§3.1

3.1.3.1 (a) **sup** 1.5
inf -1
limsup 1
liminf -1
limits 1, -1

(b) **sup** $\frac{3}{2}$
inf 0
limsup 1
liminf 1
limits 1

(c) **sup** 2
inf -3
limsup 1
liminf -3
limits -3, 1

3.1.3.2 No, the sequences don't necessarily converge. Example: $\{y_n\} = .9, .99, .999, \ldots \{z_n\} = -.9, -.99, -.999, \ldots$

Bounding the sequences doesn't help. My example sequences are bounded. Yeah, I can't even think of an example where $\{y_n\}$ and $\{z_n\}$ are *unbounded* but the sequence $x_n = y_n + z_n$ *is* bounded.

3.1.3.3 y_n is an upper bound for E because it is the limit of upper bounds. y_n is the least upper bound because it is the limit of elements x_n of E.

3.1.3.4 The union takes elements from both sets, so the suprenam can only increase. The intersection only removes elements. If $sup(B) > sup(A)$, then we know that $sup(B) \notin A$ and therefore $sup(B) \notin A \cap B$.

3.1.3.5 TODO prove first part.

Equality does not hold for $x_n = 1, 1, 1, \ldots$ and $y_n = 1, 1, 1, \ldots$. Each has a limsup of 1, and so does their shuffled sequence. Hence $limsup\{x_n + y_n\} < limsup\{x_n\} + limsup\{y_n\} \rightarrow 1 < 1 + 1$.

3.1.3.6 Yes. Members of the subsubsequence are members of the parent sequence, so one can define a subsequence selection function to extract it from the parent sequence.

3.1.3.7 Create two infinite matrices:

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & \dots & -1 & -2 & -3 & \dots \\ 0 & 1 & 2 & 3 & \dots & -1 & -2 & -3 & \dots \\ 0 & 1 & 2 & 3 & \dots & -1 & -2 & -3 & \dots \\ 0 & 1 & 2 & 3 & \dots & -1 & -2 & -3 & \dots \\ \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \end{array}$$

Diagonialize each and shuffle those resulting sequences. Now we have a sequence in which each integer appears an infinite number of times.

3.1.3.8 $\Rightarrow$: Limits of subsequences are limits of the sequence.

$\Leftarrow$: Choose the sequence itself as a subsequence. (Or delete a finite number of elements.)

3.1.3.9 Yes. Create another matrix and diagonalize it. The liminf is zero.

$$\begin{array}{cccc} 1 & \frac{1}{2} & \frac{1}{3} & \dots \\ 1 & \frac{1}{2} & \frac{1}{3} & \dots \\ 1 & \frac{1}{2} & \frac{1}{3} & \dots \\ 1 & \frac{1}{2} & \frac{1}{3} & \dots \\ \vdots & \vdots & \vdots & \end{array}$$

3.1.3.10 Limits of subsequences are limits of sequences. If an infinite number of terms in a subsequence approach a limit point, then it follows that an infinite number of terms in the sequence approach the same limit. They're the same terms!

Sequences have no limits that do not belong to their subsequences. Assume, for contradiction, that the sequence $x_n = y_n + z_n$ has a limit point l such that neither an infinite number of terms of y_n nor an infinite number of terms of z_n approach l. In other words, no more than a finite number of $y \in y_n$ and $z \in z_n$ approximate the limit. Then only finitely many $x \in x_n$ could possibly approach l, a contradiction.

Shuffling doesn't matter. The order in which subsequence terms appear in the final sequence has no bearing on whether an infinite number of sequence terms fall within some epsilon of a limit.

3.1.3.11 Rows and columns appear as subsequences, so their limits appear in the sequence. A definition for the row subsequence selection function, defined in terms of the row number, r, where the top row of the matrix is row $r = 1$:

$$f_r = 1 \text{ when } r = 0$$

$$f_r(i) = f_{r-1}(1) + r - 1 \text{ when } i = 1$$

$$f_r(i) = f_r(i-1) + i + r - 1 \text{ else}$$

Yes, one necessarily gets all limit points in this way. All the matrix terms appear in the sequence. How could you not?

3.1.3.12 Reflexivity & symmetry are easy. Any sequence differs from itself in finitely many (i.e. zero) terms, and if x differs from y in n terms then obviously y and x will be identical after n terms. For transitivity, say x and y differ by n terms and y and z differ by m terms. Then x and z differ by $max(n, m)$ terms. If z is the same sequence as y after a finite number of steps then it's obviously also identical to x after finitely many steps.

Equivalent sequences have the same limit points because they have the same behavior at infinity. They are exactly the same for an infinite number of terms.

§3.2

3.2.3.1 Let $A = (a, b)$ be the open set and let $x_1, x_2, \ldots x_n$ be the points removed from A, arranged in ascending order. We know that $a < x_1$ and $x_n < b$, so we can define the new set A' obtained by deleting all the points as the union of open intervals $(a, x_1) \cup (x_2, x_3) \cup \ldots \cup (x_n, b)$. The union of open intervals is open, thus A' is open.

The same holds if a countable number of points are removed. Let $A = \mathbb{R}$ and remove $\mathbb{N}$ from A. Now A' is the union $(-\infty, 1) \cup (1, 2) \cup (2, 3) \ldots$, which is an open set.

3.2.3.2

3.2.3.3

3.2.3.4 This seems like it follows trivially from the definitions.

3.2.3.5 If A is a closed set, x is a point in A, and $B = A - \{x\}$, then B is closed when x is not a limit point of A. Why? Because B will then contain all its limit points, just like A did, making it a closed set.

3.2.3.6

3.2.3.7

3.2.3.8

3.2.3.9

3.2.3.10

3.2.3.11

3.2.3.12

3.2.3.13

3.2.3.14 The empty set and the universe of all sets are both open and closed by definition.

3.2.3.15 Let $\bigcup \mathcal{A}$ be a union of open intervals. Replace every pair of intersecting intervals $(a, b), (c, d) \in \mathcal{A}$ by the interval $(min(a, c), max(b, d))$. What's left is to prove that this is really equivalent.

3.2.3.16 Let $\bigcup \mathcal{A}$ and $\bigcup \mathcal{B}$ be two ways of describing the same open set. Let (a_1, a_2) and (b_1, b_2) be the intervals in $\mathcal{A}$ and $\mathcal{B}$, respectively, with the smallest first element. Now $a_1 \leq b_1$, otherwise (b_1, b_2) would include the limit a_1. Likewise $b_1 \leq a_1$ because otherwise (a_1, a_2) would include b_1. Hence $a_1 = b_1$. A similar argument holds for a_2 and b_2, so the intervals must be the same. Removing these smallest intervals and applying the same argument inductively, we see that $\mathcal{A}$ must be the same set as $\mathcal{B}$.