Calendar Computations
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Did you know that March 21, 2001 was a Wednesday? Or that January 1, 2076 will be a Friday? With a bit of math, one can determine the exact day of the week of any date: past, present, or future. Calculate the weekday on which the Declaration of Independence was signed (July 4, 1776), man first set foot upon the moon (July 20, 1969), or William Shakespeare was born (April 26, 1594). Impress friends and family by naming the weekdays of important personal events.

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Intro to Modular Arithmetic

These calendar calculations are based on a concept called modular arithmetic. In modular arithmetic, numbers do not go on forever. Instead, they "reset" once they reach a certain value. This value is called the "modulus." Like the numbers on a clock or the number of days in a week, counting numbers in modular arithmetic repeat a set pattern.

For example, in a module 3 system, we would count 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2 . . . and in a module 10 system, we would count 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 1, 2, 3, . . . Notice how the numbers start off like our counting numbers do, beginning at zero and increasing one by one, but never reach the modulus. In fact, the module 10 numbers match the numbers in the ones' place of our base-10 counting numbers. Compare our counting numbers to the module 10 system in the following table:

Counting	Module 10		Counting (cont.)	Module 10 (cont.)
0	0		20	0
1	1		21	1
2	2		22	2
3	3		23	3
4	4		24	4
5	5		25	5
6	6		26	6
7	7		27	7
8	8		28	8
9	9		29	9
10	0		30	0
11	1		31	1

12	2		32	2
13	3		33	3
14	4		34	4
15	5		35	5
16	6		36	6
17	7		37	7
18	8		38	8
19	9		39	9

In fact, we can define a new elementary operation (like addition, subtraction, multiplication, and division) called the modulus. We will use the symbol % as shorthand for writing out "modulus" or the shorter "mod". The modulus of a number will be equal to that number's value in a module system. So, to convert a number into the module 10 system:

$$39 \% 10 = 9$$
$$11 \% 10 = 1$$
$$5 \% 10 = 5$$
$$400 \% 10 = 0$$
$$405 \% 10 = 5$$
$$444 \% 10 = 4$$

Notice how these equations match the table above. The number 39, in the module 10 system, has the value 9. If you were to extend the table to reach 444, you would find that the 444th value in the module 10 system is 4. But there is a faster way to find any number's value in any module system.

For any two numbers A and B, A % B is <u>the remainder</u> of A / B

For the numbers A = 444 and B = 10, 444 % 10 equals the remainder of 444 / 10. 444 / 10 = 44 remainder 4, so 444 % 10 = 4. In another example, 7 % 5 = remainder of 7 / 5. 7 / 5 = 1 remainder 2, so 7 % 5 = 2.

This is all we need to know to do the calendar computations, but if you want to learn more about modular arithmetic, check these articles at the Math Explorers' page (link) and Wikipedia (link).

Activity: compute the following modulus operations

Check your answers using google's calculator. Just search for: (number 1) % (number 2), i.e. 15 % 5

15 % 5	1 % 99	10 % 3	25 % 6	3 % 1
24 % 7	32 % 4	70 % 5	14 % 2	8 % 5

January 1st of the 21st century

Now that you know how to take the modulus of a number, lets use that skill to figure what day of the week New Years Day was—or will be—for any year from 2000 to 2099. This is a simple, quick calculation; with a little practice, you'll be able to do it accurately without paper, pencil, or calculator! More importantly, it's the first step toward finding the weekday of any date in history.

First, we need to map the days of the week to numbers in a module system. There are 7 days in a week, so we will use the module 7 system to build the mapping.

Day	Code
Sunday	0
Monday	1
Tuesday	2
Wednesday	3
Thursday	4
Friday	5
Saturday	6

Use this table for reference as you learn these techniques, and memorize it. If it helps, remember that Monday, with code 1, is the first (1) day of the week that you have to go to school. Then the other days progress one-by-one after Monday, and reset to zero (0) at Sunday.

In order to find the day of the week of any January 1st, 20ab (where 'a' and 'b' are integers):
1. Take the two-digit number 'ab' and divide it by 4. Ignore the remainder.*
2. If the year is a leap year**, subtract 1 from your answer
3. Take the modulus 7 of this value
4. The day of the week is the day that matches this value

    * Example: 25 / 3 = 8 remainder 2. Ignoring the remainder, our answer is 8.
   ** Leap years are divisible by 4. 2012 / 4 = 503 (no remainder), so 2012 is a leap year
     The modulus is a shortcut: 2012 % 4 = 0, so 2012 is a leap year
                     2014 % 4 = 2, 2 doesn't equal 0, so 2014 isn't a leap year

Here is a worked-out example for January 1st, 2067:
1. 2067 is of form 20ab, so 'ab' = 67.
   67 / 4 = 16 remainder 3. Let d = 16.
2. Is 2067 a leap year? 67 % 4 = 3, so no.
3. (ab + d) % 7 = (67 + 16) % 7
   67 + 16 = 84, 84 % 7 = 0
4. 0 maps to Sunday. Jan 1, 2067 will be a Sunday

Activity: what day of the week was / will be New Year's Day in the years
2001   2006   2020   2050   2052   2099

Any weekday in the 21ˢᵗ century

Now that we know how to find the day of Jan. 1ˢᵗ in the 21ˢᵗ century, we can find a more general rule to work for any day of the 21ˢᵗ century. This process of starting from a specific case and working towards a general rule is called "induction".

The computation here is a bit shorter than the one used for the special case above, but requires two more data tables, in addition to the day codes.

Month	Code		Year	Code
January	6 (5 in leap year)		2000	0
February	2 (1 in leap year)		2001	1
March	2		2002	2
April	5		2003	3
May	0		2004	5
June	3		2005	6
July	5		2006	0
August	1		2007	1
September	4		2008	3
October	6		2009	4
November	2		2010	5
December	4		2011	6

First, lets analyze the table of year codes. There are only twelve codes in the table, but you can easily extend it. Given the code of any year:
1. Add 1 to that code. Reset to zero if value equal 7.
2. If the next year is a leap year, add 1 to the new code

Just don't forget to "reset" the code when the value exceeds 6.

As an example, lets find the code for 2012. Given the code of 2011 = 6
1. Add 1 to code of 2011. 6 + 1 = 7. Reset to 0.
2. Is 2012 a leap year? 2012 % 4 = 0. It is a leap year. Add 1 to new code.

Code for 2012 is 1.

The code for 2013 is 2. Try working through the above steps on your own.

This is great, but what if we want the code for 2099? Are we supposed to create a data table from 2000 to 2099? We could, but there's a faster way. To determine the code for any year in the 21ˢᵗ century without creating a table:
1. Find the number of leap years before and including this year
2. Add that result to the last two digits of the year
3. Take the modulus 7 of the answer

For the year 2012, the last two digits are 12. To find the number of leap years, we can work through a multiplication table:

$$4 * 1 = 4$$
$$4 * 2 = 8$$
$$4 * 3 = 12$$

Our last two digits = 12, so we stop here, but include that result. Three (3) is the biggest number we found using the table, so we conclude there have been 3 leap years so far in the 21st century. Now we add the number of leap years and the last two digits of 2012 to get 3 + 12 = 15. Final answer = 15 % 7 = 1. You can complete the extend the year table above to verify this answer.

One more example: the year 2089. First, we need to find the number of leap years up to and including 2089. We could work through the multiplication table as in the previous example, or we could use division as a shortcut.

89 / 4 = 22 remainder 1. 4 goes in to 89 twenty-two times, we conclude there were 22 leap years before 2089.

Adding the last two digits to this answer, we get 89 + 22 = 111. Final answer = 111 % 7 = 6. The year code for 2089 is 6.

The month codes are more straightforward. Except for January and February, whose values decrease by 1 in leap years, they do not change. But memorizing the month codes presents a challenge. The following mnemonics might help you remember:

January (6): January is the first full month of winter. Winter has 6 letters & January has code 6.
February (2): It's the 2nd month, and Groundhog's Day is Feb. 2nd. February has code 2.
March (2): In like a lion, out like a lamb. One lion + one lamb = 2 animals. March has code 2.
April (5): Five letters in "April," April has code 5.
May (0): Cinco de Mayo is in May. Think "May-O". May has code 0.
June (3): June is the first of 3 months of summer vacation. June has code 3
July (5): On Independence Day, fireworks fly. Think "fireworks fly five." July has code 5
August (1): August begins with an "A", the 1st letter of the alphabet. A is #1, August has code 1.
September (4): Fall begins in September. Fall has 4 letters & September has code 4.
October (6): 666 is the number of demons. Demons play on Halloween, October has code 6
November (2): Election day is always a Tuesday in November. Tuesday, 2's-day. November, 2.
December (4): December is the holiday month. Holidays for all. 4-all. December is code 4

Putting it all together, you can get the weekday of a date by taking the modulus 7 of the sum of a date's: year code + month code + day. The month code and year code come from the tables above; the day is simply the day. To find which day of the week August 12, 2009 fell on, just solve:

$$(Year\ Code + Month\ Code + Day\ Code)\ \%\ 7$$
$$month\ code = 1$$
$$day = 12$$
$$year\ code = 4$$
$$(\ 1 + 12 + 4\ )\ \%\ 7$$
$$17\ \%\ 7 = 3$$

3 maps to Wednesday. August 12, 2009 was a Wednesday.

Any day in any century

Here it is, the main attraction: determine the day of the week of any date in the history of the world! We'll take our knowledge from the previous sections and add one last generalization.

So far we have tables for day codes, month codes, and 21st century year codes. We need a new table that will give us the year codes for the first year in any century. Let's call these "century codes" for short.

Now, we found the year codes for the 21st century by finding the day code of January 1st, 2000 and advancing that value by 1. We could apply this same rule to find the year codes for the first year of any century, but we would first need to know the day of the week that January 1st fell on each century. You can either look online and research these dates, or you can use the following formula:

For any century, let the first two digits equal C. Then the century code for January 1st of that century is: $(1 + (2 * (3 - (C \% 4)))) \% 7$.

Break it down:

1. Take the first two digits of the century to get C (For 1983, C = 19)
2. Solve C % 4
3. Subtract: 3 minus answer from step 2
4. Multiply 2 * Answer from step 3
5. Add 1 to Answer
6. Take modulus 7 for final result

Using the 21st century (2000 – 2099):

1. C = 20
2. C % 4 = 0
3. 3 – 0 = 3
4. 2 * 3 = 6
5. 6 + 1 = 7
6. 7 % 7 = 0

Our conclusion? The century code for the 21st century is 0. Notice that this number matches the year code for 2000. Also, the 6 above (result of 2 * 3 – (C%4)) is the day code of January 1st, 2000. This is no coincidence!

Another example: the 18th century (1700-1799)

1. C = 17
2. C % 4 = 1
3. 3 – 1 = 2
4. 2 * 2 = 4

5. $4 + 1 = 5$
6. $5 \% 7 = 5$

The century code for the 18th century is 5.

Activity: Compute the century codes for the 19th and 20th centuries

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Century	Code
18th (1700 - 1799)	5
19th (1800 - 1899)	
20th (1900 - 1999)	
21st (2000 - 2099)	0

We're almost done! The last step is to find the year code for any year. Recall how we found the year codes for any year in the 21st century without using a table.

1. Take the last 2 digits of the year
2. Add the number of leap years up to & including that year
3. Take the modulus 7 of the answer

Using these century codes, the formula to get the day of the week of any day in history is:

(Century Code + Year Code + Month Code + Day) % 7

One last example problem: find the day of the week of July 4, 1776.

Century Code = 5
Year Code = $(76 + (76 / 4)) \% 7 = (76 + 19) \% 7 = 95 \% 7 = 4*$
Month Code = 5
Day = 4

* I took a shortcut here. See the term 76 / 4? That's to find the number of leap years up to and including 1776. Normally, we want to ignore the remainder of this answer, but 4 divides evenly into 76, so there is no remainder. If the term were 77 / 4 = 19 remainder 1, we would drop the remainder and arrive at the same answer of 19.

Final Answer = (Century code + Year code + Month code + Day) % 7
$(5 + 4 + 5 + 4) \% 7$
$18 \% 7$
4

The day code 4 matches Thursday. Congress approved the Declaration of Independence on a Thursday!

With practice, you'll memorize the century codes, month codes, and day codes. When someone asks you the weekday of a date in history, all you'll need to do is calculate the year code and put it all

together.

Activity: Find day of the week of these dates
Oct 27, 1858   May 5, 1862   April 9, 1865   Dec. 15 1911
          April 14, 1912   Dec 7, 1941
Bonus: Research the significance of these dates

Why Does this Work?

We are mathematicians. We want to understand why things work. "Just because" is not an acceptable answer, and we don't believe in magic. We seek explanations—even if they contradict our sense of what is "right" or "normal". So why can we calculate the day that any date fell on (or will fall on)?

1. Each week has 7 days. Each year has 365 or 366 days. 365 and 366 are very close to 364, which is a multiple of 7. So any day that takes place on a Monday one year will take place on a Tuesday if the next year is not a leap year, and on a Wednesday if it is a leap year. Notice that these "steps", +1 for normal years and +2 for leap years, match the "steps" we used in the table of year codes.
   As a side note, if our years had 364 days (or any other number of days X such that X % 7 = 0), every date would fall on the same day of the week every year!

2. Modular arithmetic makes the calculations possible for any date (not just numbers less than 7). Check out the Math Explorers page and the Wikipedia article to learn more about this and other applications of modular arithmetic.