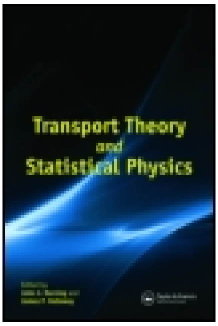

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Milestones in mathematical physics Noether's theorem

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MILESTONES IN MATHEMATICAL PHYSICS

NOETHER'S THEOREM

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The well known theorem of Emmy Noether[1] plays a role of fundamental importance in many branches of theoretical physics. Because it provides a straightforward connection between the conservation laws of a physical theory and the invariances of the variational integral whose Euler-Lagrange equations are the equations of that theory, it may be said that Noether's theorem has placed the Lagrangian formulation in a position of primacy. In addition, the theorem has brought about a situation whereby the search for conservation laws and selection rules has been reduced to the systematic study of the symmetries of a theory and the corresponding invariances of its Lagrangian.

The basic notions[2,3] underlying Noether's theorem may be paraphrased as follows. The fact that the equations of a physical theory can be derived from a variational principle is due to the postulate (i.e. Hamilton's Principle[4], or a religious belief) that the solutions of the equations are the selfsame functions that minimize (or maximize) a certain integral. The integral, whose integrand (the Lagrangian density) must be specially chosen (by hindsight) so that it provides the proper equations, is then subjected to a fixed endpoint variation and the resulting Euler-Lagrange equations are found to be the correct equations of the theory.

If we subject this same variational integral to a more general type of variation, say one which allows endpoint variations, then the result is found to include certain boundary terms along with the terms corresponding to the Euler-Lagrange equations. Since the postulate of Hamilton's Principle only applies to fixed endpoint variations, however, the results of the more general variation cannot be set equal to zero.

The contribution of Emmy Noether is the statement of a special circumstance under which a general variation of an integral may be justifiably set equal to zero. The variation must be one which is induced on the integral by the infinitesimal element of a continuous group under which the integral is invariant. Such a variation not only effectively causes endpoint variations of the dependent variables in the Lagrangian density, but may also vary the region of integration of the integral itself. Performing this variation and setting it equal to zero gives rise to certain partial differential equations (much as setting the fixed endpoint variation to zero gives rise to the Euler-Lagrange equations) which are in the form of conservation laws, i.e., a certain 4-divergence equalling zero.

Noether's theorem is applicable in both the classical and quantum domain. Classically, it may be applied to the Lagrangian of a system of point masses and it will give the conservation laws of energy, linear momentum and angular momentum, corresponding to the invariances, if they exist, under time translation, spatial translation and spatial rotation.[5] Needless to say, these conservation laws may be obtained without recourse to Noether's theorem, i.e., by direct integration of the equations of motion. Such methods, however, may not always work and do not make clear the important relation between the invariance and the conservation law. The theorem may also be applied to the Lagrangian density for a classical field theory, such as electromagnetic theory[5] or diffusion theory.[6] In these cases it is possible to derive conservation laws for the field energy, field momentum and field angular momentum. The theorem may also be used in a manner wherein its results are integrated and interpreted as first integrals of the Euler-Lagrange equations.[6] With this approach, complete solutions of the field equations may be found, under the assumption that they possess certain symmetries.

In the quantum domain, Noether's theorem may be equally well applied to the Lagrangian density for first and second quantized theories. In quantum field theory, however, for a transformation to be an invariance, it must leave the commutation (or anti-commutation) relations as well as the field equations unchanged.[7,8] Because of the fact that Lagrangian densities in quantum field theories usually contain products of the field functions and their hermitean conjugates, invariances other than those involving space-time transformations can be found. In particular, if one considers the Lagrangian density for a charged particle interacting with the electromagnetic field, it is found that it is invariant under the combined action of the so called gauge transformation of the first kind on the charged particle field and a gauge transformation of the second kind on the electromagnetic field. Noether's theorem yields the very important conservation of charge as a result of this invariance.[9] The invariance of the interacting field Lagrangian densities under gauge transformations appears to be such a basic requirement that Yang and Mills[10,11] were led to postulate the existence of new fields in these Lagrangians whose transformation properties would insure the overall invariance of the Lagrangian. Applying Noether's theorem to these gauge invariances leads to more general types of current conservation. On the basis of so strong a belief in the invariance properties of Lagrangians, one might say that the existence of the electromagnetic field is actually required to satisfy the invariance requirements of the charged particle Lagrangian.

Other important invariances and their corresponding conserved quantities are obtained by imposing more structure on the field functions. By giving the fields transformation properties in, for example, isotopic charge space,[12] it is possible to derive conserved currents that carry new kinds of "charge"[13,14] and, consequently, to find important new selection rules for interactions. Even more generally, it is possible to incorporate more structure into the fields internally, by changing the underlying number field from the complex numbers to the quaternions. In this formulation the group of automorphisms of the quaternionic field can become an invariance group of the theory. In this case, too, Noether's theorem provides a means of relating conservation laws to invariance properties.[16]

It is hoped that this brief introduction, which only scratches the surface of the applications of Emmy Noether's powerful theorem, will at least suffice to motivate its careful perusal.

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