mid \lambda x.\, e \mid \lambda (x\!:\!\tau).\, e \mid \mathsf{mon}\,(\tau \Rightarrow \tau)\, v \\ E^{\bullet} & = & [\,] \mid E^{\bullet}\ e \mid v\ E^{\bullet} \mid \langle E^{\bullet}, e\rangle \mid \langle v, E^{\bullet}\rangle \mid \\ & & op^1\ E^{\bullet} \mid op^2\ E^{\bullet}\ e \mid op^2\ v\ E^{\bullet} \mid \mathsf{chk}\ K\ E^{\bullet} \\ E & = & E^{\bullet} \mid E\ e \mid v\ E \mid \langle E, e\rangle \mid \langle v, E\rangle \mid op^1\ E \mid \\ & & op^2\ E\ e \mid op^2\ v\ E \mid \mathsf{dyn}\ \tau\ E \mid \mathsf{stat}\ \tau\ E \\ & & \mathsf{chk}\ K\ E \mid \mathsf{dyn}\ E \mid \mathsf{stat}\ E \end{array}$$

$\boxed{e\ {}_1{\lesssim}_{\mathsf{E}}\ e}$

$$\frac{}{\mathsf{Err}\ {}_1{\lesssim}_{\mathsf{E}}\ \mathsf{Err}} \qquad \frac{}{\mathsf{Err}\ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}} \qquad \frac{e^1\ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}}{\mathsf{chk}\ K\ e^1\ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}}$$

$$\frac{e^1\ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}}{\mathsf{dyn}\ e^1\ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}} \qquad \frac{e^1\ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}}{\mathsf{stat}\ e^1\ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}} \qquad \frac{}{x\ {}_1{\lesssim}_{\mathsf{E}}\ x} \qquad \frac{}{i\ {}_1{\lesssim}_{\mathsf{E}}\ i}$$

$$\frac{e^1\ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}}{\lambda x.\, e^1\ {}_1{\lesssim}_{\mathsf{E}}\ \lambda x.\, e^{\mathsf{E}}} \qquad \frac{e^1\ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}}{\lambda (x\!:\!\tau).\, e^1\ {}_1{\lesssim}_{\mathsf{E}}\ \lambda (x\!:\!\tau).\, e^{\mathsf{E}}}$$

$$\frac{e_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}} \quad e_1^1\ {}_1{\lesssim}_{\mathsf{E}}\ e_1^{\mathsf{E}}}{e_0^1\ e_1^1\ {}_1{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}}\ e_1^{\mathsf{E}}} \qquad \frac{e_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}} \quad e_1^1\ {}_1{\lesssim}_{\mathsf{E}}\ e_1^{\mathsf{E}}}{\langle e_0^1, e_1^1\rangle\ {}_1{\lesssim}_{\mathsf{E}}\ \langle e_0^{\mathsf{E}}, e_1^{\mathsf{E}}\rangle}$$

$$\frac{e_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}}}{op^1\ e_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ op^1\ e_0^{\mathsf{E}}} \qquad \frac{e_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}} \quad e_1^1\ {}_1{\lesssim}_{\mathsf{E}}\ e_1^{\mathsf{E}}}{op^2\ e_0^1\ e_1^1\ {}_1{\lesssim}_{\mathsf{E}}\ op^2\ e_0^{\mathsf{E}}\ e_1^{\mathsf{E}}}$$

$$\frac{e_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}}}{\mathsf{dyn}\ \tau\ e_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ \mathsf{dyn}\ \tau\ e_0^{\mathsf{E}}} \qquad \frac{e_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}}}{\mathsf{stat}\ \tau\ e_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ \mathsf{stat}\ \tau\ e_0^{\mathsf{E}}}$$

$\boxed{E\ {}_1{\lesssim}_{\mathsf{E}}\ E}$

$$\frac{E^1\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}}{\mathsf{chk}\ K\ E^1\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}} \qquad \frac{E^1\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}}{\mathsf{dyn}\ E^1\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}} \qquad \frac{E^1\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}}{\mathsf{stat}\ E^1\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}}$$

$$\frac{E^1\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}} \quad e^1\ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}}{E^1\ e^1\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}\ e^{\mathsf{E}}} \qquad \frac{v_0\ {}_1{\lesssim}_{\mathsf{E}}\ v_1 \quad E^1\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}}{v_0\ E^1\ {}_1{\lesssim}_{\mathsf{E}}\ v_1\ E^{\mathsf{E}}}$$

$$\frac{E^1\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}} \quad e^1\ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}}{\langle E^1, e^1\rangle\ {}_1{\lesssim}_{\mathsf{E}}\ \langle E^{\mathsf{E}}, e^{\mathsf{E}}\rangle} \qquad \frac{v_0\ {}_1{\lesssim}_{\mathsf{E}}\ v_1 \quad E^1\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}}{\langle v_0, E^1\rangle\ {}_1{\lesssim}_{\mathsf{E}}\ \langle v_1, E^{\mathsf{E}}\rangle}$$

$$\frac{E^1\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}}{op^1\ E^1\ {}_1{\lesssim}_{\mathsf{E}}\ op^1\ E^{\mathsf{E}}} \qquad \frac{E^1\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}} \quad e^1\ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}}{op^2\ E^1\ e^1\ {}_1{\lesssim}_{\mathsf{E}}\ op^2\ E^{\mathsf{E}}\ e^{\mathsf{E}}}$$

$$\frac{v_0\ {}_1{\lesssim}_{\mathsf{E}}\ v_1 \quad E^1\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}}{op^2\ v_0\ E^1\ {}_1{\lesssim}_{\mathsf{E}}\ op^2\ v_1\ E^{\mathsf{E}}} \qquad \frac{E^1\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}}{\mathsf{dyn}\ \tau\ E^1\ {}_1{\lesssim}_{\mathsf{E}}\ \mathsf{dyn}\ \tau\ E^{\mathsf{E}}}$$

$$\frac{E^1\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}}{\mathsf{stat}\ \tau\ E^1\ {}_1{\lesssim}_{\mathsf{E}}\ \mathsf{stat}\ \tau\ E^{\mathsf{E}}}$$

A.8.2 Theorems

Theorem 8.0 : Err *approximation*

If $e \in e_{\mathsf{S}}$ and $\vdash e : \tau$ then the following statements hold:
- if $e \rightarrow^*_{\mathsf{E\text{-}S}} \mathsf{Err}$ then $e \rightarrow^*_{\mathsf{1\text{-}S}} \mathsf{Err}$
- if $e \rightarrow^*_{\mathsf{1\text{-}S}} \mathsf{Err}$ then $e \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{Err}$

Proof:

QED by 1−E *approximation* and H−1 *approximation*.

□

A.8.3 Lemmas

Lemma 8.1 : 1−E *approximation*

If $e \in e_{\mathsf{S}}$ and $\vdash e : \tau$ and $e \rightarrow^*_{\mathsf{E\text{-}S}} \mathsf{Err}$ then $\vdash_1 e \rightsquigarrow e'' : \lfloor \tau \rfloor$ and $e'' \rightarrow^*_{\mathsf{1\text{-}S}} \mathsf{Err}$

Proof:
- $e''\ {}_1{\lesssim}_{\mathsf{E}}\ e$
  by 1−E *reflexivity*
- QED by 1−E *simulation*

□

Lemma 8.2 : *reflexivity*

If $\vdash e : \tau$ and $\vdash_1 e \rightsquigarrow e'' : \lfloor \tau \rfloor$ then $e''\ {}_1{\lesssim}_{\mathsf{E}}\ e$.

Proof:
- e and e'' are identical up to chk expressions
  by definition of $\rightsquigarrow$
- QED by definition of ${}_1{\lesssim}_{\mathsf{E}}$

□

Lemma 8.3 : E−1 *simulation*

If $e_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}}$ and $e_0^{\mathsf{E}} \rightarrow_{\mathsf{E\text{-}S}} e_1^{\mathsf{E}}$ and $e_0^1 \notin \mathsf{Err}$ then:
- $e_0^1 \rightarrow_{\mathsf{1\text{-}S}} \ldots \rightarrow_{\mathsf{1\text{-}S}} e_n^1$
- $\forall i \in \{1..n-1\}.\ e_i^1\ {}_1{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}}$
- $e_n^1\ {}_1{\lesssim}_{\mathsf{E}}\ e_1^{\mathsf{E}}$

Proof:

case $e_0^{\mathsf{E}} = E^{\mathsf{E}}[e_{0'}^{\mathsf{E}}]$
$\wedge\ e_{0'}^{\mathsf{E}} \rhd_{\mathsf{E\text{-}S}} e_{1'}^{\mathsf{E}}$
$\wedge\ e_1^{\mathsf{E}} = E^{\mathsf{E}}[e_{1'}^{\mathsf{E}}]$:

1. $e_0^{1} = E^{1}[e_{0'}^{1}]$
   $\wedge\ E^{1}\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}$
   $\wedge\ e_{0'}^{1}\ {}_1{\lesssim}_{\mathsf{E}}\ e_{0'}^{\mathsf{E}}$
   by 1–E *context factoring*
2. $E^{1}[e_{0'}^{1}] \rightarrow_{\mathsf{1\text{-}S}} \ldots \rightarrow_{\mathsf{1\text{-}S}} E^{1}[e_{n-1'}^{1}]$
   $\wedge\ \forall i \in \{1..n-1\}\,.\, E^{1}[e_{i'}^{1}]\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[e_{0'}^{\mathsf{E}}]$
   $\wedge\ e_{n-1}^{1} \neq \mathsf{chk}\ \tau\ e$
   $\wedge\ e_{n-1}^{1} \neq \mathsf{dyn}\ e$
   $\wedge\ e_{n-1}^{1} \neq \mathsf{stat}\ e$
   by repeated uses of 1–E *stutter* (1)
3. **if** $e_{n-1}^{1} \in \mathsf{Err}$:
   a. qed $e_0^{1} \rightarrow^{*}_{\mathsf{1\text{-}S}} \mathsf{Err}$

   else $e_{n-1}^{1} \notin \mathsf{Err}$:
   a. $E^{1}[e_{n-1}^{1}] \rightarrow_{\mathsf{1\text{-}S}} E^{1}[e_n^{1}]$
      $\wedge\ e_n^{1}\ {}_1{\lesssim}_{\mathsf{E}}\ e_{1'}^{\mathsf{E}}$
      by 1–E *step* (1, 2)
   b. qed 1–E *context congruence*

□

Lemma 8.4 : 1–E *context factoring*

If $e_0^{1}\ {}_1{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}}$ and $e_0^{\mathsf{E}} = E_0^{\mathsf{E}}[e_{1'}^{\mathsf{E}}]$ then $e_0^{1} = E_0^{1}[e_{0'}^{1}]$ and $E_0^{1}\ {}_1{\lesssim}_{\mathsf{E}}\ E_0^{\mathsf{E}}$ and $e_{0'}^{1}\ {}_1{\lesssim}_{\mathsf{E}}\ e_{0'}^{\mathsf{E}}$

Proof:

qed by structural induction on the derivation of $e_0^{1}\ {}_1{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}}$

□

Lemma 8.5 : 1–E *stutter*

If $E^{1}[e^{1}]\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[e^{\mathsf{E}}]$ and $e^{1} = E_0^{1}[\mathsf{Err}]$
$\vee\ e^{1} = E_0^{1}[\mathsf{chk}\ K\ v_0]$
$\vee\ e^{1} = E_0^{1}[\mathsf{dyn}\ v_0]$
$\vee\ e^{1} = E_0^{1}[\mathsf{stat}\ v_0]$
then $E^{1}[e^{1}] \rightarrow_{\mathsf{1\text{-}S}} E^{1}[e_1^{1}]$ and $E^{1}[e_1^{1}]\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[e^{\mathsf{E}}]$.

Proof:

case $e^{1} = E_0^{1}[\mathsf{Err}]$:
1. qed $E^{1}[e^{1}] \rightarrow_{\mathsf{1\text{-}S}} \mathsf{Err}$

case $e^{1} = E_0^{1}[\mathsf{chk}\ K\ v_0]$:

if $\mathcal{X}(K, v_0) = \mathsf{BndryErr}$:
1. qed $E^{1}[E_0^{1}[\mathsf{chk}\ K\ v_0]] \rightarrow_{\mathsf{1\text{-}S}} \mathsf{BndryErr}$

else $\mathcal{X}(K, v_0) = v_0$:
1. $E^{1}[E_0^{1}[\mathsf{chk}\ K\ v_0]] \rightarrow_{\mathsf{1\text{-}S}} E^{1}[E_0^{1}[v_0]]$
2. $E^{1}[E_0^{1}[v_0]]\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[e^{\mathsf{E}}]$
   by $E^{1}[E_0^{1}[\mathsf{chk}\ K\ v_0]]\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[e^{\mathsf{E}}]$
3. qed

case $e^{1} = E_0^{1}[\mathsf{dyn}\ v_0]$:
1. $E^{1}[E_0^{1}[\mathsf{dyn}\ v_0]] \rightarrow_{\mathsf{1\text{-}S}} E^{1}[E_0^{1}[v_0]]$
2. $E^{1}[E_0^{1}[v_0]]\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[e^{\mathsf{E}}]$
   by $E^{1}[E_0^{1}[\mathsf{dyn}\ v_0]]\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[e^{\mathsf{E}}]$
3. qed

case $e^{1} = E_0^{1}[\mathsf{stat}\ v_0]$:
1. $E^{1}[E_0^{1}[\mathsf{stat}\ v_0]] \rightarrow_{\mathsf{1\text{-}S}} E^{1}[E_0^{1}[v_0]]$
2. $E^{1}[E_0^{1}[v_0]]\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[e^{\mathsf{E}}]$
   by $E^{1}[E_0^{1}[\mathsf{stat}\ v_0]]\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[e^{\mathsf{E}}]$
3. qed

□

Lemma 8.6 : 1–E *step*

If $E^{1}[e^{1}]\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[e^{\mathsf{E}}]$ and $e^{1} \notin \{\mathsf{chk}\ \tau\ e, \mathsf{dyn}\ e, \mathsf{stat}\ e\}$ and $e^{\mathsf{E}} \rhd_{\mathsf{E\text{-}S}} e_1^{\mathsf{E}}$ then $E^{1}[e^{1}] \rightarrow_{\mathsf{1\text{-}S}} E^{1}[e_1^{1}]$ and $e_1^{1}\ {}_1{\lesssim}_{\mathsf{E}}\ e_1^{\mathsf{E}}$

Proof:

By *boundary factoring* and *static hole typing*, the inner expression e^{1} is either typed or untyped.

case $\vdash_1 e^{1}$:

qed by case analysis on $e^{\mathsf{E}} \rhd_{\mathsf{E\text{-}S}} e_1^{\mathsf{E}}$; either e^{1} steps in the same manner, or e^{1} steps to a boundary error due to the application of a typed function to an invalid argument

case $\vdash_1 e^{1} : K$:

qed by case analysis on $e^{\mathsf{E}} \rhd_{\mathsf{E\text{-}S}} e_1^{\mathsf{E}}$; note that $e_1^{\mathsf{E}} \notin \mathsf{TagErr}$ since e^{1} is well-typed.

□

Lemma 8.7 : 1–E *context congruence*

If $e^{1}\ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}$ and $E^{1}\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}$ then $E^{1}[e^{1}]\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[e^{\mathsf{E}}]$

Proof:

qed by definition of ${}_1{\lesssim}_{\mathsf{E}}$

□

Lemma 8.8 : *reflexivity*

If $\vdash e : \tau$ then $e\ {}_1{\lesssim}_{\mathsf{E}}\ e$.

Proof:

By structural induction on the derivation of $\vdash e : \tau$.
TODO

□

Lemma 8.9 : 1–H *simulation*

yolo

Proof:

nooo

□