A Spectrum of Type Soundness and Performance

Supplementary Material

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Contents

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Benchmark	Untyped LOC	Annotation LOC	# Modules
`sieve`	35	17 (49%)	2
`fsm`	182	56 (31%)	4
`morsecode`	159	38 (24%)	4
`zombie`	302	27 (9%)	4
`jpeg`	1432	165 (12%)	5
`suffixtree`	537	129 (24%)	6
`kcfa`	229	53 (23%)	7
`snake`	160	51 (32%)	8
`tetris`	246	107 (43%)	9
`synth`	835	139 (17%)	10

Fig. 1. Benchmark Size

A Benchmark Descriptions

sieve from Ben Greenman
Computes prime numbers using the sieve of Eratosthenes.

fsm from Linh Chi Nguyen
Simulates the interactions of economic agents modeled as finite-state automata.

morsecode from John B. Clements & Neil Van Dyke
Computes Levenshtein distances and morse code translations for a fixed sequence of pairs of words.

zombie from David Van Horn
Implements a game where players avoid enemies. The benchmark runs a fixed sequence of moves (representing user input).

jpeg from Andy Wingo
Parses a bytestream of JPEG data to an internal representation, then serializes the result.

suffixtree from Danny Yoo
Computes longest common subsequences between strings.

kcfa from Matt Might
Performs 1-CFA on a lambda calculus equation built from Church numerals.

snake from David Van Horn
Implements the Snake game; the benchmark replays a fixed sequence of moves.

tetris from David Van Horn
Replays a pre-recorded game of Tetris.

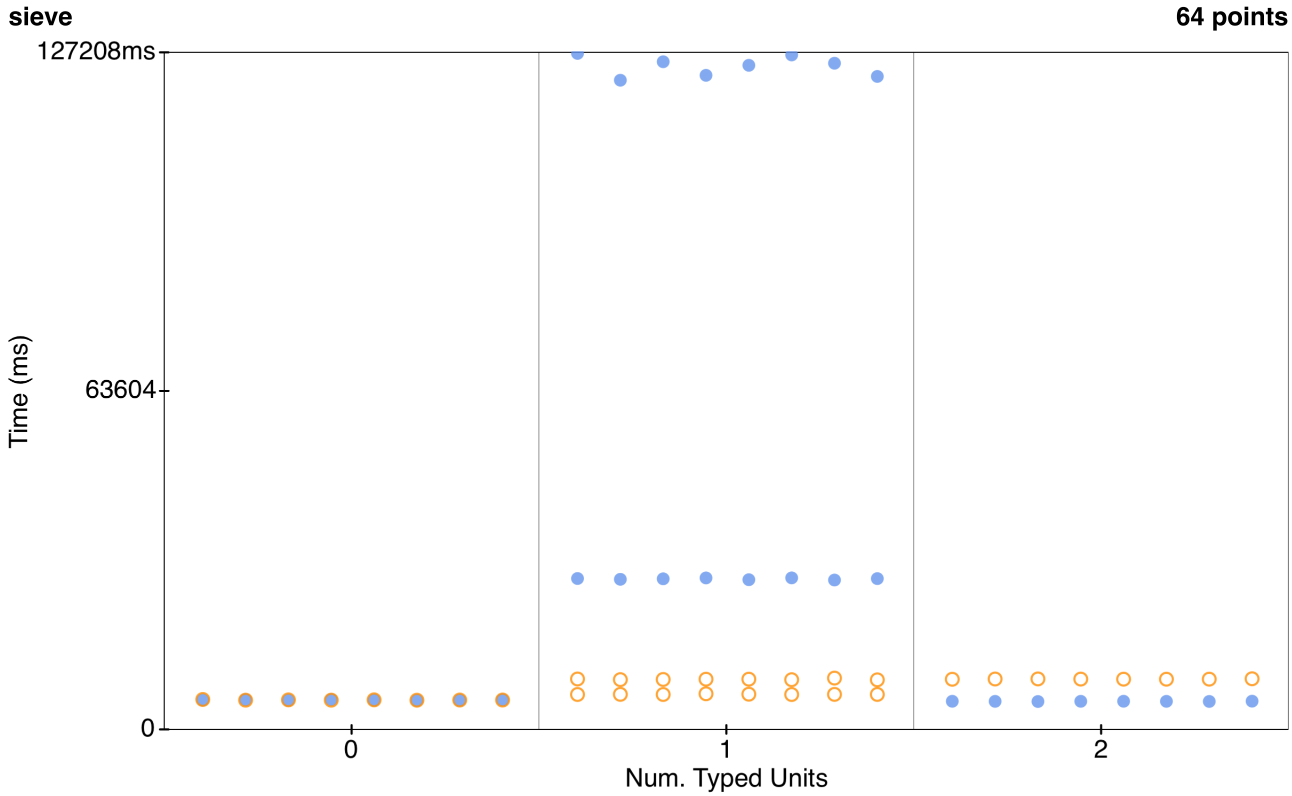
synth from Vincent St. Amour & Neil Toronto
Converts a description of notes and drum beats to WAV format.

For additional details about the benchmarks, their source code, and links to more (object-oriented) benchmarks, see: docs.racket-lang.org/gtp-benchmarks/index.html

B Performance vs. Number of Typed Modules

Figures 2, 3, 4, 5, and 6 plot every running time in the dataset. Each point represents one measurement; specifically, a point at position (X, Y) reports one running time of Y milliseconds for one configuration with X "typed units" — in this paper, one typed unit is one typed module. There are eight such points for each configuration. To make these points easier to see, the eight points for one configuration are evenly spaced along the x-axis within a bucket (delimited by solid vertical lines). The left-most point plots the running time of the first trial, the second-from-left point corresponds to the second trial, and so on.

The figures support three broad conclusions. First, the orange points for TR-1 are often lower (i.e., show better performance) than the points for TR-H. Second, the performance of TR-1 tends to degrade linearly as the number of typed modules increases. This slowdown tapers off at the right-most end because the implementation skips the codomain-check for calls to statically-typed identifiers. Third, the performance of TR-H has a steep umbrella shape. The worst performance is in the middle, but improves significantly as the number of typed modules approaches the maximum.



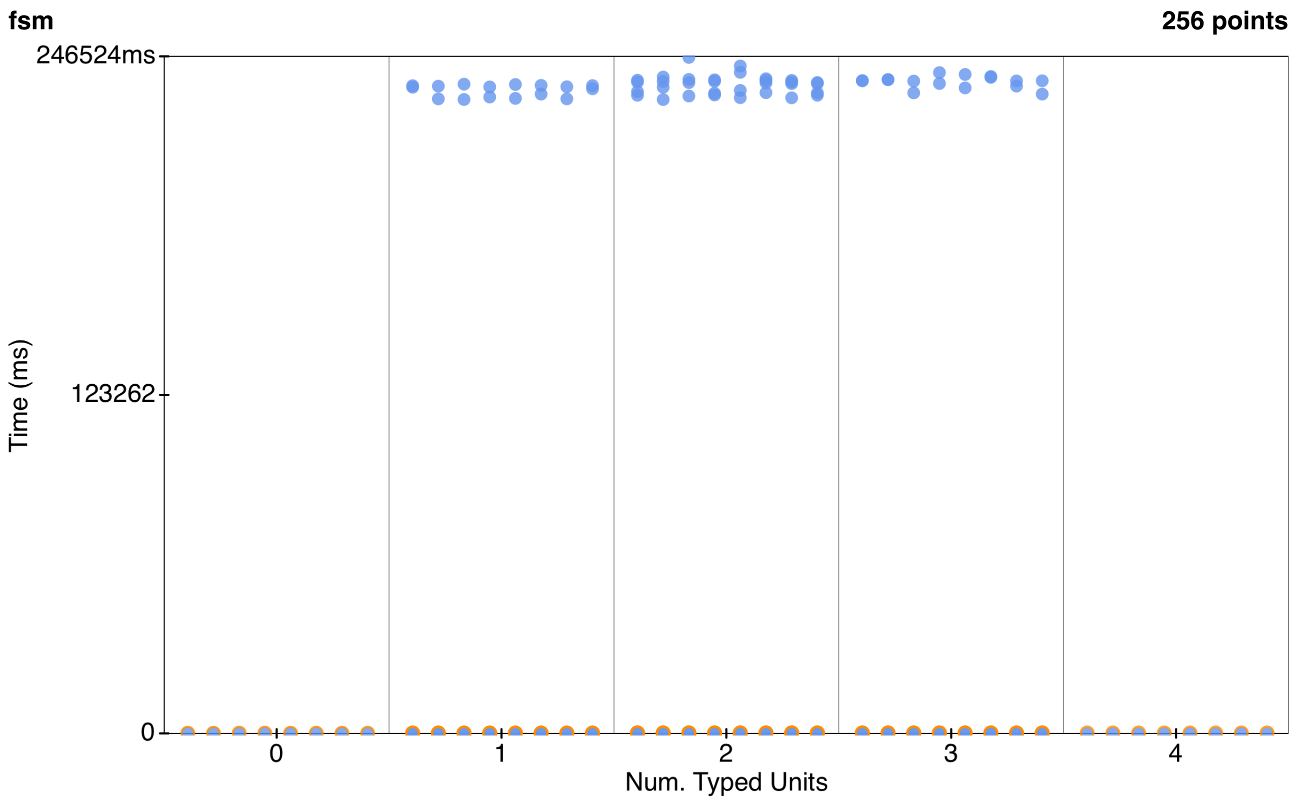


Fig. 2. Running time of TR-H configurations (blue) and TR-1 configurations (orange), part 1/5

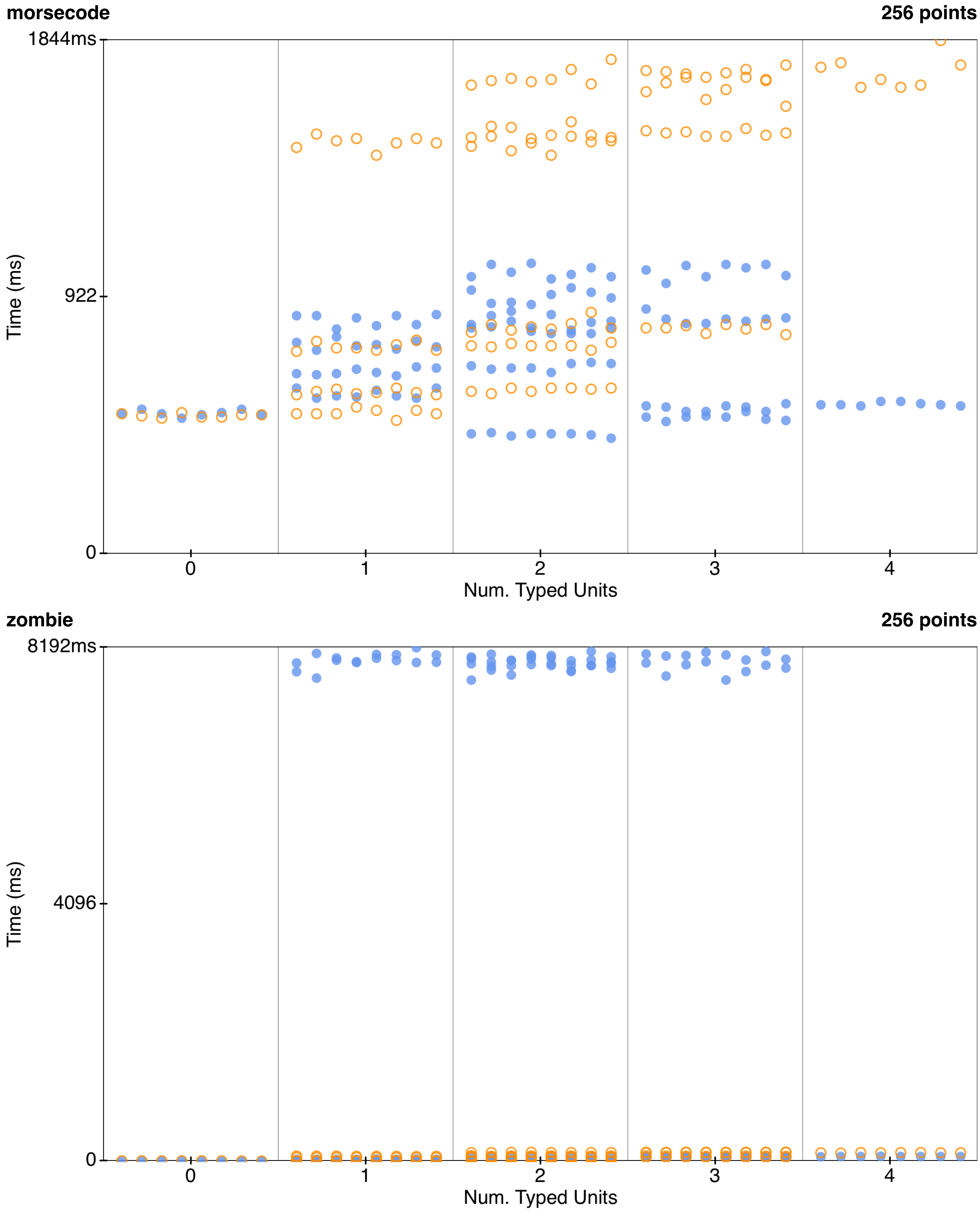


Fig. 3. Running time of TR-H configurations (blue) and TR-1 configurations (orange), part 2/5

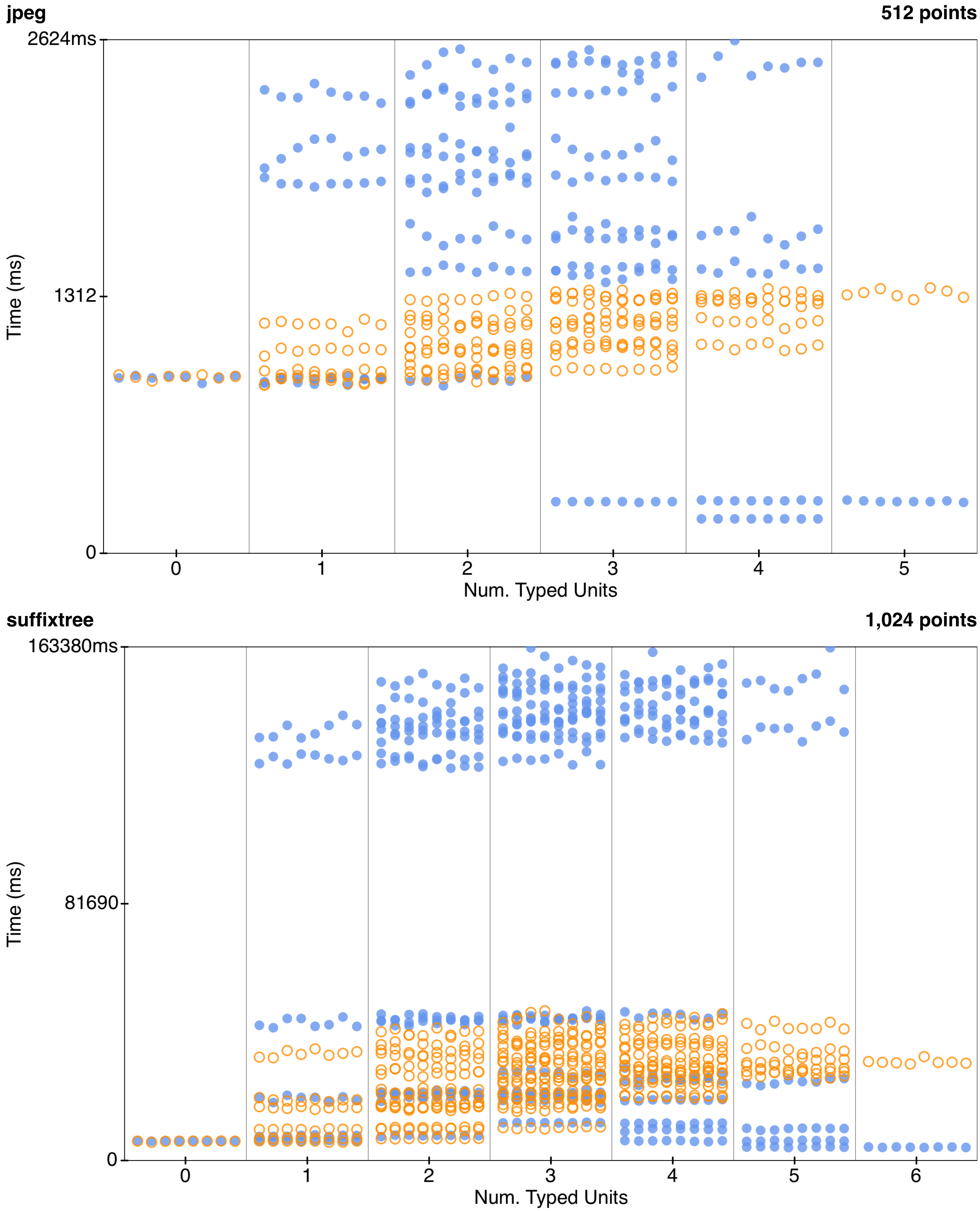
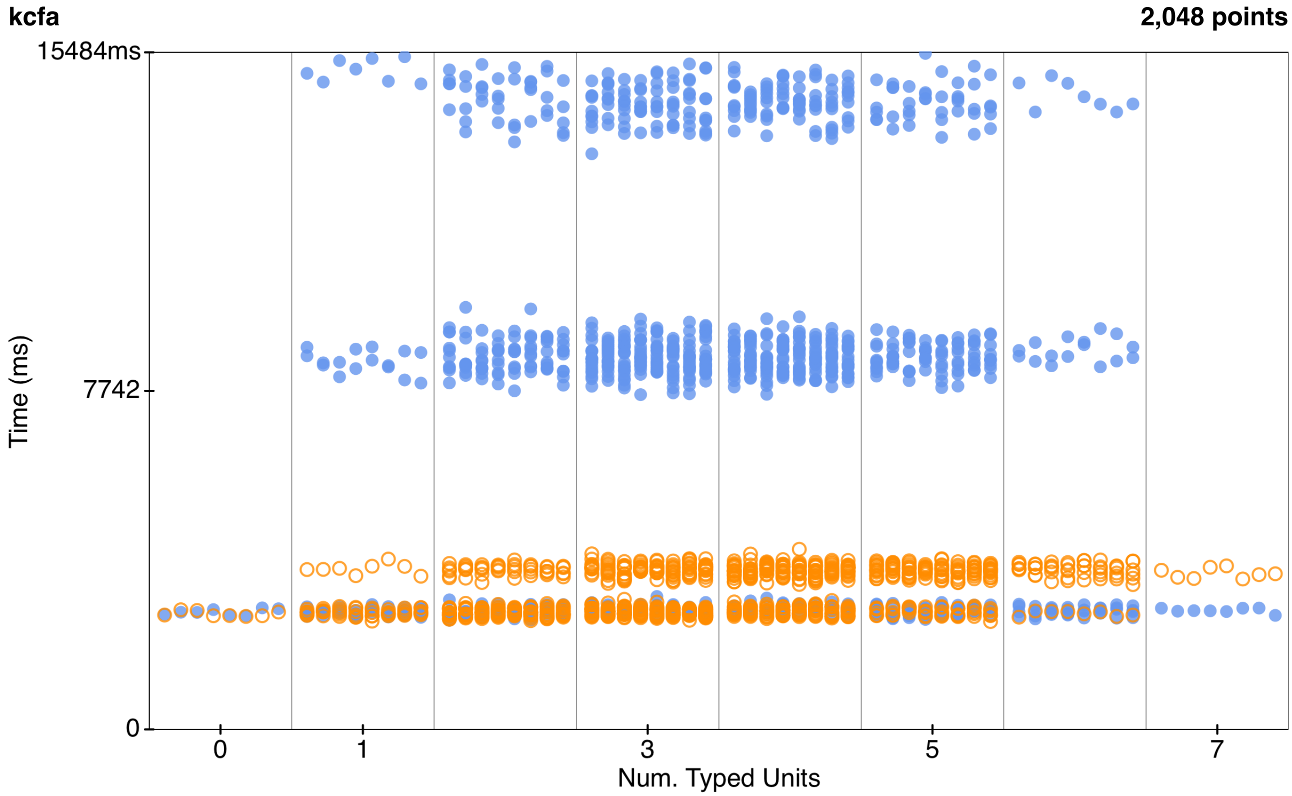


Fig. 4. Running time of TR-H configurations (blue) and TR-1 configurations (orange), part 3/5



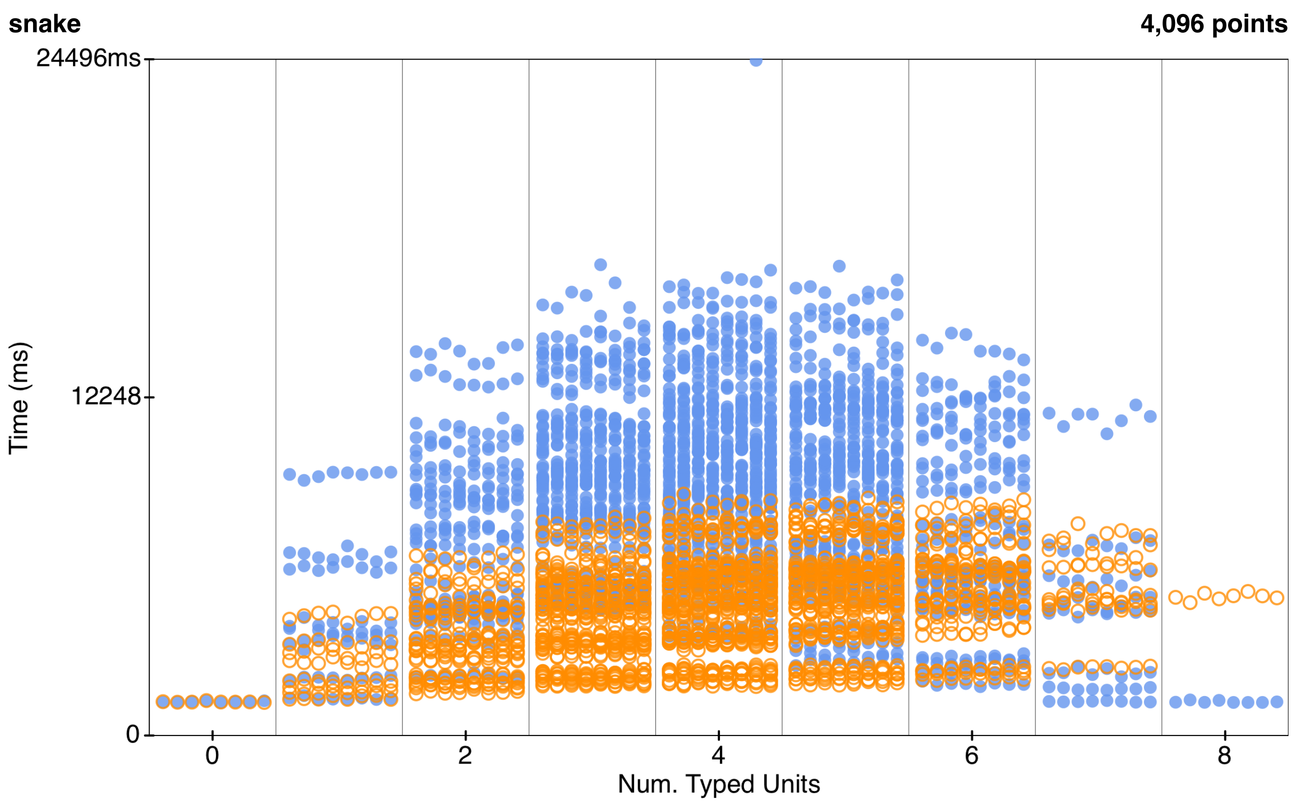
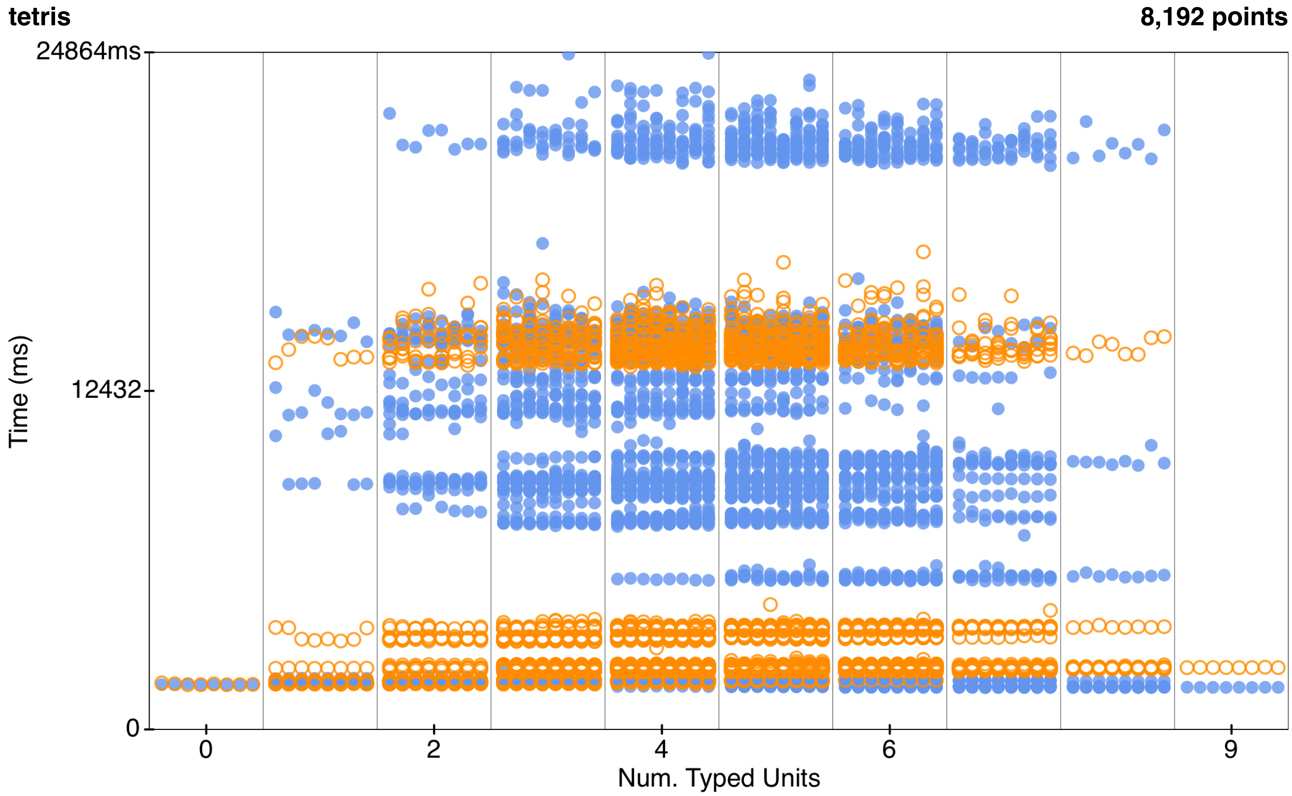


Fig. 5. Running time of TR-H configurations (blue) and TR-1 configurations (orange), part 4/5



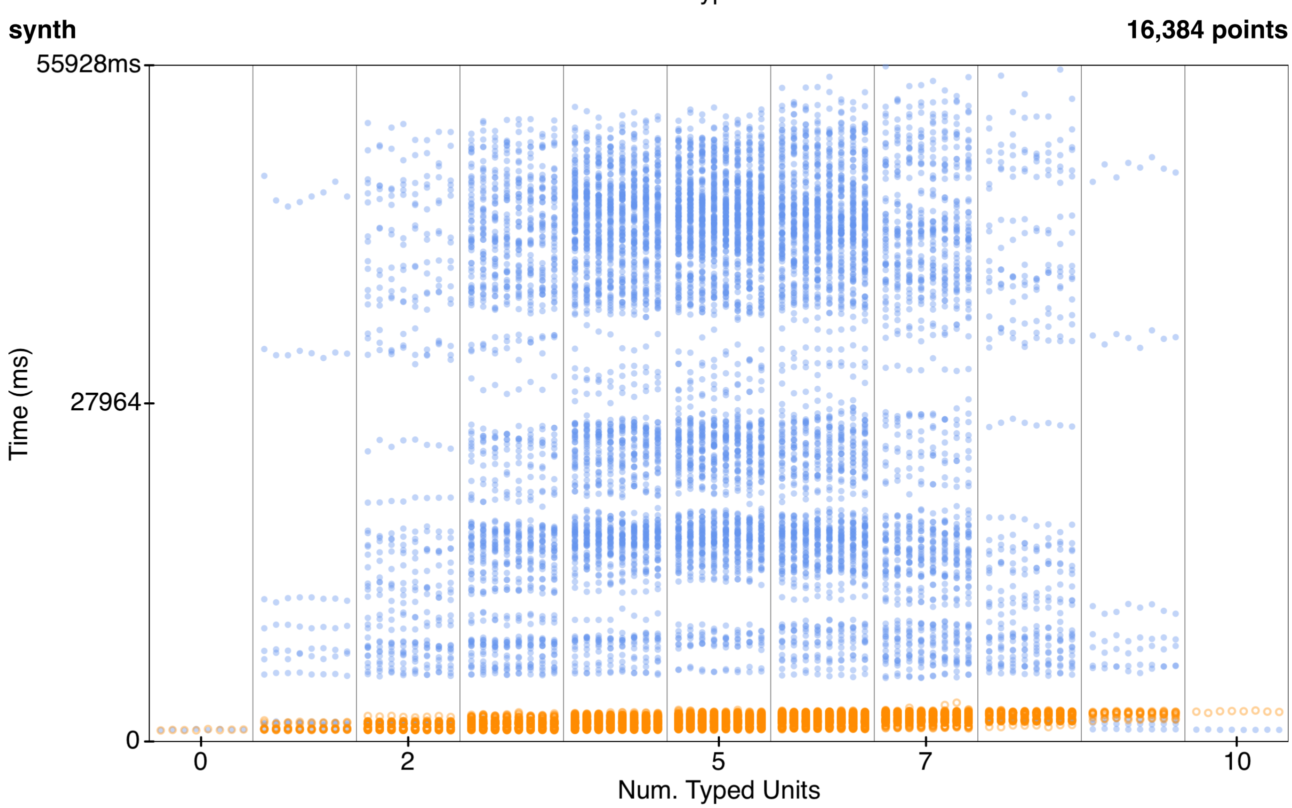


Fig. 6. Running time of TR-H configurations (blue) and TR-1 configurations (orange), part 5/5

C Implementing Tagged Racket

The high-level architecture of TR-H is to:

1. type-check a module,
2. use the type environment to generate contracts,
3. optimize the contracts for the module,
4. output Racket bytecode.

For TR-1, we modified step 2 and replaced step 3.

C.1 Generating Type-Constructor Contracts

Typed Racket defines a function `type->contract` that (1) expects a type, (2) compiles the type to a so-called *static contract*, (3) optimizes the representation of the static contract, and (4) compiles the static contract to Racket code that will generate an appropriate contract.

We modified the `type->contract` function to generate type-constructor checks by adding a new method to the internal API for static contracts. For example, the method converts the contract for a list of elements into a contract that checks the `list?` predicate.

C.2 Defending Typed Code

The TR-1 prototype replaces the Typed Racket optimizer with a completion function that adds type-constructor checks to typed code. The function implements a fold over the syntax of a type-annotated program, and performs two kinds of rewrites.

First, the completion function rewrites *most* applications `(f x)` to `(check K (f x))`, where `K` is the static type of the application. If `f` is an identifier, however, there are two exceptional cases:

- `f` may be a built-in function that is certain to return a value of the correct type constructor (e.g., `map` always returns a list); and
- `f` may be statically typed, in which case soundness guarantees that `f` returns a value that matches its static type constructor. (there is one exception: accessor functions for user-defined structs are unsafe like any other accessor, e.g., `car`),

For these exceptional cases, the completion function does not insert a type-constructor check.

Second, the completion function defends typed functions from dynamically-typed arguments by translating a function like `(λ (x) e)` to `(λ (x) (check x) e)`. The structure of the check is based on the domain type of the function.

C.3 Diff vs. Racket v6.10.1

The repository for this paper contains the TR-1 prototype and a diff between the prototype and Typed Racket v6.10.1. github.com/nuprl/tag-sound?path=src/locally-defensive.patch

D Existing Systems

This section illustrates prior work on gradual typing using the semantic framework of the paper. The goal is to demonstrate that the framework is able to express the *type boundaries* and *boundary checks* of existing systems, and to outline a formal comparison.

This section does not attempt to summarize the novelties and subtleties of each system. The interested reader must seek out the primary sources.

The subsections also give canonical forms lemmas for each system as a taste of their logical implications.

URLs All URLs accessed on 2018-06-28.

- Gradualtalk : https://pleiad.cl/research/software/gradualtalk
- Typed Racket : https://github.com/racket/typed-racket
- TPD : https://github.com/jack-williams/tpd
- StrongScript : https://plg.uwaterloo.ca/~dynjs/strongscript/
- ActionScript : https://www.adobe.com/devnet/actionscript.html
- mypy : http://mypy-lang.org/
- Flow : https://flow.org/
- Hack : http://hacklang.org/
- Pyre : https://pyre-check.org/
- Pytype : https://opensource.google.com/projects/pytype
- rtc : https://github.com/plum-umd/rtc
- Strongtalk : http://strongtalk.org/
- TypeScript : https://www.typescriptlang.org/
- Typed Clojure : http://typedclojure.org/
- Typed Lua : https://github.com/andremm/typedlua
- Pyret : https://www.pyret.org/
- Thorn : http://janvitek.org/yearly.htm
- Dart 2 : https://www.dartlang.org/dart-2
- Nom : https://www.cs.cornell.edu/~ross/publications/nomalive/
- Reticulated : https://github.com/mvitousek/reticulated
- SafeTS : https://www.microsoft.com/en-us/research/publication/safe-efficient-gradual-typing-for-typescript-3/
- TR-1 : https://github.com/bennn/typed-racket/releases/tag/ld1.0

$\boxed{\text{Thorn}}$ (sketch)

$\tau \ = \ C \mid \mathsf{like}\, C \mid \mathbf{dyn}$

$v \ = \ p \mid (\mathbf{dyn})\, p \mid (\mathsf{like}\, C)\, p$

$p \ = \ C(f = v, \ldots)$

$\boxed{\mathcal{D} : \tau \times v \longrightarrow e}$ (undefined, all values have a static type)

$\boxed{\mathcal{S} : \tau \times v \longrightarrow e}$ (via explicit type-cast)

$\mathcal{S}(C, v) \quad = p$
  if $v = p$ or $v = (\mathbf{dyn})\, p$ or $v = (\mathsf{like}\, C')\, p$
  and $p = C''(f = v, \ldots)$
  and $C'' \leqslant: C$

$\mathcal{S}(\mathsf{like}\, C, v) = (\mathsf{like}\, C)\, p$
  if $v = p$ or $v = (\mathbf{dyn})\, p$ or $v = (\mathsf{like}\, C')\, p$

$\mathcal{S}(\mathbf{dyn}, v) \quad = (\mathbf{dyn})\, p$
  if $v = p$ or $v = (\mathbf{dyn})\, p$ or $v = (\mathsf{like}\, C')\, p$

$\mathcal{S}(\tau, v) \quad = \mathsf{BndryErr}$
  otherwise

Fig. 7. Thorn types, values, and boundary functions. The $\mathcal{D}$ function is undefined for all inputs.

D.1 Thorn

Thorn (figure 7) is a nominally-typed object-oriented language. The idea is that a program may: declare typed classes, use the classes to create typed objects, and manipulate the objects in gradually-typed methods. If a method expects a dynamically-typed object, the type checker lets the method perform any operation on the object and the run-time system dynamically checks whether the operations are actually valid.

The types τ include *concrete* class names C, *like* class names ($\mathsf{like}\, C$), and a dynamic type ($\mathbf{dyn}$). The values are possibly-wrapped pointers to instances of classes; a value is either a direct pointer p, a dynamically-typed *view* to a pointer $(\mathbf{dyn})\, p$, or a like-typed view to a pointer $(\mathsf{like}\, C)\, p$. Informally, a view is a method-local pointer to an object.

One main invariant of Thorn is that every value comes with a type. In the figure, every value is an instance of a class and has the class name as its type. Because of this invariant, Thorn can efficiently check whether a value is compatible with some other type annotation at runtime. The $\mathcal{S}$ function demonstrates this compatibility check.

The $\mathcal{D}$ function is undefined for all inputs because there is no such thing as a dynamically-typed value. Put another way, the Thorn surface language is a single statically-typed language as opposed to a pair of languages. (The statically-typed language includes a dynamic to make it easy to experiment with statically-typed values, but nevertheless all values are statically typed to ensure safety and efficiency.)

Lemma 0.0 : *Thorn canonical forms*

- If $\vdash v : C$ then $v = C'(f = v_f, \ldots)$ and $C' \leqslant: C$.
- If $\vdash v : \mathsf{like}\, C$ then $v = (\mathsf{like}\, C')\, p$ and $C' \leqslant: C$
- If $\vdash v : \mathbf{dyn}$ then $v = (\mathbf{dyn})\, p$

D.2 StrongScript

StrongScript (figure 8) adapts the ideas from Thorn to a type system for JavaScript. The types τ include concrete class names ($!C$), like class names (C), a dynamic type (any), and function types ($\tau \Rightarrow \tau$). The values are objects and functions.

Every object and function comes with an intrinsic type. For an object imported from JavaScript, this type is any. (For a function imported from JavaScript, this type is presumably $\mathsf{any} \Rightarrow \mathsf{any}$.) A typed object cannot inherit from a JavaScript object, and vice-versa.

The $\mathcal{S}$ function checks the intrinsic type of a value against a type annotation. The idea is, if the check succeeds then a context may assume that the type annotation accurately describes the value.

The $\mathcal{D}$ function is undefined for all inputs because the StrongScript paper does not directly model interactions with JavaScript. Instead, a JavaScript object is modeled as an object with type any, as mentioned above.

Lemma 0.1 : StrongScript *canonical forms*

- If $\vdash v : !C$ then $v = \{(s\!:\!v) \ldots m \ldots \parallel C'\}$ and $C' \leqslant: C$
- If $\vdash v : C$ then either:
  - $v = \{(s\!:\!v) \ldots m \ldots \parallel C'\}$
  - $v = \{(s\!:\!v) \ldots m \ldots \parallel \mathsf{any}\}$
- If $\vdash v : \tau_d \Rightarrow \tau_c$ then $v = \mathsf{func}\,(x\!:\!\tau_d)\{\mathsf{return}\; e\!:\!\tau_c\}$
- If $\vdash v : \mathsf{any}$ then either:
  - $v = \{(s\!:\!v) \ldots m \ldots \parallel C'\}$
  - $v = \{(s\!:\!v) \ldots m \ldots \parallel \mathsf{any}\}$
  - $v = \mathsf{func}\,(x\!:\!\mathsf{any})\{\mathsf{return}\; e\!:\!\mathsf{any}\}$

$\boxed{\text{StrongScript}}$ (sketch)

$$
\begin{array}{lcl}
\tau & = & !C \mid C \mid \mathsf{any} \mid \tau \Rightarrow \tau \\
v & = & \{(s\!:\!v) \ldots m \ldots \parallel \tau\} \mid \mathsf{func}\,(x\!:\!\tau)\{\mathsf{return}\; e\!:\!\tau\} \\
s & \in & \mathsf{Strings} \\
m & = & x\,(x\!:\!\tau)\{\mathsf{return}\; e\!:\!\tau\}
\end{array}
$$

$\boxed{\mathcal{D} : \tau \times v \longrightarrow e}$ (undefined, all values have a static type)

$\boxed{\mathcal{S} : \tau \times v \longrightarrow e}$ (via explicit type-cast)

$$
\begin{array}{ll}
\mathcal{S}(\tau_d \Rightarrow \tau_c, v) = v' & \\
\quad \text{if } v = \mathsf{func}\,(x\!:\!\tau_x)\{\mathsf{return}\; e\!:\!\tau_v\} & \\
\quad \text{where } v' = \mathsf{func}\,(x\!:\!\tau_d)\{\mathsf{return}\,(\langle\tau_c\rangle\,(v\,(\langle\tau_x\rangle\, x)))\!:\!\tau_c\} & \\
\mathcal{S}(C, v) & = v \\
\quad \text{if } v = \{(s\!:\!v) \ldots m \ldots \parallel C'\} & \\
\quad \text{and } C' \leqslant: C & \\
\mathcal{S}(\mathsf{any}, v) & = v \\
\quad \text{if } v = \{(s\!:\!v) \ldots m \ldots \parallel \mathsf{any}\} & \\
\quad \text{or } v = \{(s\!:\!v) \ldots m \ldots \parallel C\} & \\
\mathcal{S}(\mathsf{any}, v) & = v' \\
\quad \text{if } v = \mathsf{func}\,(x\!:\!\tau_d)\{\mathsf{return}\; e\!:\!\tau_c\} & \\
\quad \text{where } v' = \mathsf{func}\,(x\!:\!\mathsf{any})\{\mathsf{return}\;\langle\mathsf{any}\rangle(v\,(\langle\tau_d\rangle x))\!:\!\mathsf{any}\} & \\
\mathcal{S}(\tau, v) & = \mathsf{BndryErr} \\
\quad \text{otherwise} &
\end{array}
$$

Fig. 8. StrongScript boundary functions. The $\mathcal{D}$ function is undefined for all inputs.

D.3 Dart 2

Dart 2 is a new language with some support for dynamic typing. For details, see: dartlang.org/dart-2

Figure 9 summarizes the key aspects of dynamic typing in Dart for a few types. The types represent integers (int), integers and decimal numbers (num), lists (List$\langle\tau\rangle$), functions ($\tau \Rightarrow \tau$), and a dynamic type (dynamic). The base values b match these types.

Dart programs do not directly interact with base values. Instead, base values are stored on a typed heap. The values v in figure 9 model this indirection by associating a base value with a compatible type.

Just like in Thorn, a typed value may be used in a context that expects a less precise type. Also like Thorn, a value of type dynamic is an object that the type checker assumes can recieve any method call. The run-time system checks that such method calls are actually safe for the given value.

The $\mathcal{S}$ boundary function checks a value against a type annotation by checking the value's associated type. The $\mathcal{D}$ function is undefined for all inputs because it is not possible to define (or interact with) an untyped value.

Lemma 0.2 : *Dart canonical forms*

- If $\vdash v : \tau$ then $v = b :: \tau'$ and $\tau' \leqslant: \tau$

Remark: a Dart function type must be declared explicitly. For example, to use the type int$\Rightarrow$int one must first define an alias:

```
typedef int IntFun(int _);
```

Then the name `IntFun` may appear in other type annotations, e.g., in a method signature.

$\boxed{\text{Dart}}$ (sketch)

$$\begin{aligned}
\tau &= \mathsf{int} \mid \mathsf{num} \mid \mathsf{List}\langle\tau\rangle \mid \tau \Rightarrow \tau \mid \mathsf{dynamic} \\
v &= b :: \tau \\
b &= i \mid d \mid [v \ldots] \mid (\tau\ x) \texttt{=>} e
\end{aligned}$$

$\boxed{\mathcal{D} : \tau \times v \longrightarrow e}$ (undefined, all values have a static type)

$\boxed{\mathcal{S} : \tau \times v \longrightarrow e}$

$$\begin{aligned}
&\mathcal{S}(\tau, b :: \tau') = b :: \tau' \\
&\quad \text{if } \tau' \leqslant: \tau \\
&\mathcal{S}(\tau, v) \qquad = \mathsf{BndryErr} \\
&\quad \text{otherwise}
\end{aligned}$$

Fig. 9. Dart boundary functions for a restricted grammar of types. The $\mathcal{D}$ function is undefined for all inputs.

D.4 Pyret

Pyret is a dynamically-typed language with optional type annotations and an optional static type checker. For details, see: pyret.org.

A type annotation in a Pyret program acts as a type-constructor check at run-time. Figure 10 illustrates this aspect of Pyret in the $\rhd_{\mathsf{S}}$ and $\rhd_{\mathsf{D}}$ notions of reduction. Both check the argument and result of a typed function against the function's type. The $\mathcal{X}$ boundary function performs the check by matching a type constructor K against a value.

One aspect of Pyret that is missing from figure 10 is the translation that maps type annotations in the source code to run-time constructor checks. This translation could be modeled with a completion function ($\rightsquigarrow$), similar to the model of the first-order embedding in the paper.

Lemma 0.3 : *Pyret assert-canonical forms*

If v is a value with the static type τ then v may be any kind of value; however, if v is assigned to a variable x with the programmer-assigned type τ, then one of the following holds:

- If $\vdash x : \tau_0 \times \tau_1$ then $v = \langle v_0, v_1 \rangle$
- If $\vdash x : \tau_d \Rightarrow \tau_c$ then either:
  - $v = \lambda y.\, e$
  - $v = \lambda((y : \tau_d') : \tau_c').\, e$
- If $\vdash x : \mathsf{Int}$ then $v \in \mathbb{Z}$
- If $\vdash x : \mathsf{Nat}$ then $v \in \mathbb{N}$

$\boxed{\text{Pyret}}$ (sketch)

$$
\begin{array}{lcl}
\tau & = & \mathsf{Int} \mid \mathsf{Nat} \mid \tau \times \tau \mid \tau \Rightarrow \tau \\
K & = & \mathsf{Int} \mid \mathsf{Nat} \mid \mathsf{Pair} \mid \mathsf{Fun} \\
v & = & i \mid \langle v, v \rangle \mid \lambda x.\, e \mid \lambda((x : \tau) : \tau).\, e \\
i & \in & \mathbb{Z} \\
e & = & \ldots \mid \mathsf{chk}\ K\ e
\end{array}
$$

$\boxed{\mathcal{D} : \tau \times v \longrightarrow e}$

$\mathcal{D}(\tau, v) = \mathcal{X}(\lfloor \tau \rfloor, v)$

$\boxed{\mathcal{S} : \tau \times v \longrightarrow e}$

$\mathcal{S}(\tau, v) = \mathcal{X}(\lfloor \tau \rfloor, v)$

$\boxed{\mathcal{X} : K \times v \longrightarrow e}$

$$
\begin{array}{ll}
\mathcal{X}(\mathsf{Fun}, v) = v & \\
\quad \text{if } v = \lambda x.\, e \text{ or } v = \lambda((x : \tau_d') : \tau_c').\, e & \\
\mathcal{X}(\mathsf{Pair}, v) = v & \\
\quad \text{if } v = \langle v_0, v_1 \rangle & \\
\mathcal{X}(\mathsf{Int}, i) & = i \\
\mathcal{X}(\mathsf{Nat}, i) & = i \\
\quad \text{if } i \in \mathbb{N} & \\
\mathcal{X}(K, v) & = \mathsf{BndryErr} \\
\quad \text{otherwise} &
\end{array}
$$

$\boxed{e \rhd_{\mathsf{S}} e}$

$$
\begin{array}{lll}
(\lambda x.\, e)\ v & \rhd_{\mathsf{S}} & e[x \leftarrow v] \\
(\lambda((x : \tau_d) : \tau_c).\, e)\ v & \rhd_{\mathsf{S}} & \mathsf{BndryErr} \\
\quad \text{if } \mathcal{X}(\lfloor \tau_d \rfloor, v) = \mathsf{BndryErr} & & \\
(\lambda((x : \tau_d) : \tau_c).\, e)\ v & \rhd_{\mathsf{S}} & \mathsf{chk}\ \lfloor \tau_c \rfloor\ (e[x \leftarrow \mathcal{X}(\lfloor \tau_d \rfloor, v)]) \\
\quad \text{if } \mathcal{X}(\lfloor \tau_d \rfloor, v) \neq \mathsf{BndryErr} & & \\
\mathsf{chk}\ K\ v & \rhd_{\mathsf{S}} & \mathcal{X}(K, v)
\end{array}
$$

$\boxed{e \rhd_{\mathsf{D}} e}$

$$
\begin{array}{lll}
(\lambda x.\, e)\ v & \rhd_{\mathsf{D}} & e[x \leftarrow v] \\
(\lambda((x : \tau_d) : \tau_c).\, e)\ v & \rhd_{\mathsf{D}} & \mathsf{BndryErr} \\
\quad \text{if } \mathcal{X}(\lfloor \tau_d \rfloor, v) = \mathsf{BndryErr} & & \\
(\lambda((x : \tau_d) : \tau_c).\, e)\ v & \rhd_{\mathsf{D}} & \mathsf{chk}\ \lfloor \tau_c \rfloor\ (e[x \leftarrow \mathcal{X}(\lfloor \tau_d \rfloor, v)]) \\
\quad \text{if } \mathcal{X}(\lfloor \tau_d \rfloor, v) \neq \mathsf{BndryErr} & & \\
\mathsf{chk}\ K\ v & \rhd_{\mathsf{D}} & \mathcal{X}(K, v)
\end{array}
$$

Fig. 10. Pyret boundary functions and semantics for a restricted grammar of types.

D.5 SafeTS

SafeTS is a core model of Safe TypeScript, which is a sound type system for JavaScript (as opposed to TypeScript).

Figure 11 demonstrates SafeTS on three types: a type for numbers (number), a type for an object with two fields ($\{\mathsf{fst}:\tau, \mathsf{snd}:\tau\}$), and a type for an object with one method ($\{\mathsf{call}(\tau) : \tau\}$). The latter types are intended to represent tuples and anonymous functions.

Every value in a SafeTS program has an intrinsic type; there is no notion of a value that is defined in dynamically-typed code and imported to statically typed code. A typed value may, however, be used in a context that expects values with a different type by means of a type cast. The different type may contain new fields, but otherwise must be a supertype of the value's intrinsic type.

The $\mathcal{S}$ boundary function illustrates the run-time checks that SafeTS performs for our number, pair, and function types. For type number, SafeTS checks that the value is a number. For a pair type, SafeTS checks that the value is a pair of compatible type and recursively transports the components. For a function type, SafeTS checks that the value is a function of compatible type.

The $\mathcal{D}$ function is undefined because the SafeTS model does not define interactions with a model of JavaScript.

Lemma 0.4 : *SafeTS canonical forms*

- If $\vdash v : \{\mathsf{fst}:\tau_0, \mathsf{snd}:\tau_1\}$ then $v = \{\mathsf{fst}:_{\tau_0'} v_0, \mathsf{snd}:_{\tau_1'} v_1\}$ and $\tau_0' \leqslant: \tau_0$ and $\tau_1' \leqslant: \tau_1$
- If $\vdash v : \{\mathsf{call}(\tau_d) : \tau_c\}$ then $v = \{\mathsf{call}(x : \tau_d') : \tau_c'\{\mathsf{return}\, e\}\}$ and $\{\mathsf{call}(\tau_d') : \tau_c'\} \leqslant: \{\mathsf{call}(\tau_d) : \tau_c\}$
- If $\vdash v : \mathsf{number}$ then $v = i$

Remark: if a SafeTS cast adds new fields to a value, the fields are recorded in an external "tag heap" of run-time type information. Figure 11 does not model the tag heap because it is not relevant to the types in the figure.

$\boxed{\text{SafeTS}}$ (sketch)

$$\tau \;=\; \mathsf{number} \mid \{\mathsf{fst}:\tau, \mathsf{snd}:\tau\} \mid \{\mathsf{call}(\tau) : \tau\} \mid \mathsf{any}$$
$$v \;=\; i \mid \{\mathsf{fst}:_\tau v, \mathsf{snd}:_\tau v\} \mid \{\mathsf{call}(x : \tau) : \tau\{\mathsf{return}\, e\}\}$$

$\boxed{\mathcal{D} : \tau \times v \longrightarrow e}$ (undefined, all values have a static type)

$\boxed{\mathcal{S} : \tau \times v \longrightarrow e}$

$$\begin{array}{ll}
\mathcal{S}(\{\mathsf{call}(\tau_d) : \tau_c\}, v) & = v \\
\quad \text{if } v = \{\mathsf{call}(x : \tau_d') : \tau_c'\{\mathsf{return}\, e\}\} & \\
\quad \text{and } \{\mathsf{call}(\tau_d') : \tau_c'\} \leqslant: \{\mathsf{call}(\tau_d) : \tau_c\} & \\
\mathcal{S}(\{\mathsf{fst}:\tau_0, \mathsf{snd}:\tau_1\}, v) & = \{\mathsf{fst}:_{\tau_0'} \mathcal{S}(\tau_0, v_0), \mathsf{snd}:_{\tau_1'} \mathcal{S}(\tau_1, v_1)\} \\
\quad \text{if } v = \{\mathsf{fst}:_{\tau_0'} v_0, \mathsf{snd}:_{\tau_1'} v_1\} & \\
\quad \text{and } \tau_0' \leqslant: \tau_0 \text{ and } \tau_1' \leqslant: \tau_1 & \\
\mathcal{S}(\mathsf{number}, i) & = i \\
\mathcal{S}(\tau, v) & = \mathsf{BndryErr} \\
\quad \text{otherwise} &
\end{array}$$

Fig. 11. SafeTS. The $\mathcal{D}$ function is undefined for all inputs.

D.6 Nom

Nom is a nominal object oriented language. Types include a top type ($\top$), class names (C), and a dynamic type (**dynamic**). Values are instances of classes. Each value has an intrinsic type; namely, the name of its class.

The $\mathcal{S}$ function checks that the intrinsic type of a value is compatible with a given type annotation. The $\mathcal{D}$ function is undefined for all inputs because there way to define or import an untyped value.

Lemma 0.5 : *Nom canonical forms*

- If $\vdash v : C$ then $v = C'(v', \ldots)$ and $C' \leqslant: C$

$\boxed{\text{Nom}}$ (sketch)

$$\tau = \top \mid C \mid \textbf{dynamic}$$
$$v = C(v, \ldots)$$

$\boxed{\mathcal{D} : \tau \times v \longrightarrow e}$ (undefined, all values have a static type)

$\boxed{\mathcal{S} : \tau \times v \longrightarrow e}$

$$\begin{array}{ll}
\mathcal{S}(\top, v) & = v \\
\mathcal{S}(C, C'(v, \ldots)) & = C'(v, \ldots) \\
\quad \text{if } C' \leqslant: C & \\
\mathcal{S}(\textbf{dynamic}, v) & = v \\
\mathcal{S}(\tau, v) & = \text{BndryErr} \\
\quad \text{otherwise} &
\end{array}$$

Fig. 12. Nom. The $\mathcal{D}$ function is undefined for all inputs.

E Models

This section contains full definitions of the languages and full proofs of our claims about each language.

Aside from the common notions in section 5.1, the definition and proofs of each model are independent and self-contained.

E.1 Preliminaries

Definition 1.0 : $\rightarrow^*$ *divergence*

Given a reduction relation $\rightarrow^*$, an expression e diverges if for all e' such that $e \rightarrow^* e'$ there exists an e'' such that $e' \rightarrow e''$.

Convention 1.1 : *variable convention*

All λ-bound variables in an expression are distinct from one another, and from any free variables in the expression.

Assumption 1.2 : $\vdash$ *permutation*

For all typing judgments and properties $\vdash$:

- If $x, x', \Gamma \vdash e$ then $x', x, \Gamma \vdash e$
- If $(x:\tau), (x':\tau'), \Gamma \vdash e$ then $(x':\tau'), (x:\tau), \Gamma \vdash e$

Definition 1.3 : $\vdash$ *boundary-free*

An expression e is *boundary free* if e does not contain a subterm of the form $(\mathsf{dyn}\ \tau'\ e')$, nor a subterm of the form $(\mathsf{stat}\ \tau'\ e')$.

Notes:

- The upcoming models use a common surface syntax and typing system, but to keep each model self-contained we reprint this system in each definition.
- The proofs are written in a structured style, typically as a list of basic steps where each step is justified by an assumption, a lemma, or a previous step. Lemma names are *italicized* and hyperlinked to the actual lemma.

E.2 (H) Higher-Order Embedding

E.2.1 Higher-Order Definitions

$\boxed{\text{Language H}}$

$$
\begin{array}{lcl}
e & = & x \mid v \mid \langle e, e\rangle \mid e\ e \mid op^1\ e \mid op^2\ e\ e \mid \\
 & & \mathsf{dyn}\ \tau\ e \mid \mathsf{stat}\ \tau\ e \mid \mathsf{Err} \\
v & = & i \mid \langle v, v\rangle \mid \lambda x.\, e \mid \lambda(x\!:\!\tau).\, e \mid \\
 & & \mathsf{mon}\,(\tau \Rightarrow \tau)\, v \\
\tau & = & \mathsf{Nat} \mid \mathsf{Int} \mid \tau \times \tau \mid \tau \Rightarrow \tau \\
\Gamma & = & \cdot \mid x, \Gamma \mid (x\!:\!\tau), \Gamma \\
\mathsf{Err} & = & \mathsf{BndryErr} \mid \mathsf{TagErr} \\
r & = & v \mid \mathsf{Err} \\
E^{\bullet} & = & [\,] \mid E^{\bullet}\ e \mid v\ E^{\bullet} \mid \langle E^{\bullet}, e\rangle \mid \langle v, E^{\bullet}\rangle \mid \\
 & & op^1\ E^{\bullet} \mid op^2\ E^{\bullet}\ e \mid op^2\ v\ E^{\bullet} \\
E & = & E^{\bullet} \mid E\ e \mid v\ E \mid \langle E, e\rangle \mid \langle v, E\rangle \mid op^1\ E \mid \\
 & & op^2\ E\ e \mid op^2\ v\ E \mid \mathsf{dyn}\ \tau\ E \mid \mathsf{stat}\ \tau\ E
\end{array}
$$

$\boxed{\Delta : op^1 \times \tau \longrightarrow \tau}$

$$
\begin{array}{l}
\Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0 \\
\Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1
\end{array}
$$

$\boxed{\Delta : op^2 \times \tau \times \tau \longrightarrow \tau}$

$$
\begin{array}{l}
\Delta(op^2, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat} \\
\Delta(op^2, \mathsf{Int}, \mathsf{Int}) = \mathsf{Int}
\end{array}
$$

$\boxed{\tau \leqslant: \tau}$

$$
\frac{}{\mathsf{Nat} \leqslant: \mathsf{Int}} \qquad
\frac{\tau_d' \leqslant: \tau_d \quad \tau_c \leqslant: \tau_c'}{\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'} \qquad
\frac{\tau_0 \leqslant: \tau_0' \quad \tau_1 \leqslant: \tau_1'}{\tau_0 \times \tau_1 \leqslant: \tau_0' \times \tau_1'}
$$

$$
\frac{}{\tau \leqslant: \tau} \qquad
\frac{\tau \leqslant: \tau' \quad \tau' \leqslant: \tau''}{\tau \leqslant: \tau''}
$$

$\boxed{\Gamma \vdash e}$

$$
\frac{x \in \Gamma}{\Gamma \vdash x} \qquad
\frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x.\, e} \qquad
\frac{}{\Gamma \vdash i} \qquad
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1\rangle}
$$

$$
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0\ e_1} \qquad
\frac{\Gamma \vdash e}{\Gamma \vdash op^1\ e} \qquad
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2\ e_0\ e_1} \qquad
\frac{}{\Gamma \vdash \mathsf{Err}}
$$

$$
\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat}\ \tau\ e}
$$

$\boxed{\Gamma \vdash e : \tau}$

$$
\frac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash x : \tau} \qquad
\frac{(x\!:\!\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c} \qquad
\frac{i \in \mathbb{N}}{\Gamma \vdash i : \mathsf{Nat}}
$$

$$
\frac{}{\Gamma \vdash i : \mathsf{Int}} \qquad
\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1\rangle : \tau_0 \times \tau_1} \qquad
\frac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash e_1 : \tau_d}{\Gamma \vdash e_0\ e_1 : \tau_c}
$$

$$
\frac{\Gamma \vdash e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash op^1\ e_0 : \tau} \qquad
\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash op^2\ e_0\ e_1 : \tau} \qquad
\frac{\Gamma \vdash e : \tau' \quad \tau' \leqslant: \tau}{\Gamma \vdash e : \tau} \qquad
\frac{}{\Gamma \vdash \mathsf{Err} : \tau}
$$

$$
\frac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau}
$$

$\boxed{\Gamma \vdash_{\mathsf{H}} e}$

$$
\frac{x \in \Gamma}{\Gamma \vdash_{\mathsf{H}} x} \qquad
\frac{x, \Gamma \vdash_{\mathsf{H}} e}{\Gamma \vdash_{\mathsf{H}} \lambda x.\, e} \qquad
\frac{}{\Gamma \vdash_{\mathsf{H}} i} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} e_0 \quad \Gamma \vdash_{\mathsf{H}} e_1}{\Gamma \vdash_{\mathsf{H}} \langle e_0, e_1\rangle}
$$

$$
\frac{\Gamma \vdash_{\mathsf{H}} e_0 \quad \Gamma \vdash_{\mathsf{H}} e_1}{\Gamma \vdash_{\mathsf{H}} e_0\ e_1} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} e}{\Gamma \vdash_{\mathsf{H}} op^1\ e} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} e_0 \quad \Gamma \vdash_{\mathsf{H}} e_1}{\Gamma \vdash_{\mathsf{H}} op^2\ e_0\ e_1} \qquad
\frac{}{\Gamma \vdash_{\mathsf{H}} \mathsf{Err}}
$$

$$
\frac{\Gamma \vdash_{\mathsf{H}} e : \tau}{\Gamma \vdash_{\mathsf{H}} \mathsf{stat}\ \tau\ e} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} v : \tau_d \Rightarrow \tau_c}{\Gamma \vdash_{\mathsf{H}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v}
$$

$\boxed{\Gamma \vdash_{\mathsf{H}} e : \tau}$

$$
\frac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash_{\mathsf{H}} x : \tau} \qquad
\frac{(x\!:\!\tau_d), \Gamma \vdash_{\mathsf{H}} e : \tau_c}{\Gamma \vdash_{\mathsf{H}} \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c} \qquad
\frac{i \in \mathbb{N}}{\Gamma \vdash_{\mathsf{H}} i : \mathsf{Nat}}
$$

$$
\frac{}{\Gamma \vdash_{\mathsf{H}} i : \mathsf{Int}} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} e_0 : \tau_0 \quad \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1}{\Gamma \vdash_{\mathsf{H}} \langle e_0, e_1\rangle : \tau_0 \times \tau_1} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash_{\mathsf{H}} e_1 : \tau_d}{\Gamma \vdash_{\mathsf{H}} e_0\ e_1 : \tau_c}
$$

$$
\frac{\Gamma \vdash_{\mathsf{H}} e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash_{\mathsf{H}} op^1\ e_0 : \tau} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} e_0 : \tau_0 \quad \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash_{\mathsf{H}} op^2\ e_0\ e_1 : \tau} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} e : \tau' \quad \tau' <: \tau}{\Gamma \vdash_{\mathsf{H}} e : \tau}
$$

$$
\frac{}{\Gamma \vdash_{\mathsf{H}} \mathsf{Err} : \tau} \qquad
\frac{\Gamma \vdash_{\mathsf{H}} e}{\Gamma \vdash_{\mathsf{H}} \mathsf{dyn}\ \tau\ e : \tau}
$$

$$
\frac{\Gamma \vdash_{\mathsf{H}} v}{\Gamma \vdash_{\mathsf{H}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v : (\tau_d \Rightarrow \tau_c)}
$$

$\boxed{\delta(op^1, v) = e}$

$$
\begin{array}{l}
\delta(\mathsf{fst}, \langle v_0, v_1\rangle) = v_0 \\
\delta(\mathsf{snd}, \langle v_0, v_1\rangle) = v_1
\end{array}
$$

$\boxed{\delta(op^2, v, v) = e}$

$$
\begin{array}{ll}
\delta(\mathsf{sum}, i_0, i_1) & = i_0 + i_1 \\
\delta(\mathsf{quotient}, i_0, 0) & = \mathsf{BndryErr} \\
\delta(\mathsf{quotient}, i_0, i_1) & = \lfloor i_0 / i_1 \rfloor \\
\quad \text{if } i_1 \neq 0 &
\end{array}
$$

$\boxed{\mathcal{D}_{\mathsf{H}} : \tau \times v \longrightarrow e}$

$$
\begin{array}{ll}
\mathcal{D}_{\mathsf{H}}(\tau_d \Rightarrow \tau_c, v) & = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v \\
\quad \text{if } v = \lambda x.\, e \text{ or } v = \mathsf{mon}\, \tau' \, v' & \\
\mathcal{D}_{\mathsf{H}}(\tau_0 \times \tau_1, \langle v_0, v_1 \rangle) & = \langle \mathsf{dyn}\ \tau_0\ v_0, \mathsf{dyn}\ \tau_1\ v_1 \rangle \\
\mathcal{D}_{\mathsf{H}}(\mathsf{Int}, i) & = i \\
\mathcal{D}_{\mathsf{H}}(\mathsf{Nat}, i) & = i \\
\quad \text{if } i \in \mathbb{N} & \\
\mathcal{D}_{\mathsf{H}}(\tau, v) & = \mathsf{BndryErr} \\
\quad \text{otherwise} &
\end{array}
$$

$\boxed{\mathcal{S}_{\mathsf{H}} : \tau \times v \longrightarrow e}$

$$
\begin{array}{ll}
\mathcal{S}_{\mathsf{H}}(\tau_d \Rightarrow \tau_c, v) & = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v \\
\mathcal{S}_{\mathsf{H}}(\tau_0 \times \tau_1, \langle v_0, v_1 \rangle) & = \langle \mathsf{stat}\ \tau_0\ v_0, \mathsf{stat}\ \tau_1\ v_1 \rangle \\
\mathcal{S}_{\mathsf{H}}(\mathsf{Int}, v) & = v \\
\mathcal{S}_{\mathsf{H}}(\mathsf{Nat}, v) & = v
\end{array}
$$

$\boxed{e \rhd_{\mathsf{H\text{-}S}} e}$

$$
\begin{array}{lll}
\mathsf{dyn}\ \tau\ v & \rhd_{\mathsf{H\text{-}S}} & \mathcal{D}_{\mathsf{H}}(\tau, v) \\
(\mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v_f)\ v & \rhd_{\mathsf{H\text{-}S}} & \mathsf{dyn}\ \tau_c\ (v_f\ (\mathsf{stat}\ \tau_d\ v)) \\
(\lambda (x : \tau).\, e)\ v & \rhd_{\mathsf{H\text{-}S}} & e[x \leftarrow v] \\
op^1\ v & \rhd_{\mathsf{H\text{-}S}} & \delta(op^1, v) \\
op^2\ v_0\ v_1 & \rhd_{\mathsf{H\text{-}S}} & \delta(op^2, v_0, v_1)
\end{array}
$$

$\boxed{e \rhd_{\mathsf{H\text{-}D}} e}$

$$
\begin{array}{lll}
\mathsf{stat}\ \tau\ v & \rhd_{\mathsf{H\text{-}D}} & \mathcal{S}_{\mathsf{H}}(\tau, v) \\
v_0\ v_1 & \rhd_{\mathsf{H\text{-}D}} & \mathsf{TagErr} \\
\quad \text{if } v_0 \in \mathbb{Z} \text{ or } v_0 = \langle v, v' \rangle & & \\
(\mathsf{mon}\ \tau_d \Rightarrow \tau_c\ v_f)\ v & \rhd_{\mathsf{H\text{-}D}} & \mathsf{stat}\ \tau_c\ (v_f\ (\mathsf{dyn}\ \tau_d\ v)) \\
(\lambda x.\, e)\ v & \rhd_{\mathsf{H\text{-}D}} & e[x \leftarrow v] \\
op^1\ v & \rhd_{\mathsf{H\text{-}D}} & \mathsf{TagErr} \\
\quad \text{if } \delta(op^1, v) \text{ is undefined} & & \\
op^1\ v & \rhd_{\mathsf{H\text{-}D}} & \delta(op^1, v) \\
op^2\ v_0\ v_1 & \rhd_{\mathsf{H\text{-}D}} & \mathsf{TagErr} \\
\quad \text{if } \delta(op^2, v_0, v_1) \text{ is undefined} & & \\
op^2\ v_0\ v_1 & \rhd_{\mathsf{H\text{-}D}} & \delta(op^2, v_0, v_1)
\end{array}
$$

$\boxed{e \rightarrow_{\mathsf{H\text{-}S}} e}$

$$
\begin{array}{lll}
E^\bullet[e] & \rightarrow_{\mathsf{H\text{-}S}} & E^\bullet[e'] \\
\quad \text{if } e \rhd_{\mathsf{H\text{-}S}} e' & & \\
E[\mathsf{stat}\ \tau\ E^\bullet[e]] & \rightarrow_{\mathsf{H\text{-}S}} & E[\mathsf{stat}\ \tau\ E^\bullet[e']] \\
\quad \text{if } e \rhd_{\mathsf{H\text{-}S}} e' & & \\
E[\mathsf{dyn}\ \tau\ E^\bullet[e]] & \rightarrow_{\mathsf{H\text{-}S}} & E[\mathsf{dyn}\ \tau\ E^\bullet[e']] \\
\quad \text{if } e \rhd_{\mathsf{H\text{-}D}} e' & & \\
E[\mathsf{Err}] & \rightarrow_{\mathsf{H\text{-}S}} & \mathsf{Err}
\end{array}
$$

$\boxed{e \rightarrow_{\mathsf{H\text{-}D}} e}$

$$
\begin{array}{lll}
E^\bullet[e] & \rightarrow_{\mathsf{H\text{-}D}} & E^\bullet[e'] \\
\quad \text{if } e \rhd_{\mathsf{H\text{-}D}} e' & & \\
E[\mathsf{stat}\ \tau\ E^\bullet[e]] & \rightarrow_{\mathsf{H\text{-}D}} & E[\mathsf{stat}\ \tau\ E^\bullet[e']] \\
\quad \text{if } e \rhd_{\mathsf{H\text{-}S}} e' & & \\
E[\mathsf{dyn}\ \tau\ E^\bullet[e]] & \rightarrow_{\mathsf{H\text{-}D}} & E[\mathsf{dyn}\ \tau\ E^\bullet[e']] \\
\quad \text{if } e \rhd_{\mathsf{H\text{-}D}} e' & & \\
E[\mathsf{Err}] & \rightarrow_{\mathsf{H\text{-}D}} & \mathsf{Err}
\end{array}
$$

$\boxed{e \rightarrow^*_{\mathsf{H\text{-}S}} e}$ reflexive, transitive closure of $\rightarrow_{\mathsf{H\text{-}S}}$

$\boxed{e \rightarrow^*_{\mathsf{H\text{-}D}} e}$ reflexive, transitive closure of $\rightarrow_{\mathsf{H\text{-}D}}$

E.2.2 Higher-Order Theorems

Theorem 2.0 : *static* H*-soundness*

If $\vdash e : \tau$ then $\vdash_{\mathsf{H}} e : \tau$ and one of the following holds:

- $e \rightarrow^{*}_{\mathsf{H\text{-}S}} v$ and $\vdash_{\mathsf{H}} v : \tau$
- $e \rightarrow^{*}_{\mathsf{H\text{-}S}} E[\mathsf{dyn}\ \tau'\ E^{\bullet}[e']]$ and $e' \rhd_{\mathsf{H\text{-}D}} \mathsf{TagErr}$
- $e \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
- e diverges

Proof:

1. $\vdash_{\mathsf{H}} e : \tau$
   by *static subset*
2. QED by H *static progress* and H *static preservation*.

□

Theorem 2.1 : *dynamic* H*-soundness*

If $\vdash e$ then $\vdash_{\mathsf{H}} e$ and one of the following holds:

- $e \rightarrow^{*}_{\mathsf{H\text{-}D}} v$ and $\vdash_{\mathsf{H}} v$
- $e \rightarrow^{*}_{\mathsf{H\text{-}D}} E[e']$ and $e' \rhd_{\mathsf{H\text{-}D}} \mathsf{TagErr}$
- $e \rightarrow^{*}_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
- e diverges

Proof:

1. $\vdash_{\mathsf{H}} e$
   by *dynamic subset*
2. QED by H *dynamic progress* and H *dynamic preservation*.

□

Corollary 2.2 : H *static soundness*

If $\vdash e : \tau$ and e is boundary-free, then one of the following holds:

- $e \rightarrow^{*}_{\mathsf{H}} v$ and $\vdash_{\mathsf{H}} v : \tau$
- $e \rightarrow^{*}_{\mathsf{H}} \mathsf{BndryErr}$
- e diverges

Proof:

Consequence of the proof for *static* H*-soundness*

□

Corollary 2.3 : H *compilation*

If $\vdash e : \tau$

and $\mathcal{D}'_{\mathsf{H}}$ extends $\mathcal{D}_{\mathsf{H}}$ with a rule to monitor a typed function:

$\mathcal{D}'_{\mathsf{H}}(\tau_d \Rightarrow \tau_c, \lambda(x\!:\!\tau).\,e) = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda(x\!:\!\tau).\,e)$

and $\rhd_{\mathsf{H\text{-}D}}{}'$ extends $\rhd_{\mathsf{H\text{-}D}}$ with a rule to apply a typed function:

$(\lambda(x\!:\!\tau).\,e)\ v\ \ \rhd_{\mathsf{H\text{-}D}}{}'\ \ e[x \leftarrow v]$

and $e \rightarrow_{\mathsf{H\text{-}D}}{}'\ e$ is defined as:

$E[e] \quad \rightarrow_{\mathsf{H\text{-}D}}{}'\ E[e']$
  if $e \rhd_{\mathsf{H\text{-}D}}{}'\ e'$

$E[\mathsf{stat}\ \tau\ v] \ \rightarrow_{\mathsf{H\text{-}D}}{}'\ E[\mathcal{D}'_{\mathsf{H}}(\tau, v)]$

$E[\mathsf{dyn}\ \tau\ v] \ \rightarrow_{\mathsf{H\text{-}D}}{}'\ E[\mathcal{D}'_{\mathsf{H}}(\tau, v)]$

$E[\mathsf{Err}] \quad \rightarrow_{\mathsf{H\text{-}D}}{}'\ \mathsf{Err}$

and $\rightarrow^{*}_{\mathsf{H\text{-}D}}{}'$ is the reflexive transitive closure of $\rightarrow_{\mathsf{H\text{-}D}}{}'$

then one of the following holds:

- $e \rightarrow^{*}_{\mathsf{H\text{-}D}}{}'\ v$ and $\vdash_{\mathsf{H}} v : \tau$
- $e \rightarrow^{*}_{\mathsf{H\text{-}D}}{}'\ \mathsf{TagErr}$
- $e \rightarrow^{*}_{\mathsf{H\text{-}D}}{}'\ \mathsf{BndryErr}$
- e diverges

Proof:

By *static* H*-soundness* and the fact that $\mathcal{S}_{\mathsf{H}}$ and $\rhd_{\mathsf{H\text{-}S}}$ are subsets of $\mathcal{D}'_{\mathsf{H}}$ and $\rhd_{\mathsf{H\text{-}D}}{}'$, respectively.

□

E.2.3 Higher-Order Lemmas

Lemma 2.4 : $\mathcal{D}_{\mathsf{H}}$ *soundness*

If $\vdash_{\mathsf{H}} v$ then $\vdash_{\mathsf{H}} \mathcal{D}_{\mathsf{H}}(\tau, v) : \tau$

Proof:

CASE $\mathcal{D}_{\mathsf{H}}(\tau_d \Rightarrow \tau_c, v) = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v$:

1. $\vdash_{\mathsf{H}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v : \tau_d \Rightarrow \tau_c$
   by $\vdash_{\mathsf{H}} v$
2. QED

CASE $v = \langle v_0, v_1 \rangle$
$\wedge\ \mathcal{D}_{\mathsf{H}}(\tau_0 \times \tau_1, v) = \langle \mathsf{dyn}\ \tau_0\ v_0, \mathsf{dyn}\ \tau_1\ v_1 \rangle$:

1. $\vdash_{\mathsf{H}} v_0$
   $\wedge \vdash_{\mathsf{H}} v_1$
   by H *inversion*
2. $\vdash_{\mathsf{H}} \mathsf{dyn}\ \tau_0\ v_0 : \tau_0$
   $\wedge \vdash_{\mathsf{H}} \mathsf{dyn}\ \tau_1\ v_1 : \tau_1$
   by (1)
3. QED (2)

CASE $v = i$
$\wedge\ \mathcal{D}_{\mathsf{H}}(\mathsf{Int}, v) = v$:

1. QED

CASE $v \in \mathbb{N}$
$\wedge\ \mathcal{D}_{\mathsf{H}}(\mathsf{Nat}, v) = v$:

1. QED

CASE $\mathcal{D}_{\mathsf{H}}(\tau, v) = \mathsf{BndryErr}$:

1. QED

□

Lemma 2.5 : $\mathcal{S}_{\mathsf{H}}$ *soundness*

If $\vdash_{\mathsf{H}} v : \tau$ then $\vdash_{\mathsf{H}} \mathcal{S}_{\mathsf{H}}(\tau, v)$

Proof:

CASE $\vdash_{\mathsf{H}} v : \tau_d \Rightarrow \tau_c$
$\wedge\ \mathcal{S}_{\mathsf{H}}(\tau_d \Rightarrow \tau_c, v) = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v$:

1. QED

CASE $\vdash_{\mathsf{H}} v : \tau_0 \times \tau_1$
$\wedge\ \mathcal{S}_{\mathsf{H}}(\tau_0 \times \tau_1, v) = \langle \mathsf{stat}\ \tau_0\ v_0, \mathsf{stat}\ \tau_1\ v_1 \rangle$:

1. $v = \langle v_0, v_1 \rangle$
   by *canonical forms*
2. $\vdash_{\mathsf{H}} v_0 : \tau_0$
   $\wedge \vdash_{\mathsf{H}} v_1 : \tau_1$
   by H *inversion* (1)
3. $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau_0\ v_0 : \tau_0$
   by the induction hypothesis (2)
4. $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau_1\ v_1 : \tau_1$
   by the induction hypothesis (2)
5. QED

CASE $\vdash_{\mathsf{H}} v : \mathsf{Int}$
$\wedge\ \mathcal{S}_{\mathsf{H}}(\mathsf{Int}, v) = v$:

1. QED

CASE $\vdash_{\mathsf{H}} v : \mathsf{Nat}$
$\wedge\ \mathcal{S}_{\mathsf{H}}(\mathsf{Nat}, v) = v$:

1. QED

□

Lemma 2.6 : H *static subset*

If $\Gamma \vdash e : \tau$ then $\Gamma \vdash_{\mathsf{H}} e : \tau$.

Proof:

By structural induction on the derivation of $\Gamma \vdash e : \tau$.

CASE $\boxed{\dfrac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash x : \tau}}$:

1. $\Gamma \vdash_{\mathsf{H}} x : \tau$
   by $(x\!:\!\tau) \in \Gamma$
2. QED

CASE $\boxed{\dfrac{(x\!:\!\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c}}$:

1. $(x\!:\!\tau_d), \Gamma \vdash_{\mathsf{H}} e : \tau_c$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{H}} \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c$
3. QED

CASE $\boxed{\dfrac{i \in \mathbb{N}}{\Gamma \vdash i : \mathsf{Nat}}}$:

1. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash i : \mathsf{Int}}}$:

1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1}}$:

1. $\Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{H}} \langle e_0, e_1 \rangle : \tau_0 \times \tau_1$
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash e_1 : \tau_d}{\Gamma \vdash e_0\ e_1 : \tau_c}}$:

1. $\Gamma \vdash_{\mathsf{H}} e_0 : \tau_d \Rightarrow \tau_c$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1 : \tau_d$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{H}} e_0\ e_1 : \tau_c$
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash op^1\ e_0 : \tau}}$:

1. $\Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{H}} op^1\ e_0 : \tau$
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash op^2\ e_0\ e_1 : \tau}}$:

1. $\Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{H}} op^2\ e_0\ e_1 : \tau$
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau' \quad \tau' <: \tau}{\Gamma \vdash e : \tau}}$:

1. $\Gamma \vdash_{\mathsf{H}} e : \tau'$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{H}} e : \tau$
3. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash \mathsf{Err} : \tau}}$:

1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau}}$:

1. $\Gamma \vdash_{\mathsf{H}} e$
   by *dynamic subset*
2. $\Gamma \vdash_{\mathsf{H}} \mathsf{dyn}\ \tau\ e : \tau$
   by (1)
3. QED

□

Lemma 2.7 : H *dynamic subset*

If $\Gamma \vdash e$ then $\Gamma \vdash_{\mathsf{H}} e$.

Proof:

By structural induction on the derivation of $\Gamma \vdash e$.

CASE $\boxed{\dfrac{x \in \Gamma}{\Gamma \vdash x}}$:

1. $\Gamma \vdash_{\mathsf{H}} x$
   by $x \in \Gamma$
2. QED

CASE $\boxed{\dfrac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x.\, e}}$:

1. $x, \Gamma \vdash_{\mathsf{H}} e$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{H}} \lambda x.\, e$
   by (1)
3. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash i}}$:

1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}}$:

1. $\Gamma \vdash_{\mathsf{H}} e_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{H}} \langle e_0, e_1 \rangle$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0\ e_1}}$:

1. $\Gamma \vdash_{\mathsf{H}} e_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{H}} e_0\ e_1$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e}{\Gamma \vdash op^1\ e}}$:

1. $\Gamma \vdash_{\mathsf{H}} e$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{H}} op^1\ e$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2\ e_0\ e_1}}$:

1. $\Gamma \vdash_{\mathsf{H}} e_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{H}} op^2\ e_0\ e_1$
   by (1)
3. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash \mathsf{Err}}}$:

1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat}\ \tau\ e}}$:

1. $\Gamma \vdash_{\mathsf{H}} e : \tau$
   by *static subset*
2. $\Gamma \vdash_{\mathsf{H}} \mathsf{stat}\ \tau\ e$
   by (1)
3. QED

□

Lemma 2.8 : H *static progress*

If $\vdash_{\mathsf{H}} e : \tau$ then one of the following holds:

- e is a value
- $e \in \mathsf{Err}$
- $e \longrightarrow_{\mathsf{H\text{-}S}} e'$
- $e \longrightarrow_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
- $e = E[\mathsf{dyn}\ \tau'\ E^{\bullet}[e']]$ and $e' \rhd_{\mathsf{H\text{-}D}} \mathsf{TagErr}$

Proof:

By the *boundary factoring* lemma, there are seven possible cases.

CASE e is a value :

1. QED

CASE $e = E^{\bullet}[v_0\ v_1]$:

1. $\vdash_{\mathsf{H}} v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\mathsf{H}} v_0 : \tau_d \Rightarrow \tau_c$
   $\wedge \vdash_{\mathsf{H}} v_1 : \tau_d$
   by H *inversion*

3. $v_0 = \lambda(x\!:\!\tau_d').\, e'$
$\vee\ v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\, v_f$
by *canonical forms*
4. **IF** $v_0 = \lambda(x\!:\!\tau_d').\, e'$:
a. $e \rightarrow_{\mathsf{H\text{-}S}} E^{\bullet}[e'[x \leftarrow v_1]]$
by $v_0\ v_1 \rhd_{\mathsf{H\text{-}S}} e'[x \leftarrow v_1]$
b. QED
ELSE $v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\, v_f$:
a. $e \rightarrow_{\mathsf{H\text{-}S}} E^{\bullet}[\mathsf{dyn}\ \tau_c'\ (v_f\ (\mathsf{stat}\ \tau_d'\ v_1))]$
by $v_0\ v_1 \rhd_{\mathsf{H\text{-}S}} \mathsf{dyn}\ \tau_c'\ (v_f\ (\mathsf{stat}\ \tau_d'\ v_1))$
b. QED

CASE $e = E^{\bullet}[op^1\ v]$:
1. $\vdash_{\mathsf{H}} op^1\ v : \tau'$
by *static hole typing*
2. $\vdash_{\mathsf{H}} v : \tau_0 \times \tau_1$
by H *inversion*
3. $v = \langle v_0, v_1 \rangle$
by *canonical forms*
4. **IF** $op^1 = \mathsf{fst}$:
a. $\delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0$
b. $e \rightarrow_{\mathsf{H\text{-}S}} E^{\bullet}[v_0]$
by $op^1\ v \rhd_{\mathsf{H\text{-}S}} v_0$
c. QED
ELSE $op^1 = \mathsf{snd}$:
a. $\delta(\mathsf{snd}, \langle v_0, v_1 \rangle) = v_1$
b. $e \rightarrow_{\mathsf{H\text{-}S}} E^{\bullet}[v_1]$
by $op^1\ v \rhd_{\mathsf{H\text{-}S}} v_1$
c. QED

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:
1. $\vdash_{\mathsf{H}} op^2\ v_0\ v_1 : \tau'$
by *static hole typing*
2. $\vdash_{\mathsf{H}} v_0 : \tau_0$
$\wedge \vdash_{\mathsf{H}} v_1 : \tau_1$
$\wedge\ \Delta(op^2, \tau_0, \tau_1) = \tau''$
by H *inversion*
3. $\delta(op^2, v_0, v_1) = e'$
by Δ *type soundness* (2)
4. $op^2\ v_0\ v_1 \rhd_{\mathsf{H\text{-}S}} e'$
by (3)
5. QED by $e \rightarrow_{\mathsf{H\text{-}S}} E^{\bullet}[e']$

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :
1. e' is a value
$\vee\ e' \in \mathsf{Err}$
$\vee\ e' \rightarrow_{\mathsf{H\text{-}D}} e''$
$\vee\ e' \rightarrow_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
$\vee\ e' = E'[e'']$ and $e'' \rhd_{\mathsf{H\text{-}D}} \mathsf{TagErr}$
by H *dynamic progress*
2. **IF** e' is a value :
a. QED $e \rightarrow_{\mathsf{H\text{-}S}} E[\mathcal{D}_{\mathsf{H}}(\tau', e')]$
IF $e' \in \mathsf{Err}$:
a. QED $e \rightarrow_{\mathsf{H\text{-}S}} e'$
IF $e' \rightarrow_{\mathsf{H\text{-}D}} e''$:
a. QED $e \rightarrow_{\mathsf{H\text{-}S}} E[\mathsf{dyn}\ \tau'\ e'']$
IF $e' \rightarrow_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$:
a. QED $e \rightarrow_{\mathsf{H\text{-}S}} E[\mathsf{dyn}\ \tau'\ \mathsf{BndryErr}]$
ELSE $e' = E'[e'']$ and $e'' \rhd_{\mathsf{H\text{-}D}} \mathsf{TagErr}$:
a. $E' \in E^{\bullet}$
by e' is boundary-free
b. QED

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :
1. e' is a value
$\vee\ e' \in \mathsf{Err}$
$\vee\ e' \rightarrow_{\mathsf{H\text{-}S}} e''$
$\vee\ e' \rightarrow_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
$\vee\ e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\mathsf{H\text{-}D}} \mathsf{TagErr}$
by H *static progress*
2. **IF** e' is a value :
a. QED $e \rightarrow_{\mathsf{H\text{-}S}} E[\mathcal{S}_{\mathsf{H}}(\tau', e')]$
IF $e' \in \mathsf{Err}$:
a. QED $e \rightarrow_{\mathsf{H\text{-}S}} e'$
IF $e' \rightarrow_{\mathsf{H\text{-}S}} e''$:
a. QED $e \rightarrow_{\mathsf{H\text{-}S}} E[\mathsf{stat}\ \tau'\ e'']$
IF $e' \rightarrow_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$:
a. QED $e \rightarrow_{\mathsf{H\text{-}S}} E[\mathsf{stat}\ \tau'\ \mathsf{BndryErr}]$
ELSE $e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\mathsf{H\text{-}D}} \mathsf{TagErr}$
:
a. Contradiction by e' is boundary-free

CASE $e = E[\mathsf{Err}]$:
1. QED $e \rightarrow_{\mathsf{H\text{-}S}} \mathsf{Err}$

□

Lemma 2.9 : H *dynamic progress*
If $\vdash_{\mathsf{H}} e$ then one of the following holds:
- e is a value
- $e \in \mathsf{Err}$
- $e \rightarrow_{\mathsf{H\text{-}D}} e'$
- $e \rightarrow_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
- $e = E[e']$ and $e' \rhd_{\mathsf{H\text{-}D}} \mathsf{TagErr}$

Proof:
By the *boundary factoring* lemma, there are seven cases.

CASE e is a value :
1. QED

CASE $e = E^{\bullet}[v_0\ v_1]$:
IF $v_0 = \lambda x.\, e'$:
1. $e \rightarrow_{\mathsf{H\text{-}D}} E^{\bullet}[e'[x \leftarrow v_1]]$
by $v_0\ v_1 \rhd_{\mathsf{H\text{-}D}} e'[x \leftarrow v_1]$
2. QED
IF $v_0 = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v_f$:
1. $e \rightarrow_{\mathsf{H\text{-}D}} E^{\bullet}[\mathsf{stat}\ \tau_c\ (v_f\ (\mathsf{dyn}\ \tau_d\ v_1))]$
by $v_0\ v_1 \rhd_{\mathsf{H\text{-}D}} \mathsf{stat}\ \tau_c\ (v_f\ (\mathsf{dyn}\ \tau_d\ v_1))$
2. QED
ELSE $v_0 = i$
$\vee\ v_0 = \langle v, v' \rangle$:
1. $e \rightarrow_{\mathsf{H\text{-}D}} \mathsf{TagErr}$
by $(v_0\ v_1) \rhd_{\mathsf{H\text{-}D}} \mathsf{TagErr}$
2. QED

CASE $e = E^{\bullet}[op^1\ v]$:
IF $\delta(op^1, v) = e'$:

1. $(op^1\ v) \rhd_{\text{H-D}} e'$
2. QED

ELSE $\delta(op^1, v)$ is undefined :

1. $e \rightarrow_{\text{H-D}} \mathsf{TagErr}$
   by $(op^1\ v) \rhd_{\text{H-D}} \mathsf{TagErr}$
2. QED

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:

IF $\delta(op^2, v_0, v_1) = e''$:

1. $op^2\ v_0\ v_1 \rhd_{\text{H-D}} e''$
2. QED

ELSE $\delta(op^2, v_0, v_1)$ is undefined :

1. $e \rightarrow_{\text{H-D}} \mathsf{TagErr}$
   by $op^2\ v_0\ v_1 \rhd_{\text{H-D}} \mathsf{TagErr}$
2. QED

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :

1. e' is a value
   $\vee\ e' \in \mathsf{Err}$
   $\vee\ e' \rightarrow_{\text{H-D}} e''$
   $\vee\ e' \rightarrow_{\text{H-D}} \mathsf{BndryErr}$
   $\vee\ e' = E[e'']$ and $e'' \rhd_{\text{H-D}} \mathsf{TagErr}$
   by H *dynamic progress*
2. IF e' is a value :
   a. QED $e \rightarrow_{\text{H-D}} E[\mathcal{D}_{\text{H}}(\tau', e')]$
   IF $e' \in \mathsf{Err}$:
   a. QED $e \rightarrow_{\text{H-D}} e'$
   IF $e' \rightarrow_{\text{H-D}} e''$:
   a. QED $e \rightarrow_{\text{H-S}} E[\mathsf{dyn}\ \tau'\ e'']$
   IF $e' \rightarrow_{\text{H-D}} \mathsf{BndryErr}$:
   a. QED $e \rightarrow_{\text{H-D}} E[\mathsf{dyn}\ \tau'\ \mathsf{BndryErr}]$
   ELSE $e' = E[e'']$ and $e'' \rhd_{\text{H-D}} \mathsf{TagErr}$:
   a. $E \in E^{\bullet}$
   by e' is boundary-free
   b. QED

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :

1. e' is a value
   $\vee\ e' \in \mathsf{Err}$
   $\vee\ e' \rightarrow_{\text{H-S}} e''$
   $\vee\ e' \rightarrow_{\text{H-S}} \mathsf{BndryErr}$
   $\vee\ e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{H-D}} \mathsf{TagErr}$
   by H *static progress*
2. IF e' is a value :
   a. QED $e \rightarrow_{\text{H-S}} E[\mathcal{S}_{\text{H}}(\tau', e')]$
   IF $e' \in \mathsf{Err}$:
   a. QED $e \rightarrow_{\text{H-S}} e'$
   IF $e' \rightarrow_{\text{H-S}} e''$:
   a. QED $e \rightarrow_{\text{H-S}} E[\mathsf{stat}\ \tau'\ e'']$
   IF $e' \rightarrow_{\text{H-S}} \mathsf{BndryErr}$:
   a. QED $e \rightarrow_{\text{H-S}} E[\mathsf{stat}\ \tau'\ \mathsf{BndryErr}]$
   ELSE $e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{H-D}} \mathsf{TagErr}$
   :
   a. Contradiction by e' is boundary-free

CASE $e = E[\mathsf{Err}]$:

1. QED $e \rightarrow_{\text{H-D}} \mathsf{Err}$

□

Lemma 2.10 : H *static preservation*

If $\vdash_{\text{H}} e : \tau$ and $e \rightarrow_{\text{H-S}} e'$ then $\vdash_{\text{H}} e' : \tau$

Proof:

By the *boundary factoring* lemma there are seven cases.

CASE e is a value :

1. Contradiction by $e \rightarrow_{\text{H-S}} e'$

CASE $e = E^{\bullet}[v_0\ v_1]$:

IF $v_0 = \lambda(x\!:\!\tau_x).\ e'$
$\wedge\ e \rightarrow_{\text{H-S}} E^{\bullet}[e'[x \leftarrow v_1]]$:

1. $\vdash_{\text{H}} v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\text{H}} v_0 : \tau_d \Rightarrow \tau_c$
   $\wedge \vdash_{\text{H}} v_1 : \tau_d$
   $\wedge\ \tau_c \leqslant: \tau'$
   by H *inversion*
3. $\tau_d \leqslant: \tau_x$
   by *canonical forms* (2)
4. $(x\!:\!\tau_x) \vdash_{\text{H}} e' : \tau_c$
   by H *inversion* (2)
5. $\vdash_{\text{H}} v_1 : \tau_x$
   by (2, 3)
6. $\vdash_{\text{H}} e'[x \leftarrow v_1] : \tau_c$
   by *substitution* (4, 5)
7. $\vdash_{\text{H}} e'[x \leftarrow v_1] : \tau'$
   by (2, 6)
8. QED by *hole substitution* (7)

ELSE $v_0 = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v_f$
$\wedge\ e \rightarrow_{\text{H-S}} E^{\bullet}[\mathsf{dyn}\ \tau_c\ (v_f\ (\mathsf{stat}\ \tau_d\ v_1))]$:

1. $\vdash_{\text{H}} v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\text{H}} v_0 : \tau_d' \Rightarrow \tau_c'$
   $\wedge \vdash_{\text{H}} v_1 : \tau_d'$
   $\wedge\ \tau_c' \leqslant: \tau'$
   by H *inversion*
3. $\vdash_{\text{H}} v_f$
   by H *inversion* (2)
4. $\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'$
   by *canonical forms* (2)
5. $\tau_d' \leqslant: \tau_d$
   $\wedge\ \tau_c \leqslant: \tau_c'$
   by (4)
6. $\vdash_{\text{H}} v_1 : \tau_d$
   by (2, 5)
7. $\vdash_{\text{H}} \mathsf{stat}\ \tau_d\ v_1$
   by (6)
8. $\vdash_{\text{H}} v_f\ (\mathsf{stat}\ \tau_d\ v_1)$
   by (3, 7)
9. $\vdash_{\text{H}} \mathsf{dyn}\ \tau_c\ v_f\ (\mathsf{stat}\ \tau_d\ v_1) : \tau_c$
   by (8)
10. $\vdash_{\text{H}} \mathsf{dyn}\ \tau_c\ v_f\ (\mathsf{stat}\ \tau_d\ v_1) : \tau'$
    by (2, 5, 9)
11. QED by *hole substitution* (10)

CASE $e = E^{\bullet}[op^1\ v]$:

IF $v = \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{fst}$
$\wedge\ e \rightarrow_{\text{H-S}} E^{\bullet}[v_0]$:
1. $\vdash_{\text{H}} \mathsf{fst}\ \langle v_0, v_1 \rangle : \tau'$
by *static hole typing*
2. $\vdash_{\text{H}} \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$
$\wedge\ \tau_0 \leqslant: \tau'$
by H *inversion* (1)
3. $\vdash_{\text{H}} v_0 : \tau_0$
by H *inversion* (2)
4. $\vdash_{\text{H}} v_0 : \tau'$
by (2, 3)
5. QED by *hole substitution* (4)

ELSE $v = \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{snd}$
$\wedge\ e \rightarrow_{\text{H-S}} E^{\bullet}[v_1]$:
1. $\vdash_{\text{H}} \mathsf{snd}\ \langle v_0, v_1 \rangle : \tau'$
by *static hole typing*
2. $\vdash_{\text{H}} \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$
$\wedge\ \tau_1 \leqslant: \tau'$
by H *inversion* (1)
3. $\vdash_{\text{H}} v_1 : \tau_1$
by H *inversion* (2)
4. $\vdash_{\text{H}} v_1 : \tau'$
by (2, 3)
5. QED by *hole substitution* (4)

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:
1. $e \rightarrow_{\text{H-S}} E^{\bullet}[\delta(op^2, v_0, v_1)]$
by $e \rightarrow_{\text{H-S}} e'$
2. $\vdash_{\text{H}} op^2\ v_0\ v_1 : \tau'$
by *static hole typing*
3. $\vdash_{\text{H}} v_0 : \tau_0$
$\wedge \vdash_{\text{H}} v_1 : \tau_1$
$\wedge\ \Delta(op^2, \tau_0, \tau_1) = \tau''$
$\wedge\ \tau'' \leqslant: \tau'$
by H *inversion* (1)
4. $\vdash_{\text{H}} \delta(op^2, v_0, v_1) : \tau''$
by Δ *type soundness* (2)
5. $\vdash_{\text{H}} \delta(op^2, v_0, v_1) : \tau'$
by (2, 3)
6. QED by *hole substitution* (4)

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :

IF e' is a value :
1. $e \rightarrow_{\text{H-S}} E[\mathcal{D}_{\text{H}}(\tau', e')]$
2. $\vdash_{\text{H}} \mathsf{dyn}\ \tau'\ e' : \tau'$
by *boundary hole typing*
3. $\vdash_{\text{H}} e'$
by H *inversion* (2)
4. $\vdash_{\text{H}} \mathcal{D}_{\text{H}}(\tau', e') : \tau'$
by $\mathcal{D}_{\text{H}}$ *soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\text{H-D}} e''$:
1. $e \rightarrow_{\text{H-S}} E[\mathsf{dyn}\ \tau'\ e'']$
2. $\vdash_{\text{H}} \mathsf{dyn}\ \tau'\ e' : \tau'$
by *boundary hole typing*
3. $\vdash_{\text{H}} e'$
by H *inversion* (2)
4. $\vdash_{\text{H}} e''$
by H *dynamic preservation* (3)
5. $\vdash_{\text{H}} \mathsf{dyn}\ \tau'\ e'' : \tau'$
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :

IF e' is a value :
1. $e \rightarrow_{\text{H-S}} E[\mathcal{S}_{\text{H}}(\tau', e')]$
2. $\vdash_{\text{H}} \mathsf{stat}\ \tau'\ e'$
by *boundary hole typing*
3. $\vdash_{\text{H}} e' : \tau'$
by H *inversion* (2)
4. $\vdash_{\text{H}} \mathcal{S}_{\text{H}}(\tau', e')$
by $\mathcal{S}_{\text{H}}$ *soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\text{H-S}} e''$:
1. $e \rightarrow_{\text{H-S}} E[\mathsf{stat}\ \tau'\ e'']$
2. $\vdash_{\text{H}} \mathsf{stat}\ \tau'\ e'$
by *boundary hole typing*
3. $\vdash_{\text{H}} e' : \tau'$
by H *inversion* (2)
4. $\vdash_{\text{H}} e'' : \tau'$
by H *static preservation* (3)
5. $\vdash_{\text{H}} \mathsf{stat}\ \tau'\ e''$
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{Err}]$:
1. $e \rightarrow_{\text{H-S}} \mathsf{Err}$
2. QED by $\vdash_{\text{H}} \mathsf{Err} : \tau$

□

Lemma 2.11 : H *dynamic preservation*
If $\vdash_{\text{H}} e$ and $e \rightarrow_{\text{H-D}} e'$ then $\vdash_{\text{H}} e'$
Proof:
By the *boundary factoring* lemma, there are seven cases.

CASE e is a value :
1. Contradiction by $e \rightarrow_{\text{H-D}} e'$

CASE $e = E^{\bullet}[v_0\ v_1]$:

IF $v_0 = \lambda x.\, e'$
$\wedge\ e \rightarrow_{\text{H-D}} E^{\bullet}[e'[x \leftarrow v_1]]$:
1. $\vdash_{\text{H}} v_0\ v_1$
by *dynamic hole typing*
2. $\vdash_{\text{H}} v_0$
$\wedge \vdash_{\text{H}} v_1$
by H *inversion* (1)
3. $x \vdash_{\text{H}} e'$
by H *inversion* (2)
4. $\vdash_{\text{H}} e'[x \leftarrow v_1]$
by *substitution* (2, 3)
5. QED *hole substitution* (4)

ELSE $v_0 = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v_f$
$\wedge\; e \rightarrow_{\text{H-D}} E^{\bullet}[\mathsf{stat}\ \tau_c\ (v_f\ (\mathsf{dyn}\ \tau_d\ v_1))]$:

1. $\vdash_{\mathrm{H}} v_0\ v_1$
   by *dynamic hole typing*
2. $\vdash_{\mathrm{H}} v_0$
   $\wedge \vdash_{\mathrm{H}} v_1$
   by H *inversion* (1)
3. $\vdash_{\mathrm{H}} v_f : \tau_d \Rightarrow \tau_c$
   by H *inversion* (2)
4. $\vdash_{\mathrm{H}} \mathsf{dyn}\ \tau_d\ v_1 : \tau_d$
   by (2)
5. $\vdash_{\mathrm{H}} v_f\ (\mathsf{dyn}\ \tau_d\ v_1) : \tau_c$
   by (3, 4)
6. $\vdash_{\mathrm{H}} \mathsf{stat}\ \tau_c\ v_f\ (\mathsf{dyn}\ \tau_d\ v_1)$
   by (5)
7. QED by *hole substitution*

CASE $e = E^{\bullet}[op^1\ v]$:

IF $v = \langle v_0, v_1\rangle$
$\wedge\; op^1 = \mathsf{fst}$
$\wedge\; e \rightarrow_{\text{H-D}} E^{\bullet}[v_0]$:

1. $\vdash_{\mathrm{H}} op^1\ v$
   by *dynamic hole typing*
2. $\vdash_{\mathrm{H}} v$
   by H *inversion* (1)
3. $\vdash_{\mathrm{H}} v_0$
   by H *inversion* (2)
4. QED by *hole substitution*

ELSE $v = \langle v_0, v_1\rangle$
$\wedge\; op^1 = \mathsf{snd}$
$\wedge\; e \rightarrow_{\text{H-D}} E^{\bullet}[v_1]$:

1. $\vdash_{\mathrm{H}} op^1\ v$
   by *dynamic hole typing*
2. $\vdash_{\mathrm{H}} v$
   by H *inversion* (1)
3. $\vdash_{\mathrm{H}} v_1$
   by H *inversion* (2)
4. QED by *hole substitution*

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:

1. $e \rightarrow_{\text{H-D}} E^{\bullet}[\delta(op^2, v_0, v_1)]$
2. $\vdash_{\mathrm{H}} op^2\ v_0\ v_1$
   by *dynamic hole typing*
3. $\vdash_{\mathrm{H}} v_0$
   $\wedge \vdash_{\mathrm{H}} v_1$
   by H *inversion* (1)
4. $\vdash_{\mathrm{H}} \delta(op^2, v_0, v_1)$
   by δ *preservation* (2)
5. QED by *hole substitution* (3)

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :

IF e' is a value :

1. $e \rightarrow_{\text{H-D}} E[\mathcal{D}_{\mathrm{H}}(\tau', e')]$
2. $\vdash_{\mathrm{H}} \mathsf{dyn}\ \tau'\ e' : \tau'$
   by *boundary hole typing*
3. $\vdash_{\mathrm{H}} e'$
   by H *inversion* (2)
4. $\vdash_{\mathrm{H}} \mathcal{D}_{\mathrm{H}}(\tau', e') : \tau'$
   by $\mathcal{D}_{\mathrm{H}}$ *soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\text{H-D}} e''$:

1. $e \rightarrow_{\text{H-D}} E[\mathsf{dyn}\ \tau'\ e'']$
2. $\vdash_{\mathrm{H}} \mathsf{dyn}\ \tau'\ e' : \tau'$
   by *boundary hole typing*
3. $\vdash_{\mathrm{H}} e'$
   $\wedge\; \tau' \leqslant: \tau''$
   by H *inversion* (2)
4. $\vdash_{\mathrm{H}} e''$
   by H *dynamic preservation* (3)
5. $\vdash_{\mathrm{H}} \mathsf{dyn}\ \tau'\ e'' : \tau'$
   by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :

IF $e' \in v$:

1. $e \rightarrow_{\text{H-D}} E[\mathcal{S}_{\mathrm{H}}(\tau', e')]$
2. $\vdash_{\mathrm{H}} \mathsf{stat}\ \tau'\ e'$
   by *boundary hole typing*
3. $\vdash_{\mathrm{H}} e' : \tau'$
   by H *inversion* (2)
4. $\vdash_{\mathrm{H}} \mathcal{S}_{\mathrm{H}}(\tau', e')$
   by $\mathcal{S}_{\mathrm{H}}$ *soundness* (3)
5. QED by *hole substitution* (5)

ELSE $e' \rightarrow_{\text{H-S}} e''$:

1. $e \rightarrow_{\text{H-D}} E[\mathsf{stat}\ \tau'\ e'']$
2. $\vdash_{\mathrm{H}} \mathsf{stat}\ \tau'\ e'$
   by *boundary hole typing*
3. $\vdash_{\mathrm{H}} e' : \tau'$
   by H *inversion* (2)
4. $\vdash_{\mathrm{H}} e'' : \tau'$
   by H *static preservation* (3)
5. $\vdash_{\mathrm{H}} \mathsf{stat}\ \tau'\ e''$
   by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{Err}]$:

1. $e \rightarrow_{\text{H-D}} \mathsf{Err}$
2. QED $\vdash_{\mathrm{H}} \mathsf{Err}$

□

Lemma 2.12 : H *static boundary factoring*

If $\vdash_{\mathrm{H}} e : \tau$ then one of the following holds:

- e is a value
- $e = E^{\bullet}[v_0\ v_1]$
- $e = E^{\bullet}[op^1\ v]$
- $e = E^{\bullet}[op^2\ v_0\ v_1]$
- $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{Err}]$

Proof:

By the *unique static evaluation contexts* lemma, there are seven cases.

CASE e is a value :

1. QED

CASE $e = E[v_0\ v_1]$:
1. $E = E^\bullet$
   $\lor\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\lor\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
   by *inner boundary*
2. **IF** $E = E^\bullet$:
   a. QED $e = E^\bullet[v_0\ v_1]$
   IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[v_0\ v_1]]$
   ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[v_0\ v_1]]$

CASE $e = E[op^1\ v]$:
1. $E = E^\bullet$
   $\lor\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\lor\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
   by *inner boundary*
2. **IF** $E = E^\bullet$:
   a. QED $e = E^\bullet[op^1\ v]$
   IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^1\ v]]$
   ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^1\ v]]$

CASE $e = E[op^2\ v_0\ v_1]$:
1. $E = E^\bullet$
   $\lor\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\lor\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
   by *inner boundary*
2. **IF** $E = E^\bullet$:
   a. QED $e = E^\bullet[op^2\ v_0\ v_1]$
   IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$
   ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$

CASE $e = E[\mathsf{dyn}\ \tau\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{stat}\ \tau\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{Err}]$:
1. QED

□

Lemma 2.13 : H *unique static evaluation contexts*

If $\vdash_{\mathrm{H}} e : \tau$ then one of the following holds:
- e is a value
- $e = E[v_0\ v_1]$
- $e = E[op^1\ v]$
- $e = E[op^2\ v_0\ v_1]$
- $e = E[\mathsf{dyn}\ \tau\ v]$
- $e = E[\mathsf{stat}\ \tau\ v]$
- $e = E[\mathsf{Err}]$

Proof:

By induction on the structure of e.

CASE $e = x$
   $\lor\ e = \lambda x.\, e'$
   $\lor\ e = \mathsf{stat}\ \tau\ e'$:
1. Contradiction by $\vdash_{\mathrm{H}} e : \tau$

CASE $e = i$
   $\lor\ e = \lambda(x\!:\!\tau_d).\, e'$
   $\lor\ e = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v$:
1. QED e is a value

CASE $e = \langle e_0, e_1 \rangle$:
IF $e_0 \notin v$:
1. $\vdash_{\mathrm{H}} e_0 : \tau_0$
   by H *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis (1)
3. $E = \langle E_0, e_1 \rangle$
4. QED $e = E[e_0']$

IF $e_0 \in v$
   $\land\ e_1 \notin v$:
1. $\vdash_{\mathrm{H}} e_1 : \tau_1$
   by H *inversion*
2. $e_1 = E_1[e_1']$
   by the induction hypothesis (1)
3. $E = \langle e_0, E_1 \rangle$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
   $\land\ e_1 \in v$:
1. $E = [\,]$
2. QED e is a value

CASE $e = e_0\ e_1$:
IF $e_0 \notin v$:
1. $\vdash_{\mathrm{H}} e_0 : \tau_0$
   by H *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis (1)
3. $E = E_0\ e_1$
4. QED $e = E[e_0']$

IF $e_0 \in v$
   $\land\ e_1 \notin v$:
1. $\vdash_{\mathrm{H}} e_1 : \tau_1$
   by H *inversion*
2. $e_1 = E_1[e_1']$
   by the induction hypothesis (1)
3. $E = e_0\ E_1$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
   $\land\ e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[e_0\ e_1]$

CASE $e = op^1\ e_0$:
IF $e_0 \notin v$:
1. $\vdash_{\mathrm{H}} e_0 : \tau_0$
   by H *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis (1)
3. $E = op^1\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:

1. $E = [\,]$
2. QED $e = E[op^1\ e_0]$

CASE $e = op^2\ e_0\ e_1$:

IF $e_0 \notin v$:

1. $\vdash_H e_0 : \tau_0$
   by H *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis (1)
3. $E = op^2\ E_0\ e_1$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\ e_1 \notin v$:

1. $\vdash_H e_1 : \tau_1$
   by H *inversion*
2. $e_1 = E_1[e_1']$
   by the induction hypothesis (1)
3. $E = op^2\ e_0\ E_1$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\ e_1 \in v$:

1. $E = [\,]$
2. QED $e = E[op^2\ e_0\ e_1]$

CASE $e = \mathsf{dyn}\ \tau\ e_0$:

IF $e_0 \notin v$:

1. $\vdash_H e_0$
   by H *inversion*
2. $e_0 = E_0[e_0']$
   by *unique dynamic evaluation contexts* (1)
3. $E = \mathsf{dyn}\ \tau\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:

1. $E = [\,]$
2. QED $e = E[\mathsf{dyn}\ \tau\ e_0]$

CASE $e = \mathsf{stat}\ \tau\ e_0$:

Contradiction by $\vdash_H e : \tau$

CASE $e = \mathsf{Err}$:

1. $E = [\,]$
2. QED $e = E[\mathsf{Err}]$

☐

Lemma 2.14 : H *inner boundary*

> For all contexts E, one of the following holds:
> - $E = E^\bullet$
> - $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
> - $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

Proof:

By induction on the structure of E.

CASE $E = E^\bullet$:

1. QED

CASE $E = E_0\ e_1$:

1. $E_0 = E^\bullet$
   $\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\vee\ E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
   by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
   a. QED E is boundary-free
   IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. $E' = E_0'\ e_1$
   b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. $E' = E_0'\ e_1$
   b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = v_0\ E_1$:

1. $E_1 = E^\bullet$
   $\vee\ E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\vee\ E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$
   by the induction hypothesis
2. **IF** $E_1 = E^\bullet$:
   a. QED E is boundary-free
   IF $E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. $E' = v_0\ E_1'$
   b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   ELSE $E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. $E' = v_0\ E_1'$
   b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = \langle E_0, e_1 \rangle$:

1. $E_0 = E^\bullet$
   $\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\vee\ E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
   by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
   a. QED E is boundary-free
   IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. $E' = \langle E_0', e_1 \rangle$
   b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. $E' = \langle E_0', e_1 \rangle$
   b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = \langle v_0, E_1 \rangle$:

1. $E_1 = E^\bullet$
   $\vee\ E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\vee\ E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$
   by the induction hypothesis
2. **IF** $E_1 = E^\bullet$:
   a. QED E is boundary-free
   IF $E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. $E' = \langle v_0, E_1' \rangle$
   b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   ELSE $E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. $E' = \langle v_0, E_1' \rangle$
   b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = op^1\ E_0$:

1. $E_0 = E^\bullet$
   $\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\vee\ E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
   by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
   a. QED E is boundary-free
   IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:

a. $E' = op^1\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = op^1\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = op^2\ E_0\ e_1$:
1. $E_0 = E^\bullet$
$\lor\ E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\lor\ E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = op^2\ E_0'\ e_1$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = op^2\ E_0'\ e_1$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = op^2\ v_0\ E_1$:
1. $E_1 = E^\bullet$
$\lor\ E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\lor\ E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_1 = E^\bullet$:
a. QED E is boundary-free
IF $E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = op^2\ v_0\ E_1'$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = op^2\ v_0\ E_1'$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = \mathsf{dyn}\ \tau\ E_0$:
1. $E_0 = E^\bullet$
$\lor\ E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$
$\lor\ E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED
IF $E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau'\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau'\ E^\bullet]$

CASE $E = \mathsf{stat}\ \tau\ E_0$:
1. $E_0 = E^\bullet$
$\lor\ E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$
$\lor\ E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED
IF $E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{stat}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau'\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{stat}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau'\ E^\bullet]$

□

Lemma 2.15 : H *dynamic boundary factoring*

If $\vdash_{\mathrm{H}} e$ then one of the following holds:
- e is a value
- $e = E^\bullet[v_0\ v_1]$
- $e = E^\bullet[op^1\ v]$
- $e = E^\bullet[op^2\ v_0\ v_1]$
- $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{Err}]$

Proof:

By the *unique dynamic evaluation contexts* lemma, there are seven cases.

CASE e is a value :
1. QED

CASE $e = E[v_0\ v_1]$:
1. $E = E^\bullet$
$\lor\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\lor\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. QED $e = E^\bullet[v_0\ v_1]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[v_0\ v_1]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[v_0\ v_1]]$

CASE $e = E[op^1\ v]$:
1. $E = E^\bullet$
$\lor\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\lor\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. QED $e = E^\bullet[op^1\ v]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^1\ v]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^1\ v]]$

CASE $e = E[op^2\ v_0\ v_1]$:
1. $E = E^\bullet$
$\lor\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\lor\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. QED $e = E^\bullet[op^2\ v_0\ v_1]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$

CASE $e = E[\mathsf{dyn}\ \tau\ v]$:
1. QED v is boundary-free

case $e = E[\mathsf{stat}\ \tau\ v]$:
1. qed v is boundary-free

case $e = E[\mathsf{Err}]$:
1. qed

□

Lemma 2.16 : H *unique dynamic evaluation contexts*

If $\vdash_{\mathsf{H}} e$ then one of the following holds:
- e is a value
- $e = E[v_0\ v_1]$
- $e = E[op^1\ v]$
- $e = E[op^2\ v_0\ v_1]$
- $e = E[\mathsf{dyn}\ \tau\ v]$
- $e = E[\mathsf{stat}\ \tau\ v]$
- $e = E[\mathsf{Err}]$

Proof:

By induction on the structure of e.

case $e = x$
$\quad \vee\ e = \lambda(x\!:\!\tau).\, e'$
$\quad \vee\ e = \mathsf{dyn}\ \tau\ e'$:
1. Contradiction by $\vdash_{\mathsf{H}} e$

case $e = i$
$\quad \vee\ e = \lambda x.\, e'$
$\quad \vee\ e = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v$:
1. qed e is a value

case $e = \mathsf{Err}$:
1. $E = [\,]$
2. qed $e = E[\mathsf{Err}]$

case $e = \langle e_0, e_1 \rangle$:

if $e_0 \notin v$:
1. $\vdash_{\mathsf{H}} e_0$
   by H *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis (1)
3. $E = \langle E_0, e_1 \rangle$
4. qed $e = E[e_0']$

if $e_0 \in v$
$\quad \wedge\ e_1 \notin v$:
1. $\vdash_{\mathsf{H}} e_1$
   by H *inversion*
2. $e_1 = E_1[e_1']$
   by the induction hypothesis (1)
3. $E = \langle e_0, E_1 \rangle$
4. qed $e = E[e_1']$

else $e_0 \in v$
$\quad \wedge\ e_1 \in v$:
1. $E = [\,]$
2. qed e is a value

case $e = e_0\ e_1$:

if $e_0 \notin v$:
1. $\vdash_{\mathsf{H}} e_0$
   by H *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis (1)
3. $E = E_0\ e_1$
4. qed $e = E[e_0']$

if $e_0 \in v$
$\quad \wedge\ e_1 \notin v$:
1. $\vdash_{\mathsf{H}} e_1$
   by H *inversion*
2. $e_1 = E_1[e_1']$
   by the induction hypothesis (1)
3. $E = e_0\ E_1$
4. qed $e = E[e_1']$

else $e_0 \in v$
$\quad \wedge\ e_1 \in v$:
1. $E = [\,]$
2. qed $e = E[e_0\ e_1]$

case $e = op^1\ e_0$:

if $e_0 \notin v$:
1. $\vdash_{\mathsf{H}} e_0$
   by H *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis (1)
3. $E = op^1\ E_0$
4. qed $e = E[e_0']$

else $e_0 \in v$:
1. $E = [\,]$
2. qed $e = E[op^1\ e_0]$

case $e = op^2\ e_0\ e_1$:

if $e_0 \notin v$:
1. $\vdash_{\mathsf{H}} e_0$
   by H *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis (1)
3. $E = op^2\ E_0\ e_1$
4. qed $e = E[e_0']$

if $e_0 \in v$
$\quad \wedge\ e_1 \notin v$:
1. $\vdash_{\mathsf{H}} e_1$
   by H *inversion*
2. $e_1 = E_1[e_1']$
   by the induction hypothesis (1)
3. $E = op^2\ e_0\ E_1$
4. qed $e = E[e_1']$

else $e_0 \in v$
$\quad \wedge\ e_1 \in v$:
1. $E = [\,]$
2. qed $e = E[op^2\ e_0\ e_1]$

case $e = \mathsf{stat}\ \tau\ e_0$:

if $e_0 \notin v$:
1. $\vdash_{\mathsf{H}} e_0$
   by H *inversion*
2. $e_0 = E_0[e_0']$
   by *unique static evaluation contexts* (1)
3. $E = \mathsf{stat}\ \tau\ E_0$
4. qed $e = E[e_0']$

else $e_0 \in v$:
1. $E = [\,]$

2. QED $e = E[\mathsf{stat}\ \tau\ e_0]$

□

Lemma 2.17 : H *static hole typing*

If $\vdash_{\mathsf{H}} E^{\bullet}[e] : \tau$ then the derivation contains a sub-term $\vdash_{\mathsf{H}} e : \tau'$

Proof:

By induction on the structure of $E^{\bullet}$.

CASE $E^{\bullet} = [\,]$:

1. QED $E^{\bullet}[e] = e$

CASE $E^{\bullet} = E^{\bullet}{}_0\ e_1$:

1. $E^{\bullet}[e] = E^{\bullet}{}_0[e]\ e_1$
2. $\vdash_{\mathsf{H}} E^{\bullet}{}_0[e] : \tau_d \Rightarrow \tau_c$ by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = v_0\ E^{\bullet}{}_1$:

1. $E^{\bullet}[e] = v_0\ E^{\bullet}{}_1[e]$
2. $\vdash_{\mathsf{H}} E^{\bullet}{}_1[e] : \tau_d$ by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = \langle E^{\bullet}{}_0, e_1\rangle$:

1. $E^{\bullet}[e] = \langle E^{\bullet}{}_0[e], e_1\rangle$
2. $\vdash_{\mathsf{H}} E^{\bullet}{}_0[e] : \tau_0$ by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = \langle v_0, E^{\bullet}{}_1\rangle$:

1. $E^{\bullet}[e] = \langle v_0, E^{\bullet}{}_1[e]\rangle$
2. $\vdash_{\mathsf{H}} E^{\bullet}{}_1[e] : \tau_1$ by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = op^1\ E^{\bullet}{}_0$:

1. $E^{\bullet}[e] = op^1\ E^{\bullet}{}_0[e]$
2. $\vdash_{\mathsf{H}} E^{\bullet}{}_0[e] : \tau_0$ by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = op^2\ E^{\bullet}{}_0\ e_1$:

1. $E^{\bullet}[e] = op^2\ E^{\bullet}{}_0[e]\ e_1$
2. $\vdash_{\mathsf{H}} E^{\bullet}{}_0[e] : \tau_0$ by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = op^2\ v_0\ E^{\bullet}{}_1$:

1. $E^{\bullet}[e] = op^2\ v_0\ E^{\bullet}{}_1[e]$
2. $\vdash_{\mathsf{H}} E^{\bullet}{}_1[e] : \tau_1$ by H *inversion*
3. QED by the induction hypothesis (2)

□

Lemma 2.18 : H *dynamic hole typing*

If $\vdash_{\mathsf{H}} E^{\bullet}[e]$ then the derivation contains a sub-term $\vdash_{\mathsf{H}} e$

Proof:

By induction on the structure of $E^{\bullet}$.

CASE $E^{\bullet} = [\,]$:

1. QED $E^{\bullet}[e] = e$

CASE $E^{\bullet} = E^{\bullet}{}_0\ e_1$:

1. $E^{\bullet}[e] = E^{\bullet}{}_0[e]\ e_1$
2. $\vdash_{\mathsf{H}} E^{\bullet}{}_0[e]$ by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = v_0\ E^{\bullet}{}_1$:

1. $E^{\bullet}[e] = v_0\ E^{\bullet}{}_1[e]$
2. $\vdash_{\mathsf{H}} E^{\bullet}{}_1[e]$ by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = \langle E^{\bullet}{}_0, e_1\rangle$:

1. $E^{\bullet}[e] = \langle E^{\bullet}{}_0[e], e_1\rangle$
2. $\vdash_{\mathsf{H}} E^{\bullet}{}_0[e]$ by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = \langle v_0, E^{\bullet}{}_1\rangle$:

1. $E^{\bullet}[e] = \langle v_0, E^{\bullet}{}_1[e]\rangle$
2. $\vdash_{\mathsf{H}} E^{\bullet}{}_1[e]$ by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = op^1\ E^{\bullet}{}_0$:

1. $E^{\bullet}[e] = op^1\ E^{\bullet}{}_0[e]$
2. $\vdash_{\mathsf{H}} E^{\bullet}{}_0[e]$ by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = op^2\ E^{\bullet}{}_0\ e_1$:

1. $E^{\bullet}[e] = op^2\ E^{\bullet}{}_0[e]\ e_1$
2. $\vdash_{\mathsf{H}} E^{\bullet}{}_0[e]$ by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = op^2\ v_0\ E^{\bullet}{}_1$:

1. $E^{\bullet}[e] = op^2\ v_0\ E^{\bullet}{}_1[e]$
2. $\vdash_{\mathsf{H}} E^{\bullet}{}_1[e]$ by H *inversion*
3. QED by the induction hypothesis (2)

□

Lemma 2.19 : H *boundary hole typing*

- If $\vdash_{\mathsf{H}} E[\mathsf{dyn}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_{\mathsf{H}} \mathsf{dyn}\ \tau\ e : \tau$
- If $\vdash_{\mathsf{H}} E[\mathsf{dyn}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_{\mathsf{H}} \mathsf{dyn}\ \tau\ e : \tau$
- If $\vdash_{\mathsf{H}} E[\mathsf{stat}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau\ e$
- If $\vdash_{\mathsf{H}} E[\mathsf{stat}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau\ e$

Proof:

By the following four lemmas: *static* dyn *hole typing*, *dynamic* dyn *hole typing*, *static* stat *hole typing*, and *dynamic* stat *hole typing*.

□

Lemma 2.20 : H *static* dyn *hole typing*

If $\vdash_{\mathsf{H}} E[\mathsf{dyn}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_{\mathsf{H}} \mathsf{dyn}\ \tau\ e : \tau$.

Proof:

By induction on the structure of E.

case $E \in E^{\bullet}$:
1. $\vdash_{\mathsf{H}} \mathsf{dyn}\ \tau\ e : \tau''$
   by *static hole typing*
2. $\vdash_{\mathsf{H}} \mathsf{dyn}\ \tau\ e : \tau$
   by H *inversion* (1)
3. qed

case $E = E_0\ e_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = E_0[\mathsf{dyn}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{dyn}\ \tau\ e] : \tau_0$
   by H *inversion*
3. qed by the induction hypothesis (2)

case $E = v_0\ E_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = v_0\ E_1[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{H}} E_1[\mathsf{dyn}\ \tau\ e] : \tau_1$
   by H *inversion*
3. qed by the induction hypothesis (2)

case $E = \langle E_0, e_1 \rangle$:
1. $E[\mathsf{dyn}\ \tau\ e] = \langle E_0[\mathsf{dyn}\ \tau\ e], e_1 \rangle$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{dyn}\ \tau\ e] : \tau_0$
   by H *inversion*
3. qed by the induction hypothesis (2)

case $E = \langle v_0, E_1 \rangle$:
1. $E[\mathsf{dyn}\ \tau\ e] = \langle v_0, E_1[\mathsf{dyn}\ \tau\ e] \rangle$
2. $\vdash_{\mathsf{H}} E_1[\mathsf{dyn}\ \tau\ e] : \tau_1$
   by H *inversion*
3. qed by the induction hypothesis (2)

case $E = op^1\ E_0$:
1. $E[\mathsf{dyn}\ \tau\ e] = op^1\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{dyn}\ \tau\ e] : \tau_0$
   by H *inversion*
3. qed by the induction hypothesis (2)

case $E = op^2\ E_0\ e_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ E_0[\mathsf{dyn}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{dyn}\ \tau\ e] : \tau_0$
   by H *inversion*
3. qed by the induction hypothesis (2)

case $E = op^2\ v_0\ E_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{H}} E_1[\mathsf{dyn}\ \tau\ e] : \tau_1$
   by H *inversion*
3. qed by the induction hypothesis (2)

case $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[\mathsf{dyn}\ \tau\ e] = \mathsf{dyn}\ \tau_0\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{dyn}\ \tau\ e]$
   by H *inversion*
3. qed by *dynamic* dyn *hole typing* (2)

case $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_{\mathsf{H}} E[\mathsf{dyn}\ \tau\ e] : \tau'$

□

Lemma 2.21 : H *dynamic* dyn *hole typing*

If $\vdash_{\mathsf{H}} E[\mathsf{dyn}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_{\mathsf{H}} \mathsf{dyn}\ \tau\ e : \tau$.

Proof:

By induction on the structure of E.

case $E \in E^{\bullet}$:
1. Contradiction by $\vdash_{\mathsf{H}} E[\mathsf{dyn}\ \tau\ e]$

case $E = E_0\ e_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = E_0[\mathsf{dyn}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{dyn}\ \tau\ e]$
   by H *inversion*
3. qed by the induction hypothesis (2)

case $E = v_0\ E_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = v_0\ E_1[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{H}} E_1[\mathsf{dyn}\ \tau\ e]$
   by H *inversion*
3. qed by the induction hypothesis (2)

case $E = \langle E_0, e_1 \rangle$:
1. $E[\mathsf{dyn}\ \tau\ e] = \langle E_0[\mathsf{dyn}\ \tau\ e], e_1 \rangle$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{dyn}\ \tau\ e]$
   by H *inversion*
3. qed by the induction hypothesis (2)

case $E = \langle v_0, E_1 \rangle$:
1. $E[\mathsf{dyn}\ \tau\ e] = \langle v_0, E_1[\mathsf{dyn}\ \tau\ e] \rangle$
2. $\vdash_{\mathsf{H}} E_1[\mathsf{dyn}\ \tau\ e]$
   by H *inversion*
3. qed by the induction hypothesis (2)

case $E = op^1\ E_0$:
1. $E[\mathsf{dyn}\ \tau\ e] = op^1\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{dyn}\ \tau\ e]$
   by H *inversion*
3. qed by the induction hypothesis (2)

case $E = op^2\ E_0\ e_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ E_0[\mathsf{dyn}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{dyn}\ \tau\ e]$
   by H *inversion*
3. qed by the induction hypothesis (2)

case $E = op^2\ v_0\ E_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{H}} E_1[\mathsf{dyn}\ \tau\ e]$
   by H *inversion*
3. qed by the induction hypothesis (2)

case $E = \mathsf{dyn}\ \tau\ E_0$:
1. Contradiction by $\vdash_{\mathsf{H}} E[\mathsf{dyn}\ \tau\ e]$

case $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[\mathsf{dyn}\ \tau\ e] = \mathsf{stat}\ \tau_0\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{H}} E_0[\mathsf{dyn}\ \tau\ e] : \tau_0$
   by H *inversion*
3. qed by *static* dyn *hole typing* (2)

□

Lemma 2.22 : H *static* stat *hole typing*

If $\vdash_{\mathsf{H}} E[\mathsf{stat}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau\ e$.

Proof:

By induction on the structure of E.

case $E \in E^{\bullet}$:
1. Contradiction by $\vdash_{\mathsf{H}} E[\mathsf{stat}\ \tau\ e] : \tau'$

case $E = E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_{H} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by H *inversion*
3. **qed** by the induction hypothesis (2)

case $E = v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{H} E_1[\mathsf{stat}\ \tau\ e] : \tau_1$
   by H *inversion*
3. **qed** by the induction hypothesis (2)

case $E = \langle E_0, e_1 \rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle E_0[\mathsf{stat}\ \tau\ e], e_1 \rangle$
2. $\vdash_{H} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by H *inversion*
3. **qed** by the induction hypothesis (2)

case $E = \langle v_0, E_1 \rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle v_0, E_1[\mathsf{stat}\ \tau\ e] \rangle$
2. $\vdash_{H} E_1[\mathsf{stat}\ \tau\ e] : \tau_1$
   by H *inversion*
3. **qed** by the induction hypothesis (2)

case $E = op^1\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = op^1\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{H} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by H *inversion*
3. **qed** by the induction hypothesis (2)

case $E = op^2\ E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_{H} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by H *inversion*
3. **qed** by the induction hypothesis (2)

case $E = op^2\ v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{H} E_1[\mathsf{stat}\ \tau\ e] : \tau_1$
   by H *inversion*
3. **qed** by the induction hypothesis (2)

case $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{dyn}\ \tau_0\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{H} E_0[\mathsf{stat}\ \tau\ e]$
   by H *inversion*
3. **qed** by *dynamic* stat *hole typing* (2)

case $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_{H} E[\mathsf{stat}\ \tau\ e] : \tau'$

□

Lemma 2.23 : H *dynamic* stat *hole typing*

If $\vdash_{H} E[\mathsf{stat}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_{H} \mathsf{stat}\ \tau\ e$.

Proof:

By induction on the structure of E.

case $E \in E^{\bullet}$:
1. **qed** by *dynamic hole typing*

case $E = E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_{H} E_0[\mathsf{stat}\ \tau\ e]$
   by H *inversion*
3. **qed** by the induction hypothesis (2)

case $E = v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{H} E_1[\mathsf{stat}\ \tau\ e]$
   by H *inversion*
3. **qed** by the induction hypothesis (2)

case $E = \langle E_0, e_1 \rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle E_0[\mathsf{stat}\ \tau\ e], e_1 \rangle$
2. $\vdash_{H} E_0[\mathsf{stat}\ \tau\ e]$
   by H *inversion*
3. **qed** by the induction hypothesis (2)

case $E = \langle v_0, E_1 \rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle v_0, E_1[\mathsf{stat}\ \tau\ e] \rangle$
2. $\vdash_{H} E_1[\mathsf{stat}\ \tau\ e]$
   by H *inversion*
3. **qed** by the induction hypothesis (2)

case $E = op^1\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = op^1\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{H} E_0[\mathsf{stat}\ \tau\ e]$
   by H *inversion*
3. **qed** by the induction hypothesis (2)

case $E = op^2\ E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_{H} E_0[\mathsf{stat}\ \tau\ e]$
   by H *inversion*
3. **qed** by the induction hypothesis (2)

case $E = op^2\ v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{H} E_1[\mathsf{stat}\ \tau\ e]$
   by H *inversion*
3. **qed** by the induction hypothesis (2)

case $E = \mathsf{dyn}\ \tau\ E_0$:
1. Contradiction by $\vdash_{H} E[\mathsf{stat}\ \tau\ e]$

case $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{stat}\ \tau_0\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{H} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by H *inversion*
3. **qed** by *static* stat *hole typing* (2)

□

Lemma 2.24 : H *static boundary-free hole substitution*

If $\vdash_{H} E^{\bullet}[e] : \tau$ and the derivation contains a sub-term $\vdash_{H} e : \tau'$ and $\vdash_{H} e' : \tau'$ then $\vdash_{H} E^{\bullet}[e'] : \tau$.

Proof:

By induction on the structure of $E^{\bullet}$

case $E^{\bullet} = [\,]$:
1. $E^{\bullet}[e] = e$
   $\wedge\ E^{\bullet}[e'] = e'$
2. $\vdash_{H} e : \tau$
   by (1)
3. $\tau' = \tau$
4. $\vdash_{H} e' : \tau$

5. QED by (1, 4)

CASE $E^{\bullet} = E^{\bullet}{}_0\ e_1$:

1. $E^{\bullet}[e] = E^{\bullet}{}_0[e]\ e_1$
   $\wedge\ E^{\bullet}[e'] = E^{\bullet}{}_0[e']\ e_1$
2. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e]\ e_1 : \tau$
3. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e] : \tau_0$
   $\wedge \vdash_{\mathrm{H}} e_1 : \tau_1$
   by H *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e']\ e_1 : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = v_0\ E^{\bullet}{}_1$:

1. $E^{\bullet}[e] = v_0\ E^{\bullet}{}_1[e]$
   $\wedge\ E^{\bullet}[e'] = v_0\ E^{\bullet}{}_1[e']$
2. $\vdash_{\mathrm{H}} v_0\ E^{\bullet}{}_1[e] : \tau$
3. $\vdash_{\mathrm{H}} v_0 : \tau_0$
   $\wedge \vdash_{\mathrm{H}} E^{\bullet}{}_1[e] : \tau_1$
   by H *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_1[e'] : \tau_1$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} v_0\ E^{\bullet}{}_1[e'] : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^1\ E^{\bullet}{}_0$:

1. $E^{\bullet}[e] = op^1\ E^{\bullet}{}_0[e]$
   $\wedge\ E^{\bullet}[e'] = op^1\ E^{\bullet}{}_0[e']$
2. $\vdash_{\mathrm{H}} op^1\ E^{\bullet}{}_0[e] : \tau$
3. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e] : \tau_0$
   by H *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} op^1\ E^{\bullet}{}_0[e'] : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \langle E^{\bullet}{}_0, e_1\rangle$:

1. $E^{\bullet}[e] = \langle E^{\bullet}{}_0[e], e_1\rangle$
   $\wedge\ E^{\bullet}[e'] = \langle E^{\bullet}{}_0[e'], e_1\rangle$
2. $\vdash_{\mathrm{H}} \langle E^{\bullet}{}_0[e], e_1\rangle : \tau$
3. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e] : \tau_0$
   $\wedge \vdash_{\mathrm{H}} e_1 : \tau_1$
   by H *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} \langle E^{\bullet}{}_0[e'], e_1\rangle : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \langle v_0, E^{\bullet}{}_1\rangle$:

1. $E^{\bullet}[e] = \langle v_0, E^{\bullet}{}_1[e]\rangle$
   $\wedge\ E^{\bullet}[e'] = \langle v_0, E^{\bullet}{}_1[e']\rangle$
2. $\vdash_{\mathrm{H}} \langle v_0, E^{\bullet}{}_1[e]\rangle : \tau$
3. $\vdash_{\mathrm{H}} v_0 : \tau_0$
   $\wedge \vdash_{\mathrm{H}} E^{\bullet}{}_1[e] : \tau_1$
   by H *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_1[e'] : \tau_1$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} \langle v_0, E^{\bullet}{}_1[e']\rangle : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ E^{\bullet}{}_0\ e_1$:

1. $E^{\bullet}[e] = op^2\ E^{\bullet}{}_0[e]\ e_1$
   $\wedge\ E^{\bullet}[e'] = op^2\ E^{\bullet}{}_0[e']\ e_1$
2. $\vdash_{\mathrm{H}} op^2\ E^{\bullet}{}_0[e]\ e_1 : \tau$
3. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e] : \tau_0$
   $\wedge \vdash_{\mathrm{H}} e_1 : \tau_1$
   by H *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} op^2\ E^{\bullet}{}_0[e']\ e_1 : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ v_0\ E^{\bullet}{}_1$:

1. $E^{\bullet}[e] = op^2\ v_0\ E^{\bullet}{}_1[e]$
   $\wedge\ E^{\bullet}[e'] = op^2\ v_0\ E^{\bullet}{}_1[e']$
2. $\vdash_{\mathrm{H}} op^2\ v_0\ E^{\bullet}{}_1[e] : \tau$
3. $\vdash_{\mathrm{H}} v_0 : \tau_0$
   $\wedge \vdash_{\mathrm{H}} E^{\bullet}{}_1[e] : \tau_1$
   by H *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_1[e'] : \tau_1$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} op^2\ v_0\ E^{\bullet}{}_1[e'] : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

□

Lemma 2.25 : H *dynamic hole substitution*

If $\vdash_{\mathrm{H}} E^{\bullet}[e]$ and $\vdash_{\mathrm{H}} e'$ then $\vdash_{\mathrm{H}} E^{\bullet}[e']$

Proof:

By induction on the structure of $E^{\bullet}$

CASE $E^{\bullet} = [\,]$:

1. QED $E^{\bullet}[e'] = e'$

CASE $E^{\bullet} = \langle E^{\bullet}{}_0, e_1\rangle$:

1. $E^{\bullet}[e] = \langle E^{\bullet}{}_0[e], e_1\rangle$
   $\wedge\ E^{\bullet}[e'] = \langle E^{\bullet}{}_0[e'], e_1\rangle$
2. $\vdash_{\mathrm{H}} \langle E^{\bullet}{}_0[e], e_1\rangle$
3. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e]$
   $\wedge \vdash_{\mathrm{H}} e_1$
   by H *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} \langle E^{\bullet}{}_0[e'], e_1\rangle$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \langle v_0, E^{\bullet}{}_1\rangle$:

1. $E^{\bullet}[e] = \langle v_0, E^{\bullet}{}_1[e]\rangle$
   $\wedge\, E^{\bullet}[e'] = \langle v_0, E^{\bullet}{}_1[e']\rangle$
2. $\vdash_{\mathrm{H}} \langle v_0, E^{\bullet}{}_1[e]\rangle$
3. $\vdash_{\mathrm{H}} v_0$
   $\wedge \vdash_{\mathrm{H}} E^{\bullet}{}_1[e]$
   by H *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_1[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} \langle v_0, E^{\bullet}{}_1[e']\rangle$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = E^{\bullet}{}_0\ e_1$:
1. $E^{\bullet}[e] = E^{\bullet}{}_0[e]\ e_1$
   $\wedge\, E^{\bullet}[e'] = E^{\bullet}{}_0[e']\ e_1$
2. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e]\ e_1$
3. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e]$
   $\wedge \vdash_{\mathrm{H}} e_1$
   by H *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e']\ e_1$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = v_0\ E^{\bullet}{}_1$:
1. $E^{\bullet}[e] = v_0\ E^{\bullet}{}_1[e]$
   $\wedge\, E^{\bullet}[e'] = v_0\ E^{\bullet}{}_1[e']$
2. $\vdash_{\mathrm{H}} v_0\ E^{\bullet}{}_1[e]$
3. $\vdash_{\mathrm{H}} v_0$
   $\wedge \vdash_{\mathrm{H}} E^{\bullet}{}_1[e]$
   by H *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_1[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} v_0\ E^{\bullet}{}_1[e']$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^1\ E^{\bullet}{}_0$:
1. $E^{\bullet}[e] = op^1\ E^{\bullet}{}_0[e]$
   $\wedge\, E^{\bullet}[e'] = op^1\ E^{\bullet}{}_0[e']$
2. $\vdash_{\mathrm{H}} op^1\ E^{\bullet}{}_0[e]$
3. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e]$
   by H *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} op^1\ E^{\bullet}{}_0[e']$
   by (4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ E^{\bullet}{}_0\ e_1$:
1. $E^{\bullet}[e] = op^2\ E^{\bullet}{}_0[e]\ e_1$
   $\wedge\, E^{\bullet}[e'] = op^2\ E^{\bullet}{}_0[e']\ e_1$
2. $\vdash_{\mathrm{H}} op^2\ E^{\bullet}{}_0[e]\ e_1$
3. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e]$
   $\wedge \vdash_{\mathrm{H}} e_1$
   by H *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} op^2\ E^{\bullet}{}_0[e']\ e_1$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ v_0\ E^{\bullet}{}_1$:
1. $E^{\bullet}[e] = op^2\ v_0\ E^{\bullet}{}_1[e]$
   $\wedge\, E^{\bullet}[e'] = op^2\ v_0\ E^{\bullet}{}_1[e']$
2. $\vdash_{\mathrm{H}} op^2\ v_0\ E^{\bullet}{}_1[e]$
3. $\vdash_{\mathrm{H}} v_0$
   $\wedge \vdash_{\mathrm{H}} E^{\bullet}{}_1[e]$
   by H *inversion*
4. $\vdash_{\mathrm{H}} E^{\bullet}{}_1[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathrm{H}} op^2\ v_0\ E^{\bullet}{}_1[e']$
   by (3, 4)
6. QED by (1, 5)

□

Lemma 2.26 : H *hole substitution*
- If $\vdash_{\mathrm{H}} E[e]$ and the derivation contains a sub-term $\vdash_{\mathrm{H}} e : \tau'$ and $\vdash_{\mathrm{H}} e' : \tau'$ then $\vdash_{\mathrm{H}} E[e']$.
- If $\vdash_{\mathrm{H}} E[e]$ and the derivation contains a sub-term $\vdash_{\mathrm{H}} e$ and $\vdash_{\mathrm{H}} e'$ then $\vdash_{\mathrm{H}} E[e']$.
- If $\vdash_{\mathrm{H}} E[e] : \tau$ and the derivation contains a sub-term $\vdash_{\mathrm{H}} e : \tau'$ and $\vdash_{\mathrm{H}} e' : \tau'$ then $\vdash_{\mathrm{H}} E[e'] : \tau$.
- If $\vdash_{\mathrm{H}} E[e] : \tau$ and the derivation contains a sub-term $\vdash_{\mathrm{H}} e$ and $\vdash_{\mathrm{H}} e'$ then $\vdash_{\mathrm{H}} E[e'] : \tau$.

Proof:
By the following four lemmas: *dynamic context static hole substitution*, *dynamic context dynamic hole substitution*, *static context static hole substitution*, and *static context dynamic hole substitution*.

□

Lemma 2.27 : H *dynamic context static hole substitution*
If $\vdash_{\mathrm{H}} E[e]$ and contains $\vdash_{\mathrm{H}} e : \tau'$, and furthermore $\vdash_{\mathrm{H}} e' : \tau'$, then $\vdash_{\mathrm{H}} E[e']$

Proof:
By induction on the structure of E.

CASE $E \in E^{\bullet}$:
1. Contradiction by $\vdash_{\mathrm{H}} E[e]$

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_{\mathrm{H}} E_0[e]$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_{\mathrm{H}} E_1[e]$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1\rangle$:
1. $E[e] = \langle E_0[e], e_1\rangle$

2. $\vdash_H E_0[e]$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_H E_1[e]$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_H E_0[e]$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_H E_0[e]$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_H E_1[e]$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau''\ E_0$:
1. Contradiction by $\vdash_H E[e]$

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{stat}\ \tau_0\ E_0[e]$
2. $\vdash_H E_0[e] : \tau_0$
   by H *inversion*
3. QED by *static context static hole substitution* (2)

□

Lemma 2.28 : H *dynamic context dynamic hole substitution*

If $\vdash_H E[e]$ and contains $\vdash_H e$, and furthermore $\vdash_H e'$, then $\vdash_H E[e']$

Proof:

By induction on the structure of E.

CASE $E \in E^\bullet$:
1. QED by *dynamic boundary-free hole substitution*

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_H E_0[e]$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_H E_1[e]$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_H E_0[e]$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_H E_1[e]$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_H E_0[e]$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_H E_0[e]$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_H E_1[e]$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau''\ E_0$:
1. Contradiction by $\vdash_H E[e]$

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{stat}\ \tau_0\ E_0[e]$
2. $\vdash_H E_0[e] : \tau_0$
   by H *inversion*
3. QED by *static context dynamic hole substitution* (2)

□

Lemma 2.29 : H *static context static hole substitution*

If $\vdash_H E[e] : \tau$ and contains $\vdash_H e : \tau'$, and furthermore $\vdash_H e' : \tau'$, then $\vdash_H E[e'] : \tau$

Proof:

By induction on the structure of E.

CASE $E \in E^\bullet$:
1. QED by *static boundary-free hole substitution*

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_H E_0[e] : \tau_0$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_H E_1[e] : \tau_1$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_H E_0[e] : \tau_0$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$

2. $\vdash_{\mathsf{H}} E_1[e] : \tau_1$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_{\mathsf{H}} E_0[e] : \tau_0$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_{\mathsf{H}} E_0[e] : \tau_0$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_{\mathsf{H}} E_1[e] : \tau_1$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{dyn}\ \tau_0\ E_0[e]$
2. $\vdash_{\mathsf{H}} E_0[e]$
   by H *inversion*
3. QED by *static* dyn *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_{\mathsf{H}} E[e] : \tau$

□

Lemma 2.30 : H *static context dynamic hole substitution*

If $\vdash_{\mathsf{H}} E[e] : \tau$ and contains $\vdash_{\mathsf{H}} e$, and furthermore $\vdash_{\mathsf{H}} e'$, then $\vdash_{\mathsf{H}} E[e'] : \tau$

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:
1. Contradiction by $\vdash_{\mathsf{H}} E[e] : \tau$

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_{\mathsf{H}} E_0[e] : \tau_0$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_{\mathsf{H}} E_1[e] : \tau_1$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_{\mathsf{H}} E_0[e] : \tau_0$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_{\mathsf{H}} E_1[e] : \tau_1$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_{\mathsf{H}} E_0[e] : \tau_0$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_{\mathsf{H}} E_0[e] : \tau_0$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_{\mathsf{H}} E_1[e] : \tau_1$
   by H *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{dyn}\ \tau_0\ E_0[e]$
2. $\vdash_{\mathsf{H}} E_0[e]$
   by H *inversion*
3. QED by *dynamic* stat *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_{\mathsf{H}} E[e] : \tau$

□

Lemma 2.31 : $\vdash_H$ *static inversion*

- If $\Gamma \vdash_{\mathsf{H}} x : \tau$ then $(x\!:\!\tau') \in \Gamma$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{H}} \lambda(x : \tau'_d).\, e' : \tau$ then $(x : \tau'_d), \Gamma \vdash_{\mathsf{H}} e' : \tau'_c$ and $\tau'_d \Rightarrow \tau'_c \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{H}} \langle e_0, e_1 \rangle : \tau$ then $\Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$ and $\Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$ and $\tau_0 \times \tau_1 \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{H}} e_0\ e_1 : \tau_c$ then $\Gamma \vdash_{\mathsf{H}} e_0 : \tau'_d \Rightarrow \tau'_c$ and $\Gamma \vdash_{\mathsf{H}} e_1 : \tau'_d$ and $\tau'_c \leqslant: \tau_c$
- If $\Gamma \vdash_{\mathsf{H}} \mathsf{fst}\ e : \tau$ then $\Gamma \vdash_{\mathsf{H}} e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0$ and $\tau_0 \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{H}} \mathsf{snd}\ e : \tau$ then $\Gamma \vdash_{\mathsf{H}} e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1$ and $\tau_1 \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{H}} op^2\ e_0\ e_1 : \tau$ then $\Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$ and $\Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau'$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{H}} \mathsf{mon}\ \tau_d \Rightarrow \tau_c\ v' : \tau$ then $\Gamma \vdash_{\mathsf{H}} v'$ and $\tau_d \Rightarrow \tau_c \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{H}} \mathsf{dyn}\ \tau'\ e' : \tau$ then $\Gamma \vdash_{\mathsf{H}} e'$ and $\tau' \leqslant: \tau$

Proof:

QED by the definition of $\Gamma \vdash_{\mathsf{H}} e : \tau$

□

Lemma 2.32 : $\vdash_H$ *dynamic inversion*

- If $\Gamma \vdash_{\mathsf{H}} x$ then $x \in \Gamma$
- If $\Gamma \vdash_{\mathsf{H}} \lambda x.\, e'$ then $x, \Gamma \vdash_{\mathsf{H}} e'$
- If $\Gamma \vdash_{\mathsf{H}} \langle e_0, e_1 \rangle$ then $\Gamma \vdash_{\mathsf{H}} e_0$ and $\Gamma \vdash_{\mathsf{H}} e_1$
- If $\Gamma \vdash_{\mathsf{H}} e_0\ e_1$ then $\Gamma \vdash_{\mathsf{H}} e_0$ and $\Gamma \vdash_{\mathsf{H}} e_1$
- If $\Gamma \vdash_{\mathsf{H}} op^1\ e_0$ then $\Gamma \vdash_{\mathsf{H}} e_0$
- If $\Gamma \vdash_{\mathsf{H}} op^2\ e_0\ e_1$ then $\Gamma \vdash_{\mathsf{H}} e_0$ and $\Gamma \vdash_{\mathsf{H}} e_1$
- If $\Gamma \vdash_{\mathsf{H}} \mathsf{mon}\ \tau_d \Rightarrow \tau_c\ v'$ then $\Gamma \vdash_{\mathsf{H}} v' : \tau_d \Rightarrow \tau_c$
- If $\Gamma \vdash_{\mathsf{H}} \mathsf{stat}\ \tau'\ e'$ then $\Gamma \vdash_{\mathsf{H}} e' : \tau'$

Proof:

QED by the definition of $\Gamma \vdash_{\mathsf{H}} e$

□

Lemma 2.33 : H *canonical forms*

- If $\vdash_{\mathsf{H}} v : \tau_0 \times \tau_1$ then $v = \langle v_0, v_1 \rangle$
- If $\vdash_{\mathsf{H}} v : \tau_d \Rightarrow \tau_c$ then either:
  - $v = \lambda(x\!:\!\tau_x).\, e'$
    $\wedge\ \tau_d \leqslant: \tau_x$
  - or $v = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\, v'$
    $\wedge\ \tau_d' \Rightarrow \tau_c' \leqslant: \tau_d \Rightarrow \tau_c$
- If $\vdash_{\mathsf{H}} v : \mathsf{Int}$ then $v \in i$
- If $\vdash_{\mathsf{H}} v : \mathsf{Nat}$ then $v \in \mathbb{N}$

Proof:

QED by definition of $\vdash_{\mathsf{H}} e : \tau$

□

Lemma 2.34 : Δ *type soundness*

If $\vdash_{\mathsf{H}} v_0 : \tau_0$ and $\vdash_{\mathsf{H}} v_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau$ then $\vdash_{\mathsf{H}} \delta(op^2, v_0, v_1) : \tau$.

Proof:

By case analysis on the definition of Δ.

CASE $\Delta(\mathsf{sum}, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat}$:

1. $v_0 = i_0, i_0 \in \mathbb{N}$
   $\wedge\ v_1 = i_1, i_1 \in \mathbb{N}$
   by *canonical forms*
2. $\delta(\mathsf{sum}, i_0, i_1) = i_0 + i_1 \in \mathbb{N}$
3. QED

CASE $\Delta(\mathsf{sum}, \mathsf{Int}, \mathsf{Int}) = \mathsf{Int}$:

1. $v_0 = i_0$
   $\wedge\ v_1 = i_1$
   by *canonical forms*
2. $\delta(\mathsf{sum}, i_0, i_1) = i_0 + i_1 \in i$
3. QED

CASE $\Delta(\mathsf{quotient}, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat}$:

1. $v_0 = i_0, i_0 \in \mathbb{N}$
   $\wedge\ v_1 = i_1, i_1 \in \mathbb{N}$
   by *canonical forms*
2. **IF** $i_1 = 0$:
   a. $\delta(\mathsf{quotient}, i_0, i_1) = \mathsf{BndryErr}$
   b. QED by $\vdash_{\mathsf{H}} \mathsf{BndryErr} : \tau$
   ELSE $i_1 \neq 0$:
   a. $\delta(\mathsf{quotient}, i_0, i_1) = \lfloor i_0 / i_1 \rfloor \in \mathbb{N}$
   b. QED

CASE $\Delta(\mathsf{quotient}, \mathsf{Int}, \mathsf{Int}) = \mathsf{Int}$:

1. $v_0 = i_0$
   $\wedge\ v_1 = i_1$
   by *canonical forms*
2. **IF** $i_1 = 0$:
   a. $\delta(\mathsf{quotient}, i_0, i_1) = \mathsf{BndryErr}$
   b. QED by $\vdash_{\mathsf{H}} \mathsf{BndryErr} : \tau$
   ELSE $i_1 \neq 0$:
   a. $\delta(\mathsf{quotient}, i_0, i_1) = \lfloor i_0 / i_1 \rfloor \in i$
   b. QED

□

Lemma 2.35 : δ *preservation*

- If $\vdash_{\mathsf{H}} v$ and $\delta(op^1, v) = e$ then $\vdash_{\mathsf{H}} e$
- If $\vdash_{\mathsf{H}} v_0$ and $\vdash_{\mathsf{H}} v_1$ and $\delta(op^2, v_0, v_1) = e$ then $\vdash_{\mathsf{H}} e$

Proof:

CASE $\delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0$:

1. $\vdash_{\mathsf{H}} v_0$
   by H *inversion*
2. QED

CASE $\delta(\mathsf{snd}, \langle v_0, v_1 \rangle) = v_1$:

1. $\vdash_{\mathsf{H}} v_1$
   by H *inversion*
2. QED

CASE $\delta(\mathsf{sum}, v_0, v_1) = v_0 + v_1$:

1. QED

CASE $\delta(\mathsf{quotient}, v_0, v_1) = \lfloor v_0 / v_1 \rfloor$:

1. QED

CASE $\delta(op^2, v_0, v_1) = \mathsf{BndryErr}$:

1. QED

□

Lemma 2.36 : H *substitution*

- If $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e$ and $\vdash_{\mathsf{H}} v : \tau_x$ then $\Gamma \vdash_{\mathsf{H}} e[x \leftarrow v]$
- If $x, \Gamma \vdash_{\mathsf{H}} e$ and $\vdash_{\mathsf{H}} v$ then $\Gamma \vdash_{\mathsf{H}} e[x \leftarrow v]$
- If $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e : \tau$ and $\vdash_{\mathsf{H}} v : \tau_x$ then $\Gamma \vdash_{\mathsf{H}} e[x \leftarrow v] : \tau$
- If $x, \Gamma \vdash_{\mathsf{H}} e : \tau$ and $\vdash_{\mathsf{H}} v$ then $\Gamma \vdash_{\mathsf{H}} e[x \leftarrow v] : \tau$

Proof:

By the following four lemmas: *dynamic context static value substitution*, *dynamic context dynamic value substitution*, *static context static value substitution*, and *static context dynamic value substitution*.

□

Lemma 2.37 : H *dynamic-static substitution*

If $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e$ and $\vdash_{\mathsf{H}} v : \tau_x$ then $\Gamma \vdash_{\mathsf{H}} e[x \leftarrow v]$

Proof:

By induction on the structure of e.

CASE $e = x$:

1. Contradiction by $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} x$

CASE $e = x'$:

1. QED by $(x'[x \leftarrow v]) = x'$

CASE $e = i$:

1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x'.\, e'$:

1. $e[x \leftarrow v] = \lambda x'.\, (e'[x \leftarrow v])$
2. $x', (x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e'$
   by H *inversion*
3. $x', \Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} \lambda x'.\, (e'[x \leftarrow v])$
   by (3)
5. QED

CASE $e = \lambda(x'\!:\!\tau').\, e'$:

1. Contradiction by $(x\!:\!\tau_x), \Gamma \vdash_{\mathsf{H}} e$

CASE $e = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'$:

1. $e[x \leftarrow v] = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'[x \leftarrow v]$

2. $(x : \tau_x), \Gamma \vdash_{H} v' : \tau_d \Rightarrow \tau_c$
   by H *inversion*
3. $\Gamma \vdash_{H} v'[x \leftarrow v] : \tau_d \Rightarrow \tau_c$
   by *static context static value substitution* (2)
4. $\Gamma \vdash_{H} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \langle e_0, e_1 \rangle$:
1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $(x : \tau_x), \Gamma \vdash_{H} e_0$
   $\wedge\ (x : \tau_x), \Gamma \vdash_{H} e_1$
   by H *inversion*
3. $\Gamma \vdash_{H} e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_{H} e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{H} \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
   by (3)
5. QED

CASE $e = e_0\ e_1$:
1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $(x : \tau_x), \Gamma \vdash_{H} e_0$
   $\wedge\ (x : \tau_x), \Gamma \vdash_{H} e_1$
   by H *inversion*
3. $\Gamma \vdash_{H} e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_{H} e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{H} e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^1\ e_0$:
1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $(x : \tau_x), \Gamma \vdash_{H} e_0$
   by H *inversion*
3. $\Gamma \vdash_{H} e_0[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{H} op^1\ e_0[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:
1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $(x : \tau_x), \Gamma \vdash_{H} e_0$
   $\wedge\ (x : \tau_x), \Gamma \vdash_{H} e_1$
   by H *inversion*
3. $\Gamma \vdash_{H} e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_{H} e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{H} op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \mathsf{stat}\ \tau'\ e'$:
1. $e[x \leftarrow v] = \mathsf{stat}\ \tau'\ e'[x \leftarrow v]$
2. $(x : \tau_x), \Gamma \vdash_{H} e' : \tau'$
   by H *inversion*
3. $\Gamma \vdash_{H} e'[x \leftarrow v] : \tau'$
   by *static context static value substitution* (2)
4. $\Gamma \vdash_{H} \mathsf{stat}\ \tau'\ e'[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \mathsf{Err}$:
1. QED $\mathsf{Err} = \mathsf{Err}[x \leftarrow v]$

□

Lemma 2.38 : H *dynamic-dynamic substitution*

If $x, \Gamma \vdash_{H} e$ and $\vdash_{H} v$ then $\Gamma \vdash_{H} e[x \leftarrow v]$

Proof:

By induction on the structure of e

CASE $e = x$:
1. $e[x \leftarrow v] = v$
2. $\Gamma \vdash_{H} v$
   by *weakening*
3. QED

CASE $e = x'$:
1. QED by $(x'[x \leftarrow v]) = x'$

CASE $e = i$:
1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x'.\, e'$:
1. $e[x \leftarrow v] = \lambda x'.\, (e'[x \leftarrow v])$
2. $x', x, \Gamma \vdash_{H} e'$
   by H *inversion*
3. $x', \Gamma \vdash_{H} e'[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{H} \lambda x'.\, (e'[x \leftarrow v])$
   by (3)
5. QED

CASE $e = \lambda(x' : \tau').\, e'$:
1. Contradiction by $x, \Gamma \vdash_{H} e$

CASE $e = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'$:
1. $e[x \leftarrow v] = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'[x \leftarrow v]$
2. $x, \Gamma \vdash_{H} v' : \tau_d \Rightarrow \tau_c$
   by H *inversion*
3. $\Gamma \vdash_{H} v'[x \leftarrow v] : \tau_d \Rightarrow \tau_c$
   by *static context dynamic value substitution* (2)
4. $\Gamma \vdash_{H} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \langle e_0, e_1 \rangle$:
1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $x, \Gamma \vdash_{H} e_0$
   $\wedge\ x, \Gamma \vdash_{H} e_1$
   by H *inversion*
3. $\Gamma \vdash_{H} e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_{H} e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{H} \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
   by (3)
5. QED

CASE $e = e_0\ e_1$:

1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} e_0$
   $\wedge\ x, \Gamma \vdash_{\mathsf{H}} e_1$
   by H *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^1\ e_0$:
1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} e_0$
   by H *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} op^1\ e_0[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:
1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} e_0$
   $\wedge\ x, \Gamma \vdash_{\mathsf{H}} e_1$
   by H *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \mathsf{stat}\ \tau'\ e'$:
1. $e[x \leftarrow v] = \mathsf{stat}\ \tau'\ e'[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} e' : \tau'$
   by H *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v] : \tau'$
   by *static context static value substitution* (2)
4. $\Gamma \vdash_{\mathsf{H}} \mathsf{stat}\ \tau'\ e'[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \mathsf{Err}$:
1. QED $\mathsf{Err} = \mathsf{Err}[x \leftarrow v]$

□

Lemma 2.39 : H *static-static substitution*

If $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} e : \tau$ and $\vdash_{\mathsf{H}} v : \tau_x$ then $\Gamma \vdash_{\mathsf{H}} e[x \leftarrow v] : \tau$

Proof:

By induction on the structure of e.

CASE $e = x$:
1. $e[x \leftarrow v] = v$
2. $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} x : \tau$
3. $\tau_x \leqslant: \tau$
   by H *inversion*
4. $\vdash_{\mathsf{H}} v : \tau$
   by (3)
5. $\Gamma \vdash_{\mathsf{H}} v : \tau$
   by *weakening*
6. QED

CASE $e = x'$:
1. QED by $(x'[x \leftarrow v]) = x'$

CASE $e = i$:
1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x'.\, e'$:
1. Contradiction by $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} e : \tau$

CASE $e = \lambda(x' : \tau').\, e'$:
1. $e[x \leftarrow v] = \lambda(x' : \tau').\, (e'[x \leftarrow v])$
2. $(x' : \tau'), (x : \tau_x), \Gamma \vdash_{\mathsf{H}} e' : \tau'_c$
   $\wedge\ \tau' \Rightarrow \tau'_c \leqslant: \tau$
3. $(x' : \tau'), \Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v] : \tau'_c$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} \lambda(x' : \tau').\, e' : \tau$
5. QED

CASE $e = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'$:
1. $e[x \leftarrow v] = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'[x \leftarrow v]$
2. $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} v'$
   by H *inversion*
3. $\Gamma \vdash_{\mathsf{H}} v'[x \leftarrow v]$
   by *dynamic context static value substitution* (2)
4. $\Gamma \vdash_{\mathsf{H}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = \langle e_0, e_1 \rangle$:
1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   $\wedge\ (x : \tau_x), \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$
   by H *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v] : \tau_1$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle : \tau$
   by (3)
5. QED

CASE $e = e_0\ e_1$:
1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   $\wedge\ (x : \tau_x), \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$
   by H *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v] : \tau_1$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]\ e_1[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = op^1\ e_0$:
1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   by H *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0$
   by the induction hypothesis (2)

4. $\Gamma \vdash_{\mathsf{H}} op^1\ e_0[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:

1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   $\wedge\ (x : \tau_x), \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$
   by H *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v] : \tau_1$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = \mathsf{dyn}\ \tau'\ e'$:

1. $e[x \leftarrow v] = \mathsf{dyn}\ \tau'\ e'[x \leftarrow v]$
2. $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} e'$
   by H *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v]$
   by *dynamic context static value substitution* (2)
4. $\Gamma \vdash_{\mathsf{H}} \mathsf{dyn}\ \tau'\ e'[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = \mathsf{Err}$:

1. QED by $\mathsf{Err} = \mathsf{Err}[x \leftarrow v]$

□

Lemma 2.40 : H *static-dynamic substitution*

If $x, \Gamma \vdash_{\mathsf{H}} e : \tau$ and $\vdash_{\mathsf{H}} v$ then $\Gamma \vdash_{\mathsf{H}} e[x \leftarrow v] : \tau$

Proof:

By induction on the structure of e.

CASE $e = x$:

1. Contradiction by $x, \Gamma \vdash_{\mathsf{H}} x : \tau$

CASE $e = x'$:

1. QED by $(x'[x \leftarrow v]) = x'$

CASE $e = i$:

1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x'.\, e'$:

1. Contradiction by $(x : \tau_x), \Gamma \vdash_{\mathsf{H}} e : \tau$

CASE $e = \lambda(x' : \tau').\, e'$:

1. $e[x \leftarrow v] = \lambda(x' : \tau').\, (e'[x \leftarrow v])$
2. $(x' : \tau'), x, \Gamma \vdash_{\mathsf{H}} e' : \tau'_c$
   $\wedge\ \tau' \Rightarrow \tau'_c \leqslant: \tau$
3. $(x' : \tau'), \Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v] : \tau'_c$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} \lambda(x' : \tau').\, e' : \tau$
5. QED

CASE $e = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'$:

1. $e[x \leftarrow v] = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} v'$
   by H *inversion*
3. $\Gamma \vdash_{\mathsf{H}} v'[x \leftarrow v]$
   by *dynamic context dynamic value substitution* (2)
4. $\Gamma \vdash_{\mathsf{H}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v'[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = \langle e_0, e_1 \rangle$:

1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $x, \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   $\wedge\ x, \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$
   by H *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v] : \tau_1$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle : \tau$
   by (3)
5. QED

CASE $e = e_0\ e_1$:

1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   $\wedge\ x, \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$
   by H *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v] : \tau_1$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v]\ e_1[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = op^1\ e_0$:

1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   by H *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} op^1\ e_0[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:

1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} e_0 : \tau_0$
   $\wedge\ x, \Gamma \vdash_{\mathsf{H}} e_1 : \tau_1$
   by H *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e_0[x \leftarrow v] : \tau_0$
   $\wedge\ \Gamma \vdash_{\mathsf{H}} e_1[x \leftarrow v] : \tau_1$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{H}} op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = \mathsf{dyn}\ \tau'\ e'$:

1. $e[x \leftarrow v] = \mathsf{dyn}\ \tau'\ e'[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{H}} e'$
   by H *inversion*
3. $\Gamma \vdash_{\mathsf{H}} e'[x \leftarrow v]$
   by *dynamic context dynamic value substitution* (2)
4. $\Gamma \vdash_{\mathsf{H}} \mathsf{dyn}\ \tau'\ e'[x \leftarrow v] : \tau$
   by (3)
5. QED

CASE $e = \mathsf{Err}$:

1. QED by $\mathsf{Err} = \mathsf{Err}[x \leftarrow v]$

□

Lemma 2.41 : *weakening*

- If $\Gamma \vdash_{\mathsf{H}} e$ then $x, \Gamma \vdash_{\mathsf{H}} e$
- If $\Gamma \vdash_{\mathsf{H}} e : \tau$ then $(x : \tau'), \Gamma \vdash_{\mathsf{H}} e : \tau$

Proof:

- e is closed under Γ
  by $\Gamma \vdash_{\mathsf{H}} e$
  $\vee\, \Gamma \vdash_{\mathsf{H}} e : \tau$
- QED x is unused in the derivation

□

E.3 (E) Erasure Embedding

E.3.1 Erasure Definitions

$\boxed{\text{Language E}}$

$$
\begin{array}{lcl}
e & = & x \mid v \mid \langle e, e\rangle \mid e\ e \mid op^1\ e \mid op^2\ e\ e \mid \\
 & & \mathsf{dyn}\ \tau\ e \mid \mathsf{stat}\ \tau\ e \mid \mathsf{Err} \\
v & = & i \mid \langle v, v\rangle \mid \lambda x.\, e \mid \lambda(x\!:\!\tau).\, e \\
\tau & = & \mathsf{Nat} \mid \mathsf{Int} \mid \tau \times \tau \mid \tau \Rightarrow \tau \\
\Gamma & = & \cdot \mid x, \Gamma \mid (x\!:\!\tau), \Gamma \\
\mathsf{Err} & = & \mathsf{BndryErr} \mid \mathsf{TagErr} \\
r & = & v \mid \mathsf{Err} \\
E^\bullet & = & [\,] \mid E^\bullet\ e \mid v\ E^\bullet \mid \langle E^\bullet, e\rangle \mid \langle v, E^\bullet\rangle \mid \\
 & & op^1\ E^\bullet \mid op^2\ E^\bullet\ e \mid op^2\ v\ E^\bullet \\
E & = & E^\bullet \mid E\ e \mid v\ E \mid \langle E, e\rangle \mid \langle v, E\rangle \mid op^1\ E \mid \\
 & & op^2\ E\ e \mid op^2\ v\ E \mid \mathsf{dyn}\ \tau\ E \mid \mathsf{stat}\ \tau\ E
\end{array}
$$

$\boxed{\Delta : op^1 \times \tau \longrightarrow \tau}$

$$
\begin{array}{l}
\Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0 \\
\Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1
\end{array}
$$

$\boxed{\Delta : op^2 \times \tau \times \tau \longrightarrow \tau}$

$$
\begin{array}{l}
\Delta(op^2, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat} \\
\Delta(op^2, \mathsf{Int}, \mathsf{Int}) = \mathsf{Int}
\end{array}
$$

$\boxed{\tau \leqslant: \tau}$

$$
\frac{}{\mathsf{Nat} \leqslant: \mathsf{Int}}
\qquad
\frac{\tau_d' \leqslant: \tau_d \quad \tau_c \leqslant: \tau_c'}{\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'}
\qquad
\frac{\tau_0 \leqslant: \tau_0' \quad \tau_1 \leqslant: \tau_1'}{\tau_0 \times \tau_1 \leqslant: \tau_0' \times \tau_1'}
$$

$$
\frac{}{\tau \leqslant: \tau}
\qquad
\frac{\tau \leqslant: \tau' \quad \tau' \leqslant: \tau''}{\tau \leqslant: \tau''}
$$

$\boxed{\Gamma \vdash e}$

$$
\frac{x \in \Gamma}{\Gamma \vdash x}
\qquad
\frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x.\, e}
\qquad
\frac{}{\Gamma \vdash i}
\qquad
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1\rangle}
$$

$$
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0\ e_1}
\qquad
\frac{\Gamma \vdash e}{\Gamma \vdash op^1\ e}
\qquad
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2\ e_0\ e_1}
\qquad
\frac{}{\Gamma \vdash \mathsf{Err}}
$$

$$
\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat}\ \tau\ e}
$$

$\boxed{\Gamma \vdash e : \tau}$

$$
\frac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash x : \tau}
\qquad
\frac{(x\!:\!\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c}
\qquad
\frac{i \in \mathbb{N}}{\Gamma \vdash i : \mathsf{Nat}}
$$

$$
\frac{}{\Gamma \vdash i : \mathsf{Int}}
\qquad
\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1\rangle : \tau_0 \times \tau_1}
\qquad
\frac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash e_1 : \tau_d}{\Gamma \vdash e_0\ e_1 : \tau_c}
$$

$$
\frac{\Gamma \vdash e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash op^1\ e_0 : \tau}
\qquad
\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash op^2\ e_0\ e_1 : \tau}
\qquad
\frac{\Gamma \vdash e : \tau' \quad \tau' \leqslant: \tau}{\Gamma \vdash e : \tau}
\qquad
\frac{}{\Gamma \vdash \mathsf{Err} : \tau}
$$

$$
\frac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau}
$$

$\boxed{\Gamma \vdash_{\mathsf{E}} e}$

$$
\frac{x \in \Gamma}{\Gamma \vdash_{\mathsf{E}} x}
\qquad
\frac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash_{\mathsf{E}} x}
\qquad
\frac{x, \Gamma \vdash_{\mathsf{E}} e}{\Gamma \vdash_{\mathsf{E}} \lambda x.\, e}
\qquad
\frac{(x\!:\!\tau), \Gamma \vdash_{\mathsf{E}} e}{\Gamma \vdash_{\mathsf{E}} \lambda(x\!:\!\tau).\, e}
\qquad
\frac{}{\Gamma \vdash_{\mathsf{E}} i}
$$

$$
\frac{\Gamma \vdash_{\mathsf{E}} e_0 \quad \Gamma \vdash_{\mathsf{E}} e_1}{\Gamma \vdash_{\mathsf{E}} \langle e_0, e_1\rangle}
\qquad
\frac{\Gamma \vdash_{\mathsf{E}} e_0 \quad \Gamma \vdash_{\mathsf{E}} e_1}{\Gamma \vdash_{\mathsf{E}} e_0\ e_1}
\qquad
\frac{\Gamma \vdash_{\mathsf{E}} e}{\Gamma \vdash_{\mathsf{E}} op^1\ e}
$$

$$
\frac{\Gamma \vdash_{\mathsf{E}} e_0 \quad \Gamma \vdash_{\mathsf{E}} e_1}{\Gamma \vdash_{\mathsf{E}} op^2\ e_0\ e_1}
\qquad
\frac{}{\Gamma \vdash_{\mathsf{E}} \mathsf{Err}}
\qquad
\frac{\Gamma \vdash_{\mathsf{E}} e}{\Gamma \vdash_{\mathsf{E}} \mathsf{dyn}\ \tau\ e}
\qquad
\frac{\Gamma \vdash_{\mathsf{E}} e}{\Gamma \vdash_{\mathsf{E}} \mathsf{stat}\ \tau\ e}
$$

$\boxed{\delta(op^1, v) = e}$

$$
\begin{array}{l}
\delta(\mathsf{fst}, \langle v_0, v_1\rangle) = v_0 \\
\delta(\mathsf{snd}, \langle v_0, v_1\rangle) = v_1
\end{array}
$$

$\boxed{\delta(op^2, v, v) = e}$

$$
\begin{array}{lcl}
\delta(\mathsf{sum}, i_0, i_1) & = & i_0 + i_1 \\
\delta(\mathsf{quotient}, i_0, 0) & = & \mathsf{BndryErr} \\
\delta(\mathsf{quotient}, i_0, i_1) & = & \lfloor i_0 / i_1 \rfloor \\
\quad \text{if } i_1 \neq 0 & &
\end{array}
$$

$\boxed{\mathcal{D}_{\mathsf{E}} : \tau \times v \longrightarrow e}$

$$\mathcal{D}_{\mathsf{E}}(\tau, v) = v$$

$\boxed{\mathcal{S}_{\mathsf{E}} : \tau \times v \longrightarrow e}$

$$\mathcal{S}_{\mathsf{E}}(\tau, v) = v$$

$\boxed{e \rhd_{\mathsf{E\text{-}S}} e}$

$$\begin{array}{lll}
\mathsf{dyn}\ \tau\ v & \rhd_{\mathsf{E\text{-}S}} & \mathcal{D}_{\mathsf{E}}(\tau, v) \\
\mathsf{stat}\ \tau\ v & \rhd_{\mathsf{E\text{-}S}} & \mathcal{S}_{\mathsf{E}}(\tau, v) \\
v_0\ v_1 & \rhd_{\mathsf{E\text{-}S}} & \mathsf{TagErr} \\
\quad \text{if } v_0 \in \mathbb{Z} \text{ or } v_0 = \langle v, v' \rangle & & \\
(\lambda(x\!:\!\tau).\, e)\ v & \rhd_{\mathsf{E\text{-}S}} & e[x \leftarrow v] \\
(\lambda x.\, e)\ v & \rhd_{\mathsf{E\text{-}S}} & e[x \leftarrow v] \\
op^1\ v & \rhd_{\mathsf{E\text{-}S}} & \mathsf{TagErr} \\
\quad \text{if } \delta(op^1, v) \text{ is undefined} & & \\
op^1\ v & \rhd_{\mathsf{E\text{-}S}} & \delta(op^1, v) \\
op^2\ v_0\ v_1 & \rhd_{\mathsf{E\text{-}S}} & \mathsf{TagErr} \\
\quad \text{if } \delta(op^2, v_0, v_1) \text{ is undefined} & & \\
op^2\ v_0\ v_1 & \rhd_{\mathsf{E\text{-}S}} & \delta(op^2, v_0, v_1)
\end{array}$$

$\boxed{e \rhd_{\mathsf{E\text{-}D}} e}$

$$\begin{array}{lll}
\mathsf{stat}\ \tau\ v & \rhd_{\mathsf{E\text{-}D}} & \mathcal{S}_{\mathsf{E}}(\tau, v) \\
\mathsf{dyn}\ \tau\ v & \rhd_{\mathsf{E\text{-}D}} & \mathcal{D}_{\mathsf{E}}(\tau, v) \\
v_0\ v_1 & \rhd_{\mathsf{E\text{-}D}} & \mathsf{TagErr} \\
\quad \text{if } v_0 \in \mathbb{Z} \text{ or } v_0 = \langle v, v' \rangle & & \\
(\lambda(x\!:\!\tau).\, e)\ v & \rhd_{\mathsf{E\text{-}D}} & e[x \leftarrow v] \\
(\lambda x.\, e)\ v & \rhd_{\mathsf{E\text{-}D}} & e[x \leftarrow v] \\
op^1\ v & \rhd_{\mathsf{E\text{-}D}} & \mathsf{TagErr} \\
\quad \text{if } \delta(op^1, v) \text{ is undefined} & & \\
op^1\ v & \rhd_{\mathsf{E\text{-}D}} & \delta(op^1, v) \\
op^2\ v_0\ v_1 & \rhd_{\mathsf{E\text{-}D}} & \mathsf{TagErr} \\
\quad \text{if } \delta(op^2, v_0, v_1) \text{ is undefined} & & \\
op^2\ v_0\ v_1 & \rhd_{\mathsf{E\text{-}D}} & \delta(op^2, v_0, v_1)
\end{array}$$

$\boxed{e \rightarrow_{\mathsf{E\text{-}S}} e}$

$$\begin{array}{lll}
E[e] & \rightarrow_{\mathsf{E\text{-}S}} & E[e'] \\
\quad \text{if } e \rhd_{\mathsf{E\text{-}S}} e' & & \\
E[\mathsf{Err}] & \rightarrow_{\mathsf{E\text{-}S}} & \mathsf{Err}
\end{array}$$

$\boxed{e \rightarrow_{\mathsf{E\text{-}D}} e}$

$$\begin{array}{lll}
E[e] & \rightarrow_{\mathsf{E\text{-}D}} & E[e'] \\
\quad \text{if } e \rhd_{\mathsf{E\text{-}D}} e' & & \\
E[\mathsf{Err}] & \rightarrow_{\mathsf{E\text{-}D}} & \mathsf{Err}
\end{array}$$

$\boxed{e \rightarrow^*_{\mathsf{E\text{-}S}} e}$ reflexive, transitive closure of $\rightarrow_{\mathsf{E\text{-}S}}$

$\boxed{e \rightarrow^*_{\mathsf{E\text{-}D}} e}$ reflexive, transitive closure of $\rightarrow_{\mathsf{E\text{-}D}}$

E.3.2 Erasure Theorems

Theorem 3.0 : *static* E*-soundness*

If $\vdash e : \tau$ then $\vdash_{\mathsf{E}} e$ and one of the following holds:

- $e \rightarrow^{*}_{\mathsf{E\text{-}S}} v$ and $\vdash_{\mathsf{E}} v$
- $e \rightarrow^{*}_{\mathsf{E\text{-}S}}$ TagErr
- $e \rightarrow^{*}_{\mathsf{E\text{-}S}}$ BndryErr
- e diverges

Proof:

1. $\vdash_{\mathsf{E}} e$
   by *static subset*
2. QED by E *progress* and E *preservation*

□

Theorem 3.1 : *dynamic* E*-soundness*

If $\vdash e$ then $\vdash_{\mathsf{E}} e$ and one of the following holds:

- $e \rightarrow^{*}_{\mathsf{E\text{-}D}} v$ and $\vdash_{\mathsf{E}} v$
- $e \rightarrow^{*}_{\mathsf{E\text{-}D}}$ TagErr
- $e \rightarrow^{*}_{\mathsf{E\text{-}D}}$ BndryErr
- e diverges

Proof:

1. $\rightarrow^{*}_{\mathsf{E\text{-}D}} = \rightarrow^{*}_{\mathsf{E\text{-}S}}$
   by definition
2. QED by *static* E *soundness*

□

Remark 3.2 : E*-compilation*

The $\rightarrow^{*}_{\mathsf{E\text{-}S}}$ and $\rightarrow^{*}_{\mathsf{E\text{-}D}}$ relations are identical. In practice, uses of $\rightarrow^{*}_{\mathsf{E\text{-}S}}$ may be replaced with $\rightarrow^{*}_{\mathsf{E\text{-}D}}$.

Theorem 3.3 : *boundary-free* E*-soundness*

If $\vdash e : \tau$ and e is boundary-free then one of the following holds:

- $e \rightarrow^{*}_{\mathsf{E\text{-}S}} v$ and $\vdash v : \tau$
- $e \rightarrow^{*}_{\mathsf{E\text{-}S}}$ BndryErr
- e diverges

Proof:

QED by *boundary-free progress* and *broundary-free preservation*.

□

E.3.3 Erasure Lemmas

Lemma 3.4 : *$\mathcal{D}_{\mathsf{E}}$ soundness*

If $\vdash_{\mathsf{E}} v$ then $\vdash_{\mathsf{E}} \mathcal{D}_{\mathsf{E}}(\tau, v)$.

Proof:

CASE $\mathcal{D}_{\mathsf{E}}(\tau, v) = v$:

1. QED

□

Lemma 3.5 : *$\mathcal{S}_{\mathsf{E}}$ soundness*

If $\vdash_{\mathsf{E}} v$ then $\vdash_{\mathsf{E}} \mathcal{S}_{\mathsf{E}}(\tau, v)$.

Proof:

CASE $\mathcal{S}_{\mathsf{E}}(\tau, v) = v$:

1. QED

□

Lemma 3.6 : *static subset*

If $\Gamma \vdash e : \tau$ then $\Gamma \vdash_{\mathsf{E}} e$.

Proof:

By structural induction on the typing relation.

CASE $\boxed{\dfrac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash x : \tau}}$:

1. $(x\!:\!\tau) \in \Gamma$
2. $\Gamma \vdash_{\mathsf{E}} x$
   by (1)
3. QED

CASE $\boxed{\dfrac{(x\!:\!\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c}}$:

1. $(x\!:\!\tau_d), \Gamma \vdash_{\mathsf{E}} e$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} \lambda(x\!:\!\tau_d).\, e$
   by (1)
3. QED

CASE $\boxed{\dfrac{i \in \mathbb{N}}{\Gamma \vdash i : \mathsf{Nat}}}$:

1. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash_{\mathsf{E}} i : \mathsf{Int}}}$:

1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1}}$:

1. $\Gamma \vdash_{\mathsf{E}} e_0$
   $\wedge\, \Gamma \vdash_{\mathsf{E}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} \langle e_0, e_1 \rangle$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash e_1 : \tau_d}{\Gamma \vdash e_0\ e_1 : \tau_c}}$:

1. $\Gamma \vdash_{\mathsf{E}} e_0$
   $\wedge\, \Gamma \vdash_{\mathsf{E}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} e_0\ e_1$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash op^1\ e_0 : \tau}}$:

1. $\Gamma \vdash_{\mathsf{E}} e_0$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} op^1\ e_0$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash op^2\ e_0\ e_1 : \tau}}$:

1. $\Gamma \vdash_{\mathsf{E}} e_0$
   $\wedge\, \Gamma \vdash_{\mathsf{E}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} op^2\ e_0\ e_1$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau' \quad \tau' <: \tau}{\Gamma \vdash e : \tau}}$:

1. $\Gamma \vdash_{\mathsf{E}} e$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash \mathsf{Err} : \tau}}$:

1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau}}$:

1. $\Gamma \vdash_{\mathsf{E}} e$
   by *dynamic subset*
2. $\Gamma \vdash_{\mathsf{E}} \mathsf{dyn}\ \tau\ e$
   by (1)
3. QED

□

Lemma 3.7 : *dynamic subset*

If $\Gamma \vdash e$ then $\Gamma \vdash_{\mathsf{E}} e$.

Proof:

By structural induction on the $\vdash$ relation.

CASE $\boxed{\dfrac{x \in \Gamma}{\Gamma \vdash x}}$:

1. $x \in \Gamma$
2. $\Gamma \vdash_{\mathsf{E}} x$
   by (1)
3. QED

CASE $\boxed{\dfrac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x.\, e}}$:

1. $x, \Gamma \vdash_{\mathsf{E}} e$
   by the induction hypothesis

2. $\Gamma \vdash_{\mathsf{E}} \lambda x.\, e$
   by (1)
3. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash i}}$:

1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}}$:

1. $\Gamma \vdash_{\mathsf{E}} e_0$
   $\wedge\ \Gamma \vdash_{\mathsf{E}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} \langle e_0, e_1 \rangle$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0\ e_1}}$:

1. $\Gamma \vdash_{\mathsf{E}} e_0$
   $\wedge\ \Gamma \vdash_{\mathsf{E}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} e_0\ e_1$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e}{\Gamma \vdash op^1\ e}}$:

1. $\Gamma \vdash_{\mathsf{E}} e_0$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} op^1\ e_0$
   by (1)
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2\ e_0\ e_1}}$:

1. $\Gamma \vdash_{\mathsf{E}} e_0$
   $\wedge\ \Gamma \vdash_{\mathsf{E}} e_1$
   by the induction hypothesis
2. $\Gamma \vdash_{\mathsf{E}} op^2\ e_0\ e_1$
   by (1)
3. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash \mathsf{Err}}}$:

1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat}\ \tau\ e}}$:

1. $\Gamma \vdash_{\mathsf{E}} e$
   by *static subset*
2. $\Gamma \vdash_{\mathsf{E}} \mathsf{stat}\ \tau\ e$
   by (1)
3. QED

☐

Lemma 3.8 : E *progress*

If $\vdash_{\mathsf{E}} e$ then one of the following holds:

- e is a value
- $e \in \mathsf{Err}$
- $e \rightarrow_{\mathsf{E\text{-}S}} e'$
- $e \rightarrow_{\mathsf{E\text{-}S}} \mathsf{TagErr}$
- $e \rightarrow_{\mathsf{E\text{-}S}} \mathsf{BndryErr}$

Proof:

By the *unique evaluation contexts* lemma, there are seven possible cases.

CASE e is a value :

1. QED

CASE $e = E[v_0\ v_1]$:

IF $v_0 = \lambda x.\, e'$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} E[e'[x \leftarrow v_1]]$
   by $v_0\ v_1 \rhd_{\mathsf{E\text{-}S}} e'[x \leftarrow v_1]$
2. QED

IF $v_0 = \lambda(x\!:\!\tau).\, e'$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} E[e'[x \leftarrow v_1]]$
   by $v_0\ v_1 \rhd_{\mathsf{E\text{-}S}} e'[x \leftarrow v_1]$
2. QED

ELSE $v_0 = i$
$\vee\ v_0 = \langle v, v' \rangle$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} \mathsf{TagErr}$
   by $v_0\ v_1 \rhd_{\mathsf{E\text{-}S}} \mathsf{TagErr}$
2. QED

CASE $e = E[op^1\ v]$:

IF $\delta(op^1, v) = e''$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} E[e'']$
   by $(op^1\ v) \rhd_{\mathsf{E\text{-}S}} e''$
2. QED

ELSE $\delta(op^1, v)$ is undefined :

1. $e \rightarrow_{\mathsf{E\text{-}S}} \mathsf{TagErr}$
   by $(op^1\ v) \rhd_{\mathsf{E\text{-}S}} \mathsf{TagErr}$
2. QED

CASE $e = E[op^2\ v_0\ v_1]$:

IF $\delta(op^2, v_0, v_1) = e''$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} E[e'']$
   by $(op^2\ v_0\ v_1) \rhd_{\mathsf{E\text{-}S}} e''$
2. QED

IF $\delta(op^2, v_0, v_1) = \mathsf{BndryErr}$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} \mathsf{BndryErr}$
   by $(op^2\ v_0\ v_1) \rhd_{\mathsf{E\text{-}S}} \mathsf{BndryErr}$
2. QED

ELSE $\delta(op^2, v_0, v_1)$ is undefined :

1. $e \rightarrow_{\mathsf{E\text{-}S}} \mathsf{TagErr}$
   by $(op^2\ v_0\ v_1) \rhd_{\mathsf{E\text{-}S}} \mathsf{TagErr}$
2. QED

CASE $e = E[\mathsf{dyn}\ \tau\ v]$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} E[\mathcal{D}_{\mathsf{E}}(\tau, v)]$
2. QED

CASE $e = E[\mathsf{stat}\ \tau\ v]$:

1. $e \rightarrow_{\mathsf{E\text{-}S}} E[\mathcal{S}_{\mathsf{E}}(\tau, v)]$
2. QED

CASE $eE[\mathsf{Err}]$:

1. $e \rightarrow_{\text{E-S}} \text{Err}$
2. QED

□

Lemma 3.9 : E *preservation*

If $\Vdash_{\text{E}} e$ and $e \rightarrow_{\text{E-S}} e'$ then $\Vdash_{\text{E}} e'$.

Proof:

By *unique evaluation contexts* there are seven cases.

CASE e is a value :

1. Contradiction by $e \rightarrow_{\text{E-S}} e'$

CASE $e = E[v_0\ v_1]$:

1. $v_0 = \lambda x.\, e'$ or $v_0 = \lambda(x\!:\!\tau).\, e'$
   $\wedge\ E[v_0\ v_1] \rightarrow_{\text{E-S}} E[e'[x \leftarrow v_1]]$
2. $\Vdash_{\text{E}} v_0\ v_1$
   by *hole typing*
3. $\Vdash_{\text{E}} v_0$
   $\wedge \Vdash_{\text{E}} v_1$
   by E *inversion* (2)
4. $x \Vdash_{\text{E}} e'$
   by E *inversion* (3)
5. $\Vdash_{\text{E}} e'[x \leftarrow v_1]$
   by *substitution* (3, 4)
6. QED by *hole substitution* (5)

CASE $e = E[op^1\ v]$:

1. $E[op^1\ v] \rightarrow_{\text{E-S}} E[v']$
   $\wedge\ \delta(op^1, v) = e''$
2. $\Vdash_{\text{E}} op^1\ v$
   by *hole typing*
3. $\Vdash_{\text{E}} v$
   by E *inversion* (2)
4. $\Vdash_{\text{E}} e''$
   by δ *preservation* (1,3)
5. QED by *hole substitution* (4)

CASE $e = E[op^2\ v_0\ v_1]$:

1. $E[op^2\ v_0\ v_1] \rightarrow_{\text{E-S}} E[v']$
   $\wedge\ \delta(op^2, v_0, v_1) = e''$
2. $\Vdash_{\text{E}} op^2\ v_0\ v_1$
   by *hole typing*
3. $\Vdash_{\text{E}} v_0$
   $\wedge \Vdash_{\text{E}} v_1$
   by E *inversion* (2)
4. $\Vdash_{\text{E}} e''$
   by δ *preservation* (3)
5. QED by *hole substitution* (4)

CASE $e = E[\text{dyn}\ \tau\ v]$:

1. $E[\text{dyn}\ \tau\ v] \rightarrow_{\text{E-S}} E[\mathcal{D}_{\text{E}}(\tau, v)]$
2. $\Vdash_{\text{E}} \text{dyn}\ \tau\ v$
   by *hole typing*
3. $\Vdash_{\text{E}} v$
   by E *inversion* (2)
4. $\Vdash_{\text{E}} \mathcal{D}_{\text{E}}(\tau, v)$
   by $\mathcal{D}_{\text{E}}$ *soundness* (3)
5. QED by *hole substitution* (4)

CASE $e = E[\text{stat}\ \tau\ v]$:

1. $E[\text{stat}\ \tau\ v] \rightarrow_{\text{E-S}} \mathcal{S}_{\text{E}}(\tau, v)$
2. $\Vdash_{\text{E}} \text{stat}\ \tau\ v$
   by *hole typing*
3. $\Vdash_{\text{E}} v$
   by E *inversion* (2)
4. $\Vdash_{\text{E}} \mathcal{S}_{\text{E}}(\tau, v)$
   by $\mathcal{S}_{\text{E}}$ *soundness* (3)
5. QED by *hole substitution* (4)

CASE $e = E[\text{Err}]$:

1. $E[\text{Err}] \rightarrow_{\text{E-S}} \text{Err}$
2. QED

□

Lemma 3.10 : E *boundary-free progress*

If $\vdash e : \tau$ and e is boundary-free, then one of the following holds:

- e is a value
- $e \in \text{Err}$
- $e \rightarrow_{\text{E-S}} e'$
- $e \rightarrow_{\text{E-S}} \text{BndryErr}$

Proof:

By the *unique static evaluation contexts* lemma, there are five cases:

CASE $e = v$:

1. QED

CASE $e = E[v_0\ v_1]$:

IF $v_0 = \lambda(x\!:\!\tau').\, e'$:

1. $e \rightarrow_{\text{E-S}} E[e'[x \leftarrow v_1]]$
   by $v_0\ v_1 \rhd_{\text{E-S}} e'[x \leftarrow v_1]$
2. QED

ELSE $v_0 = \lambda x.\, e'$
$\vee\ v_0 = i$
$\vee\ v_0 = \langle v, v' \rangle$:

1. Contradiction by $\vdash e : \tau$

CASE $e = E[op^1\ v]$:

IF $\delta(op^1, v) = e''$:

1. $e \rightarrow_{\text{E-S}} E[e'']$
   by $(op^1\ v) \rhd_{\text{E-S}} e''$
2. QED

ELSE $\delta(op^1, v)$ is undefined :

1. Contradiction by $\vdash e : \tau$

CASE $e = E[op^2\ v_0\ v_1]$:

IF $\delta(op^2, v_0, v_1) = e''$:

1. $e \rightarrow_{\text{E-S}} E[e'']$
   by $(op^2\ v_0\ v_1) \rhd_{\text{E-S}} e''$
2. QED

IF $\delta(op^2, v_0, v_1) = \text{BndryErr}$:

1. $e \rightarrow_{\text{E-S}} \text{BndryErr}$
   by $(op^2\ v_0\ v_1) \rhd_{\text{E-S}} \text{BndryErr}$
2. QED

ELSE $\delta(op^2, v_0, v_1)$ is undefined :

1. Contradiction by $\vdash e : \tau$

CASE $e = E[\text{Err}]$:

1. $E[\text{Err}] \rightarrow_{\text{E-S}} \text{Err}$
2. QED

□

Lemma 3.11 : E *boundary-free preservation*

If $\vdash e : \tau$ and e is boundary-free and $e \rightarrow_{\text{E-S}} e'$ then $\vdash e' : \tau$ and e' is boundary-free.

Proof:

By the *unique static evaluation contexts* lemma, there are five cases.

CASE e is a value :

1. Contradiction by $e \rightarrow_{\text{E-S}} e'$

CASE $e = E[v_0\ v_1]$:

IF $v_0 = \lambda(x\!:\!\tau_d).\, e'$:

1. $E[v_0\ v_1] \rightarrow_{\text{E-S}} E[e'[x \leftarrow v_1]]$
2. $\vdash v_0\ v_1 : \tau_c$
3. $\vdash v_0 : \tau_d \Rightarrow \tau_c$
   $\wedge \vdash v_1 : \tau_d$
   by (2)
4. $(x\!:\!\tau_d) \vdash e' : \tau_c$
   by (3)
5. $\vdash e'[x \leftarrow v_1] : \tau_c$
   by *substitution* (3, 4)
6. $e'[x \leftarrow v_1]$ is boundary-free
   by e' and v_1 are boundary-free
7. QED

ELSE :

1. Contradiction by $\vdash e : \tau$

CASE $e = E[op^1\ v]$:

1. $E[op^1\ v] \rightarrow_{\text{E-S}} E[v']$
   $\wedge\ \delta(op^1, v) = e''$
2. $\vdash op^1\ v : \tau'$
3. $\vdash v : \tau_0$
4. $\vdash e'' : \tau'$
   by *δ preservation* (3)
5. QED

CASE $e = E[op^2\ v_0\ v_1]$:

1. $E[op^2\ v_0\ v_1] \rightarrow_{\text{E-S}} E[v']$
   $\wedge\ \delta(op^2, v_0, v_1) = e''$
2. $\vdash op^2\ v_0\ v_1 : \tau'$
3. $\vdash v_0 : \tau_0$
   $\wedge \vdash v_1 : \tau_1$
4. $\vdash e'' : \tau'$
   by *δ preservation* (3)
5. QED

CASE $e = E[\text{Err}]$:

1. $E[\text{Err}] \rightarrow_{\text{E-S}} \text{Err}$
2. QED by $\vdash \text{Err} : \tau$

□

Lemma 3.12 : E *unique evaluation contexts*

If $\vdash_{\text{E}} e$ then one of the following holds:

- e is a value
- $e = E[v_0\ v_1]$
- $e = E[op^1\ v]$
- $e = E[op^2\ v_0\ v_1]$
- $e = E[\text{dyn}\ \tau\ v]$
- $e = E[\text{stat}\ \tau\ v]$
- $e = E[\text{Err}]$

Proof:

By induction on the structure of e.

CASE $e = x$:

1. Contradiction by $\vdash_{\text{E}} e$

CASE $e = i$
$\vee\ e = \lambda x.\, e'$
$\vee\ e = \lambda(x\!:\!\tau_d).\, e'$:

1. QED

CASE $e = \langle e_0, e_1 \rangle$:

IF $e_0 \notin v$:

1. $\vdash_{\text{E}} e_0$
   by E *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis
3. $E = \langle E_0, e_1 \rangle$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\ e_1 \notin v$:

1. $\vdash_{\text{E}} e_1$
   by E *inversion*
2. $e_1 = E_1[e_1']$
   by the induction hypothesis
3. $E = \langle e_0, E_1 \rangle$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\ e_1 \in v$:

1. QED

CASE $e = e_0\ e_1$:

IF $e_0 \notin v$:

1. $\vdash_{\text{E}} e_0$
   by E *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis
3. $E = E_0\ e_1$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\ e_1 \notin v$:

1. $\vdash_{\text{E}} e_1$
   by E *inversion*
2. $e_1 = E_1[e_1']$
   by the induction hypothesis
3. $E = e_0\ E_1$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\ e_1 \in v$:

1. $E = [\,]$

2. QED

CASE $e = op^1\ e_0$:

IF $e_0 \notin v$:

1. $\vdash_{\mathsf{E}} e_0$
   by E *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis
3. $E = op^1\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:

1. $E = [\,]$
2. QED

CASE $e = op^2\ e_0\ e_1$:

IF $e_0 \notin v$:

1. $\vdash_{\mathsf{E}} e_0$
   by E *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis
3. $E = op^2\ E_0\ e_1$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\ e_1 \notin v$:

1. $\vdash_{\mathsf{E}} e_1$
   by E *inversion*
2. $e_1 = E_1[e_1']$
   by the induction hypothesis
3. $E = op^2\ e_0\ E_1$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\ e_1 \in v$:

1. $E = [\,]$
2. QED

CASE $e = \mathsf{dyn}\ \tau\ e_0$:

IF $e_0 \notin v$:

1. $\vdash_{\mathsf{E}} e_0$
   by E *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis
3. $E = \mathsf{dyn}\ \tau\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:

1. $E = [\,]$
2. QED

CASE $e = \mathsf{stat}\ \tau\ e_0$:

IF $e_0 \notin v$:

1. $\vdash_{\mathsf{E}} e_0$
   by E *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis
3. $E = \mathsf{stat}\ \tau\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:

1. $E = [\,]$
2. QED

CASE $e = \mathsf{Err}$:

1. $E = [\,]$
2. QED

□

Lemma 3.13 : E *hole typing*

If $\vdash_{\mathsf{E}} E[e]$ then the derivation contains a sub-term $\vdash_{\mathsf{E}} e$

Proof:

By induction on the structure of E.

CASE $E = [\,]$:

1. QED $E[e] = e$

CASE $E = E_0\ e_1$:

1. $E[e] = E_0[e]\ e_1$
2. $\vdash_{\mathsf{E}} E_0[e]$
   by E *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:

1. $E[e] = v_0\ E_1[e]$
2. $\vdash_{\mathsf{E}} E_1[e]$
   by E *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:

1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_{\mathsf{E}} E_0[e]$
   by E *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:

1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_{\mathsf{E}} E_1[e]$
   by E *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:

1. $E[e] = op^1\ E_0[e]$
2. $\vdash_{\mathsf{E}} E_0[e]$
   by E *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:

1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_{\mathsf{E}} E_0[e]$
   by E *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:

1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_{\mathsf{E}} E_1[e]$
   by E *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau\ E_0$:

1. $E[e] = \mathsf{dyn}\ \tau\ E_0[e]$
2. $\vdash_{\mathsf{E}} E_0[e]$
   by E *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{stat}\ \tau\ E_0$:

1. $E[e] = \mathsf{stat}\ \tau\ E_0[e]$

2. $\vdash_{\mathsf{E}} E_0[e]$
by E *inversion*
3. QED by the induction hypothesis (2)

□

Lemma 3.14 : E *hole substitution*

If $\vdash_{\mathsf{E}} E[e]$ and $\vdash_{\mathsf{E}} e'$ then $\vdash_{\mathsf{E}} E[e']$

Proof:

By induction on the structure of E.

CASE $E = [\,]$:
1. QED $E[e'] = e'$

CASE $E = \langle E_0, e_1\rangle$:
1. $E[e] = \langle E_0[e], e_1\rangle$
$\wedge\ E[e'] = \langle E_0[e'], e_1\rangle$
2. $\vdash_{\mathsf{E}} \langle E_0[e], e_1\rangle$
3. $\vdash_{\mathsf{E}} E_0[e]$
$\wedge \vdash_{\mathsf{E}} e_1$
by E *inversion*
4. $\vdash_{\mathsf{E}} E_0[e']$
by the induction hypothesis (3)
5. $\vdash_{\mathsf{E}} \langle E_0[e'], e_1\rangle$
by (3, 4)
6. QED by (1, 5)

CASE $E = \langle v_0, E_1\rangle$:
1. $E[e] = \langle v_0, E_1[e]\rangle$
$\wedge\ E[e'] = \langle v_0, E_1[e']\rangle$
2. $\vdash_{\mathsf{E}} \langle v_0, E_1[e]\rangle$
3. $\vdash_{\mathsf{E}} v_0$
$\wedge \vdash_{\mathsf{E}} E_1[e]$
by E *inversion*
4. $\vdash_{\mathsf{E}} E_1[e']$
by the induction hypothesis (3)
5. $\vdash_{\mathsf{E}} \langle v_0, E_1[e']\rangle$
by (3, 4)
6. QED by (1, 5)

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
$\wedge\ E[e'] = E_0[e']\ e_1$
2. $\vdash_{\mathsf{E}} E_0[e]\ e_1$
3. $\vdash_{\mathsf{E}} E_0[e]$
$\wedge \vdash_{\mathsf{E}} e_1$
by E *inversion*
4. $\vdash_{\mathsf{E}} E_0[e']$
by the induction hypothesis (3)
5. $\vdash_{\mathsf{E}} E_0[e']\ e_1$
by (3, 4)
6. QED by (1, 5)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
$\wedge\ E[e'] = v_0\ E_1[e']$
2. $\vdash_{\mathsf{E}} v_0\ E_1[e]$
3. $\vdash_{\mathsf{E}} v_0$
$\wedge \vdash_{\mathsf{E}} E_1[e]$
by E *inversion*
4. $\vdash_{\mathsf{E}} E_1[e']$
by the induction hypothesis (3)
5. $\vdash_{\mathsf{E}} v_0\ E_1[e']$
by (3, 4)
6. QED by (1, 5)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
$\wedge\ E[e'] = op^1\ E_0[e']$
2. $\vdash_{\mathsf{E}} op^1\ E_0[e]$
3. $\vdash_{\mathsf{E}} E_0[e]$
by E *inversion*
4. $\vdash_{\mathsf{E}} E_0[e']$
by the induction hypothesis (3)
5. $\vdash_{\mathsf{E}} op^1\ E_0[e']$
by (3, 4)
6. QED by (1, 5)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
$\wedge\ E[e'] = op^2\ E_0[e']\ e_1$
2. $\vdash_{\mathsf{E}} op^2\ E_0[e]\ e_1$
3. $\vdash_{\mathsf{E}} E_0[e]$
$\wedge \vdash_{\mathsf{E}} e_1$
by E *inversion*
4. $\vdash_{\mathsf{E}} E_0[e']$
by the induction hypothesis (3)
5. $\vdash_{\mathsf{E}} op^2\ E_0[e']\ e_1$
by (3, 4)
6. QED by (1, 5)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
$\wedge\ E[e'] = op^2\ v_0\ E_1[e']$
2. $\vdash_{\mathsf{E}} op^2\ v_0\ E_1[e]$
3. $\vdash_{\mathsf{E}} v_0$
$\wedge \vdash_{\mathsf{E}} E_1[e]$
by E *inversion*
4. $\vdash_{\mathsf{E}} E_1[e']$
by the induction hypothesis (3)
5. $\vdash_{\mathsf{E}} op^2\ v_0\ E_1[e']$
by (3, 4)
6. QED by (1, 5)

CASE $E = \mathsf{dyn}\ \tau\ E_0$:
1. $E[e] = \mathsf{dyn}\ \tau\ E_0[e]$
$\wedge\ E[e'] = \mathsf{dyn}\ \tau\ E_0[e']$
2. $\vdash_{\mathsf{E}} \mathsf{dyn}\ \tau\ E_0[e]$
3. $\vdash_{\mathsf{E}} E_0[e]$
by E *inversion*
4. $\vdash_{\mathsf{E}} E_0[e']$
by the induction hypothesis (3)
5. $\vdash_{\mathsf{E}} \mathsf{dyn}\ \tau\ E_0[e']$
by (3, 4)
6. QED by (1, 5)

CASE $E = \mathsf{stat}\ \tau\ E_0$:
1. $E[e] = \mathsf{stat}\ \tau\ E_0[e]$
$\wedge\ E[e'] = \mathsf{stat}\ \tau\ E_0[e']$

2. $\vdash_{\mathsf{E}}$ stat τ $E_0[e]$
3. $\vdash_{\mathsf{E}} E_0[e]$
   by E *inversion*
4. $\vdash_{\mathsf{E}} E_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{E}}$ stat τ $E_0[e']$
   by (3, 4)
6. QED by (1, 5)

□

Lemma 3.15 : $\vdash_{\mathsf{E}}$ *inversion*

- If $\Gamma \vdash_{\mathsf{E}} e_0\ e_1$ then $\Gamma \vdash_{\mathsf{E}} e_0$ and $\Gamma \vdash_{\mathsf{E}} e_1$
- If $\Gamma \vdash_{\mathsf{E}} \lambda x.\, e$ then $x, \Gamma \vdash_{\mathsf{E}} e$
- If $\Gamma \vdash_{\mathsf{E}} \lambda(x\!:\!\tau).\, e$ then $(x\!:\!\tau), \Gamma \vdash_{\mathsf{E}} e$
- If $\Gamma \vdash_{\mathsf{E}} op^1\ e$ then $\Gamma \vdash_{\mathsf{E}} e$
- If $\Gamma \vdash_{\mathsf{E}} op^2\ e_0\ e_1$ then $\Gamma \vdash_{\mathsf{E}} e_0$ and $\Gamma \vdash_{\mathsf{E}} e_1$
- If $\Gamma \vdash_{\mathsf{E}}$ dyn τ e then $\Gamma \vdash_{\mathsf{E}} e$
- If $\Gamma \vdash_{\mathsf{E}}$ stat τ e then $\Gamma \vdash_{\mathsf{E}} e$

Proof:

QED by the definition of $\vdash_{\mathsf{E}}$ e.

□

Lemma 3.16 : E *substitution*

If $x, \Gamma \vdash_{\mathsf{E}} e$ or $(x\!:\!\tau), \Gamma \vdash_{\mathsf{E}} e$, and $\vdash_{\mathsf{E}} v$ then $\Gamma \vdash_{\mathsf{E}} e[x \leftarrow v]$

Proof:

By induction on the structure of e.

CASE $e = x$:
1. $e[x \leftarrow v] = v$
2. $\Gamma \vdash_{\mathsf{E}} v$
   by *weakening*
3. QED

CASE $e = x'$:
1. QED by $x'[x \leftarrow v] = x'$

CASE $e = i$:
1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x.\, e'$:
1. QED by $(\lambda x.\, e')[x \leftarrow v] = \lambda x.\, e'$

CASE $e = \lambda(x\!:\!\tau').\, e'$:
1. QED by $(\lambda(x\!:\!\tau').\, e')[x \leftarrow v] = \lambda(x\!:\!\tau').\, e'$

CASE $e = \lambda x'.\, e'$:
1. $e[x \leftarrow v] = \lambda x'.\, (e'[x \leftarrow v])$
2. $x', x, \Gamma \vdash_{\mathsf{E}} e'$
   by E *inversion*
3. $x', \Gamma \vdash_{\mathsf{E}} e'[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{E}} \lambda x'.\, e'[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \lambda(x'\!:\!\tau').\, e'$:
1. $e[x \leftarrow v] = \lambda(x'\!:\!\tau').\, (e'[x \leftarrow v])$
2. $(x'\!:\!\tau'), x, \Gamma \vdash_{\mathsf{E}} e'$
   by E *inversion*
3. $(x'\!:\!\tau'), \Gamma \vdash_{\mathsf{E}} e'[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{E}} \lambda(x'\!:\!\tau').\, (e'[x \leftarrow v])$
   by (3)
5. QED

CASE $e = \langle e_0, e_1 \rangle$:
1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $x, \Gamma \vdash_{\mathsf{E}} e_0$
   $\wedge\ x, \Gamma \vdash_{\mathsf{E}} e_1$
   by E *inversion*
3. $\Gamma \vdash_{\mathsf{E}} e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_{\mathsf{E}} e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{E}} \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
   by (3)
5. QED

CASE $e = e_0\ e_1$:
1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{E}} e_0$
   $\wedge\ x, \Gamma \vdash_{\mathsf{E}} e_1$
   by E *inversion*
3. $\Gamma \vdash_{\mathsf{E}} e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_{\mathsf{E}} e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{E}} e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^1\ e_0$:
1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{E}} e_0$
   by E *inversion*
3. $\Gamma \vdash_{\mathsf{E}} e_0[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{E}} op^1\ e_0[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:
1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{E}} e_0$
   $\wedge\ x, \Gamma \vdash_{\mathsf{E}} e_1$
   by E *inversion*
3. $\Gamma \vdash_{\mathsf{E}} e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash_{\mathsf{E}} e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{E}} op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e =$ dyn τ' e' :
1. $e[x \leftarrow v] =$ dyn τ' $e'[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{E}} e'$
   by E *inversion*
3. $\Gamma \vdash_{\mathsf{E}} e'[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{E}}$ dyn τ' $(e'[x \leftarrow v])$
   by (3)
5. QED

CASE $e = \mathsf{stat}\ \tau'\ e'$:
1. $e[x \leftarrow v] = \mathsf{stat}\ \tau'\ e'[x \leftarrow v]$
2. $x, \Gamma \vdash_{\mathsf{E}} e'$
   by E *inversion*
3. $\Gamma \vdash_{\mathsf{E}} e'[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_{\mathsf{E}} \mathsf{stat}\ \tau'\ (e'[x \leftarrow v])$
   by (3)
5. QED

CASE $e = \mathsf{Err}$:
1. QED by $\mathsf{Err}[x \leftarrow v] = \mathsf{Err}$

□

Lemma 3.17 : *δ preservation*
- If $\vdash_{\mathsf{E}} v$ and $\delta(op^1, v) = e'$ then $\vdash_{\mathsf{E}} e'$
- If $\vdash_{\mathsf{E}} v_0$ and $\vdash_{\mathsf{E}} v_1$ and $\delta(op^2, v_0, v_1) = e'$ then $\vdash_{\mathsf{E}} v'$

Proof:

CASE $\delta(\mathsf{fst}, \langle v_0, v_1\rangle) = v_0$:
1. $\vdash_{\mathsf{E}} v_0$
   by E *inversion*
2. QED

CASE $\delta(\mathsf{snd}, \langle v_0, v_1\rangle) = v_1$:
1. $\vdash_{\mathsf{E}} v_1$
   by E *inversion*
2. QED

CASE $\delta(\mathsf{sum}, v_0, v_1) = v_0 + v_1$:
1. QED

CASE $\delta(\mathsf{quotient}, v_0, v_1) = \lfloor v_0 / v_1 \rfloor$:
1. QED

CASE $\delta(\mathsf{quotient}, v_0, v_1) = \mathsf{BndryErr}$:
1. QED

□

Lemma 3.18 : *weakening*
- If $\Gamma \vdash_{\mathsf{E}} e$ then $x, \Gamma \vdash_{\mathsf{E}} e$
- If $\Gamma \vdash_{\mathsf{E}} e$ then $(x\!:\!\tau), \Gamma \vdash_{\mathsf{E}} e$

Proof:

QED because e is closed under Γ

□

Lemma 3.19 : *unique static evaluation contexts*

If $\vdash e : \tau$ then one of the following holds:
- e is a value
- $e = E[v_0\ v_1]$
- $e = E[op^1\ v]$
- $e = E[op^2\ v_0\ v_1]$
- $e = E[\mathsf{Err}]$

Proof:

By induction on the structure of e.

CASE $e = x$:
1. Contradiction by $\vdash e : \tau$

CASE $e = i$
$\quad \vee\ e = \lambda(x\!:\!\tau_d).\, e'$:
1. QED e is a value

CASE $e = \mathsf{stat}\ \tau\ e'$:
1. Contradiction by $\vdash_1 e : K$

CASE $e = \langle e_0, e_1\rangle$:

IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
   by the induction hypothesis
2. $E = \langle E_0, e_1\rangle$
3. QED by $e = E[e_0']$

IF $e_0 \in v$
$\quad \wedge\ e_1 \notin v$:
1. $e_1 = E_1[e_1']$
   by the induction hypothesis
2. $E = \langle e_0, E_1\rangle$
3. QED by $e = E[e_1']$

ELSE $e_0 \in v$
$\quad \wedge\ e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[\langle e_0, e_1\rangle]$

CASE $e = e_0\ e_1$:

IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
   by the induction hypothesis
2. $E = E_0\ e_1$
3. QED by $e = E[e_0']$

IF $e_0 \in v$
$\quad \wedge\ e_1 \notin v$:
1. $e_1 = E_1[e_1']$
   by the induction hypothesis
2. $E = e_0\ E_1$
3. QED by $e = E[e_1']$

ELSE $e_0 \in v$
$\quad \wedge\ e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[e_0\ e_1]$

CASE $e = op^1\ e_0$:

IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
   by the induction hypothesis
2. $E = op^1\ E_0$
3. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[op^1\ e_0]$

CASE $e = op^2\ e_0\ e_1$:

IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
   by the induction hypothesis
2. $E = op^2\ E_0\ e_1$
3. QED $e = E[e_0']$

IF $e_0 \in v$
$\quad \wedge\ e_1 \notin v$:
1. $e_1 = E_1[e_1']$
   by the induction hypothesis
2. $E = op^2\ e_0\ E_1$
3. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\quad \wedge\ e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[op^2\ e_0\ e_1]$

CASE $e = \mathsf{dyn}\ \tau\ e_0$:
1. Contradiction by e is boundary-free

CASE $e = \mathsf{stat}\ \tau\ e_0$:
1. Contradiction by $\vdash e : \tau$

CASE $e = \mathsf{Err}$:
1. $E = [\,]$
2. QED $e = E[\mathsf{Err}]$

□

Lemma 3.20 : $\vdash$ *static inversion*

- If $\Gamma \vdash x : \tau$ then $(x{:}\tau') \in \Gamma$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash \lambda(x{:}\tau'_d).\, e' : \tau$ then $(x{:}\tau'_d), \Gamma \vdash e' : \tau'_c$ and $\tau'_d \Rightarrow \tau'_c \leqslant: \tau$
- If $\Gamma \vdash \langle e_0, e_1 \rangle : \tau$ then $\Gamma \vdash e_0 : \tau_0$ and $\Gamma \vdash e_1 : \tau_1$ and $\tau_0 \times \tau_1 \leqslant: \tau$
- If $\Gamma \vdash e_0\ e_1 : \tau_c$ then $\Gamma \vdash e_0 : \tau'_d \Rightarrow \tau'_c$ and $\Gamma \vdash e_1 : \tau'_d$ and $\tau'_c \leqslant: \tau_c$
- If $\Gamma \vdash \mathsf{fst}\, e : \tau$ then $\Gamma \vdash e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0$ and $\tau_0 \leqslant: \tau$
- If $\Gamma \vdash \mathsf{snd}\, e : \tau$ then $\Gamma \vdash e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1$ and $\tau_1 \leqslant: \tau$
- If $\Gamma \vdash op^2\ e_0\ e_1 : \tau$ then $\Gamma \vdash e_0 : \tau_0$ and $\Gamma \vdash e_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau'$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash \mathsf{dyn}\ \tau'\ e' : \tau$ then $\Gamma \vdash e'$ and $\tau' \leqslant: \tau$

Proof:

QED by the definition of $\Gamma \vdash e : \tau$

□

Lemma 3.21 : *canonical forms*

- If $\vdash v : \tau_0 \times \tau_1$ then $v = \langle v_0, v_1 \rangle$
- If $\vdash v : \tau_d \Rightarrow \tau_c$ then $v = \lambda(x{:}\tau_x).\, e'$
  $\wedge\ \tau_d \leqslant: \tau_x$
- If $\vdash v : \mathsf{Int}$ then $v = i$
- If $\vdash v : \mathsf{Nat}$ then $v = i$ and $v \in \mathbb{N}$

Proof:

QED by definition of $\vdash e : \tau$

□

Lemma 3.22 : *substitution*

If $(x{:}\tau_x), \Gamma \vdash e : \tau$, and e is boundary-free and $\vdash v : \tau_x$ then $\Gamma \vdash e[x \leftarrow v] : \tau$

Proof:

By induction on the structure of e.

CASE $e = x$:
1. $e[x \leftarrow v] = v$
2. $\tau_x = \tau$
3. $\Gamma \vdash v : \tau$
   by *weakening*
4. QED

CASE $e = x'$:
1. QED by $x'[x \leftarrow v] = x'$

CASE $e = i$:
1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x.\, e'$:
1. Contradiction by $(x{:}\tau_x), \Gamma \vdash e : \tau$

CASE $e = \lambda(x{:}\tau').\, e'$:
1. QED by $(\lambda(x{:}\tau').\, e')[x \leftarrow v] = \lambda(x{:}\tau').\, e'$

CASE $e = \lambda(x'{:}\tau').\, e'$:
1. $e[x \leftarrow v] = \lambda(x'{:}\tau').\, (e'[x \leftarrow v])$
2. $(x'{:}\tau'), x, \Gamma \vdash e'$
   by *static inversion forms*
3. $(x'{:}\tau'), \Gamma \vdash e'[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash \lambda(x'{:}\tau').\, (e'[x \leftarrow v])$
   by (3)
5. QED

CASE $e = \langle e_0, e_1 \rangle$:
1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $x, \Gamma \vdash e_0$
   $\wedge\ x, \Gamma \vdash e_1$
   by *static inversion forms*
3. $\Gamma \vdash e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
   by (3)
5. QED

CASE $e = e_0\ e_1$:
1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash e_0$
   $\wedge\ x, \Gamma \vdash e_1$
   by *static inversion forms*
3. $\Gamma \vdash e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^1\ e_0$:
1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $x, \Gamma \vdash e_0$
   by *static inversion forms*
3. $\Gamma \vdash e_0[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash op^1\ e_0[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:
1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash e_0$
   $\wedge\ x, \Gamma \vdash e_1$
   by *static inversion forms*
3. $\Gamma \vdash e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)

5. QED

CASE $e = \mathsf{dyn}\ \tau'\ e'$:

1. Contradiction by e is boundary-free

CASE $e = \mathsf{stat}\ \tau'\ e'$:

1. Contradiction by e is boundary-free

CASE $e = \mathsf{Err}$:

1. QED $\mathsf{Err}[x \leftarrow v] = \mathsf{Err}$

□

Lemma 3.23 : *δ preservation*

- If $\vdash v$ and $\delta(op^1, v) = v'$ then $\vdash e'$
- If $\vdash v_0$ and $\vdash v_1$ and $\delta(op^2, v_0, v_1) = e'$ then $\vdash v'$

Proof:

CASE $\delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0$:

1. $\vdash v_0$
   by *static inversion forms*
2. QED

CASE $\delta(\mathsf{snd}, \langle v_0, v_1 \rangle) = v_1$:

1. $\vdash v_1$
   by *static inversion forms*
2. QED

CASE $\delta(\mathsf{sum}, v_0, v_1) = v_0 + v_1$:

1. QED

CASE $\delta(\mathsf{quotient}, v_0, v_1) = \lfloor v_0 / v_1 \rfloor$:

1. QED

CASE $\delta(\mathsf{quotient}, v_0, v_1) = \mathsf{BndryErr}$:

1. QED

□

Lemma 3.24 : *weakening*

- If $\Gamma \vdash e$ then $x, \Gamma \vdash e$
- If $\Gamma \vdash e$ then $(x : \tau), \Gamma \vdash e$

Proof:

QED because e is closed under Γ

□

E.4 (1) First-Order Embedding

E.4.1 First-Order Definitions

$\boxed{\text{Language 1}}$

$$
\begin{array}{lcl}
e & = & x \mid v \mid e\ e \mid \langle e, e\rangle \mid op^1\ e \mid op^2\ e\ e \mid \\
 & & \mathsf{dyn}\ \tau\ e \mid \mathsf{stat}\ \tau\ e \mid \mathsf{Err} \mid \mathsf{chk}\ K\ e \mid \mathsf{dyn}\ e \mid \mathsf{stat}\ e \\
v & = & i \mid \langle v, v\rangle \mid \lambda x.\ e \mid \lambda(x\!:\!\tau).\ e \\
\tau & = & \mathsf{Nat} \mid \mathsf{Int} \mid \tau\!\times\!\tau \mid \tau \Rightarrow \tau \\
K & = & \mathsf{Nat} \mid \mathsf{Int} \mid \mathsf{Pair} \mid \mathsf{Fun} \mid \mathsf{Any} \\
\Gamma & = & \cdot \mid x, \Gamma \mid (x\!:\!\tau), \Gamma \\
\mathsf{Err} & = & \mathsf{BndryErr} \mid \mathsf{TagErr} \\
r & = & v \mid \mathsf{Err} \\
E^{\bullet} & = & [\,] \mid E^{\bullet}\ e \mid v\ E^{\bullet} \mid \langle E^{\bullet}, e\rangle \mid \langle v, E^{\bullet}\rangle \mid \\
 & & op^1\ E^{\bullet} \mid op^2\ E^{\bullet}\ e \mid op^2\ v\ E^{\bullet} \mid \mathsf{chk}\ K\ E^{\bullet} \\
E & = & E^{\bullet} \mid E\ e \mid v\ E \mid \langle E, e\rangle \mid \langle v, E\rangle \mid op^1\ E \mid \\
 & & op^2\ E\ e \mid op^2\ v\ E \mid \mathsf{dyn}\ \tau\ E \mid \mathsf{stat}\ \tau\ E \mid \\
 & & \mathsf{chk}\ K\ E \mid \mathsf{dyn}\ E \mid \mathsf{stat}\ E
\end{array}
$$

$\boxed{\Delta : op^1 \times \tau \longrightarrow \tau}$

$$
\begin{array}{l}
\Delta(\mathsf{fst}, \tau_0\!\times\!\tau_1) \ = \tau_0 \\
\Delta(\mathsf{snd}, \tau_0\!\times\!\tau_1) = \tau_1
\end{array}
$$

$\boxed{\Delta : op^2 \times \tau \times \tau \longrightarrow \tau}$

$$
\begin{array}{l}
\Delta(op^2, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat} \\
\Delta(op^2, \mathsf{Int}, \mathsf{Int}) \ \ = \mathsf{Int}
\end{array}
$$

$\boxed{\tau \leqslant: \tau}$

$$
\frac{}{\mathsf{Nat} \leqslant: \mathsf{Int}}
\qquad
\frac{\tau_d' \leqslant: \tau_d \quad \tau_c \leqslant: \tau_c'}{\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'}
\qquad
\frac{\tau_0 \leqslant: \tau_0' \quad \tau_1 \leqslant: \tau_1'}{\tau_0\!\times\!\tau_1 \leqslant: \tau_0'\!\times\!\tau_1'}
$$

$$
\frac{}{\tau \leqslant: \tau}
\qquad
\frac{\tau \leqslant: \tau' \quad \tau' \leqslant: \tau''}{\tau \leqslant: \tau''}
$$

$\boxed{\Gamma \vdash e}$

$$
\frac{x \in \Gamma}{\Gamma \vdash x}
\qquad
\frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x.\ e}
\qquad
\frac{}{\Gamma \vdash i}
\qquad
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1\rangle}
$$

$$
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0\ e_1}
\qquad
\frac{\Gamma \vdash e}{\Gamma \vdash op^1\ e}
\qquad
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2\ e_0\ e_1}
\qquad
\frac{}{\Gamma \vdash \mathsf{Err}}
$$

$$
\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat}\ \tau\ e}
$$

$\boxed{\Gamma \vdash e : \tau}$

$$
\frac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash x : \tau}
\qquad
\frac{(x\!:\!\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x\!:\!\tau_d).\ e : \tau_d \Rightarrow \tau_c}
\qquad
\frac{i \in \mathbb{N}}{\Gamma \vdash i : \mathsf{Nat}}
$$

$$
\frac{}{\Gamma \vdash i : \mathsf{Int}}
\qquad
\frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_0 \\ \Gamma \vdash e_1 : \tau_1\end{array}}{\Gamma \vdash \langle e_0, e_1\rangle : \tau_0\!\times\!\tau_1}
\qquad
\frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \\ \Gamma \vdash e_1 : \tau_d\end{array}}{\Gamma \vdash e_0\ e_1 : \tau_c}
$$

$$
\frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_0 \\ \Delta(op^1, \tau_0) = \tau\end{array}}{\Gamma \vdash op^1\ e_0 : \tau}
\qquad
\frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_0 \\ \Gamma \vdash e_1 : \tau_1 \\ \Delta(op^2, \tau_0, \tau_1) = \tau\end{array}}{\Gamma \vdash op^2\ e_0\ e_1 : \tau}
\qquad
\frac{\begin{array}{c}\Gamma \vdash e : \tau' \\ \tau' \leqslant: \tau\end{array}}{\Gamma \vdash e : \tau}
\qquad
\frac{}{\Gamma \vdash \mathsf{Err} : \tau}
$$

$$
\frac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau}
$$

$\boxed{K \leqslant: K}$

$$
\frac{}{K \leqslant: \mathsf{Any}}
\qquad
\frac{}{\mathsf{Nat} \leqslant: \mathsf{Int}}
\qquad
\frac{}{K \leqslant: K}
\qquad
\frac{K \leqslant: K' \quad K' \leqslant: K''}{K \leqslant: K''}
$$

$\boxed{\lfloor \tau \rfloor = K}$

$$
\begin{array}{lcl}
\lfloor \mathsf{Nat} \rfloor & = & \mathsf{Nat} \\
\lfloor \mathsf{Int} \rfloor & = & \mathsf{Int} \\
\lfloor \tau_0\!\times\!\tau_1 \rfloor & = & \mathsf{Pair} \\
\lfloor \tau_d \Rightarrow \tau_c \rfloor & = & \mathsf{Fun}
\end{array}
$$

$\boxed{\Gamma \vdash e \rightsquigarrow e}$

$$
\frac{}{\Gamma \vdash i \rightsquigarrow i}
\qquad
\frac{\begin{array}{c}\Gamma \vdash e_0 \rightsquigarrow e_0' \\ \Gamma \vdash e_1 \rightsquigarrow e_1'\end{array}}{\Gamma \vdash \langle e_0, e_1\rangle \rightsquigarrow \langle e_0', e_1'\rangle}
\qquad
\frac{x, \Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \lambda x.\ e \rightsquigarrow \lambda x.\ e'}
$$

$$
\frac{}{\Gamma \vdash x \rightsquigarrow x}
\qquad
\frac{\begin{array}{c}\Gamma \vdash e_0 \rightsquigarrow e_0' \\ \Gamma \vdash e_1 \rightsquigarrow e_1'\end{array}}{\Gamma \vdash e_0\ e_1 \rightsquigarrow e_0'\ e_1'}
\qquad
\frac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash op^1\ e \rightsquigarrow op^1\ e'}
$$

$$
\frac{\begin{array}{c}\Gamma \vdash e_0 \rightsquigarrow e_0' \\ \Gamma \vdash e_1 \rightsquigarrow e_1'\end{array}}{\Gamma \vdash op^2\ e_0\ e_1 \rightsquigarrow op^2\ e_0'\ e_1'}
\qquad
\frac{}{\Gamma \vdash \mathsf{Err} \rightsquigarrow \mathsf{Err}}
$$

$$
\frac{\Gamma \vdash e : \tau \rightsquigarrow e'}{\Gamma \vdash \mathsf{stat}\ \tau\ e \rightsquigarrow \mathsf{stat}\ \tau\ e'}
$$

$\boxed{\Gamma \vdash e : \tau \rightsquigarrow e}$

$$\frac{}{\Gamma \vdash i : \mathsf{Nat} \rightsquigarrow i} \qquad \frac{}{\Gamma \vdash i : \mathsf{Int} \rightsquigarrow i}$$

$$\frac{\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e_0' \quad \Gamma \vdash e_1 : \tau_1 \rightsquigarrow e_1'}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1 \rightsquigarrow \langle e_0', e_1' \rangle}$$

$$\frac{(x : \tau_d), \Gamma \vdash e : \tau_c \rightsquigarrow e'}{\Gamma \vdash \lambda(x : \tau_d).\, e : \tau_d \Rightarrow \tau_c \rightsquigarrow \lambda(x : \tau_d).\, e'} \qquad \frac{}{\Gamma \vdash x : \tau \rightsquigarrow x}$$

$$\frac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \rightsquigarrow e_0' \quad \Gamma \vdash e_1 : \tau_d \rightsquigarrow e_1' \quad \lfloor \tau_c \rfloor = K}{\Gamma \vdash e_0\ e_1 : \tau_c \rightsquigarrow \mathsf{chk}\ K\ (e_0'\ e_1')}$$

$$\frac{\Gamma \vdash e : \tau_0 \times \tau_1 \rightsquigarrow e' \quad \lfloor \tau_0 \rfloor = K}{\Gamma \vdash \mathsf{fst}\ e : \tau_0 \rightsquigarrow \mathsf{chk}\ K\ (\mathsf{fst}\ e')}$$

$$\frac{\Gamma \vdash e : \tau_0 \times \tau_1 \rightsquigarrow e' \quad \lfloor \tau_1 \rfloor = K}{\Gamma \vdash \mathsf{snd}\ e : \tau_1 \rightsquigarrow \mathsf{chk}\ K\ (\mathsf{snd}\ e')}$$

$$\frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e_0' \\ \Gamma \vdash e_1 : \tau_1 \rightsquigarrow e_1'\end{array}}{\Gamma \vdash op^2\ e_0\ e_1 : \tau \rightsquigarrow op^2\ e_0'\ e_1'} \qquad \frac{\begin{array}{c}\Gamma \vdash e : \tau' \rightsquigarrow e' \\ \tau' \leqslant: \tau\end{array}}{\Gamma \vdash e : \tau \rightsquigarrow e'}$$

$$\frac{}{\Gamma \vdash \mathsf{Err} : \tau \rightsquigarrow \mathsf{Err}} \qquad \frac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau \rightsquigarrow \mathsf{dyn}\ \tau\ e'}$$

$\boxed{\Gamma \vdash_1 e}$

$$\frac{}{\Gamma \vdash_1 i} \qquad \frac{\Gamma \vdash_1 e_0 \quad \Gamma \vdash_1 e_1}{\Gamma \vdash_1 \langle e_0, e_1 \rangle} \qquad \frac{x, \Gamma \vdash_1 e}{\Gamma \vdash_1 \lambda x.\, e} \qquad \frac{(x : \tau), \Gamma \vdash_1 e : \mathsf{Any}}{\Gamma \vdash_1 \lambda(x : \tau).\, e}$$

$$\frac{x \in \Gamma}{\Gamma \vdash_1 x} \qquad \frac{(x : \tau) \in \Gamma}{\Gamma \vdash_1 x} \qquad \frac{\Gamma \vdash_1 e_0 \quad \Gamma \vdash_1 e_1}{\Gamma \vdash_1 e_0\ e_1} \qquad \frac{\Gamma \vdash_1 e}{\Gamma \vdash_1 op^1\ e}$$

$$\frac{\Gamma \vdash_1 e_0 \quad \Gamma \vdash_1 e_1}{\Gamma \vdash_1 op^2\ e_0\ e_1} \qquad \frac{}{\Gamma \vdash_1 \mathsf{Err}} \qquad \frac{\Gamma \vdash_1 e : \lfloor \tau \rfloor}{\Gamma \vdash_1 \mathsf{stat}\ \tau\ e} \qquad \frac{\Gamma \vdash_1 e : \mathsf{Any}}{\Gamma \vdash_1 \mathsf{stat}\ e}$$

$\boxed{\Gamma \vdash_1 e : K}$

$$\frac{i \in \mathbb{N}}{\Gamma \vdash_1 i : \mathsf{Nat}} \qquad \frac{}{\Gamma \vdash_1 i : \mathsf{Int}} \qquad \frac{\begin{array}{c}\Gamma \vdash_1 e_0 : \mathsf{Any} \\ \Gamma \vdash_1 e_1 : \mathsf{Any}\end{array}}{\Gamma \vdash_1 \langle e_0, e_1 \rangle : \mathsf{Pair}}$$

$$\frac{x, \Gamma \vdash_1 e}{\Gamma \vdash_1 \lambda x.\, e : \mathsf{Fun}} \qquad \frac{(x : \tau), \Gamma \vdash_1 e : \mathsf{Any}}{\Gamma \vdash_1 \lambda(x : \tau).\, e : \mathsf{Fun}} \qquad \frac{x \in \Gamma}{\Gamma \vdash_1 x : \mathsf{Any}}$$

$$\frac{\begin{array}{c}(x : \tau) \in \Gamma \\ \lfloor \tau \rfloor = K\end{array}}{\Gamma \vdash_1 x : K} \qquad \frac{\begin{array}{c}\Gamma \vdash_1 e_0 : \mathsf{Fun} \\ \Gamma \vdash_1 e_1 : \mathsf{Any}\end{array}}{\Gamma \vdash_1 e_0\ e_1 : \mathsf{Any}} \qquad \frac{\Gamma \vdash_1 e : \mathsf{Pair}}{\Gamma \vdash_1 \mathsf{fst}\ e : \mathsf{Any}}$$

$$\frac{\Gamma \vdash_1 e : \mathsf{Pair}}{\Gamma \vdash_1 \mathsf{snd}\ e : \mathsf{Any}} \qquad \frac{\begin{array}{c}\Gamma \vdash_1 e_0 : K_0 \quad \Gamma \vdash_1 e_1 : K_1 \\ \Delta(op^2, K_0, K_1) = K\end{array}}{\Gamma \vdash_1 op^2\ e_0\ e_1 : K}$$

$$\frac{\Gamma \vdash_1 e : K' \quad K' \leqslant: K}{\Gamma \vdash_1 e : K} \qquad \frac{}{\Gamma \vdash_1 \mathsf{Err} : K} \qquad \frac{\Gamma \vdash_1 e \quad \lfloor \tau \rfloor = K}{\Gamma \vdash_1 \mathsf{dyn}\ \tau\ e : K}$$

$$\frac{\Gamma \vdash_1 e}{\Gamma \vdash_1 \mathsf{dyn}\ e : \mathsf{Any}} \qquad \frac{\Gamma \vdash_1 e : \mathsf{Any}}{\Gamma \vdash_1 \mathsf{chk}\ K\ e : K}$$

$\boxed{\delta(op^1, v) = e}$

$$\begin{array}{ll}
\delta(\mathsf{fst}, \langle v_0, v_1 \rangle) & = v_0 \\
\delta(\mathsf{snd}, \langle v_0, v_1 \rangle) & = v_1
\end{array}$$

$\boxed{\delta(op^2, v, v) = e}$

$$\begin{array}{ll}
\delta(\mathsf{sum}, i_0, i_1) & = i_0 + i_1 \\
\delta(\mathsf{quotient}, i_0, 0) & = \mathsf{BndryErr} \\
\delta(\mathsf{quotient}, i_0, i_1) & = \lfloor i_0 / i_1 \rfloor \\
\quad \text{if } i_1 \neq 0 &
\end{array}$$

$\boxed{\mathcal{D}_1 : \tau \times v \longrightarrow v}$

$\mathcal{D}_1(\tau, v) = \mathcal{X}(\lfloor \tau \rfloor, v)$

$\boxed{\mathcal{S}_1 : \tau \times v \longrightarrow v}$

$\mathcal{S}_1(\tau, v) = v$

$\boxed{\mathcal{X} : K \times v \longrightarrow v}$

$$\begin{array}{ll}
\mathcal{X}(\mathsf{Fun}, \lambda x.\, e) & = \lambda x.\, e \\
\mathcal{X}(\mathsf{Fun}, \lambda(x : \tau).\, e) & = \lambda(x : \tau).\, e \\
\mathcal{X}(\mathsf{Pair}, \langle v_0, v_1 \rangle) & = \langle v_0, v_1 \rangle \\
\mathcal{X}(\mathsf{Int}, i) & = i \\
\mathcal{X}(\mathsf{Nat}, i) & = i \\
\quad \text{if } i \in \mathbb{N} & \\
\mathcal{X}(K, v) & = \mathsf{BndryErr} \\
\quad \text{otherwise} &
\end{array}$$

$\boxed{e \rhd_{\text{1-S}} e}$

$\mathsf{dyn}\ v \quad \rhd_{\text{1-S}}\ v$

$\mathsf{dyn}\ \tau\ v \quad \rhd_{\text{1-S}}\ \mathcal{D}(\tau, v)$

$\mathsf{chk}\ K\ v \quad \rhd_{\text{1-S}}\ \mathcal{X}(K, v)$

$(\lambda(x\!:\!\tau).\, e)\ v \quad \rhd_{\text{1-S}}\ \mathsf{BndryErr}$
  if $\mathcal{X}(\lfloor\tau\rfloor, v) = \mathsf{BndryErr}$

$(\lambda(x\!:\!\tau).\, e)\ v \quad \rhd_{\text{1-S}}\ e[x \leftarrow \mathcal{X}(\lfloor\tau\rfloor, v)]$

$(\lambda x.\, e)\ v \quad \rhd_{\text{1-S}}\ \mathsf{dyn}\ (e[x \leftarrow v])$

$op^1\ v \quad \rhd_{\text{1-S}}\ \delta(op^1, v)$

$op^2\ v_0\ v_1 \quad \rhd_{\text{1-S}}\ \delta(op^2, v_0, v_1)$

$\boxed{e \rhd_{\text{1-D}} e}$

$\mathsf{stat}\ v \quad \rhd_{\text{1-D}}\ v$

$\mathsf{stat}\ \tau\ v \quad \rhd_{\text{1-D}}\ \mathcal{S}(\tau, v)$

$v_0\ v_1 \quad \rhd_{\text{1-D}}\ \mathsf{TagErr}$
  if $v_0 \in \mathbb{Z}$ or $v_0 = \langle v, v' \rangle$

$(\lambda(x\!:\!\tau).\, e)\ v \quad \rhd_{\text{1-D}}\ \mathsf{BndryErr}$
  if $\mathcal{X}(\lfloor\tau\rfloor, v) = \mathsf{BndryErr}$

$(\lambda(x\!:\!\tau).\, e)\ v \quad \rhd_{\text{1-D}}\ \mathsf{stat}\ (e[x \leftarrow \mathcal{X}(\lfloor\tau\rfloor, v)])$

$(\lambda x.\, e)\ v \quad \rhd_{\text{1-D}}\ e[x \leftarrow v]$

$op^1\ v \quad \rhd_{\text{1-D}}\ \mathsf{TagErr}$
  if $\delta(op^1, v)$ is undefined

$op^1\ v \quad \rhd_{\text{1-D}}\ \delta(op^1, v)$

$op^2\ v_0\ v_1 \quad \rhd_{\text{1-D}}\ \mathsf{TagErr}$
  if $\delta(op^2, v_0, v_1)$ is undefined

$op^2\ v_0\ v_1 \quad \rhd_{\text{1-D}}\ \delta(op^2, v_0, v_1)$

$\boxed{e \rightarrow_{\text{1-S}} e}$

$E^\bullet[e] \quad \rightarrow_{\text{1-S}}\ E^\bullet[e']$
  if $e \rhd_{\text{1-S}} e'$

$E[\mathsf{stat}\ \tau\ E^\bullet[e]] \quad \rightarrow_{\text{1-S}}\ E[\mathsf{stat}\ \tau\ E^\bullet[e']]$
  if $e \rhd_{\text{1-S}} e'$

$E[\mathsf{dyn}\ \tau\ E^\bullet[e]] \quad \rightarrow_{\text{1-S}}\ E[\mathsf{dyn}\ \tau\ E^\bullet[e']]$
  if $e \rhd_{\text{1-D}} e'$

$E[\mathsf{Err}] \quad \rightarrow_{\text{1-S}}\ \mathsf{Err}$

$\boxed{e \rightarrow_{\text{1-D}} e}$

$E^\bullet[e] \quad \rightarrow_{\text{1-D}}\ E^\bullet[e']$
  if $e \rhd_{\text{1-D}} e'$

$E[\mathsf{stat}\ \tau\ E^\bullet[e]] \quad \rightarrow_{\text{1-D}}\ E[\mathsf{stat}\ \tau\ E^\bullet[e']]$
  if $e \rhd_{\text{1-S}} e'$

$E[\mathsf{dyn}\ \tau\ E^\bullet[e]] \quad \rightarrow_{\text{1-D}}\ E[\mathsf{dyn}\ \tau\ E^\bullet[e']]$
  if $e \rhd_{\text{1-D}} e'$

$E[\mathsf{Err}] \quad \rightarrow_{\text{1-D}}\ \mathsf{Err}$

$\boxed{e \rightarrow^*_{\text{1-S}} e}$ reflexive, transitive closure of $\rightarrow_{\text{1-S}}$

$\boxed{e \rightarrow^*_{\text{1-D}} e}$ reflexive, transitive closure of $\rightarrow_{\text{1-D}}$

Definition 4.0 : 1 *boundary-free*

An expression e is *boundary free* if e does not contain a subterm of the form:

- ($\mathsf{dyn}\ \tau'\ e'$),
- ($\mathsf{stat}\ \tau'\ e'$),
- ($\mathsf{dyn}\ e'$), or
- ($\mathsf{stat}\ e'$).

E.4.2 First-Order Theorems

Theorem 4.1 : *static* 1*-soundness*

If $\vdash e : \tau$ then $\vdash e : \tau \rightsquigarrow e''$ and $\vdash_1 e'' : \lfloor \tau \rfloor$ and one of the following holds:

- $e'' \rightarrow^*_{\text{1-S}} v$ and $\vdash_1 v : \lfloor \tau \rfloor$
- $e'' \rightarrow^*_{\text{1-S}} E[\mathsf{dyn}\ \tau'\ E^\bullet[e']]$ and $e' \rhd_{\text{1-D}} \mathsf{TagErr}$
- $e'' \rightarrow^*_{\text{1-S}} E[\mathsf{dyn}\ E^\bullet[e']]$ and $e' \rhd_{\text{1-D}} \mathsf{TagErr}$
- $e'' \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$
- e'' diverges

Proof:

1. $\vdash_1 e : \tau \rightsquigarrow e''$
   $\wedge \vdash_1 e'' : \lfloor \tau \rfloor$
   by $\rightsquigarrow$ *static soundness*
2. QED by 1 *static progress* and 1 *static preservation*

☐

Theorem 4.2 : *dynamic* 1*-soundness*

If $\vdash e$ then $\vdash e \rightsquigarrow e''$ and $\vdash_1 e''$ and one of the following holds:

- $e'' \rightarrow^*_{\text{1-D}} v$ and $\vdash_1 v$
- $e'' \rightarrow^*_{\text{1-D}} E[e']$ and $e' \rhd_{\text{1-D}} \mathsf{TagErr}$
- $e'' \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
- e'' diverges

Proof:

1. $\vdash_1 e \rightsquigarrow e''$
   $\wedge \vdash_1 e''$
   by $\rightsquigarrow$ *dynamic soundness*
2. QED by 1 *dynamic progress* and 1 *dynamic preservation*

☐

Theorem 4.3 : *boundary-free* 1*-soundness*

If $\vdash e : \tau$ and e is boundary-free then one of the following holds:

- $e \rightarrow^*_{\text{1-S}} v$ and $\vdash v : \tau$
- $e \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$
- e diverges

Proof:

QED by *progress* and *preservation*

☐

Theorem 4.4 : H/1 *base type equivalence*

If $\vdash e : \tau$ and all boundary terms in e are of the following four forms:

- $\mathsf{dyn}\ \mathsf{Int}\ e'$
- $\mathsf{stat}\ \mathsf{Int}\ e'$
- $\mathsf{stat}\ \mathsf{Nat}\ e'$
- $\mathsf{dyn}\ \mathsf{Nat}\ e'$

and $\vdash e : \tau \rightsquigarrow e''$, then $e \rightarrow^*_{\text{H-S}} v$ if and only if $e'' \rightarrow^*_{\text{1-S}} v$.

Proof:

1. $\mathcal{D}_{\mathsf{H}}(\mathsf{Int}, v) = \mathcal{D}_1(\mathsf{Int}, v)$
   by by definition
2. $\mathcal{D}_{\mathsf{H}}(\mathsf{Nat}, v) = \mathcal{D}_1(\mathsf{Nat}, v)$
   by by definition
3. $\mathcal{S}_{\mathsf{H}}(\mathsf{Int}, v) = \mathcal{S}_1(\mathsf{Int}, v)$
   by by definition
4. $\mathcal{S}_{\mathsf{H}}(\mathsf{Nat}, v) = \mathcal{S}_1(\mathsf{Nat}, v)$
   by by definition
5. QED

☐

Corollary 4.5 : 1 *compilation*

If $\vdash e : \tau$
and $\vdash e : \tau \rightsquigarrow e''$
and $\vdash_1 e'' : \lfloor \tau \rfloor$
and $\rhd_{\text{1-D}}'$ is similar to $\rhd_{\text{1-D}}$ but without the no-op boundaries, as follows:

$$
\begin{array}{lll}
\mathsf{chk}\ K\ v & \rhd_{\text{1-D}}' & \mathcal{X}(K, v) \\
v_0\ v_1 & \rhd_{\text{1-D}}' & \mathsf{TagErr} \\
\quad \text{if } v_0 \in \mathbb{Z} \text{ or } v_0 = \langle v, v' \rangle & & \\
(\lambda(x\!:\!\tau).\ e)\ v & \rhd_{\text{1-D}}' & \mathsf{BndryErr} \\
\quad \text{if } \mathcal{X}(\lfloor \tau \rfloor, v) = \mathsf{BndryErr} & & \\
(\lambda(x\!:\!\tau).\ e)\ v & \rhd_{\text{1-D}}' & e[x \leftarrow \mathcal{X}(\lfloor \tau \rfloor, v)] \\
(\lambda x.\ e)\ v & \rhd_{\text{1-D}}' & e[x \leftarrow v] \\
op^1\ v & \rhd_{\text{1-D}}' & \mathsf{TagErr} \\
\quad \text{if } \delta(op^1, v) \text{ is undefined} & & \\
op^1\ v & \rhd_{\text{1-D}}' & \delta(op^1, v) \\
op^2\ v_0\ v_1 & \rhd_{\text{1-D}}' & \mathsf{TagErr} \\
\quad \text{if } \delta(op^2, v_0, v_1) \text{ is undefined} & & \\
op^2\ v_0\ v_1 & \rhd_{\text{1-D}}' & \delta(op^2, v_0, v_1)
\end{array}
$$

and $e \rightarrow_{\text{1-D}}' e$ is defined as:

$$
\begin{array}{lll}
E[e] & \rightarrow_{\text{1-D}}' & E[e'] \\
\quad \text{if } e \rhd_{\text{1-D}}' e' & & \\
E[\mathsf{stat}\ \tau\ v] & \rightarrow_{\text{1-D}}' & E[\mathcal{D}_1(\tau, v)] \\
E[\mathsf{dyn}\ \tau\ v] & \rightarrow_{\text{1-D}}' & E[\mathcal{D}_1(\tau, v)] \\
E[\mathsf{Err}] & \rightarrow_{\text{1-D}}' & \mathsf{Err}
\end{array}
$$

and $\rightarrow^*_{\text{1-D}}{}'$ is the reflexive transitive closure of $\rightarrow_{\text{1-D}}'$

then one of the following holds:

- $e'' \rightarrow^*_{\text{1-D}}{}'\ v$ and $\vdash_1 v : \lfloor \tau \rfloor$
- $e'' \rightarrow^*_{\text{1-D}}{}'\ \mathsf{TagErr}$
- $e'' \rightarrow^*_{\text{1-D}}{}'\ \mathsf{BndryErr}$
- e diverges

Proof (sketch): By *static* 1*-soundness* and the fact that $\rhd_{\text{1-S}}$ is a subset of $\rhd_{\text{1-D}}'$ (modulo the $\mathsf{dyn}\ e$ and $\mathsf{stat}\ e$ boundaries).☐

E.4.3 First-Order Lemmas

Lemma 4.6 : *$\mathcal{D}_1$ soundness*

If $\vdash_1 v$ then $\vdash_1 \mathcal{D}_1(\tau, v) : \lfloor\tau\rfloor$.

Proof:

- $\mathcal{D}_1(\tau, v) = \mathcal{X}(\lfloor\tau\rfloor, v)$
- QED by *check soundness*

□

Lemma 4.7 : *$\mathcal{S}_1$ soundness*

If $\vdash_1 v : \tau$ then $\vdash_1 \mathcal{S}_1(\tau, v)$.

Proof:

- $\mathcal{S}_1(\tau, v) = \mathcal{X}(\lfloor\tau\rfloor, v)$
- QED *check soundness*

□

Lemma 4.8 : *$\rightsquigarrow$ static soundness*

If $\Gamma \vdash e : \tau$ then $\Gamma \vdash e : \tau \rightsquigarrow e'$ and $\Gamma \vdash_1 e' : \lfloor\tau\rfloor$.

Proof:

By induction on the structure of $\Gamma \vdash e : \tau$.

CASE $\boxed{\dfrac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash x : \tau}}$:

1. $\Gamma \vdash x \rightsquigarrow x$
2. $\Gamma \vdash_1 x : \lfloor\tau\rfloor$
   by $(x\!:\!\tau) \in \Gamma$
3. QED

CASE $\boxed{\dfrac{(x\!:\!\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c}}$:

1. $\Gamma \vdash e : \tau_c \rightsquigarrow e'$
   $\wedge\, (x\!:\!\tau_d), \Gamma \vdash e' : \lfloor\tau_c\rfloor$
   by the induction hypothesis
2. $(x\!:\!\tau_d), \Gamma \vdash e' : \mathsf{Any}$
   by $\lfloor\tau_c\rfloor <: \mathsf{Any}$
3. $\lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c \rightsquigarrow \lambda(x\!:\!\tau_d).\, e'$
4. $\Gamma \vdash_1 \lambda(x\!:\!\tau_d).\, e' : \mathsf{Fun}$
   by (2)
5. QED (3, 4)

CASE $\boxed{\dfrac{i \in \mathbb{N}}{\Gamma \vdash i : \mathsf{Nat}}}$:

1. $\Gamma \vdash i : \mathsf{Nat} \rightsquigarrow i$
2. QED by $\Gamma \vdash_1 i : \mathsf{Nat}$

CASE $\boxed{\dfrac{}{\Gamma \vdash i : \mathsf{Int}}}$:

1. $\Gamma \vdash i : \mathsf{Int} \rightsquigarrow i$
2. QED by $\Gamma \vdash_1 i : \mathsf{Int}$

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1\rangle : \tau_0 \times \tau_1}}$:

1. $\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e_0'$
   $\wedge\, \Gamma \vdash e_0' : \lfloor\tau_0\rfloor$
   by the induction hypothesis
2. $\Gamma \vdash e_1 : \tau_1 \rightsquigarrow e_1'$
   $\wedge\, \Gamma \vdash e_1' : \lfloor\tau_1\rfloor$
   by the induction hypothesis
3. $\Gamma \vdash_1 e_0 : \mathsf{Any}$
   by $\lfloor\tau_0\rfloor <: \mathsf{Any}$
4. $\Gamma \vdash_1 e_1 : \mathsf{Any}$
   by $\lfloor\tau_1\rfloor <: \mathsf{Any}$
5. $\Gamma \vdash \langle e_0, e_1\rangle : \tau \rightsquigarrow \langle e_0', e_1'\rangle$
   by (1, 2)
6. $\Gamma \vdash_1 \langle e_0', e_1'\rangle : \mathsf{Pair}$
   by (3, 4)
7. QED by (5, 6)

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash e_1 : \tau_d}{\Gamma \vdash e_0\ e_1 : \tau_c}}$:

1. $\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \rightsquigarrow e_0'$
   $\wedge\, \Gamma \vdash_1 e_0' : \lfloor\tau_d \Rightarrow \tau_c\rfloor$
   by the induction hypothesis
2. $\Gamma \vdash e_1 : \tau_d \rightsquigarrow e_1'$
   $\wedge\, \Gamma \vdash_1 e_1' : \lfloor\tau_c\rfloor$
   by the induction hypothesis
3. $\Gamma \vdash_1 e_0' : \mathsf{Fun}$
   by $\lfloor\tau_d \Rightarrow \tau_c\rfloor = \mathsf{Fun}$
4. $\Gamma \vdash_1 e_1' : \mathsf{Any}$
   by $\lfloor\tau_c\rfloor <: \mathsf{Any}$
5. $\Gamma \vdash e_0\ e_1 : \tau_c \rightsquigarrow \mathsf{chk}\ \lfloor\tau_c\rfloor\ (e_0'\ e_1')$
   by (1, 2)
6. $\Gamma \vdash_1 \mathsf{chk}\ \lfloor\tau_c\rfloor\ (e_0'\ e_1') : \lfloor\tau_c\rfloor$
   by (3, 4)
7. QED by (5, 6)

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash op^1\ e_0 : \tau}}$:

IF $op^1 = \mathsf{fst}$:

1. $\Delta(\mathsf{fst}, \tau_0) = \tau$
2. $\tau_0 = \tau \times \tau'$
   by *Δ inversion*
3. $\Gamma \vdash e_0 : \tau \times \tau' \rightsquigarrow e_0'$
   $\wedge\, \Gamma \vdash_1 e_0' : \lfloor\tau \times \tau'\rfloor$
   by the induction hypothesis
4. $\Gamma \vdash_1 e_0' : \mathsf{Pair}$
   by $\lfloor\tau \times \tau'\rfloor = \mathsf{Pair}$
5. $\Gamma \vdash \mathsf{fst}\, e_0 : \tau \rightsquigarrow \mathsf{chk}\ \lfloor\tau\rfloor\ (\mathsf{fst}\ e_0')$
   by (2)
6. $\Gamma \vdash_1 \mathsf{chk}\ \lfloor\tau\rfloor\ (\mathsf{fst}\ e_0') : \lfloor\tau\rfloor$
   by (3)
7. QED by 4,5

ELSE $op^1 = \mathsf{snd}$:

1. $\Delta(\mathsf{snd}, \tau_0) = \tau$
2. $\tau_0 = \tau' \times \tau$
   by *Δ inversion*
3. $\Gamma \vdash e_0 : \tau' \times \tau \rightsquigarrow e_0'$
   $\wedge\, \Gamma \vdash_1 e_0' : \lfloor\tau' \times \tau\rfloor$
   by the induction hypothesis

4. $\Gamma \vdash_1 e_0' : \mathsf{Pair}$
   by $\lfloor \tau' \times \tau \rfloor = \mathsf{Pair}$
5. $\Gamma \vdash \mathsf{snd}\, e_0 : \tau \rightsquigarrow \mathsf{chk}\, \lfloor \tau \rfloor\, (\mathsf{snd}\, e_0')$
   by (2)
6. $\Gamma \vdash_1 \mathsf{chk}\, \lfloor \tau \rfloor\, (\mathsf{snd}\, e_0') : \lfloor \tau \rfloor$
   by (3)
7. QED by 4,5

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash op^2\; e_0\; e_1 : \tau}}$:

1. $\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e_0'$
   $\wedge\, \Gamma \vdash_1 e_0' : \lfloor \tau_0 \rfloor$
   by the induction hypothesis
2. $\Gamma \vdash e_1 : \tau_1 \rightsquigarrow e_1'$
   $\wedge\, \Gamma \vdash_1 e_1' : \lfloor \tau_1 \rfloor$
   by the induction hypothesis
3. $\Delta(op^2, \lfloor \tau_0 \rfloor, \lfloor \tau_1 \rfloor) = \lfloor \tau \rfloor$
   by *Δ tag preservation*
4. $\Gamma \vdash op^2\; e_0\; e_1 : \tau \rightsquigarrow op^2\; e_0'\; e_1'$
   by (1, 2)
5. $\Gamma \vdash_1 op^2\; e_0'\; e_1' : \lfloor \tau \rfloor$
   by (1, 2, 3)
6. QED by (5, 6)

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau' \quad \tau' <: \tau}{\Gamma \vdash e : \tau}}$:

1. $\Gamma \vdash e : \tau' \rightsquigarrow e'$
   $\wedge\, \Gamma \vdash_1 e' : \lfloor \tau' \rfloor$
   by the induction hypothesis
2. $\lfloor \tau' \rfloor \leqslant: \lfloor \tau \rfloor$
   by *subtyping preservation*
3. $\Gamma \vdash_1 e' : \lfloor \tau \rfloor$
   by (2)
4. QED by (1, 3)

CASE $\boxed{\dfrac{}{\Gamma \vdash \mathsf{Err} : \tau}}$:

1. $\Gamma \vdash \mathsf{Err} : \tau \rightsquigarrow \mathsf{Err}$
2. $\Gamma \vdash_1 \mathsf{Err} : \tau$
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn}\; \tau\; e : \tau}}$:

1. $\Gamma \vdash e \rightsquigarrow e'$
   $\wedge\, \Gamma \vdash_1 e'$
   by *$\rightsquigarrow$ dynamic soundness*
2. $\Gamma \vdash \mathsf{dyn}\; \tau\; e : \tau \rightsquigarrow \mathsf{dyn}\; \tau\; e'$
   by (1)
3. $\Gamma \vdash_1 \mathsf{dyn}\; \tau\; e' : \lfloor \tau \rfloor$
   by (1)
4. QED by (2, 3)

□

Lemma 4.9 : *$\rightsquigarrow$ dynamic soundness*

If $\Gamma \vdash e$ then $\Gamma \vdash e \rightsquigarrow e'$ and $\Gamma \vdash_1 e'$

Proof:

By induction on the structure of $\Gamma \vdash e$.

CASE $\boxed{\dfrac{x \in \Gamma}{\Gamma \vdash x}}$:

1. $\Gamma \vdash x \rightsquigarrow x$
2. $\Gamma \vdash_1 x$
   by $x \in \Gamma$
3. QED

CASE $\boxed{\dfrac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x.\, e}}$:

1. $x, \Gamma \vdash e \rightsquigarrow e'$
   $\wedge\, x, \Gamma \vdash_1 e'$
   by the induction hypothesis
2. $\Gamma \vdash \lambda x.\, e \rightsquigarrow \lambda x.\, e'$
   by (1)
3. $\Gamma \vdash_1 \lambda x.\, e'$
   by (1)
4. QED by (2, 3)

CASE $\boxed{\dfrac{}{\Gamma \vdash i}}$:

1. $\Gamma \vdash i \rightsquigarrow i$
2. $\Gamma \vdash_1 i$
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}}$:

1. $\Gamma \vdash e_0 \rightsquigarrow e_0'$
   $\wedge\, \Gamma \vdash_1 e_0'$
   by the induction hypothesis
2. $\Gamma \vdash e_1 \rightsquigarrow e_1'$
   $\wedge\, \Gamma \vdash_1 e_1'$
   by the induction hypothesis
3. $\Gamma \vdash \langle e_0, e_1 \rangle \rightsquigarrow \langle e_0', e_1' \rangle$
   by (1, 2)
4. $\Gamma \vdash_1 \langle e_0', e_1' \rangle$
   by (1, 2)
5. QED by (3, 4)

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0\; e_1}}$:

1. $\Gamma \vdash e_0 \rightsquigarrow e_0'$
   $\wedge\, \Gamma \vdash_1 e_0'$
   by the induction hypothesis
2. $\Gamma \vdash e_1 \rightsquigarrow e_1'$
   $\wedge\, \Gamma \vdash_1 e_1'$
   by the induction hypothesis
3. $\Gamma \vdash e_0\; e_1 \rightsquigarrow e_0'\; e_1'$
   by (1, 2)
4. $\Gamma \vdash_1 e_0'\; e_1'$
   by (1, 2)
5. QED by (3, 4)

CASE $\boxed{\dfrac{\Gamma \vdash e}{\Gamma \vdash op^1\ e}}$:

1. $\Gamma \vdash e \rightsquigarrow e'$
   $\wedge\ \Gamma \vdash_1 e'$
   by the induction hypothesis
2. $\Gamma \vdash op^1\ e \rightsquigarrow op^1\ e'$
   by (1)
3. $\Gamma \vdash_1 op^1\ e'$
   by (1)
4. QED by (2, 3)

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2\ e_0\ e_1}}$:

1. $\Gamma \vdash e_0 \rightsquigarrow e_0'$
   $\wedge\ \Gamma \vdash_1 e_0'$
   by the induction hypothesis
2. $\Gamma \vdash e_1 \rightsquigarrow e_1'$
   $\wedge\ \Gamma \vdash_1 e_1'$
   by the induction hypothesis
3. $\Gamma \vdash op^2\ e_0\ e_1 \rightsquigarrow op^2\ e_0'\ e_1'$
   by (1, 2)
4. $\Gamma \vdash_1 op^2\ e_0'\ e_1'$
   by (1, 2)
5. QED by 3,4

CASE $\boxed{\dfrac{}{\Gamma \vdash \mathsf{Err}}}$:

1. $\Gamma \vdash \mathsf{Err} \rightsquigarrow \mathsf{Err}$
2. $\Gamma \vdash_1 \mathsf{Err}$
3. QED

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat}\ \tau\ e}}$:

1. $\Gamma \vdash e : \tau \rightsquigarrow e'$
   $\wedge\ \Gamma \vdash_1 e' : \lfloor \tau \rfloor$
   by *$\rightsquigarrow$ static soundness*
2. $\Gamma \vdash \mathsf{stat}\ \tau\ e \rightsquigarrow \mathsf{stat}\ \tau\ e'$
   by (1)
3. $\Gamma \vdash_1 \mathsf{stat}\ \tau\ e$
   by (1)
4. QED by (2,3)

□

Lemma 4.10 : 1 *static progress*

If $\vdash_1 e : K$ then one of the following holds:

- e is a value
- $e \in \mathsf{Err}$
- $e \rightarrow_{1\text{-S}} e'$
- $e \rightarrow_{1\text{-S}} \mathsf{BndryErr}$
- $e = E[\mathsf{dyn}\ \tau'\ E^\bullet[e']]$ and $e' \rightarrow_{1\text{-D}} \mathsf{TagErr}$
- $e = E[\mathsf{dyn}\ E^\bullet[e']]$ and $e' \rightarrow_{1\text{-D}} \mathsf{TagErr}$

Proof:

By the *boundary factoring* lemma, there are ten cases.

CASE e is a value :

1. QED

CASE $e = E^\bullet[v_0\ v_1]$:

1. $\vdash_1 v_0\ v_1 : K'$
   by *static hole typing*
2. $\vdash_1 v_0 : \mathsf{Fun}$
   by 1 *inversion* (1)
3. $v_0 = \lambda x.\, e'$
   $\vee\ v_0 = \lambda(x\!:\!\tau_d).\, e'$
   by *canonical forms* (2)
4. **IF** $v_0 = \lambda x.\, e'$:
   a. $e \rightarrow_{1\text{-S}} E^\bullet[\mathsf{dyn}\ (e'[x \leftarrow v_1])]$
      by $(\lambda x.\, e')\ v_1 \rhd_{1\text{-S}} (\mathsf{dyn}\ (e'[x \leftarrow v_1]))$
   b. QED

   IF $v_0 = \lambda(x\!:\!\tau_d).\, e'$
   $\wedge\ \mathcal{X}(\lfloor \tau_d \rfloor, v_1) = v_1$:
   a. $e \rightarrow_{1\text{-S}} E^\bullet[e'[x \leftarrow v_1]]$
      by $(\lambda(x\!:\!\tau_d).\, e')\ v_1 \rhd_{1\text{-S}} e'[x \leftarrow v_1]$
   b. QED

   ELSE $v_0 = \lambda(x\!:\!\tau_d).\, e'$
   $\wedge\ \mathcal{X}(\lfloor \tau_d \rfloor, v_1) = \mathsf{BndryErr}$:
   a. $e \rightarrow_{1\text{-S}} E^\bullet[\mathsf{BndryErr}]$
      by $(\lambda(x\!:\!\tau_d).\, e')\ v_1 \rhd_{1\text{-S}} \mathsf{BndryErr}$
   b. QED

CASE $e = E^\bullet[op^1\ v]$:

1. $op^1 = \mathsf{fst}$
   $\vee\ op^1 = \mathsf{snd}$
2. $\vdash_1 op^1\ v : K'$
   by *static hole typing*
3. $\vdash_1 v : \mathsf{Pair}$
   by 1 *inversion* (2)
4. $v = \langle v_0, v_1 \rangle$
   by *canonical forms* (3)
5. $\delta(op^1, v) = v_i$ where $i \in \{0, 1\}$
   by (1, 3)
6. $e \rightarrow_{1\text{-S}} E^\bullet[v_i]$
   by $(op^1\ v) \rhd_{1\text{-S}} v_i$
7. QED

CASE $e = E^\bullet[op^2\ v_0\ v_1]$:

1. $\vdash_1 op^2\ v_0\ v_1 : K'$
   by *static hole typing*
2. $\vdash_1 v_0 : K_0$
   $\wedge \vdash_1 v_1 : K_1$
   $\wedge\ \Delta(op^2, K_0, K_1) = K_2$
   by 1 *inversion* (1)
3. $\delta(op^2, v_0, v_1) = e''$
   by *Δ tag soundness*
4. QED by $e \rightarrow_{1\text{-S}} E^\bullet[e'']$

CASE $e = E^\bullet[\mathsf{chk}\ K\ v_0]$:

1. $e \rightarrow_{1\text{-S}} E^\bullet[\mathcal{X}(K, v)]$
2. QED

CASE $e = E[\mathsf{dyn}\ e']$ where e' is boundary-free :

1. e' is a value
   $\vee\ e' \in \mathsf{Err}$
   $\vee\ e' \rightarrow_{1\text{-D}} e''$

$\vee\, e' \rightarrow_{1\text{-D}}$ BndryErr
$\vee\, e' = E'[e'']$ and $e'' \rhd_{1\text{-D}}$ TagErr
by 1 *dynamic progress*
2. **IF** e' is a value :
 a. QED $e \rightarrow_{1\text{-S}} E[v]$
IF $e' \in$ Err :
 a. QED $e \rightarrow_{1\text{-S}} e'$
IF $e' \rightarrow_{1\text{-D}} e''$:
 a. QED $e \rightarrow_{1\text{-S}} E[\text{dyn } e'']$
IF $e' \rightarrow_{1\text{-D}}$ BndryErr :
 a. QED $e \rightarrow_{1\text{-S}} E[\text{dyn BndryErr}]$
ELSE $e' = E'[e'']$ and $e'' \rhd_{1\text{-D}}$ TagErr :
 a. $E' \in E^{\bullet}$
 by e' is boundary-free
 b. QED
CASE $e = E[\text{stat } e']$ where e' is boundary-free :
1. e' is a value
$\vee\, e' \in$ Err
$\vee\, e' \rightarrow_{1\text{-S}} e''$
$\vee\, e' \rightarrow_{1\text{-S}}$ BndryErr
$\vee\, e' = E''[\text{dyn } \tau'' \; E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{1\text{-D}}$ TagErr
$\vee\, e' = E''[\text{dyn } E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{1\text{-D}}$ TagErr
by 1 *static progress*
2. **IF** e' is a value :
 a. QED $e \rightarrow_{1\text{-S}} E[e']$
IF $e' \in$ Err :
 a. QED $e \rightarrow_{1\text{-S}} e'$
IF $e' \rightarrow_{1\text{-S}} e''$:
 a. QED $e \rightarrow_{1\text{-S}} E[\text{stat } e'']$
IF $e' \rightarrow_{1\text{-S}}$ BndryErr :
 a. QED $e \rightarrow_{1\text{-S}} E[\text{stat BndryErr}]$
IF $e' = E''[\text{dyn } \tau'' \; E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{1\text{-D}}$ TagErr
:
 a. Contradiction by e' is boundary-free
ELSE $e' = E''[\text{dyn } E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{1\text{-D}}$ TagErr
:
 a. Contradiction by e' is boundary-free
CASE $e = E[\text{dyn } \tau \; e']$ where e' is boundary-free :
1. e' is a value
$\vee\, e' \in$ Err
$\vee\, e' \rightarrow_{1\text{-D}} e''$
$\vee\, e' \rightarrow_{1\text{-D}}$ BndryErr
$\vee\, e' = E'[e'']$ and $e'' \rhd_{1\text{-D}}$ TagErr
by 1 *dynamic progress*
2. **IF** e' is a value :
 a. QED $e \rightarrow_{1\text{-S}} E[\mathcal{D}_1(\tau', e')]$
IF $e' \in$ Err :
 a. QED $e \rightarrow_{1\text{-S}} e'$
IF $e' \rightarrow_{1\text{-D}} e''$:
 a. QED $e \rightarrow_{1\text{-S}} E[\text{dyn } \tau' \; e'']$
IF $e' \rightarrow_{1\text{-D}}$ BndryErr :
 a. QED $e \rightarrow_{1\text{-S}} E[\text{dyn } \tau' \text{ BndryErr}]$
ELSE $e' = E'[e'']$ and $e'' \rhd_{1\text{-D}}$ TagErr :
 a. $E' \in E^{\bullet}$
 by e' is boundary-free
 b. QED
CASE $e = E[\text{stat } \tau \; e']$ where e' is boundary-free :
1. e' is a value
$\vee\, e' \in$ Err
$\vee\, e' \rightarrow_{1\text{-S}} e''$
$\vee\, e' \rightarrow_{1\text{-S}}$ BndryErr
$\vee\, e' = E''[\text{dyn } \tau'' \; E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{1\text{-D}}$ TagErr
$\vee\, e' = E''[\text{dyn } E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{1\text{-D}}$ TagErr
by 1 *static progress*
2. **IF** e' is a value :
 a. QED $e \rightarrow_{1\text{-S}} E[\mathcal{S}_1(\tau', e')]$
IF $e' \in$ Err :
 a. QED $e \rightarrow_{1\text{-S}} e'$
IF $e' \rightarrow_{1\text{-S}} e''$:
 a. QED $e \rightarrow_{1\text{-S}} E[\text{stat } \tau' \; e'']$
IF $e' \rightarrow_{1\text{-S}}$ BndryErr :
 a. QED $e \rightarrow_{1\text{-S}} E[\text{stat } \tau' \text{ BndryErr}]$
IF $e' = E''[\text{dyn } \tau'' \; E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{1\text{-D}}$ TagErr
:
 a. Contradiction by e' is boundary-free
ELSE $e' = E''[\text{dyn } E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{1\text{-D}}$ TagErr
:
 a. Contradiction by e' is boundary-free
CASE $e = E[\text{Err}]$:
1. QED $e \rightarrow_{1\text{-S}}$ Err

□

Lemma 4.11 : 1 *dynamic progress*
If $\vdash_1 e : K$ then one of the following holds:
- e is a value
- $e \in$ Err
- $e \rightarrow_{1\text{-D}} e'$
- $e \rightarrow_{1\text{-D}}$ BndryErr
- $e = E[e']$ and $e' \rightarrow_{1\text{-D}}$ TagErr

Proof:
By the *boundary factoring* lemma, there are nine cases.
CASE $e = v$:
1. QED
CASE $e = E^{\bullet}[v_0 \; v_1]$:
IF $v_0 = \lambda x.\, e_0$:
1. $e \rightarrow_{1\text{-D}} E^{\bullet}[e_0[x \leftarrow v_1]]$
 by $(\lambda x.\, e_0)\; v_1 \rhd_{1\text{-D}} e_0[x \leftarrow v_1]$
2. QED
IF $v_0 = \lambda(x\!:\!\tau_d).\, e_0$
 $\wedge\, \mathcal{X}(\lfloor \tau_d \rfloor, v_1) = v_1$:
1. $e \rightarrow_{1\text{-D}} E^{\bullet}[\text{stat } (e_0[x \leftarrow v_1])]$
 by $(\lambda(x\!:\!\tau_d).\, e_0)\; v_1 \rhd_{1\text{-D}} (\text{stat } e_0[x \leftarrow v_1])$
2. QED
IF $v_0 = \lambda(x\!:\!\tau_d).\, e_0$
 $\wedge\, \mathcal{X}(\lfloor \tau_d \rfloor, v_1) =$ BndryErr :
1. $e \rightarrow_{1\text{-D}} E^{\bullet}[\text{BndryErr}]$
 by $(\lambda(x\!:\!\tau_d).\, e_0)\; v_1 \rhd_{1\text{-D}}$ BndryErr
2. QED

else $v_0 = i$
$\vee\, v_0 = \langle v, v' \rangle$:
1. $e \rightarrow_{1\text{-D}} E^{\bullet}[\mathsf{TagErr}]$
by $v_0\ v_1 \rhd_{1\text{-D}} \mathsf{TagErr}$
2. qed

case $e = E^{\bullet}[op^1\ v]$:
if $\delta(op^1, v) = v'$:
1. $e \rightarrow_{1\text{-D}} E^{\bullet}[v']$
by $(op^1\ v) \rhd_{1\text{-D}} v'$
2. qed
else $\delta(op^1, v)$ is undefined :
1. $e \rightarrow_{1\text{-D}} \mathsf{TagErr}$
by $(op^1\ v) \rhd_{1\text{-D}} \mathsf{TagErr}$
2. qed

case $e = E^{\bullet}[op^2\ v_0\ v_1]$:
if $\delta(op^2, v_0, v_1) = e''$:
1. qed by $e \rightarrow_{1\text{-D}} E[e'']$
else $\delta(op^2, v_0, v_1)$ is undefined :
1. $e \rightarrow_{1\text{-D}} E^{\bullet}[\mathsf{TagErr}]$
by $(op^2\ v_0\ v_1) \rhd_{1\text{-D}} \mathsf{TagErr}$
2. qed

case $e = E^{\bullet}[\mathsf{chk}\ K\ v_0]$:
1. Contradiction by $\vdash_1 e$

case $e = E[\mathsf{dyn}\ v]$ where e' is boundary-free :
1. e' is a value
$\vee\, e' \in \mathsf{Err}$
$\vee\, e' \rightarrow_{1\text{-D}} e''$
$\vee\, e' \rightarrow_{1\text{-D}} \mathsf{BndryErr}$
$\vee\, e' = E'[e'']$ and $e'' \rhd_{1\text{-D}} \mathsf{TagErr}$
by 1 *dynamic progress*
2. **if** e' is a value :
a. qed $e \rightarrow_{1\text{-S}} E[v]$
if $e' \in \mathsf{Err}$:
a. qed $e \rightarrow_{1\text{-S}} e'$
if $e' \rightarrow_{1\text{-D}} e''$:
a. qed $e \rightarrow_{1\text{-S}} E[\mathsf{dyn}\ e'']$
if $e' \rightarrow_{1\text{-D}} \mathsf{BndryErr}$:
a. qed $e \rightarrow_{1\text{-S}} E[\mathsf{dyn}\ \mathsf{BndryErr}]$
else $e' = E'[e'']$ and $e'' \rhd_{1\text{-D}} \mathsf{TagErr}$:
a. $E' \in E^{\bullet}$
by e' is boundary-free
b. qed

case $e = E[\mathsf{stat}\ e']$ where e' is boundary-free :
1. e' is a value
$\vee\, e' \in \mathsf{Err}$
$\vee\, e' \rightarrow_{1\text{-S}} e''$
$\vee\, e' \rightarrow_{1\text{-S}} \mathsf{BndryErr}$
$\vee\, e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{1\text{-D}} \mathsf{TagErr}$
$\vee\, e' = E''[\mathsf{dyn}\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{1\text{-D}} \mathsf{TagErr}$
by 1 *static progress*
2. **if** e' is a value :
a. qed $e \rightarrow_{1\text{-S}} E[e']$
if $e' \in \mathsf{Err}$:
a. qed $e \rightarrow_{1\text{-S}} e'$
if $e' \rightarrow_{1\text{-S}} e''$:
a. qed $e \rightarrow_{1\text{-S}} E[\mathsf{stat}\ e'']$
if $e' \rightarrow_{1\text{-S}} \mathsf{BndryErr}$:
a. qed $e \rightarrow_{1\text{-S}} E[\mathsf{stat}\ \mathsf{BndryErr}]$
if $e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{1\text{-D}} \mathsf{TagErr}$
:
a. Contradiction by e' is boundary-free
else $e' = E''[\mathsf{dyn}\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{1\text{-D}} \mathsf{TagErr}$
:
a. Contradiction by e' is boundary-free

case $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free :
1. e' is a value
$\vee\, e' \in \mathsf{Err}$
$\vee\, e' \rightarrow_{1\text{-D}} e''$
$\vee\, e' \rightarrow_{1\text{-D}} \mathsf{BndryErr}$
$\vee\, e' = E[e'']$ and $e'' \rhd_{1\text{-D}} \mathsf{TagErr}$
by 1 *dynamic progress*
2. **if** e' is a value :
a. qed $e \rightarrow_{1\text{-D}} E[\mathcal{D}_1(\tau', e')]$
if $e' \in \mathsf{Err}$:
a. qed $e \rightarrow_{1\text{-D}} e'$
if $e' \rightarrow_{1\text{-D}} e''$:
a. qed $e \rightarrow_{1\text{-S}} E[\mathsf{dyn}\ \tau'\ e'']$
if $e' \rightarrow_{1\text{-D}} \mathsf{BndryErr}$:
a. qed $e \rightarrow_{1\text{-D}} E[\mathsf{dyn}\ \tau'\ \mathsf{BndryErr}]$
else $e' = E[e'']$ and $e'' \rhd_{1\text{-D}} \mathsf{TagErr}$:
a. $E \in E^{\bullet}$
by e' is boundary-free
b. qed

case $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free :
1. e' is a value
$\vee\, e' \in \mathsf{Err}$
$\vee\, e' \rightarrow_{1\text{-S}} e''$
$\vee\, e' \rightarrow_{1\text{-S}} \mathsf{BndryErr}$
$\vee\, e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{1\text{-D}} \mathsf{TagErr}$
$\vee\, e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{1\text{-D}} \mathsf{TagErr}$
by 1 *static progress*
2. **if** e' is a value :
a. qed $e \rightarrow_{1\text{-S}} E[\mathcal{S}_1(\tau', e')]$
if $e' \in \mathsf{Err}$:
a. qed $e \rightarrow_{1\text{-S}} e'$
if $e' \rightarrow_{1\text{-S}} e''$:
a. qed $e \rightarrow_{1\text{-S}} E[\mathsf{stat}\ \tau'\ e'']$
if $e' \rightarrow_{1\text{-S}} \mathsf{BndryErr}$:
a. qed $e \rightarrow_{1\text{-S}} E[\mathsf{stat}\ \tau'\ \mathsf{BndryErr}]$
if $e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{1\text{-D}} \mathsf{TagErr}$
:
a. Contradiction by e' is boundary-free
else $e' = E''[\mathsf{dyn}\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{1\text{-D}} \mathsf{TagErr}$
:
a. Contradiction by e' is boundary-free

case $e = E[\mathsf{Err}]$:
1. $e \rightarrow_{1\text{-D}} \mathsf{Err}$
2. qed

□

Lemma 4.12 : 1 *static preservation*

If $\vdash_1 e : K$ and $e \rightarrow_{1\text{-S}} e'$ then $\vdash_1 e' : K$

Proof:

By the *boundary factoring* lemma, there are ten cases to consider.

CASE e is a value :

1. Contradiction by $e \rightarrow_{1\text{-S}} e'$

CASE $e = E^\bullet[v_0\ v_1]$:

IF $v_0 = \lambda x.\, e'$
$\wedge\ e \rightarrow_{1\text{-S}} E^\bullet[\mathsf{dyn}\ e'[x \leftarrow v_1]]$:

1. $\vdash_1 v_0\ v_1 : \mathsf{Any}$
by *static hole typing*
2. $\vdash_1 v_0 : \mathsf{Fun}$
$\wedge \vdash_1 v_1 : \mathsf{Any}$
by *1 inversion*
3. $x \vdash_1 e'$
by *1 inversion* (2)
4. $\vdash_1 v_1$
by *static value inversion* (2)
5. $\vdash_1 e'[x \leftarrow v_1]$
by *substitution* (3, 4)
6. $\vdash_1 \mathsf{dyn}\ (e'[x \leftarrow v_1]) : \mathsf{Any}$
by (5)
7. QED by *hole substitution*

IF $v_0 = \lambda(x\!:\!\tau).\, e'$
$\wedge\ \mathcal{X}(\lfloor\tau\rfloor, v_1) = \mathsf{BndryErr}$
$\wedge\ e \rightarrow_{1\text{-D}} E^\bullet[\mathsf{BndryErr}]$:

1. $\vdash_1 v_0\ v_1 : \mathsf{Any}$
by *static hole typing*
2. $\vdash_1 \mathsf{BndryErr} : \mathsf{Any}$
3. QED by *hole substitution* (2)

ELSE $v_0 = \lambda(x\!:\!\tau).\, e'$
$\wedge\ e \rightarrow_{1\text{-S}} E^\bullet[e'[x \leftarrow \mathcal{X}(\lfloor\tau\rfloor, v_1)]]$:

1. $\vdash_1 v_0\ v_1 : \mathsf{Any}$
by *static hole typing*
2. $\vdash_1 v_0 : \mathsf{Fun}$
$\wedge \vdash_1 v_1 : \mathsf{Any}$
by *1 inversion* (1)
3. $(x\!:\!\tau) \vdash_1 e' : \mathsf{Any}$
by *1 inversion* (2)
4. $\vdash_1 \mathcal{X}(\lfloor\tau\rfloor, v_1) : \lfloor\tau\rfloor$
by *check soundness* (2)
5. $\vdash_1 e[x \leftarrow \mathcal{X}(\lfloor\tau\rfloor, v_1)] : \mathsf{Any}$
by *substitution* (3, 4)
6. QED by *hole substitution*

CASE $e = E^\bullet[op^1\ v]$
$\wedge\ \delta(op^1, v) = v'$
$\wedge\ e \rightarrow_{1\text{-S}} E^\bullet[v']$:

1. $\vdash_1 op^1\ v : \mathsf{Any}$
by *static hole typing*
2. $\vdash_1 v : \mathsf{Pair}$
by *1 inversion*
3. $v = \langle v_0, v_1\rangle$
by *canonical forms*
4. $\vdash_1 v_0 : \mathsf{Any}$
$\wedge \vdash_1 v_1 : \mathsf{Any}$
by *1 inversion* (2, 3)
5. $v' = v_0$
$\vee\ v' = v_1$
by $\delta(\mathsf{fst}, v) = v_0$
$\wedge\ \delta(\mathsf{snd}, v) = v_1$
6. QED by *hole substitution* (5)

CASE $e = E^\bullet[op^2\ v_0\ v_1]$
$\wedge\ \delta(op^2, v_0, v_1) = e''$
$\wedge\ e \rightarrow_{1\text{-S}} E^\bullet[e'']$:

1. $\vdash_1 op^2\ v_0\ v_1 : K'$
by *static hole typing*
2. $\vdash_1 v_0 : K_0$
$\wedge \vdash_1 v_1 : K_1$
$\wedge\ \Delta(op^2, K_0, K_1) = K''$
$\wedge\ K'' <: K'$
by *1 inversion* (1)
3. $\vdash_1 e'' : K''$
by *Δ tag soundness* (3)
4. $\vdash_1 e'' : K'$
by (2, 3)
5. QED by *hole substitution* (4)

CASE $e = E^\bullet[\mathsf{chk}\ K_0\ v_0]$:

1. $E^\bullet[\mathsf{chk}\ K_0\ v_0] \rightarrow_{1\text{-S}} E^\bullet[\mathcal{X}(K_0, v_0)]$
2. $\vdash_1 \mathsf{chk}\ K_0\ v : K''$
by *static hole typing*
3. $K_0 \leqslant: K''$
by *1 inversion*
4. $\vdash_1 \mathcal{X}(K_0, v_0) : K_0$
by *check soundness*
5. QED by (3, 4, *hole substitution*)

CASE $e = E[\mathsf{dyn}\ e']$ where e' is boundary-free :

IF e' is a value :

1. $e \rightarrow_{1\text{-S}} E[e']$
2. $\vdash_1 \mathsf{dyn}\ e' : \mathsf{Any}$
by *boundary hole typing*
3. $\vdash_1 e'$
by *1 inversion* (2)
4. $\vdash_1 e' : \mathsf{Any}$
by *dynamic value inversion* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{1\text{-D}} e''$:

1. $e \rightarrow_{1\text{-S}} E[\mathsf{dyn}\ e'']$
2. $\vdash_1 \mathsf{dyn}\ e' : \mathsf{Any}$
by *boundary hole typing*
3. $\vdash_1 e'$
by *1 inversion* (2)
4. $\vdash_1 e''$
by *1 dynamic preservation* (3)

5. $\vdash_1 \mathsf{dyn}\ e'' : \mathsf{Any}$
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{stat}\ e']$ where e' is boundary-free :

IF e' is a value :
1. $e \rightarrow_{1\text{-S}} E[e']$
2. $\vdash_1 \mathsf{stat}\ e'$
by *boundary hole typing*
3. $\vdash_1 e' : \mathsf{Any}$
by *1 inversion* (2)
4. $\vdash_1 e'$
by *static value inversion* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{1\text{-S}} e''$:
1. $e \rightarrow_{1\text{-S}} E[\mathsf{stat}\ e'']$
2. $\vdash_1 \mathsf{stat}\ e'$
by *boundary hole typing*
3. $\vdash_1 e' : \mathsf{Any}$
by *1 inversion* (2)
4. $\vdash_1 e'' : \mathsf{Any}$
by *1 static preservation* (3)
5. $\vdash_1 \mathsf{stat}\ e''$
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free :

IF e' is a value :
1. $e \rightarrow_{1\text{-S}} E[\mathcal{D}_1(\tau', e')]$
2. $\vdash_1 \mathsf{dyn}\ \tau'\ e' : \lfloor \tau' \rfloor$
by *boundary hole typing*
3. $\vdash_1 e'$
by *1 inversion* (2)
4. $\vdash_1 \mathcal{D}_1(\tau', e') : \lfloor \tau' \rfloor$
by *$\mathcal{D}_1$ soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{1\text{-D}} e''$:
1. $e \rightarrow_{1\text{-S}} E[\mathsf{dyn}\ \tau'\ e'']$
2. $\vdash_1 \mathsf{dyn}\ \tau'\ e' : \lfloor \tau' \rfloor$
by *boundary hole typing*
3. $\vdash_1 e'$
by *1 inversion* (2)
4. $\vdash_1 e''$
by *1 dynamic preservation* (3)
5. $\vdash_1 \mathsf{dyn}\ \tau'\ e'' : \lfloor \tau' \rfloor$
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free :

IF e' is a value :
1. $e \rightarrow_{1\text{-S}} E[\mathcal{S}_1(\tau', e')]$
2. $\vdash_1 \mathsf{stat}\ \tau'\ e'$
by *boundary hole typing*
3. $\vdash_1 e' : \lfloor \tau' \rfloor$
by *1 inversion* (2)
4. $\vdash_1 \mathcal{S}_1(\tau', e')$
by *$\mathcal{S}_1$ soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{1\text{-S}} e''$:
1. $e \rightarrow_{1\text{-S}} E[\mathsf{stat}\ \tau'\ e'']$
2. $\vdash_1 \mathsf{stat}\ \tau'\ e'$
by *boundary hole typing*
3. $\vdash_1 e' : \lfloor \tau' \rfloor$
by *1 inversion* (2)
4. $\vdash_1 e'' : \lfloor \tau' \rfloor$
by *1 static preservation* (3)
5. $\vdash_1 \mathsf{stat}\ \tau'\ e''$
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{Err}]$:
1. $e \rightarrow_{1\text{-S}} \mathsf{Err}$
2. QED $\vdash_1 \mathsf{Err} : K$

□

Lemma 4.13 : 1 *dynamic preservation*

If $\vdash_1 e$ and $e \rightarrow_{1\text{-D}} e'$ then $\vdash_1 e'$

Proof:

By *boundary factoring* there are nine cases.

CASE e is a value :
1. Contradiction by $e \rightarrow_{1\text{-D}} e'$

CASE $e = E^{\bullet}[v_0\ v_1]$:

IF $v_0 = \lambda x.\, e'$
$\wedge\ e \rightarrow_{1\text{-D}} E^{\bullet}[e'[x \leftarrow v_1]]$:
1. $\vdash_1 v_0\ v_1$
by *dynamic hole typing*
2. $\vdash_1 v_0$
$\wedge \vdash_1 v_1$
by *1 inversion* (1)
3. $x \vdash_1 e'$
by *1 inversion* (2)
4. $\vdash_1 e'[x \leftarrow v_1]$
by *substitution* (2, 3)
5. QED by *hole substitution*

IF $v_0 = \lambda(x\!:\!\tau).\, e'$
$\wedge\ \mathcal{X}(\lfloor \tau \rfloor, v_1) = \mathsf{BndryErr}$
$\wedge\ e \rightarrow_{1\text{-D}} E^{\bullet}[\mathsf{BndryErr}]$:
1. $\vdash_1 v_0\ v_1$
by *dynamic hole typing*
2. $\vdash_1 \mathsf{BndryErr}$
3. QED by *hole substitution* (2)

ELSE $v_0 = \lambda(x\!:\!\tau).\, e'$
$\wedge\ e \rightarrow_{1\text{-D}} E^{\bullet}[\mathsf{stat}\ (e'[x \leftarrow \mathcal{X}(\lfloor \tau \rfloor, v_1)])]$:
1. $\vdash_1 v_0\ v_1$
by *dynamic hole typing*
2. $\vdash_1 v_0$
$\wedge \vdash_1 v_1$
by *1 inversion* (1)
3. $(x\!:\!\tau) \vdash_1 e : \mathsf{Any}$
by *1 inversion* (2)
4. $\vdash_1 \mathcal{X}(\lfloor \tau \rfloor, v_1) : \lfloor \tau \rfloor$
by *check soundness* (2)

5. $\vdash_1 e[x \leftarrow \mathcal{X}(\lfloor\tau\rfloor, v_1)] : \mathsf{Any}$
by *substitution* (3, 4)
6. $\vdash_1 \mathsf{stat}\ (e[x \leftarrow \mathcal{X}(\lfloor\tau\rfloor, v_1)])$
by (5)
7. QED by *hole substitution* (6)

CASE $e = E^{\bullet}[op^1\ v]$
$\wedge\ \delta(op^1, v) = v'$
$\wedge\ e \rightarrow_{1\text{-D}} E^{\bullet}[v']$:
1. $\vdash_1 op^1\ v$
by *dynamic hole typing*
2. $\vdash_1 v$
by 1 *inversion* (1)
3. $\vdash_1 v'$
by *δ preservation* (2)
4. QED by *hole substitution* (3)

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$
$\wedge\ \delta(op^2, v_0, v_1) = e''$
$\wedge\ e \rightarrow_{1\text{-D}} E^{\bullet}[e'']$:
1. $\vdash_1 op^2\ v_0\ v_1$
by *dynamic hole typing*
2. $\vdash_1 v_0$
$\wedge \vdash_1 v_1$
by 1 *inversion* (1)
3. $\vdash_1 e''$
by *δ preservation* (2)
4. QED by *hole substitution* (3)

CASE $e = E[\mathsf{dyn}\ e']$ where e' is boundary-free :
IF e' is a value :
1. $e \rightarrow_{1\text{-D}} E[e']$
2. $\vdash_1 \mathsf{dyn}\ e' : \mathsf{Any}$
by *boundary hole typing*
3. $\vdash_1 e'$
by 1 *inversion* (2)
4. $\vdash_1 e' : \mathsf{Any}$
by *$\mathcal{D}_1$ soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{1\text{-D}} e''$:
1. $e \rightarrow_{1\text{-D}} E[\mathsf{dyn}\ e'']$
2. $\vdash_1 \mathsf{dyn}\ e' : \mathsf{Any}$
by *boundary hole typing*
3. $\vdash_1 e'$
by 1 *inversion* (2)
4. $\vdash_1 e''$
by 1 *dynamic preservation* (3)
5. $\vdash_1 \mathsf{dyn}\ e'' : \mathsf{Any}$
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{stat}\ e']$ where e' is boundary-free :
IF $e' \in v$:
1. $e \rightarrow_{1\text{-D}} E[e']$
2. $\vdash_1 \mathsf{stat}\ e'$
by *boundary hole typing*
3. $\vdash_1 e' : \mathsf{Any}$
by 1 *inversion* (2)
4. $\vdash_1 e'$
by *static value inversion* (3)
5. QED by *hole substitution* (5)

ELSE $e' \rightarrow_{1\text{-S}} e''$:
1. $e \rightarrow_{1\text{-D}} E[\mathsf{stat}\ e'']$
2. $\vdash_1 \mathsf{stat}\ e'$
by *boundary hole typing*
3. $\vdash_1 e' : \mathsf{Any}$
by 1 *inversion* (2)
4. $\vdash_1 e'' : \mathsf{Any}$
by 1 *static preservation* (3)
5. $\vdash_1 \mathsf{stat}\ e''$
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free :
IF e' is a value :
1. $e \rightarrow_{1\text{-D}} E[\mathcal{D}_1(\tau', e')]$
2. $\vdash_1 \mathsf{dyn}\ \tau'\ e' : \lfloor\tau'\rfloor$
by *boundary hole typing*
3. $\vdash_1 e'$
by 1 *inversion* (2)
4. $\vdash_1 \mathcal{D}_1(\tau', e') : \lfloor\tau'\rfloor$
by *$\mathcal{D}_1$ soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{1\text{-D}} e''$:
1. $e \rightarrow_{1\text{-D}} E[\mathsf{dyn}\ \tau'\ e'']$
2. $\vdash_1 \mathsf{dyn}\ \tau'\ e' : \lfloor\tau'\rfloor$
by *boundary hole typing*
3. $\vdash_1 e'$
$\wedge\ \tau' \leqslant: \tau''$
by 1 *inversion* (2)
4. $\vdash_1 e''$
by 1 *dynamic preservation* (3)
5. $\vdash_1 \mathsf{dyn}\ \tau'\ e'' : \lfloor\tau'\rfloor$
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free :
IF $e' \in v$:
1. $e \rightarrow_{1\text{-D}} E[\mathcal{S}_1(\tau', e')]$
2. $\vdash_1 \mathsf{stat}\ \tau'\ e'$
by *boundary hole typing*
3. $\vdash_1 e' : \lfloor\tau'\rfloor$
by 1 *inversion* (2)
4. $\vdash_1 \mathcal{S}_1(\tau', e')$
by *$\mathcal{S}_1$ soundness* (3)
5. QED by *hole substitution* (5)

ELSE $e' \rightarrow_{1\text{-S}} e''$:
1. $e \rightarrow_{1\text{-D}} E[\mathsf{stat}\ \tau'\ e'']$
2. $\vdash_1 \mathsf{stat}\ \tau'\ e'$
by *boundary hole typing*
3. $\vdash_1 e' : \lfloor\tau'\rfloor$
by 1 *inversion* (2)
4. $\vdash_1 e'' : \lfloor\tau'\rfloor$
by 1 *static preservation* (3)

5. $\vdash_1$ stat τ' e''
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{Err}]$:
1. $e \rightarrow_{\text{1-D}} \mathsf{Err}$
2. QED $\vdash_1 \mathsf{Err}$

□

Lemma 4.14 : *boundary-free progress*

If $\vdash e : \tau$ and e is boundary-free, then one of the following holds:
- e is a value
- $e \rightarrow_{\text{1-S}} e'$
- $e \rightarrow_{\text{1-S}} \mathsf{BndryErr}$

Proof:

By the L *unique static evaluation contexts* lemma, there are five cases.

CASE $e = v$:
1. QED

CASE $e = E^{\bullet}[v_0\ v_1]$:
IF $v_0 = \lambda(x\!:\!\tau').\, e'$:
1. $e \rightarrow_{\text{1-S}} E^{\bullet}[e'[x \leftarrow v_1]]$
by $v_0\ v_1 \rhd_{\text{1-S}} e'[x \leftarrow v_1]$
2. QED

ELSE $v_0 = \lambda x.\, e'$
$\vee\ v_0 = i$
$\vee\ v_0 = \langle v, v' \rangle$:
1. Contradiction by $\vdash e : \tau$

CASE $e = E^{\bullet}[op^1\ v]$:
IF $\delta(op^1, v) = e''$:
1. $e \rightarrow_{\text{1-S}} E^{\bullet}[e'']$
by $(op^1\ v) \rhd_{\text{1-S}} e''$
2. QED

ELSE $\delta(op^1, v)$ is undefined :
1. Contradiction by $\vdash e : \tau$

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:
IF $\delta(op^2, v_0, v_1) = e''$:
1. $e \rightarrow_{\text{1-S}} E^{\bullet}[e'']$
by $(op^2\ v_0\ v_1) \rhd_{\text{1-S}} e''$
2. QED

IF $\delta(op^2, v_0, v_1) = \mathsf{BndryErr}$:
1. $e \rightarrow_{\text{1-S}} \mathsf{BndryErr}$
by $(op^2\ v_0\ v_1) \rhd_{\text{1-S}} \mathsf{BndryErr}$
2. QED

ELSE $\delta(op^2, v_0, v_1)$ is undefined :
1. Contradiction by $\vdash e : \tau$

CASE $e = E^{\bullet}[\mathsf{Err}]$:
1. $E^{\bullet}[\mathsf{Err}] \rightarrow_{\text{1-S}} \mathsf{Err}$
2. QED

□

Lemma 4.15 : 1 *boundary-free preservation*

If $\vdash e : \tau$ and e is boundary-free and $e \rightarrow_{\text{1-S}} e'$ then $\vdash e' : \tau$ and e' is boundary-free.

Proof:

By the L *unique static evaluation contexts* lemma, there are five cases.

CASE e is a value :
1. Contradiction by $e \rightarrow_{\text{1-S}} e'$

CASE $e = E^{\bullet}[v_0\ v_1]$:
IF $v_0 = \lambda(x\!:\!\tau_d).\, e'$:
1. $E^{\bullet}[v_0\ v_1] \rightarrow_{\text{1-S}} E^{\bullet}[e'[x \leftarrow v_1]]$
2. $\vdash v_0\ v_1 : \tau_c$
3. $\vdash v_0 : \tau_d \Rightarrow \tau_c$
$\wedge \vdash v_1 : \tau_d$
by (2)
4. $(x\!:\!\tau_d) \vdash e' : \tau_c$
by (3)
5. $\vdash e'[x \leftarrow v_1] : \tau_c$
by *substitution* (3, 4)
6. $e'[x \leftarrow v_1]$ is boundary-free
by e' and v_1 are boundary-free
7. QED

ELSE :
1. Contradiction by $\vdash e : \tau$

CASE $e = E^{\bullet}[op^1\ v]$:
1. $E^{\bullet}[op^1\ v] \rightarrow_{\text{1-S}} E^{\bullet}[v']$
$\wedge\ \delta(op^1, v) = e''$
2. $\vdash op^1\ v : \tau'$
3. $\vdash v : \tau_0$
4. $\vdash e'' : \tau'$
by *δ preservation* (3)
5. QED

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:
1. $E^{\bullet}[op^2\ v_0\ v_1] \rightarrow_{\text{1-S}} E^{\bullet}[v']$
$\wedge\ \delta(op^2, v_0, v_1) = e''$
2. $\vdash op^2\ v_0\ v_1 : \tau'$
3. $\vdash v_0 : \tau_0$
$\wedge \vdash v_1 : \tau_1$
4. $\vdash e'' : \tau'$
by *δ preservation* (3)
5. QED

CASE $e = E^{\bullet}[\mathsf{Err}]$:
1. $E^{\bullet}[\mathsf{Err}] \rightarrow_{\text{1-S}} \mathsf{Err}$
2. QED by $\vdash \mathsf{Err} : \tau$

□

Lemma 4.16 : *$\mathcal{X}$ soundness*

For all K and v, $\vdash_1 \mathcal{X}(K, v) : K$.

Proof:

CASE $\vdash_1 v : K$:
1. $\mathcal{X}(K, v) = v$
2. QED

CASE $\nvdash_1 v : K$:
1. $\mathcal{X}(K, v) = \mathsf{BndryErr}$
2. QED

□

Lemma 4.17 : 1 *static boundary factoring*

If $\vdash_1 e : K$ then one of the following holds:
- e is a value
- $e = E^\bullet[v_0\ v_1]$
- $e = E^\bullet[op^1\ v]$
- $e = E^\bullet[op^2\ v_0\ v_1]$
- $e = E^\bullet[\mathsf{chk}\ K\ v]$
- $e = E[\mathsf{dyn}\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ e']$ where e' is boundary-free
- $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{Err}]$

Proof:

By the *unique evaluation contexts* lemma, there are ten cases.

CASE e is a value :
1. QED

CASE $e = E[v_0\ v_1]$:
1. $E = E^\bullet$
   $\vee\ E = E'[\mathsf{dyn}\ E^\bullet]$
   $\vee\ E = E'[\mathsf{stat}\ E^\bullet]$
   $\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
   by *inner boundary*
2. **IF** $E = E^\bullet$:
   a. QED $e = E^\bullet[v_0\ v_1]$
   IF $E = E'[\mathsf{dyn}\ E^\bullet]$:
   a. QED $e = E'[\mathsf{dyn}\ E^\bullet[v_0\ v_1]]$
   IF $E = E'[\mathsf{stat}\ E^\bullet]$:
   a. QED $e = E'[\mathsf{stat}\ E^\bullet[v_0\ v_1]]$
   IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[v_0\ v_1]]$
   ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[v_0\ v_1]]$

CASE $e = E[op^1\ v]$:
1. $E = E^\bullet$
   $\vee\ E = E'[\mathsf{dyn}\ E^\bullet]$
   $\vee\ E = E'[\mathsf{stat}\ E^\bullet]$
   $\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
   by *inner boundary*
2. **IF** $E = E^\bullet$:
   a. QED $e = E^\bullet[op^1\ v]$
   IF $E = E'[\mathsf{dyn}\ E^\bullet]$:
   a. QED $e = E'[\mathsf{dyn}\ E^\bullet[op^1\ v]]$
   IF $E = E'[\mathsf{stat}\ E^\bullet]$:
   a. QED $e = E'[\mathsf{stat}\ E^\bullet[op^1\ v]]$
   IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^1\ v]]$
   ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^1\ v]]$

CASE $e = E[op^2\ v_0\ v_1]$:
1. $E = E^\bullet$
   $\vee\ E = E'[\mathsf{dyn}\ E^\bullet]$
   $\vee\ E = E'[\mathsf{stat}\ E^\bullet]$
   $\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
   by *inner boundary*
2. **IF** $E = E^\bullet$:
   a. QED $e = E^\bullet[op^2\ v_0\ v_1]$
   IF $E = E'[\mathsf{dyn}\ E^\bullet]$:
   a. QED $e = E'[\mathsf{dyn}\ E^\bullet[op^2\ v_0\ v_1]]$
   IF $E = E'[\mathsf{stat}\ E^\bullet]$:
   a. QED $e = E'[\mathsf{stat}\ E^\bullet[op^2\ v_0\ v_1]]$
   IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$
   ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$

CASE $e = E[\mathsf{dyn}\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{stat}\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{dyn}\ \tau\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{stat}\ \tau\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{Err}]$:
1. QED

□

Lemma 4.18 : 1 *unique static evaluation contexts*

If $\vdash_1 e : K$ then one of the following holds:
- e is a value
- $e = E[v_0\ v_1]$
- $e = E[op^1\ v]$
- $e = E[op^2\ v_0\ v_1]$
- $e = E[\mathsf{chk}\ K\ v]$
- $e = E[\mathsf{dyn}\ v]$
- $e = E[\mathsf{stat}\ v]$
- $e = E[\mathsf{dyn}\ \tau\ v]$
- $e = E[\mathsf{stat}\ \tau\ v]$
- $e = E[\mathsf{Err}]$

Proof:

By induction on the structure of e.

CASE $e = x$:
1. Contradiction by $\vdash_1 e : K$

CASE $e = i$
$\vee\ e = \lambda x.\,e'$
$\vee\ e = \lambda(x\!:\!\tau_d).\,e'$:
1. QED e is a value

CASE $e = \langle e_0, e_1\rangle$:

IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
   by the induction hypothesis
2. $E = \langle E_0, e_1\rangle$
3. QED by $e = E[e_0']$

IF $e_0 \in v$
$\wedge\ e_1 \notin v$:
1. $e_1 = E_1[e_1']$
   by the induction hypothesis

2. $E = \langle e_0, E_1 \rangle$
3. QED by $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[\langle e_0, e_1 \rangle]$

CASE $e = e_0\ e_1$:

IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
   by the induction hypothesis
2. $E = E_0\ e_1$
3. QED by $e = E[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:
1. $e_1 = E_1[e_1']$
   by the induction hypothesis
2. $E = e_0\ E_1$
3. QED by $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[e_0\ e_1]$

CASE $e = op^1\ e_0$:
1. **IF** $e_0 \notin v$:
   a. $e_0 = E_0[e_0']$
      by the induction hypothesis
   b. $E = op^1\ E_0$
   c. QED $e = E[e_0']$
2. **ELSE** $e_0 \in v$:
   a. $E = [\,]$
   b. QED $e = E[op^1\ e_0]$

CASE $e = op^2\ e_0\ e_1$:

IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
   by the induction hypothesis
2. $E = op^2\ E_0\ e_1$
3. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:
1. $e_1 = E_1[e_1']$
   by the induction hypothesis
2. $E = op^2\ e_0\ E_1$
3. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[op^2\ e_0\ e_1]$

CASE $e = \mathsf{chk}\ K\ e_0$:

IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
   by the induction hypothesis
2. $E = \mathsf{chk}\ K\ E_0$
3. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[\mathsf{chk}\ K\ e_0]$

CASE $e = \mathsf{dyn}\ e_0$:

IF $e_0 \notin v$:
1. $\vdash_1 e_0$
   by *1 inversion*
2. $e_0 = E_0[e_0']$
   by *unique evaluation contexts* (1)
3. $E = \mathsf{dyn}\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[\mathsf{dyn}\ e_0]$

CASE $e = \mathsf{stat}\ e_0$:
1. Contradiction by $\vdash_1 e : K$

CASE $e = \mathsf{dyn}\ \tau\ e_0$:

IF $e_0 \notin v$:
1. $\vdash_1 e_0$
   by *1 inversion*
2. $e_0 = E_0[e_0']$
   by *unique evaluation contexts* (1)
3. $E = \mathsf{dyn}\ \tau\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[\mathsf{dyn}\ \tau\ e_0]$

CASE $e = \mathsf{stat}\ K'\ e_0$:
1. Contradiction by $\vdash_1 e : K$

CASE $e = \mathsf{Err}$:
1. $E = [\,]$
2. QED $e = E[\mathsf{Err}]$

□

Lemma 4.19 : 1 *inner boundary*

For all contexts E, one of the following holds:
- $E = E^\bullet$
- $E = E'[\mathsf{dyn}\ v]$
- $E = E'[\mathsf{stat}\ v]$
- $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
- $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

Proof:
By induction on the structure of E.

CASE $E = E^\bullet$:
1. QED

CASE $E = E_0\ e_1$:
1. $E_0 = E^\bullet$
   $\vee\, E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$
   $\vee\, E_0 = E_0'[\mathsf{stat}\ E^\bullet]$
   $\vee\, E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
   $\vee\, E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
   by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
   a. QED E is boundary-free

   IF $E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$:
   a. $E' = E_0'\ e_1$

b. QED $E = E'[\text{dyn } E^\bullet]$
IF $E_0 = E_0'[\text{stat } E^\bullet]$:
a. $E' = E_0'\ e_1$
b. QED $E = E'[\text{stat } E^\bullet]$
IF $E_0 = E_0'[\text{dyn } \tau\ E^\bullet]$:
a. $E' = E_0'\ e_1$
b. QED $E = E'[\text{dyn } \tau\ E^\bullet]$
ELSE $E_0 = E_0'[\text{stat } \tau\ E^\bullet]$:
a. $E' = E_0'\ e_1$
b. QED $E = E'[\text{stat } \tau\ E^\bullet]$
CASE $E = v_0\ E_1$:
1. $E_1 = E^\bullet$
$\vee\ E_1 = E_1'[\text{dyn } E^\bullet]$
$\vee\ E_1 = E_1'[\text{stat } E^\bullet]$
$\vee\ E_1 = E_1'[\text{dyn } \tau\ E^\bullet]$
$\vee\ E_1 = E_1'[\text{stat } \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_1 = E^\bullet$:
a. QED E is boundary-free
IF $E_1 = E_1'[\text{dyn } E^\bullet]$:
a. $E' = v_0\ E_1'$
b. QED $E = E'[\text{dyn } E^\bullet]$
IF $E_1 = E_1'[\text{stat } E^\bullet]$:
a. $E' = v_0\ E_1'$
b. QED $E = E'[\text{stat } E^\bullet]$
IF $E_1 = E_1'[\text{dyn } \tau\ E^\bullet]$:
a. $E' = v_0\ E_1'$
b. QED $E = E'[\text{dyn } \tau\ E^\bullet]$
ELSE $E_1 = E_1'[\text{stat } \tau\ E^\bullet]$:
a. $E' = v_0\ E_1'$
b. QED $E = E'[\text{stat } \tau\ E^\bullet]$
CASE $E = \langle E_0, e_1 \rangle$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\text{dyn } E^\bullet]$
$\vee\ E_0 = E_0'[\text{stat } E^\bullet]$
$\vee\ E_0 = E_0'[\text{dyn } \tau\ E^\bullet]$
$\vee\ E_0 = E_0'[\text{stat } \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E_0'[\text{dyn } E^\bullet]$:
a. $E' = \langle E_0', e_1 \rangle$
b. QED $E = E'[\text{dyn } E^\bullet]$
IF $E_0 = E_0'[\text{stat } E^\bullet]$:
a. $E' = \langle E_0', e_1 \rangle$
b. QED $E = E'[\text{stat } E^\bullet]$
IF $E_0 = E_0'[\text{dyn } \tau\ E^\bullet]$:
a. $E' = \langle E_0', e_1 \rangle$
b. QED $E = E'[\text{dyn } \tau\ E^\bullet]$
ELSE $E_0 = E_0'[\text{stat } \tau\ E^\bullet]$:
a. $E' = \langle E_0', e_1 \rangle$
b. QED $E = E'[\text{stat } \tau\ E^\bullet]$
CASE $E = \langle v_0, E_1 \rangle$:
1. $E_1 = E^\bullet$
$\vee\ E_0 = E_0'[\text{dyn } E^\bullet]$
$\vee\ E_0 = E_0'[\text{stat } E^\bullet]$
$\vee\ E_0 = E_0'[\text{dyn } \tau\ E^\bullet]$
$\vee\ E_0 = E_0'[\text{stat } \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_1 = E^\bullet$:
a. QED E is boundary-free
IF $E_1 = E_1'[\text{dyn } E^\bullet]$:
a. $E' = \langle v_0, E_1' \rangle$
b. QED $E = E'[\text{dyn } E^\bullet]$
IF $E_1 = E_1'[\text{stat } E^\bullet]$:
a. $E' = \langle v_0, E_1' \rangle$
b. QED $E = E'[\text{stat } E^\bullet]$
IF $E_1 = E_1'[\text{dyn } \tau\ E^\bullet]$:
a. $E' = \langle v_0, E_1' \rangle$
b. QED $E = E'[\text{dyn } \tau\ E^\bullet]$
ELSE $E_1 = E_1'[\text{stat } \tau\ E^\bullet]$:
a. $E' = \langle v_0, E_1' \rangle$
b. QED $E = E'[\text{stat } \tau\ E^\bullet]$
CASE $E = op^1\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\text{dyn } E^\bullet]$
$\vee\ E_0 = E_0'[\text{stat } E^\bullet]$
$\vee\ E_0 = E_0'[\text{dyn } \tau\ E^\bullet]$
$\vee\ E_0 = E_0'[\text{stat } \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E_0'[\text{dyn } E^\bullet]$:
a. $E' = op^1\ E_0'$
b. QED $E = E'[\text{dyn } E^\bullet]$
IF $E_0 = E_0'[\text{stat } E^\bullet]$:
a. $E' = op^1\ E_0'$
b. QED $E = E'[\text{stat } E^\bullet]$
IF $E_0 = E_0'[\text{dyn } \tau\ E^\bullet]$:
a. $E' = op^1\ E_0'$
b. QED $E = E'[\text{dyn } \tau\ E^\bullet]$
ELSE $E_0 = E_0'[\text{stat } \tau\ E^\bullet]$:
a. $E' = op^1\ E_0'$
b. QED $E = E'[\text{stat } \tau\ E^\bullet]$
CASE $E = op^2\ E_0\ e_1$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\text{dyn } E^\bullet]$
$\vee\ E_0 = E_0'[\text{stat } E^\bullet]$
$\vee\ E_0 = E_0'[\text{dyn } \tau\ E^\bullet]$
$\vee\ E_0 = E_0'[\text{stat } \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E_0'[\text{dyn } E^\bullet]$:
a. $E' = op^2\ E_0'\ e_1$
b. QED $E = E'[\text{dyn } E^\bullet]$

IF $E_0 = E_0'[\mathsf{stat}\ E^\bullet]$:
a. $E' = op^2\ E_0'\ e_1$
b. QED $E = E'[\mathsf{stat}\ E^\bullet]$
IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = op^2\ E_0'\ e_1$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = op^2\ E_0'\ e_1$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = op^2\ v_0\ E_1$:
1. $E_1 = E^\bullet$
$\vee\ E_1 = E_1'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E_1 = E_1'[\mathsf{stat}\ E^\bullet]$
$\vee\ E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_1 = E^\bullet$:
a. QED E is boundary-free
IF $E_1 = E_1'[\mathsf{dyn}\ E^\bullet]$:
a. $E' = op^2\ v_0\ E_1'$
b. QED $E = E'[\mathsf{dyn}\ E^\bullet]$
IF $E_1 = E_1'[\mathsf{stat}\ E^\bullet]$:
a. $E' = op^2\ v_0\ E_1'$
b. QED $E = E'[\mathsf{stat}\ E^\bullet]$
IF $E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = op^2\ v_0\ E_1'$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = op^2\ v_0\ E_1'$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = \mathsf{dyn}\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED
IF $E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ E^\bullet]$
IF $E_0 = E_0'[\mathsf{stat}\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ E_0'$
b. QED $E = E'[\mathsf{stat}\ E^\bullet]$
IF $E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau'\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau'\ E^\bullet]$

CASE $E = \mathsf{stat}\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED
IF $E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$:
a. $E' = \mathsf{stat}\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ E^\bullet]$
IF $E_0 = E_0'[\mathsf{stat}\ E^\bullet]$:
a. $E' = \mathsf{stat}\ E_0'$
b. QED $E = E'[\mathsf{stat}\ E^\bullet]$
IF $E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{stat}\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau'\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{stat}\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau'\ E^\bullet]$

CASE $E = \mathsf{dyn}\ \tau\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED
IF $E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ E^\bullet]$
IF $E_0 = E_0'[\mathsf{stat}\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{stat}\ E^\bullet]$
IF $E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau'\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau'\ E^\bullet]$

CASE $E = \mathsf{stat}\ \tau\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED
IF $E_0 = E_0'[\mathsf{dyn}\ E^\bullet]$:
a. $E' = \mathsf{stat}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ E^\bullet]$
IF $E_0 = E_0'[\mathsf{stat}\ E^\bullet]$:

a. $E' = \mathsf{stat}\ \tau\ E'_0$
b. QED $E = E'[\mathsf{stat}\ E^\bullet]$
IF $E_0 = E'_0[\mathsf{dyn}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{stat}\ \tau\ E'_0$
b. QED $E = E'[\mathsf{dyn}\ \tau'\ E^\bullet]$
ELSE $E_0 = E'_0[\mathsf{stat}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{stat}\ \tau\ E'_0$
b. QED $E = E'[\mathsf{stat}\ \tau'\ E^\bullet]$
CASE $E = \mathsf{chk}\ K_0\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E'_0[\mathsf{dyn}\ E^\bullet]$
$\vee\ E_0 = E'_0[\mathsf{stat}\ E^\bullet]$
$\vee\ E_0 = E'_0[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E_0 = E'_0[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E'_0[\mathsf{dyn}\ E^\bullet]$:
a. $E' = \mathsf{chk}\ K_0\ E'_0$
b. QED $E = E'[\mathsf{dyn}\ E^\bullet]$
IF $E_0 = E'_0[\mathsf{stat}\ E^\bullet]$:
a. $E' = \mathsf{chk}\ K_0\ E'_0$
b. QED $E = E'[\mathsf{stat}\ E^\bullet]$
IF $E_0 = E'_0[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = \mathsf{chk}\ K_0\ E'_0$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_0 = E'_0[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = \mathsf{chk}\ K_0\ E'_0$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
□

Lemma 4.20 : 1 *dynamic boundary factoring*
If $\vdash_1 e$ then one of the following holds:
- e is a value
- $e = E^\bullet[v_0\ v_1]$
- $e = E^\bullet[op^1\ v]$
- $e = E^\bullet[op^2\ v_0\ v_1]$
- $e = E[\mathsf{dyn}\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ e']$ where e' is boundary-free
- $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{Err}]$

Proof:
By the *unique evaluation contexts* lemma, there are ten cases.
CASE e is a value :
1. QED
CASE $e = E[v_0\ v_1]$:
1. $E = E^\bullet$
$\vee\ E = E'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E = E'[\mathsf{stat}\ E^\bullet]$
$\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. QED $e = E^\bullet[v_0\ v_1]$
IF $E = E'[\mathsf{dyn}\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ E^\bullet[v_0\ v_1]]$
IF $E = E'[\mathsf{stat}\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ E^\bullet[v_0\ v_1]]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[v_0\ v_1]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[v_0\ v_1]]$
CASE $e = E[op^1\ v]$:
1. $E = E^\bullet$
$\vee\ E = E'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E = E'[\mathsf{stat}\ E^\bullet]$
$\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. QED $e = E^\bullet[op^1\ v]$
IF $E = E'[\mathsf{dyn}\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ E^\bullet[op^1\ v]]$
IF $E = E'[\mathsf{stat}\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ E^\bullet[op^1\ v]]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^1\ v]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^1\ v]]$
CASE $e = E[op^2\ v_0\ v_1]$:
1. $E = E^\bullet$
$\vee\ E = E'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E = E'[\mathsf{stat}\ E^\bullet]$
$\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. QED $e = E^\bullet[op^2\ v_0\ v_1]$
IF $E = E'[\mathsf{dyn}\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ E^\bullet[op^2\ v_0\ v_1]]$
IF $E = E'[\mathsf{stat}\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ E^\bullet[op^2\ v_0\ v_1]]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$
CASE $e = E[\mathsf{chk}\ K'\ v]$:
1. $E = E^\bullet$
$\vee\ E = E'[\mathsf{dyn}\ E^\bullet]$
$\vee\ E = E'[\mathsf{stat}\ E^\bullet]$
$\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
by *inner boundary*
2. **IF** $E = E^\bullet$:
a. QED $e = E^\bullet[\mathsf{chk}\ K'\ v]$
IF $E = E'[\mathsf{dyn}\ E^\bullet]$:
a. QED $e = E'[\mathsf{dyn}\ E^\bullet[\mathsf{chk}\ K'\ v]]$

IF $E = E'[\mathsf{stat}\ E^{\bullet}]$:
a. QED $e = E'[\mathsf{stat}\ E^{\bullet}[\mathsf{chk}\ K'\ v]]$
IF $E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$:
a. QED $e = E'[\mathsf{dyn}\ \tau\ E^{\bullet}[\mathsf{chk}\ K'\ v]]$
ELSE $E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$:
a. QED $e = E'[\mathsf{stat}\ \tau\ E^{\bullet}[\mathsf{chk}\ K'\ v]]$
CASE $e = E[\mathsf{dyn}\ v]$:
1. QED v is boundary-free
CASE $e = E[\mathsf{stat}\ v]$:
1. QED v is boundary-free
CASE $e = E[\mathsf{dyn}\ \tau\ v]$:
1. QED v is boundary-free
CASE $e = E[\mathsf{stat}\ \tau\ v]$:
1. QED v is boundary-free
CASE $e = E[\mathsf{Err}]$:
1. QED

□

Lemma 4.21 : 1 *unique dynamic evaluation contexts*
If $\vdash_1 e$ then one of the following holds:
- e is a value
- $e = E[v_0\ v_1]$
- $e = E[op^1\ v]$
- $e = E[op^2\ v_0\ v_1]$
- $e = E[\mathsf{chk}\ K\ v]$
- $e = E[\mathsf{dyn}\ v]$
- $e = E[\mathsf{stat}\ v]$
- $e = E[\mathsf{dyn}\ \tau\ v]$
- $e = E[\mathsf{stat}\ \tau\ v]$
- $e = E[\mathsf{Err}]$

Proof:
By induction on the structure of e.
CASE $e = x$:
1. Contradiction by $\vdash_1 e$

CASE $e = i$
$\lor\ e = \lambda x.\, e'$
$\lor\ e = \lambda(x\!:\!\tau_d).\, e'$:
1. QED e is a value

CASE $e = \langle e_0, e_1\rangle$:
IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
by the induction hypothesis
2. $E = \langle E_0, e_1\rangle$
3. QED $e = E[e_0']$

IF $e_0 \in v$
$\land\ e_1 \notin v$:
1. $e_1 = E_1[e_1']$
by the induction hypothesis
2. $E = \langle e_0, E_1\rangle$
3. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\land\ e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[\langle e_0, e_1\rangle]$

CASE $e = e_0\ e_1$:
IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
by the induction hypothesis
2. $E = E_0\ e_1$
3. QED $e = E[e_0']$

IF $e_0 \in v$
$\land\ e_1 \notin v$:
1. $e_1 = E_1[e_1']$
by the induction hypothesis
2. $E = e_0\ E_1$
3. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\land\ e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[e_0\ e_1]$

CASE $e = op^1\ e_0$:
IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
by the induction hypothesis
2. $E = op^1\ E_0$
3. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[op^1\ e_0]$

CASE $e = op^2\ e_0\ e_1$:
IF $e_0 \notin v$:
1. $e_0 = E_0[e_0']$
by the induction hypothesis
2. $E = op^2\ E_0\ e_1$
3. QED $e = E[e_0']$

IF $e_0 \in v$
$\land\ e_1 \notin v$:
1. $e_1 = E_1[e_1']$
by the induction hypothesis
2. $E = op^2\ e_0\ E_1$
3. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\land\ e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[op^2\ e_0\ e_1]$

CASE $e = \mathsf{chk}\ K\ e'$:
1. Contradiction by $\vdash_1 e$

CASE $e = \mathsf{dyn}\ e_0$:
1. Contradiction by $\vdash_1 e$

CASE $e = \mathsf{stat}\ e_0$:
IF $e_0 \notin v$:
1. $\vdash_1 e_0$
by 1 *inversion*
2. $e_0 = E_0[e_0']$
by *unique evaluation contexts* (1)
3. $E = \mathsf{stat}\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$

2. QED $e = E[\mathsf{stat}\ e_0]$

CASE $e = \mathsf{dyn}\ e_0$:
Contradiction by $\vdash_1 e$

CASE $e = \mathsf{stat}\ K_0\ e_0$:

IF $e_0 \notin v$:
1. $\vdash_1 e_0$
by 1 *inversion*
2. $e_0 = E_0[e_0']$
by *unique evaluation contexts* (1)
3. $E = \mathsf{stat}\ \tau\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[\mathsf{stat}\ \tau\ e_0]$

□

Lemma 4.22 : 1 *static hole typing*

If $\vdash_1 E^\bullet[e] : K$ then the typing derivation contains a sub-term $\vdash_1 e : K'$ for some K'.

Proof:

By induction on the structure of $E^\bullet$.

CASE $E^\bullet = [\,]$:
1. QED $E^\bullet[e] = e$

CASE $E^\bullet = E^\bullet{}_0\ e_1$:
1. $E^\bullet[e] = E^\bullet{}_0[e]\ e_1$
2. $\vdash_1 E^\bullet{}_0[e] : \mathsf{Fun}$
by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = v_0\ E^\bullet{}_1$:
1. $E^\bullet[e] = v_0\ E^\bullet{}_1[e]$
2. $\vdash_1 E^\bullet{}_1[e] : \mathsf{Any}$
by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = \langle E^\bullet{}_0, e_1\rangle$:
1. $E^\bullet[e] = \langle E^\bullet{}_0[e], e_1\rangle$
2. $\vdash_1 E^\bullet{}_0[e] : \mathsf{Any}$
by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = \langle v_0, E^\bullet{}_1\rangle$:
1. $E^\bullet[e] = \langle v_0, E^\bullet{}_1[e]\rangle$
2. $\vdash_1 E^\bullet{}_1[e] : \mathsf{Any}$
by 1 *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = op^1\ E^\bullet{}_0$:
1. $E^\bullet[e] = op^1\ E^\bullet{}_0[e]$
2. $\vdash_1 E^\bullet{}_0[e] : \mathsf{Pair}$
by 1 *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = op^2\ E^\bullet{}_0\ e_1$:
1. $E^\bullet[e] = op^2\ E^\bullet{}_0[e]\ e_1$
2. $\vdash_1 E^\bullet{}_0[e] : K_0$
by 1 *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = op^2\ v_0\ E^\bullet{}_1$:
1. $E^\bullet[e] = op^2\ v_0\ E^\bullet{}_1[e]$
2. $\vdash_1 E^\bullet{}_1[e] : K_1$
by 1 *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = \mathsf{chk}\ K\ E^\bullet{}_0$:
1. $E^\bullet[e] = \mathsf{chk}\ K\ E^\bullet{}_0[e]$
2. $\vdash_1 E^\bullet{}_0[e] : \mathsf{Any}$
by 1 *inversion*
3. QED the induction hypothesis (2)

□

Lemma 4.23 : 1 *dynamic hole typing*

If $\vdash_1 E^\bullet[e]$ then the derivation contains a sub-term $\vdash_1 e$

Proof:

By induction on the structure of $E^\bullet$.

CASE $E^\bullet = [\,]$:
1. QED $E^\bullet[e] = e$

CASE $E^\bullet = E^\bullet{}_0\ e_1$:
1. $E^\bullet[e] = E^\bullet{}_0[e]\ e_1$
2. $\vdash_1 E^\bullet{}_0[e]$
by 1 *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = v_0\ E^\bullet{}_1$:
1. $E^\bullet[e] = v_0\ E^\bullet{}_1[e]$
2. $\vdash_1 E^\bullet{}_1[e]$
by 1 *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = \langle E^\bullet{}_0, e_1\rangle$:
1. $E^\bullet[e] = \langle E^\bullet{}_0[e], e_1\rangle$
2. $\vdash_1 E^\bullet{}_0[e]$
by 1 *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = \langle v_0, E^\bullet{}_1\rangle$:
1. $E^\bullet[e] = \langle v_0, E^\bullet{}_1[e]\rangle$
2. $\vdash_1 E^\bullet{}_1[e]$
by 1 *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = op^1\ E^\bullet{}_0$:
1. $E^\bullet[e] = op^1\ E^\bullet{}_0[e]$
2. $\vdash_1 E^\bullet{}_0[e]$
by 1 *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = op^2\ E^\bullet{}_0\ e_1$:
1. $E^\bullet[e] = op^2\ E^\bullet{}_0[e]\ e_1$
2. $\vdash_1 E^\bullet{}_0[e]$
by 1 *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = op^2\ v_0\ E^\bullet{}_1$:
1. $E^\bullet[e] = op^2\ v_0\ E^\bullet{}_1[e]$
2. $\vdash_1 E^\bullet{}_1[e]$
by 1 *inversion*
3. QED the induction hypothesis (2)

CASE $E^\bullet = \mathsf{chk}\ K\ E^\bullet{}_0$:
1. Contradiction by $\vdash_1 E^\bullet[e]$

□

Lemma 4.24 : 1 *boundary hole typing*

- If $\vdash_1 E[\mathsf{dyn}\ e]$ then the derivation contains a sub-term $\vdash_1 \mathsf{dyn}\ e : \mathsf{Any}$
- If $\vdash_1 E[\mathsf{dyn}\ e] : K'$ then the derivation contains a sub-term $\vdash_1 \mathsf{dyn}\ e : \mathsf{Any}$
- If $\vdash_1 E[\mathsf{stat}\ e]$ then the derivation contains a sub-term $\vdash_1 \mathsf{stat}\ e$
- If $\vdash_1 E[\mathsf{stat}\ e] : K'$ then the derivation contains a sub-term $\vdash_1 \mathsf{stat}\ e$
- If $\vdash_1 E[\mathsf{dyn}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_1 \mathsf{dyn}\ \tau\ e : \lfloor \tau \rfloor$
- If $\vdash_1 E[\mathsf{dyn}\ \tau\ e] : K'$ then the derivation contains a sub-term $\vdash_1 \mathsf{dyn}\ \tau\ e : \lfloor \tau \rfloor$
- If $\vdash_1 E[\mathsf{stat}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_1 \mathsf{stat}\ \tau\ e$
- If $\vdash_1 E[\mathsf{stat}\ \tau\ e] : K'$ then the derivation contains a sub-term $\vdash_1 \mathsf{stat}\ \tau\ e$

Proof:

By the following four lemmas: *static* dyn *hole typing*, *dynamic* dyn *hole typing*, *static* stat *hole typing*, and *dynamic* stat *hole typing*.

□

Lemma 4.25 : 1 *static* dyn *hole typing*

If $\vdash_1 E[\mathsf{dyn}\ \tau\ e] : K'$ then the derivation contains a sub-term $\vdash_1 \mathsf{dyn}\ \tau\ e : \lfloor \tau \rfloor$.

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:
1. $\vdash_1 \mathsf{dyn}\ \tau\ e : K''$
   by *static hole typing*
2. $\vdash_1 \mathsf{dyn}\ \tau\ e : \lfloor \tau \rfloor$
   by 1 *inversion* (1)
3. QED

CASE $E = E_0\ e_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = E_0[\mathsf{dyn}\ \tau\ e]\ e_1$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e] : K_0$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = v_0\ E_1[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_1 E_1[\mathsf{dyn}\ \tau\ e] : K_1$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[\mathsf{dyn}\ \tau\ e] = \langle E_0[\mathsf{dyn}\ \tau\ e], e_1 \rangle$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e] : K_0$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[\mathsf{dyn}\ \tau\ e] = \langle v_0, E_1[\mathsf{dyn}\ \tau\ e] \rangle$
2. $\vdash_1 E_1[\mathsf{dyn}\ \tau\ e] : K_1$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[\mathsf{dyn}\ \tau\ e] = op^1\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e] : K_0$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ E_0[\mathsf{dyn}\ \tau\ e]\ e_1$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e] : K_0$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_1 E_1[\mathsf{dyn}\ \tau\ e] : K_1$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ E_0$:
1. $E[\mathsf{dyn}\ \tau\ e] = \mathsf{dyn}\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e]$
   by 1 *inversion*
3. QED by *dynamic* dyn *hole typing* (2)

CASE $E = \mathsf{stat}\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{dyn}\ \tau\ e] : \tau'$

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[\mathsf{dyn}\ \tau\ e] = \mathsf{dyn}\ \tau_0\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e]$
   by 1 *inversion*
3. QED by *dynamic* dyn *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{dyn}\ \tau\ e] : \tau'$

CASE $E = \mathsf{chk}\ K_0\ E_0$:
1. $E[\mathsf{dyn}\ \tau\ e] = \mathsf{chk}\ K_0\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e] : \mathsf{Any}$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

□

Lemma 4.26 : 1 *dynamic* dyn *hole typing*

If $\vdash_1 E[\mathsf{dyn}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_1 \mathsf{dyn}\ \tau\ e : \lfloor \tau \rfloor$.

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:
1. Contradiction by $\vdash_1 E[\mathsf{dyn}\ \tau\ e]$

CASE $E = E_0\ e_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = E_0[\mathsf{dyn}\ \tau\ e]\ e_1$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = v_0\ E_1[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_1 E_1[\mathsf{dyn}\ \tau\ e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:

1. $E[\mathsf{dyn}\ \tau\ e] = \langle E_0[\mathsf{dyn}\ \tau\ e], e_1\rangle$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1\rangle$:
1. $E[\mathsf{dyn}\ \tau\ e] = \langle v_0, E_1[\mathsf{dyn}\ \tau\ e]\rangle$
2. $\vdash_1 E_1[\mathsf{dyn}\ \tau\ e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[\mathsf{dyn}\ \tau\ e] = op^1\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ E_0[\mathsf{dyn}\ \tau\ e]\ e_1$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_1 E_1[\mathsf{dyn}\ \tau\ e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{dyn}\ \tau\ e]$

CASE $E = \mathsf{stat}\ E_0$:
1. $E[\mathsf{dyn}\ \tau\ e] = \mathsf{stat}\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e] : \lfloor\tau_0\rfloor$
   by 1 *inversion*
3. QED by *static* dyn *hole typing* (2)

CASE $E = \mathsf{dyn}\ \tau\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{dyn}\ \tau\ e]$

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[\mathsf{dyn}\ \tau\ e] = \mathsf{stat}\ \tau_0\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{dyn}\ \tau\ e] : \lfloor\tau_0\rfloor$
   by 1 *inversion*
3. QED by *static* dyn *hole typing* (2)

CASE $E = \mathsf{chk}\ K_0\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{dyn}\ \tau\ e]$

□

Lemma 4.27 : 1 *static* stat *hole typing*

If $\vdash_1 E[\mathsf{stat}\ \tau\ e] : K'$ then the derivation contains a sub-term $\vdash_1 \mathsf{stat}\ \tau\ e$.

Proof:

By induction on the structure of E.

CASE $E \in E^\bullet$:
1. Contradiction by $\vdash_1 E[\mathsf{stat}\ \tau\ e] : \tau'$

CASE $E = E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e] : K_0$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_1[\mathsf{stat}\ \tau\ e] : K_1$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1\rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle E_0[\mathsf{stat}\ \tau\ e], e_1\rangle$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e] : K_0$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1\rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle v_0, E_1[\mathsf{stat}\ \tau\ e]\rangle$
2. $\vdash_1 E_1[\mathsf{stat}\ \tau\ e] : K_1$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = op^1\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e] : K_0$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e] : K_0$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_1[\mathsf{stat}\ \tau\ e] : K_1$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{dyn}\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e]$
   by 1 *inversion*
3. QED by *dynamic* stat *hole typing* (2)

CASE $E = \mathsf{stat}\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{stat}\ \tau\ e] : \tau'$

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{dyn}\ \tau_0\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e]$
   by 1 *inversion*
3. QED by *dynamic* stat *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{stat}\ \tau\ e] : \tau'$

CASE $E = \mathsf{chk}\ K_0\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{chk}\ K_0\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e] : \mathsf{Any}$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

□

Lemma 4.28 : 1 *dynamic* stat *hole typing*

If $\vdash_1 E[\mathsf{stat}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_1 \mathsf{stat}\ \tau\ e$.

Proof:

By induction on the structure of E.

CASE $E \in E^\bullet$:
1. QED by *dynamic hole typing*

CASE $E = E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_1[\mathsf{stat}\ \tau\ e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1\rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle E_0[\mathsf{stat}\ \tau\ e], e_1\rangle$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1\rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle v_0, E_1[\mathsf{stat}\ \tau\ e]\rangle$
2. $\vdash_1 E_1[\mathsf{stat}\ \tau\ e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = op^1\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_1[\mathsf{stat}\ \tau\ e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{stat}\ \tau\ e]$

CASE $E = \mathsf{stat}\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{stat}\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e] : \lfloor\tau_0\rfloor$
   by 1 *inversion*
3. QED by *static stat hole typing* (2)

CASE $E = \mathsf{dyn}\ \tau\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{stat}\ \tau\ e]$

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{stat}\ \tau_0\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_1 E_0[\mathsf{stat}\ \tau\ e] : \lfloor\tau_0\rfloor$
   by 1 *inversion*
3. QED by *static stat hole typing* (2)

CASE $E = \mathsf{chk}\ K_0\ E_0$:
1. Contradiction by $\vdash_1 E[\mathsf{stat}\ \tau\ e]$

□

Lemma 4.29 : 1 *static hole substitution*

If $\vdash_1 E^\bullet[e] : K$ and the derivation contains a sub-term $\vdash_1 e : K'$ and $\vdash_1 e' : K'$, then $\vdash_1 E^\bullet[e'] : K$

Proof:

By induction on the structure of $E^\bullet$.

CASE $E^\bullet = [\,]$:
1. $E^\bullet[e] = e$
   $\wedge\ E^\bullet[e'] = e'$
2. $\vdash_1 e : K$
   by (1)
3. $K' = K$
4. QED

CASE $E^\bullet = \langle E^\bullet{}_0, e_1\rangle$:
1. $E^\bullet[e] = \langle E^\bullet{}_0[e], e_1\rangle$
   $\wedge\ E^\bullet[e'] = \langle E^\bullet{}_0[e'], e_1\rangle$
2. $\vdash_1 \langle E^\bullet{}_0[e], e_1\rangle : K$
3. $\vdash_1 E^\bullet{}_0[e] : K_0$
   $\wedge \vdash_1 e_1 : K_1$
   by 1 *inversion*
4. $\vdash_1 E^\bullet{}_0[e'] : K_0$
   by the induction hypothesis (3)
5. $\vdash_1 \langle E^\bullet{}_0[e'], e_1\rangle : K$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^\bullet = \langle v_0, E^\bullet{}_1\rangle$:
1. $E^\bullet[e] = \langle v_0, E^\bullet{}_1[e]\rangle$
   $\wedge\ E^\bullet[e'] = \langle v_0, E^\bullet{}_1[e']\rangle$
2. $\vdash_1 \langle v_0, E^\bullet{}_1[e]\rangle : K$
3. $\vdash_1 v_0 : K_0$
   $\wedge \vdash_1 E^\bullet{}_1[e] : K_1$
   by 1 *inversion*
4. $\vdash_1 E^\bullet{}_1[e'] : K_1$
   by the induction hypothesis (3)
5. $\vdash_1 \langle v_0, E^\bullet{}_1[e']\rangle : K$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^\bullet = E^\bullet{}_0\ e_1$:
1. $E^\bullet[e] = E^\bullet{}_0[e]\ e_1$
   $\wedge\ E^\bullet[e'] = E^\bullet{}_0[e']\ e_1$
2. $\vdash_1 E^\bullet{}_0[e]\ e_1 : K$
3. $\vdash_1 E^\bullet{}_0[e] : K_0$
   $\wedge \vdash_1 e_1 : K_1$
   by 1 *inversion*
4. $\vdash_1 E^\bullet{}_0[e'] : K_0$
   by the induction hypothesis (3)
5. $\vdash_1 E^\bullet{}_0[e']\ e_1 : K$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^\bullet = v_0\ E^\bullet{}_1$:
1. $E^\bullet[e] = v_0\ E^\bullet{}_1[e]$
   $\wedge\ E^\bullet[e'] = v_0\ E^\bullet{}_1[e']$
2. $\vdash_1 v_0\ E^\bullet{}_1[e] : K$

3. $\vdash_1 v_0 : K_0$
   $\wedge \vdash_1 E^{\bullet}{}_1[e] : K_1$
   by 1 *inversion*
4. $\vdash_1 E^{\bullet}{}_1[e'] : K_1$
   by the induction hypothesis (3)
5. $\vdash_1 v_0\ E^{\bullet}{}_1[e'] : K$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^1\ E^{\bullet}{}_0$:

1. $E^{\bullet}[e] = op^1\ E^{\bullet}{}_0[e]$
   $\wedge E^{\bullet}[e'] = op^1\ E^{\bullet}{}_0[e']$
2. $\vdash_1 op^1\ E^{\bullet}{}_0[e] : K$
3. $\vdash_1 E^{\bullet}{}_0[e] : K_0$
   by 1 *inversion*
4. $\vdash_1 E^{\bullet}{}_0[e'] : K_0$
   by the induction hypothesis (3)
5. $\vdash_1 op^1\ E^{\bullet}{}_0[e'] : K$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ E^{\bullet}{}_0\ e_1$:

1. $E^{\bullet}[e] = op^2\ E^{\bullet}{}_0[e]\ e_1$
   $\wedge E^{\bullet}[e'] = op^2\ E^{\bullet}{}_0[e']\ e_1$
2. $\vdash_1 op^2\ E^{\bullet}{}_0[e]\ e_1 : K$
3. $\vdash_1 E^{\bullet}{}_0[e] : K_0$
   $\wedge \vdash_1 e_1 : K_1$
   by 1 *inversion*
4. $\vdash_1 E^{\bullet}{}_0[e'] : K_0$
   by the induction hypothesis (3)
5. $\vdash_1 op^2\ E^{\bullet}{}_0[e']\ e_1 : K$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ v_0\ E^{\bullet}{}_1$:

1. $E^{\bullet}[e] = op^2\ v_0\ E^{\bullet}{}_1[e]$
   $\wedge E^{\bullet}[e'] = op^2\ v_0\ E^{\bullet}{}_1[e']$
2. $\vdash_1 op^2\ v_0\ E^{\bullet}{}_1[e] : K$
3. $\vdash_1 v_0 : K_0$
   $\wedge \vdash_1 E^{\bullet}{}_1[e] : K_1$
   by 1 *inversion*
4. $\vdash_1 E^{\bullet}{}_1[e'] : K_1$
   by the induction hypothesis (3)
5. $\vdash_1 op^2\ v_0\ E^{\bullet}{}_1[e'] : K$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \mathsf{chk}\ K_c\ E^{\bullet}{}_0$:

1. $E^{\bullet}[e] = \mathsf{chk}\ K_c\ E^{\bullet}{}_0[e]$
   $\wedge E^{\bullet}[e'] = \mathsf{chk}\ K_c\ E^{\bullet}{}_0[e']$
2. $\vdash_1 \mathsf{chk}\ K_c\ E^{\bullet}{}_0[e] : K$
3. $\vdash_1 E^{\bullet}{}_0[e] : K_0$
   by 1 *inversion*
4. $\vdash_1 E^{\bullet}{}_0[e'] : K_0$
   by the induction hypothesis (3)
5. $\vdash_1 \mathsf{chk}\ K_c\ E^{\bullet}{}_0[e'] : K$
   by (2, 3, 4)
6. QED by (1, 5)

□

Lemma 4.30 : 1 *dynamic hole substitution*

If $\vdash_1 E^{\bullet}[e]$ and $\vdash_1 e'$ then $\vdash_1 E^{\bullet}[e']$

Proof:

By induction on the structure of $E^{\bullet}$.

CASE $E^{\bullet} = [\,]$:

1. QED $E^{\bullet}[e'] = e'$

CASE $E^{\bullet} = \langle E^{\bullet}{}_0, e_1\rangle$:

1. $E^{\bullet}[e] = \langle E^{\bullet}{}_0[e], e_1\rangle$
   $\wedge E^{\bullet}[e'] = \langle E^{\bullet}{}_0[e'], e_1\rangle$
2. $\vdash_1 \langle E^{\bullet}{}_0[e], e_1\rangle$
3. $\vdash_1 E^{\bullet}{}_0[e]$
   $\wedge \vdash_1 e_1$
   by 1 *inversion*
4. $\vdash_1 E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_1 \langle E^{\bullet}{}_0[e'], e_1\rangle$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \langle v_0, E^{\bullet}{}_1\rangle$:

1. $E^{\bullet}[e] = \langle v_0, E^{\bullet}{}_1[e]\rangle$
   $\wedge E^{\bullet}[e'] = \langle v_0, E^{\bullet}{}_1[e']\rangle$
2. $\vdash_1 \langle v_0, E^{\bullet}{}_1[e]\rangle$
3. $\vdash_1 v_0$
   $\wedge \vdash_1 E^{\bullet}{}_1[e]$
   by 1 *inversion*
4. $\vdash_1 E^{\bullet}{}_1[e']$
   by the induction hypothesis (3)
5. $\vdash_1 \langle v_0, E^{\bullet}{}_1[e']\rangle$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = E^{\bullet}{}_0\ e_1$:

1. $E^{\bullet}[e] = E^{\bullet}{}_0[e]\ e_1$
   $\wedge E^{\bullet}[e'] = E^{\bullet}{}_0[e']\ e_1$
2. $\vdash_1 E^{\bullet}{}_0[e]\ e_1$
3. $\vdash_1 E^{\bullet}{}_0[e]$
   $\wedge \vdash_1 e_1$
   by 1 *inversion*
4. $\vdash_1 E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_1 E^{\bullet}{}_0[e']\ e_1$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = v_0\ E^{\bullet}{}_1$:

1. $E^{\bullet}[e] = v_0\ E^{\bullet}{}_1[e]$
   $\wedge E^{\bullet}[e'] = v_0\ E^{\bullet}{}_1[e']$
2. $\vdash_1 v_0\ E^{\bullet}{}_1[e]$
3. $\vdash_1 v_0$
   $\wedge \vdash_1 E^{\bullet}{}_1[e]$
   by 1 *inversion*
4. $\vdash_1 E^{\bullet}{}_1[e']$
   by the induction hypothesis (3)

5. $\vdash_1 v_0\ E^{\bullet}{}_1[e']$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^1\ E^{\bullet}{}_0$:

1. $E^{\bullet}[e] = op^1\ E^{\bullet}{}_0[e]$
   $\wedge\ E^{\bullet}[e'] = op^1\ E^{\bullet}{}_0[e']$
2. $\vdash_1 op^1\ E^{\bullet}{}_0[e]$
3. $\vdash_1 E^{\bullet}{}_0[e]$
   by *1 inversion*
4. $\vdash_1 E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_1 op^1\ E^{\bullet}{}_0[e']$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ E^{\bullet}{}_0\ e_1$:

1. $E^{\bullet}[e] = op^2\ E^{\bullet}{}_0[e]\ e_1$
   $\wedge\ E^{\bullet}[e'] = op^2\ E^{\bullet}{}_0[e']\ e_1$
2. $\vdash_1 op^2\ E^{\bullet}{}_0[e]\ e_1$
3. $\vdash_1 E^{\bullet}{}_0[e]$
   $\wedge \vdash_1 e_1$
   by *1 inversion*
4. $\vdash_1 E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_1 op^2\ E^{\bullet}{}_0[e']\ e_1$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ v_0\ E^{\bullet}{}_1$:

1. $E^{\bullet}[e] = op^2\ v_0\ E^{\bullet}{}_1[e]$
   $\wedge\ E^{\bullet}[e'] = op^2\ v_0\ E^{\bullet}{}_1[e']$
2. $\vdash_1 op^2\ v_0\ E^{\bullet}{}_1[e]$
3. $\vdash_1 v_0$
   $\wedge \vdash_1 E^{\bullet}{}_1[e]$
   by *1 inversion*
4. $\vdash_1 E^{\bullet}{}_1[e']$
   by the induction hypothesis (3)
5. $\vdash_1 op^2\ v_0\ E^{\bullet}{}_1[e']$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \mathsf{chk}\ K_c\ E^{\bullet}{}_0$:

1. Contradiction by $\vdash_1 E^{\bullet}[e]$

□

Lemma 4.31 : 1 *hole substitution*

- If $\vdash_1 E[e] : K$ and the derivation contains a sub-term $\vdash_1 e : K'$ and $\vdash_1 e' : K'$ then $\vdash_1 E[e'] : K$.
- If $\vdash_1 E[e] : K$ and the derivation contains a sub-term $\vdash_1 e$ and $\vdash_1 e'$ then $\vdash_1 E[e'] : K$.
- If $\vdash_1 E[e]$ and the derivation contains a sub-term $\vdash_1 e : K'$ and $\vdash_1 e' : K'$ then $\vdash_1 E[e']$.
- If $\vdash_1 E[e]$ and the derivation contains a sub-term $\vdash_1 e$ and $\vdash_1 e'$ then $\vdash_1 E[e']$.

Proof:

By the following four lemmas: *dynamic context static hole substitution*, *dynamic context dynamic hole substitution*, *static context static hole substitution*, and *static context dynamic hole substitution*.

□

Lemma 4.32 : 1 *dynamic context static hole substitution*

If $\vdash_1 E[e]$ and contains $\vdash_1 e : K'$, and furthermore $\vdash_1 e' : K'$, then $\vdash_1 E[e']$

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:

1. Contradiction by $\vdash_1 E[e]$

CASE $E = E_0\ e_1$:

1. $E[e] = E_0[e]\ e_1$
2. $\vdash_1 E_0[e]$
   by *1 inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:

1. $E[e] = v_0\ E_1[e]$
2. $\vdash_1 E_1[e]$
   by *1 inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1\rangle$:

1. $E[e] = \langle E_0[e], e_1\rangle$
2. $\vdash_1 E_0[e]$
   by *1 inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1\rangle$:

1. $E[e] = \langle v_0, E_1[e]\rangle$
2. $\vdash_1 E_1[e]$
   by *1 inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:

1. $E[e] = op^1\ E_0[e]$
2. $\vdash_1 E_0[e]$
   by *1 inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:

1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_1 E_0[e]$
   by *1 inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:

1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_1 E_1[e]$
   by *1 inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ E_0$:

1. Contradiction by $\vdash_1 E[e]$

CASE $E = \mathsf{stat}\ E_0$:

1. $E[e] = \mathsf{stat}\ E_0[e]$
2. $\vdash_1 E_0[e] : \mathsf{Any}$
   by *1 inversion*
3. QED by *static context static hole substitution* (2)

CASE $E = \mathsf{dyn}\ \tau''\ E_0$:

1. Contradiction by $\vdash_1 E[e]$

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{stat}\ \tau_0\ E_0[e]$
2. $\vdash_1 E_0[e] : \lfloor \tau_0 \rfloor$
   by 1 *inversion*
3. QED by *static context static hole substitution* (2)

CASE $E = \mathsf{chk}\ K_0\ E_0$:
1. Contradiction by $\vdash_1 E[e]$

☐

Lemma 4.33 : 1 *dynamic context dynamic hole substitution*

If $\vdash_1 E[e]$ and contains $\vdash_1 e$, and furthermore $\vdash_1 e'$, then $\vdash_1 E[e']$

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:
1. QED by *dynamic boundary-free hole substitution*

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_1 E_0[e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_1 E_1[e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_1 E_0[e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_1 E_1[e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_1 E_0[e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_1 E_0[e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_1 E_1[e]$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ E_0$:
1. Contradiction by $\vdash_1 E[e]$

CASE $E = \mathsf{stat}\ E_0$:
1. $E[e] = \mathsf{stat}\ E_0[e]$
2. $\vdash_1 E_0[e] : \mathsf{Any}$
   by 1 *inversion*
3. QED by *static context dynamic hole substitution* (2)

CASE $E = \mathsf{dyn}\ \tau''\ E_0$:
1. Contradiction by $\vdash_1 E[e]$

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{stat}\ \tau_0\ E_0[e]$
2. $\vdash_1 E_0[e] : \lfloor \tau_0 \rfloor$
   by 1 *inversion*
3. QED by *static context dynamic hole substitution* (2)

CASE $E = \mathsf{chk}\ K_0\ E_0$:
1. Contradiction by $\vdash_1 E[e]$

☐

Lemma 4.34 : 1 *static context static hole substitution*

If $\vdash_1 E[e] : K$ and contains $\vdash_1 e : K'$, and furthermore $\vdash_1 e' : K'$, then $\vdash_1 E[e'] : K$

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:
1. QED by *static boundary-free hole substitution*

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_1 E_0[e] : K_0$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_1 E_1[e] : K_1$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_1 E_0[e] : K_0$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_1 E_1[e] : K_1$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_1 E_0[e] : K_0$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_1 E_0[e] : K_0$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$

2. $\vdash_1 E_1[e] : K_1$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ E_0$:
1. $E[e] = \mathsf{dyn}\ E_0[e]$
2. $\vdash_1 E_0[e]$
   by 1 *inversion*
3. QED by *static* dyn *hole typing* (2)

CASE $E = \mathsf{stat}\ E_0$:
1. Contradiction by $\vdash_1 E[e] : K$

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{dyn}\ \tau_0\ E_0[e]$
2. $\vdash_1 E_0[e]$
   by 1 *inversion*
3. QED by *static* dyn *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_1 E[e] : K$

CASE $E = \mathsf{chk}\ K_0\ E_0$:
1. $E[e] = \mathsf{chk}\ K_0\ E_0[e]$
2. $\vdash_1 E_0[e] : \mathsf{Any}$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

□

Lemma 4.35 : 1 *static context dynamic hole substitution*

If $\vdash_1 E[e] : K$ and contains $\vdash_1 e$, and furthermore $\vdash_1 e'$, then $\vdash_1 E[e'] : K$

Proof:

By induction on the structure of E.

CASE $E \in E^\bullet$:
1. Contradiction by $\vdash_1 E[e] : K$

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_1 E_0[e] : K_0$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_1 E_1[e] : K_1$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_1 E_0[e] : K_0$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_1 E_1[e] : K_1$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_1 E_0[e] : K_0$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_1 E_0[e] : K_0$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_1 E_1[e] : K_1$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ E_0$:
1. $E[e] = \mathsf{dyn}\ E_0[e]$
2. $\vdash_1 E_0[e]$
   by 1 *inversion*
3. QED by *dynamic* stat *hole typing* (2)

CASE $E = \mathsf{stat}\ E_0$:
1. Contradiction by $\vdash_1 E[e] : K$

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{dyn}\ \tau_0\ E_0[e]$
2. $\vdash_1 E_0[e]$
   by 1 *inversion*
3. QED by *dynamic* stat *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_1 E[e] : K$

CASE $E = \mathsf{chk}\ K_0\ E_0$:
1. $E[e] = \mathsf{chk}\ K_0\ E_0[e]$
2. $\vdash_1 E_0[e] : \mathsf{Any}$
   by 1 *inversion*
3. QED by the induction hypothesis (2)

□

Lemma 4.36 : 1 *static inversion*

- If $\vdash_1 \langle e_0, e_1 \rangle : K$ then $\vdash_1 e_0 : \mathsf{Any}$ and $\vdash_1 e_1 : \mathsf{Any}$
- If $\vdash_1 \lambda x.\, e : K$ then $x \vdash_1 e$
- If $\vdash_1 \lambda(x\!:\!\tau).\, e : K$ then $(x\!:\!\tau) \vdash_1 e : \mathsf{Any}$
- If $\vdash_1 e_0\ e_1 : K$ then $K = \mathsf{Any}$ and $\vdash_1 e_0 : \mathsf{Fun}$ and $\vdash_1 e_1 : \mathsf{Any}$
- If $\vdash_1 op^1\ e_0 : K$ then $K = \mathsf{Any}$ and $\vdash_1 e_0 : \mathsf{Pair}$
- If $\vdash_1 op^2\ e_0\ e_1 : K$ then $\vdash_1 e_0 : K_0$ and $\vdash_1 e_1 : K_1$ and $\Delta(op^2, K_0, K_1) = K'$ and $K' <: K$
- If $\vdash_1 \mathsf{dyn}\ \tau\ e : K$ then $\vdash_1 e$ and $\lfloor \tau \rfloor \leqslant: K$
- If $\vdash_1 \mathsf{chk}\ K'\ e_0 : K$ then $\vdash_1 e_0 : \mathsf{Any}$ and $K' \leqslant: K$

Proof:

QED by the definition of $\Gamma \vdash_1 e : \tau$

□

Lemma 4.37 : 1 *dynamic inversion*

- If $\vdash_1 \langle e_0, e_1 \rangle$ then $\vdash_1 e_0$ and $\vdash_1 e_1$
- If $\vdash_1 \lambda x.\, e$ then $x \vdash_1 e$
- If $\vdash_1 \lambda(x\!:\!\tau).\, e$ then $(x\!:\!\tau) \vdash_1 e : \mathsf{Any}$
- If $\vdash_1 e_0\ e_1$ then $\vdash_1 e_0$ and $\vdash_1 e_1$
- If $\vdash_1 op^1\ e_0$ then $\vdash_1 e_0$
- If $\vdash_1 op^2\ e_0\ e_1$ then $\vdash_1 e_0$ and $\vdash_1 e_1$
- If $\vdash_1 \mathsf{stat}\ \tau\ e$ then $\vdash_1 e : \lfloor \tau \rfloor$
- If $\vdash_1 \mathsf{stat}\ e$ then $\vdash_1 e : \mathsf{Any}$

Proof:

QED by the definition of $\vdash_1 e$.

□

Lemma 4.38 : 1 *canonical forms*

- If $\vdash_1 v : \mathsf{Pair}$ then $v = \langle v_0, v_1 \rangle$
- If $\vdash_1 v : \mathsf{Fun}$ then $v = \lambda x.\, e'$ or $v = \lambda(x\!:\!\tau_d).\, e'$
- If $\vdash_1 v : \mathsf{Int}$ then $v = i$
- If $\vdash_1 v : \mathsf{Nat}$ then $v \in \mathbb{N}$

Proof:

QED by definition of $\vdash_1 \cdot : K$

□

Lemma 4.39 : Δ *tag soundness*

If $\vdash_1 v_0 : K_0$ and $\vdash_1 v_1 : K_1$ and $\Delta(op^2, K_0, K_1) = K$ then $\vdash_1 \delta(op^2, v_0, v_1) : K$.

Proof:

By case analysis on Δ.

CASE $\Delta(\mathsf{sum}, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat}$:

1. $v_0 = i_0, i_0 \in \mathbb{N}$
   $\wedge\ v_1 = i_1, i_1 \in \mathbb{N}$
   by *canonical forms*
2. $\delta(\mathsf{sum}, i_0, i_1) = i_0 + i_1 \in \mathbb{N}$
3. QED

CASE $\Delta(\mathsf{sum}, \mathsf{Int}, \mathsf{Int}) = \mathsf{Int}$:

1. $v_0 = i_0$
   $\wedge\ v_1 = i_1$
   by *canonical forms*
2. $\delta(\mathsf{sum}, i_0, i_1) = i_0 + i_1 \in i$
3. QED

CASE $\Delta(\mathsf{quotient}, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat}$:

1. $v_0 = i_0, i_0 \in \mathbb{N}$
   $\wedge\ v_1 = i_1, i_1 \in \mathbb{N}$
   by *canonical forms*
2. **IF** $i_1 = 0$:
   a. $\delta(\mathsf{quotient}, i_0, i_1) = \mathsf{BndryErr}$
   b. QED by $\vdash_1 \mathsf{BndryErr} : K$
   ELSE $i_1 \neq 0$:
   a. $\delta(\mathsf{quotient}, i_0, i_1) = \lfloor i_0 / i_1 \rfloor \in \mathbb{N}$
   b. QED

CASE $\Delta(\mathsf{quotient}, \mathsf{Int}, \mathsf{Int}) = \mathsf{Int}$:

1. $v_0 = i_0$
   $\wedge\ v_1 = i_1$
   by *canonical forms*
2. **IF** $i_1 = 0$:
   a. $\delta(\mathsf{quotient}, i_0, i_1) = \mathsf{BndryErr}$
   b. QED by $\vdash_1 \mathsf{BndryErr} : K$
   ELSE $i_1 \neq 0$:
   a. $\delta(\mathsf{quotient}, i_0, i_1) = \lfloor i_0 / i_1 \rfloor \in i$
   b. QED

□

Lemma 4.40 : δ *preservation*

- If $\vdash_1 v$ and $\delta(op^1, v) = e$ then $\vdash_1 e$
- If $\vdash_1 v_0$ and $\vdash_1 v_1$ and $\delta(op^2, v_0, v_1) = e$ then $\vdash_1 e$

Proof:

CASE $\delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0$:

1. $\vdash_1 v_0$
   by 1 *inversion*
2. QED

CASE $\delta(\mathsf{snd}, \langle v_0, v_1 \rangle) = v_1$:

1. $\vdash_1 v_1$
   by 1 *inversion*
2. QED

CASE $\delta(\mathsf{sum}, v_0, v_1) = v_0 + v_1$:

1. QED

CASE $\delta(\mathsf{quotient}, v_0, v_1) = \lfloor v_0 / v_1 \rfloor$:

1. QED

CASE $\delta(op^2, v_0, v_1) = \mathsf{BndryErr}$:

1. QED

□

Lemma 4.41 : Δ *preservation*

If $\Delta(op^2, \tau_0, \tau_1) = \tau$ then $\Delta(op^2, \lfloor \tau_0 \rfloor, \lfloor \tau_1 \rfloor) = \lfloor \tau \rfloor$.

Proof:

By case analysis on the definition of Δ

CASE $\Delta(op^2, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat}$:

1. QED by $\lfloor \mathsf{Nat} \rfloor = \mathsf{Nat}$

CASE $\Delta(op^2, \mathsf{Int}, \mathsf{Int}) = \mathsf{Int}$:

1. QED by $\lfloor \mathsf{Int} \rfloor = \mathsf{Int}$

□

Lemma 4.42 : Δ *inversion*

- If $\Delta(\mathsf{fst}, \tau) = \tau'$ then $\tau = \tau_0 \times \tau_1$ and $\tau' = \tau_0$
- If $\Delta(\mathsf{snd}, \tau) = \tau'$ then $\tau = \tau_0 \times \tau_1$ and $\tau' = \tau_1$

Proof:

QED by the definition of Δ

□

Lemma 4.43 : $<:$ *preservation*

If $\tau <: \tau'$ then $\lfloor \tau \rfloor \leqslant: \lfloor \tau' \rfloor$

Proof:

By case analysis on the last rule used to show $\tau <: \tau'$.

CASE $\mathsf{Nat} <: \mathsf{Int}$:

1. QED $\lfloor \mathsf{Nat} \rfloor <: \lfloor \mathsf{Int} \rfloor$

CASE $\tau_d \Rightarrow \tau_c <: \tau_d' \Rightarrow \tau_c'$:

1. $\lfloor \tau_d \Rightarrow \tau_c \rfloor = \mathsf{Fun}$
   $\wedge\ \lfloor \tau_d' \Rightarrow \tau_c' \rfloor = \mathsf{Fun}$
2. QED

CASE $\tau_0 \times \tau_1 <: \tau_0' \times \tau_1'$:

1. $\lfloor \tau_0 \times \tau_1 \rfloor = \mathsf{Pair}$
   $\wedge\ \lfloor \tau_0' \times \tau_1' \rfloor = \mathsf{Pair}$
2. QED

□

Lemma 4.44 : 1 *static value inversion*

If $\vdash_1 v : \mathsf{Any}$ then $\vdash_1 v$

Proof:

By induction on the structure of v.

CASE $v = i$:

1. QED by $\vdash_1 v$

CASE $v = \langle v_0, v_1 \rangle$:

1. $\vdash_1 v_0 : \mathsf{Any}$
 $\land \vdash_1 v_1 : \mathsf{Any}$
 by 1 *inversion*
2. $\vdash_1 v_0$
 $\land \vdash_1 v_1$
 by the induction hypothesis
3. QED by (2)

CASE $v = \lambda x.\, e$:

1. $x \vdash_1 e$
 by 1 *inversion*
2. QED

CASE $v = \lambda(x\!:\!\tau).\, e$:

1. $(x\!:\!\tau) \vdash_1 e : \mathsf{Any}$
 by 1 *inversion*
2. QED

□

Lemma 4.45 : 1 *dynamic value inversion*

If $\vdash_1 v$ then $\vdash_1 v : \mathsf{Any}$

Proof:

By induction on the structure of v.

CASE $v = i$:

1. $\vdash_1 v : \mathsf{Int}$
2. QED by $\mathsf{Int} <: \mathsf{Any}$

CASE $v = \langle v_0, v_1 \rangle$:

1. $\vdash_1 v_0$
 $\land \vdash_1 v_1$
 by 1 *inversion*
2. $\vdash_1 v_0 : \mathsf{Any}$
 $\land \vdash_1 v_1 : \mathsf{Any}$
 by the induction hypothesis
3. $\vdash_1 \langle v_0, v_1 \rangle : \mathsf{Pair}$
 by (2)
4. QED by $\mathsf{Pair} <: \mathsf{Any}$

CASE $v = \lambda x.\, e$:

1. $x \vdash_1 e$
 by 1 *inversion*
2. $\vdash_1 \lambda x.\, e : \mathsf{Fun}$
 by (1)
3. QED by $\mathsf{Fun} <: \mathsf{Any}$

CASE $v = \lambda(x\!:\!\tau).\, e$:

1. $(x\!:\!\tau) \vdash_1 e : \mathsf{Any}$
 by 1 *inversion*
2. $\vdash_1 \lambda(x\!:\!\tau).\, e : \mathsf{Fun}$
 by (1)
3. QED by $\mathsf{Fun} <: \mathsf{Any}$

□

Lemma 4.46 : 1 *substitution*

- If $(x\!:\!\tau), \Gamma \vdash_1 e$ and $\vdash_1 v : \lfloor \tau \rfloor$ then $\Gamma \vdash_1 e[x \leftarrow v]$
- If $x, \Gamma \vdash_1 e$ and $\vdash_1 v$ then $\Gamma \vdash_1 e[x \leftarrow v]$
- If $(x\!:\!\tau_x), \Gamma \vdash_1 e : K$ and $\vdash_1 v : \lfloor \tau_x \rfloor$ then $\Gamma \vdash_1 e[x \leftarrow v] : K$
- If $x, \Gamma \vdash_1 e : K$ and $\vdash_1 v$ then $\Gamma \vdash_1 e[x \leftarrow v] : K$

Proof:

By the following four lemmas: *dynamic context static value substitution*, *dynamic context dynamic value substitution*, *static context static value substitution*, and *static context dynamic value substitution*.

□

Lemma 4.47 : 1 *dynamic-static substitution*

If $(x\!:\!\tau), \Gamma \vdash_1 e$ and $\vdash_1 v : \lfloor \tau \rfloor$ then $\Gamma \vdash_1 e[x \leftarrow v]$

Proof:

By induction on the structure of e.

CASE $e = x$:

1. $e[x \leftarrow v] = v$
2. $\vdash_1 v : \mathsf{Any}$
 by $\lfloor \tau \rfloor <: \mathsf{Any}$
3. $\vdash_1 v$
 by *static value inversion* (2)
4. $\Gamma \vdash_1 v$
 by *weakening* (3)
5. QED

CASE $e = x'$:

1. QED by $x'[x \leftarrow v] = x'$

CASE $e = i$:

1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x.\, e'$:

1. QED by $(\lambda x.\, e')[x \leftarrow v] = \lambda x.\, e'$

CASE $e = \lambda(x\!:\!\tau').\, e'$:

1. QED by $(\lambda(x\!:\!\tau').\, e')[x \leftarrow v] = \lambda(x\!:\!\tau').\, e'$

CASE $e = \lambda x'.\, e'$:

1. $e[x \leftarrow v] = \lambda x'.\, (e'[x \leftarrow v])$
2. $x', (x\!:\!\tau), \Gamma \vdash_1 e'$
 by 1 *inversion*
3. $x', \Gamma \vdash_1 e'[x \leftarrow v]$
 by *dynamic context static value substitution*
4. $\Gamma \vdash_1 \lambda x'.\, e'[x \leftarrow v]$
 by (3)
5. QED

CASE $e = \lambda(x'\!:\!\tau').\, e'$:

1. $e[x \leftarrow v] = \lambda(x'\!:\!\tau').\, (e'[x \leftarrow v])$
2. $(x'\!:\!\tau'), (x\!:\!\tau), \Gamma \vdash_1 e' : \mathsf{Any}$
 by 1 *inversion*
3. $(x'\!:\!\tau'), \Gamma \vdash_1 e'[x \leftarrow v] : \mathsf{Any}$
 by *static context static value substitution*
4. $\Gamma \vdash_1 \lambda(x'\!:\!\tau').\, (e'[x \leftarrow v])$
5. QED

CASE $e = \langle e_0, e_1 \rangle$:

1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $(x\!:\!\tau), \Gamma \vdash_1 e_0$
 $\land (x\!:\!\tau), \Gamma \vdash_1 e_1$
 by 1 *inversion*

3. $\Gamma \vdash_1 e_0[x \leftarrow v]$
   $\wedge\, \Gamma \vdash_1 e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
   by (3)
5. QED

CASE $e = e_0\ e_1$:

1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $(x : \tau), \Gamma \vdash_1 e_0$
   $\wedge\, (x : \tau), \Gamma \vdash_1 e_1$
   by *1 inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v]$
   $\wedge\, \Gamma \vdash_1 e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^1\ e_0$:

1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $(x : \tau), \Gamma \vdash_1 e_0$
   by *1 inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 op^1\ e_0[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:

1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $(x : \tau), \Gamma \vdash_1 e_0$
   $\wedge\, (x : \tau), \Gamma \vdash_1 e_1$
   by *1 inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v]$
   $\wedge\, \Gamma \vdash_1 e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \mathsf{dyn}\ \tau'\ e'$
   $\vee\, e = \mathsf{dyn}\ e'$
   $\vee\, e = \mathsf{chk}\ K'\ e'$:

1. Contradiction by $(x : \tau), \Gamma \vdash_1 e$

CASE $e = \mathsf{stat}\ \tau'\ e'$:

1. $e[x \leftarrow v] = \mathsf{stat}\ \tau'\ e'[x \leftarrow v]$
2. $(x : \tau), \Gamma \vdash_1 e' : \lfloor \tau' \rfloor$
   by *1 inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v] : \lfloor \tau' \rfloor$
   by *static context static value substitution* (2)
4. $\Gamma \vdash_1 \mathsf{stat}\ \tau'\ (e'[x \leftarrow v])$
   by (3)
5. QED

CASE $e = \mathsf{stat}\ e'$:

1. $e[x \leftarrow v] = \mathsf{stat}\ e'[x \leftarrow v]$
2. $(x : \tau), \Gamma \vdash_1 e' : \mathsf{Any}$
   by *1 inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v] : \mathsf{Any}$
   by *static context static value substitution* (2)
4. $\Gamma \vdash_1 \mathsf{stat}\ (e'[x \leftarrow v])$
   by (3)
5. QED

□

Lemma 4.48 : 1 *dynamic-dynamic substitution*

If $x, \Gamma \vdash_1 e$ and $\vdash_1 v$ then $\Gamma \vdash_1 e[x \leftarrow v]$

Proof:

By induction on the structure of e.

CASE $e = x$:

1. $e[x \leftarrow v] = v$
2. $\Gamma \vdash_1 v$
   by *weakening* (3)
3. QED

CASE $e = x'$:

1. QED by $x'[x \leftarrow v] = x'$

CASE $e = i$:

1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x.\, e'$:

1. QED by $(\lambda x.\, e')[x \leftarrow v] = \lambda x.\, e'$

CASE $e = \lambda(x : \tau').\, e'$:

1. QED by $(\lambda(x : \tau').\, e')[x \leftarrow v] = \lambda(x : \tau').\, e'$

CASE $e = \lambda x'.\, e'$:

1. $e[x \leftarrow v] = \lambda x'.\, (e'[x \leftarrow v])$
2. $x', x, \Gamma \vdash_1 e'$
   by *1 inversion*
3. $x', \Gamma \vdash_1 e'[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 \lambda x'.\, e'[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \lambda(x' : \tau').\, e'$:

1. $e[x \leftarrow v] = \lambda(x' : \tau').\, (e'[x \leftarrow v])$
2. $(x' : \tau'), x, \Gamma \vdash_1 e' : \mathsf{Any}$
   by *1 inversion*
3. $(x' : \tau'), \Gamma \vdash_1 e'[x \leftarrow v] : \mathsf{Any}$
   by *static context dynamic value substitution*
4. $\Gamma \vdash_1 \lambda(x' : \tau').\, (e'[x \leftarrow v])$
5. QED

CASE $e = \langle e_0, e_1 \rangle$:

1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $x, \Gamma \vdash_1 e_0$
   $\wedge\, x, \Gamma \vdash_1 e_1$
   by *1 inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v]$
   $\wedge\, \Gamma \vdash_1 e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
   by (3)
5. QED

CASE $e = e_0\ e_1$:

1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$

2. $x, \Gamma \vdash_1 e_0$
   $\wedge\, x, \Gamma \vdash_1 e_1$
   by 1 *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v]$
   $\wedge\, \Gamma \vdash_1 e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^1\ e_0$:
1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e_0$
   by 1 *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 op^1\ e_0[x \leftarrow v]$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:
1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e_0$
   $\wedge\, x, \Gamma \vdash_1 e_1$
   by 1 *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v]$
   $\wedge\, \Gamma \vdash_1 e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. QED

CASE $e = \mathsf{dyn}\ \tau'\ e'$
$\vee\, e = \mathsf{dyn}\ e'$
$\vee\, e = \mathsf{chk}\ K'\ e'$:
1. Contradiction by $\Gamma \vdash_1 e$

CASE $e = \mathsf{stat}\ \tau'\ e'$:
1. $e[x \leftarrow v] = \mathsf{stat}\ \tau'\ e'[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e' : \lfloor \tau' \rfloor$
   by 1 *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v] : \lfloor \tau' \rfloor$
   by *static context dynamic value substitution* (2)
4. $\Gamma \vdash_1 \mathsf{stat}\ \tau'\ (e'[x \leftarrow v])$
   by (3)
5. QED

CASE $e = \mathsf{stat}\ e'$:
1. $e[x \leftarrow v] = \mathsf{stat}\ e'[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e' : \mathsf{Any}$
   by 1 *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v] : \mathsf{Any}$
   by *static context dynamic value substitution* (2)
4. $\Gamma \vdash_1 \mathsf{stat}\ (e'[x \leftarrow v])$
   by (3)
5. QED

□

Lemma 4.49 : 1 *static-static substitution*

If $(x : \tau), \Gamma \vdash_1 e : K$ and $\vdash_1 v : \lfloor \tau \rfloor$ then $\Gamma \vdash_1 e[x \leftarrow v] : K$

Proof:

By induction on the structure of e.

CASE $e = x$:
1. $\lfloor \tau \rfloor \leqslant: K$
   by $(x : \tau), \Gamma \vdash_1 x : K$
2. $e[x \leftarrow v] = v$
3. $\vdash_1 v : K$
   by (1)
4. $\Gamma \vdash_1 v : K$
   by *weakening* (3)
5. QED

CASE $e = x'$:
1. QED by $(x'[x \leftarrow v]) = x'$

CASE $e = i$:
1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x.\, e'$:
1. QED by $(\lambda x.\, e')[x \leftarrow v] = \lambda x.\, e'$

CASE $e = \lambda x'.\, e'$:
1. $e[x \leftarrow v] = \lambda x'.\, (e'[x \leftarrow v])$
2. $x', (x : \tau), \Gamma \vdash_1 e'$
   by 1 *inversion*
3. $x', \Gamma \vdash_1 e'[x \leftarrow v]$
   by *dynamic context static value substitution*
4. $\Gamma \vdash_1 \lambda x'.\, e'[x \leftarrow v] : K$
   by (3)
5. QED

CASE $e = \lambda(x' : \tau').\, e'$:
1. $e[x \leftarrow v] = \lambda(x' : \tau').\, (e'[x \leftarrow v])$
2. $(x' : \tau'), (x : \tau), \Gamma \vdash_1 e' : \mathsf{Any}$
   by 1 *inversion*
3. $(x' : \tau'), \Gamma \vdash_1 e'[x \leftarrow v] : \mathsf{Any}$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 \lambda(x' : \tau').\, (e'[x \leftarrow v]) : K$
5. QED

CASE $e = \langle e_0, e_1 \rangle$:
1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $(x : \tau), \Gamma \vdash_1 e_0 : \mathsf{Any}$
   $\wedge\, (x : \tau), \Gamma \vdash_1 e_1 : \mathsf{Any}$
   by 1 *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v] : \mathsf{Any}$
   $\wedge\, \Gamma \vdash_1 e_1[x \leftarrow v] : \mathsf{Any}$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle : K$
   by (3)
5. QED

CASE $e = e_0\ e_1$:
1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $(x : \tau), \Gamma \vdash_1 e_0 : K_0$
   $\wedge\, (x : \tau), \Gamma \vdash_1 e_1 : K_1$
   by 1 *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$
   $\wedge\, \Gamma \vdash_1 e_1[x \leftarrow v] : K_1$
   by the induction hypothesis (2)

4. $\Gamma \vdash_1 e_0[x \leftarrow v]\ e_1[x \leftarrow v] : K$
   by (3)
5. QED

CASE $e = op^1\ e_0$:
1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $(x:\tau), \Gamma \vdash_1 e_0 : K_0$
   by 1 *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 op^1\ e_0[x \leftarrow v] : K$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:
1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $(x:\tau), \Gamma \vdash_1 e_0 : K_0$
   $\wedge\ (x:\tau), \Gamma \vdash_1 e_1 : K_1$
   by 1 *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$
   $\wedge\ \Gamma \vdash_1 e_1[x \leftarrow v] : K_1$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v] : K$
   by (3)
5. QED

CASE $e = \mathsf{dyn}\ \tau'\ e'$:
1. $e[x \leftarrow v] = \mathsf{dyn}\ \tau'\ e'[x \leftarrow v]$
2. $(x:\tau), \Gamma \vdash_1 e'$
   by 1 *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v]$
   by *dynamic context static value substitution* (2)
4. $\Gamma \vdash_1 \mathsf{dyn}\ \tau'\ (e'[x \leftarrow v]) : K$
   by (3)
5. QED

CASE $e = \mathsf{dyn}\ e'$:
1. $e[x \leftarrow v] = \mathsf{dyn}\ e'[x \leftarrow v]$
2. $(x:\tau), \Gamma \vdash_1 e'$
   by 1 *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v]$
   by *dynamic context static value substitution* (2)
4. $\Gamma \vdash_1 \mathsf{dyn}\ (e'[x \leftarrow v]) : K$
   by (3)
5. QED

CASE $e = \mathsf{chk}\ K'\ e'$:
1. $e[x \leftarrow v] = \mathsf{chk}\ K'\ (e'[x \leftarrow v])$
2. $(x:\tau), \Gamma \vdash_1 e' : \mathsf{Any}$
   by 1 *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v] : \mathsf{Any}$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 \mathsf{chk}\ K'\ (e'[x \leftarrow v]) : K$
   by (3)
5. QED

CASE $e = \mathsf{stat}\ \tau'\ e'$
$\vee\ e = \mathsf{stat}\ e'$:
1. Contradiction by $\Gamma \vdash_1 e : K$

□

Lemma 4.50 : 1 *static-dynamic substitution*
If $x, \Gamma \vdash_1 e : K$ and $\vdash_1 v$ then $\Gamma \vdash_1 e[x \leftarrow v] : K$

Proof:
By induction on the structure of e.

CASE $e = x$:
1. $K = \mathsf{Any}$
   by $x, \Gamma \vdash_1 x : K$
2. $e[x \leftarrow v] = v$
3. $\vdash_1 v : K$
   by *dynamic value inversion*
4. $\vdash_1 v : \mathsf{Any}$
   by $K \leqslant: \mathsf{Any}$
5. $\Gamma \vdash_1 v : \mathsf{Any}$
   by *weakening* (3)
6. QED

CASE $e = x'$:
1. QED by $x'[x \leftarrow v] = x'$

CASE $e = i$:
1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x.\, e'$:
1. QED by $(\lambda x.\, e')[x \leftarrow v] = \lambda x.\, e'$

CASE $e = \lambda(x:\tau').\, e'$:
1. QED by $(\lambda(x:\tau').\, e')[x \leftarrow v] = \lambda(x:\tau').\, e'$

CASE $e = \lambda x'.\, e'$:
1. $e[x \leftarrow v] = \lambda x'.\, (e'[x \leftarrow v])$
2. $x', x, \Gamma \vdash_1 e'$
   by 1 *inversion*
3. $x', \Gamma \vdash_1 e'[x \leftarrow v]$
   by *dynamic context dynamic value substitution*
4. $\Gamma \vdash_1 \lambda x'.\, e'[x \leftarrow v] : K$
   by (3)
5. QED

CASE $e = \lambda(x':\tau').\, e'$:
1. $e[x \leftarrow v] = \lambda(x':\tau').\, (e'[x \leftarrow v])$
2. $(x':\tau'), x, \Gamma \vdash_1 e' : \mathsf{Any}$
   by 1 *inversion*
3. $(x':\tau'), \Gamma \vdash_1 e'[x \leftarrow v] : \mathsf{Any}$
   by *static context dynamic value substitution*
4. $\Gamma \vdash_1 \lambda(x':\tau').\, (e'[x \leftarrow v]) : K$
5. QED

CASE $e = \langle e_0, e_1 \rangle$:
1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $x, \Gamma \vdash_1 e_0 : \mathsf{Any}$
   $\wedge\ x, \Gamma \vdash_1 e_1 : \mathsf{Any}$
   by 1 *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v] : \mathsf{Any}$
   $\wedge\ \Gamma \vdash_1 e_1[x \leftarrow v] : \mathsf{Any}$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle : K$
   by (3)
5. QED

CASE $e = e_0\ e_1$:

1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e_0 : K_0$
   $\wedge\ x, \Gamma \vdash_1 e_1 : K_1$
   by 1 *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$
   $\wedge\ \Gamma \vdash_1 e_1[x \leftarrow v] : K_1$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 e_0[x \leftarrow v]\ e_1[x \leftarrow v] : K$
   by (3)
5. QED

CASE $e = op^1\ e_0$:
1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e_0 : K_0$
   by 1 *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 op^1\ e_0[x \leftarrow v] : K$
   by (3)
5. QED

CASE $e = op^2\ e_0\ e_1$:
1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e_0 : K_0$
   $\wedge\ x, \Gamma \vdash_1 e_1 : K_1$
   by 1 *inversion*
3. $\Gamma \vdash_1 e_0[x \leftarrow v] : K_0$
   $\wedge\ \Gamma \vdash_1 e_1[x \leftarrow v] : K_1$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v] : K$
   by (3)
5. QED

CASE $e = \mathsf{dyn}\ \tau'\ e'$:
1. $e[x \leftarrow v] = \mathsf{dyn}\ \tau'\ e'[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e'$
   by 1 *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v]$
   by *dynamic context dynamic value substitution* (2)
4. $\Gamma \vdash_1 \mathsf{dyn}\ \tau'\ (e'[x \leftarrow v]) : K$
   by (3)
5. QED

CASE $e = \mathsf{dyn}\ e'$:
1. $e[x \leftarrow v] = \mathsf{dyn}\ e'[x \leftarrow v]$
2. $x, \Gamma \vdash_1 e'$
   by 1 *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v]$
   by *dynamic context dynamic value substitution* (2)
4. $\Gamma \vdash_1 \mathsf{dyn}\ (e'[x \leftarrow v]) : K$
   by (3)
5. QED

CASE $e = \mathsf{chk}\ K'\ e'$:
1. $e[x \leftarrow v] = \mathsf{chk}\ K'\ (e'[x \leftarrow v])$
2. $x, \Gamma \vdash_1 e' : \mathsf{Any}$
   by 1 *inversion*
3. $\Gamma \vdash_1 e'[x \leftarrow v] : \mathsf{Any}$
   by the induction hypothesis (2)
4. $\Gamma \vdash_1 \mathsf{chk}\ K'\ (e'[x \leftarrow v]) : K$
   by (3)
5. QED

CASE $e = \mathsf{stat}\ \tau'\ e'$
   $\vee\ e = \mathsf{stat}\ e'$:
1. Contradiction by $\Gamma \vdash_1 e : K$

□

Lemma 4.51 : *weakening*

- If $\Gamma \vdash_1 e$ then $x, \Gamma \vdash_1 e$
- If $\Gamma \vdash_1 e : \tau$ then $(x{:}\tau'), \Gamma \vdash_1 e : \tau$

Proof:
- e is closed under Γ
  by $\Gamma \vdash_1 e$
  $\vee\ \Gamma \vdash_1 e : \tau$ QED

□

Lemma 4.52 : *unique static evaluation contexts*

If $\vdash e : \tau$ and e is boundary-free then one of the following holds:
- e is a value
- $e = E^{\bullet}[v_0\ v_1]$
- $e = E^{\bullet}[op^1\ v]$
- $e = E^{\bullet}[op^2\ v_0\ v_1]$
- $e = E^{\bullet}[\mathsf{Err}]$

Proof:

By induction on the structure of e.

CASE $e = x$:
1. Contradiction by $\vdash e : \tau$

CASE $e = i$
   $\vee\ e = \lambda(x{:}\tau_d).\ e'$:
1. QED e is a value

CASE $e = \langle e_0, e_1 \rangle$:

IF $e_0 \notin v$:
1. $e_0 = E^{\bullet}{}_0[e_0']$
   by the induction hypothesis
2. $E^{\bullet} = \langle E^{\bullet}{}_0, e_1 \rangle$
3. QED by $e = E^{\bullet}[e_0']$

IF $e_0 \in v$
   $\wedge\ e_1 \notin v$:
1. $e_1 = E^{\bullet}{}_1[e_1']$
   by the induction hypothesis
2. $E^{\bullet} = \langle e_0, E^{\bullet}{}_1 \rangle$
3. QED by $e = E^{\bullet}[e_1']$

ELSE $e_0 \in v$
   $\wedge\ e_1 \in v$:
1. $E^{\bullet} = [\,]$
2. QED $e = E^{\bullet}[\langle e_0, e_1 \rangle]$

CASE $e = e_0\ e_1$:

IF $e_0 \notin v$:
1. $e_0 = E^{\bullet}{}_0[e_0']$
   by the induction hypothesis
2. $E^{\bullet} = E^{\bullet}{}_0\ e_1$
3. QED by $e = E^{\bullet}[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:
1. $e_1 = E^{\bullet}{}_1[e_1']$
by the induction hypothesis
2. $E^{\bullet} = e_0\ E^{\bullet}{}_1$
3. QED by $e = E^{\bullet}[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E^{\bullet} = [\,]$
2. QED $e = E^{\bullet}[e_0\ e_1]$

CASE $e = op^1\ e_0$:

IF $e_0 \notin v$:
1. $e_0 = E^{\bullet}{}_0[e_0']$
by the induction hypothesis
2. $E^{\bullet} = op^1\ E^{\bullet}{}_0$
3. QED $e = E^{\bullet}[e_0']$

ELSE $e_0 \in v$:
1. $E^{\bullet} = [\,]$
2. QED $e = E^{\bullet}[op^1\ e_0]$

CASE $e = op^2\ e_0\ e_1$:

IF $e_0 \notin v$:
1. $e_0 = E^{\bullet}{}_0[e_0']$
by the induction hypothesis
2. $E^{\bullet} = op^2\ E^{\bullet}{}_0\ e_1$
3. QED $e = E^{\bullet}[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:
1. $e_1 = E^{\bullet}{}_1[e_1']$
by the induction hypothesis
2. $E^{\bullet} = op^2\ e_0\ E^{\bullet}{}_1$
3. QED $e = E^{\bullet}[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E^{\bullet} = [\,]$
2. QED $e = E^{\bullet}[op^2\ e_0\ e_1]$

CASE $e = \mathsf{chk}\ K'\ e'$:
1. Contradiction by $\vdash e : \tau$

CASE $e = \mathsf{dyn}\ e_0$:
1. Contradiction by $\vdash e : \tau$

CASE $e = \mathsf{stat}\ e'$:
1. Contradiction by $\vdash e : \tau$

CASE $e = \mathsf{dyn}\ \tau\ e_0$:
1. QED e is boundary-free

CASE $e = \mathsf{stat}\ \tau\ e'$:
1. Contradiction by $\vdash e : \tau$

CASE $e = \mathsf{Err}$:
1. $E^{\bullet} = [\,]$
2. QED

□

Lemma 4.53 : $\vdash$ *static inversion*

- If $\Gamma \vdash x : \tau$ then $(x\!:\!\tau') \in \Gamma$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash \lambda(x\!:\!\tau_d').\, e' : \tau$ then $(x\!:\!\tau_d'), \Gamma \vdash e' : \tau_c'$ and $\tau_d' \Rightarrow \tau_c' \leqslant: \tau$
- If $\Gamma \vdash \langle e_0, e_1\rangle : \tau$ then $\Gamma \vdash e_0 : \tau_0$ and $\Gamma \vdash e_1 : \tau_1$ and $\tau_0 \times \tau_1 \leqslant: \tau$
- If $\Gamma \vdash e_0\ e_1 : \tau_c$ then $\Gamma \vdash e_0 : \tau_d' \Rightarrow \tau_c'$ and $\Gamma \vdash e_1 : \tau_d'$ and $\tau_c' \leqslant: \tau_c$
- If $\Gamma \vdash \mathsf{fst}\, e : \tau$ then $\Gamma \vdash e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0$ and $\tau_0 \leqslant: \tau$
- If $\Gamma \vdash \mathsf{snd}\, e : \tau$ then $\Gamma \vdash e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1$ and $\tau_1 \leqslant: \tau$
- If $\Gamma \vdash op^2\ e_0\ e_1 : \tau$ then $\Gamma \vdash e_0 : \tau_0$ and $\Gamma \vdash e_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau'$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash \mathsf{dyn}\ \tau'\ e' : \tau$ then $\Gamma \vdash e'$ and $\tau' \leqslant: \tau$

Proof:
QED by the definition of $\Gamma \vdash e : \tau$
□

Lemma 4.54 : *canonical forms*

- If $\vdash v : \tau_0 \times \tau_1$ then $v = \langle v_0, v_1\rangle$
- If $\vdash v : \tau_d \Rightarrow \tau_c$ then $v = \lambda(x\!:\!\tau_x).\, e'$
$\wedge\, \tau_d \leqslant: \tau_x$
- If $\vdash v : \mathsf{Int}$ then $v = i$
- If $\vdash v : \mathsf{Nat}$ then $v = i$ and $v \in \mathbb{N}$

Proof:
QED by definition of $\vdash e : \tau$
□

Lemma 4.55 : *substitution*

If $(x\!:\!\tau_x), \Gamma \vdash e : \tau$, and e is boundary-free and $\vdash v : \tau_x$ then $\Gamma \vdash e[x \leftarrow v] : \tau$

Proof:
By induction on the structure of e.

CASE $e = x$:
1. $e[x \leftarrow v] = v$
2. $\tau_x = \tau$
3. $\Gamma \vdash v : \tau$
by *weakening*
4. QED

CASE $e = x'$:
1. QED by $x'[x \leftarrow v] = x'$

CASE $e = i$:
1. QED by $i[x \leftarrow v] = i$

CASE $e = \lambda x.\, e'$:
1. Contradiction by $(x\!:\!\tau_x), \Gamma \vdash e : \tau$

CASE $e = \lambda(x\!:\!\tau').\, e'$:
1. QED by $(\lambda(x\!:\!\tau').\, e')[x \leftarrow v] = \lambda(x\!:\!\tau').\, e'$

CASE $e = \lambda(x'\!:\!\tau').\, e'$:
1. $e[x \leftarrow v] = \lambda(x'\!:\!\tau').\, (e'[x \leftarrow v])$
2. $(x'\!:\!\tau'), x, \Gamma \vdash e'$
by *static inversion forms*
3. $(x'\!:\!\tau'), \Gamma \vdash e'[x \leftarrow v]$
by the induction hypothesis (2)
4. $\Gamma \vdash \lambda(x'\!:\!\tau').\, (e'[x \leftarrow v])$
by (3)
5. QED

case $e = \langle e_0, e_1 \rangle$:

1. $e[x \leftarrow v] = \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
2. $x, \Gamma \vdash e_0$
   $\wedge\ x, \Gamma \vdash e_1$
   by *static inversion forms*
3. $\Gamma \vdash e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash \langle e_0[x \leftarrow v], e_1[x \leftarrow v] \rangle$
   by (3)
5. qed

case $e = e_0\ e_1$:

1. $e[x \leftarrow v] = e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash e_0$
   $\wedge\ x, \Gamma \vdash e_1$
   by *static inversion forms*
3. $\Gamma \vdash e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. qed

case $e = op^1\ e_0$:

1. $e[x \leftarrow v] = op^1\ e_0[x \leftarrow v]$
2. $x, \Gamma \vdash e_0$
   by *static inversion forms*
3. $\Gamma \vdash e_0[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash op^1\ e_0[x \leftarrow v]$
   by (3)
5. qed

case $e = op^2\ e_0\ e_1$:

1. $e[x \leftarrow v] = op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
2. $x, \Gamma \vdash e_0$
   $\wedge\ x, \Gamma \vdash e_1$
   by *static inversion forms*
3. $\Gamma \vdash e_0[x \leftarrow v]$
   $\wedge\ \Gamma \vdash e_1[x \leftarrow v]$
   by the induction hypothesis (2)
4. $\Gamma \vdash op^2\ e_0[x \leftarrow v]\ e_1[x \leftarrow v]$
   by (3)
5. qed

case $e = \mathsf{chk}\ K'\ e'$:

1. Contradiction by $\vdash e : \tau$

case $e = \mathsf{dyn}\ e'$:

1. Contradiction by $\vdash e : \tau$

case $e = \mathsf{stat}\ e'$:

1. Contradiction by $\vdash e : \tau$

case $e = \mathsf{dyn}\ \tau'\ e'$:

1. Contradiction by e is boundary-free

case $e = \mathsf{stat}\ \tau'\ e'$:

1. Contradiction by $\vdash e : \tau$

case $e = \mathsf{Err}$:

1. qed $\mathsf{Err}[x \leftarrow v] = \mathsf{Err}$

□

Lemma 4.56 : *δ preservation*

- If $\vdash v$ and $\delta(op^1, v) = v'$ then $\vdash e'$
- If $\vdash v_0$ and $\vdash v_1$ and $\delta(op^2, v_0, v_1) = e'$ then $\vdash v'$

Proof:

case $\delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0$:

1. $\vdash v_0$
   by *static inversion forms*
2. qed

case $\delta(\mathsf{snd}, \langle v_0, v_1 \rangle) = v_1$:

1. $\vdash v_1$
   by *static inversion forms*
2. qed

case $\delta(\mathsf{sum}, v_0, v_1) = v_0 + v_1$:

1. qed

case $\delta(\mathsf{quotient}, v_0, v_1) = \lfloor v_0 / v_1 \rfloor$:

1. qed

case $\delta(\mathsf{quotient}, v_0, v_1) = \mathsf{BndryErr}$:

1. qed

□

Lemma 4.57 : *weakening*

- If $\Gamma \vdash e$ then $x, \Gamma \vdash e$
- If $\Gamma \vdash e$ then $(x : \tau), \Gamma \vdash e$

Proof:

qed because e is closed under Γ

□

E.5 (HC) Co-Natural Embedding

E.5.1 Co-Natural Definitions

$\boxed{\text{Language HC}}$

$$\begin{array}{lcl}
e & = & x \mid v \mid \langle e, e\rangle \mid e\ e \mid op^1\ e \mid op^2\ e\ e \mid \\
 & & \mathsf{dyn}\ \tau\ e \mid \mathsf{stat}\ \tau\ e \mid \mathsf{Err} \\
v & = & i \mid \langle v, v\rangle \mid \lambda x.\, e \mid \lambda(x\!:\!\tau).\, e \\
 & & \mathsf{mon}\,(\tau \Rightarrow \tau)\, v \mid \mathsf{mon}\,(\tau \times \tau)\, v \\
\tau & = & \mathsf{Nat} \mid \mathsf{Int} \mid \tau \times \tau \mid \tau \Rightarrow \tau \\
\Gamma & = & \cdot \mid x, \Gamma \mid (x\!:\!\tau), \Gamma \\
\mathsf{Err} & = & \mathsf{BndryErr} \mid \mathsf{TagErr} \\
r & = & v \mid \mathsf{Err} \\
E^\bullet & = & [\,] \mid E^\bullet\ e \mid v\ E^\bullet \mid \langle E^\bullet, e\rangle \mid \langle v, E^\bullet\rangle \mid \\
 & & op^1\ E^\bullet \mid op^2\ E^\bullet\ e \mid op^2\ v\ E^\bullet \\
E & = & E^\bullet \mid E\ e \mid v\ E \mid \langle E, e\rangle \mid \langle v, E\rangle \mid op^1\ E \mid \\
 & & op^2\ E\ e \mid op^2\ v\ E \mid \mathsf{dyn}\ \tau\ E \mid \mathsf{stat}\ \tau\ E
\end{array}$$

$\boxed{\Delta : op^1 \times \tau \longrightarrow \tau}$

$$\begin{array}{l}
\Delta(\mathsf{fst}, \tau_0 \times \tau_1) \ = \tau_0 \\
\Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1
\end{array}$$

$\boxed{\Delta : op^2 \times \tau \times \tau \longrightarrow \tau}$

$$\begin{array}{l}
\Delta(op^2, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat} \\
\Delta(op^2, \mathsf{Int}, \mathsf{Int}) \ \ = \mathsf{Int}
\end{array}$$

$\boxed{\tau \leqslant: \tau}$

$$\frac{}{\mathsf{Nat} \leqslant: \mathsf{Int}} \qquad \frac{\tau_d' \leqslant: \tau_d \quad \tau_c \leqslant: \tau_c'}{\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'} \qquad \frac{\tau_0 \leqslant: \tau_0' \quad \tau_1 \leqslant: \tau_1'}{\tau_0 \times \tau_1 \leqslant: \tau_0' \times \tau_1'}$$

$$\frac{}{\tau \leqslant: \tau} \qquad \frac{\tau \leqslant: \tau' \quad \tau' \leqslant: \tau''}{\tau \leqslant: \tau''}$$

$\boxed{\Gamma \vdash e}$

$$\frac{x \in \Gamma}{\Gamma \vdash x} \qquad \frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x.\, e} \qquad \frac{}{\Gamma \vdash i} \qquad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1\rangle}$$

$$\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0\ e_1} \qquad \frac{\Gamma \vdash e}{\Gamma \vdash op^1\ e} \qquad \frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2\ e_0\ e_1} \qquad \frac{}{\Gamma \vdash \mathsf{Err}}$$

$$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat}\ \tau\ e}$$

$\boxed{\Gamma \vdash e : \tau}$

$$\frac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash x : \tau} \qquad \frac{(x\!:\!\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c} \qquad \frac{i \in \mathbb{N}}{\Gamma \vdash i : \mathsf{Nat}}$$

$$\frac{}{\Gamma \vdash i : \mathsf{Int}} \qquad \frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_0 \\ \Gamma \vdash e_1 : \tau_1\end{array}}{\Gamma \vdash \langle e_0, e_1\rangle : \tau_0 \times \tau_1} \qquad \frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \\ \Gamma \vdash e_1 : \tau_d\end{array}}{\Gamma \vdash e_0\ e_1 : \tau_c}$$

$$\frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_0 \\ \Delta(op^1, \tau_0) = \tau\end{array}}{\Gamma \vdash op^1\ e_0 : \tau} \qquad \frac{\begin{array}{c}\Gamma \vdash e_0 : \tau_0 \\ \Gamma \vdash e_1 : \tau_1 \\ \Delta(op^2, \tau_0, \tau_1) = \tau\end{array}}{\Gamma \vdash op^2\ e_0\ e_1 : \tau} \qquad \frac{\begin{array}{c}\Gamma \vdash e : \tau' \\ \tau' \leqslant: \tau\end{array}}{\Gamma \vdash e : \tau} \qquad \frac{}{\Gamma \vdash \mathsf{Err} : \tau}$$

$$\frac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau}$$

$\boxed{\Gamma \vdash_{\mathsf{C}} e}$

$$\frac{x \in \Gamma}{\Gamma \vdash_{\mathsf{C}} x} \qquad \frac{x, \Gamma \vdash_{\mathsf{C}} e}{\Gamma \vdash_{\mathsf{C}} \lambda x.\, e} \qquad \frac{}{\Gamma \vdash_{\mathsf{C}} i} \qquad \frac{\Gamma \vdash_{\mathsf{C}} e_0 \quad \Gamma \vdash_{\mathsf{C}} e_1}{\Gamma \vdash_{\mathsf{C}} \langle e_0, e_1\rangle}$$

$$\frac{\Gamma \vdash_{\mathsf{C}} e_0 \quad \Gamma \vdash_{\mathsf{C}} e_1}{\Gamma \vdash_{\mathsf{C}} e_0\ e_1} \qquad \frac{\Gamma \vdash_{\mathsf{C}} e}{\Gamma \vdash_{\mathsf{C}} op^1\ e} \qquad \frac{\Gamma \vdash_{\mathsf{C}} e_0 \quad \Gamma \vdash_{\mathsf{C}} e_1}{\Gamma \vdash_{\mathsf{C}} op^2\ e_0\ e_1} \qquad \frac{}{\Gamma \vdash_{\mathsf{C}} \mathsf{Err}}$$

$$\frac{\Gamma \vdash_{\mathsf{C}} e : \tau}{\Gamma \vdash_{\mathsf{C}} \mathsf{stat}\ \tau\ e} \qquad \frac{\Gamma \vdash_{\mathsf{C}} v : \tau_0 \times \tau_1}{\Gamma \vdash_{\mathsf{C}} \mathsf{mon}\,(\tau_0 \times \tau_1)\, v} \qquad \frac{\Gamma \vdash_{\mathsf{C}} v : \tau_d \Rightarrow \tau_c}{\Gamma \vdash_{\mathsf{C}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v}$$

$\boxed{\Gamma \vdash_{\mathsf{C}} e : \tau}$

$$\frac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash_{\mathsf{C}} x : \tau} \qquad \frac{(x\!:\!\tau_d), \Gamma \vdash_{\mathsf{C}} e : \tau_c}{\Gamma \vdash_{\mathsf{C}} \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c} \qquad \frac{i \in \mathbb{N}}{\Gamma \vdash_{\mathsf{C}} i : \mathsf{Nat}}$$

$$\frac{}{\Gamma \vdash_{\mathsf{C}} i : \mathsf{Int}} \qquad \frac{\begin{array}{c}\Gamma \vdash_{\mathsf{C}} e_0 : \tau_0 \\ \Gamma \vdash_{\mathsf{C}} e_1 : \tau_1\end{array}}{\Gamma \vdash_{\mathsf{C}} \langle e_0, e_1\rangle : \tau_0 \times \tau_1} \qquad \frac{\begin{array}{c}\Gamma \vdash_{\mathsf{C}} e_0 : \tau_d \Rightarrow \tau_c \\ \Gamma \vdash_{\mathsf{C}} e_1 : \tau_d\end{array}}{\Gamma \vdash_{\mathsf{C}} e_0\ e_1 : \tau_c}$$

$$\frac{\begin{array}{c}\Gamma \vdash_{\mathsf{C}} e_0 : \tau_0 \\ \Delta(op^1, \tau_0) = \tau\end{array}}{\Gamma \vdash_{\mathsf{C}} op^1\ e_0 : \tau} \qquad \frac{\begin{array}{c}\Gamma \vdash_{\mathsf{C}} e_0 : \tau_0 \\ \Gamma \vdash_{\mathsf{C}} e_1 : \tau_1 \\ \Delta(op^2, \tau_0, \tau_1) = \tau\end{array}}{\Gamma \vdash_{\mathsf{C}} op^2\ e_0\ e_1 : \tau} \qquad \frac{\begin{array}{c}\Gamma \vdash_{\mathsf{C}} e : \tau' \\ \tau' <: \tau\end{array}}{\Gamma \vdash_{\mathsf{C}} e : \tau}$$

$$\frac{}{\Gamma \vdash_{\mathsf{C}} \mathsf{Err} : \tau} \qquad \frac{\Gamma \vdash_{\mathsf{C}} e}{\Gamma \vdash_{\mathsf{C}} \mathsf{dyn}\ \tau\ e : \tau}$$

$$\frac{\Gamma \vdash_{\mathsf{C}} v}{\Gamma \vdash_{\mathsf{C}} \mathsf{mon}\,(\tau_0 \times \tau_1)\, v : (\tau_0 \times \tau_1)}$$

$$\frac{\Gamma \vdash_{\mathsf{C}} v}{\Gamma \vdash_{\mathsf{C}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v : (\tau_d \Rightarrow \tau_c)}$$

$\boxed{\delta(op^1, v) = e}$

$\delta(\mathsf{fst}, \langle v_0, v_1\rangle) = v_0$

$\delta(\mathsf{snd}, \langle v_0, v_1\rangle) = v_1$

$\boxed{\delta(op^2, v, v) = e}$

$\delta(\mathsf{sum}, i_0, i_1) = i_0 + i_1$

$\delta(\mathsf{quotient}, i_0, 0) = \mathsf{BndryErr}$

$\delta(\mathsf{quotient}, i_0, i_1) = \lfloor i_0/i_1 \rfloor$
  if $i_1 \neq 0$

$\boxed{\mathcal{D}_{\mathsf{C}} : \tau \times v \longrightarrow e}$

$\mathcal{D}_{\mathsf{C}}(\tau_d \Rightarrow \tau_c, v) = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v$
  if $v = \lambda x.\, e$ or $v = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\, v'$

$\mathcal{D}_{\mathsf{C}}(\tau_0 \times \tau_1, v) = \mathsf{mon}\,(\tau_0 \times \tau_1)\, v$
  if $v = \langle v_0, v_1\rangle$ or $v = \mathsf{mon}\,(\tau_0' \times \tau_1')\, v'$

$\mathcal{D}_{\mathsf{C}}(\mathsf{Int}, i) = i$

$\mathcal{D}_{\mathsf{C}}(\mathsf{Nat}, i) = i$
  if $i \in \mathbb{N}$

$\mathcal{D}_{\mathsf{C}}(\tau, v) = \mathsf{BndryErr}$
  otherwise

$\boxed{\mathcal{S}_{\mathsf{C}} : \tau \times v \longrightarrow e}$

$\mathcal{S}_{\mathsf{C}}(\tau_d \Rightarrow \tau_c, v) = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v$

$\mathcal{S}_{\mathsf{C}}(\tau_0 \times \tau_1, v) = \mathsf{mon}\,(\tau_0 \times \tau_1)\, v$

$\mathcal{S}_{\mathsf{C}}(\tau, v) = v$
  otherwise

$\boxed{e \rhd_{\mathsf{S\text{-}C}} e}$

$\mathsf{dyn}\ \tau\ v \rhd_{\mathsf{S\text{-}C}} \mathcal{D}_{\mathsf{C}}(\tau, v)$

$(\mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v_f)\ v \rhd_{\mathsf{S\text{-}C}} \mathsf{dyn}\ \tau_c\ (v_f\ e')$
  where $e' = \mathsf{stat}\ \tau_d\ v$

$(\lambda(x{:}\tau).\, e)\ v \rhd_{\mathsf{S\text{-}C}} e[x \leftarrow v]$

$\mathsf{fst}\,(\mathsf{mon}\,(\tau_0 \times \tau_1)\, v) \rhd_{\mathsf{S\text{-}C}} \mathsf{dyn}\ \tau_0\ (\mathsf{fst}\, v)$

$\mathsf{snd}\,(\mathsf{mon}\,(\tau_0 \times \tau_1)\, v) \rhd_{\mathsf{S\text{-}C}} \mathsf{dyn}\ \tau_1\ (\mathsf{snd}\, v)$

$op^1\ v \rhd_{\mathsf{S\text{-}C}} \delta(op^1, v)$

$op^2\ v_0\ v_1 \rhd_{\mathsf{S\text{-}C}} \delta(op^2, v_0, v_1)$

$\boxed{e \rhd_{\mathsf{D\text{-}C}} e}$

$\mathsf{stat}\ \tau\ v \rhd_{\mathsf{D\text{-}C}} \mathcal{S}_{\mathsf{C}}(\tau, v)$

$v_0\ v_1 \rhd_{\mathsf{D\text{-}C}} \mathsf{TagErr}$
  if $v_0 \in \mathbb{Z}$ or $v_0 = \langle v, v'\rangle$

$(\mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v_f)\ v \rhd_{\mathsf{D\text{-}C}} \mathsf{stat}\ \tau_c\ (v_f\ e')$
  where $e' = \mathsf{dyn}\ \tau_d\ v$

$(\lambda x.\, e)\ v \rhd_{\mathsf{D\text{-}C}} e[x \leftarrow v]$

$\mathsf{fst}\,(\mathsf{mon}\,(\tau_0 \times \tau_1)\, v) \rhd_{\mathsf{D\text{-}C}} \mathsf{stat}\ \tau_0\ (\mathsf{fst}\, v)$

$\mathsf{snd}\,(\mathsf{mon}\,(\tau_0 \times \tau_1)\, v) \rhd_{\mathsf{D\text{-}C}} \mathsf{stat}\ \tau_1\ (\mathsf{snd}\, v)$

$op^1\ v \rhd_{\mathsf{D\text{-}C}} \mathsf{TagErr}$
  if $\delta(op^1, v)$ is undefined

$op^1\ v \rhd_{\mathsf{D\text{-}C}} \delta(op^1, v)$

$op^2\ v_0\ v_1 \rhd_{\mathsf{D\text{-}C}} \mathsf{TagErr}$
  if $\delta(op^2, v_0, v_1)$ is undefined

$op^2\ v_0\ v_1 \rhd_{\mathsf{D\text{-}C}} \delta(op^2, v_0, v_1)$

$\boxed{e \rightarrow_{\mathsf{C\text{-}S}} e}$

$E^\bullet[e] \rightarrow_{\mathsf{C\text{-}S}} E^\bullet[e']$
  if $e \rhd_{\mathsf{S\text{-}C}} e'$

$E[\mathsf{stat}\ \tau\ E^\bullet[e]] \rightarrow_{\mathsf{C\text{-}S}} E[\mathsf{stat}\ \tau\ E^\bullet[e']]$
  if $e \rhd_{\mathsf{S\text{-}C}} e'$

$E[\mathsf{dyn}\ \tau\ E^\bullet[e]] \rightarrow_{\mathsf{C\text{-}S}} E[\mathsf{dyn}\ \tau\ E^\bullet[e']]$
  if $e \rhd_{\mathsf{D\text{-}C}} e'$

$E[\mathsf{Err}] \rightarrow_{\mathsf{C\text{-}S}} \mathsf{Err}$

$\boxed{e \rightarrow_{\mathsf{C\text{-}D}} e}$

$E^\bullet[e] \rightarrow_{\mathsf{C\text{-}D}} E^\bullet[e']$
  if $e \rhd_{\mathsf{D\text{-}C}} e'$

$E[\mathsf{stat}\ \tau\ E^\bullet[e]] \rightarrow_{\mathsf{C\text{-}D}} E[\mathsf{stat}\ \tau\ E^\bullet[e']]$
  if $e \rhd_{\mathsf{S\text{-}C}} e'$

$E[\mathsf{dyn}\ \tau\ E^\bullet[e]] \rightarrow_{\mathsf{C\text{-}D}} E[\mathsf{dyn}\ \tau\ E^\bullet[e']]$
  if $e \rhd_{\mathsf{D\text{-}C}} e'$

$E[\mathsf{Err}] \rightarrow_{\mathsf{C\text{-}D}} \mathsf{Err}$

$\boxed{e \rightarrow^*_{\mathsf{C\text{-}S}} e}$ reflexive, transitive closure of $\rightarrow_{\mathsf{C\text{-}S}}$

$\boxed{e \rightarrow^*_{\mathsf{C\text{-}D}} e}$ reflexive, transitive closure of $\rightarrow_{\mathsf{C\text{-}D}}$

E.5.2 Co-Natural Theorems

Theorem 5.0 : *static* HC *soundness*

If $\vdash e : \tau$ then $\vdash_{C} e : \tau$ and one of the following holds:

- $e \rightarrow^{*}_{\text{C-S}} v$ and $\vdash_{C} v : \tau$
- $e \rightarrow^{*}_{\text{C-S}} E[\mathsf{dyn}\ \tau'\ e']$ and $e' \rhd_{\text{D-C}} \mathsf{TagErr}$
- $e \rightarrow^{*}_{\text{C-S}} \mathsf{BndryErr}$
- e diverges

Proof:

1. $\vdash_{C} e : \tau$
   by *static subset*
2. QED by *static progress* and *static preservation*.

□

Theorem 5.1 : *dynamic* HC-*soundness*

If $\vdash e$ then $\vdash_{C} e$ and one of the following holds:

- $e \rightarrow^{*}_{\text{C-D}} v$ and $\vdash_{C} v$
- $e \rightarrow^{*}_{\text{C-D}} E[e']$ and $e' \rhd_{\text{D-C}} \mathsf{TagErr}$
- $e \rightarrow^{*}_{\text{C-D}} \mathsf{BndryErr}$
- e diverges

Proof:

1. $\vdash_{C} e$
   by *dynamic subset*
2. QED by *dynamic progress* and *dynamic preservation*.

□

Corollary 5.2 : HC *static soundness*

If $\vdash e : \tau$ and e is boundary-free, then one of the following holds:

- $e \rightarrow^{*}_{\text{C-S}} v$ and $\vdash_{C} v : \tau$
- $e \rightarrow^{*}_{\text{C-S}} \mathsf{BndryErr}$
- e diverges

Proof:

Consequence of the proof for *static* HC-*soundness*

□

E.5.3 Co-Natural Lemmas

Lemma 5.3 : *$\mathcal{D}_C$ soundness*

If $\vdash_C v$ then $\vdash_C \mathcal{D}_C(\tau, v) : \tau$

Proof:

CASE $\mathcal{D}_C(\tau_d \Rightarrow \tau_c, v) = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v$:

1. $\vdash_C \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v : \tau_d \Rightarrow \tau_c$
   by $\vdash_C v$
2. QED

CASE $\mathcal{D}_C(\tau_0 \times \tau_1, v) = \mathsf{mon}\,(\tau_0 \times \tau_1)\, v$:

1. $\vdash_C \mathsf{mon}\,(\tau_0 \times \tau_1)\, v : \tau_0 \times \tau_1$
   by $\vdash_C v$
2. QED

CASE $v = i$
$\wedge\ \mathcal{D}_C(\mathsf{Int}, v) = v$:

1. QED

CASE $v \in \mathbb{N}$
$\wedge\ \mathcal{D}_C(\mathsf{Nat}, v) = v$:

1. QED

CASE $\mathcal{D}_C(\tau, v) = \mathsf{BndryErr}$:

1. QED

□

Lemma 5.4 : *$\mathcal{S}_C$ soundness*

If $\vdash_C v : \tau$ then $\vdash_C \mathcal{S}_C(\tau, v)$

Proof:

CASE $\vdash_C v : \tau_d \Rightarrow \tau_c$
$\wedge\ \mathcal{S}_C(\tau_d \Rightarrow \tau_c, v) = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v$:

1. QED

CASE $\vdash_C v : \tau_0 \times \tau_1$
$\wedge\ \mathcal{S}_C(\tau_0 \times \tau_1, v) = \mathsf{mon}\,(\tau_0 \times \tau_1)\, v$:

1. QED

CASE $\vdash_C v : \mathsf{Int}$
$\wedge\ \mathcal{S}_C(\mathsf{Int}, v) = v$:

1. QED

CASE $\vdash_C v : \mathsf{Nat}$
$\wedge\ \mathcal{S}_C(\mathsf{Nat}, v) = v$:

1. QED

□

Corollary 5.5 : HC *static subset*

If $\Gamma \vdash e : \tau$ then $\Gamma \vdash_C e : \tau$.

Proof:

Consequence of the proof for the higher-order *static subset* lemma; both $\vdash_C$ and $\vdash_H$ have the same typing rules for surface-language expressions.

□

Corollary 5.6 : HC *dynamic subset*

If $\Gamma \vdash e$ then $\Gamma \vdash_C e$.

Proof:

Consequence of the proof for the higher-order *dynamic subset* lemma.

□

Lemma 5.7 : HC *static progress*

If $\vdash_C e : \tau$ then one of the following holds:

- e is a value
- $e \in \mathsf{Err}$
- $e \rightarrow_{C\text{-}S} e'$
- $e \rightarrow_{C\text{-}S} \mathsf{BndryErr}$
- $e = E[\mathsf{dyn}\ \tau'\ e']$ and $e' \rightarrow_{C\text{-}D} \mathsf{TagErr}$

Proof:

By the *boundary factoring* lemma, there are seven possible cases.

CASE e is a value :

1. QED

CASE $e = E^{\bullet}[v_0\ v_1]$:

1. $\vdash_C v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_C v_0 : \tau_d \Rightarrow \tau_c$
   $\wedge \vdash_C v_1 : \tau_d$
   by *inversion*
3. $v_0 = \lambda(x\!:\!\tau_d').\, e'$
   $\vee\ v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\, v_f$
   by *canonical forms*
4. **IF** $v_0 = \lambda(x\!:\!\tau_d').\, e'$:
   a. $e \rightarrow_{C\text{-}S} E^{\bullet}[e'[x \leftarrow v_1]]$
      by $v_0\ v_1 \rhd_{S\text{-}C} e'[x \leftarrow v_1]$
   b. QED

   ELSE $v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\, v_f$:
   a. $e \rightarrow_{C\text{-}S} E^{\bullet}[\mathsf{dyn}\ \tau_c'\ (v_f\ (\mathsf{stat}\ \tau_d'\ v_1))]$
      by $v_0\ v_1 \rhd_{S\text{-}C} \mathsf{dyn}\ \tau_c'\ (v_f\ (\mathsf{stat}\ \tau_d'\ v_1))$
   b. QED

CASE $e = E^{\bullet}[op^1\ v]$:

1. $\vdash_C op^1\ v : \tau'$
   by *static hole typing*
2. $\vdash_C v : \tau_0 \times \tau_1$
   by *inversion*
3. $v = \langle v_0, v_1 \rangle$
   $\vee\ v = \mathsf{mon}\,(\tau_0 \times \tau_1)\, v'$
   by *canonical forms*
4. **IF** $v = \langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \mathsf{fst}$:
   a. $\delta(op^1, \langle v_0, v_1 \rangle) = v_0$
      by definition
   b. $e \rightarrow_{C\text{-}S} E^{\bullet}[v_0]$
      by $op^1\ v \rhd_{S\text{-}C} v_0$
   c. QED

   IF $v = \langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \mathsf{snd}$:
   a. $\delta(op^1, \langle v_0, v_1 \rangle) = v_1$
      by definition
   b. $e \rightarrow_{C\text{-}S} E^{\bullet}[v_1]$
      by $op^1\ v \rhd_{S\text{-}C} v_1$
   c. QED

   IF $v = \mathsf{mon}\,(\tau_0 \times \tau_1)\, v'$
   $\wedge\ op^1 = \mathsf{fst}$:
   a. $e \rightarrow_{C\text{-}S} E^{\bullet}[\mathsf{dyn}\ \tau_0\ (op^1\ v')]$
      by definition

b. QED

ELSE $v = \mathsf{mon}\,(\tau_0{\times}\tau_1)\,v'$
$\land\ op^1 = \mathsf{snd}$:

a. $e \rightarrow_{\text{C-S}} E^\bullet[\mathsf{dyn}\ \tau_1\ (op^1\ v')]$
by definition

b. QED

CASE $e = E^\bullet[op^2\ v_0\ v_1]$:

1. $\vdash_{\mathsf{C}} op^2\ v_0\ v_1 : \tau'$
by *static hole typing*
2. $\vdash_{\mathsf{C}} v_0 : \tau_0$
$\land \vdash_{\mathsf{C}} v_1 : \tau_1$
$\land\ \Delta(op^2, \tau_0, \tau_1) = \tau''$
by *inversion*
3. $\delta(op^2, v_0, v_1) = e'$
by *Δ type soundness* (2)
4. $op^2\ v_0\ v_1 \rhd_{\text{S-C}} e'$
by (3)
5. QED by $e \rightarrow_{\text{C-S}} E^\bullet[e']$

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :

1. e' is a value
$\lor\ e' \in \mathsf{Err}$
$\lor\ e' \rightarrow_{\text{C-D}} e''$
$\lor\ e' \rightarrow_{\text{C-D}} \mathsf{BndryErr}$
$\lor\ e' = E'[e'']$ and $e'' \rhd_{\text{D-C}} \mathsf{TagErr}$
by *dynamic progress*
2. **IF** e' is a value :
a. QED $e \rightarrow_{\text{C-S}} E[\mathcal{D}_{\mathsf{C}}(\tau', e')]$
IF $e' \in \mathsf{Err}$:
a. QED $e \rightarrow_{\text{C-S}} e'$
IF $e' \rightarrow_{\text{C-D}} e''$:
a. QED $e \rightarrow_{\text{C-S}} E[\mathsf{dyn}\ \tau'\ e'']$
IF $e' \rightarrow_{\text{C-D}} \mathsf{BndryErr}$:
a. QED $e \rightarrow_{\text{C-S}} E[\mathsf{dyn}\ \tau'\ \mathsf{BndryErr}]$
ELSE $e' = E'[e'']$ and $e'' \rhd_{\text{D-C}} \mathsf{TagErr}$:
a. $E' \in E^\bullet$
by e' is boundary-free
b. QED

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :

1. e' is a value
$\lor\ e' \in \mathsf{Err}$
$\lor\ e' \rightarrow_{\text{C-S}} e''$
$\lor\ e' \rightarrow_{\text{C-S}} \mathsf{BndryErr}$
$\lor\ e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{D-C}} \mathsf{TagErr}$
by *static progress*
2. **IF** e' is a value :
a. QED $e \rightarrow_{\text{C-S}} E[\mathcal{S}_{\mathsf{C}}(\tau', e')]$
IF $e' \in \mathsf{Err}$:
a. QED $e \rightarrow_{\text{C-S}} e'$
IF $e' \rightarrow_{\text{C-S}} e''$:
a. QED $e \rightarrow_{\text{C-S}} E[\mathsf{stat}\ \tau'\ e'']$
IF $e' \rightarrow_{\text{C-S}} \mathsf{BndryErr}$:
a. QED $e \rightarrow_{\text{C-S}} E[\mathsf{stat}\ \tau'\ \mathsf{BndryErr}]$
ELSE $e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{D-C}} \mathsf{TagErr}$
:
a. Contradiction by e' is boundary-free

CASE $e = E[\mathsf{Err}]$:

1. QED $e \rightarrow_{\text{C-S}} \mathsf{Err}$

□

Lemma 5.8 : HC *dynamic progress*

If $\vdash_{\mathsf{C}} e$ then one of the following holds:

- e is a value
- $e \in \mathsf{Err}$
- $e \rightarrow_{\text{C-D}} e'$
- $e \rightarrow_{\text{C-D}} \mathsf{BndryErr}$
- $e \rightarrow_{\text{C-D}} \mathsf{TagErr}$

Proof:

By the *boundary factoring* lemma, there are seven cases.

CASE e is a value :

1. QED

CASE $e = E^\bullet[v_0\ v_1]$:

IF $v_0 = \lambda x.\, e'$:
1. $e \rightarrow_{\text{C-D}} E^\bullet[e'[x \leftarrow v_1]]$
by $v_0\ v_1 \rhd_{\text{D-C}} e'[x \leftarrow v_1]$
2. QED

IF $v_0 = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v_f$:
1. $e \rightarrow_{\text{C-D}} E^\bullet[\mathsf{stat}\ \tau_c\ (v_f\ (\mathsf{dyn}\ \tau_d\ v_1))]$
by $v_0\ v_1 \rhd_{\text{D-C}} \mathsf{stat}\ \tau_c\ (v_f\ (\mathsf{dyn}\ \tau_d\ v_1))$
2. QED

ELSE $v_0 = i$
$\lor\ v_0 = \langle v, v' \rangle$:
1. $e \rightarrow_{\text{C-D}} \mathsf{TagErr}$
by $(v_0\ v_1) \rhd_{\text{D-C}} \mathsf{TagErr}$
2. QED

CASE $e = E^\bullet[op^1\ v]$:

IF $v = \mathsf{mon}\,(\tau_0{\times}\tau_1)\,v'$
$\land\ op^1 = \mathsf{fst}$:
1. $e \rightarrow_{\text{C-D}} E^\bullet[\mathsf{stat}\ \tau_0\ op^1\ v']$
by $op^1\ v \rhd_{\text{D-C}} \mathsf{stat}\ \tau_0\ op^1\ v'$
2. QED

IF $v = \mathsf{mon}\,(\tau_0{\times}\tau_1)\,v'$
$\land\ op^1 = \mathsf{snd}$:
1. $e \rightarrow_{\text{C-D}} E^\bullet[\mathsf{stat}\ \tau_1\ op^1\ v']$
by $op^1\ v \rhd_{\text{D-C}} \mathsf{stat}\ \tau_1\ op^1\ v'$
2. QED

IF $\delta(op^1, v) = e'$:
1. $(op^1\ v) \rhd_{\text{D-C}} e'$
2. QED

ELSE $\delta(op^1, v)$ is undefined :
1. $e \rightarrow_{\text{C-D}} \mathsf{TagErr}$
by $(op^1\ v) \rhd_{\text{D-C}} \mathsf{TagErr}$
2. QED

CASE $e = E^\bullet[op^2\ v_0\ v_1]$:

IF $\delta(op^2, v_0, v_1) = e''$:
1. $op^2\ v_0\ v_1 \rhd_{\text{D-C}} e''$
2. QED

ELSE $\delta(op^2, v_0, v_1)$ is undefined :

1. $e \rightarrow_{\text{C-D}} \mathsf{TagErr}$
   by $op^2\ v_0\ v_1 \rhd_{\text{D-C}} \mathsf{TagErr}$
2. QED

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :

1. e' is a value
   $\vee\ e' \in \mathsf{Err}$
   $\vee\ e' \rightarrow_{\text{C-D}} e''$
   $\vee\ e' \rightarrow_{\text{C-D}} \mathsf{BndryErr}$
   $\vee\ e' = E[e'']$ and $e'' \rhd_{\text{D-C}} \mathsf{TagErr}$
   by *dynamic progress*
2. **IF** e' is a value :
   a. QED $e \rightarrow_{\text{C-D}} E[\mathcal{D}_{\text{C}}(\tau', e')]$
   IF $e' \in \mathsf{Err}$:
   a. QED $e \rightarrow_{\text{C-D}} e'$
   IF $e' \rightarrow_{\text{C-D}} e''$:
   a. QED $e \rightarrow_{\text{C-S}} E[\mathsf{dyn}\ \tau'\ e'']$
   IF $e' \rightarrow_{\text{C-D}} \mathsf{BndryErr}$:
   a. QED $e \rightarrow_{\text{C-D}} E[\mathsf{dyn}\ \tau'\ \mathsf{BndryErr}]$
   ELSE $e' = E[e'']$ and $e'' \rhd_{\text{D-C}} \mathsf{TagErr}$:
   a. $E \in E^{\bullet}$
      by e' is boundary-free
   b. QED

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :

1. e' is a value
   $\vee\ e' \in \mathsf{Err}$
   $\vee\ e' \rightarrow_{\text{C-S}} e''$
   $\vee\ e' \rightarrow_{\text{C-S}} \mathsf{BndryErr}$
   $\vee\ e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{D-C}} \mathsf{TagErr}$
   by *static progress*
2. **IF** e' is a value :
   a. QED $e \rightarrow_{\text{C-S}} E[\mathcal{S}_{\text{C}}(\tau', e')]$
   IF $e' \in \mathsf{Err}$:
   a. QED $e \rightarrow_{\text{C-S}} e'$
   IF $e' \rightarrow_{\text{C-S}} e''$:
   a. QED $e \rightarrow_{\text{C-S}} E[\mathsf{stat}\ \tau'\ e'']$
   IF $e' \rightarrow_{\text{C-S}} \mathsf{BndryErr}$:
   a. QED $e \rightarrow_{\text{C-S}} E[\mathsf{stat}\ \tau'\ \mathsf{BndryErr}]$
   ELSE $e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{D-C}} \mathsf{TagErr}$
   :
   a. Contradiction by e' is boundary-free

CASE $e = E[\mathsf{Err}]$:

1. QED $e \rightarrow_{\text{C-D}} \mathsf{Err}$

$\square$

Lemma 5.9 : HC *static preservation*

If $\vdash_{\text{C}} e : \tau$ and $e \rightarrow_{\text{C-S}} e'$ then $\vdash_{\text{C}} e' : \tau$

Proof:

By the *boundary factoring* lemma there are seven cases.

CASE e is a value :

1. Contradiction by $e \rightarrow_{\text{C-S}} e'$

CASE $e = E^{\bullet}[v_0\ v_1]$:

IF $v_0 = \lambda(x\!:\!\tau_x).\,e'$
$\wedge\ e \rightarrow_{\text{C-S}} E^{\bullet}[e'[x \leftarrow v_1]]$:

1. $\vdash_{\text{C}} v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\text{C}} v_0 : \tau_d \Rightarrow \tau_c$
   $\wedge \vdash_{\text{C}} v_1 : \tau_d$
   $\wedge\ \tau_c \leqslant: \tau'$
   by *inversion*
3. $\tau_d \leqslant: \tau_x$
   by *canonical forms* (2)
4. $(x\!:\!\tau_x) \vdash_{\text{C}} e' : \tau_c$
   by *inversion* (2)
5. $\vdash_{\text{C}} v_1 : \tau_x$
   by (2, 3)
6. $\vdash_{\text{C}} e'[x \leftarrow v_1] : \tau_c$
   by *substitution* (4, 5)
7. $\vdash_{\text{C}} e'[x \leftarrow v_1] : \tau'$
   by (2, 6)
8. QED by *hole substitution* (7)

ELSE $v_0 = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v_f$
$\wedge\ e \rightarrow_{\text{C-S}} E^{\bullet}[\mathsf{dyn}\ \tau_c\ (v_f\ (\mathsf{stat}\ \tau_d\ v_1))]$:

1. $\vdash_{\text{C}} v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\text{C}} v_0 : \tau_d' \Rightarrow \tau_c'$
   $\wedge \vdash_{\text{C}} v_1 : \tau_d'$
   $\wedge\ \tau_c' \leqslant: \tau'$
   by *inversion*
3. $\vdash_{\text{C}} v_f$
   by *inversion* (2)
4. $\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'$
   by *canonical forms* (2)
5. $\tau_d' \leqslant: \tau_d$
   $\wedge\ \tau_c \leqslant: \tau_c'$
   by (4)
6. $\vdash_{\text{C}} v_1 : \tau_d$
   by (2, 5)
7. $\vdash_{\text{C}} \mathsf{stat}\ \tau_d\ v_1$
   by (6)
8. $\vdash_{\text{C}} v_f\ (\mathsf{stat}\ \tau_d\ v_1)$
   by (3, 7)
9. $\vdash_{\text{C}} \mathsf{dyn}\ \tau_c\ (v_f\ (\mathsf{stat}\ \tau_d\ v_1)) : \tau_c$
   by (8)
10. $\vdash_{\text{C}} \mathsf{dyn}\ \tau_c\ (v_f\ (\mathsf{stat}\ \tau_d\ v_1)) : \tau'$
    by (2, 5, 9)
11. QED by *hole substitution* (10)

CASE $e = E^{\bullet}[op^1\ v]$:

IF $v = \mathsf{mon}\,(\tau_0 \times \tau_1)\,v'$
$\wedge\ op^1 = \mathsf{fst}$
$\wedge\ e \rightarrow_{\text{C-S}} E^{\bullet}[\mathsf{dyn}\ \tau_0\ (\mathsf{fst}\ v')]$:

1. $\vdash_{\text{C}} \mathsf{fst}\ v : \tau'$
   by *static hole typing*
2. $\vdash_{\text{C}} v : \tau_0' \times \tau_1'$
   $\wedge\ \tau_0' <: \tau'$
   by *inversion*
3. $\vdash_{\text{C}} v'$
   by *inversion* (2)

4. $\tau_0 \times \tau_1 \leqslant: \tau_0' \times \tau_1'$
by *canonical forms* (2)
5. $\tau_0 \leqslant: \tau_0'$
6. $\vdash_{\mathrm{C}} \mathsf{fst}\, v'$
by (3)
7. $\vdash_{\mathrm{C}} \mathsf{dyn}\ \tau_0\ (\mathsf{fst}\, v') : \tau_0$
by (6)
8. $\vdash_{\mathrm{C}} \mathsf{dyn}\ \tau_0\ (\mathsf{fst}\, v') : \tau'$
by (2, 5, 7)
9. QED by *hole substitution*

IF $v = \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{snd}$
$\wedge\ e \rightarrow_{\text{C-S}} E^{\bullet}[\mathsf{dyn}\ \tau_0\ (\mathsf{snd}\, v')]$:

1. $\vdash_{\mathrm{C}} \mathsf{snd}\, v : \tau'$
by *static hole typing*
2. $\vdash_{\mathrm{C}} v : \tau_0' \times \tau_1'$
$\wedge\ \tau_1' <: \tau'$
by *inversion*
3. $\vdash_{\mathrm{C}} v'$
by *inversion* (2)
4. $\tau_0 \times \tau_1 \leqslant: \tau_0' \times \tau_1'$
by *canonical forms* (2)
5. $\tau_1 \leqslant: \tau_1'$
6. $\vdash_{\mathrm{C}} \mathsf{snd}\, v'$
by (3)
7. $\vdash_{\mathrm{C}} \mathsf{dyn}\ \tau_0\ (\mathsf{snd}\, v') : \tau_0$
by (5)
8. $\vdash_{\mathrm{C}} \mathsf{dyn}\ \tau_0\ (\mathsf{snd}\, v') : \tau'$
by (2, 5, 7)
9. QED by *hole substitution*

IF $v = \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{fst}$
$\wedge\ e \rightarrow_{\text{C-S}} E^{\bullet}[v_0]$:

1. $\vdash_{\mathrm{C}} \mathsf{fst}\,\langle v_0, v_1 \rangle : \tau'$
by *static hole typing*
2. $\vdash_{\mathrm{C}} \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$
$\wedge\ \tau_0 \leqslant: \tau'$
by *inversion* (1)
3. $\vdash_{\mathrm{C}} v_0 : \tau_0$
by *inversion* (2)
4. $\vdash_{\mathrm{C}} v_0 : \tau'$
by (2, 3)
5. QED by *hole substitution* (4)

ELSE $v = \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{snd}$
$\wedge\ e \rightarrow_{\text{C-S}} E^{\bullet}[v_1]$:

1. $\vdash_{\mathrm{C}} \mathsf{snd}\,\langle v_0, v_1 \rangle : \tau'$
by *static hole typing*
2. $\vdash_{\mathrm{C}} \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$
$\wedge\ \tau_1 \leqslant: \tau'$
by *inversion* (1)
3. $\vdash_{\mathrm{C}} v_1 : \tau_1$
by *inversion* (2)
4. $\vdash_{\mathrm{C}} v_1 : \tau'$
by (2, 3)
5. QED by *hole substitution* (4)

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:

1. $e \rightarrow_{\text{C-S}} E^{\bullet}[\delta(op^2, v_0, v_1)]$
by $e \rightarrow_{\text{C-S}} e'$
2. $\vdash_{\mathrm{C}} op^2\ v_0\ v_1 : \tau'$
by *static hole typing*
3. $\vdash_{\mathrm{C}} v_0 : \tau_0$
$\wedge \vdash_{\mathrm{C}} v_1 : \tau_1$
$\wedge\ \Delta(op^2, \tau_0, \tau_1) = \tau''$
$\wedge\ \tau'' \leqslant: \tau'$
by *inversion* (1)
4. $\vdash_{\mathrm{C}} \delta(op^2, v_0, v_1) : \tau''$
by *Δ type soundness* (2)
5. $\vdash_{\mathrm{C}} \delta(op^2, v_0, v_1) : \tau'$
by (2, 3)
6. QED by *hole substitution* (4)

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :

IF e' is a value :

1. $e \rightarrow_{\text{C-S}} E[\mathcal{D}_{\mathrm{C}}(\tau', e')]$
2. $\vdash_{\mathrm{C}} \mathsf{dyn}\ \tau'\ e' : \tau'$
by *boundary hole typing*
3. $\vdash_{\mathrm{C}} e'$
by *inversion* (2)
4. $\vdash_{\mathrm{C}} \mathcal{D}_{\mathrm{C}}(\tau', e') : \tau'$
by *$\mathcal{D}_{\mathrm{C}}$ soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\text{C-D}} e''$:

1. $e \rightarrow_{\text{C-S}} E[\mathsf{dyn}\ \tau'\ e'']$
2. $\vdash_{\mathrm{C}} \mathsf{dyn}\ \tau'\ e' : \tau'$
by *boundary hole typing*
3. $\vdash_{\mathrm{C}} e'$
by *inversion* (2)
4. $\vdash_{\mathrm{C}} e''$
by *dynamic preservation* (3)
5. $\vdash_{\mathrm{C}} \mathsf{dyn}\ \tau'\ e'' : \tau'$
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :

IF e' is a value :

1. $e \rightarrow_{\text{C-S}} E[\mathcal{S}_{\mathrm{C}}(\tau', e')]$
2. $\vdash_{\mathrm{C}} \mathsf{stat}\ \tau'\ e'$
by *boundary hole typing*
3. $\vdash_{\mathrm{C}} e' : \tau'$
by *inversion* (2)
4. $\vdash_{\mathrm{C}} \mathcal{S}_{\mathrm{C}}(\tau', e')$
by *$\mathcal{S}_{\mathrm{C}}$ soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\text{C-S}} e''$:

1. $e \rightarrow_{\text{C-S}} E[\mathsf{stat}\ \tau'\ e'']$
2. $\vdash_{\mathrm{C}} \mathsf{stat}\ \tau'\ e'$
by *boundary hole typing*

3. $\vdash_C e' : \tau'$
   by *inversion* (2)
4. $\vdash_C e'' : \tau'$
   by *static preservation* (3)
5. $\vdash_C \mathsf{stat}\ \tau'\ e''$
   by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{Err}]$:

1. $e \rightarrow_{\text{C-S}} \mathsf{Err}$
2. QED by $\vdash_C \mathsf{Err} : \tau$

□

Lemma 5.10 : HC *dynamic preservation*

If $\vdash_C e$ and $e \rightarrow_{\text{C-D}} e'$ then $\vdash_C e'$

Proof:

By the *boundary factoring* lemma, there are seven cases.

CASE e is a value :

1. Contradiction by $e \rightarrow_{\text{C-D}} e'$

CASE $e = E^{\bullet}[v_0\ v_1]$:

IF $v_0 = \lambda x.\, e'$
$\wedge\ e \rightarrow_{\text{C-D}} E^{\bullet}[e'[x \leftarrow v_1]]$:

1. $\vdash_C v_0\ v_1$
   by *dynamic hole typing*
2. $\vdash_C v_0$
   $\wedge \vdash_C v_1$
   by *inversion* (1)
3. $x \vdash_C e'$
   by *inversion* (2)
4. $\vdash_C e'[x \leftarrow v_1]$
   by *substitution* (2, 3)
5. QED *hole substitution* (4)

ELSE $v_0 = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v_f$
$\wedge\ e \rightarrow_{\text{C-D}} E^{\bullet}[\mathsf{stat}\ \tau_c\ (v_f\ (\mathsf{dyn}\ \tau_d\ v_1))]$:

1. $\vdash_C v_0\ v_1$
   by *dynamic hole typing*
2. $\vdash_C v_0$
   $\wedge \vdash_C v_1$
   by *inversion* (1)
3. $\vdash_C v_f : \tau_d \Rightarrow \tau_c$
   by *inversion* (2)
4. $\vdash_C \mathsf{dyn}\ \tau_d\ v_1 : \tau_d$
   by (2)
5. $\vdash_C v_f\ (\mathsf{dyn}\ \tau_d\ v_1) : \tau_c$
   by (3, 4)
6. $\vdash_C \mathsf{stat}\ \tau_c\ (v_f\ (\mathsf{dyn}\ \tau_d\ v_1))$
   by (5)
7. QED by *hole substitution*

CASE $e = E^{\bullet}[op^1\ v]$:

IF $v = \mathsf{mon}\,(\tau_0 \times \tau_1)\, v'$
$\wedge\ op^1 = \mathsf{fst}$
$\wedge\ e \rightarrow_{\text{C-D}} E[\mathsf{stat}\ \tau_0\ (\mathsf{fst}\ v')]$:

1. $\vdash_C op^1\ v$
   by *dynamic hole typing*
2. $\vdash_C \mathsf{mon}\,(\tau_0 \times \tau_1)\, v'$
   by *inversion* (1)
3. $\vdash_C v' : \tau_0 \times \tau_1$
   by *inversion* (2)
4. $\vdash_C \mathsf{fst}\ v' : \tau_0$
   by (3)
5. $\vdash_C \mathsf{stat}\ \tau_0\ (\mathsf{fst}\ v')$
   by (4)
6. QED by *hole substitution*

IF $v = \mathsf{mon}\,(\tau_0 \times \tau_1)\, v'$
$\wedge\ op^1 = \mathsf{snd}$
$\wedge\ e \rightarrow_{\text{C-D}} E[\mathsf{stat}\ \tau_0\ (\mathsf{snd}\ v')]$:

1. $\vdash_C op^1\ v$
   by *dynamic hole typing*
2. $\vdash_C \mathsf{mon}\,(\tau_0 \times \tau_1)\, v'$
   by *inversion* (1)
3. $\vdash_C v' : \tau_0 \times \tau_1$
   by *inversion* (2)
4. $\vdash_C \mathsf{snd}\ v' : \tau_1$
   by (3)
5. $\vdash_C \mathsf{stat}\ \tau_1\ (\mathsf{snd}\ v')$
   by (4)
6. QED by *hole substitution*

IF $v = \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{fst}$
$\wedge\ e \rightarrow_{\text{C-D}} E^{\bullet}[v_0]$:

1. $\vdash_C op^1\ v$
   by *dynamic hole typing*
2. $\vdash_C v$
   by *inversion* (1)
3. $\vdash_C v_0$
   by *inversion* (2)
4. QED by *hole substitution*

ELSE $v = \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{snd}$
$\wedge\ e \rightarrow_{\text{C-D}} E^{\bullet}[v_1]$:

1. $\vdash_C op^1\ v$
   by *dynamic hole typing*
2. $\vdash_C v$
   by *inversion* (1)
3. $\vdash_C v_1$
   by *inversion* (2)
4. QED by *hole substitution*

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:

1. $e \rightarrow_{\text{C-D}} E^{\bullet}[\delta(op^2, v_0, v_1)]$
2. $\vdash_C op^2\ v_0\ v_1$
   by *dynamic hole typing*
3. $\vdash_C v_0$
   $\wedge \vdash_C v_1$
   by *inversion* (1)
4. $\vdash_C \delta(op^2, v_0, v_1)$
   by *δ preservation* (2)
5. QED by *hole substitution* (3)

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :

IF e' is a value :

1. $e \rightarrow_{\text{C-D}} E[\mathcal{D}_{\text{C}}(\tau', e')]$
2. $\vdash_{\text{C}} \text{dyn}\ \tau'\ e' : \tau'$ by *boundary hole typing*
3. $\vdash_{\text{C}} e'$ by *inversion* (2)
4. $\vdash_{\text{C}} \mathcal{D}_{\text{C}}(\tau', e') : \tau'$ by *$\mathcal{D}_{\text{C}}$ soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\text{C-D}} e''$:

1. $e \rightarrow_{\text{C-D}} E[\text{dyn}\ \tau'\ e'']$
2. $\vdash_{\text{C}} \text{dyn}\ \tau'\ e' : \tau'$ by *boundary hole typing*
3. $\vdash_{\text{C}} e'$ $\wedge\ \tau' \leqslant: \tau''$ by *inversion* (2)
4. $\vdash_{\text{C}} e''$ by *dynamic preservation* (3)
5. $\vdash_{\text{C}} \text{dyn}\ \tau'\ e'' : \tau'$ by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\text{stat}\ \tau'\ e']$ and e' is boundary-free :

IF $e' \in v$:

1. $e \rightarrow_{\text{C-D}} E[\mathcal{S}_{\text{C}}(\tau', e')]$
2. $\vdash_{\text{C}} \text{stat}\ \tau'\ e'$ by *boundary hole typing*
3. $\vdash_{\text{C}} e' : \tau'$ by *inversion* (2)
4. $\vdash_{\text{C}} \mathcal{S}_{\text{C}}(\tau', e')$ by *$\mathcal{S}_{\text{C}}$ soundness* (3)
5. QED by *hole substitution* (5)

ELSE $e' \rightarrow_{\text{C-S}} e''$:

1. $e \rightarrow_{\text{C-D}} E[\text{stat}\ \tau'\ e'']$
2. $\vdash_{\text{C}} \text{stat}\ \tau'\ e'$ by *boundary hole typing*
3. $\vdash_{\text{C}} e' : \tau'$ by *inversion* (2)
4. $\vdash_{\text{C}} e'' : \tau'$ by *static preservation* (3)
5. $\vdash_{\text{C}} \text{stat}\ \tau'\ e''$ by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\text{Err}]$:

1. $e \rightarrow_{\text{C-D}} \text{Err}$
2. QED $\vdash_{\text{C}} \text{Err}$

☐

Lemma 5.11 : HC *static boundary factoring*

If $\vdash_{\text{C}} e : \tau$ then one of the following holds:

- e is a value
- $e = E^{\bullet}[v_0\ v_1]$
- $e = E^{\bullet}[op^1\ v]$
- $e = E^{\bullet}[op^2\ v_0\ v_1]$
- $e = E[\text{dyn}\ \tau\ e']$ where e' is boundary-free
- $e = E[\text{stat}\ \tau\ e']$ where e' is boundary-free
- $e = E[\text{Err}]$

Proof:

By the *boundary factoring* lemma for the higher-order embedding. (The only difference is the meaning of *e is a value*.)

☐

Lemma 5.12 : HC *dynamic boundary factoring*

If $\vdash_{\text{C}} e$ then one of the following holds:

- e is a value
- $e = E^{\bullet}[v_0\ v_1]$
- $e = E^{\bullet}[op^1\ v]$
- $e = E^{\bullet}[op^2\ v_0\ v_1]$
- $e = E[\text{dyn}\ \tau\ e']$ where e' is boundary-free
- $e = E[\text{stat}\ \tau\ e']$ where e' is boundary-free
- $e = E[\text{Err}]$

Proof:

By the *boundary factoring* lemma for the higher-order embedding.

☐

Lemma 5.13 : HC *static hole typing*

If $\vdash_{\text{C}} E^{\bullet}[e] : \tau$ then the derivation contains a sub-term $\vdash_{\text{C}} e : \tau'$

Proof (sketch): Similar to the *static hole typing* lemma for the higher-order embedding. ☐

Lemma 5.14 : HC *dynamic hole typing*

If $\vdash_{\text{C}} E^{\bullet}[e]$ then the derivation contains a sub-term $\vdash_{\text{C}} e$

Proof (sketch): Similar to the *static hole typing* lemma for the higher-order embedding. ☐

Lemma 5.15 : HC *boundary hole typing*

- If $\vdash_{\text{C}} E[\text{dyn}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_{\text{C}} \text{dyn}\ \tau\ e : \tau$
- If $\vdash_{\text{C}} E[\text{dyn}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_{\text{C}} \text{dyn}\ \tau\ e : \tau$
- If $\vdash_{\text{C}} E[\text{stat}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_{\text{C}} \text{stat}\ \tau\ e$
- If $\vdash_{\text{C}} E[\text{stat}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_{\text{C}} \text{stat}\ \tau\ e$

Proof (sketch): Similar to the proof for the higher-order *boundary hole typing* lemma. ☐

Lemma 5.16 : HC *hole substitution*

- If $\vdash_{\text{C}} E[e]$ and the derivation contains a sub-term $\vdash_{\text{C}} e : \tau'$ and $\vdash_{\text{C}} e' : \tau'$ then $\vdash_{\text{C}} E[e']$.
- If $\vdash_{\text{C}} E[e]$ and the derivation contains a sub-term $\vdash_{\text{C}} e$ and $\vdash_{\text{C}} e'$ then $\vdash_{\text{C}} E[e']$.
- If $\vdash_{\text{C}} E[e] : \tau$ and the derivation contains a sub-term $\vdash_{\text{C}} e : \tau'$ and $\vdash_{\text{C}} e' : \tau'$ then $\vdash_{\text{C}} E[e'] : \tau$.
- If $\vdash_{\text{C}} E[e] : \tau$ and the derivation contains a sub-term $\vdash_{\text{C}} e$ and $\vdash_{\text{C}} e'$ then $\vdash_{\text{C}} E[e'] : \tau$.

Proof (sketch): Similar to the proof of the higher-order *hole substitution* lemma, just replacing $\vdash_H$ with $\vdash_C$. □

Lemma 5.17 : $\vdash_C$ *static inversion*

- If $\Gamma \vdash_C x : \tau$ then $(x{:}\tau') \in \Gamma$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash_C \lambda(x : \tau'_d).\, e' : \tau$ then $(x : \tau'_d), \Gamma \vdash_C e' : \tau'_c$ and $\tau'_d \Rightarrow \tau'_c \leqslant: \tau$
- If $\Gamma \vdash_C \langle e_0, e_1 \rangle : \tau_0 \times \tau_1$ then $\Gamma \vdash_C e_0 : \tau'_0$ and $\Gamma \vdash_C e_1 : \tau'_1$ and $\tau'_0 \leqslant: \tau_0$ and $\tau'_1 \leqslant: \tau_1$
- If $\Gamma \vdash_C e_0\ e_1 : \tau_c$ then $\Gamma \vdash_C e_0 : \tau'_d \Rightarrow \tau'_c$ and $\Gamma \vdash_C e_1 : \tau'_d$ and $\tau'_c \leqslant: \tau_c$
- If $\Gamma \vdash_C \mathsf{fst}\, e : \tau$ then $\Gamma \vdash_C e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0$ and $\tau_0 \leqslant: \tau$
- If $\Gamma \vdash_C \mathsf{snd}\, e : \tau$ then $\Gamma \vdash_C e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1$ and $\tau_1 \leqslant: \tau$
- If $\Gamma \vdash_C op^2\ e_0\ e_1 : \tau$ then $\Gamma \vdash_C e_0 : \tau_0$ and $\Gamma \vdash_C e_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau'$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash_C \mathsf{mon}\ \tau'_0 \times \tau'_1\ v' : \tau_0 \times \tau_1$ then $\Gamma \vdash_C v'$ and $\tau'_0 \times \tau'_1 \leqslant: \tau_0 \times \tau_1$
- If $\Gamma \vdash_C \mathsf{mon}\ \tau'_d \Rightarrow \tau'_c\ v' : \tau_d \Rightarrow \tau_c$ then $\Gamma \vdash_C v'$ and $\tau'_d \Rightarrow \tau'_c \leqslant: \tau_d \Rightarrow \tau_c$
- If $\Gamma \vdash_C \mathsf{dyn}\ \tau'\ e' : \tau$ then $\Gamma \vdash_C e'$ and $\tau' \leqslant: \tau$

Proof:

QED by the definition of $\Gamma \vdash_C e : \tau$

□

Lemma 5.18 : $\vdash_C$ *dynamic inversion*

- If $\Gamma \vdash_C x$ then $x \in \Gamma$
- If $\Gamma \vdash_C \lambda x.\, e'$ then $x, \Gamma \vdash_C e'$
- If $\Gamma \vdash_C \langle e_0, e_1 \rangle$ then $\Gamma \vdash_C e_0$ and $\Gamma \vdash_C e_1$
- If $\Gamma \vdash_C e_0\ e_1$ then $\Gamma \vdash_C e_0$ and $\Gamma \vdash_C e_1$
- If $\Gamma \vdash_C op^1\ e_0$ then $\Gamma \vdash_C e_0$
- If $\Gamma \vdash_C op^2\ e_0\ e_1$ then $\Gamma \vdash_C e_0$ and $\Gamma \vdash_C e_1$
- If $\Gamma \vdash_C \mathsf{mon}\ \tau_d \Rightarrow \tau_c\ v'$ then $\Gamma \vdash_C v' : \tau_d \Rightarrow \tau_c$
- If $\Gamma \vdash_C \mathsf{mon}\ \tau_0 \times \tau_1\ v'$ then $\Gamma \vdash_C v' : \tau_0 \times \tau_1$
- If $\Gamma \vdash_C \mathsf{stat}\ \tau'\ e'$ then $\Gamma \vdash_C e' : \tau'$

Proof:

QED by the definition of $\Gamma \vdash_C e$

□

Lemma 5.19 : HC *canonical forms*

- If $\vdash_C v : \tau_0 \times \tau_1$ then either:
  - $v = \langle v_0, v_1 \rangle$
  - or $v = \mathsf{mon}\,(\tau'_0 \times \tau'_1)\, v'$ $\wedge\ \tau'_0 \times \tau'_1 \leqslant: \tau_0 \times \tau_1$
- If $\vdash_C v : \tau_d \Rightarrow \tau_c$ then either:
  - $v = \lambda(x{:}\tau_x).\, e'$ $\wedge\ \tau_d \leqslant: \tau_x$
  - or $v = \mathsf{mon}\,(\tau'_d \Rightarrow \tau'_c)\, v'$ $\wedge\ \tau'_d \Rightarrow \tau'_c \leqslant: \tau_d \Rightarrow \tau_c$
- If $\vdash_C v : \mathsf{Int}$ then $v = i$
- If $\vdash_C v : \mathsf{Nat}$ then $v = i$ and $v \in \mathbb{N}$

Proof:

QED by definition of $\vdash_C e : \tau$

□

Lemma 5.20 : Δ *type soundness*

If $\vdash_C v_0 : \tau_0$ and $\vdash_C v_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau$ then one of the following holds:

- $\delta(op^2, v_0, v_1) = v$ and $\vdash_C v : \tau$, or
- $\delta(op^2, v_0, v_1) = \mathsf{BndryErr}$

Proof (sketch): Similar to the proof for the higher-order *Δ type soundness* lemma. □

Lemma 5.21 : δ *preservation*

- If $\vdash_C v$ and $\delta(op^1, v) = v'$ then $\vdash_C v'$
- If $\vdash_C v_0$ and $\vdash_C v_1$ and $\delta(op^2, v_0, v_1) = v'$ then $\vdash_C v'$

Proof (sketch): Similar to the proof for the higher-order *δ preservation* lemma. □

Lemma 5.22 : HC *substitution*

- If $(x{:}\tau_x), \Gamma \vdash_C e$ and $\vdash_C v : \tau_x$ then $\Gamma \vdash_C e[x \leftarrow v]$
- If $x, \Gamma \vdash_C e$ and $\vdash_C v$ then $\Gamma \vdash_C e[x \leftarrow v]$
- If $(x{:}\tau_x), \Gamma \vdash_C e : \tau$ and $\vdash_C v : \tau_x$ then $\Gamma \vdash_C e[x \leftarrow v] : \tau$
- If $x, \Gamma \vdash_C e : \tau$ and $\vdash_C v$ then $\Gamma \vdash_C e[x \leftarrow v] : \tau$

Proof (sketch): Similar to the proof for the higher-order *substitution* lemma. □

Lemma 5.23 : *weakening*

- If $\Gamma \vdash_C e$ then $x, \Gamma \vdash_C e$
- If $\Gamma \vdash_C e : \tau$ then $(x{:}\tau'), \Gamma \vdash_C e : \tau$

Proof:

QED because e is closed under Γ

□

E.6 (HF) Forgetful Embedding

E.6.1 Forgetful Definitions

$\boxed{\text{Language HF}}$

$$
\begin{array}{lcl}
e & = & x \mid v \mid \langle e, e \rangle \mid e\ e \mid op^1\ e \mid op^2\ e\ e \mid \\
 & & \mathsf{dyn}\ \tau\ e \mid \mathsf{stat}\ \tau\ e \mid \mathsf{Err} \mid \mathsf{chk}\ \tau\ e \\
v & = & i \mid \langle v, v \rangle \mid \lambda x.\, e \mid \lambda(x\!:\!\tau).\, e \mid \\
 & & \mathsf{mon}\,(\tau \Rightarrow \tau)\,(\lambda x.\, e) \mid \mathsf{mon}\,(\tau \Rightarrow \tau)\,(\lambda(x\!:\!\tau).\, e) \mid \\
 & & \mathsf{mon}\,(\tau \times \tau)\,\langle v, v \rangle \\
\tau & = & \mathsf{Nat} \mid \mathsf{Int} \mid \tau \times \tau \mid \tau \Rightarrow \tau \\
\Gamma & = & \cdot \mid x, \Gamma \mid (x\!:\!\tau), \Gamma \\
\mathsf{Err} & = & \mathsf{BndryErr} \mid \mathsf{TagErr} \\
r & = & v \mid \mathsf{Err} \\
E^{\bullet} & = & [\,] \mid E^{\bullet}\ e \mid v\ E^{\bullet} \mid \langle E^{\bullet}, e \rangle \mid \langle v, E^{\bullet} \rangle \mid \\
 & & op^1\ E^{\bullet} \mid op^2\ E^{\bullet}\ e \mid op^2\ v\ E^{\bullet} \\
E & = & E^{\bullet} \mid E\ e \mid v\ E \mid \langle E, e \rangle \mid \langle v, E \rangle \mid op^1\ E \mid \\
 & & op^2\ E\ e \mid op^2\ v\ E \mid \mathsf{dyn}\ \tau\ E \mid \mathsf{stat}\ \tau\ E
\end{array}
$$

$\boxed{\Delta : op^1 \times \tau \longrightarrow \tau}$

$$
\begin{array}{l}
\Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0 \\
\Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1
\end{array}
$$

$\boxed{\Delta : op^2 \times \tau \times \tau \longrightarrow \tau}$

$$
\begin{array}{l}
\Delta(op^2, \mathsf{Nat}, \mathsf{Nat}) = \mathsf{Nat} \\
\Delta(op^2, \mathsf{Int}, \mathsf{Int}) = \mathsf{Int}
\end{array}
$$

$\boxed{\tau \leqslant: \tau}$

$$
\frac{}{\mathsf{Nat} \leqslant: \mathsf{Int}} \qquad
\frac{\tau_d' \leqslant: \tau_d \quad \tau_c \leqslant: \tau_c'}{\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'} \qquad
\frac{\tau_0 \leqslant: \tau_0' \quad \tau_1 \leqslant: \tau_1'}{\tau_0 \times \tau_1 \leqslant: \tau_0' \times \tau_1'}
$$

$$
\frac{}{\tau \leqslant: \tau} \qquad
\frac{\tau \leqslant: \tau' \quad \tau' \leqslant: \tau''}{\tau \leqslant: \tau''}
$$

$\boxed{\Gamma \vdash e}$

$$
\frac{x \in \Gamma}{\Gamma \vdash x} \qquad
\frac{x, \Gamma \vdash e}{\Gamma \vdash \lambda x.\, e} \qquad
\frac{}{\Gamma \vdash i} \qquad
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash \langle e_0, e_1 \rangle}
$$

$$
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash e_0\ e_1} \qquad
\frac{\Gamma \vdash e}{\Gamma \vdash op^1\ e} \qquad
\frac{\Gamma \vdash e_0 \quad \Gamma \vdash e_1}{\Gamma \vdash op^2\ e_0\ e_1} \qquad
\frac{}{\Gamma \vdash \mathsf{Err}}
$$

$$
\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{stat}\ \tau\ e}
$$

$\boxed{\Gamma \vdash e : \tau}$

$$
\frac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash x : \tau} \qquad
\frac{(x\!:\!\tau_d), \Gamma \vdash e : \tau_c}{\Gamma \vdash \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c} \qquad
\frac{i \in \mathbb{N}}{\Gamma \vdash i : \mathsf{Nat}}
$$

$$
\frac{}{\Gamma \vdash i : \mathsf{Int}} \qquad
\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1}{\Gamma \vdash \langle e_0, e_1 \rangle : \tau_0 \times \tau_1} \qquad
\frac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash e_1 : \tau_d}{\Gamma \vdash e_0\ e_1 : \tau_c}
$$

$$
\frac{\Gamma \vdash e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash op^1\ e_0 : \tau} \qquad
\frac{\Gamma \vdash e_0 : \tau_0 \quad \Gamma \vdash e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash op^2\ e_0\ e_1 : \tau} \qquad
\frac{\Gamma \vdash e : \tau' \quad \tau' \leqslant: \tau}{\Gamma \vdash e : \tau} \qquad
\frac{}{\Gamma \vdash \mathsf{Err} : \tau}
$$

$$
\frac{\Gamma \vdash e}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau}
$$

$\boxed{\Gamma \models_{\mathsf{F}} e}$

$$
\frac{x \in \Gamma}{\Gamma \models_{\mathsf{F}} x} \qquad
\frac{x, \Gamma \models_{\mathsf{F}} e}{\Gamma \models_{\mathsf{F}} \lambda x.\, e} \qquad
\frac{}{\Gamma \models_{\mathsf{F}} i} \qquad
\frac{\Gamma \models_{\mathsf{F}} e_0 \quad \Gamma \models_{\mathsf{F}} e_1}{\Gamma \models_{\mathsf{F}} \langle e_0, e_1 \rangle}
$$

$$
\frac{\Gamma \models_{\mathsf{F}} e_0 \quad \Gamma \models_{\mathsf{F}} e_1}{\Gamma \models_{\mathsf{F}} e_0\ e_1} \qquad
\frac{\Gamma \models_{\mathsf{F}} e}{\Gamma \models_{\mathsf{F}} op^1\ e} \qquad
\frac{\Gamma \models_{\mathsf{F}} e_0 \quad \Gamma \models_{\mathsf{F}} e_1}{\Gamma \models_{\mathsf{F}} op^2\ e_0\ e_1} \qquad
\frac{}{\Gamma \models_{\mathsf{F}} \mathsf{Err}}
$$

$$
\frac{\Gamma \models_{\mathsf{F}} e : \tau}{\Gamma \models_{\mathsf{F}} \mathsf{stat}\ \tau\ e} \qquad
\frac{\Gamma \models_{\mathsf{F}} v_0 : \tau_0' \quad \Gamma \models_{\mathsf{F}} v_1 : \tau_1'}{\Gamma \models_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1 \rangle}
$$

$$
\frac{\Gamma \models_{\mathsf{F}} v_0 \quad \Gamma \models_{\mathsf{F}} v_1}{\Gamma \models_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1 \rangle} \qquad
\frac{\Gamma \models_{\mathsf{F}} \lambda x.\, e}{\Gamma \models_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,\lambda x.\, e}
$$

$$
\frac{\Gamma \models_{\mathsf{F}} \lambda(x\!:\!\tau_d').\, e : \tau_d' \Rightarrow \tau_c'}{\Gamma \models_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,\lambda(x\!:\!\tau_d').\, e}
$$

$\boxed{\Gamma \vdash_{\mathsf{F}} e : \tau}$

$$\frac{(x\!:\!\tau) \in \Gamma}{\Gamma \vdash_{\mathsf{F}} x : \tau} \qquad \frac{(x\!:\!\tau_d), \Gamma \vdash_{\mathsf{F}} e : \tau_c}{\Gamma \vdash_{\mathsf{F}} \lambda(x\!:\!\tau_d).\, e : \tau_d \Rightarrow \tau_c} \qquad \frac{i \in \mathbb{N}}{\Gamma \vdash_{\mathsf{F}} i : \mathsf{Nat}}$$

$$\frac{}{\Gamma \vdash_{\mathsf{F}} i : \mathsf{Int}} \qquad \frac{\Gamma \vdash_{\mathsf{F}} e_0 : \tau_0 \quad \Gamma \vdash_{\mathsf{F}} e_1 : \tau_1}{\Gamma \vdash_{\mathsf{F}} \langle e_0, e_1 \rangle : \tau_0 \times \tau_1} \qquad \frac{\Gamma \vdash_{\mathsf{F}} e_0 : \tau_d \Rightarrow \tau_c \quad \Gamma \vdash_{\mathsf{F}} e_1 : \tau_d}{\Gamma \vdash_{\mathsf{F}} e_0\ e_1 : \tau_c}$$

$$\frac{\Gamma \vdash_{\mathsf{F}} e_0 : \tau_0 \quad \Delta(op^1, \tau_0) = \tau}{\Gamma \vdash_{\mathsf{F}} op^1\ e_0 : \tau} \qquad \frac{\Gamma \vdash_{\mathsf{F}} e_0 : \tau_0 \quad \Gamma \vdash_{\mathsf{F}} e_1 : \tau_1 \quad \Delta(op^2, \tau_0, \tau_1) = \tau}{\Gamma \vdash_{\mathsf{F}} op^2\ e_0\ e_1 : \tau} \qquad \frac{\Gamma \vdash_{\mathsf{F}} e : \tau' \quad \tau' <: \tau}{\Gamma \vdash_{\mathsf{F}} e : \tau}$$

$$\frac{}{\Gamma \vdash_{\mathsf{F}} \mathsf{Err} : \tau} \qquad \frac{\Gamma \vdash_{\mathsf{F}} e}{\Gamma \vdash_{\mathsf{F}} \mathsf{dyn}\ \tau\ e : \tau}$$

$$\frac{\Gamma \vdash_{\mathsf{F}} v_0 : \tau_0' \quad \Gamma \vdash_{\mathsf{F}} v_1 : \tau_1'}{\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\, \langle v_0, v_1 \rangle : (\tau_0 \times \tau_1)}$$

$$\frac{\Gamma \vdash_{\mathsf{F}} v_0 \quad \Gamma \vdash_{\mathsf{F}} v_1}{\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\, \langle v_0, v_1 \rangle : (\tau_0 \times \tau_1)}$$

$$\frac{\Gamma \vdash_{\mathsf{F}} \lambda x.\, e}{\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, \lambda x.\, e : (\tau_d \Rightarrow \tau_c)}$$

$$\frac{\Gamma \vdash_{\mathsf{F}} \lambda(x\!:\!\tau_d').\, e : \tau_d' \Rightarrow \tau_c'}{\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, \lambda(x\!:\!\tau_d').\, e : (\tau_d \Rightarrow \tau_c)} \qquad \frac{\Gamma \vdash_{\mathsf{F}} e : \tau'}{\Gamma \vdash_{\mathsf{F}} \mathsf{chk}\ \tau\ e : \tau}$$

$\boxed{\delta(op^1, v) = e}$

$$\begin{array}{ll}
\delta(\mathsf{fst}, \langle v_0, v_1 \rangle) & = v_0 \\
\delta(\mathsf{snd}, \langle v_0, v_1 \rangle) & = v_1
\end{array}$$

$\boxed{\delta(op^2, v, v) = e}$

$$\begin{array}{ll}
\delta(\mathsf{sum}, i_0, i_1) & = i_0 + i_1 \\
\delta(\mathsf{quotient}, i_0, 0) & = \mathsf{BndryErr} \\
\delta(\mathsf{quotient}, i_0, i_1) & = \lfloor i_0 / i_1 \rfloor \\
\quad \text{if } i_1 \neq 0 &
\end{array}$$

$\boxed{\mathcal{D}_{\mathsf{F}} : \tau \times v \longrightarrow e}$

$$\mathcal{D}_{\mathsf{F}}(\tau, v) = \mathcal{X}(\tau, v)$$

$\boxed{\mathcal{S}_{\mathsf{F}} : \tau \times v \longrightarrow e}$

$$\mathcal{S}_{\mathsf{F}}(\tau, v) = \mathcal{X}(\tau, v)$$

$\boxed{\mathcal{X} : \tau \times v \longrightarrow e}$

$$\begin{array}{ll}
\mathcal{X}(\tau_d \Rightarrow \tau_c, \lambda x.\, e) & = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda x.\, e) \\
\mathcal{X}(\tau_d \Rightarrow \tau_c, \lambda(x\!:\!\tau).\, e) & = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda(x\!:\!\tau).\, e) \\
\mathcal{X}(\tau_d \Rightarrow \tau_c, \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\, v') & = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v' \\
\mathcal{X}(\tau_0 \times \tau_1, \langle v_0, v_1 \rangle) & = \mathsf{mon}\,(\tau_0 \times \tau_1)\, \langle v_0, v_1 \rangle \\
\mathcal{X}(\tau_0 \times \tau_1, \mathsf{mon}\,(\tau_0' \times \tau_1')\, v') & = \mathsf{mon}\,(\tau_0 \times \tau_1)\, v' \\
\mathcal{X}(\mathsf{Int}, i) & = i \\
\mathcal{X}(\mathsf{Nat}, i) & = i \\
\quad \text{if } i \in \mathbb{N} & \\
\mathcal{X}(\tau, v) & = \mathsf{BndryErr} \\
\quad \text{otherwise} &
\end{array}$$

$\boxed{e \rhd_{\text{S-1}} e}$

$$\begin{array}{ll}
\mathsf{dyn}\ \tau\ v & \rhd_{\text{S-1}} \mathcal{D}_{\mathsf{F}}(\tau, v) \\
\mathsf{chk}\ \tau\ v & \rhd_{\text{S-1}} \mathcal{X}(\tau, v) \\
(\mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda x.\, e))\ v & \rhd_{\text{S-1}} \mathsf{dyn}\ \tau_c\ e' \\
\quad \text{where } e' = (\lambda x.\, e)\ (\mathcal{X}(\tau_d, v)) & \\
(\mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda(x\!:\!\tau).\, e))\ v & \rhd_{\text{S-1}} \mathsf{chk}\ \tau_c\ e' \\
\quad \text{where } e' = (\lambda(x\!:\!\tau).\, e)\ (\mathcal{X}(\tau, v)) & \\
\mathsf{fst}\,(\mathsf{mon}\,(\tau_0 \times \tau_1)\, \langle v_0, v_1 \rangle) & \rhd_{\text{S-1}} \mathcal{X}(\tau_0, v_0) \\
\mathsf{snd}\,(\mathsf{mon}\,(\tau_0 \times \tau_1)\, \langle v_0, v_1 \rangle) & \rhd_{\text{S-1}} \mathcal{X}(\tau_1, v_1) \\
(\lambda(x\!:\!\tau).\, e)\ v & \rhd_{\text{S-1}} e[x \leftarrow v] \\
op^1\ v & \rhd_{\text{S-1}} \delta(op^1, v) \\
op^2\ v_0\ v_1 & \rhd_{\text{S-1}} \delta(op^2, v_0, v_1)
\end{array}$$

$\boxed{e \rhd_{\text{D-1}} e}$

$$\begin{array}{ll}
\mathsf{stat}\ \tau\ v & \rhd_{\text{D-1}} \mathcal{S}_{\mathsf{F}}(\tau, v) \\
v_0\ v_1 & \rhd_{\text{D-1}} \mathsf{TagErr} \\
\quad \text{if } v_0 \in \mathbb{Z} \text{ or } v_0 = \langle v, v' \rangle & \\
(\mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda x.\, e))\ v & \rhd_{\text{D-1}} (\lambda x.\, e)\ v \\
(\mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda(x\!:\!\tau).\, e))\ v & \rhd_{\text{D-1}} \mathsf{stat}\ \tau_c\ e' \\
\quad \text{where } e' = \mathsf{chk}\ \tau_c\ ((\lambda(x\!:\!\tau).\, e)\ (\mathcal{X}(\tau, v))) & \\
(\lambda x.\, e)\ v & \rhd_{\text{H-D}} e[x \leftarrow v] \\
\mathsf{fst}\,(\mathsf{mon}\,(\tau_0 \times \tau_1)\, \langle v_0, v_1 \rangle) & \rhd_{\text{D-1}} \mathcal{X}(\tau_0, v_0) \\
\mathsf{snd}\,(\mathsf{mon}\,(\tau_0 \times \tau_1)\, \langle v_0, v_1 \rangle) & \rhd_{\text{D-1}} \mathcal{X}(\tau_1, v_1) \\
op^1\ v & \rhd_{\text{H-D}} \mathsf{TagErr} \\
\quad \text{if } \delta(op^1, v) \text{ is undefined} & \\
op^1\ v & \rhd_{\text{H-D}} \delta(op^1, v) \\
op^2\ v_0\ v_1 & \rhd_{\text{H-D}} \mathsf{TagErr} \\
\quad \text{if } \delta(op^2, v_0, v_1) \text{ is undefined} & \\
op^2\ v_0\ v_1 & \rhd_{\text{H-D}} \delta(op^2, v_0, v_1)
\end{array}$$

$\boxed{e \rightarrow_{\text{F-S}} e}$

$$\begin{array}{lll}
E^{\bullet}[e] & \rightarrow_{\text{F-S}} & E^{\bullet}[e'] \\
\quad \text{if } e \rhd_{\text{S-1}} e' & & \\
E[\mathsf{stat}\ \tau\ E^{\bullet}[e]] & \rightarrow_{\text{F-S}} & E[\mathsf{stat}\ \tau\ E^{\bullet}[e']] \\
\quad \text{if } e \rhd_{\text{S-1}} e' & & \\
E[\mathsf{dyn}\ \tau\ E^{\bullet}[e]] & \rightarrow_{\text{F-S}} & E[\mathsf{dyn}\ \tau\ E^{\bullet}[e']] \\
\quad \text{if } e \rhd_{\text{D-1}} e' & & \\
E[\mathsf{Err}] & \rightarrow_{\text{F-S}} & \mathsf{Err}
\end{array}$$

$\boxed{e \rightarrow_{\text{F-D}} e}$

$E^{\bullet}[e] \quad \rightarrow_{\text{F-D}} \quad E^{\bullet}[e']$
  if $e \rhd_{\text{D-1}} e'$

$E[\text{stat}\ \tau\ E^{\bullet}[e]] \quad \rightarrow_{\text{F-D}} \quad E[\text{stat}\ \tau\ E^{\bullet}[e']]$
  if $e \rhd_{\text{S-1}} e'$

$E[\text{dyn}\ \tau\ E^{\bullet}[e]] \quad \rightarrow_{\text{F-D}} \quad E[\text{dyn}\ \tau\ E^{\bullet}[e']]$
  if $e \rhd_{\text{D-1}} e'$

$E[\text{Err}] \quad \rightarrow_{\text{F-D}} \quad \text{Err}$

$\boxed{e \rightarrow^{*}_{\text{F-S}} e}$ reflexive, transitive closure of $\rightarrow_{\text{F-S}}$

$\boxed{e \rightarrow^{*}_{\text{F-D}} e}$ reflexive, transitive closure of $\rightarrow_{\text{F-D}}$

E.6.2 Forgetful Theorems

Theorem 6.0 : *static* HF-*soundness*

> If $\vdash e : \tau$ then $\vdash_{\mathsf{F}} e : \tau$ and one of the following holds:
> - $e \rightarrow^{*}_{\mathsf{F\text{-}S}} v$ and $\vdash_{\mathsf{F}} v : \tau$
> - $e \rightarrow^{*}_{\mathsf{F\text{-}S}} E[\mathsf{dyn}\ \tau'\ e']$ and $e' \rhd_{\mathsf{D\text{-}1}} \mathsf{TagErr}$
> - $e \rightarrow^{*}_{\mathsf{F\text{-}S}} \mathsf{BndryErr}$
> - e diverges

Proof:

1. $\vdash_{\mathsf{F}} e : \tau$
   by *static subset*
2. QED by *static progress* and *static preservation*.

□

Theorem 6.1 : *dynamic* HF-*soundness*

> If $\vdash e$ then $\vdash_{\mathsf{F}} e$ and one of the following holds:
> - $e \rightarrow^{*}_{\mathsf{F\text{-}D}} v$ and $\vdash_{\mathsf{F}} v$
> - $e \rightarrow^{*}_{\mathsf{F\text{-}D}} E[e']$ and $e' \rhd_{\mathsf{D\text{-}1}} \mathsf{TagErr}$
> - $e \rightarrow^{*}_{\mathsf{F\text{-}D}} \mathsf{BndryErr}$
> - e diverges

Proof:

1. $\vdash_{\mathsf{F}} e$
   by *dynamic subset*
2. QED by *dynamic progress* and *dynamic preservation*.

□

Corollary 6.2 : HF *static soundness*

> If $\vdash e : \tau$ and e is boundary-free, then one of the following holds:
> - $e \rightarrow^{*}_{\mathsf{F\text{-}S}} v$ and $\vdash_{\mathsf{F}} v : \tau$
> - $e \rightarrow^{*}_{\mathsf{F\text{-}S}} \mathsf{BndryErr}$
> - e diverges

Proof:

Consequence of the proof for *static* HF-*soundness*

□

E.6.3 Forgetful Lemmas

Lemma 6.3 : $\mathcal{X}(\cdot,\cdot)$ *soundness*

> If $\Gamma \vdash_{\mathsf{F}} v$ or $\Gamma \vdash_{\mathsf{F}} v : \tau$
> and $\mathcal{X}(\tau', v) = v'$,
> then $\Gamma \vdash_{\mathsf{F}} v'$ and $\Gamma \vdash_{\mathsf{F}} v' : \tau'$

Proof:

By case analysis of the definition of $\mathcal{X}(\cdot,\cdot)$.

CASE $\mathcal{X}(\tau_d \Rightarrow \tau_c, v) = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v$:

IF $v = \lambda x.\, e$ $\wedge\ \Gamma \vdash_{\mathsf{F}} v$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v$ by $\Gamma \vdash_{\mathsf{F}} v$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v : \tau_d \Rightarrow \tau_c$ by $\Gamma \vdash_{\mathsf{F}} v$
3. QED

ELSE $v = \lambda(x\!:\!\tau_x).\, e$ $\wedge\ \Gamma \vdash_{\mathsf{F}} v : \tau_d' \Rightarrow \tau_c'$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v$ by $\Gamma \vdash_{\mathsf{F}} v : \tau_d' \Rightarrow \tau_c'$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v : \tau_d \Rightarrow \tau_c$ by $\Gamma \vdash_{\mathsf{F}} v : \tau_d' \Rightarrow \tau_c'$
3. QED

CASE $\mathcal{X}(\tau_d \Rightarrow \tau_c, \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,v') = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v'$:

IF $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,v'$:

IF $v' = \lambda x.\, e'$ $\wedge\ \Gamma \vdash_{\mathsf{F}} v'$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v'$ by $\Gamma \vdash_{\mathsf{F}} v'$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v' : \tau_d \Rightarrow \tau_c$ by $\Gamma \vdash_{\mathsf{F}} v'$
3. QED

ELSE $v' = \lambda(x\!:\!\tau_x).\, e'$ $\wedge\ \Gamma \vdash_{\mathsf{F}} v' : \tau_d'' \Rightarrow \tau_c''$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v'$ by $\Gamma \vdash_{\mathsf{F}} v' : \tau_d'' \Rightarrow \tau_c''$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v' : \tau_d \Rightarrow \tau_c$ by $\Gamma \vdash_{\mathsf{F}} v' : \tau_d'' \Rightarrow \tau_c''$
3. QED

ELSE $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,v' : \tau_d' \Rightarrow \tau_c'$:

IF $v' = \lambda x.\, e'$ $\wedge\ \Gamma \vdash_{\mathsf{F}} v'$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v'$ by $\Gamma \vdash_{\mathsf{F}} v'$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v' : \tau_d \Rightarrow \tau_c$ by $\Gamma \vdash_{\mathsf{F}} v'$
3. QED

ELSE $v' = \lambda(x\!:\!\tau_x).\, e'$ $\wedge\ \Gamma \vdash_{\mathsf{F}} v' : \tau_d'' \Rightarrow \tau_c''$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v'$ by $\Gamma \vdash_{\mathsf{F}} v' : \tau_d'' \Rightarrow \tau_c''$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,v' : \tau_d \Rightarrow \tau_c$ by $\Gamma \vdash_{\mathsf{F}} v' : \tau_d'' \Rightarrow \tau_c''$
3. QED

CASE $\mathcal{X}(\tau_0 \times \tau_1, \langle v_0, v_1\rangle) = \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1\rangle$:

IF $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1\rangle$ by $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1\rangle : \tau_0 \times \tau_1$ by $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle$
3. QED

ELSE $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle : \tau_0' \times \tau_1'$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1\rangle$ by $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle : \tau_0' \times \tau_1'$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1\rangle : \tau_0 \times \tau_1$ by $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle : \tau_0' \times \tau_1'$
3. QED

CASE $\mathcal{X}(\tau_0 \times \tau_1, \mathsf{mon}\,(\tau_0' \times \tau_1')\,\langle v_0, v_1\rangle) = \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1\rangle$:

IF $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1\rangle$ by $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1\rangle : \tau_0 \times \tau_1$ by $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle$
3. QED

ELSE $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle : \tau_0'' \times \tau_1''$:

1. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1\rangle$ by $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle : \tau_0'' \times \tau_1''$
2. $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1\rangle : \tau_0 \times \tau_1$ by $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1\rangle : \tau_0'' \times \tau_1''$
3. QED

CASE $\mathcal{X}(\mathsf{Int}, i) = i$:

1. $\Gamma \vdash_{\mathsf{F}} i$
2. $\Gamma \vdash_{\mathsf{F}} i : \mathsf{Int}$
3. QED

CASE $\mathcal{X}(\mathsf{Nat}, i) = i$:

1. $\Gamma \vdash_{\mathsf{F}} i$
2. $\Gamma \vdash_{\mathsf{F}} i : \mathsf{Nat}$ by $i \in \mathbb{N}$
3. QED

□

Corollary 6.4 : $\mathcal{D}_{\mathsf{F}}$ *soundness*

> If $\vdash_{\mathsf{F}} v$ then $\vdash_{\mathsf{F}} \mathcal{D}_{\mathsf{F}}(\tau, v) : \tau$

Proof:

QED by $\mathcal{X}$ *soundness*

□

Corollary 6.5 : $\mathcal{S}_{\mathsf{F}}$ *soundness*

> If $\vdash_{\mathsf{F}} v : \tau$ then $\vdash_{\mathsf{F}} \mathcal{S}_{\mathsf{F}}(\tau, v)$

Proof:

QED by $\mathcal{X}$ *soundness*

□

Corollary 6.6 : HF *static subset*

> If $\Gamma \vdash e : \tau$ then $\Gamma \vdash_{\mathsf{F}} e : \tau$.

Proof:

Consequence of the proof for the higher-order *static subset* lemma; both $\vdash_{\mathsf{F}}$ and $\vdash_{\mathsf{H}}$ have the same typing rules for surface-language expressions.

$\square$

Corollary 6.7 : HF *dynamic subset*

If $\Gamma \vdash e$ then $\Gamma \vdash_{\mathsf{F}} e$.

Proof:

Consequence of the proof for the higher-order *dynamic subset* lemma.

$\square$

Lemma 6.8 : HF *static progress*

If $\vdash_{\mathsf{F}} e : \tau$ then one of the following holds:

- e is a value
- $e \in \mathsf{Err}$
- $e \rightarrow_{\mathsf{F\text{-}S}} e'$
- $e \rightarrow_{\mathsf{F\text{-}S}} \mathsf{BndryErr}$
- $e = E[\mathsf{dyn}\ \tau'\ e']$ and $e' \rightarrow_{\mathsf{F\text{-}D}} \mathsf{TagErr}$

Proof:

By the *boundary factoring* lemma, there are eight possible cases.

CASE e is a value :

1. QED

CASE $e = E^{\bullet}[v_0\ v_1]$:

1. $\vdash_{\mathsf{F}} v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\mathsf{F}} v_0 : \tau_d \Rightarrow \tau_c$
   $\wedge \vdash_{\mathsf{F}} v_1 : \tau_d$
   by *inversion*
3. $v_0 = \lambda(x\!:\!\tau_d').\, e'$
   $\vee\ v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,\lambda x.\, e'$
   $\vee\ v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,\lambda(x\!:\!\tau_x).\, e'$
   by *canonical forms*
4. **IF** $v_0 = \lambda(x\!:\!\tau_d').\, e'$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[e'[x \leftarrow v_1]]$
      by $v_0\ v_1 \rhd_{\mathsf{S\text{-}1}} e'[x \leftarrow v_1]$
   b. QED

   IF $v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,\lambda x.\, e'$
   $\wedge\ \mathcal{X}(\tau_d', v_1) = v_1'$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[\mathsf{dyn}\ \tau_c'\ (e'[x \leftarrow v_1'])]$
      by $v_0\ v_1 \rhd_{\mathsf{S\text{-}1}} \mathsf{dyn}\ \tau_c'\ (e'[x \leftarrow v_1'])$
   b. QED

   IF $v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,\lambda x.\, e'$
   $\wedge\ \mathcal{X}(\tau_d', v_1) = \mathsf{BndryErr}$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} \mathsf{BndryErr}$
      by $v_0\ v_1 \rhd_{\mathsf{S\text{-}1}} \mathsf{BndryErr}$
   b. QED

   IF $v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,\lambda(x\!:\!\tau_x).\, e'$
   $\wedge\ \mathcal{X}(\tau_x, v_1) = v_1'$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[\mathsf{stat}\ \tau_c'\ (\mathsf{chk}\ \tau_c'\ e'[x \leftarrow v_1'])]$
      by $v_0\ v_1 \rhd_{\mathsf{S\text{-}1}} \mathsf{stat}\ \tau_c'\ (\mathsf{chk}\ \tau_c'\ e'[x \leftarrow v_1'])$
   b. QED

   ELSE $v_0 = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\,\lambda(x\!:\!\tau_x).\, e'$
   $\wedge\ \mathcal{X}(\tau_x, v_1) = \mathsf{BndryErr}$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} \mathsf{BndryErr}$
      by $v_0\ v_1 \rhd_{\mathsf{S\text{-}1}} \mathsf{BndryErr}$
   b. QED

CASE $e = E^{\bullet}[op^1\ v]$:

1. $\vdash_{\mathsf{F}} op^1\ v : \tau'$
   by *static hole typing*
2. $\vdash_{\mathsf{F}} v : \tau_0 \times \tau_1$
   by *inversion*
3. $v = \langle v_0, v_1 \rangle$
   $\vee\ v = \mathsf{mon}\,(\tau_0' \times \tau_1')\,\langle v_0, v_1 \rangle$
   by *canonical forms*
4. **IF** $v = \langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \mathsf{fst}$:
   a. $\delta(\mathsf{fst}, \langle v_0, v_1 \rangle) = v_0$
   b. $e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[v_0]$
      by $op^1\ v \rhd_{\mathsf{S\text{-}1}} v_0$
   c. QED

   IF $v = \langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \mathsf{snd}$:
   a. $\delta(\mathsf{snd}, \langle v_0, v_1 \rangle) = v_1$
   b. $e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[v_1]$
      by $op^1\ v \rhd_{\mathsf{S\text{-}1}} v_1$
   c. QED

   IF $v = \mathsf{mon}\,(\tau_0' \times \tau_1')\,\langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \mathsf{fst}$
   $\wedge\ \mathcal{X}(\tau_0', v_0) = v_0'$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[v_0']$
      by $\mathsf{fst}\ v \rhd_{\mathsf{S\text{-}1}} v_0'$
   b. QED

   IF $v = \mathsf{mon}\,(\tau_0' \times \tau_1')\,\langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \mathsf{fst}$
   $\wedge\ \mathcal{X}(\tau_0', v_0) = \mathsf{BndryErr}$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} \mathsf{BndryErr}$
      by $\mathsf{fst}\ v \rhd_{\mathsf{S\text{-}1}} \mathsf{BndryErr}$
   b. QED

   IF $v = \mathsf{mon}\,(\tau_0' \times \tau_1')\,\langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \mathsf{snd}$
   $\wedge\ \mathcal{X}(\tau_1', v_1) = v_1'$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} E^{\bullet}[v_1']$
      by $\mathsf{snd}\ v \rhd_{\mathsf{S\text{-}1}} v_1'$
   b. QED

   ELSE $v = \mathsf{mon}\,(\tau_0' \times \tau_1')\,\langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \mathsf{snd}$
   $\wedge\ \mathcal{X}(\tau_1', v_1) = \mathsf{BndryErr}$:
   a. $e \rightarrow_{\mathsf{F\text{-}S}} \mathsf{BndryErr}$
      by $\mathsf{snd}\ v \rhd_{\mathsf{S\text{-}1}} \mathsf{BndryErr}$
   b. QED

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:

1. $\vdash_{\mathsf{F}} op^2\ v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\mathsf{F}} v_0 : \tau_0$
   $\wedge \vdash_{\mathsf{F}} v_1 : \tau_1$
   $\wedge\ \Delta(op^2, \tau_0, \tau_1) = \tau''$
   by *inversion*
3. $\delta(op^2, v_0, v_1) = e'$
   by Δ *type soundness* (2)

4. $op^2\ v_0\ v_1 \rhd_{\text{S-1}} e'$
   by (3)
5. QED by $e \rightarrow_{\text{F-S}} E^{\bullet}[e']$

CASE $e = E[\text{dyn}\ \tau'\ e']$ and e' is boundary-free :

1. e' is a value
   $\vee\ e' \in \text{Err}$
   $\vee\ e' \rightarrow_{\text{F-D}} e''$
   $\vee\ e' \rightarrow_{\text{F-D}} \text{BndryErr}$
   $\vee\ e' = E'[e'']$ and $e'' \rhd_{\text{D-1}} \text{TagErr}$
   by *dynamic progress*
2. **IF** e' is a value :
   a. QED $e \rightarrow_{\text{F-S}} E[\mathcal{D}_{\text{F}}(\tau', e')]$
   IF $e' \in \text{Err}$:
   a. QED $e \rightarrow_{\text{F-S}} e'$
   IF $e' \rightarrow_{\text{F-D}} e''$:
   a. QED $e \rightarrow_{\text{F-S}} E[\text{dyn}\ \tau'\ e'']$
   IF $e' \rightarrow_{\text{F-D}} \text{BndryErr}$:
   a. QED $e \rightarrow_{\text{F-S}} E[\text{dyn}\ \tau'\ \text{BndryErr}]$
   ELSE $e' = E'[e'']$ and $e'' \rhd_{\text{D-1}} \text{TagErr}$:
   a. $E' \in E^{\bullet}$
      by e' is boundary-free
   b. QED

CASE $e = E[\text{stat}\ \tau'\ e']$ and e' is boundary-free :

1. e' is a value
   $\vee\ e' \in \text{Err}$
   $\vee\ e' \rightarrow_{\text{F-S}} e''$
   $\vee\ e' \rightarrow_{\text{F-S}} \text{BndryErr}$
   $\vee\ e' = E''[\text{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{D-1}} \text{TagErr}$
   by *static progress*
2. **IF** e' is a value :
   a. QED $e \rightarrow_{\text{F-S}} E[\mathcal{S}_{\text{F}}(\tau', e')]$
   IF $e' \in \text{Err}$:
   a. QED $e \rightarrow_{\text{F-S}} e'$
   IF $e' \rightarrow_{\text{F-S}} e''$:
   a. QED $e \rightarrow_{\text{F-S}} E[\text{stat}\ \tau'\ e'']$
   IF $e' \rightarrow_{\text{F-S}} \text{BndryErr}$:
   a. QED $e \rightarrow_{\text{F-S}} E[\text{stat}\ \tau'\ \text{BndryErr}]$
   ELSE $e' = E''[\text{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{D-1}} \text{TagErr}$
   :
   a. Contradiction by e' is boundary-free

CASE $e = E[\text{Err}]$:

1. QED $e \rightarrow_{\text{F-S}} \text{Err}$

CASE $e = E^{\bullet}[\text{chk}\ \tau'\ v]$:

IF $\mathcal{X}(\tau, v) = v'$:
1. $e \rightarrow_{\text{F-S}} E^{\bullet}[v']$
   by $(\text{chk}\ \tau\ v) \rhd_{\text{S-1}} v'$
2. QED

ELSE $\mathcal{X}(\tau, v) = \text{BndryErr}$:
1. $e \rightarrow_{\text{F-S}} \text{BndryErr}$
   by $(\text{chk}\ \tau\ v) \rhd_{\text{S-1}} \text{BndryErr}$
2. QED

☐

Lemma 6.9 : HF *dynamic progress*

If $\vdash_{\text{F}} e$ then one of the following holds:
- e is a value
- $e \in \text{Err}$
- $e \rightarrow_{\text{F-D}} e'$
- $e \rightarrow_{\text{F-D}} \text{BndryErr}$
- $e \rightarrow_{\text{F-D}} \text{TagErr}$

Proof:

By the *boundary factoring* lemma, there are seven cases.

CASE e is a value :
1. QED

CASE $e = E^{\bullet}[v_0\ v_1]$:

IF $v_0 = \lambda x.\, e'$:
1. $e \rightarrow_{\text{F-D}} E^{\bullet}[e'[x \leftarrow v_1]]$
   by $v_0\ v_1 \rhd_{\text{D-1}} e'[x \leftarrow v_1]$
2. QED

IF $v_0 = \text{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda x.\, e')$:
1. $e \rightarrow_{\text{F-D}} E^{\bullet}[e'[x \leftarrow v_1]]$
   by $v_0\ v_1 \rhd_{\text{D-1}} e'[x \leftarrow v_1]$
2. QED

IF $v_0 = \text{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda(x\!:\!\tau_x).\, e')$
   $\wedge\ \mathcal{X}(\tau_x, v_1) = v_1'$:
1. $e \rightarrow_{\text{F-D}} E^{\bullet}[\text{stat}\ \tau_c\ (\text{chk}\ \tau_c\ e'[x \leftarrow v_1'])]$
   by $v_0\ v_1 \rhd_{\text{D-1}} \text{stat}\ \tau_c\ (\text{chk}\ \tau_c\ e'[x \leftarrow v_1'])$
2. QED

IF $v_0 = \text{mon}\,(\tau_d \Rightarrow \tau_c)\,(\lambda(x\!:\!\tau_x).\, e')$
   $\wedge\ \mathcal{X}(\tau_x, v_1) = \text{BndryErr}$:
1. $e \rightarrow_{\text{F-D}} \text{BndryErr}$
   by $v_0\ v_1 \rhd_{\text{D-1}} \text{BndryErr}$
2. QED

IF $v_0 = \lambda(x\!:\!\tau_x).\, e'$:
1. Contradiction by $\vdash_{\text{F}} e$

ELSE $v_0 = i$
   $\vee\ v_0 = \langle v, v' \rangle$
   $\vee\ v_0 = \text{mon}\ \tau_0 \times \tau_1\ v'$:
1. $e \rightarrow_{\text{F-D}} \text{TagErr}$
   by $(v_0\ v_1) \rhd_{\text{D-1}} \text{TagErr}$
2. QED

CASE $e = E^{\bullet}[op^1\ v]$:

IF $v = \text{mon}\ \tau_0 \times \tau_1\ \langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \text{fst}$
   $\wedge\ \mathcal{X}(\tau_0, v_0) = v_0'$:
1. $e \rightarrow_{\text{F-D}} E^{\bullet}[v_0']$
   by $op^1\ v \rhd_{\text{D-1}} v_0'$
2. QED

IF $v = \text{mon}\ \tau_0 \times \tau_1\ \langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \text{fst}$
   $\wedge\ \mathcal{X}(\tau_0, v_0) = \text{BndryErr}$:
1. $e \rightarrow_{\text{F-D}} \text{BndryErr}$
   by $op^1\ v \rhd_{\text{D-1}} \text{BndryErr}$
2. QED

IF $v = \text{mon}\ \tau_0 \times \tau_1\ \langle v_0, v_1 \rangle$
   $\wedge\ op^1 = \text{snd}$
   $\wedge\ \mathcal{X}(\tau_1, v_1) = v_1'$:

1. $e \rightarrow_{\text{F-D}} E^{\bullet}[v_1']$
   by $op^1\ v \rhd_{\text{D-1}} v_1'$
2. QED

IF $v = \mathsf{mon}\ \tau_0{\times}\tau_1\ \langle v_0, v_1\rangle$
$\wedge\ op^1 = \mathsf{snd}$
$\wedge\ \mathcal{X}(\tau_1, v_1) = \mathsf{BndryErr}$:
1. $e \rightarrow_{\text{F-D}} \mathsf{BndryErr}$
   by $op^1\ v \rhd_{\text{D-1}} \mathsf{BndryErr}$
2. QED

IF $\delta(op^1, v) = e'$:
1. $(op^1\ v) \rhd_{\text{D-1}} e'$
2. QED by $e \rightarrow_{\text{F-D}} E^{\bullet}[e']$

ELSE $\delta(op^1, v)$ is undefined :
1. $e \rightarrow_{\text{F-D}} \mathsf{TagErr}$
   by $(op^1\ v) \rhd_{\text{D-1}} \mathsf{TagErr}$
2. QED

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$:

IF $\delta(op^2, v_0, v_1) = e''$:
1. $op^2\ v_0\ v_1 \rhd_{\text{D-1}} e''$
2. QED

ELSE $\delta(op^2, v_0, v_1)$ is undefined :
1. $e \rightarrow_{\text{F-D}} \mathsf{TagErr}$
   by $op^2\ v_0\ v_1 \rhd_{\text{D-1}} \mathsf{TagErr}$
2. QED

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :
1. e' is a value
   $\vee\ e' \in \mathsf{Err}$
   $\vee\ e' \rightarrow_{\text{F-D}} e''$
   $\vee\ e' \rightarrow_{\text{F-D}} \mathsf{BndryErr}$
   $\vee\ e' = E^{\bullet}[e'']$ and $e'' \rhd_{\text{D-1}} \mathsf{TagErr}$
   by *dynamic progress*
2. **IF** e' is a value :
   a. QED $e \rightarrow_{\text{F-D}} E[\mathcal{D}_{\text{F}}(\tau', e')]$

   IF $e' \in \mathsf{Err}$:
   a. QED $e \rightarrow_{\text{F-D}} e'$

   IF $e' \rightarrow_{\text{F-D}} e''$:
   a. QED $e \rightarrow_{\text{F-S}} E[\mathsf{dyn}\ \tau'\ e'']$

   IF $e' \rightarrow_{\text{F-D}} \mathsf{BndryErr}$:
   a. QED $e \rightarrow_{\text{F-D}} E[\mathsf{dyn}\ \tau'\ \mathsf{BndryErr}]$

   ELSE $e' = E[e'']$ and $e'' \rhd_{\text{D-1}} \mathsf{TagErr}$:
   a. $E \in E^{\bullet}$
      by e' is boundary-free
   b. QED

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :
1. e' is a value
   $\vee\ e' \in \mathsf{Err}$
   $\vee\ e' \rightarrow_{\text{F-S}} e''$
   $\vee\ e' \rightarrow_{\text{F-S}} \mathsf{BndryErr}$
   $\vee\ e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{D-1}} \mathsf{TagErr}$
   by *static progress*
2. **IF** e' is a value :
   a. QED $e \rightarrow_{\text{F-S}} E[\mathcal{S}_{\text{F}}(\tau', e')]$

   IF $e' \in \mathsf{Err}$:
   a. QED $e \rightarrow_{\text{F-S}} e'$

   IF $e' \rightarrow_{\text{F-S}} e''$:
   a. QED $e \rightarrow_{\text{F-S}} E[\mathsf{stat}\ \tau'\ e'']$

   IF $e' \rightarrow_{\text{F-S}} \mathsf{BndryErr}$:
   a. QED $e \rightarrow_{\text{F-S}} E[\mathsf{stat}\ \tau'\ \mathsf{BndryErr}]$

   ELSE $e' = E''[\mathsf{dyn}\ \tau''\ E^{\bullet\prime\prime}[e'']]$ and $e'' \rhd_{\text{D-1}} \mathsf{TagErr}$
   :
   a. Contradiction by e' is boundary-free

CASE $e = E[\mathsf{Err}]$:
1. QED $e \rightarrow_{\text{F-D}} \mathsf{Err}$

□

Lemma 6.10 : HF *static preservation*

If $\vdash_{\text{F}} e : \tau$ and $e \rightarrow_{\text{F-S}} e'$ then $\vdash_{\text{F}} e' : \tau$

Proof:

By the *boundary factoring* lemma there are eight cases.

CASE e is a value :
1. Contradiction by $e \rightarrow_{\text{F-S}} e'$

CASE $e = E^{\bullet}[v_0\ v_1]$:

IF $v_0 = \lambda(x{:}\tau_x).\ e'$
$\wedge\ e \rightarrow_{\text{F-S}} e'[x \leftarrow v_1]$:
1. $\vdash_{\text{F}} v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\text{F}} v_0 : \tau_d \Rightarrow \tau_c$
   $\wedge \vdash_{\text{F}} v_1 : \tau_d$
   $\wedge\ \tau_c \leqslant: \tau'$
   by *inversion*
3. $(x{:}\tau_x) \vdash_{\text{F}} e' : \tau_c$
   by *inversion* (2)
4. $\tau_d \leqslant: \tau_x$
   by *canonical forms* (2)
5. $\vdash_{\text{F}} v_1 : \tau_x$
   by (2, 4)
6. $\vdash_{\text{F}} e'[x \leftarrow v_1] : \tau_c$
   by *substitution* (3, 5)
7. $\vdash_{\text{F}} e'[x \leftarrow v_1] : \tau'$
   by (2, 6)
8. QED by *hole substitution*

IF $v_0 = \mathsf{mon}\ \tau_d \Rightarrow \tau_c\ \lambda x.\ e'$
$\wedge\ e \rightarrow_{\text{F-S}} E^{\bullet}[\mathsf{dyn}\ \tau_c\ ((\lambda x.\ e')\ (\mathcal{X}(\tau_d, v_1))]$:
1. $\vdash_{\text{F}} v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\text{F}} v_0 : \tau_d' \Rightarrow \tau_c'$
   $\wedge\ \tau_c' \leqslant: \tau'$
   by *inversion*
3. $\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'$
   by *canonical forms* (2)
4. $\vdash_{\text{F}} \lambda x.\ e'$
   by *inversion* (2)
5. $\tau_d' \leqslant: \tau_d$
   $\wedge\ \tau_c \leqslant: \tau_c'$
   by (3)
6. $\vdash_{\text{F}} \mathcal{X}(\tau_d, v_1)$
   by *$\mathcal{X}$ soundness*

7. $\vdash_F (\lambda x.\, e')\, \mathcal{X}(\tau_d, v_1)$
by (4, 6)
8. $\vdash_F \mathsf{dyn}\ \tau_c\ ((\lambda x.\, e')\, \mathcal{X}(\tau_d, v_1)) : \tau_c$
by (7)
9. $\vdash_F \mathsf{dyn}\ \tau_c\ ((\lambda x.\, e')\, \mathcal{X}(\tau_d, v_1)) : \tau'$
by (2, 5, 8)
10. QED by *hole substitution*

ELSE $v_0 = \mathsf{mon}\ \tau_d \Rightarrow \tau_c\ (\lambda(x\!:\!\tau_x).\, e')$
$\wedge e \rightarrow_{\text{F-S}} E^\bullet[\mathsf{stat}\ \tau_c\ (\mathsf{chk}\ \tau_c\ ((\lambda(x\!:\!\tau_x).\, e')\, \mathcal{X}(\tau_x, v_1)))]$
:
1. $\vdash_F v_0\ v_1 : \tau'$
by *static hole typing*
2. $\vdash_F v_0 : \tau_d' \Rightarrow \tau_c'$
$\wedge \vdash_F v_1 : \tau_d'$
$\wedge\ \tau_c' \leqslant: \tau'$
by *inversion*
3. $\vdash_F \lambda(x\!:\!\tau_x).\, e' : \tau_x \Rightarrow \tau_x'$
by *inversion*
4. $\tau_d \Rightarrow \tau_c \leqslant: \tau_d' \Rightarrow \tau_c'$
by *canonical forms* (2)
5. $\tau_c \leqslant: \tau_c'$
by (4)
6. $\vdash_F \mathcal{X}(\tau_x, v_1) : \tau_x$
by $\mathcal{X}$ *soundness*
7. $\vdash_F (\lambda(x\!:\!\tau_x).\, e')\, \mathcal{X}(\tau_x, v_1) : \tau_x'$
by (3, 6)
8. $\vdash_F \mathsf{chk}\ \tau_c\ ((\lambda(x\!:\!\tau_x).\, e')\, \mathcal{X}(\tau_x, v_1)) : \tau_c$
by (7)
9. $\vdash_F \mathsf{stat}\ \tau_c\ (\mathsf{chk}\ \tau_c\ ((\lambda(x\!:\!\tau_x).\, e')\, \mathcal{X}(\tau_x, v_1))) : \tau_c$
by (8)
10. $\vdash_F \mathsf{stat}\ \tau_c\ (\mathsf{chk}\ \tau_c\ ((\lambda(x\!:\!\tau_x).\, e')\, \mathcal{X}(\tau_x, v_1))) : \tau'$
by (2, 5, 9)
11. QED by *hole substitution*

CASE $e = E^\bullet[op^1\ v]$:

IF $v = \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{fst}$
$\wedge\ e \rightarrow_{\text{F-S}} E^\bullet[v_0]$:
1. $\vdash_F \mathsf{fst}\ \langle v_0, v_1 \rangle : \tau'$
by *static hole typing*
2. $\vdash_F \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$
$\wedge\ \tau_0 \leqslant: \tau'$
by *inversion*
3. $\vdash_F v_0 : \tau_0$
by *inversion*
4. $\vdash_F v_0 : \tau'$
by (2)
5. QED by *hole substitution*

IF $v = \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{snd}$
$\wedge\ e \rightarrow_{\text{F-S}} E^\bullet[v_1]$:
1. $\vdash_F \mathsf{snd}\ \langle v_0, v_1 \rangle : \tau'$
by *static hole typing*
2. $\vdash_F \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$
$\wedge\ \tau_1 \leqslant: \tau'$
by *inversion*
3. $\vdash_F v_1 : \tau_1$
by *inversion*
4. $\vdash_F v_1 : \tau'$
by (2)
5. QED by *hole substitution*

IF $v = \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{fst}$
$\wedge\ e \rightarrow_{\text{F-S}} E^\bullet[\mathcal{X}(\tau_0, v_0)]$:
1. $\vdash_F \mathsf{fst}\ v : \tau'$
by *static hole typing*
2. $\vdash_F v : \tau_0' \times \tau_1'$
$\wedge\ \tau_0' <: \tau'$
by *inversion* (1)
3. $\tau_0 \leqslant: \tau_0'$
by *canonical forms* (2)
4. $\vdash_F \langle v_0, v_1 \rangle$
$\vee \vdash_F \langle v_0, v_1 \rangle : \tau''$
by *inversion* (2)
5. $\vdash_F v_0$
$\vee \vdash_F v_0 : \tau'''$
by *inversion* (4)
6. $\vdash_F \mathcal{X}(\tau_0, v_0) : \tau_0$
by $\mathcal{X}$ *soundness* (5)
7. $\vdash_F \mathcal{X}(\tau_0, v_0) : \tau'$
by (2, 3, 6)
8. QED by *hole substitution*

ELSE $v = \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{snd}$
$\wedge\ e \rightarrow_{\text{F-S}} E^\bullet[\mathcal{X}(\tau_1, v_1)]$:
1. $\vdash_F \mathsf{snd}\ v : \tau'$
by *static hole typing*
2. $\vdash_F v : \tau_0' \times \tau_1'$
$\wedge\ \tau_1' <: \tau'$
by *inversion* (1)
3. $\tau_1 \leqslant: \tau_1'$
by *canonical forms* (2)
4. $\vdash_F \langle v_0, v_1 \rangle$
$\vee \vdash_F \langle v_0, v_1 \rangle : \tau''$
by *inversion* (2)
5. $\vdash_F v_1$
$\vee \vdash_F v_1 : \tau'''$
by *inversion* (4)
6. $\vdash_F \mathcal{X}(\tau_1, v_1) : \tau_1$
by $\mathcal{X}$ *soundness* (5)
7. $\vdash_F \mathcal{X}(\tau_1, v_1) : \tau'$
by (2, 3, 6)
8. QED by *hole substitution*

CASE $e = E^\bullet[op^2\ v_0\ v_1]$
$\wedge\ \delta(op^2, v_0, v_1) = v$
$\wedge\ e \rightarrow_{\text{F-S}} E^\bullet[v]$:

1. $\vdash_{\mathsf{F}} op^2\ v_0\ v_1 : \tau'$
   by *static hole typing*
2. $\vdash_{\mathsf{F}} v_0 : \tau_0$
   $\wedge \vdash_{\mathsf{F}} v_1 : \tau_1$
   $\wedge\ \Delta(op^2, \tau_0, \tau_1) = \tau''$
   $\wedge\ \tau'' \leqslant: \tau'$
   by *inversion*
3. $\vdash_{\mathsf{F}} v : \tau''$
   by *Δ type soundness* (2)
4. $\vdash_{\mathsf{F}} v : \tau'$
   by (2, 3)
5. QED by *hole substitution* (4)

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :

IF e' is a value :
1. $e \rightarrow_{\mathsf{F\text{-}S}} E[\mathcal{D}_{\mathsf{F}}(\tau', e')]$
2. $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau'\ e' : \tau'$
   by *boundary hole typing*
3. $\vdash_{\mathsf{F}} e'$
   by *inversion* (2)
4. $\vdash_{\mathsf{F}} \mathcal{D}_{\mathsf{F}}(\tau', e') : \tau'$
   by *$\mathcal{D}_{\mathsf{F}}$ soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\mathsf{F\text{-}D}} e''$:
1. $e \rightarrow_{\mathsf{F\text{-}S}} E[\mathsf{dyn}\ \tau'\ e'']$
2. $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau'\ e' : \tau'$
   by *boundary hole typing*
3. $\vdash_{\mathsf{F}} e'$
   by *inversion* (2)
4. $\vdash_{\mathsf{F}} e''$
   by *dynamic preservation* (3)
5. $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau'\ e'' : \tau'$
   by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :

IF e' is a value :
1. $e \rightarrow_{\mathsf{F\text{-}S}} E[\mathcal{S}_{\mathsf{F}}(\tau', e')]$
2. $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau'\ e'$
   by *boundary hole typing*
3. $\vdash_{\mathsf{F}} e' : \tau'$
   by *inversion* (2)
4. $\vdash_{\mathsf{F}} \mathcal{S}_{\mathsf{F}}(\tau', e')$
   by *$\mathcal{S}_{\mathsf{F}}$ soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\mathsf{F\text{-}S}} e''$:
1. $e \rightarrow_{\mathsf{F\text{-}S}} E[\mathsf{stat}\ \tau'\ e'']$
2. $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau'\ e'$
   by *boundary hole typing*
3. $\vdash_{\mathsf{F}} e' : \tau'$
   by *inversion* (2)
4. $\vdash_{\mathsf{F}} e'' : \tau'$
   by *static preservation* (3)
5. $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau'\ e''$
   by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{Err}]$:
1. $e \rightarrow_{\mathsf{F\text{-}S}} \mathsf{Err}$
2. QED by $\vdash_{\mathsf{F}} \mathsf{Err} : \tau$

CASE $e = E^{\bullet}[\mathsf{chk}\ \tau\ v]$:
1. $\vdash_{\mathsf{F}} \mathsf{chk}\ \tau\ v : \tau$
   by *static hole typing*
2. $\vdash_{\mathsf{F}} v : \tau'$
   by *inversion* (1)
3. $\vdash_{\mathsf{F}} \mathcal{X}(\tau, v') : \tau$
   by *$\mathcal{X}$ soundness* (2)
4. QED by *hole substitution* (3)

$\square$

Lemma 6.11 : HF *dynamic preservation*

If $\vdash_{\mathsf{F}} e$ and $e \rightarrow_{\mathsf{F\text{-}D}} e'$ then $\vdash_{\mathsf{F}} e'$

Proof:

By the *boundary factoring* lemma, there are seven cases.

CASE e is a value :
1. Contradiction by $e \rightarrow_{\mathsf{F\text{-}D}} e'$

CASE $e = E^{\bullet}[v_0\ v_1]$:

IF $v_0 = \lambda x.\, e'$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}D}} E^{\bullet}[e'[x \leftarrow v_1]]$:
1. $\vdash_{\mathsf{F}} v_0\ v_1$
   by *dynamic hole typing*
2. $\vdash_{\mathsf{F}} v_0$
   $\wedge \vdash_{\mathsf{F}} v_1$
   by *inversion* (1)
3. $x \vdash_{\mathsf{F}} e'$
   by *inversion* (2)
4. $\vdash_{\mathsf{F}} e'[x \leftarrow v_1]$
   by *substitution* (2, 3)
5. QED *hole substitution* (4)

IF $v_0 = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,\lambda x.\, e'$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}D}} E^{\bullet}[(\lambda x.\, e')\ v_1]$:
1. $\vdash_{\mathsf{F}} v_0\ v_1$
   by *dynamic hole typing*
2. $\vdash_{\mathsf{F}} v_0$
   $\wedge \vdash_{\mathsf{F}} v_1$
   by *inversion* (1)
3. $\vdash_{\mathsf{F}} (\lambda x.\, e')\ v_1$
   by (2)
4. QED *hole substitution* (5)

ELSE $v_0 = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\,\lambda(x\!:\!\tau_x).\, e'$
$\wedge e \rightarrow_{\mathsf{F\text{-}D}} E^{\bullet}[\mathsf{stat}\ \tau_c\ (\mathsf{chk}\ \tau_c\ ((\lambda(x\!:\!\tau_x).\, e')\ \mathcal{X}(\tau_x, v_1)))]$
:
1. $\vdash_{\mathsf{F}} v_0\ v_1$
   by *dynamic hole typing*
2. $\vdash_{\mathsf{F}} v_0$
   $\wedge \vdash_{\mathsf{F}} v_1$
   by *inversion*
3. $\vdash_{\mathsf{F}} \lambda(x\!:\!\tau_x).\, e' : \tau_x \Rightarrow \tau_x'$
   by *inversion* (2)

4. $\vdash_{\mathsf{F}} \mathcal{X}(\tau_x, v_1) : \tau_x$
by *$\mathcal{X}$ soundness* (2)
5. $\vdash_{\mathsf{F}} ((\lambda(x\!:\!\tau_x).\, e')\ \mathcal{X}(\tau_x, v_1)) : \tau_x'$
by (3, 4)
6. $\vdash_{\mathsf{F}} \mathsf{chk}\ \tau_c\ ((\lambda(x\!:\!\tau_x).\, e')\ \mathcal{X}(\tau_x, v_1)) : \tau_c$
by (5)
7. $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau_c\ (\mathsf{chk}\ \tau_c\ ((\lambda(x\!:\!\tau_x).\, e')\ \mathcal{X}(\tau_x, v_1)))$
by (6)
8. QED *hole substitution*

CASE $e = E^{\bullet}[op^1\ v]$:
IF $v = \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{fst}$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}D}} E^{\bullet}[v_0]$:
1. $\vdash_{\mathsf{F}} op^1\ v$
by *dynamic hole typing*
2. $\vdash_{\mathsf{F}} v$
by *inversion* (1)
3. $\vdash_{\mathsf{F}} v_0$
by *inversion* (2)
4. QED by *hole substitution*

IF $v = \langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{snd}$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}D}} E^{\bullet}[v_1]$:
1. $\vdash_{\mathsf{F}} op^1\ v$
by *dynamic hole typing*
2. $\vdash_{\mathsf{F}} v$
by *inversion* (1)
3. $\vdash_{\mathsf{F}} v_1$
by *inversion* (2)
4. QED by *hole substitution*

IF $v = \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{fst}$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}D}} E^{\bullet}[\mathcal{X}(\tau_0, v_0)]$:
1. $\vdash_{\mathsf{F}} op^1\ v$
by *dynamic hole typing*
2. $\vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1 \rangle$
by *inversion* (1)
3. $\vdash_{\mathsf{F}} v_0$
$\vee \vdash_{\mathsf{F}} v_0 : \tau_0'$
by *inversion* (2)
4. $\vdash_{\mathsf{F}} \mathcal{X}(\tau_0, v_0)$
by *$\mathcal{X}$ soundness* (3)
5. QED by *hole substitution*

ELSE $v = \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1 \rangle$
$\wedge\ op^1 = \mathsf{snd}$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}D}} E^{\bullet}[\mathcal{X}(\tau_1, v_1)]$:
1. $\vdash_{\mathsf{F}} op^1\ v$
by *dynamic hole typing*
2. $\vdash_{\mathsf{F}} \mathsf{mon}\,(\tau_0 \times \tau_1)\,\langle v_0, v_1 \rangle$
by *inversion* (1)
3. $\vdash_{\mathsf{F}} v_1$
$\vee \vdash_{\mathsf{F}} v_1 : \tau_1'$
by *inversion* (2)
4. $\vdash_{\mathsf{F}} \mathcal{X}(\tau_1, v_1)$
by *$\mathcal{X}$ soundness* (3)
5. QED by *hole substitution*

CASE $e = E^{\bullet}[op^2\ v_0\ v_1]$
$\wedge\ \delta(op^2, v_0, v_1) = v$
$\wedge\ e \rightarrow_{\mathsf{F\text{-}D}} E^{\bullet}[v]$:
1. $\vdash_{\mathsf{F}} op^2\ v_0\ v_1$
by *dynamic hole typing*
2. $\vdash_{\mathsf{F}} v_0$
$\wedge \vdash_{\mathsf{F}} v_1$
by *inversion* (1)
3. $\vdash_{\mathsf{F}} v$
by *δ preservation* (2)
4. QED by *hole substitution* (3)

CASE $e = E[\mathsf{dyn}\ \tau'\ e']$ and e' is boundary-free :
IF e' is a value :
1. $e \rightarrow_{\mathsf{F\text{-}D}} E[\mathcal{D}_{\mathsf{F}}(\tau', e')]$
2. $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau'\ e' : \tau'$
by *boundary hole typing*
3. $\vdash_{\mathsf{F}} e'$
by *inversion* (2)
4. $\vdash_{\mathsf{F}} \mathcal{D}_{\mathsf{F}}(\tau', e') : \tau'$
by *$\mathcal{D}_{\mathsf{F}}$ soundness* (3)
5. QED by *hole substitution* (4)

ELSE $e' \rightarrow_{\mathsf{F\text{-}D}} e''$:
1. $e \rightarrow_{\mathsf{F\text{-}D}} E[\mathsf{dyn}\ \tau'\ e'']$
2. $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau'\ e' : \tau'$
by *boundary hole typing*
3. $\vdash_{\mathsf{F}} e'$
$\wedge\ \tau' \leqslant: \tau''$
by *inversion* (2)
4. $\vdash_{\mathsf{F}} e''$
by *dynamic preservation* (3)
5. $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau'\ e'' : \tau'$
by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{stat}\ \tau'\ e']$ and e' is boundary-free :
IF $e' \in v$:
1. $e \rightarrow_{\mathsf{F\text{-}D}} E[\mathcal{S}_{\mathsf{F}}(\tau', e')]$
2. $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau'\ e'$
by *boundary hole typing*
3. $\vdash_{\mathsf{F}} e' : \tau'$
by *inversion* (2)
4. $\vdash_{\mathsf{F}} \mathcal{S}_{\mathsf{F}}(\tau', e')$
by *$\mathcal{S}_{\mathsf{F}}$ soundness* (3)
5. QED by *hole substitution* (5)

ELSE $e' \rightarrow_{\mathsf{F\text{-}S}} e''$:
1. $e \rightarrow_{\mathsf{F\text{-}D}} E[\mathsf{stat}\ \tau'\ e'']$
2. $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau'\ e'$
by *boundary hole typing*
3. $\vdash_{\mathsf{F}} e' : \tau'$
by *inversion* (2)
4. $\vdash_{\mathsf{F}} e'' : \tau'$
by *static preservation* (3)

5. $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau'\ e''$
   by (4)
6. QED by *hole substitution* (5)

CASE $e = E[\mathsf{Err}]$:
1. $e \rightarrow_{\mathsf{F\text{-}D}} \mathsf{Err}$
2. QED $\vdash_{\mathsf{F}} \mathsf{Err}$

□

Lemma 6.12 : HF *static boundary factoring*

If $\vdash_{\mathsf{F}} e : \tau$ then one of the following holds:
- e is a value
- $e = E^{\bullet}[v_0\ v_1]$
- $e = E^{\bullet}[op^1\ v]$
- $e = E^{\bullet}[op^2\ v_0\ v_1]$
- $e = E^{\bullet}[\mathsf{chk}\ \tau\ v]$
- $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{Err}]$

Proof:

By the *unique static evaluation contexts* lemma, there are eight cases.

CASE e is a value :
1. QED

CASE $e = E[v_0\ v_1]$:
1. $E = E^{\bullet}$
   $\vee\ E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$
   $\vee\ E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$
   by *inner boundary*
2. **IF** $E = E^{\bullet}$:
   a. QED $e = E^{\bullet}[v_0\ v_1]$
   IF $E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$:
   a. QED $e = E'[\mathsf{dyn}\ \tau\ E^{\bullet}[v_0\ v_1]]$
   ELSE $E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$:
   a. QED $e = E'[\mathsf{stat}\ \tau\ E^{\bullet}[v_0\ v_1]]$

CASE $e = E[op^1\ v]$:
1. $E = E^{\bullet}$
   $\vee\ E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$
   $\vee\ E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$
   by *inner boundary*
2. **IF** $E = E^{\bullet}$:
   a. QED $e = E^{\bullet}[op^1\ v]$
   IF $E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$:
   a. QED $e = E'[\mathsf{dyn}\ \tau\ E^{\bullet}[op^1\ v]]$
   ELSE $E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$:
   a. QED $e = E'[\mathsf{stat}\ \tau\ E^{\bullet}[op^1\ v]]$

CASE $e = E[op^2\ v_0\ v_1]$:
1. $E = E^{\bullet}$
   $\vee\ E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$
   $\vee\ E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$
   by *inner boundary*
2. **IF** $E = E^{\bullet}$:
   a. QED $e = E^{\bullet}[op^2\ v_0\ v_1]$
   IF $E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$:
   a. QED $e = E'[\mathsf{dyn}\ \tau\ E^{\bullet}[op^2\ v_0\ v_1]]$
   ELSE $E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$:
   a. QED $e = E'[\mathsf{stat}\ \tau\ E^{\bullet}[op^2\ v_0\ v_1]]$

CASE $e = E[\mathsf{chk}\ \tau\ v]$:
1. $E = E^{\bullet}$
   $\vee\ E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$
   $\vee\ E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$
   by *inner boundary*
2. **IF** $E = E^{\bullet}$:
   a. QED $e = E^{\bullet}[\mathsf{chk}\ \tau\ v]$
   IF $E = E'[\mathsf{dyn}\ \tau\ E^{\bullet}]$:
   a. Contradiction by $\vdash_{\mathsf{F}} e : \tau$
   ELSE $E = E'[\mathsf{stat}\ \tau\ E^{\bullet}]$:
   a. QED $e = E'[\mathsf{stat}\ \tau\ E^{\bullet}[\mathsf{chk}\ \tau\ v]]$

CASE $e = E[\mathsf{dyn}\ \tau\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{stat}\ \tau\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{Err}]$:
1. QED

□

Lemma 6.13 : HF *unique static evaluation contexts*

If $\vdash_{\mathsf{F}} e : \tau$ then one of the following holds:
- e is a value
- $e = E[v_0\ v_1]$
- $e = E[op^1\ v]$
- $e = E[op^2\ v_0\ v_1]$
- $e = E[\mathsf{chk}\ \tau\ v]$
- $e = E[\mathsf{dyn}\ \tau\ v]$
- $e = E[\mathsf{stat}\ \tau\ v]$
- $e = E[\mathsf{Err}]$

Proof:

By induction on the structure of e.

CASE $e = x$
   $\vee\ e = \lambda x.\, e'$
   $\vee\ e = \mathsf{stat}\ \tau\ e'$:
1. Contradiction by $\vdash_{\mathsf{F}} e : \tau$

CASE $e = i$
   $\vee\ e = \lambda (x\!:\!\tau_d).\, e'$
   $\vee\ e = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v$:
1. QED e is a value

CASE $e = \langle e_0, e_1 \rangle$:

IF $e_0 \notin v$:
1. $\vdash_{\mathsf{F}} e_0 : \tau_0$
   by *inversion*
2. $e_0 = E_0[e_0']$
   by the induction hypothesis (1)
3. $E = \langle E_0, e_1 \rangle$
4. QED $e = E[e_0']$

IF $e_0 \in v$
   $\wedge\ e_1 \notin v$:
1. $\vdash_{\mathsf{F}} e_1 : \tau_1$
   by *inversion*
2. $e_1 = E_1[e_1']$
   by the induction hypothesis (1)
3. $E = \langle e_0, E_1 \rangle$

4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:

1. $E = [\,]$
2. QED e is a value

CASE $e = e_0\ e_1$:

IF $e_0 \notin v$:

1. $\vdash_F e_0 : \tau_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis (1)
3. $E = E_0\ e_1$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:

1. $\vdash_F e_1 : \tau_1$
by *inversion*
2. $e_1 = E_1[e_1']$
by the induction hypothesis (1)
3. $E = e_0\ E_1$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:

1. $E = [\,]$
2. QED $e = E[e_0\ e_1]$

CASE $e = op^1\ e_0$:

IF $e_0 \notin v$:

1. $\vdash_F e_0 : \tau_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis (1)
3. $E = op^1\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:

1. $E = [\,]$
2. QED $e = E[op^1\ e_0]$

CASE $e = op^2\ e_0\ e_1$:

IF $e_0 \notin v$:

1. $\vdash_F e_0 : \tau_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis (1)
3. $E = op^2\ E_0\ e_1$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:

1. $\vdash_F e_1 : \tau_1$
by *inversion*
2. $e_1 = E_1[e_1']$
by the induction hypothesis (1)
3. $E = op^2\ e_0\ E_1$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:

1. $E = [\,]$
2. QED $e = E[op^2\ e_0\ e_1]$

CASE $e = \mathsf{chk}\ \tau\ e_0$:

IF $e_0 \notin v$:

1. $\vdash_F e_0 : \tau_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis (1)
3. $E = \mathsf{chk}\ \tau\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:

1. $E = [\,]$
2. QED $e = E[\mathsf{chk}\ \tau\ e_0]$

CASE $e = \mathsf{dyn}\ \tau\ e_0$:

IF $e_0 \notin v$:

1. $\vdash_F e_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by *unique dynamic evaluation contexts* (1)
3. $E = \mathsf{dyn}\ \tau\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:

1. $E = [\,]$
2. QED $e = E[\mathsf{dyn}\ \tau\ e_0]$

CASE $e = \mathsf{Err}$:

1. $E = [\,]$
2. QED $e = E[\mathsf{Err}]$

□

Lemma 6.14 : HF *inner boundary*

For all contexts E, one of the following holds:

- $E = E^\bullet$
- $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
- $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

Proof:

By induction on the structure of E.

CASE $E = E^\bullet$:

1. QED

CASE $E = E_0\ e_1$:

1. $E_0 = E^\bullet$
$\vee\, E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\, E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
   a. QED E is boundary-free

   IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:
   a. $E' = E_0'\ e_1$
   b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$

   ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
   a. $E' = E_0'\ e_1$
   b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = v_0\ E_1$:

1. $E_1 = E^\bullet$
$\vee\, E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$

$\vee\ E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_1 = E^\bullet$:
a. QED E is boundary-free
IF $E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = v_0\ E_1'$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = v_0\ E_1'$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = \langle E_0, e_1 \rangle$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = \langle E_0', e_1 \rangle$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = \langle E_0', e_1 \rangle$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = \langle v_0, E_1 \rangle$:
1. $E_1 = E^\bullet$
$\vee\ E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_1 = E^\bullet$:
a. QED E is boundary-free
IF $E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = \langle v_0, E_1' \rangle$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = \langle v_0, E_1' \rangle$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = op^1\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = op^1\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = op^1\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = op^2\ E_0\ e_1$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = op^2\ E_0'\ e_1$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = op^2\ E_0'\ e_1$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = op^2\ v_0\ E_1$:
1. $E_1 = E^\bullet$
$\vee\ E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_1 = E^\bullet$:
a. QED E is boundary-free
IF $E_1 = E_1'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = op^2\ v_0\ E_1'$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_1 = E_1'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = op^2\ v_0\ E_1'$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = \mathsf{chk}\ \tau\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED E is boundary-free
IF $E_0 = E_0'[\mathsf{dyn}\ \tau\ E^\bullet]$:
a. $E' = \mathsf{chk}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau\ E^\bullet]$:
a. $E' = \mathsf{chk}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$

CASE $E = \mathsf{dyn}\ \tau\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$
$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
a. QED
IF $E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{dyn}\ \tau'\ E^\bullet]$
ELSE $E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$:
a. $E' = \mathsf{dyn}\ \tau\ E_0'$
b. QED $E = E'[\mathsf{stat}\ \tau'\ E^\bullet]$

CASE $E = \mathsf{stat}\ \tau\ E_0$:
1. $E_0 = E^\bullet$
$\vee\ E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$

$\vee\ E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$
by the induction hypothesis
2. **IF** $E_0 = E^\bullet$:
  a. QED
  IF $E_0 = E_0'[\mathsf{dyn}\ \tau'\ E^\bullet]$:
  a. $E' = \mathsf{stat}\ \tau\ E_0'$
  b. QED $E = E'[\mathsf{dyn}\ \tau'\ E^\bullet]$
  ELSE $E_0 = E_0'[\mathsf{stat}\ \tau'\ E^\bullet]$:
  a. $E' = \mathsf{stat}\ \tau\ E_0'$
  b. QED $E = E'[\mathsf{stat}\ \tau'\ E^\bullet]$

☐

Lemma 6.15 : HF *dynamic boundary factoring*
If $\vdash_{\mathsf{F}} e$ then one of the following holds:
- e is a value
- $e = E^\bullet[v_0\ v_1]$
- $e = E^\bullet[op^1\ v]$
- $e = E^\bullet[op^2\ v_0\ v_1]$
- $e = E[\mathsf{dyn}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{stat}\ \tau\ e']$ where e' is boundary-free
- $e = E[\mathsf{Err}]$

Proof:
By the *unique dynamic evaluation contexts* lemma, there are eight cases.
CASE e is a value :
1. QED

CASE $e = E[v_0\ v_1]$:
1. $E = E^\bullet$
 $\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
 $\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
 by *inner boundary*
2. **IF** $E = E^\bullet$:
  a. QED $e = E^\bullet[v_0\ v_1]$
  IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
  a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[v_0\ v_1]]$
  ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
  a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[v_0\ v_1]]$

CASE $e = E[op^1\ v]$:
1. $E = E^\bullet$
 $\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
 $\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
 by *inner boundary*
2. **IF** $E = E^\bullet$:
  a. QED $e = E^\bullet[op^1\ v]$
  IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
  a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^1\ v]]$
  ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
  a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^1\ v]]$

CASE $e = E[op^2\ v_0\ v_1]$:
1. $E = E^\bullet$
 $\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
 $\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
 by *inner boundary*
2. **IF** $E = E^\bullet$:
  a. QED $e = E^\bullet[op^2\ v_0\ v_1]$
  IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
  a. QED $e = E'[\mathsf{dyn}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$
  ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
  a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[op^2\ v_0\ v_1]]$

CASE $e = E[\mathsf{chk}\ \tau'\ v]$:
1. $E = E^\bullet$
 $\vee\ E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$
 $\vee\ E = E'[\mathsf{stat}\ \tau\ E^\bullet]$
 by *inner boundary*
2. **IF** $E = E^\bullet$:
  a. Contradiction by $\vdash_{\mathsf{F}} e$
  IF $E = E'[\mathsf{dyn}\ \tau\ E^\bullet]$:
  a. Contradiction by $\vdash_{\mathsf{F}} e$
  ELSE $E = E'[\mathsf{stat}\ \tau\ E^\bullet]$:
  a. QED $e = E'[\mathsf{stat}\ \tau\ E^\bullet[\mathsf{chk}\ \tau'\ v]]$

CASE $e = E[\mathsf{dyn}\ \tau\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{stat}\ \tau\ v]$:
1. QED v is boundary-free

CASE $e = E[\mathsf{Err}]$:
1. QED

☐

Lemma 6.16 : HF *unique dynamic evaluation contexts*
If $\vdash_{\mathsf{F}} e$ then one of the following holds:
- e is a value
- $e = E[v_0\ v_1]$
- $e = E[op^1\ v]$
- $e = E[op^2\ v_0\ v_1]$
- $e = E[\mathsf{chk}\ \tau\ v]$
- $e = E[\mathsf{dyn}\ \tau\ v]$
- $e = E[\mathsf{stat}\ \tau\ v]$
- $e = E[\mathsf{Err}]$

Proof:
By induction on the structure of e.
CASE $e = x$
 $\vee\ e = \lambda(x{:}\tau).\ e'$
 $\vee\ e = \mathsf{dyn}\ \tau\ e'$:
1. Contradiction by $\vdash_{\mathsf{F}} e$

CASE $e = i$
 $\vee\ e = \lambda x.\ e'$
 $\vee\ e = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v$:
1. QED e is a value

CASE $e = \mathsf{Err}$:
1. $E = [\,]$
2. QED $e = E[\mathsf{Err}]$

CASE $e = \langle e_0, e_1\rangle$:
 IF $e_0 \notin v$:
 1. $\vdash_{\mathsf{F}} e_0$
  by *inversion*
 2. $e_0 = E_0[e_0']$
  by the induction hypothesis (1)
 3. $E = \langle E_0, e_1\rangle$
 4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:
1. $\vdash_{\mathsf{F}} e_1$
by *inversion*
2. $e_1 = E_1[e_1']$
by the induction hypothesis (1)
3. $E = \langle e_0, E_1 \rangle$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E = [\,]$
2. QED e is a value

CASE $e = e_0\ e_1$:

IF $e_0 \notin v$:
1. $\vdash_{\mathsf{F}} e_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis (1)
3. $E = E_0\ e_1$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:
1. $\vdash_{\mathsf{F}} e_1$
by *inversion*
2. $e_1 = E_1[e_1']$
by the induction hypothesis (1)
3. $E = e_0\ E_1$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[e_0\ e_1]$

CASE $e = op^1\ e_0$:

IF $e_0 \notin v$:
1. $\vdash_{\mathsf{F}} e_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis (1)
3. $E = op^1\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[op^1\ e_0]$

CASE $e = op^2\ e_0\ e_1$:

IF $e_0 \notin v$:
1. $\vdash_{\mathsf{F}} e_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by the induction hypothesis (1)
3. $E = op^2\ E_0\ e_1$
4. QED $e = E[e_0']$

IF $e_0 \in v$
$\wedge\, e_1 \notin v$:
1. $\vdash_{\mathsf{F}} e_1$
by *inversion*
2. $e_1 = E_1[e_1']$
by the induction hypothesis (1)
3. $E = op^2\ e_0\ E_1$
4. QED $e = E[e_1']$

ELSE $e_0 \in v$
$\wedge\, e_1 \in v$:
1. $E = [\,]$
2. QED $e = E[op^2\ e_0\ e_1]$

CASE $e = \mathsf{chk}\ \tau\ e_0$:
Contradiction by $\vdash_{\mathsf{F}} e$

CASE $e = \mathsf{stat}\ \tau\ e_0$:

IF $e_0 \notin v$:
1. $\vdash_{\mathsf{F}} e_0$
by *inversion*
2. $e_0 = E_0[e_0']$
by *unique static evaluation contexts* (1)
3. $E = \mathsf{stat}\ \tau\ E_0$
4. QED $e = E[e_0']$

ELSE $e_0 \in v$:
1. $E = [\,]$
2. QED $e = E[\mathsf{stat}\ \tau\ e_0]$

□

Lemma 6.17 : HF *static hole typing*

If $\vdash_{\mathsf{F}} E^{\bullet}[e] : \tau$ then the derivation contains a sub-term $\vdash_{\mathsf{F}} e : \tau'$

Proof:

By induction on the structure of $E^{\bullet}$.

CASE $E^{\bullet} = [\,]$:
1. QED $E^{\bullet}[e] = e$

CASE $E^{\bullet} = E^{\bullet}{}_0\ e_1$:
1. $E^{\bullet}[e] = E^{\bullet}{}_0[e]\ e_1$
2. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e] : \tau_d \Rightarrow \tau_c$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = v_0\ E^{\bullet}{}_1$:
1. $E^{\bullet}[e] = v_0\ E^{\bullet}{}_1[e]$
2. $\vdash_{\mathsf{F}} E^{\bullet}{}_1[e] : \tau_d$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = \langle E^{\bullet}{}_0, e_1 \rangle$:
1. $E^{\bullet}[e] = \langle E^{\bullet}{}_0[e], e_1 \rangle$
2. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e] : \tau_0$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = \langle v_0, E^{\bullet}{}_1 \rangle$:
1. $E^{\bullet}[e] = \langle v_0, E^{\bullet}{}_1[e] \rangle$
2. $\vdash_{\mathsf{F}} E^{\bullet}{}_1[e] : \tau_1$
by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^{\bullet} = op^1\ E^{\bullet}{}_0$:
1. $E^{\bullet}[e] = op^1\ E^{\bullet}{}_0[e]$

2. $\vdash_F E^\bullet{}_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = op^2\ E^\bullet{}_0\ e_1$:
1. $E^\bullet[e] = op^2\ E^\bullet{}_0[e]\ e_1$
2. $\vdash_F E^\bullet{}_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = op^2\ v_0\ E^\bullet{}_1$:
1. $E^\bullet[e] = op^2\ v_0\ E^\bullet{}_1[e]$
2. $\vdash_F E^\bullet{}_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = \mathsf{chk}\ \tau\ E^\bullet{}_0$:
1. $E^\bullet[e] = \mathsf{chk}\ \tau\ E^\bullet{}_0[e]$
2. $\vdash_F E^\bullet{}_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

□

Lemma 6.18 : HF *dynamic hole typing*

If $\vdash_F E^\bullet[e]$ then the derivation contains a sub-term $\vdash_F e$

Proof:

By induction on the structure of $E^\bullet$.

CASE $E^\bullet = [\,]$:
1. QED $E^\bullet[e] = e$

CASE $E^\bullet = E^\bullet{}_0\ e_1$:
1. $E^\bullet[e] = E^\bullet{}_0[e]\ e_1$
2. $\vdash_F E^\bullet{}_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = v_0\ E^\bullet{}_1$:
1. $E^\bullet[e] = v_0\ E^\bullet{}_1[e]$
2. $\vdash_F E^\bullet{}_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = \langle E^\bullet{}_0, e_1\rangle$:
1. $E^\bullet[e] = \langle E^\bullet{}_0[e], e_1\rangle$
2. $\vdash_F E^\bullet{}_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = \langle v_0, E^\bullet{}_1\rangle$:
1. $E^\bullet[e] = \langle v_0, E^\bullet{}_1[e]\rangle$
2. $\vdash_F E^\bullet{}_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = op^1\ E^\bullet{}_0$:
1. $E^\bullet[e] = op^1\ E^\bullet{}_0[e]$
2. $\vdash_F E^\bullet{}_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = op^2\ E^\bullet{}_0\ e_1$:
1. $E^\bullet[e] = op^2\ E^\bullet{}_0[e]\ e_1$
2. $\vdash_F E^\bullet{}_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E^\bullet = op^2\ v_0\ E^\bullet{}_1$:
1. $E^\bullet[e] = op^2\ v_0\ E^\bullet{}_1[e]$
2. $\vdash_F E^\bullet{}_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

□

Lemma 6.19 : HF *boundary hole typing*

- If $\vdash_F E[\mathsf{dyn}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_F \mathsf{dyn}\ \tau\ e : \tau$
- If $\vdash_F E[\mathsf{dyn}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_F \mathsf{dyn}\ \tau\ e : \tau$
- If $\vdash_F E[\mathsf{stat}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_F \mathsf{stat}\ \tau\ e$
- If $\vdash_F E[\mathsf{stat}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_F \mathsf{stat}\ \tau\ e$

Proof:

By the following four lemmas: *static* dyn *hole typing*, *dynamic* dyn *hole typing*, *static* stat *hole typing*, and *dynamic* stat *hole typing*.

□

Lemma 6.20 : HF *static* dyn *hole typing*

If $\vdash_F E[\mathsf{dyn}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_F \mathsf{dyn}\ \tau\ e : \tau$.

Proof:

By induction on the structure of E.

CASE $E \in E^\bullet$:
1. $\vdash_F \mathsf{dyn}\ \tau\ e : \tau''$
   by *static hole typing*
2. $\vdash_F \mathsf{dyn}\ \tau\ e : \tau$
   by *inversion* (1)
3. QED

CASE $E = E_0\ e_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = E_0[\mathsf{dyn}\ \tau\ e]\ e_1$
2. $\vdash_F E_0[\mathsf{dyn}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[\mathsf{dyn}\ \tau\ e] = v_0\ E_1[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_F E_1[\mathsf{dyn}\ \tau\ e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1\rangle$:
1. $E[\mathsf{dyn}\ \tau\ e] = \langle E_0[\mathsf{dyn}\ \tau\ e], e_1\rangle$
2. $\vdash_F E_0[\mathsf{dyn}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1\rangle$:
1. $E[\mathsf{dyn}\ \tau\ e] = \langle v_0, E_1[\mathsf{dyn}\ \tau\ e]\rangle$
2. $\vdash_F E_1[\mathsf{dyn}\ \tau\ e] : \tau_1$
   by *inversion*

3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:

1. $E[\mathsf{dyn}\ \tau\ e] = op^1\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:

1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ E_0[\mathsf{dyn}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:

1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{dyn}\ \tau\ e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{chk}\ \tau''\ E_0$:

1. $E[\mathsf{dyn}\ \tau\ e] = \mathsf{chk}\ \tau''\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:

1. $E[\mathsf{dyn}\ \tau\ e] = \mathsf{dyn}\ \tau_0\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by *dynamic* dyn *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:

1. Contradiction by $\vdash_{\mathsf{F}} E[\mathsf{dyn}\ \tau\ e] : \tau'$

□

Lemma 6.21 : HF *dynamic* dyn *hole typing*

If $\vdash_{\mathsf{F}}\ E[\mathsf{dyn}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_{\mathsf{F}} \mathsf{dyn}\ \tau\ e : \tau$.

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:

1. Contradiction by $\vdash_{\mathsf{F}} E[\mathsf{dyn}\ \tau\ e]$

CASE $E = E_0\ e_1$:

1. $E[\mathsf{dyn}\ \tau\ e] = E_0[\mathsf{dyn}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:

1. $E[\mathsf{dyn}\ \tau\ e] = v_0\ E_1[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1\rangle$:

1. $E[\mathsf{dyn}\ \tau\ e] = \langle E_0[\mathsf{dyn}\ \tau\ e], e_1\rangle$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1\rangle$:

1. $E[\mathsf{dyn}\ \tau\ e] = \langle v_0, E_1[\mathsf{dyn}\ \tau\ e]\rangle$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:

1. $E[\mathsf{dyn}\ \tau\ e] = op^1\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:

1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ E_0[\mathsf{dyn}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:

1. $E[\mathsf{dyn}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{dyn}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{chk}\ \tau''\ E_0$:

1. Contradiction by $\vdash_{\mathsf{F}} E[\mathsf{dyn}\ \tau\ e]$

□

CASE $E = \mathsf{dyn}\ \tau\ E_0$:

1. Contradiction by $\vdash_{\mathsf{F}} E[\mathsf{dyn}\ \tau\ e]$

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:

1. $E[\mathsf{dyn}\ \tau\ e] = \mathsf{stat}\ \tau_0\ E_0[\mathsf{dyn}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{dyn}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by *static* dyn *hole typing* (2)

}

Lemma 6.22 : HF *static* stat *hole typing*

If $\vdash_{\mathsf{F}} E[\mathsf{stat}\ \tau\ e] : \tau'$ then the derivation contains a sub-term $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau\ e$.

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:

1. Contradiction by $\vdash_{\mathsf{F}} E[\mathsf{stat}\ \tau\ e] : \tau'$

CASE $E = E_0\ e_1$:

1. $E[\mathsf{stat}\ \tau\ e] = E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:

1. $E[\mathsf{stat}\ \tau\ e] = v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{stat}\ \tau\ e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1\rangle$:

1. $E[\mathsf{stat}\ \tau\ e] = \langle E_0[\mathsf{stat}\ \tau\ e], e_1\rangle$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1\rangle$:

1. $E[\mathsf{stat}\ \tau\ e] = \langle v_0, E_1[\mathsf{stat}\ \tau\ e]\rangle$

2. $\vdash_{\mathsf{F}} E_1[\mathsf{stat}\ \tau\ e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = op^1\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{stat}\ \tau\ e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{chk}\ \tau''\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{chk}\ \tau''\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{dyn}\ \tau_0\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by *dynamic* stat *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_{\mathsf{F}} E[\mathsf{stat}\ \tau\ e] : \tau'$

□

Lemma 6.23 : HF *dynamic* stat *hole typing*

If $\vdash_{\mathsf{F}} E[\mathsf{stat}\ \tau\ e]$ then the derivation contains a sub-term $\vdash_{\mathsf{F}} \mathsf{stat}\ \tau\ e$.

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:
1. QED by *dynamic hole typing*

CASE $E = E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1\rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle E_0[\mathsf{stat}\ \tau\ e], e_1\rangle$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1\rangle$:
1. $E[\mathsf{stat}\ \tau\ e] = \langle v_0, E_1[\mathsf{stat}\ \tau\ e]\rangle$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = op^1\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ E_0[\mathsf{stat}\ \tau\ e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[\mathsf{stat}\ \tau\ e] = op^2\ v_0\ E_1[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_1[\mathsf{stat}\ \tau\ e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{chk}\ \tau''\ E_0$:
1. Contradiction by $\vdash_{\mathsf{F}} E[\mathsf{stat}\ \tau\ e]$

CASE $E = \mathsf{dyn}\ \tau\ E_0$:
1. Contradiction by $\vdash_{\mathsf{F}} E[\mathsf{stat}\ \tau\ e]$

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[\mathsf{stat}\ \tau\ e] = \mathsf{stat}\ \tau_0\ E_0[\mathsf{stat}\ \tau\ e]$
2. $\vdash_{\mathsf{F}} E_0[\mathsf{stat}\ \tau\ e] : \tau_0$
   by *inversion*
3. QED by *static* stat *hole typing* (2)

□

Lemma 6.24 : HF *static boundary-free hole substitution*

If $\vdash_{\mathsf{F}} E^{\bullet}[e] : \tau$ and the derivation contains a sub-term $\vdash_{\mathsf{F}} e : \tau'$ and $\vdash_{\mathsf{F}} e' : \tau'$ then $\vdash_{\mathsf{F}} E^{\bullet}[e'] : \tau$.

Proof:

By induction on the structure of $E^{\bullet}$

CASE $E^{\bullet} = [\,]$:
1. $E^{\bullet}[e] = e$
   $\wedge\ E^{\bullet}[e'] = e'$
2. $\vdash_{\mathsf{F}} e : \tau$
   by (1)
3. $\tau' = \tau$
4. $\vdash_{\mathsf{F}} e' : \tau$
5. QED by (1, 4)

CASE $E^{\bullet} = E^{\bullet}{}_0\ e_1$:
1. $E^{\bullet}[e] = E^{\bullet}{}_0[e]\ e_1$
   $\wedge\ E^{\bullet}[e'] = E^{\bullet}{}_0[e']\ e_1$
2. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e]\ e_1 : \tau$
3. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e] : \tau_0$
   $\wedge \vdash_{\mathsf{F}} e_1 : \tau_1$
   by *inversion*
4. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e']\ e_1 : \tau$
   by (2, 3, 4)

6. QED by (1, 5)

CASE $E^{\bullet} = v_0\ E^{\bullet}{}_1$:
1. $E^{\bullet}[e] = v_0\ E^{\bullet}{}_1[e]$
   $\wedge\ E^{\bullet}[e'] = v_0\ E^{\bullet}{}_1[e']$
2. $\vdash_{\mathsf{F}} v_0\ E^{\bullet}{}_1[e] : \tau$
3. $\vdash_{\mathsf{F}} v_0 : \tau_0$
   $\wedge \vdash_{\mathsf{F}} E^{\bullet}{}_1[e] : \tau_1$
   by *inversion*
4. $\vdash_{\mathsf{F}} E^{\bullet}{}_1[e'] : \tau_1$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{F}} v_0\ E^{\bullet}{}_1[e'] : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^1\ E^{\bullet}{}_0$:
1. $E^{\bullet}[e] = op^1\ E^{\bullet}{}_0[e]$
   $\wedge\ E^{\bullet}[e'] = op^1\ E^{\bullet}{}_0[e']$
2. $\vdash_{\mathsf{F}} op^1\ E^{\bullet}{}_0[e] : \tau$
3. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e] : \tau_0$
   by *inversion*
4. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{F}} op^1\ E^{\bullet}{}_0[e'] : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \langle E^{\bullet}{}_0, e_1\rangle$:
1. $E^{\bullet}[e] = \langle E^{\bullet}{}_0[e], e_1\rangle$
   $\wedge\ E^{\bullet}[e'] = \langle E^{\bullet}{}_0[e'], e_1\rangle$
2. $\vdash_{\mathsf{F}} \langle E^{\bullet}{}_0[e], e_1\rangle : \tau$
3. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e] : \tau_0$
   $\wedge \vdash_{\mathsf{F}} e_1 : \tau_1$
   by *inversion*
4. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{F}} \langle E^{\bullet}{}_0[e'], e_1\rangle : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \langle v_0, E^{\bullet}{}_1\rangle$:
1. $E^{\bullet}[e] = \langle v_0, E^{\bullet}{}_1[e]\rangle$
   $\wedge\ E^{\bullet}[e'] = \langle v_0, E^{\bullet}{}_1[e']\rangle$
2. $\vdash_{\mathsf{F}} \langle v_0, E^{\bullet}{}_1[e]\rangle : \tau$
3. $\vdash_{\mathsf{F}} v_0 : \tau_0$
   $\wedge \vdash_{\mathsf{F}} E^{\bullet}{}_1[e] : \tau_1$
   by *inversion*
4. $\vdash_{\mathsf{F}} E^{\bullet}{}_1[e'] : \tau_1$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{F}} \langle v_0, E^{\bullet}{}_1[e']\rangle : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ E^{\bullet}{}_0\ e_1$:
1. $E^{\bullet}[e] = op^2\ E^{\bullet}{}_0[e]\ e_1$
   $\wedge\ E^{\bullet}[e'] = op^2\ E^{\bullet}{}_0[e']\ e_1$
2. $\vdash_{\mathsf{F}} op^2\ E^{\bullet}{}_0[e]\ e_1 : \tau$
3. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e] : \tau_0$
   $\wedge \vdash_{\mathsf{F}} e_1 : \tau_1$
   by *inversion*
4. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{F}} op^2\ E^{\bullet}{}_0[e']\ e_1 : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ v_0\ E^{\bullet}{}_1$:
1. $E^{\bullet}[e] = op^2\ v_0\ E^{\bullet}{}_1[e]$
   $\wedge\ E^{\bullet}[e'] = op^2\ v_0\ E^{\bullet}{}_1[e']$
2. $\vdash_{\mathsf{F}} op^2\ v_0\ E^{\bullet}{}_1[e] : \tau$
3. $\vdash_{\mathsf{F}} v_0 : \tau_0$
   $\wedge \vdash_{\mathsf{F}} E^{\bullet}{}_1[e] : \tau_1$
   by *inversion*
4. $\vdash_{\mathsf{F}} E^{\bullet}{}_1[e'] : \tau_1$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{F}} op^2\ v_0\ E^{\bullet}{}_1[e'] : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \mathsf{chk}\ \tau''\ E^{\bullet}{}_0$:
1. $E^{\bullet}[e] = \mathsf{chk}\ \tau''\ E^{\bullet}{}_0[e]$
   $\wedge\ E^{\bullet}[e'] = \mathsf{chk}\ \tau''\ E^{\bullet}{}_0[e']$
2. $\vdash_{\mathsf{F}} \mathsf{chk}\ \tau''\ E^{\bullet}{}_0[e] : \tau$
3. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e] : \tau_0$
   by *inversion*
4. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e'] : \tau_0$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{F}} \mathsf{chk}\ \tau''\ E^{\bullet}{}_0[e'] : \tau$
   by (2, 3, 4)
6. QED by (1, 5)

□

Lemma 6.25 : HF *dynamic hole substitution*

If $\vdash_{\mathsf{F}} E^{\bullet}[e]$ and $\vdash_{\mathsf{F}} e'$ then $\vdash_{\mathsf{F}} E^{\bullet}[e']$

Proof:

By induction on the structure of $E^{\bullet}$

CASE $E^{\bullet} = [\,]$:
1. QED $E^{\bullet}[e'] = e'$

CASE $E^{\bullet} = \langle E^{\bullet}{}_0, e_1\rangle$:
1. $E^{\bullet}[e] = \langle E^{\bullet}{}_0[e], e_1\rangle$
   $\wedge\ E^{\bullet}[e'] = \langle E^{\bullet}{}_0[e'], e_1\rangle$
2. $\vdash_{\mathsf{F}} \langle E^{\bullet}{}_0[e], e_1\rangle$
3. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e]$
   $\wedge \vdash_{\mathsf{F}} e_1$
   by *inversion*
4. $\vdash_{\mathsf{F}} E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{\mathsf{F}} \langle E^{\bullet}{}_0[e'], e_1\rangle$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \langle v_0, E^{\bullet}{}_1\rangle$:
1. $E^{\bullet}[e] = \langle v_0, E^{\bullet}{}_1[e]\rangle$
   $\wedge\ E^{\bullet}[e'] = \langle v_0, E^{\bullet}{}_1[e']\rangle$

2. $\vdash_{F} \langle v_0, E^{\bullet}{}_1[e] \rangle$
3. $\vdash_{F} v_0$
   $\wedge \vdash_{F} E^{\bullet}{}_1[e]$
   by *inversion*
4. $\vdash_{F} E^{\bullet}{}_1[e']$
   by the induction hypothesis (3)
5. $\vdash_{F} \langle v_0, E^{\bullet}{}_1[e'] \rangle$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = E^{\bullet}{}_0\ e_1$:

1. $E^{\bullet}[e] = E^{\bullet}{}_0[e]\ e_1$
   $\wedge E^{\bullet}[e'] = E^{\bullet}{}_0[e']\ e_1$
2. $\vdash_{F} E^{\bullet}{}_0[e]\ e_1$
3. $\vdash_{F} E^{\bullet}{}_0[e]$
   $\wedge \vdash_{F} e_1$
   by *inversion*
4. $\vdash_{F} E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{F} E^{\bullet}{}_0[e']\ e_1$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = v_0\ E^{\bullet}{}_1$:

1. $E^{\bullet}[e] = v_0\ E^{\bullet}{}_1[e]$
   $\wedge E^{\bullet}[e'] = v_0\ E^{\bullet}{}_1[e']$
2. $\vdash_{F} v_0\ E^{\bullet}{}_1[e]$
3. $\vdash_{F} v_0$
   $\wedge \vdash_{F} E^{\bullet}{}_1[e]$
   by *inversion*
4. $\vdash_{F} E^{\bullet}{}_1[e']$
   by the induction hypothesis (3)
5. $\vdash_{F} v_0\ E^{\bullet}{}_1[e']$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^1\ E^{\bullet}{}_0$:

1. $E^{\bullet}[e] = op^1\ E^{\bullet}{}_0[e]$
   $\wedge E^{\bullet}[e'] = op^1\ E^{\bullet}{}_0[e']$
2. $\vdash_{F} op^1\ E^{\bullet}{}_0[e]$
3. $\vdash_{F} E^{\bullet}{}_0[e]$
   by *inversion*
4. $\vdash_{F} E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{F} op^1\ E^{\bullet}{}_0[e']$
   by (4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ E^{\bullet}{}_0\ e_1$:

1. $E^{\bullet}[e] = op^2\ E^{\bullet}{}_0[e]\ e_1$
   $\wedge E^{\bullet}[e'] = op^2\ E^{\bullet}{}_0[e']\ e_1$
2. $\vdash_{F} op^2\ E^{\bullet}{}_0[e]\ e_1$
3. $\vdash_{F} E^{\bullet}{}_0[e]$
   $\wedge \vdash_{F} e_1$
   by *inversion*
4. $\vdash_{F} E^{\bullet}{}_0[e']$
   by the induction hypothesis (3)
5. $\vdash_{F} op^2\ E^{\bullet}{}_0[e']\ e_1$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = op^2\ v_0\ E^{\bullet}{}_1$:

1. $E^{\bullet}[e] = op^2\ v_0\ E^{\bullet}{}_1[e]$
   $\wedge E^{\bullet}[e'] = op^2\ v_0\ E^{\bullet}{}_1[e']$
2. $\vdash_{F} op^2\ v_0\ E^{\bullet}{}_1[e]$
3. $\vdash_{F} v_0$
   $\wedge \vdash_{F} E^{\bullet}{}_1[e]$
   by *inversion*
4. $\vdash_{F} E^{\bullet}{}_1[e']$
   by the induction hypothesis (3)
5. $\vdash_{F} op^2\ v_0\ E^{\bullet}{}_1[e']$
   by (3, 4)
6. QED by (1, 5)

CASE $E^{\bullet} = \mathsf{chk}\ \tau_0\ E^{\bullet}{}_0$:

1. Contradiction by $\vdash_{F} E^{\bullet}[e]$

☐

Lemma 6.26 : HF *hole substitution*

- If $\vdash_{F} E[e]$ and the derivation contains a sub-term $\vdash_{F} e : \tau'$ and $\vdash_{F} e' : \tau'$ then $\vdash_{F} E[e']$.
- If $\vdash_{F} E[e]$ and the derivation contains a sub-term $\vdash_{F} e$ and $\vdash_{F} e'$ then $\vdash_{F} E[e']$.
- If $\vdash_{F} E[e] : \tau$ and the derivation contains a sub-term $\vdash_{F} e : \tau'$ and $\vdash_{F} e' : \tau'$ then $\vdash_{F} E[e'] : \tau$.
- If $\vdash_{F} E[e] : \tau$ and the derivation contains a sub-term $\vdash_{F} e$ and $\vdash_{F} e'$ then $\vdash_{F} E[e'] : \tau$.

Proof:

By the following four lemmas: *dynamic context static hole substitution*, *dynamic context dynamic hole substitution*, *static context static hole substitution*, and *static context dynamic hole substitution*.

☐

Lemma 6.27 : HF *dynamic context static hole substitution*

If $\vdash_{F} E[e]$ and contains $\vdash_{F} e : \tau'$, and furthermore $\vdash_{F} e' : \tau'$, then $\vdash_{F} E[e']$

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:

1. Contradiction by $\vdash_{F} E[e]$

CASE $E = E_0\ e_1$:

1. $E[e] = E_0[e]\ e_1$
2. $\vdash_{F} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:

1. $E[e] = v_0\ E_1[e]$
2. $\vdash_{F} E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:

1. $E[e] = \langle E_0[e], e_1 \rangle$

2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_{\mathsf{F}} E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_{\mathsf{F}} E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{chk}\ \tau''\ E_0$:
1. $E[e] = \mathsf{chk}\ \tau''\ E_0[e]$
2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau''\ E_0$:
1. Contradiction by $\vdash_{\mathsf{F}} E[e]$

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{stat}\ \tau_0\ E_0[e]$
2. $\vdash_{\mathsf{F}} E_0[e] : \tau_0$
   by *inversion*
3. QED by *static context static hole substitution* (2)

□

Lemma 6.28 : HF *dynamic context dynamic hole substitution*

If $\vdash_{\mathsf{F}} E[e]$ and contains $\vdash_{\mathsf{F}} e$, and furthermore $\vdash_{\mathsf{F}} e'$, then $\vdash_{\mathsf{F}} E[e']$

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:
1. QED by *dynamic boundary-free hole substitution*

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_{\mathsf{F}} E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_{\mathsf{F}} E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_{\mathsf{F}} E_1[e]$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{chk}\ \tau''\ E_0$:
1. Contradiction by $\vdash_{\mathsf{F}} E[e]$

CASE $E = \mathsf{dyn}\ \tau''\ E_0$:
1. Contradiction by $\vdash_{\mathsf{F}} E[e]$

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{stat}\ \tau_0\ E_0[e]$
2. $\vdash_{\mathsf{F}} E_0[e] : \tau_0$
   by *inversion*
3. QED by *static context dynamic hole substitution* (2)

□

Lemma 6.29 : HF *static context static hole substitution*

If $\vdash_{\mathsf{F}} E[e] : \tau$ and contains $\vdash_{\mathsf{F}} e : \tau'$, and furthermore $\vdash_{\mathsf{F}} e' : \tau'$, then $\vdash_{\mathsf{F}} E[e'] : \tau$

Proof:

By induction on the structure of E.

CASE $E \in E^{\bullet}$:
1. QED by *static boundary-free hole substitution*

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_{\mathsf{F}} E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_{\mathsf{F}} E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[e] = \langle E_0[e], e_1 \rangle$

2. $\vdash_F E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_F E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_F E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_F E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_F E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{chk}\ \tau''\ E_0$:
1. $E[e] = \mathsf{chk}\ \tau''\ E_0[e]$
2. $\vdash_F E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{dyn}\ \tau_0\ E_0[e]$
2. $\vdash_F E_0[e]$
   by *inversion*
3. QED by *static* dyn *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_F E[e] : \tau$

□

Lemma 6.30 : HF *static context dynamic hole substitution*

If $\vdash_F E[e] : \tau$ and contains $\vdash_F e$, and furthermore $\vdash_F e'$, then $\vdash_F E[e'] : \tau$

Proof:

By induction on the structure of E.

CASE $E \in E^\bullet$:
1. Contradiction by $\vdash_F E[e] : \tau$

CASE $E = E_0\ e_1$:
1. $E[e] = E_0[e]\ e_1$
2. $\vdash_F E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = v_0\ E_1$:
1. $E[e] = v_0\ E_1[e]$
2. $\vdash_F E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle E_0, e_1 \rangle$:
1. $E[e] = \langle E_0[e], e_1 \rangle$
2. $\vdash_F E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \langle v_0, E_1 \rangle$:
1. $E[e] = \langle v_0, E_1[e] \rangle$
2. $\vdash_F E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^1\ E_0$:
1. $E[e] = op^1\ E_0[e]$
2. $\vdash_F E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ E_0\ e_1$:
1. $E[e] = op^2\ E_0[e]\ e_1$
2. $\vdash_F E_0[e] : \tau_0$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = op^2\ v_0\ E_1$:
1. $E[e] = op^2\ v_0\ E_1[e]$
2. $\vdash_F E_1[e] : \tau_1$
   by *inversion*
3. QED by the induction hypothesis (2)

CASE $E = \mathsf{chk}\ \tau''\ E_0$:
1. Contradiction by $\vdash_F E[e] : \tau$

CASE $E = \mathsf{dyn}\ \tau_0\ E_0$:
1. $E[e] = \mathsf{dyn}\ \tau_0\ E_0[e]$
2. $\vdash_F E_0[e]$
   by *inversion*
3. QED by *dynamic* stat *hole typing* (2)

CASE $E = \mathsf{stat}\ \tau_0\ E_0$:
1. Contradiction by $\vdash_F E[e] : \tau$

□

Lemma 6.31 : $\vdash_F$ *static inversion*

- If $\Gamma \vdash_{\mathsf{F}} x : \tau$ then $(x{:}\tau') \in \Gamma$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{F}} \lambda(x{:}\tau'_d).\, e' : \tau$ then $(x{:}\tau'_d), \Gamma \vdash_{\mathsf{F}} e' : \tau'_c$ and $\tau'_d \Rightarrow \tau'_c \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{F}} \langle e_0, e_1 \rangle : \tau_0 \times \tau_1$ then $\Gamma \vdash_{\mathsf{F}} e_0 : \tau'_0$ and $\Gamma \vdash_{\mathsf{F}} e_1 : \tau'_1$ and $\tau'_0 \leqslant: \tau_0$ and $\tau'_1 \leqslant: \tau_1$
- If $\Gamma \vdash_{\mathsf{F}} e_0\ e_1 : \tau_c$ then $\Gamma \vdash_{\mathsf{F}} e_0 : \tau'_d \Rightarrow \tau'_c$ and $\Gamma \vdash_{\mathsf{F}} e_1 : \tau'_d$ and $\tau'_c \leqslant: \tau_c$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{fst}\, e : \tau$ then $\Gamma \vdash_{\mathsf{F}} e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{fst}, \tau_0 \times \tau_1) = \tau_0$ and $\tau_0 \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{snd}\, e : \tau$ then $\Gamma \vdash_{\mathsf{F}} e : \tau_0 \times \tau_1$ and $\Delta(\mathsf{snd}, \tau_0 \times \tau_1) = \tau_1$ and $\tau_1 \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{F}} op^2\ e_0\ e_1 : \tau$ then $\Gamma \vdash_{\mathsf{F}} e_0 : \tau_0$ and $\Gamma \vdash_{\mathsf{F}} e_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau'$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\, \tau'_0 \times \tau'_1\, \langle v_0, v_1 \rangle : \tau_0 \times \tau_1$ then either:
  - $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle$
  - or $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle : \tau$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\, \tau'_d \Rightarrow \tau'_c\, \lambda x.\, e : \tau_d \Rightarrow \tau_c$ then $\Gamma \vdash_{\mathsf{F}} \lambda x.\, e$ and $\tau'_d \Rightarrow \tau'_c \leqslant: \tau_d \Rightarrow \tau_c$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\, \tau'_d \Rightarrow \tau'_c\, \lambda(x : \tau''_d).\, e : \tau_d \Rightarrow \tau_c$ then $\Gamma \vdash_{\mathsf{F}} \lambda(x : \tau_x).\, e : \tau''_d \Rightarrow \tau''_c$ and $\tau'_d \Rightarrow \tau'_c \leqslant: \tau_d \Rightarrow \tau_c$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{dyn}\, \tau'\, e' : \tau$ then $\Gamma \vdash_{\mathsf{F}} e'$ and $\tau' \leqslant: \tau$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{chk}\, \tau\, e' : \tau$ then $\Gamma \vdash_{\mathsf{F}} e' : \tau'$

Proof:

QED by the definition of $\Gamma \vdash_{\mathsf{F}} e : \tau$

□

Lemma 6.32 : $\vdash_{\mathsf{F}}$ *dynamic inversion*

- If $\Gamma \vdash_{\mathsf{F}} x$ then $x \in \Gamma$
- If $\Gamma \vdash_{\mathsf{F}} \lambda x.\, e'$ then $x, \Gamma \vdash_{\mathsf{F}} e'$
- If $\Gamma \vdash_{\mathsf{F}} \langle e_0, e_1 \rangle$ then $\Gamma \vdash_{\mathsf{F}} e_0$ and $\Gamma \vdash_{\mathsf{F}} e_1$
- If $\Gamma \vdash_{\mathsf{F}} e_0\ e_1$ then $\Gamma \vdash_{\mathsf{F}} e_0$ and $\Gamma \vdash_{\mathsf{F}} e_1$
- If $\Gamma \vdash_{\mathsf{F}} op^1\ e_0$ then $\Gamma \vdash_{\mathsf{F}} e_0$
- If $\Gamma \vdash_{\mathsf{F}} op^2\ e_0\ e_1$ then $\Gamma \vdash_{\mathsf{F}} e_0$ and $\Gamma \vdash_{\mathsf{F}} e_1$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\, (\tau_d \Rightarrow \tau_c)\, \lambda x.\, e$ then $\Gamma \vdash_{\mathsf{F}} \lambda x.\, e$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\, (\tau_d \Rightarrow \tau_c)\, \lambda(x{:}\tau_x).\, e$ then $\Gamma \vdash_{\mathsf{F}} \lambda(x{:}\tau_x).\, e : \tau_x \Rightarrow \tau'_x$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{mon}\, (\tau_0 \times \tau_1)\, \langle v_0, v_1 \rangle$ then either:
  - $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle$
  - or $\Gamma \vdash_{\mathsf{F}} \langle v_0, v_1 \rangle : \tau'$
- If $\Gamma \vdash_{\mathsf{F}} \mathsf{stat}\, \tau'\, e'$ then $\Gamma \vdash_{\mathsf{F}} e' : \tau'$

Proof:

QED by the definition of $\Gamma \vdash_{\mathsf{F}} e$

□

Lemma 6.33 : HF *canonical forms*

- If $\vdash_{\mathsf{F}} v : \tau_0 \times \tau_1$ then either:
  - $v = \langle v_0, v_1 \rangle$
  - or $v = \mathsf{mon}\, \tau'_0 \times \tau'_1\, \langle v_0, v_1 \rangle$ $\wedge\ \tau'_0 \times \tau'_1 \leqslant: \tau_0 \times \tau_1$
- If $\vdash_{\mathsf{F}} v : \tau_d \Rightarrow \tau_c$ then either:
  - $v = \lambda(x{:}\tau_x).\, e'$ $\wedge\ \tau_d \leqslant: \tau_x$
  - or $v = \mathsf{mon}\, (\tau'_d \Rightarrow \tau'_c)\, (\lambda x.\, e$ $\wedge\ \tau'_d \Rightarrow \tau'_c \leqslant: \tau_d \Rightarrow \tau_c$
  - or $v = \mathsf{mon}\, (\tau'_d \Rightarrow \tau'_c)\, \lambda(x{:}\tau_x).\, e$ $\wedge\ \tau'_d \Rightarrow \tau'_c \leqslant: \tau_d \Rightarrow \tau_c$
- If $\vdash_{\mathsf{F}} v : \mathsf{Int}$ then $v = i$
- If $\vdash_{\mathsf{F}} v : \mathsf{Nat}$ then $v = i$ and $v \in \mathbb{N}$

Proof:

QED by definition of $\vdash_{\mathsf{F}} \cdot : \tau$

□

Lemma 6.34 : Δ *type soundness*

If $\vdash_{\mathsf{F}} v_0 : \tau_0$ and $\vdash_{\mathsf{F}} v_1 : \tau_1$ and $\Delta(op^2, \tau_0, \tau_1) = \tau$ then one of the following holds:

- $\delta(op^2, v_0, v_1) = v$ and $\vdash_{\mathsf{F}} v : \tau$, or
- $\delta(op^2, v_0, v_1) = \mathsf{BndryErr}$

Proof (sketch): Similar to the proof for the higher-order Δ *type soundness* lemma. □

Lemma 6.35 : δ *preservation*

- If $\vdash_{\mathsf{F}} v$ and $\delta(op^1, v) = v'$ then $\vdash_{\mathsf{F}} v'$
- If $\vdash_{\mathsf{F}} v_0$ and $\vdash_{\mathsf{F}} v_1$ and $\delta(op^2, v_0, v_1) = v'$ then $\vdash_{\mathsf{F}} v'$

Proof:

Similar to the proof for the higher-order δ *preservation* lemma.

□

Lemma 6.36 : HF *substitution*

- If $(x{:}\tau_x), \Gamma \vdash_{\mathsf{F}} e$ and $\vdash_{\mathsf{F}} v : \tau_x$ then $\Gamma \vdash_{\mathsf{F}} e[x \leftarrow v]$
- If $x, \Gamma \vdash_{\mathsf{F}} e$ and $\vdash_{\mathsf{F}} v$ then $\Gamma \vdash_{\mathsf{F}} e[x \leftarrow v]$
- If $(x{:}\tau_x), \Gamma \vdash_{\mathsf{F}} e : \tau$ and $\vdash_{\mathsf{F}} v : \tau_x$ then $\Gamma \vdash_{\mathsf{F}} e[x \leftarrow v] : \tau$
- If $x, \Gamma \vdash_{\mathsf{F}} e : \tau$ and $\vdash_{\mathsf{F}} v$ then $\Gamma \vdash_{\mathsf{F}} e[x \leftarrow v] : \tau$

Proof:

Similar to the proof for the higher-order *substitution* lemma.

□

Lemma 6.37 : *weakening*

- If $\Gamma \vdash_{\mathsf{F}} e$ then $x, \Gamma \vdash_{\mathsf{F}} e$
- If $\Gamma \vdash_{\mathsf{F}} e : \tau$ then $(x{:}\tau'), \Gamma \vdash_{\mathsf{F}} e : \tau$

Proof:

QED because e is closed under Γ

□

E.7 Embeddings Summary

The paragraphs in this section summarize the five embeddings with four slogans. Each slogan pertains to one aspect of the embedding:

1. What kinds of checks does the embedding perform when a value reaches a type boundary?
2. When, if ever, does the embedding wrap a value in a monitor?
3. If an ill-typed value reaches a type boundary, when does the embedding signal an error?
4. How do types affect behavior?

These embeddings are ordered on a speculative scale from "most guarantees" to "least guarantees".

Higher-Order embedding

1. recursively check read-only values;
2. monitor functional and mutable values;
3. detect boundary errors as early as possible;
4. types globally constrain behavior.

Co-Natural embedding

1. tag-check all values;
2. monitor all data structures and functions;
3. detect boundary errors as late as possible;
4. types globally constrain behavior

Forgetful embedding

1. tag-check all values;
2. apply at most one monitor to each value;
3. detect boundary errors as late as possible;
4. types (of values) locally constrain behavior.

First-Order embedding

1. tag-check all values;
2. never allocate a monitor;
3. detect boundary errors as late as possible;
4. types (of contexts) locally constrain behavior.

Erasure embedding

1. never check values;
2. never allocate a monitor;
3. never detect a type boundary error;
4. types do not affect behavior

E.8 Simulation Lemmas

E.8.1 Definitions

$\boxed{\text{Combined Language}}$

$$
\begin{array}{lcl}
e & = & x \mid v \mid \langle e, e\rangle \mid e\ e \mid op^1\ e \mid op^2\ e\ e \mid \\
 & & \mathsf{dyn}\ \tau\ e \mid \mathsf{stat}\ \tau\ e \mid \mathsf{Err} \mid \mathsf{chk}\ K\ e \mid \mathsf{dyn}\ e \mid \mathsf{stat}\ e \\
v & = & i \mid \langle v, v\rangle \mid \lambda x.\, e \mid \lambda(x\!:\!\tau).\, e \mid \mathsf{mon}\,(\tau \Rightarrow \tau)\, v \\
E^{\bullet} & = & [\,] \mid E^{\bullet}\ e \mid v\ E^{\bullet} \mid \langle E^{\bullet}, e\rangle \mid \langle v, E^{\bullet}\rangle \mid \\
 & & op^1\ E^{\bullet} \mid op^2\ E^{\bullet}\ e \mid op^2\ v\ E^{\bullet} \mid \mathsf{chk}\ K\ E^{\bullet} \\
E & = & E^{\bullet} \mid E\ e \mid v\ E \mid \langle E, e\rangle \mid \langle v, E\rangle \mid op^1\ E \mid \\
 & & op^2\ E\ e \mid op^2\ v\ E \mid \mathsf{dyn}\ \tau\ E \mid \mathsf{stat}\ \tau\ E \\
 & & \mathsf{chk}\ K\ E \mid \mathsf{dyn}\ E \mid \mathsf{stat}\ E
\end{array}
$$

$\boxed{e \;{}_{1}{\lesssim}_{\mathsf{E}}\; e}$

$$
\frac{}{\mathsf{Err} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e^{\mathsf{E}}}
\qquad
\frac{e^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e^{\mathsf{E}}}{\mathsf{chk}\ K\ e^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e^{\mathsf{E}}}
\qquad
\frac{e^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e^{\mathsf{E}}}{\mathsf{dyn}\ e^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e^{\mathsf{E}}}
$$

$$
\frac{e^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e^{\mathsf{E}}}{\mathsf{stat}\ e^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e^{\mathsf{E}}}
\qquad
\frac{}{x \;{}_{1}{\lesssim}_{\mathsf{E}}\; x}
\qquad
\frac{}{i \;{}_{1}{\lesssim}_{\mathsf{E}}\; i}
\qquad
\frac{e^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e^{\mathsf{E}}}{\lambda x.\, e^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; \lambda x.\, e^{\mathsf{E}}}
$$

$$
\frac{e^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e^{\mathsf{E}}}{\lambda(x\!:\!\tau).\, e^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; \lambda(x\!:\!\tau).\, e^{\mathsf{E}}}
\qquad
\frac{e_0^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e_0^{\mathsf{E}} \quad e_1^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e_1^{\mathsf{E}}}{e_0^{1}\ e_1^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e_0^{\mathsf{E}}\ e_1^{\mathsf{E}}}
$$

$$
\frac{e_0^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e_0^{\mathsf{E}} \quad e_1^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e_1^{\mathsf{E}}}{\langle e_0^{1}, e_1^{1}\rangle \;{}_{1}{\lesssim}_{\mathsf{E}}\; \langle e_0^{\mathsf{E}}, e_1^{\mathsf{E}}\rangle}
\qquad
\frac{e_0^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e_0^{\mathsf{E}}}{op^1\ e_0^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; op^1\ e_0^{\mathsf{E}}}
$$

$$
\frac{e_0^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e_0^{\mathsf{E}} \quad e_1^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e_1^{\mathsf{E}}}{op^2\ e_0^{1}\ e_1^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; op^2\ e_0^{\mathsf{E}}\ e_1^{\mathsf{E}}}
\qquad
\frac{e_0^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e_0^{\mathsf{E}}}{\mathsf{dyn}\ \tau\ e_0^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; \mathsf{dyn}\ \tau\ e_0^{\mathsf{E}}}
$$

$$
\frac{e_0^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e_0^{\mathsf{E}}}{\mathsf{stat}\ \tau\ e_0^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; \mathsf{stat}\ \tau\ e_0^{\mathsf{E}}}
\qquad
\frac{}{\mathsf{Err} \;{}_{1}{\lesssim}_{\mathsf{E}}\; \mathsf{Err}}
$$

$\boxed{E \;{}_{1}{\lesssim}_{\mathsf{E}}\; E}$

$$
\frac{}{[\,] \;{}_{1}{\lesssim}_{\mathsf{E}}\; [\,]}
\qquad
\frac{E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}}{\mathsf{chk}\ K\ E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}}
\qquad
\frac{E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}}{\mathsf{dyn}\ E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}}
$$

$$
\frac{E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}}{\mathsf{stat}\ E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}}
\qquad
\frac{E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}} \quad e^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e^{\mathsf{E}}}{E^{1}\ e^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}\ e^{\mathsf{E}}}
$$

$$
\frac{v^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}} \quad E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}}{v^{1}\ E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}}\ E^{\mathsf{E}}}
\qquad
\frac{E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}} \quad e^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e^{\mathsf{E}}}{\langle E^{1}, e^{1}\rangle \;{}_{1}{\lesssim}_{\mathsf{E}}\; \langle E^{\mathsf{E}}, e^{\mathsf{E}}\rangle}
$$

$$
\frac{v^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}} \quad E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}}{\langle v^{1}, E^{1}\rangle \;{}_{1}{\lesssim}_{\mathsf{E}}\; \langle v^{\mathsf{E}}, E^{\mathsf{E}}\rangle}
\qquad
\frac{E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}}{op^1\ E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; op^1\ E^{\mathsf{E}}}
$$

$$
\frac{E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}} \quad e^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; e^{\mathsf{E}}}{op^2\ E^{1}\ e^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; op^2\ E^{\mathsf{E}}\ e^{\mathsf{E}}}
\qquad
\frac{v^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}} \quad E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}}{op^2\ v^{1}\ E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; op^2\ v^{\mathsf{E}}\ E^{\mathsf{E}}}
$$

$$
\frac{E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}}{\mathsf{dyn}\ \tau\ E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; \mathsf{dyn}\ \tau\ E^{\mathsf{E}}}
\qquad
\frac{E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}}{\mathsf{stat}\ \tau\ E^{1} \;{}_{1}{\lesssim}_{\mathsf{E}}\; \mathsf{stat}\ \tau\ E^{\mathsf{E}}}
$$

$\boxed{e \;{}_{\mathsf{H}}{\lesssim}_{1}\; e}$

$$\frac{e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e^{1}}{e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; \mathsf{chk}\; K\; e^{1}} \qquad \frac{}{\mathsf{Err} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e^{1}} \qquad \frac{e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e^{1}}{\mathsf{dyn}\; \tau\; e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; \mathsf{dyn}\; \tau\; e^{1}}$$

$$\frac{e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e^{1}}{\mathsf{dyn}\; \tau\; e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; \mathsf{dyn}\; e^{1}} \qquad \frac{e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e^{1}}{\mathsf{dyn}\; \tau_0\; (\mathsf{stat}\; \tau_1\; e^{\mathsf{H}}) \;{}_{\mathsf{H}}{\lesssim}_{1}\; e^{1}}$$

$$\frac{e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e^{1}}{\mathsf{stat}\; \tau\; e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; \mathsf{stat}\; \tau\; e^{1}} \qquad \frac{e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e^{1}}{\mathsf{stat}\; \tau\; e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; \mathsf{stat}\; e^{1}}$$

$$\frac{e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e^{1}}{\mathsf{stat}\; \tau_0\; (\mathsf{dyn}\; \tau_1\; e^{\mathsf{H}}) \;{}_{\mathsf{H}}{\lesssim}_{1}\; e^{1}} \qquad \frac{e_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e_0^{1} \quad e_1^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e_1^{1}}{e_0^{\mathsf{H}}\; e_1^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e_0^{1}\; e_1^{1}}$$

$$\frac{e_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e_0^{1} \quad e_1^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e_1^{1}}{\langle e_0^{\mathsf{H}}, e_1^{\mathsf{H}} \rangle \;{}_{\mathsf{H}}{\lesssim}_{1}\; \langle e_0^{1}, e_1^{1} \rangle} \qquad \frac{e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e^{1}}{op^1\; e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; op^1\; e^{1}}$$

$$\frac{e_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e_0^{1} \quad e_1^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e_1^{1}}{op^2\; e_0^{\mathsf{H}}\; e_1^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; op^2\; e_0^{1}\; e_1^{1}} \qquad \frac{}{x \;{}_{\mathsf{H}}{\lesssim}_{1}\; x} \qquad \frac{}{i \;{}_{\mathsf{H}}{\lesssim}_{1}\; i}$$

$$\frac{e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e^{1}}{\lambda x.\, e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; \lambda x.\, e^{1}} \qquad \frac{e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e^{1}}{\lambda(x\!:\!\tau).\, e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; \lambda(x\!:\!\tau).\, e^{1}}$$

$$\frac{v^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; v^{1}}{\mathsf{mon}\; \tau\; v^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; v^{1}} \qquad \frac{}{\mathsf{Err} \;{}_{\mathsf{H}}{\lesssim}_{1}\; \mathsf{Err}}$$

$\boxed{E \;{}_{\mathsf{H}}{\lesssim}_{1}\; E}$

$$\frac{}{[\,] \;{}_{1}{\lesssim}_{\mathsf{E}}\; [\,]} \qquad \frac{E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; E^{1}}{E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; \mathsf{chk}\; K\; E^{1}} \qquad \frac{E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; E^{1} \quad e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e^{1}}{E^{\mathsf{H}}\; e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; E^{1}\; e^{1}}$$

$$\frac{v^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; v^{1} \quad E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; E^{1}}{v^{\mathsf{H}}\; E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; v^{1}\; E^{1}} \qquad \frac{E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; E^{1} \quad e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e^{1}}{\langle E^{\mathsf{H}}, e^{\mathsf{H}} \rangle \;{}_{\mathsf{H}}{\lesssim}_{1}\; \langle E^{1}, e^{1} \rangle}$$

$$\frac{v^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; v^{1} \quad E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; E^{1}}{\langle v^{\mathsf{H}}, E^{\mathsf{H}} \rangle \;{}_{\mathsf{H}}{\lesssim}_{1}\; \langle v^{1}, E^{1} \rangle} \qquad \frac{E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; E^{1}}{op^1\; E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; op^1\; E^{1}}$$

$$\frac{E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; E^{1} \quad e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; e^{1}}{op^2\; E^{\mathsf{H}}\; e^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; op^2\; E^{1}\; e^{1}} \qquad \frac{v^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; v^{1} \quad E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; E^{1}}{op^2\; v^{\mathsf{H}}\; E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; op^2\; v^{1}\; E^{1}}$$

$$\frac{E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; E^{1}}{\mathsf{dyn}\; \tau\; E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; \mathsf{dyn}\; \tau\; E^{1}} \qquad \frac{E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; E^{1}}{\mathsf{dyn}\; \tau\; E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; \mathsf{dyn}\; E^{1}}$$

$$\frac{E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; E^{1}}{\mathsf{dyn}\; \tau_0\; (\mathsf{stat}\; \tau_1\; E^{\mathsf{H}}) \;{}_{\mathsf{H}}{\lesssim}_{1}\; E^{1}} \qquad \frac{E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; E^{1}}{\mathsf{stat}\; \tau\; E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; \mathsf{stat}\; \tau\; E^{1}}$$

$$\frac{E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; E^{1}}{\mathsf{stat}\; \tau\; E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; \mathsf{stat}\; E^{1}} \qquad \frac{E^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_{1}\; E^{1}}{\mathsf{stat}\; \tau_0\; (\mathsf{dyn}\; \tau_1\; E^{\mathsf{H}}) \;{}_{\mathsf{H}}{\lesssim}_{1}\; E^{1}}$$

E.8.2 Theorems

Theorem 8.0 : Err *approximation*

If $e \in e_S$ and $\vdash e : \tau$ then the following statements hold:

- if $e \rightarrow^*_{\text{E-S}} \text{Err}$ then $e \rightarrow^*_{\text{1-S}} \text{Err}$
- if $e \rightarrow^*_{\text{1-S}} \text{Err}$ then $e \rightarrow^*_{\text{H-S}} \text{Err}$

Proof:

QED by 1–E *approximation* and H–1 *approximation*.

□

E.8.3 Lemmas

Lemma 8.1 : 1–E *approximation*

If $e \in e_S$ and $\vdash e : \tau$ and $e \rightarrow^*_{\text{E-S}} \text{Err}$ then:

- $\vdash_1 e : \lfloor\tau\rfloor \rightsquigarrow e''$
- $e'' \rightarrow^*_{\text{1-S}} \text{Err}$

Proof:

- $e'' \;{}_1{\lesssim}_{\text{E}}\; e$
  by 1–E *static reflexivity*
- QED by 1–E *simulation*

□

Lemma 8.2 : 1–E *static reflexivity*

If $\Gamma \vdash e : \tau$ and $\Gamma \vdash e : \tau \rightsquigarrow e''$ then $e'' \;{}_1{\lesssim}_{\text{E}}\; e$.

Proof:

By structural induction on the $\Gamma \vdash e : \tau \rightsquigarrow e''$ judgment.

CASE $\boxed{\dfrac{}{\Gamma \vdash i : \text{Nat} \rightsquigarrow i}}$:

1. QED $i \;{}_1{\lesssim}_{\text{E}}\; i$

CASE $\boxed{\dfrac{}{\Gamma \vdash i : \text{Int} \rightsquigarrow i}}$:

1. QED $i \;{}_1{\lesssim}_{\text{E}}\; i$

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e_0' \quad \Gamma \vdash e_1 : \tau_1 \rightsquigarrow e_1'}{\Gamma \vdash \langle e_0, e_1\rangle : \tau_0 \times \tau_1 \rightsquigarrow \langle e_0', e_1'\rangle}}$:

1. $e_0' \;{}_1{\lesssim}_{\text{E}}\; e_0$
   $\wedge\, e_1' \;{}_1{\lesssim}_{\text{E}}\; e_1$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{(x{:}\tau_d), \Gamma \vdash e : \tau_c \rightsquigarrow e'}{\Gamma \vdash \lambda(x{:}\tau_d).\, e : \tau_d \Rightarrow \tau_c \rightsquigarrow \lambda(x{:}\tau_d).\, e'}}$:

1. $e' \;{}_1{\lesssim}_{\text{E}}\; e$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash x : \tau \rightsquigarrow x}}$:

1. QED $x \;{}_1{\lesssim}_{\text{E}}\; x$

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \rightsquigarrow e_0' \quad \Gamma \vdash e_1 : \tau_d \rightsquigarrow e_1' \quad \lfloor\tau_c\rfloor = K}{\Gamma \vdash e_0\; e_1 : \tau_c \rightsquigarrow \text{chk}\; K\; (e_0'\; e_1')}}$

:

1. $e_0' \;{}_1{\lesssim}_{\text{E}}\; e_0$
   $\wedge\, e_1' \;{}_1{\lesssim}_{\text{E}}\; e_1$
   by the induction hypothesis
2. $e_0'\; e_1' \;{}_1{\lesssim}_{\text{E}}\; e_0\; e_1$
3. QED $\text{chk}\; K\; e_0'\; e_1' \;{}_1{\lesssim}_{\text{E}}\; e_0\; e_1$

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau_0 \times \tau_1 \rightsquigarrow e' \quad \lfloor\tau_0\rfloor = K}{\Gamma \vdash \text{fst}\; e : \tau_0 \rightsquigarrow \text{chk}\; K\; (\text{fst}\; e')}}$:

1. $e' \;{}_1{\lesssim}_{\text{E}}\; e$
   by the induction hypothesis
2. $\text{fst}\; e' \;{}_1{\lesssim}_{\text{E}}\; \text{fst}\; e$
3. QED $\text{chk}\; K\; \text{fst}\; e' \;{}_1{\lesssim}_{\text{E}}\; \text{fst}\; e$

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau_0 \times \tau_1 \rightsquigarrow e' \quad \lfloor \tau_1 \rfloor = K}{\Gamma \vdash \mathsf{snd}\ e : \tau_1 \rightsquigarrow \mathsf{chk}\ K\ (\mathsf{snd}\ e')}}$:

1. $e' \ {}_1\!\lesssim_{\mathsf{E}}\ e$
   by the induction hypothesis
2. $\mathsf{snd}\ e' \ {}_1\!\lesssim_{\mathsf{E}}\ \mathsf{snd}\ e$
3. QED $\mathsf{chk}\ K\ \mathsf{snd}\ e' \ {}_1\!\lesssim_{\mathsf{E}}\ \mathsf{snd}\ e$

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e_0' \quad \Gamma \vdash e_1 : \tau_1 \rightsquigarrow e_1'}{\Gamma \vdash op^2\ e_0\ e_1 : \tau \rightsquigarrow op^2\ e_0'\ e_1'}}$:

1. $e_0' \ {}_1\!\lesssim_{\mathsf{E}}\ e_0$
   $\wedge\ e_1' \ {}_1\!\lesssim_{\mathsf{E}}\ e_1$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau' \rightsquigarrow e' \quad \tau' \leqslant: \tau}{\Gamma \vdash e : \tau \rightsquigarrow e'}}$:

1. QED by the induction hypothesis

CASE $\boxed{\dfrac{}{\Gamma \vdash \mathsf{Err} : \tau \rightsquigarrow \mathsf{Err}}}$:

1. QED $\mathsf{Err} \ {}_1\!\lesssim_{\mathsf{E}}\ \mathsf{Err}$

CASE $\boxed{\dfrac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau \rightsquigarrow \mathsf{dyn}\ \tau\ e'}}$:

1. $e' \ {}_1\!\lesssim_{\mathsf{E}}\ e$
   by the induction hypothesis
2. QED

□

Lemma 8.3 : 1–E *dynamic reflexivity*

If $\Gamma \vdash e$ and $\Gamma \vdash e \rightsquigarrow e''$ then $e'' \ {}_1\!\lesssim_{\mathsf{E}}\ e$.

Proof:

By structural induction on the $\Gamma \vdash e \rightsquigarrow e''$ judgment.

CASE $\boxed{\dfrac{}{\Gamma \vdash i \rightsquigarrow i}}$:

1. QED $i \ {}_1\!\lesssim_{\mathsf{E}}\ i$

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \rightsquigarrow e_0' \quad \Gamma \vdash e_1 \rightsquigarrow e_1'}{\Gamma \vdash \langle e_0, e_1 \rangle \rightsquigarrow \langle e_0', e_1' \rangle}}$:

1. $e_0' \ {}_1\!\lesssim_{\mathsf{E}}\ e_0$
   $\wedge\ e_1' \ {}_1\!\lesssim_{\mathsf{E}}\ e_1$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{x, \Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \lambda x.\, e \rightsquigarrow \lambda x.\, e'}}$:

1. $e' \ {}_1\!\lesssim_{\mathsf{E}}\ e$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash x \rightsquigarrow x}}$:

1. QED $x \ {}_1\!\lesssim_{\mathsf{E}}\ x$

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \rightsquigarrow e_0' \quad \Gamma \vdash e_1 \rightsquigarrow e_1'}{\Gamma \vdash e_0\ e_1 \rightsquigarrow e_0'\ e_1'}}$:

1. $e_0' \ {}_1\!\lesssim_{\mathsf{E}}\ e_0$
   $\wedge\ e_1' \ {}_1\!\lesssim_{\mathsf{E}}\ e_1$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash op^1\ e \rightsquigarrow op^1\ e'}}$:

1. $e' \ {}_1\!\lesssim_{\mathsf{E}}\ e$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \rightsquigarrow e_0' \quad \Gamma \vdash e_1 \rightsquigarrow e_1'}{\Gamma \vdash op^2\ e_0\ e_1 \rightsquigarrow op^2\ e_0'\ e_1'}}$:

1. $e_0' \ {}_1\!\lesssim_{\mathsf{E}}\ e_0$
   $\wedge\ e_1' \ {}_1\!\lesssim_{\mathsf{E}}\ e_1$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash \mathsf{Err} \rightsquigarrow \mathsf{Err}}}$:

1. QED $\mathsf{Err} \ {}_1\!\lesssim_{\mathsf{E}}\ \mathsf{Err}$

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau \rightsquigarrow e'}{\Gamma \vdash \mathsf{stat}\ \tau\ e \rightsquigarrow \mathsf{stat}\ \tau\ e'}}$:

1. $e' \ {}_1\!\lesssim_{\mathsf{E}}\ e$
   by 1–E *static reflexivity*
2. QED

□

Lemma 8.4 : 1–E *simulation*

If $E^1[e_0^1] \ {}_1\!\lesssim_{\mathsf{E}}\ E^{\mathsf{E}}[e_0^{\mathsf{E}}]$ and $E^{\mathsf{E}}[e_0^{\mathsf{E}}] \rightarrow_{\mathsf{E\text{-}S}} E^{\mathsf{E}}[e_1^{\mathsf{E}}]$
then $E^1[e_0^1] \rightarrow^*_{\mathsf{1\text{-}S}} E^1[e_1^1]$ and $E^1[e_1^1] \ {}_1\!\lesssim_{\mathsf{E}}\ E^{\mathsf{E}}[e_1^{\mathsf{E}}]$

Proof:

By case analysis on $e_0^{\mathsf{E}} \rhd_{\mathsf{E\text{-}S}} e_1^{\mathsf{E}}$.

CASE $\mathsf{dyn}\ \tau_0\ v_0^{\mathsf{E}} \rhd_{\mathsf{E\text{-}S}} v_0^{\mathsf{E}}$:

1. $e_0^1 = \mathsf{dyn}\ \tau_0\ e_{0'}^1$
   by definition ${}_1\!\lesssim_{\mathsf{E}}$
2. $e_{0'}^1 \ {}_1\!\lesssim_{\mathsf{E}}\ v_0^{\mathsf{E}}$
   by (1)
3. $\vdash_1 e_{0'}^1$
   by 1 *inversion*
4. $e_{0'}^1 \rightarrow^*_{\mathsf{1\text{-}D}} v_0^1$ and $v_0^1 \ {}_1\!\lesssim_{\mathsf{E}}\ v_0^{\mathsf{E}}$
   $\vee\ e_{0'}^1 \rightarrow^*_{\mathsf{1\text{-}D}} \mathsf{BndryErr}$
   by 1–E *dynamic value stutter*
5. **IF** $e_{0'}^1 \rightarrow^*_{\mathsf{1\text{-}D}} \mathsf{BndryErr}$:
   a. $e_0^1 \rightarrow^*_{\mathsf{1\text{-}S}} \mathsf{BndryErr}$
   b. $E^1[\mathsf{BndryErr}] \ {}_1\!\lesssim_{\mathsf{E}}\ E^{\mathsf{E}}[e_1^{\mathsf{E}}]$
      by 1–E *hole substitution*
   c. QED

ELSE $e^1_{0'} \rightarrow^*_{1\text{-D}} v^1_0$:
 a. $e^1_0 \rightarrow^*_{1\text{-S}} v^1_0$
 b. $E^1[v^1_0] \; {}_1{\lesssim}_{E} \; E^E[v^E_0]$
  by 1–E *hole substitution*
 c. QED

CASE $\mathsf{stat}\ \tau_0\ v^E_0 \rhd_{E\text{-S}} v^E_0$:
1. $e^1_0 = \mathsf{stat}\ \tau_0\ e^1_{0'}$
 by definition ${}_1{\lesssim}_{E}$
2. $e^1_{0'} \; {}_1{\lesssim}_{E} \; v^E_0$
 by (1)
3. $\vdash_1 e^1_{0'} : \lfloor \tau_0 \rfloor$
 by 1 *inversion*
4. $e^1_{0'} \rightarrow^*_{1\text{-S}} v^1_0$ and $v^1_0 \; {}_1{\lesssim}_{E} \; v^E_0$
 $\vee\ e^1_{0'} \rightarrow^*_{1\text{-S}} \mathsf{BndryErr}$
 by 1–E *dynamic value stutter*
5. **IF** $e^1_{0'} \rightarrow^*_{1\text{-S}} \mathsf{BndryErr}$:
 a. $e^1_0 \rightarrow^*_{1\text{-D}} \mathsf{BndryErr}$
 b. $E^1[\mathsf{BndryErr}] \; {}_1{\lesssim}_{E} \; E^E[e^E_1]$
  by 1–E *hole substitution*
 c. QED
 ELSE $e^1_{0'} \rightarrow^*_{1\text{-S}} v^1_0$:
 a. $e^1_0 \rightarrow^*_{1\text{-D}} v^1_0$
 b. $E^1[v^1_0] \; {}_1{\lesssim}_{E} \; E^E[v^E_0]$
  by 1–E *hole substitution*
 c. QED

CASE $v^E_0\ v^E_1 \rhd_{E\text{-S}} \mathsf{TagErr}$
 $\wedge \vdash_1 e^1_0 : K_0$:
1. $e^1_0 = e^1_{0'}\ e^1_{1'}$
 by $e^1_0 \; {}_1{\lesssim}_{E} \; e^E_0$
2. $e^1_{0'} \rightarrow^*_{1\text{-S}} v^1_{0'}$ and $v^1_{0'} \; {}_1{\lesssim}_{E} \; v^E_0$
 $\vee\ e^1_{0'} \rightarrow^*_{1\text{-S}} \mathsf{BndryErr}$
 by 1–E *static value stutter*
3. **IF** $e^1_{0'} \rightarrow^*_{1\text{-S}} \mathsf{BndryErr}$:
 a. $e^1 \rightarrow^*_{1\text{-S}} \mathsf{BndryErr}$
 b. QED $E^1[\mathsf{BndryErr}] \; {}_1{\lesssim}_{E} \; E^E[\mathsf{TagErr}]$
4. $e^1_{1'} \rightarrow^*_{1\text{-S}} v^1_{1'}$ and $v^1_{1'} \; {}_1{\lesssim}_{E} \; v^E_1$
 $\vee\ e^1_{1'} \rightarrow^*_{1\text{-S}} \mathsf{BndryErr}$
 by 1–E *static value stutter*
5. **IF** $e^1_{1'} \rightarrow^*_{1\text{-S}} \mathsf{BndryErr}$:
 a. $e^1 \rightarrow^*_{1\text{-S}} \mathsf{BndryErr}$
 b. QED $E^1[\mathsf{BndryErr}] \; {}_1{\lesssim}_{E} \; E^E[\mathsf{TagErr}]$
6. $v^E_0 \in \mathbb{Z}$
 $\vee\ v^E_0 \in \langle v, v' \rangle$
 by definition $\rhd_{E\text{-S}}$
7. **IF** $v^E_0 \in \mathbb{Z}$:
 a. $v^1_{0'} \in \mathbb{Z}$
  by (2)
 b. $\vdash_1 v^1_{0'}\ v^1_{1'} : K_0$
  by 1 *static preservation*
 c. Contradiction by (a, b)
 ELSE $v^E_0 \in \langle v, v' \rangle$:
 a. $v^1_{0'} \in \langle v, v' \rangle$
  by (2)
 b. $\vdash_1 v^1_{0'}\ v^1_{1'} : K_0$
  by 1 *static preservation*
 c. Contradiction by (a, b)

CASE $v^E_0\ v^E_1 \rhd_{E\text{-S}} \mathsf{TagErr}$
 $\wedge \vdash_1 e^1_0$:
1. $e^1_0 = e^1_{0'}\ e^1_{1'}$
 by $e^1_0 \; {}_1{\lesssim}_{E} \; e^E_0$
2. $e^1_{0'} \rightarrow^*_{1\text{-D}} v^1_{0'}$ and $v^1_{0'} \; {}_1{\lesssim}_{E} \; v^E_0$
 $\vee\ e^1_{0'} \rightarrow^*_{1\text{-D}} \mathsf{BndryErr}$
 by 1–E *dynamic value stutter*
3. **IF** $e^1_{0'} \rightarrow^*_{1\text{-D}} \mathsf{BndryErr}$:
 a. $e^1 \rightarrow^*_{1\text{-D}} \mathsf{BndryErr}$
 b. QED $E^1[\mathsf{BndryErr}] \; {}_1{\lesssim}_{E} \; E^E[\mathsf{TagErr}]$
4. $e^1_{1'} \rightarrow^*_{1\text{-D}} v^1_{1'}$ and $v^1_{1'} \; {}_1{\lesssim}_{E} \; v^E_1$
 $\vee\ e^1_{1'} \rightarrow^*_{1\text{-D}} \mathsf{BndryErr}$
 by 1–E *static value stutter*
5. **IF** $e^1_{1'} \rightarrow^*_{1\text{-D}} \mathsf{BndryErr}$:
 a. $e^1 \rightarrow^*_{1\text{-D}} \mathsf{BndryErr}$
 b. QED $E^1[\mathsf{BndryErr}] \; {}_1{\lesssim}_{E} \; E^E[\mathsf{TagErr}]$
6. $v^E_0 \in \mathbb{Z}$
 $\vee\ v^E_0 \in \langle v, v' \rangle$
 by definition $\rhd_{E\text{-S}}$
7. $v^1_{0'} \in \mathbb{Z}$
 $\vee\ v^1_{0'} \in \langle v, v' \rangle$
 by (2)
8. $e^1 \rightarrow^*_{1\text{-D}} \mathsf{TagErr}$
9. QED $E^1[\mathsf{TagErr}] \; {}_1{\lesssim}_{E} \; E^E[\mathsf{TagErr}]$

CASE $(\lambda x.\, e^E)\ v^E_1 \rhd_{E\text{-S}} e^E[x \leftarrow v^E_1]$:
1. $e^1_0 = e^1_{0'}\ e^1_{1'}$
 by $e^1_0 \; {}_1{\lesssim}_{E} \; e^E_0$
2. $e^1_{0'} \rightarrow^*_{1\text{-D}} v^1_0$ and $v^1_0 \; {}_1{\lesssim}_{E} \; v^E_0$
 $\vee\ e^1_{0'} \rightarrow^*_{1\text{-D}} \mathsf{BndryErr}$
 by 1–E *dynamic value stutter*
3. **IF** $e^1_{0'} \rightarrow^*_{1\text{-D}} \mathsf{BndryErr}$:
 a. $e^1 \rightarrow^*_{1\text{-D}} \mathsf{BndryErr}$
 b. QED $E^1[\mathsf{BndryErr}] \; {}_1{\lesssim}_{E} \; E^E[\mathsf{TagErr}]$
4. $e^1_{1'} \rightarrow^*_{1\text{-D}} v^1_1$ and $v^1_1 \; {}_1{\lesssim}_{E} \; v^E_1$
 $\vee\ e^1_{1'} \rightarrow^*_{1\text{-D}} \mathsf{BndryErr}$
 by 1–E *static value stutter*
5. **IF** $e^1_{1'} \rightarrow^*_{1\text{-D}} \mathsf{BndryErr}$:
 a. $e^1 \rightarrow^*_{1\text{-D}} \mathsf{BndryErr}$
 b. QED $E^1[\mathsf{BndryErr}] \; {}_1{\lesssim}_{E} \; E^E[\mathsf{TagErr}]$
6. $v^1_0 = \lambda x.\, e^1$ and $e^1 \; {}_1{\lesssim}_{E} \; e^E$
 by (2)
7. $v^1_0\ v^1_1 \rhd_{1\text{-D}} e^1[x \leftarrow v^1_1]$
 $\vee\ v^1_0\ v^1_1 \rhd_{1\text{-S}} e^1[x \leftarrow v^1_1]$
8. $e^1[x \leftarrow v^1_1] \; {}_1{\lesssim}_{E} \; e^E[x \leftarrow v^E_1]$
 by 1–E *substitution*
9. $E^1[e^1[x \leftarrow v^1_1]] \; {}_1{\lesssim}_{E} \; E^E[e^E[x \leftarrow v^E_1]]$
 by 1–E *hole substitution*
10. QED

CASE $(\lambda(x\!:\!\tau).\, e^E_0)\ v^E_1 \rhd_{E\text{-S}} e^E_0[x \leftarrow v^E_1]$:

1. $e^1_0 = e^1_{0'}\ e^1_{1'}$
   by $e^1_0 \ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}_0$
2. $e^1_{0'} \rightarrow^*_{\text{1-D}} v^1_0$ and $v^1_0 \ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}_0$
   $\vee\ e^1_{0'} \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
   by 1–E *dynamic value stutter*
3. **IF** $e^1_{0'} \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$:
   a. $e^1 \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
   b. QED $E^1[\mathsf{BndryErr}] \ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[\mathsf{TagErr}]$
4. $e^1_{1'} \rightarrow^*_{\text{1-D}} v^1_1$ and $v^1_1 \ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}_1$
   $\vee\ e^1_{1'} \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
   by 1–E *static value stutter*
5. **IF** $e^1_{1'} \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$:
   a. $e^1 \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
   b. QED $E^1[\mathsf{BndryErr}] \ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[\mathsf{TagErr}]$
6. $v^1_0 = \lambda(x\!:\!\tau).\, e^1$ and $e^1 \ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}$
   by (2)
7. $v^1_0\ v^1_1 \rhd_{\text{1-D}} e^1[x \leftarrow v^1_1]$
   $\vee v^1_0\ v^1_1 \rhd_{\text{1-S}} \mathsf{BndryErr}$ and $\mathcal{X}(\lfloor\tau\rfloor, v^1_1) = \mathsf{BndryErr}$
   $\vee\ v^1_0\ v^1_1 \rhd_{\text{1-S}} e^1[x \leftarrow \mathcal{X}(\lfloor\tau\rfloor, v^1_1)]$
8. **IF** $v^1_0\ v^1_1 \rhd_{\text{1-D}} e^1[x \leftarrow v^1_1]$:
   a. $e^1[x \leftarrow v^1_1] \ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}[x \leftarrow v^{\mathsf{E}}_1]$
      by 1–E *substitution*
   b. $E^1[e^1[x \leftarrow v^1_1]] \ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[e^{\mathsf{E}}[x \leftarrow v^{\mathsf{E}}_1]]$
      by 1–E *hole substitution*
   c. QED
   IF $v^1_0\ v^1_1 \rhd_{\text{1-S}} \mathsf{BndryErr}$:
   a. $e^1 \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$
   b. QED $E^1[\mathsf{BndryErr}] \ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[\mathsf{BndryErr}]$
   ELSE $v^1_0\ v^1_1 \rhd_{\text{1-S}} e^1[x \leftarrow \mathcal{X}(\lfloor\tau\rfloor, v^1_1)]$:
   a. $\mathcal{X}(\lfloor\tau\rfloor, v^1_1) = v^1_1$
   b. $e^1[x \leftarrow v^1_1] \ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}[x \leftarrow v^{\mathsf{E}}_1]$
      by 1–E *substitution*
   c. $E^1[e^1[x \leftarrow v^1_1]] \ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[e^{\mathsf{E}}[x \leftarrow v^{\mathsf{E}}_1]]$
      by 1–E *hole substitution*
   d. QED

CASE $op^1\ v^{\mathsf{E}}_0 \rhd_{\text{E-S}} \mathsf{TagErr}$:
1. $e^1_0 = op^1\ e^1_{0'}$
   by $e^1_0 \ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}_0$
2. $e^1_{0'} \rightarrow^*_{\text{1-D}} v^1_0$ and $v^1_0 \ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}_0$
   $\vee\ e^1_{0'} \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
   $\vee\ e^1_{0'} \rightarrow^*_{\text{1-S}} v^1_0$ and $v^1_0 \ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}_0$
   $\vee\ e^1_{0'} \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$
   by 1–E *dynamic value stutter* or 1–E *static value stutter*
3. **IF** $e^1_{0'} \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
   $\vee\ e^1_{0'} \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$:
   a. $e^1 \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
   b. QED $E^1[\mathsf{BndryErr}] \ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[\mathsf{TagErr}]$
   IF $e^1_{0'} \rightarrow^*_{\text{1-D}} v^1_0$:
   a. $v^{\mathsf{E}}_0 \notin \langle v, v\rangle$
      by definition $\rhd_{\text{E-S}}$
   b. $v^1_0 \notin \langle v, v\rangle$
      by $v^1_0 \ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}_0$
   c. $op^1\ v^1_0 \rhd_{\text{1-D}} \mathsf{TagErr}$
      by definition $\rhd_{\text{1-D}}$
   d. QED $E^1[\mathsf{TagErr}] \ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[\mathsf{TagErr}]$
   ELSE $e^1_{0'} \rightarrow^*_{\text{1-S}} v^1_0$:
   a. $\vdash_1 op^1\ e^1_{0'} : K_0$
   b. $\vdash_1 op^1\ v^1_0 : K_0$
      by 1 *static preservation*
   c. $v^{\mathsf{E}}_0 \notin \langle v, v\rangle$
      by definition $\rhd_{\text{E-S}}$
   d. $v^1_0 \notin \langle v, v\rangle$
      by $v^1_0 \ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}_0$
   e. Contradiction by (b)

CASE $op^1\ v^{\mathsf{E}}_0 \rhd_{\text{E-S}} \delta(op^1, v^{\mathsf{E}}_0)$:
1. $e^1_0 = op^1\ e^1_{0'}$
   by $e^1_0 \ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}_0$
2. $e^1_{0'} \rightarrow^*_{\text{1-D}} v^1_0$ and $v^1_0 \ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}_0$
   $\vee\ e^1_{0'} \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
   $\vee\ e^1_{0'} \rightarrow^*_{\text{1-S}} v^1_0$ and $v^1_0 \ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}_0$
   $\vee\ e^1_{0'} \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$
   by 1–E *dynamic value stutter* or 1–E *static value stutter*
3. **IF** $e^1_{0'} \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
   $\vee\ e^1_{0'} \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$:
   a. QED $E^1[\mathsf{BndryErr}] \ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[\mathsf{TagErr}]$
   IF $e^1_{0'} \rightarrow^*_{\text{1-D}} v^1_0$:
   a. $v^{\mathsf{E}}_0 \in \langle v, v\rangle$
      by definition $\rhd_{\text{E-S}}$
   b. $v^1_0 \in \langle v, v\rangle$
      by $v^1_0 \ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}_0$
   c. $op^1\ v^1_0 \rhd_{\text{1-D}} \delta(op^1, v^1_0)$
      by definition $\rhd_{\text{1-D}}$
   d. $\delta(op^1, v^1_0) \ {}_1{\lesssim}_{\mathsf{E}}\ \delta(op^1, v^{\mathsf{E}}_0)$
      by 1–E *delta*
   e. QED $E^1[\delta(op^1, v^1_0)] \ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[\delta(op^1, v^{\mathsf{E}}_0)]$
   ELSE $e^1_{0'} \rightarrow^*_{\text{1-S}} v^1_0$:
   a. $v^{\mathsf{E}}_0 \in \langle v, v\rangle$
      by definition $\rhd_{\text{E-S}}$
   b. $v^1_0 \in \langle v, v\rangle$
      by $v^1_0 \ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}_0$
   c. $op^1\ v^1_0 \rhd_{\text{1-S}} \delta(op^1, v^1_0)$
      by definition $\rhd_{\text{1-S}}$
   d. $\delta(op^1, v^1_0) \ {}_1{\lesssim}_{\mathsf{E}}\ \delta(op^1, v^{\mathsf{E}}_0)$
      by 1–E *delta*
   e. QED $E^1[\delta(op^1, v^1_0)] \ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[\delta(op^1, v^{\mathsf{E}}_0)]$

CASE $op^2\ v^{\mathsf{E}}_0\ v^{\mathsf{E}}_1 \rhd_{\text{E-S}} \mathsf{TagErr}$:
1. $e^1_0 = op^2\ e^1_{0'}\ e^1_{1'}$
   by $e^1_0 \ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}_0$
2. $e^1_{0'} \rightarrow^*_{\text{1-D}} v^1_0$ and $v^1_0 \ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}_0$
   $\vee\ e^1_{0'} \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
   $\vee\ e^1_{0'} \rightarrow^*_{\text{1-S}} v^1_0$ and $v^1_0 \ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}_0$

$\vee\ e^1_{0'} \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$
by 1–E *dynamic value stutter* or 1–E *static value stutter*

3. $e^1_{1'} \rightarrow^*_{\text{1-D}} v^1_1$ and $v^1_1 \;{}_1{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}}_1$
$\vee\ e^1_{1'} \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
$\vee\ e^1_{1'} \rightarrow^*_{\text{1-S}} v^1_1$ and $v^1_1 \;{}_1{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}}_1$
$\vee\ e^1_{1'} \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$
by 1–E *dynamic value stutter* or 1–E *static value stutter*
4. **IF** $e^1_{0'} \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
$\vee\ e^1_{0'} \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$
$\vee\ e^1_{1'} \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
$\vee\ e^1_{1'} \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$:
a. QED $E^1[\mathsf{BndryErr}] \;{}_1{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}[\mathsf{TagErr}]$
IF $e^1_{0'} \rightarrow^*_{\text{1-D}} v^1_0$
$\wedge\ e^1_{1'} \rightarrow^*_{\text{1-D}} v^1_1$:
a. $v^{\mathsf{E}}_0 \notin \mathbb{Z}$
$\vee\ v^{\mathsf{E}}_1 \notin \mathbb{Z}$
by definition $\rhd_{\text{E-S}}$
b. $v^1_0 \notin \mathbb{Z}$
$\vee\ v^1_1 \notin \mathbb{Z}$
by (a)
c. $op^2\ v^1_0\ v^1_1 \rhd_{\text{1-D}} \mathsf{TagErr}$
by definition $\rhd_{\text{1-D}}$
d. QED $E^1[\mathsf{TagErr}] \;{}_1{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}[\mathsf{TagErr}]$
ELSE $e^1_{0'} \rightarrow^*_{\text{1-S}} v^1_0$
$\wedge\ e^{\mathsf{E}}_{1'} \rightarrow^*_{\text{1-S}} v^1_1$:
a. $\vdash_1 op^2\ e^1_{0'}\ e^1_{1'} : K_0$
b. $\vdash_1 op^2\ v^1_0\ v^1_1 : K_0$
by 1 *static preservation*
c. $v^{\mathsf{E}}_0 \notin \mathbb{Z}$
$\vee\ v^{\mathsf{E}}_1 \notin \mathbb{Z}$
by definition $\rhd_{\text{E-S}}$
d. $v^1_0 \notin \mathbb{Z}$
$\vee\ v^1_1 \notin \mathbb{Z}$
by (c)
e. Contradiction by (b)

CASE $op^2\ v^{\mathsf{E}}_0\ v^{\mathsf{E}}_1 \rhd_{\text{E-S}} \delta(op^2, v^{\mathsf{E}}_0, v^{\mathsf{E}}_1)$:
1. $e^1_0 = op^2\ e^1_{0'}\ e^1_{1'}$
by $e^1_0 \;{}_1{\lesssim}_{\mathsf{E}}\; e^{\mathsf{E}}_0$
2. $e^1_{0'} \rightarrow^*_{\text{1-D}} v^1_0$ and $v^1_0 \;{}_1{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}}_0$
$\vee\ e^1_{0'} \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
$\vee\ e^1_{0'} \rightarrow^*_{\text{1-S}} v^1_0$ and $v^1_0 \;{}_1{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}}_0$
$\vee\ e^1_{0'} \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$
by 1–E *dynamic value stutter* or 1–E *static value stutter*
3. $e^1_{1'} \rightarrow^*_{\text{1-D}} v^1_1$ and $v^1_1 \;{}_1{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}}_1$
$\vee\ e^1_{1'} \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
$\vee\ e^1_{1'} \rightarrow^*_{\text{1-S}} v^1_1$ and $v^1_1 \;{}_1{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}}_1$
$\vee\ e^1_{1'} \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$
by 1–E *dynamic value stutter* or 1–E *static value stutter*
4. **IF** $e^1_{0'} \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
$\vee\ e^1_{0'} \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$
$\vee\ e^1_{1'} \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
$\vee\ e^1_{1'} \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$:
a. QED $E^1[\mathsf{BndryErr}] \;{}_1{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}[\mathsf{TagErr}]$
IF $e^1_{0'} \rightarrow^*_{\text{1-D}} v^1_0$
$\wedge\ e^1_{1'} \rightarrow^*_{\text{1-D}} v^1_1$:
a. $v^{\mathsf{E}}_0 \in \mathbb{Z}$
$\wedge\ v^{\mathsf{E}}_1 \in \mathbb{Z}$
by definition $\rhd_{\text{E-S}}$
b. $v^1_0 \in \mathbb{Z}$
by $v^1_0 \;{}_1{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}}_0$
c. $v^1_1 \in \mathbb{Z}$
by $v^1_1 \;{}_1{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}}_1$
d. $op^2\ v^1_0\ v^1_1 \rhd_{\text{1-D}} \delta(op^2, v^1_0, v^1_1)$
by definition $\rhd_{\text{1-D}}$
e. $\delta(op^2, v^1_0, v^1_1) \;{}_1{\lesssim}_{\mathsf{E}}\; \delta(op^2, v^{\mathsf{E}}_0, v^{\mathsf{E}}_1)$
by 1–E *delta*
f. QED $E^1[\delta(op^2, v^1_0, v^1_1)] \;{}_1{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}[\delta(op^2, v^{\mathsf{E}}_0, v^{\mathsf{E}}_1)]$
ELSE $e^1_{0'} \rightarrow^*_{\text{1-S}} v^1_0$
$\wedge\ e^{\mathsf{E}}_{1'} \rightarrow^*_{\text{1-S}} v^1_1$:
a. $v^{\mathsf{E}}_0 \in \mathbb{Z}$
$\wedge\ v^{\mathsf{E}}_1 \in \mathbb{Z}$
by definition $\rhd_{\text{E-S}}$
b. $v^1_0 \in \mathbb{Z}$
by $v^1_0 \;{}_1{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}}_0$
c. $v^1_1 \in \mathbb{Z}$
by $v^1_1 \;{}_1{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}}_1$
d. $op^2\ v^1_0\ v^1_1 \rhd_{\text{1-S}} \delta(op^2, v^1_0, v^1_1)$
by definition $\rhd_{\text{1-S}}$
e. $\delta(op^2, v^1_0, v^1_1) \;{}_1{\lesssim}_{\mathsf{E}}\; \delta(op^2, v^{\mathsf{E}}_0, v^{\mathsf{E}}_1)$
by 1–E *delta*
f. QED $E^1[\delta(op^2, v^1_0, v^1_1)] \;{}_1{\lesssim}_{\mathsf{E}}\; E^{\mathsf{E}}[\delta(op^2, v^{\mathsf{E}}_0, v^{\mathsf{E}}_1)]$

□

Lemma 8.5 : 1–E *static value stutter*

If $e^1 \;{}_1{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}}$ and $\vdash_1 e^1 : K$ then one of the following holds:
- $e^1 \rightarrow^*_{\text{1-S}} v^1$ and $v^1 \;{}_1{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}}$
- $e^1 \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$

Proof:

By induction on the structure of e^1, and case analysis on $e^1 \;{}_1{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}}$.

CASE e^1 is a value :
1. QED

CASE $e^1 = \mathsf{chk}\ K\ e^1_0$:
1. $e^1_0 \;{}_1{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}}$
by definition ${}_1{\lesssim}_{\mathsf{E}}$
2. $\vdash_1 e^1_0 : \mathsf{Any}$
3. $e^1_0 \rightarrow^*_{\text{1-S}} v^1_0$ and $v^1_0 \;{}_1{\lesssim}_{\mathsf{E}}\; v^{\mathsf{E}}$
$\vee\ e^1_0 \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$
by the induction hypothesis
4. **IF** $e^1_0 \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$:
a. QED $e^1 \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$

IF $e_0^1 \rightarrow^*_{\text{1-S}} v_0^1$
$\wedge\ \mathcal{X}(K, e_0^1) = \mathsf{BndryErr}$:
a. QED $e^1 \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$
ELSE $e_0^1 \rightarrow^*_{\text{1-S}} v_0^1$
$\wedge\ \mathcal{X}(K, e_0^1) = v_0^1$:
a. QED $e^1 \rightarrow^*_{\text{1-S}} v_0^1$
CASE $e^1 = \mathsf{dyn}\ e_0^1$:
1. $e_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}$
by definition ${}_1{\lesssim}_{\mathsf{E}}$
2. $\vdash_1 e_0^1$
3. $e_0^1 \rightarrow^*_{\text{1-D}} v_0^1$ and $v_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}$
$\vee\ e_0^1 \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
by 1–E *dynamic value stutter*
4. **IF** $e_0^1 \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$:
a. QED $e^1 \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$
ELSE :
a. QED $e^1 \rightarrow^*_{\text{1-S}} v_0^1$
CASE $e^1 = \mathsf{stat}\ v_0^1$:
1. Contradiction by $\vdash_1 e^1 : K$

□

Lemma 8.6 : 1–E *dynamic value stutter*

If $e^1\ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}$ and $\vdash_1 e^1$ then one of the following holds:
- $e^1 \rightarrow^*_{\text{1-D}} v^1$ and $v^1\ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}$
- $e^1 \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$

Proof:
By induction on the structure of e^1, and case analysis on $e^1\ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}$.
CASE e^1 is a value :
1. QED
CASE $e^1 = \mathsf{chk}\ K\ e_0^1$:
1. Contradiction by $\vdash_1 e^1$
CASE $e^1 = \mathsf{dyn}\ e_0^1$:
1. Contradiction by $\vdash_1 e^1$
CASE $e^1 = \mathsf{stat}\ v_0^1$:
1. $e_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}$
by definition ${}_1{\lesssim}_{\mathsf{E}}$
2. $\vdash_1 e_0^1 : \mathsf{Any}$
3. $e_0^1 \rightarrow^*_{\text{1-S}} v_0^1$ and $v_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}$
$\vee\ e_0^1 \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$
by 1–E *static value stutter*
4. **IF** $e_0^1 \rightarrow^*_{\text{1-S}} \mathsf{BndryErr}$:
a. QED $e^1 \rightarrow^*_{\text{1-D}} \mathsf{BndryErr}$
ELSE :
a. QED $e^1 \rightarrow^*_{\text{1-D}} v_0^1$

□

Lemma 8.7 : 1–E *hole substitution*

If $e^1\ {}_1{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}$ and $E^1\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}$ then $E^1[e^1]\ {}_1{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[e^{\mathsf{E}}]$

Proof:
By induction on the structure of E^1.
CASE $E^1 = [\,]$:
1. $e^{\mathsf{E}} = [\,]$
2. $E^1[e^1] = e^1$
$\wedge\ E^{\mathsf{E}}[e^{\mathsf{E}}] = e^{\mathsf{E}}$
3. QED
CASE $E^1 = E_0^1\ e_1^1$:
1. $E^{\mathsf{E}} = E_0^{\mathsf{E}}\ e_1^{\mathsf{E}}$
$\wedge\ E_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ E_0^{\mathsf{E}}$
$\wedge\ e_1^1\ {}_1{\lesssim}_{\mathsf{E}}\ e_1^{\mathsf{E}}$
by definition ${}_1{\lesssim}_{\mathsf{E}}$
2. $E_0^1[e^1]\ {}_1{\lesssim}_{\mathsf{E}}\ E_0^{\mathsf{E}}[e^{\mathsf{E}}]$
by the induction hypothesis
3. $E^1[e^1] = E_0^1[e_0^1]\ e_1^1$
4. $E^{\mathsf{E}}[e^{\mathsf{E}}] = E_0^{\mathsf{E}}[e_0^{\mathsf{E}}]\ e_1^{\mathsf{E}}$
5. QED by (1, 2)
CASE $E^1 = v_0^1\ E_1^1$:
1. $E^{\mathsf{E}} = e_0^{\mathsf{E}}\ E_1^{\mathsf{E}}$
$\wedge\ e_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}}$
$\wedge\ E_1^1\ {}_1{\lesssim}_{\mathsf{E}}\ E_1^{\mathsf{E}}$
by definition ${}_1{\lesssim}_{\mathsf{E}}$
2. $E_1^1[e^1]\ {}_1{\lesssim}_{\mathsf{E}}\ E_1^{\mathsf{E}}[e^{\mathsf{E}}]$
by the induction hypothesis
3. $E^1[e^1] = e_0^1\ E_1^1[e_1^1]$
4. $E^{\mathsf{E}}[e^{\mathsf{E}}] = e_0^{\mathsf{E}}\ E_1^{\mathsf{E}}[e_1^{\mathsf{E}}]$
5. QED by (1, 2)
CASE $E^1 = \langle E_0^1, e_1^1 \rangle$:
1. $E^{\mathsf{E}} = \langle E_0^{\mathsf{E}}, e_1^{\mathsf{E}} \rangle$
$\wedge\ E_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ E_0^{\mathsf{E}}$
$\wedge\ e_1^1\ {}_1{\lesssim}_{\mathsf{E}}\ e_1^{\mathsf{E}}$
by definition ${}_1{\lesssim}_{\mathsf{E}}$
2. $E_0^1[e^1]\ {}_1{\lesssim}_{\mathsf{E}}\ E_0^{\mathsf{E}}[e^{\mathsf{E}}]$
by the induction hypothesis
3. $E^1[e^1] = \langle E_0^1[e_0^1], e_1^1 \rangle$
4. $E^{\mathsf{E}}[e^{\mathsf{E}}] = \langle E_0^{\mathsf{E}}[e_0^{\mathsf{E}}], e_1^{\mathsf{E}} \rangle$
5. QED by (1, 2)
CASE $E^1 = \langle v_0^1, E_1^1 \rangle$:
1. $E^{\mathsf{E}} = \langle e_0^{\mathsf{E}}, E_1^{\mathsf{E}} \rangle$
$\wedge\ e_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}}$
$\wedge\ E_1^1\ {}_1{\lesssim}_{\mathsf{E}}\ E_1^{\mathsf{E}}$
by definition ${}_1{\lesssim}_{\mathsf{E}}$
2. $E_1^1[e^1]\ {}_1{\lesssim}_{\mathsf{E}}\ E_1^{\mathsf{E}}[e^{\mathsf{E}}]$
by the induction hypothesis
3. $E^1[e^1] = \langle e_0^1, E_1^1[e_1^1] \rangle$
4. $E^{\mathsf{E}}[e^{\mathsf{E}}] = \langle e_0^{\mathsf{E}}, E_1^{\mathsf{E}}[e_1^{\mathsf{E}}] \rangle$
5. QED by (1, 2)
CASE $E^1 = op^1\ E_0^1$:
1. $E^{\mathsf{E}} = op^1\ E_0^{\mathsf{E}}$
$\wedge\ E_0^1\ {}_1{\lesssim}_{\mathsf{E}}\ E_0^{\mathsf{E}}$
by definition ${}_1{\lesssim}_{\mathsf{E}}$
2. $E_0^1[e^1]\ {}_1{\lesssim}_{\mathsf{E}}\ E_0^{\mathsf{E}}[e^{\mathsf{E}}]$
by the induction hypothesis
3. $E^1[e^1] = op^1\ E_0^1[e_0^1]$
4. $E^{\mathsf{E}}[e^{\mathsf{E}}] = op^1\ E_0^{\mathsf{E}}[e_0^{\mathsf{E}}]$
5. QED by (1, 2)
CASE $E^1 = op^2\ E_0^1\ e_1^1$:

1. $E^{\mathsf{E}} = op^2\ E_0^{\mathsf{E}}\ e_1^{\mathsf{E}}$
   $\wedge\ E_0^1\ {}_{1}{\lesssim}_{\mathsf{E}}\ E_0^{\mathsf{E}}$
   $\wedge\ e_1^1\ {}_{1}{\lesssim}_{\mathsf{E}}\ e_1^{\mathsf{E}}$
   by definition ${}_{1}{\lesssim}_{\mathsf{E}}$
2. $E_0^1[e^1]\ {}_{1}{\lesssim}_{\mathsf{E}}\ E_0^{\mathsf{E}}[e^{\mathsf{E}}]$
   by the induction hypothesis
3. $E^1[e^1] = op^2\ E_0^1[e_0^1]\ e_1^1$
4. $E^{\mathsf{E}}[e^{\mathsf{E}}] = op^2\ E_0^{\mathsf{E}}[e_0^{\mathsf{E}}]\ e_1^{\mathsf{E}}$
5. QED by (1, 2)

CASE $E^1 = op^2\ v_0^1\ E_1^1$:
1. $E^{\mathsf{E}} = op^2\ e_0^{\mathsf{E}}\ E_1^{\mathsf{E}}$
   $\wedge\ e_0^1\ {}_{1}{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}}$
   $\wedge\ E_1^1\ {}_{1}{\lesssim}_{\mathsf{E}}\ E_1^{\mathsf{E}}$
   by definition ${}_{1}{\lesssim}_{\mathsf{E}}$
2. $E_1^1[e^1]\ {}_{1}{\lesssim}_{\mathsf{E}}\ E_1^{\mathsf{E}}[e^{\mathsf{E}}]$
   by the induction hypothesis
3. $E^1[e^1] = op^2\ e_0^1\ E_1^1[e_1^1]$
4. $E^{\mathsf{E}}[e^{\mathsf{E}}] = op^2\ e_0^{\mathsf{E}}\ E_1^{\mathsf{E}}[e_1^{\mathsf{E}}]$
5. QED by (1, 2)

CASE $E^1 = \mathsf{dyn}\ \tau\ E_0^1$:
1. $E^{\mathsf{E}} = \mathsf{dyn}\ \tau\ E_0^{\mathsf{E}}$
   $\wedge\ E_0^1\ {}_{1}{\lesssim}_{\mathsf{E}}\ E_0^{\mathsf{E}}$
   by definition ${}_{1}{\lesssim}_{\mathsf{E}}$
2. $E_0^1[e^1]\ {}_{1}{\lesssim}_{\mathsf{E}}\ E_0^{\mathsf{E}}[e^{\mathsf{E}}]$
   by the induction hypothesis
3. $E^1[e^1] = \mathsf{dyn}\ \tau\ E_0^1[e_0^1]$
4. $E^{\mathsf{E}}[e^{\mathsf{E}}] = \mathsf{dyn}\ \tau\ E_0^{\mathsf{E}}[e_0^{\mathsf{E}}]$
5. QED by (1, 2)

CASE $E^1 = \mathsf{stat}\ \tau\ E_0^1$:
1. $E^{\mathsf{E}} = \mathsf{stat}\ \tau\ E_0^{\mathsf{E}}$
   $\wedge\ E_0^1\ {}_{1}{\lesssim}_{\mathsf{E}}\ E_0^{\mathsf{E}}$
   by definition ${}_{1}{\lesssim}_{\mathsf{E}}$
2. $E_0^1[e^1]\ {}_{1}{\lesssim}_{\mathsf{E}}\ E_0^{\mathsf{E}}[e^{\mathsf{E}}]$
   by the induction hypothesis
3. $E^1[e^1] = \mathsf{stat}\ \tau\ E_0^1[e_0^1]$
4. $E^{\mathsf{E}}[e^{\mathsf{E}}] = \mathsf{stat}\ \tau\ E_0^{\mathsf{E}}[e_0^{\mathsf{E}}]$
5. QED by (1, 2)

CASE $E^1 = \mathsf{chk}\ K\ E_0^1$:
1. $E_0^1\ {}_{1}{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}$
2. $E_0^1[e^1]\ {}_{1}{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[e^{\mathsf{E}}]$
   by the induction hypothesis (1)
3. $E^1[e^1] = \mathsf{chk}\ K\ E_0^1[e^1]$
4. QED (2)

CASE $E^1 = \mathsf{dyn}\ E_0^1$:
1. $E_0^1\ {}_{1}{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}$
2. $E_0^1[e^1]\ {}_{1}{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[e^{\mathsf{E}}]$
   by the induction hypothesis (1)
3. $E^1[e^1] = \mathsf{dyn}\ E_0^1[e^1]$
4. QED (2)

CASE $E^1 = \mathsf{stat}\ E_0^1$:
1. $E_0^1\ {}_{1}{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}$
2. $E_0^1[e^1]\ {}_{1}{\lesssim}_{\mathsf{E}}\ E^{\mathsf{E}}[e^{\mathsf{E}}]$
   by the induction hypothesis (1)
3. $E^1[e^1] = \mathsf{stat}\ E_0^1[e^1]$
4. QED (2)

□

Lemma 8.8 : 1–E *substitution*

If $e^1\ {}_{1}{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}$ and $v^1\ {}_{1}{\lesssim}_{\mathsf{E}}\ v^{\mathsf{E}}$ then $e^1[x\leftarrow v^1]\ {}_{1}{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}[x\leftarrow v^{\mathsf{E}}]$

Proof:

By induction on the structure of e^1.

CASE $e^1 = x$:
1. $e^{\mathsf{E}} = x$
   by $e^1\ {}_{1}{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}$
2. $e^1[x\leftarrow v^1] = v^1$
   $\wedge\ e^{\mathsf{E}}[x\leftarrow v^{\mathsf{E}}] = v^{\mathsf{E}}$
3. QED

CASE $e^1 = y$:
1. $e^{\mathsf{E}} = y$
   by $e^1\ {}_{1}{\lesssim}_{\mathsf{E}}\ e^{\mathsf{E}}$
2. $e^1[x\leftarrow v^1] = y$
   $\wedge\ e^{\mathsf{E}}[x\leftarrow v^{\mathsf{E}}] = y$
3. QED (1)

CASE $e^1 = i$:
1. $e^{\mathsf{E}} = i$
   by definition ${}_{1}{\lesssim}_{\mathsf{E}}$
2. $e^1[x\leftarrow v^1] = i$
   $\wedge\ e^{\mathsf{E}}[x\leftarrow v^{\mathsf{E}}] = i$
3. QED

CASE $e^1 = \lambda x.\, e_0^1$:
1. $e^{\mathsf{E}} = \lambda x.\, e_0^{\mathsf{E}}$
   $\wedge\ e_0^1\ {}_{1}{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}}$
   by definition ${}_{1}{\lesssim}_{\mathsf{E}}$
2. $e^1[x\leftarrow v^1] = \lambda x.\, e_0^1$
   $\wedge\ e^{\mathsf{E}}[x\leftarrow v^{\mathsf{E}}] = \lambda x.\, e_0^{\mathsf{E}}$
3. QED

CASE $e^1 = \lambda(x\!:\!\tau).\, e_0^1$:
1. $e^{\mathsf{E}} = \lambda(x\!:\!\tau).\, e_0^{\mathsf{E}}$
   $\wedge\ e_0^1\ {}_{1}{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}}$
   by definition ${}_{1}{\lesssim}_{\mathsf{E}}$
2. $e^1[x\leftarrow v^1] = \lambda(x\!:\!\tau).\, e_0^1$
   $\wedge\ e^{\mathsf{E}}[x\leftarrow v^{\mathsf{E}}] = \lambda(x\!:\!\tau).\, e_0^{\mathsf{E}}$
3. QED

CASE $e^1 = \lambda y.\, e_0^1$:
1. $e^{\mathsf{E}} = \lambda y.\, e_0^{\mathsf{E}}$
   $\wedge\ e_0^1\ {}_{1}{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}}$
   by definition ${}_{1}{\lesssim}_{\mathsf{E}}$
2. $e_0^1[x\leftarrow v^1]\ {}_{1}{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}}[x\leftarrow v^{\mathsf{E}}]$
   by the induction hypothesis (1)
3. $e^1[x\leftarrow v^1] = \lambda y.\, e_0^1[x\leftarrow v^{\mathsf{E}}]$
   $\wedge\ e^{\mathsf{E}}[x\leftarrow v^{\mathsf{E}}] = \lambda y.\, e_0^{\mathsf{E}}[x\leftarrow v^{\mathsf{E}}]$
4. QED (2, 3)

CASE $e^1 = \lambda(y\!:\!\tau).\, e_0^1$:
1. $e^{\mathsf{E}} = \lambda(y\!:\!\tau).\, e_0^{\mathsf{E}}$
   $\wedge\ e_0^1\ {}_{1}{\lesssim}_{\mathsf{E}}\ e_0^{\mathsf{E}}$
   by definition ${}_{1}{\lesssim}_{\mathsf{E}}$

2. $e_0^1[x \leftarrow v^1] \;{}_1{\lesssim}_E\; e_0^E[x \leftarrow v^E]$
   by the induction hypothesis (1)
3. $e^1[x \leftarrow v^1] = \lambda(y:\tau).\, e_0^1[x \leftarrow v^E]$
   $\wedge\; e^E[x \leftarrow v^E] = \lambda(y:\tau).\, e_0^E[x \leftarrow v^E]$
4. QED (2, 3)

CASE $e^1 = \mathsf{mon}\ \tau\ v_0^1$:

1. Contradiction by $e^1 \;{}_1{\lesssim}_E\; e^E$

CASE $e^1 = \langle e_0^1, e_1^1 \rangle$:

1. $e^E = \langle e_0^E, e_1^E \rangle$
2. $e_0^1[x \leftarrow v^1] \;{}_1{\lesssim}_E\; e_0^E[x \leftarrow v^E]$
   $\wedge\; e_1^1[x \leftarrow v^1] \;{}_1{\lesssim}_E\; e_1^E[x \leftarrow v^E]$
   by the induction hypothesis
3. $e^1[x \leftarrow v^1] = \langle e_0^1[x \leftarrow v^1], e_1^1[x \leftarrow v^1] \rangle$
   $\wedge\; e^E[x \leftarrow v^E] = \langle e_0^E[x \leftarrow v^E], e_1^E[x \leftarrow v^E] \rangle$
4. QED (2, 3)

CASE $e^1 = e_0^1\ e_1^1$:

1. $e^E = e_0^E\ e_1^E$
2. $e_0^1[x \leftarrow v^1] \;{}_1{\lesssim}_E\; e_0^E[x \leftarrow v^E]$
   $\wedge\; e_1^1[x \leftarrow v^1] \;{}_1{\lesssim}_E\; e_1^E[x \leftarrow v^E]$
   by the induction hypothesis
3. $e^1[x \leftarrow v^1] = e_0^1[x \leftarrow v^1]\ e_1^1[x \leftarrow v^1]$
   $\wedge\; e^E[x \leftarrow v^E] = e_0^E[x \leftarrow v^E]\ e_1^E[x \leftarrow v^E]$
4. QED (2, 3)

CASE $e^1 = op^1\ e_0^1$:

1. $e^E = op^1\ e_0^E$
2. $e_0^1[x \leftarrow v^1] \;{}_1{\lesssim}_E\; e_0^E[x \leftarrow v^E]$
   by the induction hypothesis
3. $e^1[x \leftarrow v^1] = op^1\ e_0^1[x \leftarrow v^1]$
   $\wedge\; e^E[x \leftarrow v^E] = op^1\ e_0^E[x \leftarrow v^E]$
4. QED (2, 3)

CASE $e^1 = op^2\ e_0^1\ e_1^1$:

1. $e^E = op^2\ e_0^E\ e_1^E$
2. $e_0^1[x \leftarrow v^1] \;{}_1{\lesssim}_E\; e_0^E[x \leftarrow v^E]$
   $\wedge\; e_1^1[x \leftarrow v^1] \;{}_1{\lesssim}_E\; e_1^E[x \leftarrow v^E]$
   by the induction hypothesis
3. $e^1[x \leftarrow v^1] = op^2\ e_0^1[x \leftarrow v^1]\ e_1^1[x \leftarrow v^1]$
   $\wedge\; e^E[x \leftarrow v^E] = op^2\ e_0^E[x \leftarrow v^E]\ e_1^E[x \leftarrow v^E]$
4. QED (2, 3)

CASE $e^1 = \mathsf{dyn}\ \tau\ e_0^1$:

1. $e^E = \mathsf{dyn}\ \tau\ e_0^E$
2. $e_0^1[x \leftarrow v^1] \;{}_1{\lesssim}_E\; e_0^E[x \leftarrow v^E]$
   by the induction hypothesis
3. $e^1[x \leftarrow v^1] = \mathsf{dyn}\ \tau\ e_0^1[x \leftarrow v^1]$
   $\wedge\; e^E[x \leftarrow v^E] = \mathsf{dyn}\ \tau\ e_0^E[x \leftarrow v^E]$
4. QED (2, 3)

CASE $e^1 = \mathsf{stat}\ \tau\ e_0^1$:

1. $e^E = \mathsf{stat}\ \tau\ e_0^E$
2. $e_0^1[x \leftarrow v^1] \;{}_1{\lesssim}_E\; e_0^E[x \leftarrow v^E]$
   by the induction hypothesis
3. $e^1[x \leftarrow v^1] = \mathsf{stat}\ \tau\ e_0^1[x \leftarrow v^1]$
   $\wedge\; e^E[x \leftarrow v^E] = \mathsf{stat}\ \tau\ e_0^E[x \leftarrow v^E]$
4. QED (2, 3)

CASE $e^1 = \mathsf{Err}$:

1. QED $e^1[x \leftarrow v^1] = \mathsf{Err}$

CASE $e^1 = \mathsf{chk}\ K\ e_0^1$:

1. $e_0^1 \;{}_1{\lesssim}_E\; e^E$
2. $e_0^1[x \leftarrow v^1] \;{}_1{\lesssim}_E\; e^E[x \leftarrow v^E]$
   by the induction hypothesis
3. $e^1[x \leftarrow v^1] = \mathsf{chk}\ K\ e_0^1[x \leftarrow v^1]$
4. QED (2, 3)

CASE $e^1 = \mathsf{dyn}\ e_0^1$:

1. $e_0^1 \;{}_1{\lesssim}_E\; e^E$
2. $e_0^1[x \leftarrow v^1] \;{}_1{\lesssim}_E\; e^E[x \leftarrow v^E]$
   by the induction hypothesis
3. $e^1[x \leftarrow v^1] = \mathsf{dyn}\ e_0^1[x \leftarrow v^1]$
4. QED (2, 3)

CASE $e^1 = \mathsf{stat}\ e_0^1$:

1. $e_0^1 \;{}_1{\lesssim}_E\; e^E$
2. $e_0^1[x \leftarrow v^1] \;{}_1{\lesssim}_E\; e^E[x \leftarrow v^E]$
   by the induction hypothesis
3. $e^1[x \leftarrow v^1] = \mathsf{stat}\ e_0^1[x \leftarrow v^1]$
4. QED (2, 3)

□

Lemma 8.9 : 1–E *δ-preservation*

- If $v^1 \;{}_1{\lesssim}_E\; v^E$ and $\delta(op^1, v^1)$ is defined then $\delta(op^1, v^1) \;{}_1{\lesssim}_E\; \delta(op^1, v^E)$
- If $v_0^1 \;{}_1{\lesssim}_E\; v_0^E$ and $v_1^1 \;{}_1{\lesssim}_E\; v_1^E$ and $\delta(op^2, v_0^1, v_1^1)$ is defined then $\delta(op^2, v_0^1, v_1^1) \;{}_1{\lesssim}_E\; \delta(op^2, v_0^E, v_1^E)$

Proof:

CASE $op^1 = \mathsf{fst}$:

1. $v^1 = \langle v_0^1, v_1^1 \rangle$
   by $\delta(\mathsf{fst}, v^1)$ is defined
2. $v^E = \langle v_0^E, v_1^E \rangle$
   $\wedge\; v_0^1 \;{}_1{\lesssim}_E\; v_0^1$ and $v_1^1 \;{}_1{\lesssim}_E\; v_1^E$
   by ${}_1{\lesssim}_E$
3. $\delta(\mathsf{fst}, v^1) = v_0^1$
   $\wedge\; \delta(\mathsf{fst}, v^E) = v_0^E$
4. QED (2)

CASE $op^1 = \mathsf{snd}$:

1. $v^1 = \langle v_0^1, v_1^1 \rangle$
   by $\delta(\mathsf{snd}, v^1)$ is defined
2. $v^E = \langle v_0^E, v_1^E \rangle$
   $\wedge\; v_0^1 \;{}_1{\lesssim}_E\; v_0^E$ and $v_1^1 \;{}_1{\lesssim}_E\; v_1^E$
   by ${}_1{\lesssim}_E$
3. $\delta(\mathsf{snd}, v^1) = v_1^1$
   $\wedge\; \delta(\mathsf{snd}, v^E) = v_1^E$
4. QED (2)

CASE $op^2 = \mathsf{sum}$:

1. $v_0^1 \in \mathbb{Z}$
   $\wedge\; v_1^1 \in \mathbb{Z}$
   by $\delta(op^2, v_0^1, v_1^1)$ is defined
2. $v_0^1 = v_0^E$
   $\wedge\; v_1^1 = v_1^E$
   by ${}_1{\lesssim}_E$
3. QED

CASE op^2 = quotient :

1. $v_0^1 \in \mathbb{Z}$
   $\wedge\ v_1^1 \in \mathbb{Z}$
   by $\delta(op^2, v_0^1, v_1^1)$ is defined
2. $v_0^1 = v_0^{\mathsf{E}}$
   $\wedge\ v_1^1 = v_1^{\mathsf{E}}$
   by ${}_1\!\lesssim_{\mathsf{E}}$
3. QED

□

Lemma 8.10 : H−1 *approximation*

If $e \in e_{\mathsf{S}}$ and $\vdash e : \tau$ and $\vdash_1 e : \lfloor\tau\rfloor \rightsquigarrow e''$ and $e'' \rightarrow^*_{\text{1-S}}$ Err then $e \rightarrow^*_{\text{H-S}}$ Err

Proof:

- $e\ {}_{\mathsf{H}}\!\lesssim_1 e''$
  by H−1 *static reflexivity*
- QED by H−1 *simulation*

□

Lemma 8.11 : H−1 *static reflexivity*

If $\Gamma \vdash e : \tau$
$\wedge\ \Gamma \vdash_1 e : \lfloor\tau\rfloor \rightsquigarrow e''$
then $e\ {}_{\mathsf{H}}\!\lesssim_1 e''$.

Proof:

By structural induction on the $\Gamma \vdash_1 e : \lfloor\tau\rfloor \rightsquigarrow e''$ judgment.

CASE $\boxed{\dfrac{}{\Gamma \vdash i : \mathsf{Nat} \rightsquigarrow i}}$:

1. QED $i\ {}_{\mathsf{H}}\!\lesssim_1 i$

CASE $\boxed{\dfrac{}{\Gamma \vdash i : \mathsf{Int} \rightsquigarrow i}}$:

1. QED $i\ {}_{\mathsf{H}}\!\lesssim_1 i$

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e_0' \quad \Gamma \vdash e_1 : \tau_1 \rightsquigarrow e_1'}{\Gamma \vdash \langle e_0, e_1\rangle : \tau_0 \times \tau_1 \rightsquigarrow \langle e_0', e_1'\rangle}}$:

1. $e_0'\ {}_{\mathsf{H}}\!\lesssim_1 e_0$
   $\wedge\ e_1'\ {}_{\mathsf{H}}\!\lesssim_1 e_1$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{(x : \tau_d), \Gamma \vdash e : \tau_c \rightsquigarrow e'}{\Gamma \vdash \lambda(x : \tau_d).\, e : \tau_d \Rightarrow \tau_c \rightsquigarrow \lambda(x : \tau_d).\, e'}}$:

1. $e'\ {}_{\mathsf{H}}\!\lesssim_1 e$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash x : \tau \rightsquigarrow x}}$:

1. QED $x\ {}_{\mathsf{H}}\!\lesssim_1 x$

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \rightsquigarrow e_0' \quad \Gamma \vdash e_1 : \tau_d \rightsquigarrow e_1' \quad \lfloor\tau_c\rfloor = K}{\Gamma \vdash e_0\ e_1 : \tau_c \rightsquigarrow \mathsf{chk}\ K\ (e_0'\ e_1')}}$

:

1. $e_0'\ {}_{\mathsf{H}}\!\lesssim_1 e_0$
   $\wedge\ e_1'\ {}_{\mathsf{H}}\!\lesssim_1 e_1$
   by the induction hypothesis
2. $e_0'\ e_1'\ {}_{\mathsf{H}}\!\lesssim_1 e_0\ e_1$
3. QED $\mathsf{chk}\ K\ e_0'\ e_1'\ {}_{\mathsf{H}}\!\lesssim_1 e_0\ e_1$

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau_0 \times \tau_1 \rightsquigarrow e' \quad \lfloor\tau_0\rfloor = K}{\Gamma \vdash \mathsf{fst}\ e : \tau_0 \rightsquigarrow \mathsf{chk}\ K\ (\mathsf{fst}\ e')}}$:

1. $e'\ {}_{\mathsf{H}}\!\lesssim_1 e$
   by the induction hypothesis
2. $\mathsf{fst}\ e'\ {}_{\mathsf{H}}\!\lesssim_1 \mathsf{fst}\ e$
3. QED $\mathsf{chk}\ K\ \mathsf{fst}\ e'\ {}_{\mathsf{H}}\!\lesssim_1 \mathsf{fst}\ e$

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau_0 \times \tau_1 \rightsquigarrow e' \quad \lfloor \tau_1 \rfloor = K}{\Gamma \vdash \mathsf{snd}\ e : \tau_1 \rightsquigarrow \mathsf{chk}\ K\ (\mathsf{snd}\ e')}}$:

1. $e' \ {}_{\mathsf{H}}\!\lesssim_1 e$
   by the induction hypothesis
2. $\mathsf{snd}\ e' \ {}_{\mathsf{H}}\!\lesssim_1 \mathsf{snd}\ e$
3. QED $\mathsf{chk}\ K\ \mathsf{snd}\ e' \ {}_{\mathsf{H}}\!\lesssim_1 \mathsf{snd}\ e$

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_0 \rightsquigarrow e_0' \quad \Gamma \vdash e_1 : \tau_1 \rightsquigarrow e_1'}{\Gamma \vdash op^2\ e_0\ e_1 : \tau \rightsquigarrow op^2\ e_0'\ e_1'}}$:

1. $e_0' \ {}_{\mathsf{H}}\!\lesssim_1 e_0$
   $\wedge\ e_1' \ {}_{\mathsf{H}}\!\lesssim_1 e_1$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau' \rightsquigarrow e' \quad \tau' \leqslant: \tau}{\Gamma \vdash e : \tau \rightsquigarrow e'}}$:

1. QED by the induction hypothesis

CASE $\boxed{\dfrac{}{\Gamma \vdash \mathsf{Err} : \tau \rightsquigarrow \mathsf{Err}}}$:

1. QED $\mathsf{Err} \ {}_{\mathsf{H}}\!\lesssim_1 \mathsf{Err}$

CASE $\boxed{\dfrac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \mathsf{dyn}\ \tau\ e : \tau \rightsquigarrow \mathsf{dyn}\ \tau\ e'}}$:

1. $e' \ {}_{\mathsf{H}}\!\lesssim_1 e$
   by the induction hypothesis
2. QED

□

Lemma 8.12 : H–1 *dynamic reflexivity*

> If $\Gamma \vdash e$
> $\wedge\ \Gamma \vdash_1 e \rightsquigarrow e''$
> then $e \ {}_{\mathsf{H}}\!\lesssim_1 e''$.

Proof:

By structural induction on the $\Gamma \vdash_1 e \rightsquigarrow e''$ judgment.

CASE $\boxed{\dfrac{}{\Gamma \vdash i \rightsquigarrow i}}$:

1. QED $i \ {}_{\mathsf{H}}\!\lesssim_1 i$

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \rightsquigarrow e_0' \quad \Gamma \vdash e_1 \rightsquigarrow e_1'}{\Gamma \vdash \langle e_0, e_1 \rangle \rightsquigarrow \langle e_0', e_1' \rangle}}$:

1. $e_0' \ {}_{\mathsf{H}}\!\lesssim_1 e_0$
   $\wedge\ e_1' \ {}_{\mathsf{H}}\!\lesssim_1 e_1$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{x, \Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash \lambda x.\, e \rightsquigarrow \lambda x.\, e'}}$:

1. $e' \ {}_{\mathsf{H}}\!\lesssim_1 e$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash x \rightsquigarrow x}}$:

1. QED $x \ {}_{\mathsf{H}}\!\lesssim_1 x$

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \rightsquigarrow e_0' \quad \Gamma \vdash e_1 \rightsquigarrow e_1'}{\Gamma \vdash e_0\ e_1 \rightsquigarrow e_0'\ e_1'}}$:

1. $e_0' \ {}_{\mathsf{H}}\!\lesssim_1 e_0$
   $\wedge\ e_1' \ {}_{\mathsf{H}}\!\lesssim_1 e_1$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{\Gamma \vdash e \rightsquigarrow e'}{\Gamma \vdash op^1\ e \rightsquigarrow op^1\ e'}}$:

1. $e' \ {}_{\mathsf{H}}\!\lesssim_1 e$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{\Gamma \vdash e_0 \rightsquigarrow e_0' \quad \Gamma \vdash e_1 \rightsquigarrow e_1'}{\Gamma \vdash op^2\ e_0\ e_1 \rightsquigarrow op^2\ e_0'\ e_1'}}$:

1. $e_0' \ {}_{\mathsf{H}}\!\lesssim_1 e_0$
   $\wedge\ e_1' \ {}_{\mathsf{H}}\!\lesssim_1 e_1$
   by the induction hypothesis
2. QED

CASE $\boxed{\dfrac{}{\Gamma \vdash \mathsf{Err} \rightsquigarrow \mathsf{Err}}}$:

1. QED $\mathsf{Err} \ {}_{\mathsf{H}}\!\lesssim_1 \mathsf{Err}$

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau \rightsquigarrow e'}{\Gamma \vdash \mathsf{stat}\ \tau\ e \rightsquigarrow \mathsf{stat}\ \tau\ e'}}$:

1. $e' \ {}_{\mathsf{H}}\!\lesssim_1 e$
   by 1–E *static reflexivity*
2. QED

□

Lemma 8.13 : H–1 *simulation*

> If $\vdash_{\mathsf{H}} E^{\mathsf{H}}[e_0^{\mathsf{H}}] : \tau$
> $\wedge\ \vdash_1 E^1[e_0^1] : \lfloor \tau \rfloor$
> $\wedge\ E^{\mathsf{H}}[e_0^{\mathsf{H}}] \ {}_{\mathsf{H}}\!\lesssim_1 E^1[e_0^1]$
> $\wedge\ E^1[e_0^1] \rightarrow_{1\text{-}\mathsf{S}} E^1[e_1^1]$
> then $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H}\text{-}\mathsf{S}} E^{\mathsf{H}}[e_n^{\mathsf{H}}]$ and $E^{\mathsf{H}}[e_n^{\mathsf{H}}] \ {}_{\mathsf{H}}\!\lesssim_1 E^1[e_1^1]$

Proof:

By case analysis on $E^1[e_0^1] \rightarrow_{1\text{-}\mathsf{S}} E^1[e_1^1]$

CASE $E^1[\mathsf{dyn}\ v_0^1] \rightarrow_{1\text{-}\mathsf{S}} E^1[v_0^1]$:

1. $e_0^{\mathsf{H}} = \mathsf{dyn}\ \tau_0\ e_{0'}^{\mathsf{H}}$
   $\wedge\ e_{0'}^{\mathsf{H}} \ {}_{\mathsf{H}}\!\lesssim_1 v_0^1$
   by ${}_{\mathsf{H}}\!\lesssim_1$
2. $\vdash_{\mathsf{H}} e_{0'}^{\mathsf{H}}$
   by $\vdash_{\mathsf{H}} E^{\mathsf{H}}[e_0^{\mathsf{H}}] : \tau$
3. $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H}\text{-}\mathsf{S}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}} \ {}_{\mathsf{H}}\!\lesssim_1 v_0^1$
   $\vee\ e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H}\text{-}\mathsf{S}} \mathsf{BndryErr}$
   by H–1 *static value stutter* (1, 2)
4. **IF** $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H}\text{-}\mathsf{S}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}} \ {}_{\mathsf{H}}\!\lesssim_1 v_0^1$:

a. $\vdash_{\mathsf{H}} v_0^{\mathsf{H}}$
by H *static preservation*
b. $\mathsf{dyn}\ \tau_0\ v_0^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} e_{1'}^{\mathsf{H}}$
$\wedge\ e_{1'}^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$
$\wedge\ e_{1'}^{\mathsf{H}} \in v$ or $e_1^{\mathsf{H}} = \mathsf{BndryErr}$
by H−1 *boundary checking*
c. QED $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} E^{\mathsf{H}}[e_1^{\mathsf{H}}]$
ELSE $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$:
a. QED $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

CASE $E^{1}[\mathsf{stat}\ v_0^{1}] \rightarrow_{\mathsf{1\text{-}S}} E^{1}[v_0^{1}]$:
1. $e_0^{\mathsf{H}} = \mathsf{stat}\ \tau_0\ e_{0'}^{\mathsf{H}}$
$\wedge\ e_{0'}^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$
by ${}_{\mathsf{H}}{\lesssim}_{1}$
2. $\vdash_{\mathsf{H}} e_{0'}^{\mathsf{H}} : \tau_0$
by $\vdash_{\mathsf{H}} E^{\mathsf{H}}[e_0^{\mathsf{H}}] : \tau$
3. $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$
$\vee\ e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
by H−1 *dynamic value stutter* (1, 2)
4. **IF** $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$:
a. $\vdash_{\mathsf{H}} v_0^{\mathsf{H}} : \tau_0$
by H *dynamic preservation*
b. $\mathsf{stat}\ \tau_0\ v_0^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} e_{1'}^{\mathsf{H}}$
$\wedge\ e_{1'}^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$
$\wedge\ e_{1'}^{\mathsf{H}} \in v$ or $e_1^{\mathsf{H}} = \mathsf{BndryErr}$
by H−1 *boundary checking*
c. QED $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} E^{\mathsf{H}}[e_1^{\mathsf{H}}]$
ELSE $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$:
a. QED $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

CASE $E^{1}[\mathsf{dyn}\ \tau_0\ v_0^{1}] \rightarrow_{\mathsf{1\text{-}S}} E^{1}[\mathsf{BndryErr}]$:
1. $e_0^{\mathsf{H}} = \mathsf{dyn}\ \tau_0\ e_{0'}^{\mathsf{H}}$
$\wedge\ e_{0'}^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$
by ${}_{\mathsf{H}}{\lesssim}_{1}$
2. $\vdash_{\mathsf{H}} e_{0'}^{\mathsf{H}}$
by $\vdash_{\mathsf{H}} E^{\mathsf{H}}[e_0^{\mathsf{H}}] : \tau$
3. $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$
$\vee\ e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
by H−1 *static value stutter* (1, 2)
4. **IF** $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$:
a. $\mathcal{X}(\lfloor\tau_0\rfloor, v_0^{1}) = \mathsf{BndryErr}$
by $E^{1}[\mathsf{dyn}\ \tau_0\ v_0^{1}] \rightarrow_{\mathsf{1\text{-}S}} E^{1}[\mathsf{BndryErr}]$
b. $\mathcal{D}_{\mathsf{H}}(\tau_0, v_0^{\mathsf{H}}) = \mathsf{BndryErr}$
by $\mathcal{X}$ *inversion*
c. QED $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
ELSE $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$:
a. QED $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

CASE $E^{1}[\mathsf{dyn}\ \tau_0\ v_0^{1}] \rightarrow_{\mathsf{1\text{-}S}} E^{1}[v_0^{1}]$:
1. $e_0^{\mathsf{H}} = \mathsf{dyn}\ \tau_0\ e_{0'}^{\mathsf{H}}$
$\wedge\ e_{0'}^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$
by ${}_{\mathsf{H}}{\lesssim}_{1}$
2. $\vdash_{\mathsf{H}} e_{0'}^{\mathsf{H}}$
by $\vdash_{\mathsf{H}} E^{\mathsf{H}}[e_0^{\mathsf{H}}] : \tau$
3. $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$
$\vee\ e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
by H−1 *static value stutter* (1, 2)
4. **IF** $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$:
a. $\vdash_{\mathsf{H}} v_0^{\mathsf{H}}$
by H *static preservation*
b. $\mathsf{dyn}\ \tau_0\ v_0^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} e_{1'}^{\mathsf{H}}$
$\wedge\ e_{1'}^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$
$\wedge\ e_{1'}^{\mathsf{H}} \in v$ or $e_1^{\mathsf{H}} = \mathsf{BndryErr}$
by H−1 *boundary checking*
c. QED $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} E^{\mathsf{H}}[e_1^{\mathsf{H}}]$
ELSE $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$:
a. QED $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

CASE $E^{1}[\mathsf{stat}\ \tau_0\ v_0^{1}] \rightarrow_{\mathsf{1\text{-}S}} E^{1}[v_0^{1}]$:
1. $e_0^{\mathsf{H}} = \mathsf{stat}\ \tau_0\ e_{0'}^{\mathsf{H}}$
$\wedge\ e_{0'}^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$
by ${}_{\mathsf{H}}{\lesssim}_{1}$
2. $\vdash_{\mathsf{H}} e_{0'}^{\mathsf{H}} : \tau_0$
by $\vdash_{\mathsf{H}} E^{\mathsf{H}}[e_0^{\mathsf{H}}] : \tau$
3. $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$
$\vee\ e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
by H−1 *dynamic value stutter* (1, 2)
4. **IF** $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$:
a. $\vdash_{\mathsf{H}} v_0^{\mathsf{H}} : \tau_0$
by H *dynamic preservation*
b. $\mathsf{stat}\ \tau_0\ v_0^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} e_{1'}^{\mathsf{H}}$
$\wedge\ e_{1'}^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$
$\wedge\ e_{1'}^{\mathsf{H}} \in v$ or $e_1^{\mathsf{H}} = \mathsf{BndryErr}$
by H−1 *boundary checking*
c. QED $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} E^{\mathsf{H}}[e_1^{\mathsf{H}}]$
ELSE $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$:
a. QED $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

CASE $E^{1}[\mathsf{chk}\ K_0\ v_0^{1}] \rightarrow_{\mathsf{1\text{-}S}} E^{1}[\mathsf{BndryErr}]$:
1. $\vdash_{1} \mathsf{chk}\ K_0\ v_0^{1} : K_0$
by $\vdash_{1} E^{1}[e_0^{1}] : K$
2. $\vdash_{\mathsf{H}} e_0^{\mathsf{H}} : \tau_0$
by H−1 *static hole typing*
3. $e_0^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$
$\vee\ e_0^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
by H−1 *static value stutter* (2)
4. **IF** $e_0^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}$:
a. $\vdash_{1} v_0^{1} : \lfloor\tau_0\rfloor$
by H−1 *value inversion* (2, 3)
b. $\lfloor\tau_0\rfloor = K_0$
by chk *inversion*, H−1 *static reflexivity*, H *static preservation*, and 1 *static preservation*
c. $\mathsf{chk}\ K_0\ v_0^{1} \rhd_{\mathsf{1\text{-}S}} v_0^{1}$
by definition $\rhd_{\mathsf{1\text{-}S}}$
d. Contradiction by $E^{1}[\mathsf{chk}\ K_0\ v_0^{1}] \rightarrow_{\mathsf{1\text{-}S}} E^{1}[\mathsf{BndryErr}]$ (b)
ELSE $e_0^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$:
a. QED $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

case $E^1[\mathsf{chk}\ K_0\ v_0^1] \rightarrow_{1\text{-S}} E^1[v_0^1]$:
1. $\vdash_1 \mathsf{chk}\ K_0\ v_0^1 : K_0$
   by $\vdash_1 E^1[e_0^1] : K$
2. $\vdash_{\mathsf{H}} e_0^{\mathsf{H}} : \tau_0$
   by H−1 *static hole typing*
3. $e_0^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_0^1$
   $\vee\ e_0^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
   by H−1 *static value stutter* (2)
4. **if** $e_0^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_0^1$:
   a. qed $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} v_0^{\mathsf{H}}$
   else $e_0^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$:
   a. qed $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

case $E^1[v_0^1\ v_1^1] \rightarrow_{1\text{-S}} E^1[e_2^1]$
$\wedge \vdash_1 v_0^1\ v_1^1 : K'$:
1. $\vdash_{\mathsf{H}} e_0^{\mathsf{H}} : \tau_0$
   by H−1 *static hole typing*
2. $e_0^{\mathsf{H}} = e_{0'}^{\mathsf{H}}\ e_{1'}^{\mathsf{H}}$
   by ${}_{\mathsf{H}}\!\lesssim_1$
3. $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_0^1$
   $\vee\ e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
   by H−1 *static value stutter* (1, 2)
4. $e_{1'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} v_1^{\mathsf{H}}$ and $v_1^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_1^1$
   $\vee\ e_{1'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
   by H−1 *static value stutter* (1, 2)
5. **if** $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} v_0^{\mathsf{H}}$
   $\wedge\ e_{1'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} v_1^{\mathsf{H}}$:
   a. $v_0^{\mathsf{H}}\ v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} e_2^{\mathsf{H}}$
      $\wedge\ e_2^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 e_2^1$
      by H−1 *static application* (1)
   b. qed $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} E^{\mathsf{H}}[e_2^{\mathsf{H}}]$
   if $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$:
   a. qed $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
   else $e_{1'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$:
   a. qed $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

case $E^1[v_0^1\ v_1^1] \rightarrow_{1\text{-S}} E^1[e_2^1]$
$\wedge \vdash_1 v_0^1\ v_1^1$:
1. $\vdash_{\mathsf{H}} e_0^{\mathsf{H}}$
   by H−1 *static hole typing*
2. $e_0^{\mathsf{H}} = e_{0'}^{\mathsf{H}}\ e_{1'}^{\mathsf{H}}$
   by ${}_{\mathsf{H}}\!\lesssim_1$
3. $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_0^1$
   $\vee\ e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
   by H−1 *dynamic value stutter* (1, 2)
4. $e_{1'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} v_1^{\mathsf{H}}$ and $v_1^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_1^1$
   $\vee\ e_{1'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
   by H−1 *dynamic value stutter* (1, 2)
5. **if** $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} v_0^{\mathsf{H}}$
   $\wedge\ e_{1'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} v_1^{\mathsf{H}}$:
   a. $v_0^{\mathsf{H}}\ v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} e_2^{\mathsf{H}}$
      $\wedge\ e_2^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 e_2^1$
      by H−1 *dynamic application* (1)
   b. qed $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}D}} E^{\mathsf{H}}[e_2^{\mathsf{H}}]$
   if $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$:
   a. qed $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
   else $e_{1'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$:
   a. qed $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

case $E^1[op^1\ v_0^1] \rightarrow_{1\text{-S}} E^1[\delta(op^1, v_0^1)]$
$\wedge \vdash_1 op^1\ v_0^1 : K_0$:
1. $\vdash_{\mathsf{H}} e_0^{\mathsf{H}} : \tau_0$
   by H−1 *static hole typing*
2. $e_0^{\mathsf{H}} = op^1\ e_{0'}^{\mathsf{H}}$
   by ${}_{\mathsf{H}}\!\lesssim_1$
3. $\vdash_{\mathsf{H}} e_{0'}^{\mathsf{H}} : \tau_{0'}$
   by H *inversion* (1, 2)
4. $e_{0'}^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_0^1$
   by ${}_{\mathsf{H}}\!\lesssim_1$
5. $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_0^1$
   $\vee\ e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
   by H−1 *static value stutter*
6. **if** $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} v_0^{\mathsf{H}}$:
   a. $op^1\ v_0^{\mathsf{H}} \rhd_{\mathsf{H\text{-}S}} \delta(op^1, v_0^{\mathsf{H}}$
      by $\rhd_{\mathsf{H\text{-}S}}$
   b. $\delta(op^1, v_0^{\mathsf{H}})\ {}_{\mathsf{H}}\!\lesssim_1 \delta(op^1, v_0^1)$
      by H−1 *δ-preservation*
   c. qed $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} E^{\mathsf{H}}[\delta(op^1, v_0^{\mathsf{H}})]$
   else $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$:
   a. qed $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

case $E^1[op^1\ v_0^1] \rightarrow_{1\text{-S}} E^1[\delta(op^1, v_0^1)]$
$\wedge \vdash_1 op^1\ v_0^1$:
1. $\vdash_{\mathsf{H}} e_0^{\mathsf{H}}$
   by H−1 *dynamic hole typing*
2. $e_0^{\mathsf{H}} = op^1\ e_{0'}^{\mathsf{H}}$
   by ${}_{\mathsf{H}}\!\lesssim_1$
3. $\vdash_{\mathsf{H}} e_{0'}^{\mathsf{H}} : \tau_{0'}$
   by H *inversion* (1, 2)
4. $e_{0'}^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_0^1$
   by ${}_{\mathsf{H}}\!\lesssim_1$
5. $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} v_0^{\mathsf{H}}$ and $v_0^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_0^1$
   $\vee\ e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
   by H−1 *dynamic value stutter*
6. **if** $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} v_0^{\mathsf{H}}$:
   a. $op^1\ v_0^{\mathsf{H}} \rhd_{\mathsf{H\text{-}D}} \delta(op^1, v_0^{\mathsf{H}})$
      by $\rhd_{\mathsf{H\text{-}D}}$
   b. $\delta(op^1, v_0^{\mathsf{H}})\ {}_{\mathsf{H}}\!\lesssim_1 \delta(op^1, v_0^1)$
      by H−1 *δ-preservation*
   c. qed $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} E^{\mathsf{H}}[\delta(op^1, v_0^{\mathsf{H}})]$
   else $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$:
   a. qed $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

case $E^1[op^1\ v_0^1] \rightarrow_{1\text{-S}} E^1[\mathsf{TagErr}]$:
1. $\vdash_1 op^1\ v_0^1$
   by definition $\rightarrow_{1\text{-S}}$
2. $\vdash_{\mathsf{H}} e_0^{\mathsf{H}}$
   by H−1 *dynamic hole typing* (1)

3. $e^{\mathsf{H}}_{0} = op^1\ e^{\mathsf{H}}_{0'}$
by ${}_{\mathsf{H}}{\lesssim}_{1}$
4. $\vdash_{\mathsf{H}} e^{\mathsf{H}}_{0'} : \tau_{0'}$
by H *inversion* (2, 3)
5. $e^{\mathsf{H}}_{0'}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}$
by ${}_{\mathsf{H}}{\lesssim}_{1}$
6. $e^{\mathsf{H}}_{0'} \rightarrow^{*}_{\mathsf{H\text{-}D}} v^{\mathsf{H}}_{0}$ and $v^{\mathsf{H}}_{0}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}$
$\vee\ e^{\mathsf{H}}_{0'} \rightarrow^{*}_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
by H−1 *dynamic value stutter*
7. **IF** $e^{\mathsf{H}}_{0'} \rightarrow^{*}_{\mathsf{H\text{-}D}} v^{\mathsf{H}}_{0}$:
   a. $\delta(op^1, v^{1}_{0}$ is undefined
      by definition $\rhd_{\mathsf{H\text{-}D}}$
   b. $v^{1}_{0} \notin \langle v, v\rangle$
      by (a)
   c. $v^{\mathsf{H}}_{0} \notin \langle v, v\rangle$
      by $v^{\mathsf{H}}_{0}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}$
   d. QED $E^{\mathsf{H}}[e^{\mathsf{H}}_{0}] \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{TagErr}$
   ELSE $e^{\mathsf{H}}_{0'} \rightarrow^{*}_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$:
   a. QED $E^{\mathsf{H}}[e^{\mathsf{H}}_{0}] \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

CASE $E^{1}[op^2\ v^{1}_{0}\ v^{1}_{1}] \rightarrow_{\mathsf{1\text{-}S}} E^{1}[\delta(op^2, v^{1}_{0}, v^{1}_{1})]$
$\wedge \vdash_{1} op^2\ v^{1}_{0}\ v^{1}_{1} : K_0$:
1. $\vdash_{\mathsf{H}} e^{\mathsf{H}}_{0} : \tau_{0}$
by H−1 *static hole typing*
2. $e^{\mathsf{H}}_{0} = op^2\ e^{\mathsf{H}}_{0'}\ e^{\mathsf{H}}_{1'}$
by ${}_{\mathsf{H}}{\lesssim}_{1}$
3. $\vdash_{\mathsf{H}} e^{\mathsf{H}}_{0'} : \tau_{0'}$
$\wedge \vdash_{\mathsf{H}} e^{\mathsf{H}}_{1'} : \tau_{1'}$
by H *inversion* (1, 2)
4. $e^{\mathsf{H}}_{0'}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}$
$\wedge\ e^{\mathsf{H}}_{1'}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{1}$
by ${}_{\mathsf{H}}{\lesssim}_{1}$
5. $e^{\mathsf{H}}_{0'} \rightarrow^{*}_{\mathsf{H\text{-}S}} v^{\mathsf{H}}_{0}$ and $v^{\mathsf{H}}_{0}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}$
$\vee\ e^{\mathsf{H}}_{0'} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
by H−1 *static value stutter*
6. $e^{\mathsf{H}}_{1'} \rightarrow^{*}_{\mathsf{H\text{-}S}} v^{\mathsf{H}}_{1}$ and $v^{\mathsf{H}}_{1}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{1}$
$\vee\ e^{\mathsf{H}}_{1'} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
by H−1 *static value stutter*
7. **IF** $e^{\mathsf{H}}_{0'} \rightarrow^{*}_{\mathsf{H\text{-}S}} v^{\mathsf{H}}_{0}$
   $\wedge\ e^{\mathsf{H}}_{1'} \rightarrow^{*}_{\mathsf{H\text{-}S}} v^{\mathsf{H}}_{1}$:
   a. $op^2\ v^{\mathsf{H}}_{0}\ v^{\mathsf{H}}_{1} \rhd_{\mathsf{H\text{-}S}} \delta(op^2, v^{\mathsf{H}}_{0}, v^{\mathsf{H}}_{1})$
      by $\rhd_{\mathsf{H\text{-}D}}$
   b. $\delta(op^2, v^{\mathsf{H}}_{0}, v^{\mathsf{H}}_{1})\ {}_{\mathsf{H}}{\lesssim}_{1}\ \delta(op^2, v^{1}_{0}, v^{1}_{1})$
      by H−1 *δ-preservation*
   c. QED $E^{\mathsf{H}}[e^{\mathsf{H}}_{0}] \rightarrow^{*}_{\mathsf{H\text{-}S}} E^{\mathsf{H}}[\delta(op^2, v^{\mathsf{H}}_{0}, v^{\mathsf{H}}_{1})]$
   IF $e^{\mathsf{H}}_{0'} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$:
   a. QED $E^{\mathsf{H}}[e^{\mathsf{H}}_{0}] \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
   ELSE $e^{\mathsf{H}}_{1'} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$:
   a. QED $E^{\mathsf{H}}[e^{\mathsf{H}}_{0}] \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

CASE $E^{1}[op^2\ v^{1}_{0}\ v^{1}_{1}] \rightarrow_{\mathsf{1\text{-}S}} E^{1}[\delta(op^2, v^{1}_{0}, v^{1}_{1})]$
$\wedge \vdash_{1} op^2\ v^{1}_{0}\ v^{1}_{1}$:
1. $\vdash_{\mathsf{H}} e^{\mathsf{H}}_{0}$
by H−1 *static hole typing*
2. $e^{\mathsf{H}}_{0} = op^2\ e^{\mathsf{H}}_{0'}\ e^{\mathsf{H}}_{1'}$
by ${}_{\mathsf{H}}{\lesssim}_{1}$
3. $\vdash_{\mathsf{H}} e^{\mathsf{H}}_{0'}$
$\wedge \vdash_{\mathsf{H}} e^{\mathsf{H}}_{1'}$
by H *inversion* (1, 2)
4. $e^{\mathsf{H}}_{0'}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}$
$\wedge\ e^{\mathsf{H}}_{1'}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{1}$
by ${}_{\mathsf{H}}{\lesssim}_{1}$
5. $e^{\mathsf{H}}_{0'} \rightarrow^{*}_{\mathsf{H\text{-}D}} v^{\mathsf{H}}_{0}$ and $v^{\mathsf{H}}_{0}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}$
$\vee\ e^{\mathsf{H}}_{0'} \rightarrow^{*}_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
by H−1 *dynamic value stutter*
6. $e^{\mathsf{H}}_{1'} \rightarrow^{*}_{\mathsf{H\text{-}D}} v^{\mathsf{H}}_{1}$ and $v^{\mathsf{H}}_{1}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{1}$
$\vee\ e^{\mathsf{H}}_{1'} \rightarrow^{*}_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
by H−1 *dynamic value stutter*
7. **IF** $e^{\mathsf{H}}_{0'} \rightarrow^{*}_{\mathsf{H\text{-}D}} v^{\mathsf{H}}_{0}$
   $\wedge\ e^{\mathsf{H}}_{1'} \rightarrow^{*}_{\mathsf{H\text{-}D}} v^{\mathsf{H}}_{1}$:
   a. $op^2\ v^{\mathsf{H}}_{0}\ v^{\mathsf{H}}_{1} \rhd_{\mathsf{H\text{-}D}} \delta(op^2, v^{\mathsf{H}}_{0}, v^{\mathsf{H}}_{1})$
      by $\rhd_{\mathsf{H\text{-}D}}$
   b. $\delta(op^2, v^{\mathsf{H}}_{0}, v^{\mathsf{H}}_{1})\ {}_{\mathsf{H}}{\lesssim}_{1}\ \delta(op^2, v^{1}_{0}, v^{1}_{1})$
      by H−1 *δ-preservation*
   c. QED $E^{\mathsf{H}}[e^{\mathsf{H}}_{0}] \rightarrow^{*}_{\mathsf{H\text{-}S}} E^{\mathsf{H}}[\delta(op^2, v^{\mathsf{H}}_{0}, v^{\mathsf{H}}_{1})]$
   IF $e^{\mathsf{H}}_{0'} \rightarrow^{*}_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$:
   a. QED $E^{\mathsf{H}}[e^{\mathsf{H}}_{0}] \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
   ELSE $e^{\mathsf{H}}_{1'} \rightarrow^{*}_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$:
   a. QED $E^{\mathsf{H}}[e^{\mathsf{H}}_{0}] \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

CASE $E^{1}[op^2\ v^{1}_{0}\ v^{1}_{1}] \rightarrow_{\mathsf{1\text{-}S}} E^{1}[\mathsf{TagErr}]$:
1. $\vdash_{1} op^2\ v^{1}_{0}\ v^{1}_{1}$
by definition $\rightarrow_{\mathsf{1\text{-}S}}$
2. $\vdash_{\mathsf{H}} e^{\mathsf{H}}_{0}$
by H−1 *dynamic hole typing* (1)
3. $e^{\mathsf{H}}_{0} = op^2\ e^{\mathsf{H}}_{0'}\ e^{\mathsf{H}}_{1'}$
by ${}_{\mathsf{H}}{\lesssim}_{1}$
4. $\vdash_{\mathsf{H}} e^{\mathsf{H}}_{0'} : \tau_{0'}$
$\wedge \vdash_{\mathsf{H}} e^{\mathsf{H}}_{1'} : \tau_{1'}$
by H *inversion* (2, 3)
5. $e^{\mathsf{H}}_{0'}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}$
$\wedge\ e^{\mathsf{H}}_{1'}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{1}$
by ${}_{\mathsf{H}}{\lesssim}_{1}$
6. $e^{\mathsf{H}}_{0'} \rightarrow^{*}_{\mathsf{H\text{-}D}} v^{\mathsf{H}}_{0}$ and $v^{\mathsf{H}}_{0}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}$
$\vee\ e^{\mathsf{H}}_{0'} \rightarrow^{*}_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
by H−1 *dynamic value stutter*
7. $e^{\mathsf{H}}_{1'} \rightarrow^{*}_{\mathsf{H\text{-}D}} v^{\mathsf{H}}_{1}$ and $v^{\mathsf{H}}_{1}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{1}$
$\vee\ e^{\mathsf{H}}_{1'} \rightarrow^{*}_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
by H−1 *dynamic value stutter*
8. **IF** $e^{\mathsf{H}}_{0'} \rightarrow^{*}_{\mathsf{H\text{-}D}} v^{\mathsf{H}}_{0}$
   $\wedge\ e^{\mathsf{H}}_{1'} \rightarrow^{*}_{\mathsf{H\text{-}D}} v^{\mathsf{H}}_{1}$:
   a. $\delta(op^2, v^{1}_{0}, v^{1}_{1}$ is undefined
      by definition $\rhd_{\mathsf{H\text{-}D}}$
   b. $v^{1}_{0} \notin \mathbb{Z}$
      $\wedge\ v^{1}_{1} \notin \mathbb{Z}$
      by (a)

c. $v_0^{\mathsf{H}} \notin \mathbb{Z}$
by $v_0^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_0^1$
d. $v_1^{\mathsf{H}} \notin \mathbb{Z}$
by $v_1^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_1^1$
e. QED $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{TagErr}$
IF $e_{0'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$:
a. QED $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
ELSE $e_{1'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$:
a. QED $E^{\mathsf{H}}[e_0^{\mathsf{H}}] \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

☐

Lemma 8.14 : H−1 *static application*

If $v_0^{\mathsf{H}}\, v_1^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_0^1\, v_1^1$
$\wedge \vdash_{\mathsf{H}} v_0^{\mathsf{H}}\, v_1^{\mathsf{H}} : \tau$
$\wedge \vdash_{1} v_0^1\, v_1^1 : \mathsf{Any}$
$\wedge\, v_0^1\, v_1^1 \rightarrow_{\mathsf{1\text{-}S}} e_2^1$ then $v_0^{\mathsf{H}}\, v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} e_2^{\mathsf{H}}$ and $e_2^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 e_2^1$

Proof:

By induction on the number of monitors in v_0^{H}, proceeding by case analysis on $v_0^1\, v_1^1 \rightarrow_{\mathsf{1\text{-}S}} e_2^1$.

CASE $(\lambda(x\!:\!\tau_0).\, e_0^1)\, v_1^1 \rightarrow_{\mathsf{1\text{-}S}} \mathsf{BndryErr}$
$\wedge\, v_0^{\mathsf{H}} = \lambda(x\!:\!\tau_0).\, e_0^{\mathsf{H}}$:
1. $\mathcal{X}(\lfloor\tau_0\rfloor, v_1^1) = \mathsf{BndryErr}$
by definition $\rhd_{\mathsf{1\text{-}S}}$
2. $\vdash_{\mathsf{H}} v_1^{\mathsf{H}} : \tau_1$
$\wedge\, \tau_1 \leqslant: \tau_0$
by H *inversion*
3. $v_1^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_1^1$
by $v_0^{\mathsf{H}}\, v_1^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_0^1\, v_1^1$
4. $\vdash_1 v_1^1 : \lfloor\tau_1\rfloor$
by H−1 *value inversion* (3)
5. $\vdash_1 v_1^1 : \lfloor\tau_0\rfloor$
by *subtyping preservation* (2)
6. $\mathcal{X}(\lfloor\tau_0\rfloor, v_1^1) = v_1^1$
by (5)
7. Contradiction by (1, 6)

CASE $(\lambda(x\!:\!\tau_0).\, e_0^1)\, v_1^1 \rightarrow_{\mathsf{1\text{-}S}} \mathsf{BndryErr}$
$\wedge\, v_0^{\mathsf{H}} = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v_{0'}^{\mathsf{H}}$:
1. $v_{0'}^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_0^1$
$\wedge\, v_1^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_1^1$
by $v_0^{\mathsf{H}}\, v_1^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_0^1\, v_1^1$
2. $v_0^{\mathsf{H}}\, v_1^{\mathsf{H}} \rhd_{\mathsf{H\text{-}S}} \mathsf{dyn}\ \tau_c\ (v_{0'}^{\mathsf{H}}\ (\mathsf{stat}\ \tau_d\ v_1^{\mathsf{H}}))$
by definition $\rhd_{\mathsf{H\text{-}S}}$
3. $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau_d\ v_1^{\mathsf{H}}$
by H *static preservation*
4. $\mathsf{stat}\ \tau_d\ v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} v_{1'}^{\mathsf{H}}$ and $v_{1'}^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_1^1$
by H−1 *boundary checking* (1, 3)
5. $v_{0'}^{\mathsf{H}} = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\, v_{0''}^{\mathsf{H}}$ and $v_{0''}^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_0^1$
$\vee\, v_{0'}^{\mathsf{H}} = \lambda(x\!:\!\tau_0).\, e^{\mathsf{H}}$ and $e^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 e_0^1$
by (1)
6. **IF** $v_{0'}^{\mathsf{H}} = \lambda(x\!:\!\tau_0).\, e^{\mathsf{H}}$:
a. Contradiction by $\vdash_{\mathsf{H}} v_0^{\mathsf{H}}\, v_1^{\mathsf{H}} : \tau$
7. $v_{0'}^{\mathsf{H}}\, v_{1'}^{\mathsf{H}} \rhd_{\mathsf{H\text{-}D}} \mathsf{stat}\ \tau_c'\ (v_{0''}^{\mathsf{H}}\ (\mathsf{dyn}\ \tau_d'\ v_{1'}^{\mathsf{H}}))$
8. $\mathsf{dyn}\ \tau_d'\ v_{1'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} v_{1''}^{\mathsf{H}}$ and $v_{1''}^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_1^1$
$\vee\, \mathsf{dyn}\ \tau_d'\ v_{1'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
by H−1 *boundary checking*
9. **IF** $\mathsf{dyn}\ \tau_d'\ v_{1'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$:
a. QED $v_0^{\mathsf{H}}\, v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
10. $\mathsf{dyn}\ \tau_c\ (\mathsf{stat}\ \tau_c'\ [\,])\ {}_{\mathsf{H}}\!\lesssim_1 [\,]$
by definition ${}_{\mathsf{H}}\!\lesssim_1$
11. $v_{0''}^{\mathsf{H}}\, v_{1''}^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_0^1\, v_1^1$
12. $\vdash_{\mathsf{H}} v_{0''}^{\mathsf{H}}\, v_{1''}^{\mathsf{H}} : \tau_c'$
by H *static preservation*
13. $v_{0''}^{\mathsf{H}}\, v_{1''}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} e_2^{\mathsf{H}}$
$\wedge\, e_2^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 \mathsf{BndryErr}$
by the induction hypothesis
14. QED $v_0^{\mathsf{H}}\, v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

CASE $(\lambda(x\!:\!\tau_0).\, e_0^1)\, v_1^1 \rightarrow_{\mathsf{1\text{-}S}} e_0^1[x \leftarrow \mathcal{X}(\lfloor\tau_0\rfloor, v_1^1)]$
$\wedge\, v_0^{\mathsf{H}} = \lambda(x\!:\!\tau_0).\, e_0^{\mathsf{H}}$:
1. $\mathcal{X}(\lfloor\tau_0\rfloor, v_1^1) = v_1^1$
by definition $\rhd_{\mathsf{1\text{-}S}}$
2. $\vdash_{\mathsf{H}} v_1^{\mathsf{H}} : \tau_1$
$\wedge\, \tau_1 \leqslant: \tau_0$
by H *inversion*
3. $v_1^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_1^1$
$\wedge\, e_0^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 e_0^1$
by $v_0^{\mathsf{H}}\, v_1^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_0^1\, v_1^1$
4. $(\lambda(x\!:\!\tau_0).\, e_0^{\mathsf{H}})\, v_1^{\mathsf{H}} \rhd_{\mathsf{H\text{-}S}} e_0^{\mathsf{H}}[x \leftarrow v_1^{\mathsf{H}}]$
5. $e_0^{\mathsf{H}}[x \leftarrow v_1^{\mathsf{H}}] \;{}_{\mathsf{H}}\!\lesssim_1 e_0^1[x \leftarrow v_1^1]$
by H−1 *substitution*
6. QED $v_0^{\mathsf{H}}\, v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} e_0^{\mathsf{H}}[x \leftarrow v_1^{\mathsf{H}}]$

CASE $(\lambda(x\!:\!\tau_0).\, e_0^1)\, v_1^1 \rightarrow_{\mathsf{1\text{-}S}} e_0^1[x \leftarrow \mathcal{X}(\lfloor\tau_0\rfloor, v_1^1)]$
$\wedge\, v_0^{\mathsf{H}} = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v_{0'}^{\mathsf{H}}$:
1. $v_{0'}^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_0^1$
$\wedge\, v_1^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_1^1$
by $v_0^{\mathsf{H}}\, v_1^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_0^1\, v_1^1$
2. $v_0^{\mathsf{H}}\, v_1^{\mathsf{H}} \rhd_{\mathsf{H\text{-}S}} \mathsf{dyn}\ \tau_c\ (v_{0'}^{\mathsf{H}}\ (\mathsf{stat}\ \tau_d\ v_1^{\mathsf{H}}))$
by definition $\rhd_{\mathsf{H\text{-}S}}$
3. $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau_d\ v_1^{\mathsf{H}}$
by H *static preservation*
4. $\mathsf{stat}\ \tau_d\ v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} v_{1'}^{\mathsf{H}}$ and $v_{1'}^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_1^1$
by H−1 *boundary checking* (1, 3)
5. $v_{0'}^{\mathsf{H}} = \mathsf{mon}\,(\tau_d' \Rightarrow \tau_c')\, v_{0''}^{\mathsf{H}}$ and $v_{0''}^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 v_0^1$
$\vee\, v_{0'}^{\mathsf{H}} = \lambda(x\!:\!\tau_0).\, e^{\mathsf{H}}$ and $e^{\mathsf{H}} \;{}_{\mathsf{H}}\!\lesssim_1 e_0^1$
by (1)
6. **IF** $v_{0'}^{\mathsf{H}} = \lambda(x\!:\!\tau_0).\, e^{\mathsf{H}}$:
a. $\vdash_{\mathsf{H}} \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v_{0'}^{\mathsf{H}} : \tau_d \Rightarrow \tau_c$
by $\vdash_{\mathsf{H}} v_0^{\mathsf{H}}\, v_1^{\mathsf{H}} : \tau$
b. $\vdash_{\mathsf{H}} v_{0'}^{\mathsf{H}}$
by H *inversion* (a)
c. Contradiction by $\vdash_{\mathsf{H}} \lambda(x\!:\!\tau_0).\, e^{\mathsf{H}}$
7. $v_{0'}^{\mathsf{H}}\, v_{1'}^{\mathsf{H}} \rhd_{\mathsf{H\text{-}D}} \mathsf{stat}\ \tau_c'\ (v_{0''}^{\mathsf{H}}\ (\mathsf{dyn}\ \tau_d'\ v_{1'}^{\mathsf{H}}))$
8. $\vdash_{\mathsf{H}} \mathsf{dyn}\ \tau_d'\ v_{1'}^{\mathsf{H}} : \tau_d'$
by H *dynamic preservation*

9. $\mathsf{dyn}\ \tau'_d\ v^{\mathsf{H}}_{1'} \rightarrow^{*}_{\mathsf{H\text{-}S}} v^{\mathsf{H}}_{1''}$ and $v^{\mathsf{H}}_{1''}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{1}$
$\vee\ \mathsf{dyn}\ \tau'_d\ v^{\mathsf{H}}_{1'} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
by H−1 *boundary checking*
10. **IF** $\mathsf{dyn}\ \tau'_d\ v^{\mathsf{H}}_{1'} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$:
a. QED $v^{\mathsf{H}}_{0}\ v^{\mathsf{H}}_{1} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
11. $\mathsf{dyn}\ \tau_c\ (\mathsf{stat}\ \tau'_c\ [\,])\ {}_{\mathsf{H}}{\lesssim}_{1}\ [\,]$
by definition ${}_{\mathsf{H}}{\lesssim}_{1}$
12. $v^{\mathsf{H}}_{0''}\ v^{\mathsf{H}}_{1''}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}\ v^{1}_{1}$
13. $\vdash_{\mathsf{H}} v^{\mathsf{H}}_{0''}\ v^{\mathsf{H}}_{1''} : \tau'_c$
by H *static preservation*
14. $v^{\mathsf{H}}_{0''}\ v^{\mathsf{H}}_{1''} \rightarrow^{*}_{\mathsf{H\text{-}S}} e^{\mathsf{H}}_{2}$
$\wedge\ e^{\mathsf{H}}_{2}\ {}_{\mathsf{H}}{\lesssim}_{1}\ e^{1}_{2}$
by the induction hypothesis
15. QED $v^{\mathsf{H}}_{0}\ v^{\mathsf{H}}_{1} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{dyn}\ \tau_c\ (\mathsf{stat}\ \tau'_c\ e^{\mathsf{H}}_{2})$

CASE $(\lambda x.\, e^{1}_{0})\ v^{1}_{1} \rightarrow_{\mathsf{1\text{-}S}} \mathsf{dyn}\ e^{1}_{0}[x\leftarrow v^{1}_{1}]$
$\wedge\ v^{\mathsf{H}}_{0} = \lambda x.\, e^{\mathsf{H}}_{0}$:
1. Contradiction by $\vdash_{\mathsf{H}} v^{\mathsf{H}}_{0}\ v^{\mathsf{H}}_{1} : \tau$

CASE $(\lambda x.\, e^{1}_{0})\ v^{1}_{1} \rightarrow_{\mathsf{1\text{-}S}} \mathsf{dyn}\ e^{1}_{0}[x\leftarrow v^{1}_{1}]$
$\wedge\ v^{\mathsf{H}}_{0} = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v^{\mathsf{H}}_{0'}$:
1. $v^{\mathsf{H}}_{0'}\ {}_{\mathsf{H}}{\lesssim}_{1}\ (\lambda x.\, e^{1}_{0})$
$\wedge\ v^{\mathsf{H}}_{1}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{1}$
by $v^{\mathsf{H}}_{0}\ v^{\mathsf{H}}_{1}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}\ v^{1}_{1}$
2. $v^{\mathsf{H}}_{0}\ v^{\mathsf{H}}_{1} \rhd_{\mathsf{H\text{-}S}} \mathsf{dyn}\ \tau_c\ (v^{\mathsf{H}}_{0'}\ (\mathsf{stat}\ \tau_d\ v^{\mathsf{H}}_{1}))$
by definition $\rhd_{\mathsf{H\text{-}S}}$
3. $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau_d\ v^{\mathsf{H}}_{1}$
by H *static preservation*
4. $\mathsf{stat}\ \tau_d\ v^{\mathsf{H}}_{1} \rightarrow^{*}_{\mathsf{H\text{-}D}} v^{\mathsf{H}}_{1'}$ and $v^{\mathsf{H}}_{1'}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{1}$
by H−1 *boundary checking*
5. $\mathsf{dyn}\ \tau_c\ [\,]\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{dyn}\ [\,]$
by definition ${}_{\mathsf{H}}{\lesssim}_{1}$
6. $\vdash_{\mathsf{H}} v^{\mathsf{H}}_{0'}\ v^{\mathsf{H}}_{1'}$
by H *static preservation* (2, 3)
7. $v^{\mathsf{H}}_{0'}\ v^{\mathsf{H}}_{1'}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}\ v^{1}_{1}$
8. $v^{\mathsf{H}}_{0'}\ v^{\mathsf{H}}_{1'} \rightarrow^{*}_{\mathsf{H\text{-}D}} e^{\mathsf{H}}_{2}$
$\wedge\ e^{\mathsf{H}}_{2}\ {}_{\mathsf{H}}{\lesssim}_{1}\ e^{1}_{2}$
by H−1 *dynamic application* (6, 7)
9. QED $v^{\mathsf{H}}_{0}\ v^{\mathsf{H}}_{1} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{dyn}\ \tau_c\ e^{\mathsf{H}}_{2}$
$\wedge\ \mathsf{dyn}\ \tau_c\ e^{\mathsf{H}}_{2}\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{dyn}\ e^{1}_{2}$

□

Lemma 8.15 : H−1 *dynamic application*

If $v^{\mathsf{H}}_{0}\ v^{\mathsf{H}}_{1}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}\ v^{1}_{1}$
$\wedge \vdash_{\mathsf{H}} v^{\mathsf{H}}_{0}\ v^{\mathsf{H}}_{1}$
$\wedge \vdash_{1} v^{1}_{0}\ v^{1}_{1}$
$\wedge\ v^{1}_{0}\ v^{1}_{1} \rightarrow_{\mathsf{1\text{-}D}} e^{1}_{2}$ then $v^{\mathsf{H}}_{0}\ v^{\mathsf{H}}_{1} \rightarrow^{*}_{\mathsf{H\text{-}D}} e^{\mathsf{H}}_{2}$ and $e^{\mathsf{H}}_{2}\ {}_{\mathsf{H}}{\lesssim}_{1}\ e^{1}_{2}$

Proof:

By induction on the number of monitors in v^{H}_{0}, proceeding by case analysis on $v^{1}_{0}\ v^{1}_{1} \rightarrow_{\mathsf{1\text{-}D}} e^{1}_{2}$.

CASE $v^{1}_{0}\ v^{1}_{1} \rhd_{\mathsf{1\text{-}D}} \mathsf{TagErr}$:
1. $v^{1}_{0} \in \mathbb{Z}$
$\vee\ v^{1}_{0} \in \langle v, v\rangle$
by definition $\rhd_{\mathsf{1\text{-}D}}$
2. $v^{\mathsf{H}}_{0} \in \mathbb{Z}$
$\vee\ v^{\mathsf{H}}_{0} \in \langle v, v\rangle$
by $v^{\mathsf{H}}_{0}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}$
3. QED $v^{\mathsf{H}}_{0}\ v^{\mathsf{H}}_{1} \rhd_{\mathsf{H\text{-}D}} \mathsf{TagErr}$

CASE $(\lambda(x\!:\!\tau_0).\, e^{1}_{0})\ v^{1}_{1} \rightarrow_{\mathsf{1\text{-}D}} \mathsf{BndryErr}$
$\wedge\ v^{\mathsf{H}}_{0} = \lambda(x\!:\!\tau_0).\, e^{\mathsf{H}}_{0}$:
1. Contradiction by $\vdash_{\mathsf{H}} v^{\mathsf{H}}_{0}\ v^{\mathsf{H}}_{1}$

CASE $(\lambda(x\!:\!\tau_0).\, e^{1}_{0})\ v^{1}_{1} \rightarrow_{\mathsf{1\text{-}D}} \mathsf{BndryErr}$
$\wedge\ v^{\mathsf{H}}_{0} = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\, v^{\mathsf{H}}_{0'}$:
1. $v^{\mathsf{H}}_{0'}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}$
$\wedge\ v^{\mathsf{H}}_{1}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{1}$
by $v^{\mathsf{H}}_{0}\ v^{\mathsf{H}}_{1}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}\ v^{1}_{1}$
2. $v^{\mathsf{H}}_{0}\ v^{\mathsf{H}}_{1} \rhd_{\mathsf{H\text{-}D}} \mathsf{stat}\ \tau_c\ (v^{\mathsf{H}}_{0'}\ (\mathsf{dyn}\ \tau_d\ v^{\mathsf{H}}_{1}))$
by definition $\rhd_{\mathsf{H\text{-}D}}$
3. $\vdash_{\mathsf{H}} \mathsf{dyn}\ \tau_d\ v^{\mathsf{H}}_{1} : \tau_d$
by H *dynamic preservation*
4. $\mathsf{dyn}\ \tau_d\ v^{\mathsf{H}}_{1} \rightarrow^{*}_{\mathsf{H\text{-}S}} v^{\mathsf{H}}_{1'}$ and $v^{\mathsf{H}}_{1'}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{1}$
$\vee\ \mathsf{dyn}\ \tau_d\ v^{\mathsf{H}}_{1} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
by H−1 *boundary checking* (1, 3)
5. **IF** $\mathsf{dyn}\ \tau_d\ v^{\mathsf{H}}_{1} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$:
a. QED $v^{\mathsf{H}}_{0}\ v^{\mathsf{H}}_{1} \rightarrow^{*}_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
6. $v^{\mathsf{H}}_{0'} = \mathsf{mon}\,(\tau'_d \Rightarrow \tau'_c)\, v^{\mathsf{H}}_{0''}$ and $v^{\mathsf{H}}_{0''}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}$
$\vee\ v^{\mathsf{H}}_{0'} = \lambda(x\!:\!\tau_0).\, e^{\mathsf{H}}$ and $e^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ e^{1}_{0}$
by (1)
7. **IF** $v^{\mathsf{H}}_{0'} = \lambda(x\!:\!\tau_0).\, e^{\mathsf{H}}$:
a. $\mathcal{X}(\lfloor\tau_0\rfloor, v^{1}_{1}) = \mathsf{BndryErr}$
by definition $\rhd_{\mathsf{1\text{-}D}}$
b. $\vdash_{\mathsf{H}} v^{\mathsf{H}}_{1'} : \tau_1$
$\wedge\ \tau_1 \leqslant: \tau_0$
by H *inversion*
c. $\vdash_{1} v^{1}_{1} : \lfloor\tau_1\rfloor$
by H−1 *value inversion* (3)
d. $\vdash_{1} v^{1}_{1} : \lfloor\tau_0\rfloor$
by *subtyping preservation* (c)
e. $\mathcal{X}(\lfloor\tau_0\rfloor, v^{1}_{1}) = v^{1}_{1}$
by (d)
f. Contradiction by (a)
8. $v^{\mathsf{H}}_{0'}\ v^{\mathsf{H}}_{1'} \rhd_{\mathsf{H\text{-}S}} \mathsf{dyn}\ \tau'_c\ (v^{\mathsf{H}}_{0''}\ (\mathsf{stat}\ \tau'_d\ v^{\mathsf{H}}_{1'}))$
9. $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau'_d\ v^{\mathsf{H}}_{1'}$
by H *static preservation*
10. $\mathsf{stat}\ \tau'_d\ v^{\mathsf{H}}_{1'} \rightarrow^{*}_{\mathsf{H\text{-}D}} v^{\mathsf{H}}_{1''}$ and $v^{\mathsf{H}}_{1''}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{1}$
by H−1 *boundary checking*
11. $\mathsf{stat}\ \tau_c\ (\mathsf{dyn}\ \tau'_c\ [\,])\ {}_{\mathsf{H}}{\lesssim}_{1}\ [\,]$
by definition ${}_{\mathsf{H}}{\lesssim}_{1}$
12. $v^{\mathsf{H}}_{0''}\ v^{\mathsf{H}}_{1''}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}_{0}\ v^{1}_{1}$
13. $\vdash_{\mathsf{H}} v^{\mathsf{H}}_{0''}\ v^{\mathsf{H}}_{1''}$
by H *dynamic preservation*
14. $v^{\mathsf{H}}_{0''}\ v^{\mathsf{H}}_{1''} \rightarrow^{*}_{\mathsf{H\text{-}D}} e^{\mathsf{H}}_{2}$
$\wedge\ e^{\mathsf{H}}_{2}\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{BndryErr}$
by the induction hypothesis
15. QED $v^{\mathsf{H}}_{0}\ v^{\mathsf{H}}_{1} \rightarrow^{*}_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$

CASE $(\lambda(x\!:\!\tau_0).\, e_0^1)\ v_1^1 \rightarrow_{1\text{-D}} \mathsf{stat}\ e_0^1[x \leftarrow \mathcal{X}(\lfloor\tau_0\rfloor, v_1^1)]$
$\wedge\ v_0^{\mathsf{H}} = \lambda(x\!:\!\tau_0).\, e_0^{\mathsf{H}}$:

1. $\vdash_{\mathsf{H}} v_0^{\mathsf{H}}$
   by H *inversion*
2. Contradiction by $\vdash_{\mathsf{H}} \lambda(x\!:\!\tau_0).\, e_0^{\mathsf{H}}$

CASE $(\lambda(x\!:\!\tau_0).\, e_0^1)\ v_1^1 \rightarrow_{1\text{-D}} \mathsf{stat}\ e_0^1[x \leftarrow \mathcal{X}(\lfloor\tau_0\rfloor, v_1^1)]$
$\wedge\ v_0^{\mathsf{H}} = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\ v_{0'}^{\mathsf{H}}$:

1. $v_{0'}^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_0^1$
   $\wedge\ v_1^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_1^1$
   by $v_0^{\mathsf{H}}\ v_1^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_0^1\ v_1^1$
2. $v_0^{\mathsf{H}}\ v_1^{\mathsf{H}} \rhd_{\mathsf{H\text{-}D}} \mathsf{stat}\ \tau_c\ (v_{0'}^{\mathsf{H}}\ (\mathsf{dyn}\ \tau_d\ v_1^{\mathsf{H}}))$
   by definition $\rhd_{\mathsf{H\text{-}D}}$
3. $\vdash_{\mathsf{H}} \mathsf{dyn}\ \tau_d\ v_1^{\mathsf{H}} : \tau_d$
   by H *dynamic preservation*
4. $\mathsf{dyn}\ \tau_d\ v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} v_{1'}^{\mathsf{H}}$ and $v_{1'}^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_1^1$
   $\vee\ \mathsf{dyn}\ \tau_d\ v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
   by H−1 *boundary checking* (1, 3)
5. **IF** $\mathsf{dyn}\ \tau_d\ v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$:
   a. QED $v_0^{\mathsf{H}}\ v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
6. $\mathsf{stat}\ \tau_c\ [\,]\ {}_{\mathsf{H}}\!\lesssim_1 \mathsf{stat}\ [\,]$
   by definition of ${}_{\mathsf{H}}\!\lesssim_1$
7. $\vdash_{\mathsf{H}} v_{0'}^{\mathsf{H}}\ v_{1'}^{\mathsf{H}} : \tau_c$
   by H *dynamic preservation*
8. $v_{0'}^{\mathsf{H}}\ v_{1'}^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_0^1\ v_1^1$
9. $v_{0'}^{\mathsf{H}}\ v_{1'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} e_2^{\mathsf{H}}$
   $\wedge\ e_2^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 e_2^1$
   by H−1 *static application* (7, 8)
10. QED $v_0^{\mathsf{H}}\ v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} \mathsf{stat}\ \tau_c\ e_2^{\mathsf{H}}$
    $\wedge\ \mathsf{stat}\ \tau_c\ e_2^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 \mathsf{stat}\ e_2^1$

CASE $(\lambda x.\, e_0^1)\ v_1^1 \rightarrow_{1\text{-S}} e_0^1[x \leftarrow v_1^1]$
$\wedge\ v_0^{\mathsf{H}} = \lambda x.\, e_0^{\mathsf{H}}$:

1. $e_0^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 e_0^{\mathsf{H}}$
   $\wedge\ v_1^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_1^1$
   by $v_0^{\mathsf{H}}\ v_1^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_0^1\ v_1^1$
2. $v_0^{\mathsf{H}}\ v_1^{\mathsf{H}} \rhd_{\mathsf{H\text{-}D}} e_0^{\mathsf{H}}[x \leftarrow v_1^{\mathsf{H}}]$
   by definition $\rhd_{\mathsf{H\text{-}D}}$
3. $e_0^{\mathsf{H}}[x \leftarrow v_1^{\mathsf{H}}]\ {}_{\mathsf{H}}\!\lesssim_1 e_0^1[x \leftarrow v_1^1]$
   by H−1 *substitution*
4. QED

CASE $(\lambda x.\, e_0^1)\ v_1^1 \rightarrow_{1\text{-D}} e_0^1[x \leftarrow v_1^1]$
$\wedge\ v_0^{\mathsf{H}} = \mathsf{mon}\,(\tau_d \Rightarrow \tau_c)\ v_{0'}^{\mathsf{H}}$:

1. $v_{0'}^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 (\lambda x.\, e_0^1) v_1^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_1^1$
   by $v_0^{\mathsf{H}}\ v_1^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_0^1\ v_1^1$
2. $v_0^{\mathsf{H}}\ v_1^{\mathsf{H}} \rhd_{\mathsf{H\text{-}D}} \mathsf{stat}\ \tau_c\ (v_{0'}^{\mathsf{H}}\ (\mathsf{dyn}\ \tau_d\ v_1^{\mathsf{H}}))$
   by definition $\rhd_{\mathsf{H\text{-}D}}$
3. $\vdash_{\mathsf{H}} \mathsf{dyn}\ \tau_d\ v_1^{\mathsf{H}} : \tau_d$
   by H *dynamic preservation*
4. $\mathsf{dyn}\ \tau_d\ v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} v_{1'}^{\mathsf{H}}$ and $v_{1'}^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_1^1$
   $\vee\ \mathsf{dyn}\ \tau_d\ v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
   by H−1 *boundary checking* (1, 3)
5. **IF** $\mathsf{dyn}\ \tau_d\ v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$:
   a. QED $v_0^{\mathsf{H}}\ v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$
6. $v_{0'}^{\mathsf{H}} = \mathsf{mon}\,(\tau'_d \Rightarrow \tau'_c)\ v_{0''}^{\mathsf{H}}$ and $v_{0''}^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_0^1$
   $\vee\ v_{0'}^{\mathsf{H}} = \lambda x.\, e^{\mathsf{H}}$ and $e^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 e_0^1$
   by (1)
7. **IF** $v_{0'}^{\mathsf{H}} = \lambda x.\, e^{\mathsf{H}}$:
   a. $\vdash_{\mathsf{H}} v_{0'}^{\mathsf{H}} : \tau_0$
      $\wedge\ \tau_0 \leqslant: \tau_d \Rightarrow \tau_c$
      by H *inversion*
   b. Contradiction by $\vdash_{\mathsf{H}} \lambda x.\, e^{\mathsf{H}} : \tau_0$
8. $v_{0'}^{\mathsf{H}}\ v_{1'}^{\mathsf{H}} \rhd_{\mathsf{H\text{-}S}} \mathsf{dyn}\ \tau'_c\ (\mathsf{stat}\ \tau'_d\ v_{1'}^{\mathsf{H}})$
   by definition $\rhd_{\mathsf{H\text{-}S}}$
9. $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau'_d\ v_{1'}^{\mathsf{H}} : \tau'_d$
   by H *static preservation*
10. $\mathsf{stat}\ \tau'_d\ v_{1'}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} v_{1''}^{\mathsf{H}}$ and $v_{1''}^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_1^1$
    by H−1 *boundary checking*
11. $\mathsf{stat}\ \tau_c\ (\mathsf{dyn}\ \tau'_c\ [\,])\ {}_{\mathsf{H}}\!\lesssim_1 [\,]$
    by definition ${}_{\mathsf{H}}\!\lesssim_1$
12. $v_{0''}^{\mathsf{H}}\ v_{1''}^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v_0^1\ v_1^1$
13. $\vdash_{\mathsf{H}} v_{0''}^{\mathsf{H}}\ v_{1''}^{\mathsf{H}}$
    by H *dynamic preservation*
14. $v_{0''}^{\mathsf{H}}\ v_{1''}^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}D}} e_2^{\mathsf{H}}$
    $\wedge\ e_2^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 e_2^1$
    by the induction hypothesis
15. QED $v_0^{\mathsf{H}}\ v_1^{\mathsf{H}} \rightarrow^*_{\mathsf{H\text{-}S}} \mathsf{stat}\ \tau_c\ (\mathsf{dyn}\ \tau'_c\ e_2^{\mathsf{H}})$

□

Lemma 8.16 : $\mathcal{X}$ *inversion*

If $\mathcal{X}(\lfloor\tau\rfloor, v^1) = \mathsf{BndryErr}$ and $v^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_1 v^1$ then $\mathcal{D}_{\mathsf{H}}(\tau, v^{\mathsf{H}}) = \mathsf{BndryErr}$

Proof:

By case analysis on τ.

CASE $\tau = \mathsf{Nat}$:

1. $v^1 \notin \mathbb{N}$
   by $\mathcal{X}(\lfloor\tau\rfloor, v^1) = \mathsf{BndryErr}$
2. $v^{\mathsf{H}} \notin \mathbb{N}$
   by (1)
3. $\mathcal{D}_{\mathsf{H}}(\tau, v^{\mathsf{H}}) = \mathsf{BndryErr}$
   by (2)
4. QED

CASE $\tau = \mathsf{Int}$:

1. $v^1 \notin \mathbb{Z}$
   by $\mathcal{X}(\lfloor\tau\rfloor, v^1) = \mathsf{BndryErr}$
2. $v^{\mathsf{H}} \notin \mathbb{Z}$
   by (1)
3. $\mathcal{D}_{\mathsf{H}}(\tau, v^{\mathsf{H}}) = \mathsf{BndryErr}$
   by (2)
4. QED

CASE $\tau = \tau_0 \times \tau_1$:

1. $v^1 \notin \langle v, v\rangle$
   by $\mathcal{X}(\lfloor\tau\rfloor, v^1) = \mathsf{BndryErr}$
2. $v^{\mathsf{H}} \notin \langle v, v\rangle$
   by (1)
3. $\mathcal{D}_{\mathsf{H}}(\tau, v^{\mathsf{H}}) = \mathsf{BndryErr}$
   by (2)
4. QED

case $\tau = \tau_d \Rightarrow \tau_c$:
1. $v^1 \in \mathbb{Z}$
   $\vee\ v^1 \in \langle v, v \rangle$
   by $\mathcal{X}(\lfloor \tau \rfloor, v^1) = \mathsf{BndryErr}$
2. $v^{\mathsf{H}} \in \mathbb{Z}$
   $\vee\ v^{\mathsf{H}} \in \langle v, v \rangle$
   by (1)
3. $\mathcal{D}_{\mathsf{H}}(\tau, v^{\mathsf{H}}) = \mathsf{BndryErr}$
   by (2)
4. QED

□

Lemma 8.17 : H−1 *static value stutter*

If $e^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^1$ and $\vdash_{\mathsf{H}} e^{\mathsf{H}} : \tau$ then one of the following holds:
- $e^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}S}} v^{\mathsf{H}}$ and $v^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^1$
- $e^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

Proof:

By induction on the number of boundary terms in e^{H}, and case analysis on $e^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^1$.

case e^{H} is a value :
1. QED

case $e^{\mathsf{H}} = \mathsf{dyn}\ \tau\ (\mathsf{stat}\ \tau'\ e_2^{\mathsf{H}})$:
1. $e_2^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^1$
   by ${}_{\mathsf{H}}{\lesssim}_{1}$
2. $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau'\ e_2^{\mathsf{H}}$
   $\wedge \vdash_{\mathsf{H}} e_2^{\mathsf{H}} : \tau'$
   by H *inversion* and H *inversion*
3. $e_2^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}S}} v_2^{\mathsf{H}}$
   $\wedge\ v_2^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^1$
   by the induction hypothesis (1, 2)
4. $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau'\ v_2^{\mathsf{H}}$
   by H *static preservation* (2, 3)
5. $\mathsf{stat}\ \tau'\ v_2^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}S}} v_3^{\mathsf{H}}$ and $v_3^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^1$
   $\vee\ \mathsf{stat}\ \tau'\ v_2^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
   by H−1 *boundary checking* (4)
6. **if** $\mathsf{stat}\ \tau'\ v_2^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}S}} v_3^{\mathsf{H}}$:
   a. $\vdash_{\mathsf{H}} \mathsf{dyn}\ \tau\ v_3^{\mathsf{H}} : \tau$
      by H *static preservation* (5)
   b. $\mathsf{dyn}\ \tau\ v_3^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}S}} v_4^{\mathsf{H}}$ and $v_4^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^1$
      $\vee\ \mathsf{dyn}\ \tau\ v_3^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
      by H−1 *boundary checking* (a)
   c. QED (b)

   else $\mathsf{stat}\ \tau'\ v_2^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$:
   a. QED $\mathsf{dyn}\ \tau\ (\mathsf{stat}\ \tau'\ e_2^{\mathsf{H}}) \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

case $e^{\mathsf{H}} = \mathsf{stat}\ \tau\ (\mathsf{dyn}\ \tau'\ e_2^{\mathsf{H}})$:
1. Contradiction by $\vdash_{\mathsf{H}} e^{\mathsf{H}} : \tau$

□

Lemma 8.18 : H−1 *dynamic value stutter*

If $e^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^1$ and $\vdash_{\mathsf{H}} e^{\mathsf{H}}$ then one of the following holds:
- $e^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}D}} v^{\mathsf{H}}$ and $v^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^1$
- $e^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}D}} \mathsf{BndryErr}$

Proof:

By induction on the number of boundary terms in e^{H}, and case analysis on $e^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^1$.

case e^{H} is a value :
1. QED

case $e^{\mathsf{H}} = \mathsf{dyn}\ \tau\ (\mathsf{stat}\ \tau'\ e_2^{\mathsf{H}})$:
1. Contradiction by $\vdash_{\mathsf{H}} e^{\mathsf{H}} : \tau$

case $e^{\mathsf{H}} = \mathsf{stat}\ \tau\ (\mathsf{dyn}\ \tau'\ e_2^{\mathsf{H}})$:
1. $e_2^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^1$
   by ${}_{\mathsf{H}}{\lesssim}_{1}$
2. $\vdash_{\mathsf{H}} \mathsf{dyn}\ \tau'\ e_2^{\mathsf{H}} : \tau'$
   $\wedge \vdash_{\mathsf{H}} e_2^{\mathsf{H}}$
   by H *inversion* and H *inversion*
3. $e_2^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}D}} v_2^{\mathsf{H}}$
   $\wedge\ v_2^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^1$
   by the induction hypothesis (1, 2)
4. $\vdash_{\mathsf{H}} \mathsf{dyn}\ \tau'\ v_2^{\mathsf{H}}$
   by H *static preservation* (2, 3)
5. $\mathsf{dyn}\ \tau'\ v_2^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}S}} v_3^{\mathsf{H}}$ and $v_3^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^1$
   $\vee\ \mathsf{dyn}\ \tau'\ v_2^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
   by H−1 *boundary checking* (4)
6. **if** $\mathsf{dyn}\ \tau'\ v_2^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}S}} v_3^{\mathsf{H}}$:
   a. $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau\ v_3^{\mathsf{H}} : \tau$
      by H *static preservation* (5)
   b. $\mathsf{stat}\ \tau\ v_3^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}S}} v_4^{\mathsf{H}}$ and $v_4^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^1$
      $\vee\ \mathsf{stat}\ \tau\ v_3^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
      by H−1 *boundary checking* (a)
   c. QED (b)

   else $\mathsf{dyn}\ \tau'\ v_2^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$:
   a. QED $\mathsf{stat}\ \tau\ (\mathsf{dyn}\ \tau'\ e_2^{\mathsf{H}}) \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$

□

Lemma 8.19 : H−1 *static hole typing*

If $\vdash_{\mathsf{H}} E^{\mathsf{H}}[e^{\mathsf{H}}] : \tau$
$\wedge \vdash_{1} E^{1}[e^{1}] : K$
$\wedge\ E^{\mathsf{H}}[e^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ E^{1}[e^{1}]$
then one of the following holds:
- $\vdash_{\mathsf{H}} e^{\mathsf{H}} : \tau'$
  $\wedge \vdash_{1} e^{1} : K'$
- $\vdash_{\mathsf{H}} e^{\mathsf{H}}$
  $\wedge \vdash_{1} e^{1}$

Proof:

By induction on the structure of $E^{\mathsf{H}}[e^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ E^{1}[e^{1}]$ judgment.

case $[\,]\ {}_{\mathsf{H}}{\lesssim}_{1}\ [\,]$:
1. $E^{\mathsf{H}}[e^{\mathsf{H}}] = e^{\mathsf{H}}$
2. $E^{1}[e^{1}] = e^{1}$
3. QED

case $E^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{chk}\ K\ E_0^{1}$:
1. $E^{1}[e^{1}] = \mathsf{chk}\ K\ E_0^{1}[e^{1}]$
2. $\vdash_{1} E_0^{1}[e^{1}] : \mathsf{Any}$
   by 1 *inversion*
3. $E^{\mathsf{H}}[e^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}[e^{1}]$
4. QED by the induction hypothesis

case $E_0^{\mathsf{H}}\ e_1^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}\ e_1^{1}$:

1. $E^{\mathsf{H}}[e^{\mathsf{H}}] = E_0^{\mathsf{H}}[e^{\mathsf{H}}]\ e_1^{\mathsf{H}}$
   $\wedge\ E^{1}[e^{1}] = E_0^{1}[e^{1}]\ e_1^{1}$
2. $\vdash_{\mathsf{H}} E_0^{\mathsf{H}}[e^{\mathsf{H}}] : \tau_d \Rightarrow \tau_c$
   by H *inversion*
3. $\vdash_{1} E_0^{1}[e^{1}] : \mathsf{Fun}$
4. $E_0^{\mathsf{H}}[e^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}[e^{1}]$
5. QED by the induction hypothesis

CASE $v_0^{\mathsf{H}}\ E_1^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}\ E_1^{1}$:
1. $E^{\mathsf{H}}[e^{\mathsf{H}}] = v_0^{\mathsf{H}}\ E_1^{\mathsf{H}}[e^{\mathsf{H}}]$
   $\wedge\ E^{1}[e^{1}] = v_0^{1}\ E_1^{1}[e^{1}]$
2. $\vdash_{\mathsf{H}} E_1^{\mathsf{H}}[e^{\mathsf{H}}] : \tau_d$
   by H *inversion*
3. $\vdash_{1} E_1^{1}[e^{1}] : \mathsf{Any}$
4. $E_1^{\mathsf{H}}[e^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_1^{1}[e^{1}]$
5. QED by the induction hypothesis

CASE $\langle E_0^{\mathsf{H}}, e_1^{\mathsf{H}}\rangle\ {}_{\mathsf{H}}{\lesssim}_{1}\ \langle E_0^{1}, e_1^{1}\rangle$:
1. $E^{\mathsf{H}}[e^{\mathsf{H}}] = \langle E_0^{\mathsf{H}}[e^{\mathsf{H}}], e_1^{\mathsf{H}}\rangle$
   $\wedge\ E^{1}[e^{1}] = \langle E_0^{1}[e^{1}], e_1^{1}\rangle$
2. $\vdash_{\mathsf{H}} E_0^{\mathsf{H}}[e^{\mathsf{H}}] : \tau_0$
   by H *inversion*
3. $\vdash_{1} E_0^{1}[e^{1}] : \mathsf{Any}$
4. $E_0^{\mathsf{H}}[e^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}[e^{1}]$
5. QED by the induction hypothesis

CASE $\langle v_0^{\mathsf{H}}, E_1^{\mathsf{H}}\rangle\ {}_{\mathsf{H}}{\lesssim}_{1}\ \langle v_0^{1}, E_1^{1}\rangle$:
1. $E^{\mathsf{H}}[e^{\mathsf{H}}] = \langle v_0^{\mathsf{H}}, E_1^{\mathsf{H}}[e^{\mathsf{H}}]\rangle$
   $\wedge\ E^{1}[e^{1}] = \langle v_0^{1}, E_1^{1}[e^{1}]\rangle$
2. $\vdash_{\mathsf{H}} E_1^{\mathsf{H}}[e^{\mathsf{H}}] : \tau_1$
   by H *inversion*
3. $\vdash_{1} E_1^{1}[e^{1}] : \mathsf{Any}$
4. $E_1^{\mathsf{H}}[e^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_1^{1}[e^{1}]$
5. QED by the induction hypothesis

CASE $op^1\ E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ op^1\ E_0^{1}$:
1. $E^{\mathsf{H}}[e^{\mathsf{H}}] = op^1\ E_0^{\mathsf{H}}[e^{\mathsf{H}}]$
   $\wedge\ E^{1}[e^{1}] = op^1\ E_0^{1}[e^{1}]$
2. $\vdash_{\mathsf{H}} E_0^{\mathsf{H}}[e^{\mathsf{H}}] : \tau_0 \times \tau_1$
   by H *inversion*
3. $\vdash_{1} E_0^{1}[e^{1}] : \mathsf{Pair}$
4. $E_0^{\mathsf{H}}[e^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}[e^{1}]$
5. QED by the induction hypothesis

CASE $op^2\ E_0^{\mathsf{H}}\ e_1^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ op^2\ E_0^{1}\ e_1^{1}$:
1. $E^{\mathsf{H}}[e^{\mathsf{H}}] = op^2\ E_0^{\mathsf{H}}[e^{\mathsf{H}}]\ e_1^{\mathsf{H}}$
   $\wedge\ E^{1}[e^{1}] = op^2\ E_0^{1}[e^{1}]\ e_1^{1}$
2. $\vdash_{\mathsf{H}} E_0^{\mathsf{H}}[e^{\mathsf{H}}] : \tau_0$
   by H *inversion*
3. $\vdash_{1} E_0^{1}[e^{1}] : K_0$
4. $E_0^{\mathsf{H}}[e^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}[e^{1}]$
5. QED by the induction hypothesis

CASE $op^2\ v_0^{\mathsf{H}}\ E_1^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ op^2\ v_0^{1}\ E_1^{1}$:
1. $E^{1}[e^{1}] = op^2\ v_0^{\mathsf{H}}\ E_1^{1}[e^{1}]$
2. $\vdash_{\mathsf{H}} E_1^{\mathsf{H}}[e^{\mathsf{H}}] : \tau_1$
   by H *inversion*
3. $\vdash_{1} E_1^{1}[e^{1}] : K_1$
4. $E_1^{\mathsf{H}}[e^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_1^{1}[e^{1}]$
5. QED by the induction hypothesis

CASE $\mathsf{dyn}\ \tau_0\ E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{dyn}\ \tau_0\ E_0^{1}$:
1. $E^{\mathsf{H}}[e^{\mathsf{H}}] = \mathsf{dyn}\ \tau_0\ E_0^{\mathsf{H}}[e^{\mathsf{H}}]$
   $\wedge\ E^{1}[e^{1}] = \mathsf{dyn}\ \tau_0\ E_0^{1}[e^{1}]$
2. $\vdash_{\mathsf{H}} E_0^{\mathsf{H}}[e^{\mathsf{H}}]$
   $\wedge \vdash_{1} E_0^{1}[e^{1}]$
3. $E_0^{\mathsf{H}}[e^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}[e^{1}]$
4. QED H−1 *dynamic hole typing*

CASE $\mathsf{dyn}\ \tau_0\ E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{dyn}\ E_0^{1}$:
1. $E^{\mathsf{H}}[e^{\mathsf{H}}] = \mathsf{dyn}\ \tau_0\ E_0^{\mathsf{H}}[e^{\mathsf{H}}]$
   $\wedge\ E^{1}[e^{1}] = \mathsf{dyn}\ E_0^{1}[e^{1}]$
2. $\vdash_{\mathsf{H}} E_0^{\mathsf{H}}[e^{\mathsf{H}}]$
   $\wedge \vdash_{1} E_0^{1}[e^{1}]$
3. $E_0^{\mathsf{H}}[e^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}[e^{1}]$
4. QED H−1 *dynamic hole typing*

CASE $\mathsf{dyn}\ \tau_0\ (\mathsf{stat}\ \tau_1\ E_0^{\mathsf{H}})\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}$:
1. $E^{\mathsf{H}}[e^{\mathsf{H}}] = \mathsf{dyn}\ \tau_0\ (\mathsf{stat}\ \tau_1\ E_0^{\mathsf{H}}[e^{\mathsf{H}}])$
2. $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau_1\ E_0^{\mathsf{H}}[e^{\mathsf{H}}]$
3. $\vdash_{\mathsf{H}} E_0^{\mathsf{H}}[e^{\mathsf{H}}] : \tau_1$
4. $E_0^{\mathsf{H}}[e^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ E^{1}[e^{1}]$
5. QED by the induction hypothesis

CASE $\mathsf{stat}\ \tau_0\ E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{stat}\ \tau_0\ E_0^{1}$:
1. Contradiction by $\vdash_{\mathsf{H}} E^{\mathsf{H}}[e^{\mathsf{H}}] : \tau_0$

CASE $\mathsf{stat}\ \tau_0\ E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{stat}\ E_0^{1}$:
1. Contradiction by $\vdash_{\mathsf{H}} E^{\mathsf{H}}[e^{\mathsf{H}}] : \tau_0$

CASE $\mathsf{stat}\ \tau_0\ (\mathsf{dyn}\ \tau_1\ E_0^{\mathsf{H}})\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}$:
1. Contradiction by $\vdash_{\mathsf{H}} E^{\mathsf{H}}[e^{\mathsf{H}}] : \tau$

□

Lemma 8.20 : H−1 *dynamic hole typing*

If $\vdash_{\mathsf{H}} E^{\mathsf{H}}[e^{\mathsf{H}}]$
$\wedge \vdash_{1} E^{1}[e^{1}]$
$\wedge\ E^{\mathsf{H}}[e^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ E^{1}[e^{1}]$
then one of the following holds:
- $\vdash_{\mathsf{H}} e^{\mathsf{H}} : \tau'$
  $\wedge \vdash_{1} e^{1} : K$
- $\vdash_{\mathsf{H}} e^{\mathsf{H}}$
  $\wedge \vdash_{1} e^{1}$

Proof:

By induction on the structure of the $E^{\mathsf{H}}[e^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ E^{1}[e^{1}]$ judgment.

CASE $[\,]\ {}_{\mathsf{H}}{\lesssim}_{1}\ [\,]$:
1. $E^{\mathsf{H}}[e^{\mathsf{H}}] = e^{\mathsf{H}}$
2. $E^{1}[e^{1}] = e^{1}$
3. QED

CASE $E^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{chk}\ K\ E_0^{1}$:
1. Contradiction by $\vdash_{1} E^{1}[e^{1}]$

CASE $E_0^{\mathsf{H}}\ e_1^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}\ e_1^{1}$:
1. $E^{\mathsf{H}}[e^{\mathsf{H}}] = E_0^{\mathsf{H}}[e^{\mathsf{H}}]\ e_1^{\mathsf{H}}$
   $\wedge\ E^{1}[e^{1}] = E_0^{1}[e^{1}]\ e_1^{1}$
2. $\vdash_{\mathsf{H}} E_0^{\mathsf{H}}[e^{\mathsf{H}}]$
   by H *inversion*

3. $\vdash_1 E_0^1[e^1]$
4. $E_0^{H}[e^{H}] \;{}_{H}\!\lesssim_1 E_0^1[e^1]$
5. QED by the induction hypothesis

CASE $v_0^{H}\ E_1^{H} \;{}_{H}\!\lesssim_1 v_0^1\ E_1^1$:
1. $E^{H}[e^{H}] = v_0^{H}\ E_1^{H}[e^{H}]$
   $\wedge\ E^1[e^1] = v_0^1\ E_1^1[e^1]$
2. $\vdash_H E_1^{H}[e^{H}]$
   by H *inversion*
3. $\vdash_1 E_1^1[e^1]$
4. $E_1^{H}[e^{H}] \;{}_{H}\!\lesssim_1 E_1^1[e^1]$
5. QED by the induction hypothesis

CASE $\langle E_0^{H}, e_1^{H}\rangle \;{}_{H}\!\lesssim_1 \langle E_0^1, e_1^1\rangle$:
1. $E^{H}[e^{H}] = \langle E_0^{H}[e^{H}], e_1^{H}\rangle$
   $\wedge\ E^1[e^1] = \langle E_0^1[e^1], e_1^1\rangle$
2. $\vdash_H E_0^{H}[e^{H}]$
   by H *inversion*
3. $\vdash_1 E_0^1[e^1]$
4. $E_0^{H}[e^{H}] \;{}_{H}\!\lesssim_1 E_0^1[e^1]$
5. QED by the induction hypothesis

CASE $\langle v_0^{H}, E_1^{H}\rangle \;{}_{H}\!\lesssim_1 \langle v_0^1, E_1^1\rangle$:
1. $E^{H}[e^{H}] = \langle v_0^{H}, E_1^{H}[e^{H}]\rangle$
   $\wedge\ E^1[e^1] = \langle v_0^1, E_1^1[e^1]\rangle$
2. $\vdash_H E_1^{H}[e^{H}]$
   by H *inversion*
3. $\vdash_1 E_1^1[e^1]$
4. $E_1^{H}[e^{H}] \;{}_{H}\!\lesssim_1 E_1^1[e^1]$
5. QED by the induction hypothesis

CASE $op^1\ E_0^{H} \;{}_{H}\!\lesssim_1 op^1\ E_0^1$:
1. $E^{H}[e^{H}] = op^1\ E_0^{H}[e^{H}]$
   $\wedge\ E^1[e^1] = op^1\ E_0^1[e^1]$
2. $\vdash_H E_0^{H}[e^{H}]$
   by H *inversion*
3. $\vdash_1 E_0^1[e^1]$
4. $E_0^{H}[e^{H}] \;{}_{H}\!\lesssim_1 E_0^1[e^1]$
5. QED by the induction hypothesis

CASE $op^2\ E_0^{H}\ e_1^{H} \;{}_{H}\!\lesssim_1 op^2\ E_0^1\ e_1^1$:
1. $E^{H}[e^{H}] = op^2\ E_0^{H}[e^{H}]\ e_1^{H}$
   $\wedge\ E^1[e^1] = op^2\ E_0^1[e^1]\ e_1^1$
2. $\vdash_H E_0^{H}[e^{H}]$
   by H *inversion*
3. $\vdash_1 E_0^1[e^1]$
4. $E_0^{H}[e^{H}] \;{}_{H}\!\lesssim_1 E_0^1[e^1]$
5. QED by the induction hypothesis

CASE $op^2\ v_0^{H}\ E_1^{H} \;{}_{H}\!\lesssim_1 op^2\ v_0^1\ E_1^1$:
1. $E^{H}[e^{H}] = op^2\ e_0^{H}\ E_1^{H}[e^{H}]$
   $\wedge\ E^1[e^1] = op^2\ e_0^1\ E_1^1[e^1]$
2. $\vdash_H E_1^{H}[e^{H}]$
   by H *inversion*
3. $\vdash_1 e_0^1$
   $\wedge \vdash_1 E_1^1[e^1]$
4. $E_1^{H}[e^{H}] \;{}_{H}\!\lesssim_1 E_1^1[e^1]$
5. QED by the induction hypothesis

CASE $\mathsf{dyn}\ \tau_0\ E_0^{H} \;{}_{H}\!\lesssim_1 \mathsf{dyn}\ \tau_0\ E_0^1$:
1. Contradiction by $\vdash_H E^{H}[e^{H}]$

CASE $\mathsf{dyn}\ \tau_0\ E_0^{H} \;{}_{H}\!\lesssim_1 \mathsf{dyn}\ E_0^1$:
1. Contradiction by $\vdash_H E^{H}[e^{H}]$

CASE $\mathsf{dyn}\ \tau_0\ (\mathsf{stat}\ \tau_1\ E_0^{H}) \;{}_{H}\!\lesssim_1 E_0^1$:
1. Contradiction by $\vdash_H E^{H}[e^{H}]$

CASE $\mathsf{stat}\ \tau_0\ E_0^{H} \;{}_{H}\!\lesssim_1 \mathsf{stat}\ \tau_0\ E_0^1$:
1. $E^{H}[e^{H}] = \mathsf{stat}\ \tau_0\ E_0^{H}[e^{H}]$
   $\wedge\ E^1[e^1] = \mathsf{stat}\ \tau_0\ E_0^1[e^1]$
2. $\vdash_H E_0^{H}[e^{H}] : \tau_0$
   $\wedge \vdash_1 E_0^1[e^1] : \lfloor \tau_0 \rfloor$
3. $E_0^{H}[e^{H}] \;{}_{H}\!\lesssim_1 E_0^1[e^1]$
4. QED H−1 *static hole typing*

CASE $\mathsf{stat}\ \tau_0\ E_0^{H} \;{}_{H}\!\lesssim_1 \mathsf{stat}\ E_0^1$:
1. $E^{H}[e^{H}] = \mathsf{stat}\ \tau_0\ E_0^{H}[e^{H}]$
   $\wedge\ E^1[e^1] = \mathsf{stat}\ E_0^1[e^1]$
2. $\vdash_H E_0^{H}[e^{H}] : \tau_0$
   $\wedge \vdash_1 E_0^1[e^1]$
3. $E_0^{H}[e^{H}] \;{}_{H}\!\lesssim_1 E_0^1[e^1]$
4. QED H−1 *static hole typing*

CASE $\mathsf{stat}\ \tau_0\ (\mathsf{dyn}\ \tau_1\ E_0^{H}) \;{}_{H}\!\lesssim_1 E_0^1$:
1. $E^{H}[e^{H}] = \mathsf{stat}\ \tau_0\ (\mathsf{dyn}\ \tau_1\ E_0^{H}[e^{H}])$
2. $\vdash_H \mathsf{dyn}\ \tau_1\ E_0^{H}[e^{H}] : \tau_1$
3. $\vdash_H E_0^{H}[e^{H}]$
4. $E_0^{H}[e^{H}] \;{}_{H}\!\lesssim_1 E^1[e^1]$
5. QED by the induction hypothesis

□

Lemma 8.21 : H−1 *value inversion*

If $\vdash_H v^{H} : \tau$
$\wedge\ v^{H} \;{}_{H}\!\lesssim_1 v^1$
then $\vdash_1 v^1 : \lfloor \tau \rfloor$

Proof:

By induction on the structure of the $v^{H} \;{}_{H}\!\lesssim_1 v^1$ judgment.

CASE $\langle v_0^{H}, v_1^{H}\rangle \;{}_{H}\!\lesssim_1 \langle v_0^1, v_1^1\rangle$:
1. $\tau = \tau_0 \times \tau_1$
   by H *inversion*
2. $\lfloor \tau \rfloor = \mathsf{Pair}$
3. QED $\vdash_1 \langle v_0^1, v_1^1\rangle : \mathsf{Pair}$

$i \;{}_{H}\!\lesssim_1 i$

by **IF** $\tau = \mathsf{Nat}$:
1. $i \in \mathbb{N}$
   by H *inversion*
2. QED $\vdash_1 i\mathsf{Nat}$

ELSE $\tau = \mathsf{Int}$:
1. QED $\vdash_1 i : \mathsf{Int}$

$\lambda x.\, e^{H} \;{}_{H}\!\lesssim_1 \lambda x.\, e^1$

by Contradiction by $\vdash_H \lambda x.\, e^1 : \tau$

$\lambda(x\!:\!\tau).\, e^{H} \;{}_{H}\!\lesssim_1 \lambda(x\!:\!\tau).\, e^1$

by $\tau = \tau_d \Rightarrow \tau_c$

by H *inversion*$\lfloor \tau \rfloor = \mathsf{Fun}$QED $\vdash_1 \lambda(x\!:\!\tau).\, e^1 : \mathsf{Fun}$

$\mathsf{mon}\ \tau\ v^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}$
by $v^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}$QED by the induction hypothesis
□

Lemma 8.22 : H−1 *boundary checking*

- If $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau\ v^{\mathsf{H}}$ and $v^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}$ then $\mathsf{stat}\ \tau\ v^{\mathsf{H}} \rightarrow^{*}_{\mathsf{1\text{-}D}} v^{\mathsf{H}}_{1}$ $\wedge\ v^{\mathsf{H}}_{1}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}$
- If $\vdash_{\mathsf{H}} \mathsf{dyn}\ \tau\ v^{\mathsf{H}} : \tau$ and $v^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}$ then one of the following holds:
  - $\mathsf{dyn}\ \tau\ v^{\mathsf{H}} \rightarrow^{*}_{\mathsf{1\text{-}S}} \mathsf{BndryErr}$
  - $\mathsf{dyn}\ \tau\ v^{\mathsf{H}} \rightarrow^{*}_{\mathsf{1\text{-}S}} v^{\mathsf{H}}_{1}$ $\wedge\ v^{\mathsf{H}}_{1}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}$

Proof:
By the following two lemmas: H−1 stat *checking* and H−1 dyn *checking*.
□

Lemma 8.23 : H−1 stat *checking*

If $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau\ v^{\mathsf{H}}$
$\wedge\ v^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}$
then $\mathsf{stat}\ \tau\ v^{\mathsf{H}} \rightarrow^{*}_{\mathsf{1\text{-}D}} v^{\mathsf{H}}_{1}$
$\wedge\ v^{\mathsf{H}}_{1}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}$

Proof:
By induction on the structure of τ.

CASE $\tau = \mathsf{Nat}$:
1. $\mathsf{stat}\ \tau\ v^{\mathsf{H}} \rhd_{\mathsf{H\text{-}D}} v^{\mathsf{H}}$
   by definition $\rhd_{\mathsf{H\text{-}D}}$
2. QED

CASE $\tau = \mathsf{Int}$:
1. $\mathsf{stat}\ \tau\ v^{\mathsf{H}} \rhd_{\mathsf{H\text{-}D}} v^{\mathsf{H}}$
   by definition $\rhd_{\mathsf{H\text{-}D}}$
2. QED

CASE $\tau = \tau_0 \times \tau_1$:
1. $v^{\mathsf{H}} = \langle v^{\mathsf{H}}_{0}, v^{\mathsf{H}}_{1} \rangle$
   by H *inversion* and *canonical forms*
2. $v^{1} = \langle v^{1}_{0}, v^{1}_{1} \rangle$
   $\wedge\ v^{\mathsf{H}}_{0}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}_{0}$
   $\wedge\ v^{\mathsf{H}}_{1}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}_{1}$
   by ${}_{\mathsf{H}}\!\lesssim_{1}$
3. $\mathsf{stat}\ \tau\ v^{\mathsf{H}} \rhd_{\mathsf{H\text{-}D}} \langle \mathsf{stat}\ \tau_0\ v^{\mathsf{H}}_{0}, \mathsf{stat}\ \tau_1\ v^{\mathsf{H}}_{1} \rangle$
   by definition $\rhd_{\mathsf{H\text{-}D}}$
4. $\vdash_{\mathsf{H}} \mathsf{stat}\ \tau_0\ v^{\mathsf{H}}_{0}$
   $\wedge \vdash_{\mathsf{H}} \mathsf{stat}\ \tau_1\ v^{\mathsf{H}}_{1}$
   by H *static preservation* and H *inversion*
5. $\mathsf{stat}\ \tau_0\ v^{\mathsf{H}}_{0} \rightarrow^{*}_{\mathsf{1\text{-}D}} v^{\mathsf{H}}_{0'}$
   $\wedge\ v^{\mathsf{H}}_{0'}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}_{0}$
   $\wedge\ \mathsf{stat}\ \tau_1\ v^{\mathsf{H}}_{1} \rightarrow^{*}_{\mathsf{1\text{-}D}} v^{\mathsf{H}}_{1'}$
   $\wedge\ v^{\mathsf{H}}_{1'}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}_{1}$
   by the induction hypothesis (2, 4)
6. $\mathsf{stat}\ \tau\ v^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}D}} \langle v^{\mathsf{H}}_{0'}, v^{\mathsf{H}}_{1'} \rangle$
   $\wedge\ \langle v^{\mathsf{H}}_{0'}, v^{\mathsf{H}}_{1'} \rangle\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}$
   by (5)
7. QED

CASE $\tau = \tau_d \Rightarrow \tau_c$:
1. $\mathsf{stat}\ \tau\ v^{\mathsf{H}} \rhd_{\mathsf{H\text{-}D}} \mathsf{mon}\ \tau\ v^{\mathsf{H}}$
2. $\mathsf{mon}\ \tau\ v^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}$
   by $v^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}$
3. QED

□

Lemma 8.24 : H−1 dyn *checking*

If $\vdash_{\mathsf{H}} \mathsf{dyn}\ \tau\ v^{\mathsf{H}} : \tau$
$\wedge\ v^{\mathsf{H}}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}$
then one of the following holds:
- $\mathsf{dyn}\ \tau\ v^{\mathsf{H}} \rightarrow^{*}_{\mathsf{1\text{-}S}} \mathsf{BndryErr}$
- $\mathsf{dyn}\ \tau\ v^{\mathsf{H}} \rightarrow^{*}_{\mathsf{1\text{-}S}} v^{\mathsf{H}}_{1}$
  $\wedge\ v^{\mathsf{H}}_{1}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}$

Proof:
By induction on the structure of τ.

CASE $\tau = \mathsf{Nat}$:

IF $v^{\mathsf{H}} \in \natural$:
1. $\mathsf{dyn}\ \tau\ v^{\mathsf{H}} \rhd_{\mathsf{H\text{-}S}} v^{\mathsf{H}}$
   by definition $\rhd_{\mathsf{H\text{-}S}}$
2. QED

ELSE $v^{\mathsf{H}} \notin \natural$:
1. $\mathsf{dyn}\ \tau\ v^{\mathsf{H}} \rhd_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
2. QED

CASE $\tau = \mathsf{Int}$:

IF $v^{\mathsf{H}} \in \mathbb{Z}$:
1. $\mathsf{dyn}\ \tau\ v^{\mathsf{H}} \rhd_{\mathsf{H\text{-}S}} v^{\mathsf{H}}$
   by definition $\rhd_{\mathsf{H\text{-}S}}$
2. QED

ELSE $v^{\mathsf{H}} \notin \mathbb{Z}$:
1. $\mathsf{dyn}\ \tau\ v^{\mathsf{H}} \rhd_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
2. QED

CASE $\tau = \tau_0 \times \tau_1$:

IF $v^{\mathsf{H}} = \langle v^{\mathsf{H}}_{0}, v^{\mathsf{H}}_{1} \rangle$:
1. $v^{1} = \langle v^{1}_{0}, v^{1}_{1} \rangle$
   $\wedge\ v^{\mathsf{H}}_{0}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}_{0}$
   $\wedge\ v^{\mathsf{H}}_{1}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}_{1}$
   by ${}_{\mathsf{H}}\!\lesssim_{1}$
2. $\mathsf{dyn}\ \tau\ v^{\mathsf{H}} \rhd_{\mathsf{H\text{-}S}} \langle \mathsf{dyn}\ \tau_0\ v^{\mathsf{H}}_{0}, \mathsf{dyn}\ \tau_1\ v^{\mathsf{H}}_{1} \rangle$
   by definition $\rhd_{\mathsf{H\text{-}S}}$
3. $\vdash_{\mathsf{H}} \mathsf{dyn}\ \tau_0\ v^{\mathsf{H}}_{0} : \tau_0$
   $\wedge \vdash_{\mathsf{H}} \mathsf{dyn}\ \tau_1\ v^{\mathsf{H}}_{1} : \tau_1$
   by H *static preservation* and H *inversion*
4. $\mathsf{dyn}\ \tau_0\ v^{\mathsf{H}}_{0} \rightarrow^{*}_{\mathsf{1\text{-}D}} e^{\mathsf{H}}_{0}$
   $\wedge\ e^{\mathsf{H}}_{0}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}_{0}$
   $\wedge\ e^{\mathsf{H}}_{0} = v^{\mathsf{H}}_{0'}$ or $e^{\mathsf{H}}_{0} = \mathsf{BndryErr}$
   $\wedge\ \mathsf{dyn}\ \tau_1\ v^{\mathsf{H}}_{1} \rightarrow^{*}_{\mathsf{1\text{-}D}} e^{\mathsf{H}}_{1}$
   $\wedge\ e^{\mathsf{H}}_{1}\ {}_{\mathsf{H}}\!\lesssim_{1}\ v^{1}_{1} e^{\mathsf{H}}_{1} = v^{\mathsf{H}}_{1'}$ or $e^{\mathsf{H}}_{1} = \mathsf{BndryErr}$
   by the induction hypothesis (2, 4)
5. $\mathsf{dyn}\ \tau\ v^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
   $\vee\ \mathsf{dyn}\ \tau\ v^{\mathsf{H}} \rightarrow^{*}_{\mathsf{H\text{-}S}} \langle v^{\mathsf{H}}_{0'}, v^{\mathsf{H}}_{1'} \rangle$ and $\langle v^{\mathsf{H}}_{0'}, v^{\mathsf{H}}_{1'} \rangle\ {}_{\mathsf{H}}\!\lesssim_{1}$ v^{1}
   by (5)
6. QED

ELSE $v^{\mathsf{H}} \notin \langle v, v \rangle$:

1. $\mathsf{dyn}\ \tau\ v^{\mathsf{H}} \rhd_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
2. QED

CASE $\tau = \tau_d \Rightarrow \tau_c$:

IF $v^{\mathsf{H}} = \lambda x.\, e^{\mathsf{H}}$:

1. $\mathsf{dyn}\ \tau\ v^{\mathsf{H}} \rhd_{\mathsf{H\text{-}S}} \mathsf{mon}\ \tau\ v^{\mathsf{H}}$
2. $\mathsf{mon}\ \tau\ v^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}$
   by $v^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}$
3. QED

IF $v^{\mathsf{H}} = \mathsf{mon}\ \tau_0\ v_0^{\mathsf{H}}$:

1. $\mathsf{dyn}\ \tau\ v^{\mathsf{H}} \rhd_{\mathsf{H\text{-}S}} \mathsf{mon}\ \tau\ v^{\mathsf{H}}$
2. $\mathsf{mon}\ \tau\ v^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}$
   by $v^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v^{1}$
3. QED

ELSE $v^{\mathsf{H}} \in i \vee v^{\mathsf{H}} \in \langle v, v\rangle$:

1. $\mathsf{dyn}\ \tau\ v^{\mathsf{H}} \rhd_{\mathsf{H\text{-}S}} \mathsf{BndryErr}$
2. QED

□

Lemma 8.25 : H−1 *hole substitution*

If $E^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E^{1}$
$\wedge\ e^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ e^{1}$
then $E^{\mathsf{H}}[e^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ E^{1}[e^{1}]$

Proof:

By induction on the structure of the $E^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E^{1}$ judgment.

CASE $[\,]\ {}_{\mathsf{H}}{\lesssim}_{1}\ [\,]$:

1. $E^{\mathsf{H}}[e^{\mathsf{H}}] = e^{\mathsf{H}}$
2. $E^{1}[e^{1}] = e^{1}$
3. QED

CASE $E^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{chk}\ K\ E_0^{1}$:

1. $E^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}$
2. QED by the induction hypothesis

CASE $E_0^{\mathsf{H}}\ e_1^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}\ e_1^{1}$:

1. $E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}$
2. QED by the induction hypothesis

CASE $v_0^{\mathsf{H}}\ E_1^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ v_0^{1}\ E_1^{1}$:

1. $E_1^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_1^{1}$
2. QED by the induction hypothesis

CASE $\langle E_0^{\mathsf{H}}, e_1^{\mathsf{H}}\rangle\ {}_{\mathsf{H}}{\lesssim}_{1}\ \langle E_0^{1}, e_1^{1}\rangle$:

1. $E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}$
2. QED by the induction hypothesis

CASE $\langle v_0^{\mathsf{H}}, E_1^{\mathsf{H}}\rangle\ {}_{\mathsf{H}}{\lesssim}_{1}\ \langle v_0^{1}, E_1^{1}\rangle$:

1. $E_1^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_1^{1}$
2. QED by the induction hypothesis

CASE $op^{1}\ E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ op^{1}\ E_0^{1}$:

1. $E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}$
2. QED by the induction hypothesis

CASE $op^{2}\ E_0^{\mathsf{H}}\ e_1^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ op^{2}\ E_0^{1}\ e_1^{1}$:

1. $E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}$
2. QED by the induction hypothesis

CASE $op^{2}\ v_0^{\mathsf{H}}\ E_1^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ op^{2}\ v_0^{1}\ E_1^{1}$:

1. $E_1^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_1^{1}$
2. QED by the induction hypothesis

CASE $\mathsf{dyn}\ \tau_0\ E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{dyn}\ \tau_0\ E_0^{1}$:

1. $E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}$
2. QED by the induction hypothesis

CASE $\mathsf{dyn}\ \tau_0\ E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{dyn}\ E_0^{1}$:

1. $E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}$
2. QED by the induction hypothesis

CASE $\mathsf{dyn}\ \tau_0\ (\mathsf{stat}\ \tau_1\ E_0^{\mathsf{H}})\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}$:

1. $E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}$
2. QED by the induction hypothesis

CASE $\mathsf{stat}\ \tau_0\ E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{stat}\ \tau_0\ E_0^{1}$:

1. $E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}$
2. QED by the induction hypothesis

CASE $\mathsf{stat}\ \tau_0\ E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{stat}\ E_0^{1}$:

1. $E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}$
2. QED by the induction hypothesis

CASE $\mathsf{stat}\ \tau_0\ (\mathsf{dyn}\ \tau_1\ E_0^{\mathsf{H}})\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}$:

1. $E_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ E_0^{1}$
2. QED by the induction hypothesis

□

Lemma 8.26 : H−1 *substitution*

If $e^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ e^{1}$ and $v^{\mathsf{H}}\ {}_{1}{\lesssim}_{\mathsf{E}}\ v^{1}$ then $e^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ e^{1}[x \leftarrow v^{1}]$

Proof:

By induction on the structure of the $e^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ e^{1}$ judgment.

CASE $e^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{chk}\ K_0\ e_0^{1}$:

1. $e^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ e_0^{1}$
2. $e^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ e_0^{1}[x \leftarrow v^{1}]$
   by the induction hypothesis
3. $\mathsf{chk}\ K_0\ e_0^{1}[x \leftarrow v^{1}] = \mathsf{chk}\ K_0\ e_0^{1}[x \leftarrow v^{1}]$
4. QED

CASE $\mathsf{Err}\ {}_{\mathsf{H}}{\lesssim}_{1}\ e^{1}$:

1. $\mathsf{Err}[x \leftarrow v^{\mathsf{H}}] = \mathsf{Err}$
2. QED $\mathsf{Err}\ {}_{\mathsf{H}}{\lesssim}_{1}\ e^{1}[x \leftarrow v^{1}]$

CASE $\mathsf{dyn}\ \tau_0\ e_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{dyn}\ \tau_0\ e_0^{1}$:

1. $e_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ e_0^{1}$
2. $e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ e_0^{1}[x \leftarrow v^{1}]$
   by the induction hypothesis
3. $\mathsf{dyn}\ \tau_0\ e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] = \mathsf{dyn}\ \tau_0\ e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}]$
4. $\mathsf{dyn}\ \tau_0\ e_0^{1}[x \leftarrow v^{1}] = \mathsf{dyn}\ \tau_0\ e_0^{1}[x \leftarrow v^{1}]$
5. QED

CASE $\mathsf{dyn}\ \tau_0\ e_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{dyn}\ e_0^{1}$:

1. $e_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ e_0^{1}$
2. $e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ e_0^{1}[x \leftarrow v^{1}]$
   by the induction hypothesis
3. $\mathsf{dyn}\ \tau_0\ e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] = \mathsf{dyn}\ \tau_0\ e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}]$
4. $\mathsf{dyn}\ e_0^{1}[x \leftarrow v^{1}] = \mathsf{dyn}\ e_0^{1}[x \leftarrow v^{1}]$
5. QED

CASE $\mathsf{dyn}\ \tau_0\ (\mathsf{stat}\ \tau_1\ e_0^{\mathsf{H}})\ {}_{\mathsf{H}}{\lesssim}_{1}\ e^{1}$:

1. $e_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ e^{1}$
2. $e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}]\ {}_{\mathsf{H}}{\lesssim}_{1}\ e^{1}[x \leftarrow v^{1}]$
   by the induction hypothesis
3. $\mathsf{dyn}\ \tau_0\ (\mathsf{stat}\ \tau_1\ e_0^{\mathsf{H}})[x \leftarrow v^{\mathsf{H}}] = \mathsf{dyn}\ \tau_0\ (\mathsf{stat}\ \tau_1\ e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}])$
4. QED

CASE $\mathsf{stat}\ \tau_0\ e_0^{\mathsf{H}}\ {}_{\mathsf{H}}{\lesssim}_{1}\ \mathsf{stat}\ \tau_0\ e_0^{1}$:

1. $e_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; e_0^{1}$
2. $e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] \;{}_{\mathsf{H}}{\lesssim}_1\; e_0^{1}[x \leftarrow v^{1}]$
   by the induction hypothesis
3. $\mathsf{stat}\;\tau_0\; e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] = \mathsf{stat}\;\tau_0\; e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}]$
4. QED

CASE $\mathsf{stat}\;\tau_0\; e_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; \mathsf{stat}\; e_0^{1}$:
1. $e_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; e_0^{1}$
2. $e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] \;{}_{\mathsf{H}}{\lesssim}_1\; e_0^{1}[x \leftarrow v^{1}]$
   by the induction hypothesis
3. $\mathsf{stat}\;\tau_0\; e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] = \mathsf{stat}\;\tau_0\; e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}]$
4. $\mathsf{stat}\; e_0^{1}[x \leftarrow v^{1}] = \mathsf{stat}\; e_0^{1}[x \leftarrow v^{1}]$
5. QED

CASE $\mathsf{stat}\;\tau_0\;(\mathsf{dyn}\;\tau_1\; e_0^{\mathsf{H}}) \;{}_{\mathsf{H}}{\lesssim}_1\; e^{1}$:
1. $e_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; e^{1}$
2. $e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] \;{}_{\mathsf{H}}{\lesssim}_1\; e^{1}[x \leftarrow v^{1}]$
   by the induction hypothesis
3. $\mathsf{stat}\;\tau_0\;(\mathsf{dyn}\;\tau_1\; e_0^{\mathsf{H}})[x \leftarrow v^{\mathsf{H}}] = \mathsf{stat}\;\tau_0\;(\mathsf{dyn}\;\tau_1\; e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}])$
4. QED

CASE $e_0^{\mathsf{H}}\; e_1^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; e_0^{1}\; e_1^{1}$:
1. $e_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; e_0^{1}$
   $\wedge\; e_1^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; e_1^{1}$
2. $e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] \;{}_{\mathsf{H}}{\lesssim}_1\; e_0^{1}[x \leftarrow v^{1}]$
   $\wedge\; e_1^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] \;{}_{\mathsf{H}}{\lesssim}_1\; e_1^{1}[x \leftarrow v^{1}]$
   by the induction hypothesis
3. $e_0^{\mathsf{H}}\; e_1^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] = e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}]\; e_1^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}]$
4. $e_0^{1}\; e_1^{1}[x \leftarrow v^{1}] = e_0^{1}[x \leftarrow v^{1}]\; e_1^{1}[x \leftarrow v^{1}]$
5. QED

CASE $\langle e_0^{\mathsf{H}}, e_1^{\mathsf{H}} \rangle \;{}_{\mathsf{H}}{\lesssim}_1\; \langle e_0^{1}, e_1^{1} \rangle$:
1. $e_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; e_0^{1}$
   $\wedge\; e_1^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; e_1^{1}$
2. $e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] \;{}_{\mathsf{H}}{\lesssim}_1\; e_0^{1}[x \leftarrow v^{1}]$
   $\wedge\; e_1^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] \;{}_{\mathsf{H}}{\lesssim}_1\; e_1^{1}[x \leftarrow v^{1}]$
   by the induction hypothesis
3. $\langle e_0^{\mathsf{H}}, e_1^{\mathsf{H}} \rangle[x \leftarrow v^{\mathsf{H}}] = \langle e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}], e_1^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] \rangle$
4. $\langle e_0^{1}, e_1^{1} \rangle[x \leftarrow v^{1}] = \langle e_0^{1}[x \leftarrow v^{1}], e_1^{1}[x \leftarrow v^{1}] \rangle$
5. QED

CASE $op^1\; e_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; op^1\; e_0^{1}$:
1. $e_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; e_0^{1}$
2. $e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] \;{}_{\mathsf{H}}{\lesssim}_1\; e_0^{1}[x \leftarrow v^{1}]$
   by the induction hypothesis
3. $op^1\; e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] = op^1\; e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}]$
4. $op^1\; e_0^{1}[x \leftarrow v^{1}] = op^1\; e_0^{1}[x \leftarrow v^{1}]$
5. QED

CASE $op^2\; e_0^{\mathsf{H}}\; e_1^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; op^2\; e_0^{1}\; e_1^{1}$:
1. $e_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; e_0^{1}$
   $\wedge\; e_1^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; e_1^{1}$
2. $e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] \;{}_{\mathsf{H}}{\lesssim}_1\; e_0^{1}[x \leftarrow v^{1}]$
   $\wedge\; e_1^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] \;{}_{\mathsf{H}}{\lesssim}_1\; e_1^{1}[x \leftarrow v^{1}]$
   by the induction hypothesis
3. $op^2\; e_0^{\mathsf{H}}\; e_1^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] = op^2\; e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}]\; e_1^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}]$
4. $op^2\; e_0^{1}\; e_1^{1}[x \leftarrow v^{1}] = op^2\; e_0^{1}[x \leftarrow v^{1}]\; e_1^{1}[x \leftarrow v^{1}]$
5. QED

CASE $x_0 \;{}_{\mathsf{H}}{\lesssim}_1\; x_0$:
 IF $x_0 = x$:
 1. QED $x_0[x \leftarrow v^{\mathsf{H}}] = v^{\mathsf{H}}$
    $\wedge\; x_0[x \leftarrow v^{1}] = v^{1}$

 ELSE $x_0 \neq x$:
 1. QED $x_0[x \leftarrow v^{\mathsf{H}}] = x_0$
    $\wedge\; x_0[x \leftarrow v^{1}] = x_0$

CASE $i \;{}_{\mathsf{H}}{\lesssim}_1\; i$:
1. QED $i[x \leftarrow v^{\mathsf{H}}] = i$
   $\wedge\; i[x \leftarrow v^{1}] = i$

CASE $\lambda x_0.\, e_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; \lambda x_0.\, e_0^{1}$:
1. **IF** $x_0 = x$:
   a. QED $\lambda x_0.\, e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] = \lambda x_0.\, e_0^{\mathsf{H}}$
      $\wedge\; \lambda x_0.\, e_0^{1}[x \leftarrow v^{1}] = \lambda x_0.\, e_0^{1}$
2. **ELSE** $x_0 \neq x$:
   a. $e_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; e_0^{1}$
   b. $e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] \;{}_{\mathsf{H}}{\lesssim}_1\; e_0^{1}[x \leftarrow v^{1}]$
      by the induction hypothesis
   c. $\lambda x_0.\, e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] = \lambda x_0.\, e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}]$
      $\wedge\; \lambda x_0.\, e_0^{1}[x \leftarrow v^{1}] = \lambda x_0.\, e_0^{1}[x \leftarrow v^{1}]$
   d. QED

CASE $\lambda(x_0\!:\!\tau_0).\, e_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; \lambda(x_0\!:\!\tau_0).\, e_0^{1}$:
1. **IF** $x_0 = x$:
   a. QED $\lambda(x_0\!:\!\tau_0).\, e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] = \lambda(x_0\!:\!\tau_0).\, e_0^{\mathsf{H}}$
      $\wedge\; \lambda(x_0\!:\!\tau_0).\, e_0^{1}[x \leftarrow v^{1}] = \lambda(x_0\!:\!\tau_0).\, e_0^{1}$
2. **ELSE** $x_0 \neq x$:
   a. $e_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; e_0^{1}$
   b. $e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] \;{}_{\mathsf{H}}{\lesssim}_1\; e_0^{1}[x \leftarrow v^{1}]$
      by the induction hypothesis
   c. $\lambda(x_0 : \tau_0).\, e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] = \lambda(x_0 : \tau_0).\, e_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}]$
      $\wedge\; \lambda(x_0 : \tau_0).\, e_0^{1}[x \leftarrow v^{1}] = \lambda(x_0 : \tau_0).\, e_0^{1}[x \leftarrow v^{1}]$
   d. QED

CASE $\mathsf{mon}\;\tau_0\; v_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; v_0^{1}$:
1. $v_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; v_0^{1}$
2. $v_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] \;{}_{\mathsf{H}}{\lesssim}_1\; v_0^{1}[x \leftarrow v^{1}]$
   by the induction hypothesis
3. $\mathsf{mon}\;\tau_0\; v_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}] = \mathsf{mon}\;\tau_0\; v_0^{\mathsf{H}}[x \leftarrow v^{\mathsf{H}}]$
4. QED

CASE $\mathsf{Err} \;{}_{\mathsf{H}}{\lesssim}_1\; \mathsf{Err}$:
1. QED $\mathsf{Err}[x \leftarrow v^{\mathsf{H}}] = \mathsf{Err}$

□

Lemma 8.27 : chk *inversion*

If $\Gamma \vdash_1 \mathsf{chk}\; K\; e' : K$ and $\Gamma \vdash e : \tau \rightsquigarrow \mathsf{chk}\; K\; e'$ then $K = \lfloor \tau \rfloor$.

Proof:

By case analysis on $\rightsquigarrow$.

CASE $\boxed{\dfrac{\Gamma \vdash e_0 : \tau_d \Rightarrow \tau_c \rightsquigarrow e_0' \quad \Gamma \vdash e_1 : \tau_d \rightsquigarrow e_1' \quad \lfloor \tau_c \rfloor = K}{\Gamma \vdash e_0\; e_1 : \tau_c \rightsquigarrow \mathsf{chk}\; K\; (e_0'\; e_1')}}$:
1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau_0{\times}\tau_1 \rightsquigarrow e' \quad \lfloor \tau_0 \rfloor = K}{\Gamma \vdash \mathsf{fst}\ e : \tau_0 \rightsquigarrow \mathsf{chk}\ K\ (\mathsf{fst}\ e')}}$:

1. QED

CASE $\boxed{\dfrac{\Gamma \vdash e : \tau_0{\times}\tau_1 \rightsquigarrow e' \quad \lfloor \tau_1 \rfloor = K}{\Gamma \vdash \mathsf{snd}\ e : \tau_1 \rightsquigarrow \mathsf{chk}\ K\ (\mathsf{snd}\ e')}}$:

1. QED

□

Lemma 8.28 : H−1 *δ-preservation*

- If $v^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; v^1$ and $\delta(op^1, v^{\mathsf{H}})$ is defined then $\delta(op^1, v^{\mathsf{H}}) \;{}_{\mathsf{H}}{\lesssim}_1\; \delta(op^1, v^1)$
- If $v_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; v_0^1$ and $v_1^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; v_1^1$ and $\delta(op^2, v_0^{\mathsf{H}}, v_1^{\mathsf{H}})$ is defined then $\delta(op^2, v_0^{\mathsf{H}}, v_1^{\mathsf{H}}) \;{}_{\mathsf{H}}{\lesssim}_1\; \delta(op^2, v_0^1, v_1^1)$

Proof:

CASE $op^1 = \mathsf{fst}$:

1. $v^{\mathsf{H}} = \langle v_0^{\mathsf{H}}, v_1^{\mathsf{H}} \rangle$
   by $\delta(\mathsf{fst}, v^{\mathsf{H}})$ is defined
2. $v^1 = \langle v_0^1, v_1^1 \rangle$
   $\wedge\ v_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; v_0^1$ and $v_1^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; v_1^1$
   by ${}_{\mathsf{H}}{\lesssim}_1$
3. $\delta(\mathsf{fst}, v^{\mathsf{H}}) = v_0^{\mathsf{H}}$
   $\wedge\ \delta(\mathsf{fst}, v^1) = v_0^1$
4. QED (2)

CASE $op^1 = \mathsf{snd}$:

1. $v^{\mathsf{H}} = \langle v_0^{\mathsf{H}}, v_1^{\mathsf{H}} \rangle$
   by $\delta(\mathsf{snd}, v^{\mathsf{H}})$ is defined
2. $v^1 = \langle v_0^1, v_1^1 \rangle$
   $\wedge\ v_0^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; v_0^1$ and $v_1^{\mathsf{H}} \;{}_{\mathsf{H}}{\lesssim}_1\; v_1^1$
   by ${}_{\mathsf{H}}{\lesssim}_1$
3. $\delta(\mathsf{snd}, v^{\mathsf{H}}) = v_1^{\mathsf{H}}$
   $\wedge\ \delta(\mathsf{snd}, v^1) = v_1^1$
4. QED (2)

CASE $op^2 = \mathsf{sum}$:

1. $v_0^{\mathsf{H}} \in \mathbb{Z}$
   $\wedge\ v_1^{\mathsf{H}} \in \mathbb{Z}$
   by $\delta(op^2, v_0^{\mathsf{H}}, v_1^{\mathsf{H}})$ is defined
2. $v_0^{\mathsf{H}} = v_0^1$
   $\wedge\ v_1^{\mathsf{H}} = v_1^1$
   by ${}_{\mathsf{H}}{\lesssim}_1$
3. QED

CASE $op^2 = \mathsf{quotient}$:

1. $v_0^{\mathsf{H}} \in \mathbb{Z}$
   $\wedge\ v_1^{\mathsf{H}} \in \mathbb{Z}$
   by $\delta(op^2, v_0^{\mathsf{H}}, v_1^{\mathsf{H}})$ is defined
2. $v_0^{\mathsf{H}} = v_0^1$
   $\wedge\ v_1^{\mathsf{H}} = v_1^1$
   by ${}_{\mathsf{H}}{\lesssim}_1$
3. QED

□