Highly Acyclic Groups, Hypergraph Covers, and the Guarded Fragment

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We construct finite groups whose Cayley graphs have large girth even with respect to a discounted distance measure that contracts arbitrarily long sequences of edges from the same color class (subgroup), and only counts transitions between color classes (cosets). These groups are shown to be useful in the construction of finite bisimilar hypergraph covers that avoid any small cyclic configurations. We present two applications to the finite model theory of the guarded fragment: a strengthening of the known finite model property for GF and the characterization of GF as the guarded bisimulation invariant fragment of first-order logic in the sense of finite model theory.

Categories and Subject Descriptors: E.1 [**Data Structures**]: *Graphs and networks*; F.4.1 [**Mathematical Logic and Formal Languages**]: Mathematical Logic—*Model theory*; G.2.2 [**Discrete Mathematics**]: Graph Theory—*Hypergraphs*

General Terms: Theory

Additional Key Words and Phrases: Finite model theory, groups of large girth, guarded fragment, expressive completeness

ACM Reference Format:
Otto, M. 2012. Highly acyclic groups, hypergraph covers, and the guarded fragment. J. ACM 59, 1, Article 5 (February 2012), 40 pages.
DOI = 10.1145/2108242.2108247 http://doi.acm.org/10.1145/2108242.2108247

1. INTRODUCTION

Although degrees of acyclicity—of Cayley groups and of finite hypergraph covers—form the technical backbone of this article, it is motivated in the broader sense by the study of structural transformations and model constructions that are compatible with certain forms of bisimulation equivalence (hypergraph bisimulation and guarded bisimulation). Such constructions have their place in the combinatorial exploration of hypergraphs and relational structures. In particular, they play an important role in the model theoretic analysis of modal and guarded logics, whose semantics is preserved under these equivalences and transformations. We therefore start this introduction with an informal discussion of these bisimulation equivalences—as interesting and natural notions from the point of view of discrete mathematics, of model theory and logic, and of certain application domains in computer science, where the associated logics also play an important role.[1]

[1]The rest of this introduction will then be devoted to informal encounters with more specific technical themes in sections devoted to hypergraph acyclicity (Section 1.1) and an introduction to the guarded fragment

This research was supported by a DFG grant “Strukturkonstruktionen und modelltheoretische Spiele.”
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DOI 10.1145/2108242.2108247 http://doi.acm.org/10.1145/2108242.2108247

The notions of *hypergraph bisimulation* and of *guarded bisimulation* extend the well-known concept of bisimulation equivalence from graph-like structures, transition systems and Kripke structures to the more general setting of hypergraphs and relational structures with not just binary relations.

The concept of bisimulation itself is familiar as the quintessential notion of back & forth equivalence that captures behavioral equivalence between states—viz., behavioral equivalence in terms of the available transition patterns. Its broad usefulness is witnessed by its many applications in domains ranging from the step-by-step analysis of computation devices or of reactive systems to the strategy analysis in game graphs and to the analysis of knowledge states in epistemic systems. Bisimulation equivalence is intimately related to the model theory of the relevant modal logics in these settings, as their semantics is preserved under bisimulation. In fact, bisimulation invariance is the characteristic semantic feature of modal logics and accounts for much of their smooth model theory and for their algorithmic tractability.

Hypergraphs extend the notion of graphs in allowing hyperedges that consist of more than just two nodes instead of the edges in graphs; just as graphs describe the accessibility and connectivity patterns of, for example, transition systems, hypergraphs capture the overlap and connectivity patterns of more general relational structures. Such patterns arise and are of interest, for instance, in the syntactic analysis of conjunctive queries or of constraint satisfaction problems, where the nodes are variables and hyperedges are formed by clusters of variables that occur together in a relational atom or in an atomic constraint; they also arise in the structural analysis of database instances, where the hyperedges are the abstractions of the individual tuples that form the entries in the tables of the database [Beeri et al. 1983]. Similarly, one can associate with an arbitrary relational structure the hypergraph of its *guarded subsets*, whose hyperedges are the subsets formed by the components of tuples in the given relations. This hypergraph of guarded subsets abstracts from the relational structure just the carrier sets of its tuples, in the same sense that the Gaifman graph of a relational structure abstracts just the information about co-existence of nodes within tuples. In fact, the Gaifman graph of a relational structure is itself induced by the hypergraph of guarded subsets.[2] In this sense, therefore, the hypergraph of guarded subsets in general contains more complex information than the Gaifman graph about the overlap and connectivity pattern that a relational structure induces on its set of elements.

Just as bisimulation is based on a back & forth analysis of a graph-like link structure, hypergraph bisimulation explores the hypergraph link structure. Whereas bisimulation equivalence over graph-like structures is defined in terms of transitions from one node to another along one of the available edges, hypergraph bisimulation is centered on the transitions from one hyperedge to another which may fix some nodes in the overlap of these hyperedges.

In the context of relational structures, the concept of hypergraph bisimulation further extends in a straightforward manner to the very natural notion of guarded bisimulation equivalence between relational structures. Guarded bisimulations are best thought of as hypergraph bisimulations between the associated hypergraphs of guarded subsets that respect not just the hypergraph link structure but also the actual relational content. This is achieved by treating the guarded subsets not just as sets but as induced substructures, and thus by working with local isomorphisms rather than just local

(Section 1.2). Section 1.3 gives a brief guide to the overall structure of the article, which also details parts that may be of independent interest and accessible in isolation, for instance, to readers whose interest lies more with the discrete mathematics of hypergraphs than with guarded logics.

[2]Here we refer to the usual manner in which a hypergraph induces an associated graph on the same carrier set where every hyperedge gives rise to a clique formed by its member nodes.

bijections. This notion of guarded bisimulation supports a particularly useful level of analysis of relational structures, which allows for local access to the actual relational content but is globally tamed by its restriction to the hypergraph link pattern induced by the guarded subsets.

Apart from its natural motivation in terms of the combinatorics of relational structures, guarded bisimulation is of particular interest in logic because of its central role in the model theoretic analysis of the *guarded fragment* GF $\subseteq$ FO of first-order logic [Andréka et al. 1998]. In fact, guarded bisimulation is the characteristic notion of semantic invariance for the guarded fragment of first-order logic. In this way, guarded bisimulation between relational structures occupies the same central position for the guarded fragment that ordinary bisimulation occupies for modal logic.

Maybe the most important technical contribution of this article, from the point of view of discrete mathematics, concerns the construction of bisimilar coverings, in the sense of hypergraph bisimulation or guarded bisimulation, of given finite hypergraphs or relational structures. Such coverings project homomorphically onto the given base structure in such a manner that the projection also induces a bisimulation; the basic intuition involved in such coverings is that of a discrete analogue of topological coverings (which may have singularities) that come with a projection that locally matches layers of the covering space to their projection in the base space. Similar, essentially discrete analogues of topological coverings and the topological analysis of combinatorial coverings, have been studied in connection with simplicial complexes rather than hypergraphs, under the name of *branched coverings* or *coverings with singularities*, (see, e.g., Fox [1957]).

Our goal here are coverings in the hypergraph setting that, despite being finite, guarantee certain qualified levels of hypergraph acyclicity that may not have been present in the base structure. As one essential application of a model theoretic nature, suitable coverings of this kind will in particular be applied to the analysis of the expressive power of the guarded fragment GF over finite structures. The combinatorial and group theoretic constructions used to obtain these finite covers, however, could well be of independent interest. They produce, for instance, highly uniform and homogeneous finite hypergraphs of qualified acyclicity from novel constructions of finite Cayley groups satisfying strong acyclicity criteria, which go considerably beyond the familiar notion of large girth.

1.1. Hypergraph Acyclicity

Acyclicity of hypergraphs [Berge 1973] or relational structures has long been recognized as an important structural property because of its relation with tree decomposability [Beeri et al. 1983]. Acyclicity criteria, often also in the more liberal form of bounds on tree width (and generalizations), play an important role in the delineation of well-behaved problem instances, for example, for model checking or query answering [Courcelle 1990; Gottlob et al. 2001; Flum et al. 2002; Frick and Grohe 2001; Grohe 2008]. But also from a purely model theoretic point of view, tree decomposable models—and again, more liberally, models of bounded tree width—are of interest because of their interpretability in actual trees, which makes them amenable, for instance, to automata theoretic techniques. The *generalized tree model property* for the guarded fragment GF $\subseteq$ FO in Grädel [1999], which is responsible for a range of decidability results and complexity bounds, is an important case in point. The natural notion of unfolding of relational structures, which is compatible with guarded bisimulation equivalence, produces models that are tree-like not just in the sense of bounded tree width, but in the stronger sense of tree decomposability of the hypergraph of guarded subsets of the model. Similar phenomena are well known from graph-like structures (especially transition systems), which can be unfolded into bisimilar tree structures.

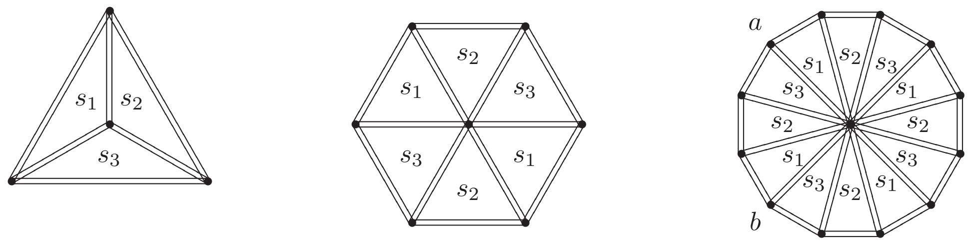

Fig. 1. The 3-spoke cartwheel (or top of the tetrahedron) unfolded into 6 and 12 spokes.

If we want to stick with finite structures, then these tree unfoldings are not available since, even in the graph case, any cycle in the original structure can only be unfolded into an infinite path. For some purposes, however, it suffices to achieve some measure of local acyclicity or acyclicity in bounded configurations rather than global acyclicity. In the case of graph-like structures, constructions of bisimilar covers by finite, *locally acyclic* structures are available [Otto 2004] and have been used for constructive alternative proofs of expressive completeness results for modal logics [Otto 2004, 2006, 2011]. Unlike the classical proof methods, which are based on compactness arguments, these techniques also work in the context of finite model theory and other non-elementary classes of structures where first-order compactness fails [Dawar and Otto 2009].

The situation for hypergraphs or relational structures of width greater than 2, as opposed to graphs or graph-like structures, has proved a major challenge in this respect. As outlined, it is quite clear what the natural notion of bisimilar *hypergraph covers* or *guarded covers* ought to be: in the case of hypergraphs, for instance, we are looking for hypergraphs that project onto the given hypergraph through a hypergraph homomorphism which at the same time induces a hypergraph bisimulation between the covering hypergraph and the base hypergraph; it is not at all obvious, however, which measure of hypergraph acyclicity can be achieved in covers of finite hypergraphs by finite hypergraphs (and similarly for finite guarded covers of finite relational structures). The immediate analogue of the graph case is ruled out: *local acyclicity*—in the sense that the induced hypergraph structures on ℓ-neighborhoods must be acyclic—cannot be achieved, not even for width 3 hypergraphs and for $\ell = 1$, as Example 1.1 shows. What, then, can be hoped for?

Classical hypergraph theory [Berge 1973; Beeri et al. 1983] characterizes acyclicity and tree-decomposability of hypergraphs in terms of two independent requirements: *conformality* and *chordality*. The former forbids cliques in the Gaifman graph other than those that are induced by individual hyperedges; the latter forbids chordless cycles in the Gaifman graph.

Conformal finite covers were constructed in Hodkinson and Otto [2003] and employed in a simplified proof of the finite model property for the clique-guarded fragment. The following example shows that, in contrast, chordal finite covers are not generally available; it rules out local chordality even in 1-neighborhoods.

Example 1.1. Consider a cartwheel hypergraph consisting of at least three 3-hyperedges that all share one pivot vertex and form a cycle with respect to the edges formed by the remaining two vertices in each hyperedge. It is clear that any finite cover needs to contain a necessarily cyclic finite cover of the perimeter cycle in the 1-neighborhood of any vertex representing the pivot vertex. See Figure 1 for a two- and four-fold unfolding of the cartwheel with 3 spokes/hyperedges into one with 6 or 12 spokes/hyperedges. The configurations in this figure will be used as illustrations again. Note that such a cartwheel configuration occurs in the regular full 3-hypergraph on

4 vertices (a tetrahedron with its faces as hyperedges, the boundary of the 3-simplex), which itself is chordal but not conformal. This tetrahedron hypergraph is probably the simplest example of a hypergraph for which it is not obvious what measure of acyclicity may be achieved in finite covers, if any; it is clear, however, that every finite cover will have to have chordless cycles in the 1-neighborhood of every vertex.

A natural relaxation would forbid just *short* chordless cycles (of lengths up to N say), which we call *N-chordality*; with *N-conformality* similarly defined in terms of cliques up to size N, their combination, *N-acyclicity*, captures the condition that all subconfigurations of size up to N are acyclic and tree-decomposable.

A construction of finite hypergraph covers in which short cycles in the cover become chordally decomposable in projection to the base hypergraph (weak N-chordality) was obtained in Otto [2009]; an alternative, much more explicit construction of such covers with reasonable size bounds is presented in Bárány et al. [2010]. These approaches notwithstanding, the question whether finite hypergraphs generally admit finite N-acyclic covers remained wide open.

With methods entirely different from those in Otto [2009] or Bárány et al. [2010], we now obtain conformal N-chordal finite hypergraph covers. Our main theorem in this respect is Theorem 3.8, which in turn is based on the combinatorial main result, about highly acyclic Cayley groups, presented as Corollary 2.12. Both results are technically rather involved.

These results point us to the class of N-acyclic relational structures, which seems to be very smooth from a model theoretic point of view. For instance, it supports a natural notion of *bounded convex hulls*. The analysis of suitable models in this class also leads to a positive resolution of one of the key open questions in the finite model theory of the guarded fragment.

1.2. The Guarded Fragment

The guarded fragment GF of first-order logic can be seen as an extension of modal logic to the richer setting of relational structures of any width, rather than the graph-like structures of width 2 (Kripke structures, transition systems) that modal logic deals with. Where modal logic restricts first-order quantification to relativized quantification along edges, the guarded fragment allows quantification over *guarded* tuples of elements, so that it accesses transitions between overlapping relational patches (the guarded subsets induced by the tuples in the basic relations). GF combines a natural level of expressiveness that fits many applications in computer science, especially in database theory and description logics (cf. discussion and references in Bárány et al. [2010] and Baader et al. [2003] for background). On the theoretical side, it retains many of the well-known good model-theoretic and algorithmic properties of modal logics at a higher level of expressiveness and over richer structures. Just as the model theory of modal logic is governed by the notion of bisimulation equivalence, the guarded fragment is governed by a corresponding notion of guarded bisimulation equivalence [Andréka et al. 1998; Grädel 1999; Grädel et al. 2002]. As previously indicated, the intuitive presentation of guarded bisimulation in terms of a back & forth game deals with challenges and responses with respect to transitions between accessible patches (the guarded subsets). If we disregard the local atomic relational content and just focus on the combinatorial pattern of available moves, guarded bisimulation stems from an underlying bisimulation between the hypergraphs of guarded subsets—just as modal bisimulation is a bisimulation between graphs.

Together with the introduction of GF, Andréka et al. [1998] gave a classical, compactness-based proof via suitably saturated infinite structures that GF $\subseteq$ FO is expressively complete for guarded bisimulation invariant properties: a first-order property of relational structures is definable in GF if, and only if, it does not distinguish between structures that are guarded bisimilar. In symbols, we write GF $\equiv$ FO/$\sim_{\mathrm{g}}$ to indicate this match in expressiveness between GF and the guarded bisimulation invariant fragment of first-order logic. This match is the guarded analogue of van Benthem's characterization of basic modal logic as the bisimulation invariant fragment of FO, ML $\equiv$ FO/$\sim$ [van Benthem 1983]. The latter match has been known to be good also as a characterization in the sense of finite model theory since Rosen's proof [Rosen 1997] (also compare Otto [2004, 2006, 2011] for alternative proofs and ramifications). In contrast, the finite model theory status of GF $\equiv$ FO/$\sim_{\mathrm{g}}$ has prominently remained open.

Characterizations of the expressive power of fragments of first-order logic in terms of semantic invariances generally are of great systematic value, as witnessed by the many preservation theorems of classical model theory. If the underlying invariance, like (guarded) bisimulation invariance, is semantically well motivated in its own right and is at the core of a structural understanding of good model theoretic properties, then an expressive completeness result also provides effective syntax for the otherwise ineffective class of first-order properties displaying that invariance—and shows the logic at hand to be just right for the purpose. For many natural applications, however, the finite model theory version, rather than the classical version, of such a characterization addresses these concerns. This is certainly the case whenever, as in databases, the intended models are meant to be finite.

The crux of a finite model theory argument for expressive completeness, as required here, lies in a proof that guarded bisimulation invariance of some $\varphi \in$ FO already implies its invariance under one of the finitary approximations $\sim_{\mathrm{g}}^{\ell}$. Ehrenfeucht–Fraïssé analysis links these finite approximations $\sim_{\mathrm{g}}^{\ell}$ to equivalence in GF up to nesting depth ℓ. Clearly, invariance under some level $\sim_{\mathrm{g}}^{\ell}$ over the class of structures at hand is a necessary condition for expressibility in GF over that class—but it is also sufficient, by a straightforward consequence of the Ehrenfeucht–Fraïssé analysis.

We therefore seek an upgrading of suitable levels $\sim_{\mathrm{g}}^{\ell}$ between finite relational structures to levels $\equiv_q$ of first-order equivalence (up to some quantifier rank q) in *finite* structures that are guarded bisimilar to the given structures. Suitable guarded bisimilar companion structures are here obtained as guarded covers. It is clear that some level of hypergraph acyclicity needs to be achieved in these covers in order to avoid low-level first-order distinctions: while the existence of certain short chordless cycles is obviously first-order expressible in relational structures, the length of such cycles is not guarded bisimulation invariant and their existence is not expressible in GF. The three hypergraphs in Figure 1 (taken as relational structures with three ternary relations to distinguish the three distinct triangle colors s_i, say) are guarded bisimilar and GF equivalent.

We here show that the guarded fragment is indeed expressively complete for all first-order properties that are invariant under guarded bisimulation in finite models, see Theorem 4.7. The proof uses N-acyclic covers and further applications of our main combinatorial result, together with a general analysis of N-acyclic structures to allow for an Ehrenfeucht–Fraïssé game based upgrading. It gives an essentially constructive expressive completeness argument that is totally different from the classical variant [Andréka et al. 1998], and it shows the guarded fragment to behave in beautiful analogy with the modal fragment in yet another way.

1.3. Organization of the Article

The article is organized in three separate parts, each starting with a short review of the relevant technical notions so that these parts may to a reasonable extent be read independently.

The first part, Section 2, deals with the combinatorial group and graph constructions that are the main technical tool in the following; it culminates in Corollary 2.12; the Cayley groups obtained may well be of independent interest and useful in other contexts.

The second part, Section 3, builds on Section 2 and develops the construction of N-acyclic hypergraph covers as stated in Theorem 3.8. Richer variants of these covers that also boost multiplicities in a generic manner are discussed in Section 3.6. Section 3.7 provides some structural analysis of N-acyclicity in its own right. Like the first part, this part may be of independent interest to readers curious about the combinatorics and discrete mathematics of finite hypergraphs.

Both these parts are presented without any essential input from logic or the model theory of the guarded fragment.

The third part, Section 4, deals with the applications to the model theory of the guarded fragment. As an immediate consequence, we obtain a strengthened finite model property for GF in Corollary 4.6; substantially more work is required to prove the expressive completeness result for GF in Theorem 4.7.

For this third part, the analysis of N-acyclic structures based on Section 3.7 is essential; the results about guarded covers from the first two parts of this article, though crucial, may be treated as combinatorial imports (in the form of mere existence guarantees for suitable bisimilar companions of finite structures) for the finite model theory of the guarded fragment.

2. HIGHLY ACYCLIC GROUPS

We aim to construct finite, regularly edge-colored, homogeneous graphs that do not realize short cycles, or even cycles that would be short when subjected to certain contractions of paths running within the same group of colors.

Figure 1 gives a first idea why not just short cycles in the usual sense, but cycles that are short with respect to discounted distances, need to be controlled. An unfolding of the 3-cartwheel (as, e.g., in the neighborhood of any one vertex in the tetrahedron hypergraph) into some $3n$-cartwheel allows us to make any number of consecutive transitions from hyperedges s_i to s_{i+1} without making any progress along a cycle that is just visiting the pivot vertex in this cartwheel—a correspondingly discounted distance measure will have to count just two steps, one for entering and another for leaving this cartwheel. Compare, for instance, a path segment from a to b via the pivot vertex in the rightmost cartwheel hypergraph of Figure 1, viewed as a local configuration in some cover of the tetrahedron say.

2.1. Cayley Groups and Graphs

Regular Graphs of Large Girth. A simple example is the following construction of k-regular graphs of girth greater than N, for arbitrary given k and N, see Alon [1995]. Let T be the regularly k-colored undirected tree, in which every node has precisely one neighbor across each one of the k edge-colors $e_1, \dots, e_k$; designate one node λ in this tree as its root and truncate the whole tree at depth N from the root. Each color e_i induces a permutation π_i of the vertex set of this finite tree, if we let π_i swap each pair of vertices that are linked by an e_i-edge.

This operation is a well-defined bijection since every vertex is incident with at most one edge of color e_i. Each π_i is in fact an involution: $\pi_i \circ \pi_i = \mathrm{id}$. In the truncated trees we use here, every leaf is fixed by all but one of the π_i, and there are no other fixed points.

Let G be the group generated by $(\pi_i)_{1\leqslant i\leqslant k}$ in the full symmetric group of the vertex set.[3] We obtain the desired graph as the Cayley graph of the group G: its vertices are the group elements $g \in G$; g and g' are linked by an edge (of color e_i) if $g' = g \circ \pi_i$ (equivalently: $g = g' \circ \pi_i$, as π_i is involutive). Call a sequence of generators *reduced* if it contains no factors of the form $\pi_i\pi_i$ (repetitions of the same generator, which cancel in G due to their involutive nature). It is clear that no reduced sequence of generators of length up to N can represent the neutral element $1 \in G$: just observe its operation on the root λ to see that λ is moved precisely one step away from the root by each new generator application, whence a sequence of up to N generators cannot operate as the identity transformation. It follows that the Cayley graph has girth greater than N (also $> 2N + 1$ is easy to see).

In the following, we shall modify the basic idea in this construction to yield finite graphs displaying a much stronger form of acyclicity with respect to to discounted distance measures along cycles. For the rest of this section, let E be a finite set of *edge colors*. A subset $\alpha \subseteq E$ is regarded as a *color class*. We deal with E-colored undirected graphs in which every node is incident with at most one edge of any fixed color e. We call such graphs *E-graphs*. The class of E-graphs is closed under subgraphs, in the sense of weak substructures, as well as under reducts. Recall the usual model theoretic terminology: unless specified otherwise, the substructure relationship $\mathfrak{A} \subseteq \mathfrak{B}$ says that $\mathfrak{A}$ is the structure obtained by simultaneous restriction of all the relations of $\mathfrak{B}$ to the subset $A \subseteq B$ so that $R^{\mathfrak{A}} = R^{\mathfrak{B}} \cap A^r$ if $R^{\mathfrak{B}}$ is a relation of arity r (i.e., substructures in our sense are *induced substructures*). Note that the notion of a *weak substructure*, as in the common graph theoretic interpretation of the term subgraph, is more general in allowing passage to some subset $R^{\mathfrak{A}} \subseteq R^{\mathfrak{B}} \cap A^r$, if $R^{\mathfrak{B}}$ is a relation of arity r. A *reduct* $\mathfrak{A} = \mathfrak{B}\restriction\tau_0$ on the other hand has the same universe as $\mathfrak{B}$, $A = B$, and the same relations $R^{\mathfrak{A}} = R^{\mathfrak{B}}$ for all relations $R \in \tau_0$ but drops the interpretation of all relations $R \notin \tau_0$.

In E-graphs, connected components with respect to subsets $\alpha \subseteq E$ are defined as usual. We here regard an α-component of an E-graph as an α-graph, in the sense of an implicit passage to the α-reduct, with all edges of colors $e \notin \alpha$ deleted. In particular, we shall look at Cayley graphs of groups generated by a finite set of pairwise distinct involutive generators $e \in E$; in the following, we just speak of generators $e \in E$ and of *groups with generator set E*. In any such group G, we associate with the word $w = e_1 \cdots e_n$ over E the group element $[w]^G = e_1 \circ \cdots \circ e_n$. We think of the letters e_i also as edge labels along a path w from 1 to $[w]^G$ in the Cayley graph of G; in the natural fashion we let G operate on its Cayley graph from the right, so that $e_i = [e_i]^G$ translates g into $g\circ e_i$. We denote by w^{-1} the word $w^{-1} = e_n \cdots e_1$ obtained by reversing $w = e_1 \cdots e_n$; clearly, $[w^{-1}]^G = ([w]^G)^{-1}$ because of the involutive nature of the generators.

For any such group G, we also denote its Cayley graph by G, which is a regular E-graph. For a subset $\alpha \subseteq E$, we look at the subgroup $G_\alpha := G\restriction\alpha \subseteq G$ generated by this subset and at the α-components in the Cayley graph G. Then, the Cayley graph of the subgroup $G_\alpha \subseteq G$ is naturally isomorphic to the α-component of 1 in the Cayley graph of G, which is an α-graph (as well as an E-graph). The α-component of an arbitrary group element g, correspondingly, is described by the coset $gG_\alpha = \{g \circ h \colon h \in G_\alpha\}$; for simplicity, we also just speak of α-cosets.

If H is any E-graph, we write $\mathrm{sym}(H)$ for the Cayley group induced by the natural operation of edge colors $e \in E$ as involutions, as discussed before.[4] We denote the operation of $g \in \mathrm{sym}(H)$ on $v \in H$ by $v \cdot g$ as before. If G is itself a Cayley graph of a

[3]We here think of the group operation as composition in the sense of an action from the right, so that $\pi_i \circ \pi_j$ stands for "π_i followed by π_j" and an application to a vertex v of T would be denoted as $v \cdot (\pi_i \circ \pi_j) = (v \cdot \pi_i) \cdot \pi_j$ $(= \pi_j(\pi_i(v))$ in more conventional functional notation).

[4]The lower-case $\mathrm{sym}(H)$ distinguishes this subgroup from the full symmetric group of the vertex set H.

group with generator set E, then the group G is reproduced as $\mathrm{sym}(G)$. For this just note that $e(g) = g \cdot e = g \circ e$ for any $g \in G$ (viewed as a vertex of the Cayley graph) and $e \in E$ (viewed as a generator of $\mathrm{sym}(G)$).

We sum up the key points as follows.

Definition 2.1. An *E-graph* is an undirected, irreflexive graph with edge colors from the finite set E such that this coloring is a partition of the edge set and every vertex is incident with at most one edge of color e, for every $e \in E$. For an E-graph H, we denote by $\mathrm{sym}(H)$ the subgroup of the full symmetric group on its vertex set that is generated by the involutions $e \in E$; here $e \in E$ operates on H by swapping the two members of every e-colored edge. We regard $\mathrm{sym}(H)$ as a Cayley group with generator set E, and the corresponding Cayley graph, also denoted $\mathrm{sym}(H)$, as a (regular) E-graph.

The following will be an important compatibility criterion when we produce richer Cayley groups $\mathrm{sym}(H)$ through augmentations in the underlying graphs H.

Definition 2.2. Let G be a group with generator set E.

(i) An E-graph H is *compatible* with G if for all words w over E:
$[w]^G = 1 \;\Rightarrow\; [w]^{\mathrm{sym}(H)} = 1$.

(ii) *G reflects intersections* if, for all $\alpha, \beta \subseteq E$: $G_\alpha \cap G_\beta = G_{\alpha\cap\beta}$.

Example 2.3. The graph consisting of a single e-edge is compatible with G if, and only if, every generator word w that represents $1 \in G$ has an even number of letters e. If $G = \mathrm{sym}(H)$, then every connected component of H is trivially compatible with G. Conversely, if H' is compatible with $G = \mathrm{sym}(H)$, then $G \simeq \mathrm{sym}(H \dot{\cup} H')$, that is, G is unaffected by adjoining any compatible graph as a disjoint component.

Given a sequence of generators $w = e_1 \cdots e_n$ as a representation of the group element $h = \prod_i e_i = [w]^G \in G$ and a subset $\alpha \subseteq E$, we write $w \restriction \alpha$ for the projection of w to generators in α, that is, the sequence obtained by deletion of all generators $e_i \notin \alpha$. In general different representations of the same group element $[w]^G = [w']^G$ would have different projections to the subgroup G_α. If G_α is compatible with G, however, then $[w]^G = [w']^G \Leftrightarrow [w(w')^{-1}]^G = 1$ further implies that $[w(w')^{-1}]^{G_\alpha} = [(w(w')^{-1}) \restriction \alpha]^G = 1$, which implies that $[w \restriction \alpha]^G = [w' \restriction \alpha]^G$. So in this situation

$$h \restriction \alpha := [w \restriction \alpha] \text{ for any representation } [w]^G = h$$

is well defined in terms of h. In fact, G_α is compatible with G if, and only if, for all w

$$[w]^G = 1 \;\;\Rightarrow\;\; [w \restriction \alpha]^G = 1.$$

The following lemma shows that G reflects intersections if (the Cayley graphs of) its subgroups G_α are compatible with G, for all $\alpha \subseteq E$.

LEMMA 2.4. *Suppose G_α is compatible with G for every $\alpha \subseteq E$. Then, $G = \mathrm{sym}(G) = \mathrm{sym}(G \cup \dot{\bigcup}_\alpha G_\alpha)$ and for any $h = \prod_i e_i$ and $\alpha \subseteq E$:*

$$h \in G_\alpha \quad\Rightarrow\quad h = h \restriction \alpha := \textstyle\prod_{i\,:\, e_i \in \alpha} e_i,$$

with all $e_i \notin \alpha$ deleted. It follows that G reflects intersections.

PROOF. Let $h = \prod_i e_i \in G_\alpha$, and put $h \restriction \alpha := \prod_{i\,:\, e_i \in \alpha} e_i$. We want to show that $h = h \restriction \alpha$. We let $H := G \dot{\cup} G_\alpha$. By compatibility, $G = \mathrm{sym}(G) = \mathrm{sym}(H)$. It suffices to show that $\prod_i e_i$ and $\prod_{i\,:\, e_i \in \alpha} e_i$ have the same effect on every node $v \in H$. Since $h \in G_\alpha$, the target node $v \cdot h$ lies in the α-component of v. Consider a corresponding node v' in the isomorphic copy G_α of this α-component. We see that $v' \cdot h = v' \cdot (h \restriction \alpha)$, since all $e_i \notin \alpha$ operate trivially within this component, which does not have any edges of such colours.

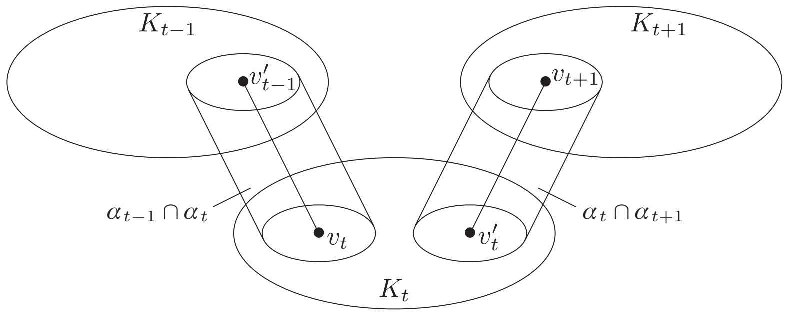

Fig. 2. Amalgamation: Merging chains of components.

Since the α-components of v and v' are isomorphic, and as both $h \in G_\alpha$ and $h \restriction \alpha \in G_\alpha$ operate within α-components, we find that $v \cdot h = v \cdot (h \restriction \alpha)$. As this holds for all $v \in H$, $h = h \restriction \alpha$ is an identity in G.

Applying this argument to $h \in G_\alpha \cap G_\beta$, we find that $h = h \restriction \alpha = (h \restriction \alpha) \restriction \beta \in G_{\alpha \cap \beta}$. $\square$

2.2. Amalgamation: Merging Chains of Components

Consider any two E-graphs K and K' with distinguished nodes $v \in K, v' \in K'$ and a distinguished subset $\alpha \subseteq E$. Assume that the α-components of v and v' are isomorphic via some isomorphism ρ that maps v to v'. Recall that α-components are regarded as α-graphs (reducts to colors in α). We let

$$K \ \frac{v=v'}{\alpha} \ K'$$

be the result of gluing K and K' according to the isomorphism ρ. If α is the intersection of the color classes of K and K', then this merged graph is again an E-graph.

In the following, we shall build chains by merging α-components of the Cayley graph of a group G. In this case, there always is, for any nodes $g \in G_\alpha$ and $g' \in G'_\alpha$, a unique isomorphism between the $(\alpha \cap \alpha')$-components of $g \in G_\alpha$ and of $g' \in G_{\alpha'}$ (both isomorphic to $G_{\alpha\cap\alpha'}$) that maps g to g'.

In merging a sequence of graphs $(K_t)_{1 \leqslant t \leqslant n}$, each with designated nodes to be identified with corresponding nodes in the left and right neighbors, we perform these identifications simultaneously, that is, apply the isomorphisms between matching components in any pair of neighbors along the sequence. A simple sufficient condition that guarantees that the resulting graph is again an E-graph is the following: we require the two patches in K_t that are joined with patches in K_{t-1} and K_{t+1}, respectively, to be disjoint. In this manner, no identifications are carried through any three or more consecutive members in the merged chain and no node can be incident with more than one e-edge, for any $e \in E$. (Compare Figure 2, also for the following.)

Definition 2.5. Consider a sequence $(K_t, v_t, v'_t)_{1 \leqslant t \leqslant n}$ of pairwise disjoint graphs isomorphic to α_t-components of G, where $K_t, v_t, v'_t \simeq G_{\alpha_t}, h_t, h'_t$ for $1 \leqslant t \leqslant n$.

This sequence is called *simple* if, for all $1 < t < n$, the connected components in K_t of v_t with respect to α_{t-1} and of v'_t with respect to α_{t+1} are disjoint.

In terms of the isomorphic representation of K_t, v_t, v'_t as G_{α_t}, h_t, h'_t, simplicity means that the α_{t-1}-component of h_t is disjoint from the α_{t+1}-component of h'_t in G_{α_t}:

$$h_t G_{\alpha_{t-1}\cap\alpha_t} \ \cap \ h'_t G_{\alpha_t \cap \alpha_{t+1}} = \emptyset;$$

or that

$$(h_t)^{-1} \circ h'_t \ \notin \ G_{\alpha_{t-1}\cap\alpha_t} \circ G_{\alpha_t\cap\alpha_{t+1}}.$$

Remark 2.6. Simplicity of the sequence $(K_t, v_t, v'_t)_{1 \leqslant t \leqslant n}$ implies that the *merged chain* obtained as

$$\sum_{t=1}^{n}(K_t, v_t, v'_t) := K_1 \; \frac{v'_1 = v_2}{\alpha_1 \cap \alpha_2} \; K_2 \quad \cdots \quad \frac{v'_{n-1} = v_n}{\alpha_{n-1} \cap \alpha_n} \; K_n$$

is an E-graph. The simplicity condition also rules out inclusion relationships between the color classes of K_t and K_{t+1} (other than at the ends, where an inclusion results in a trivial absorption): $\alpha_{t+1} \subseteq \alpha_t$ implies that K_{t+1} is contained in the α_t-component of v_{t+1}; this rules out a continuation beyond K_{t+1}, because any v'_{t+1} would itself lie in the α_t-component of v_{t+1}. The merging between K_t and K_{t+1} in this case is trivial in the sense that it is isomorphic to just K_t (absorption; cf. Figure 2).

The merged chains of simple sequences to be considered in the following will typically be of the form that $\alpha_t = \alpha \cap \beta_t$ for some sequence of subsets $\beta_t \subseteq E$ and a fixed subset $\alpha \subseteq E$. For simplicity we shall often write just $\alpha\beta$ instead of $\alpha \cap \beta$, especially when speaking of components and subgroups with respect to $\alpha \cap \beta$. For example, $G_{\alpha\beta}$ stands for $G_{\alpha\cap\beta}$.

Definition 2.7. Let $G' \subseteq G$ be any subgroup, $\alpha \subseteq E$. We say that G' *respects chains of* $(G_{\alpha\beta})_{\beta\subseteq E}$ *up to length* N, if every merged chain of a simple sequence of length up to N of components of the form $G_{\alpha\beta}$ for $\beta \subseteq E$ is compatible with G'.

2.3. Discounted Lengths: Avoiding Not Just Short Cycles

We want to measure the length of certain cycles in E-graphs in such a way as to reflect distances that discount repeated moves within the same $\alpha \subseteq E$. We present these notions in terms of Cayley groups, but they could analogously be introduced in terms of E-graphs. We deal with cyclic words w of group elements, that is, words $w = g_0 \cdots g_{n-1} = (g_t)_{t\in\mathbb{Z}_n}$, cyclically indexed modulo n.

Definition 2.8. Let G be a group with generator set E, with subgroups G_α for subsets $\alpha \subseteq E$ as previously described. A *nontrivial colored cycle of length* n in G is any cyclic tuple $(g_t)_{t\in\mathbb{Z}_n}$ in G together with a coloring $\sigma : \mathbb{Z}_n \to \mathcal{P}(E)$ such that

(i) $\prod_{t\in\mathbb{Z}_n} g_t = g_0 \circ \cdots \circ g_{n-1} = 1$,
(ii) $g_t \in G_{\sigma(t)}$,
(iii) $g_t \notin G_{\sigma(t-1)\sigma(t)} \circ G_{\sigma(t)\sigma(t+1)}$.

G is called N*-acyclic* if all subgroups G_α for $\alpha \subseteq E$ are compatible with G and G has no nontrivial colored cycles of lengths $n \leqslant N$.

The point of this notion is the way in which lengths of cycles in the Cayley graph of G are measured: we effectively count factors in subgroups G_α rather than the length of generator sequences that produce these factors. Therefore, the usual graph theoretic length of a colored cycle of length n is a priori unbounded in terms of the underlying cycle of generator edges.

Condition (iii) concerns a property of the factors g_t in the subgroups $G_{\sigma(t)}$: it says that within this subgroup g_t is not equal to any product of two elements from the two subgroups $G_{\sigma(t)\sigma(t\pm1)} \subseteq G_{\sigma(t)}$. Intuitively, this condition says that the effect of factor g_t cannot be absorbed via variations in the immediate predecessor and successor factors; this closely matches the condition on simple chains in Definition 2.5 (cf. Figures 2 and 3). Together with (ii), (iii) also rules out inclusions between adjacent color classes: $\sigma(t) \not\subseteq \sigma(t \pm 1)$.

The following is just a simple reformulation but will be useful in this form later.

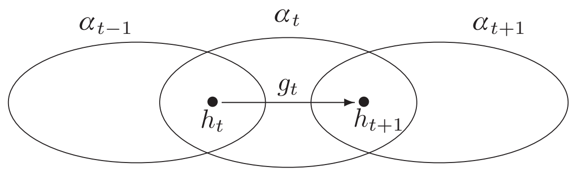

Fig. 3. Amalgamation: Overlap between cosets.

OBSERVATION 2.9. *Given any $(h_t)_{t\in\mathbb{Z}_n}$ in G, put $g_t := (h_t)^{-1} \circ h_{t+1}$ for $t \in \mathbb{Z}_n$. Then clearly, $\prod_{t\in\mathbb{Z}_n} g_t = 1$. If $g_t \in G_{\sigma(t)}$ for $t \in \mathbb{Z}_n$, then $(g_t)_{t\in\mathbb{Z}_n}$ is nontrivially colored by σ, that is, also satisfies condition (iii) of Definition 2.8, if, and only if,*

$$h_t G_{\sigma(t-1)\sigma(t)} \;\cap\; h_{t+1} G_{\sigma(t)\sigma(t+1)} = \emptyset.$$

PROOF. It suffices to observe that $g_t = k_1 \circ k_2$ for some $k_i \in G_{\alpha_i}$ implies $h_t \circ k_1 = h_{t+1} \circ (k_2)^{-1} \in h_t G_{\alpha_1} \cap h_{t+1} G_{\alpha_2}$; and that any element k of this intersection in turn gives rise to a decomposition of $g_t = (h_t)^{-1} \circ h_{t+1}$ as $g_t = k_1 \circ (k_2)^{-1}$ where the $k_i \in G_{\alpha_i}$ are such that $k = h_t \circ k_1 = h_{t+1} \circ k_2$. □

LEMMA 2.10. *Let G be a group with generator set E as before, $k \in \mathbb{N}$. Assume that, for every $\alpha \subseteq E$ with $|\alpha| < k$, the subgroup G_α*

(a) respects chains of $(G_{\alpha\beta})_{\beta\subseteq E}$ up to length N, and
(b) has no nontrivial colored cycles of length up to N.

Then there is a finite group G^* with generator set E such that:

(i) for every $\alpha \subseteq E$ with $|\alpha| < k$, $G^*_\alpha \;\simeq\; G_\alpha$,

and for all $\alpha \subseteq E$ with $|\alpha| \leqslant k$, the subgroups G^*_α

(ii) respect chains of $(G^*_{\alpha\beta})_{\beta\subseteq E}$ up to length N, and
(iii) have no nontrivial colored cycles of length up to N.

Compare Definition 2.2 and Lemma 2.4 for the following.

Remark 2.11. In the special case that $k = |E|$ and for $\alpha = E$, (ii) implies in particular that G^* is compatible with its β-components for all $\beta \subseteq E$. Because $G^* \simeq \mathrm{sym}(G^*)$, it follows that G^* is compatible with its subgroups G^*_α for $\alpha \subseteq E$ and, by Lemma 2.4, reflects intersections.

PROOF OF THE LEMMA. We construct G^* as $G^* := \mathrm{sym}(H)$ for a graph $H = G \cup H^0$ consisting of the disjoint union of the Cayley graph of G and certain merged chains of components of G.

Consider any simple sequence $(K_t, v_t, v'_t)_{1\leqslant t\leqslant n}$ of length $n \leqslant N$ of components $K_t, v_t, v'_t \simeq G_{\alpha\beta_t}, h_t, h'_t$ with $|\alpha| \leqslant k$. For any such sequence, we put the corresponding merged chain

$$\sum_{t=1}^{n} (K_t, v_t, v'_t) := K_1 \;\frac{v'_1 = v_2}{\alpha\beta_1\beta_2}\; \cdots \;\frac{v'_{n-1} = v_n}{\alpha\beta_{n-1}\beta_n}\; K_n \qquad (*)$$

as a separate connected component in H^0.

By construction, $G^* = \mathrm{sym}(G \cup H^0)$ respects chains of $(G_{\alpha\beta})_{\beta\subseteq E}$ up to length N. Together with (i), this implies that G^* respects chains of $(G^*_{\alpha\beta})_{\beta\subseteq E}$ for the following reason. If the chain in question is such that all components $G^*_{\alpha\beta}$ have $|\alpha \cap \beta| < k$, (i) tells us that $G^*_{\alpha\beta} \simeq G_{\alpha\beta}$. If, on the other hand, some component $G^*_{\alpha\beta}$ has $|\alpha \cap \beta| = k$, then it must be that $|\alpha| = k$ and $\beta \supseteq \alpha$ and the merged chain is isomorphic to G^*_α (cf. Remark 2.6); so, in this case, the claim boils down to G^*_α respects G^*_α, which is trivially true.

Towards (i) we claim that each one of the new connected components K as in $(*)$ is compatible with all $G_{\alpha'}$ for $|\alpha'| < k$. Let K as in $(*)$ and fix some $|\alpha'| < k$. Compatibility of K with $G_{\alpha'}$ depends only on the isomorphism types of α'-components of K. Every such component is obtained as a merged chain of a simple sequence of components of type $G_{\alpha'\alpha\beta_t}$ for t from some subinterval of $[1, n]$. Since $|\alpha'| < k$, assumption (a) implies that this component is compatible with $G_{\alpha'}$.

It follows that $G^* = \mathrm{sym}(G \cup H^0)$ is compatible with all $G_{\alpha'}$ for $|\alpha'| < k$, whence $G^*_{\alpha'} \simeq G_{\alpha'}$ for $|\alpha'| < k$ (cf. comments in Example 2.3).

For (iii), it remains to argue that G^*_α does not have nontrivial colored cycles of lengths $n \leqslant N$ whenever $|\alpha| \leqslant k$. Let $|\alpha| \leqslant k$ and let $((g_t)_{t\in\mathbb{Z}_n}, \sigma)$ be a nontrivial colored cycle in G^*_α. We need to show that $n > N$.

As a consequence of condition (iii) of Definition 2.8, $\sigma(t) \not\supseteq \sigma(t-1)$ whence $\sigma(t) \subsetneq \alpha$ for all t. It follows that $|\alpha \cap \sigma(t)| < k$. Let $g_t = [u_t]^{G^*_\alpha}$ for a word u_t over $\alpha \cap \sigma(t)$, and put $w := u_1 \cdots u_n$. We want to show that $\prod_t g_t = [w]^{G^*_\alpha} \neq 1$ if $n \leqslant N$. It suffices to find an element of H on which w does not act as the identity. An element in a component of H^0 obtained as a suitable merged chain of components $G_{\alpha\sigma(t)}$ will serve this purpose. We look at the sequence

$$K_t, v_t, v'_t \simeq G_{\alpha\sigma(t)}, h_t, h'_t \simeq G^*_{\alpha\sigma(t)}, h_t, h'_t$$

with $h_t := 1$ and $h'_t := [u_t]^G$ for $t \in \mathbb{Z}_n$.

The sequence of these K_t, v_t, v'_t is simple in the sense of Definition 2.5, by condition (iii) in Definition 2.8. The corresponding merged chain $K := \sum_t (K_t, v_t, v'_t)$ is a component of H provided $n \leqslant N$. But the element corresponding to $1 \in K_1$ is mapped by $[w]^{G^*}$ to the element corresponding to $h'_n \in K_n$, which is distinct from all elements represented in the components K_t for $t < n$ and in particular from $1 \in K_1$. It follows that, if $n \leqslant N$, $[w]^{G^*} \neq 1$, so that $(g_t)_{t\in\mathbb{Z}_n}$ cannot be a cycle in G^*_α. □

By iterated application of the lemma starting with $k = 1$ such that conditions (a) and (b) are trivially fulfilled (for $\alpha = \emptyset$!), we obtain the following, which technically is one of our key results.

COROLLARY 2.12. *For every finite set E and every $N \in \mathbb{N}$, there is a finite N-acyclic group with generator set E. That is, there is a finite Cayley group G with E as its set of involutive generators such that all the subgroups G_α generated by subsets $\alpha \subseteq E$ are compatible with G (and in particular $G_\alpha \cap G_\beta = G_{\alpha\cap\beta}$ for all $\alpha, \beta \subseteq E$) and such that G has no nontrivial colored cycles of length up to N.*

3. HYPERGRAPH COVERS OF QUALIFIED ACYCLICITY

In this part, we deal with hypergraphs and hypergraph covers. We first review some basic terminology.

3.1. Hypergraphs and Acyclicity

A hypergraph is a structure $\mathfrak{A} = (A, S)$ consisting of a (finite) universe A together with a set of hyperedges $S \subseteq \mathcal{P}(A)$. The *width* of $\mathfrak{A}$ is the maximal cardinality among its hyperedges. With the hypergraph $\mathfrak{A} = (A, S)$ we associate its *Gaifman graph*, which is an undirected graph over the vertex set A with edges linking any pair of distinct vertices that are members of the same hyperedge $s \in S$ (a clique for every hyperedge of $\mathfrak{A}$). The notion of (induced) subhypergraph is the natural one: think of removing all elements not in the designated subset from both the universe and from every hyperedge. We shall not look at weak substructure relationships between hypergraphs because these do not preserve the acyclicity features we are mostly interested in.

A hypergraph is *conformal* if every clique in its Gaifman graph is contained in some hyperedge; in analogy with guardedness in relational structures, cf. Section 4, we also say that every clique must be *guarded* by a hyperedge. More generally, a configuration of nodes is said to be guarded if it is contained in some hyperedge.

An *n-cycle* in a hypergraph is a cycle of length n in its Gaifman graph (which is a homomorphic image of the standard n-cycle with vertex set $\mathbb{Z}_n$ and edges between next neighbors). A *chord* in an n-cycle is an edge between vertices of the cycle that are not next neighbors in the cycle. A hypergraph and its Gaifman graph are called *chordal* if every cycle of length greater than 3 has a chord, that is, if there are no chordless cycles of length greater than 3 in the Gaifman graph. We use cyclic words $(a_t)_{t\in\mathbb{Z}_n}$ to denote cycles (indexing modulo n understood); this cycle is *chordless* if $\{a_i, a_j\}$ is not guarded unless $i = j, j \pm 1$. (Note that, since we do not necessarily require a cycle to be injectively embedded, we also regard double points as constituting a chord; while $a_i = a_{i+1}$ is ruled out, since (a_i, a_i) is not an edge of the Gaifman graph, $a_i = a_j$ for $j \neq i \pm 1$ is possible, but would be a chord in any n-cycle for $n \geqslant 4$.)

It is known from classical hypergraph theory, cf. Berge [1973] and Beeri et al. [1983], that a hypergraph is tree-decomposable (also called acyclic) if, and only if, it is both conformal and chordal. $\mathfrak{A} = (A, S)$ is tree-decomposable if it admits a *tree decomposition* $\mathcal{T} = (T, \delta)$: T is a tree and $\delta\colon v \mapsto \delta(v) \in S$ maps the nodes of T to hyperedges of $\mathfrak{A}$ in such a manner that $\mathrm{im}(\delta) = S$ and, for every node $a \in \mathfrak{A}$, the subset $\{v \in T : a \in \delta(v)\}$ is connected in T. An equivalent characterization requires that $\mathfrak{A}$ can be reduced to the empty hypergraph by repeated application of two kinds of reduction steps: removal of a node that is covered by at most one hyperedge, and removal of a hyperedge that is fully contained in some other hyperedge.

The bounded variants of acyclicity and its constituents, which are relevant to us, are the following. The N-bounded version of each one of these properties precisely captures the requirement that every induced subhypergraph of size up to N has the unqualified property.

Definition 3.1. Let $N \in \mathbb{N}$. A hypergraph $\mathfrak{A}$ is called

(i) *N-conformal* if it does not have any unguarded cliques up to size N.
(ii) *N-chordal* if it does not have any chordless cycles of lengths n for $4 \leqslant n \leqslant N$.
(iii) *N-acyclic* if it is both N-conformal and N-chordal.

OBSERVATION 3.2. *For $N > w$, any hypergraph of width w that is N-conformal is in fact conformal.*

The following example links N-acyclicity of Cayley groups to N-acyclicity of an associated hypergraph of cosets. We sketch the argument for N-acyclicity because some of the underlying proof ideas will re-appear in the more technical proofs related to the construction of N-acyclic hypergraph covers.

OBSERVATION 3.3. *Let G be an N-acyclic group with generator set E. Then, the following hypergraph of cosets in G, $\mathfrak{A}[G] = (A[G], S[G])$, is N-acyclic:*

$$A[G] := \{gG_\alpha : g \in G, \alpha \subseteq E\}$$

$$S[G] := \{[g] : g \in G\} \textit{ where } [g] := \{gG_\alpha : \alpha \subseteq E\}.$$

Note that the hyperedge $[g]$ consists precisely of all α-cosets (α-components in the Cayley graph of G) that are incident with g.

PROOF. N-chordality of $\mathfrak{A}[G]$ is rather straightforward. Let $(a_t = h_t G_{\alpha_t})_{t\in\mathbb{Z}_n}$ be a chordless cycle of length $n > 3$ in $\mathfrak{A}$ with hyperedges $([h_t])_{t\in\mathbb{Z}_n}$ linking a_{t-1} and a_t (without loss of generality we use these same h_t as the representatives for $a_t = h_t G_{\alpha_t}$ as $a_t \in [h_t]$ means that h_t is an element of the coset a_t). As also $a_t \in [h_{t+1}]$, we have $h_{t+1} \in h_t G_{\alpha_t}$ and thus

$$g_t := (h_t)^{-1} \circ h_{t+1} \in G_{\alpha_t} \text{ for } t \in \mathbb{Z}_n.$$

Clearly $\prod_{t\in\mathbb{Z}_n} g_t = 1$. We want to show that $\sigma(t) := \alpha_t$ induces a coloring satisfying condition (iii) of Definition 2.8, which by Observation 2.9 is equivalent to

$$h_t G_{\alpha_{t-1}\alpha_t} \cap h_{t+1} G_{\alpha_t\alpha_{t+1}} = \emptyset.$$

But clearly, a violation of this emptiness assertion would mean that there is a hyperedge in $\mathfrak{A}$ that links a_{t-1} to a_{t+1} so that $(a_t)_{t\in\mathbb{Z}_n}$ would not be chordless. As $(g_t)_{t\in\mathbb{Z}_n}$ therefore gives rise to a nontrivial colored cycle of length n in G, N-acyclicity of G implies that $n > N$.

We turn to N-conformality of $\mathfrak{A}[G]$. Suppose that $\{a_t : t \in \mathbb{Z}_n\} \subseteq A[G]$ forms an unguarded clique in $\mathfrak{A}[G]$ that is minimal in the sense that every subclique of $n-1$ of these vertices is guarded in $\mathfrak{A}[G]$. We aim to show that again this implies $n > N$. For $t \in \mathbb{Z}_n$ let $[h_t] \in S[G]$ be such that $\{a_s : s \in \mathbb{Z}_n \setminus \{t\}\} \subseteq [h_t]$. This implies that, if we let a_t be an α_t-coset, then this coset contains all the elements h_s for $s \neq t$ so that $a_t = h_s G_{\alpha_t}$ for every $s \neq t$. Since h_t and h_{t+1} are members of the same α_s-coset for all $s \neq t, t+1$, it follows that

$$g_t := (h_t)^{-1} \circ h_{t+1} \in \textstyle\bigcap_{s\neq t,t+1} G_{\alpha_s}.$$

Clearly $\prod_{t\in\mathbb{Z}_n} g_t = 1$, and again we seek to show that $\sigma(t) := \bigcap_{s\neq t,t+1} \alpha_s$ induces a nontrivial coloring of this cycle. Suppose that, contradicting condition (iii) of Definition 2.8, $g_t \in G_{\sigma(t-1)\sigma(t)} \circ G_{\sigma(t)\sigma(t+1)}$, or that (cf. Observation 2.9)

$$h_t G_{\sigma(t-1)\sigma(t)} \cap h_{t+1} G_{\sigma(t)\sigma(t+1)} \neq \emptyset.$$

Note that $\sigma(t-1) \cap \sigma(t) = \bigcap_{s\neq t} \alpha_s$ and $\sigma(t) \cap \sigma(t+1) = \bigcap_{s\neq t+1} \alpha_s$.

For any element $h \in h_t G_{\sigma(t-1)\sigma(t)} \cap h_{t+1} G_{\sigma(t)\sigma(t+1)}$, however, we see that $a_s \in [h]$ for all $s \in \mathbb{Z}_n$, so that $[h]$ would guard the whole clique, contrary to our assumptions.

For $s \neq t$, $a_s \in [h]$ since $h \in h_t G_{\sigma(t-1)\sigma(t)} \subseteq h_t G_{\alpha_s} = a_s$ (recall that $a_s = h_t G_{\alpha_s}$ for every $t \neq s$). For $s \neq t+1$, $a_s \in [h]$ since $h \in h_{t+1} G_{\sigma(t)\sigma(t+1)} \subseteq h_{t+1} G_{\alpha_s} = a_s$.

Therefore, our assumptions imply that $(g_t)_{t\in\mathbb{Z}_n}$ induces a nontrivial colored cycle of length n in G, whence $n > N$ follows from N-acyclicity of G. □

3.2. Hypergraph Covers

The notion of hypergraph bisimulation is the natural generalization of bisimulation between graph-like structures. It captures the idea of a back & forth correspondence whose individual matches are bijections between individual hyperedges and whose back & forth requirements ensure that the overlap patterns between hyperedges in one hypergraph can be simulated in the other. In this sense, hypergraph bisimulation is also at the combinatorial core of guarded bisimulation (as if stripped of the relational information within relational hyperedges). Here we discuss a special case, viz. hypergraph bisimulations induced by a hypergraph homomorphism from one (covering) hypergraph onto another.

Definition 3.4. A map $\pi : \hat{\mathfrak{A}} \to \mathfrak{A}$ between hypergraphs $\hat{\mathfrak{A}} = (\hat{A}, \hat{S})$ and $\mathfrak{A} = (A, S)$ is a *hypergraph homomorphism* if $\pi \restriction \hat{s}$ is a bijection between the hyperedge $\hat{s}$ and its image $\pi(\hat{s}) \in S$, for every $\hat{s} \in \hat{S}$.

A homomorphism $\pi\colon \hat{\mathfrak{A}} \rightarrow \mathfrak{A}$ is a (bisimilar) *hypergraph cover* if it satisfies the following *back*-property: for every $s \in S$, there is some $\hat{s} \in \hat{S}$ such that $\pi(\hat{s}) = s$, and, whenever $\pi(\hat{s}) = s$ and $s' \in S$, then there is some $\hat{s}' \in \hat{S}$ such that $\pi(\hat{s}') = s'$ and $\pi(\hat{s} \cap \hat{s}') = s \cap s'$.

The (conformal and) N-acyclic hypergraph covers to be constructed in this section are hypergraph covers $\pi\colon \hat{\mathfrak{A}} \rightarrow \mathfrak{A}$ by (conformal and) N-acyclic hypergraphs $\hat{A}$.

Note that the homomorphism requirement for covers settles the *forth*-property in the back & forth view. With respect to the branching between hyperedges, however, this definition poses no constraints, as the following illustrates.

Remark 3.5. A covering hypergraph according to this definition may have a richer local branching structure than the base hypergraph. For instance, every single hyperedge s of $\mathfrak{A}$ may be covered by a cluster of hyperedges of the form $\hat{s}_\sigma = \{(a, \sigma(a))\colon a \in s\}$, that is, graphs of functions $\sigma\colon s \rightarrow \{1, \ldots, k\}$, where $\pi\colon (a, i) \mapsto a$ is the natural projection to the first component. For $k > 1$, any subset $t \subseteq s$ occurs as the π-image of the intersection of two covering hyperedges for s.

In fact, the conformal finite hypergraph covers of Hodkinson and Otto [2003] can be regarded as controlled restrictions of free coverings as in this remark. We state the result, which will be used later. More succinct finite conformal covers are obtained in Bárány et al. [2010], which relies on a more intricate construction.

THEOREM 3.6. [HODKINSON AND OTTO 2003]. *Every finite hypergraph admits a cover by a finite conformal hypergraph.*

Just as hypergraphs generalize (undirected) graphs, hypergraph covers generalize graph covers or bisimilar covers of graphs. In graphs just as in hypergraphs, these notions and constructions extend naturally to relational structures: to relational structures of width 2 (possibly directed, edge- and vertex-colored graphs, Kripke structures or transition systems) in the graph case; and to relational structures with relations of arbitrary arity in the hypergraph case; hypergraph bisimulations turn into guarded bisimulations in this view (cf. Section 4).

As mentioned in the introduction, finite locally acyclic covers are available in the graph case and thus for transition systems, Kripke structures or relational structures of width 2. A graph, or hypergraph of width 2, is *N-locally acyclic* if it has no cycles of length up to $2N + 1$. The N-locally acyclic covers constructed in Otto [2004] even preserve the degree; that is, in these covers the branching in the cover is locally the same as in the base graph. We state this result from Otto [2004], whose core will also be the base case for our inductive approach to N-acyclic hypergraph covers in widths greater than two.

PROPOSITION 3.7. *For every $N \in \mathbb{N}$, every finite graph admits a cover by a finite graph that is N-locally acyclic. Moreover, the cover can be chosen to preserve the degree of vertices.*

The construction of these covers can be based on a straightforward product between the given graph and a Cayley group of large girth, as outlined in the following.

Let $\mathfrak{A} = (A, E)$ be a finite, undirected and irreflexive graph; G a Cayley group with involutive generators $e \in E$ of girth greater than $2N + 1$ (i.e., the Cayley graph has no nontrivial cycles of length up to $2N + 1$).

Let $\hat{\mathfrak{A}} := \mathfrak{A} \otimes G := (A \times G, \hat{E})$ be the graph with edge relation

$$\hat{E} := \big\{\{(a, g), (a', g \circ e)\}\colon e = \{a, a'\} \in E\big\}.$$

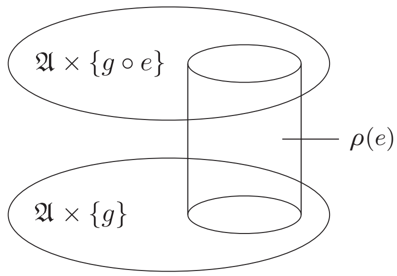

Fig. 4. Identification between layers in $\mathfrak{A} \times_\rho G$.

It is clear that the natural projection $\pi : \hat{\mathfrak{A}} \to \mathfrak{A}$, which maps (a, g) to a, is a degree-preserving graph cover: the edge $e = \{a, a'\}$ lifts to any $(a, g) \in \pi^{-1}(a)$ as the unique edge linking (a, g) to $(a', g \circ e)$. Moreover, $\hat{\mathfrak{A}}$ is N-locally acyclic in the sense that the N-neighborhood of any vertex is acyclic. In the second component, any cycle in $\hat{\mathfrak{A}}$ projects to a cycle in G; as G does not admit nontrivial cycles of length up to $2N + 1$, $\hat{\mathfrak{A}}$ cannot have nondegenerate cycles of length up to $2N + 1$ and it follows that $\hat{\mathfrak{A}}$ is N-locally acyclic. For $N \geqslant 1$, $\hat{\mathfrak{A}}$ is in particular triangle-free, hence also conformal.

The goal here is the following, which is our second main technical result.

THEOREM 3.8. *For every $N \in \mathbb{N}$, every finite hypergraph admits a bisimilar cover by some finite conformal and N-acyclic hypergraph.*

The proof of this theorem will be completed in Section 3.5. It involves a local-to-global construction, which uses localizations for a reduction with respect to width (Section 3.4) and a reduced product between a hypergraph and a group (Section 3.3) towards the global completion of covers. The reduced product construction, which seems to be new here, serves to glue layers locally so as to turn a stack of isomorphic hypergraphs into a “millefeuilles of hypergraphs”; the N-acyclic groups from Section 2 are used in this construction to maintain degrees of acyclicity in the passage form the layers to the reduced product.

3.3. Millefeuilles of Hypergraphs

Let $\mathfrak{A} = (A, S)$ be a finite hypergraph, and let the colors $e \in E$ be associated with guarded subsets of $\mathfrak{A}$ through a map $\rho : e \mapsto \rho(e) \subseteq A$ such that $\rho(e) \subseteq s$ for some $s \in S$. We consider stacks of copies of the hypergraph $\mathfrak{A}$ that are selectively joined in the subsets $\rho(e)$, as indicated in Figure 4.

For $a \in A$, let $\alpha_a := \{e \in E : a \in \rho(e)\}$.

For a group G with generator set E, we write G_a for the subgroup generated by α_a; $G_{aa'}$ for the subgroup generated by $\alpha_{aa'} := \alpha_a \cap \alpha_{a'} = \{e \in E : a, a' \in \rho(e)\}$, etc. Note that for $a \notin \bigcup\{\rho(e) : e \in E\}$, $\alpha_a = \emptyset$ and $G_a = \{1\}$.

On $A \times G$, consider the equivalence relation

$$(a, g) \approx (a, g') \quad :\Leftrightarrow \quad g^{-1} \circ g' \in G_a.$$

We write $[a, g]$ for the equivalence class of (a, g) with respect to $\approx$, and lift this notation to tuples and sets of elements as, for example, in $[s, g] := \{[a, g] : a \in s\}$. We put

$$\mathfrak{A} \times_\rho G := (\hat{A}, \hat{S}) \quad \text{with} \quad \begin{aligned} \hat{A} &:= (A \times G)/\approx, \\ \hat{S} &:= \{[s, g] : s \in S, g \in G\}. \end{aligned}$$

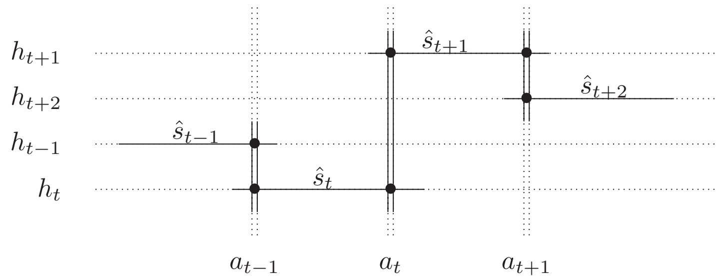

Fig. 5. Layers with connecting hyperedges along some path or cycle.

The definitions of $\approx$ and $\hat{S}$ imply that

$$\begin{aligned}[a, g] \in [s, h] &\text{ iff } a \in s \text{ and } g^{-1} \circ h \in G_a \\ &\text{ iff } a \in s \text{ and } [a, g] = [a, h].\end{aligned}$$

Note that $\approx$ is trivial in restriction to $A\times\{g\}$, whence $(A\times\{g\})/\approx$ is naturally identified with $A \times \{g\}$ and carries the hypergraph structure of $\mathfrak{A}$. We refer to these embedded isomorphic copies of $\mathfrak{A}$ as the *layers* of $\mathfrak{A}\times_\rho G$ and denote them as $\mathfrak{A}\times\{g\}$. Different layers of the millefeuille are locally joined, through identification according to $\approx$, whence they are not disjoint.

We say that a clique or a cycle is contained in a layer $\mathfrak{A} \times \{g\}$ if its vertices are all represented in this layer.

Note that the natural projection $\pi : \mathfrak{A} \times_\rho G \to \mathfrak{A}$ is a cover.

PROPOSITION 3.9. *Let G be N-acyclic with generator set E, $\mathfrak{A} \times_\rho G$ as before:*

(i) Any chordless cycle of length up to N in $\mathfrak{A} \times_\rho G$ must be contained within a single layer of $\mathfrak{A} \times_\rho G$.
(ii) Any unguarded clique of size up to N in $\mathfrak{A} \times_\rho G$ must be contained within a single layer of $\mathfrak{A} \times_\rho G$.

So N-conformality and N-chordality are preserved in the passage from $\mathfrak{A}$ to $\mathfrak{A} \times_\rho G$. Moreover, if $\mathfrak{A}$ is conformal and of width $w < N$, then $\mathfrak{A} \times_\rho G$ is also conformal.

PROOF. For the proof of (i) assume that $(\hat{a}_t)_{t\in\mathbb{Z}_n}$ is a chordless cycle of length n, $3 < n \leqslant N$, in $\hat{\mathfrak{A}} = \mathfrak{A} \times_\rho G = (\hat{A}, \hat{S})$. We let $\hat{s}_t = [s_t, h_t] \in \hat{S}$ be a sequence of linking hyperedges such that $\hat{a}_t \in \hat{s}_t \cap \hat{s}_{t+1}$ as in Figure 5 and assume that the (s_t, h_t) are chosen such that the number of jumps between distinct layers $\mathfrak{A} \times \{h_t\}$ is minimal: with $J := \{t : h_t \neq h_{t+1}\}$, the (s_t, h_t) have been chosen so as to minimize $|J|$. Clearly $0 \leqslant |J| \leqslant n$, and our goal is to show that this minimization implies $J = \emptyset$, so that indeed the whole cycle is represented in a single layer and hence a cycle of $\mathfrak{A}$. Put

$$\begin{aligned} g_t &:= h_t^{-1} \circ h_{t+1}, \\ \sigma(t) &:= \alpha_{a_t} = \{e \in E : a_t \in \rho(e)\}. \end{aligned}$$

Then, $\hat{a}_t \in \hat{s}_t \cap \hat{s}_{t+1}$ implies that $g_t \in G_{\sigma(t)}$. Clearly, $g_t \neq 1$ iff $t \in J$, and

$$\textstyle\prod_t g_t = \prod_{t\in J} g_t = 1.$$

That the cycle of the $\hat{a}_t$ is chordless implies that, for $t' \neq t \pm 1$:

$$\begin{array}{l}\text{if } \hat{a}_t \text{ and } \hat{a}_{t'} \text{ are represented in the same layer of } \hat{\mathfrak{A}}, \\ \text{then } \{a_t, a_{t'}\} \text{ is not guarded in } \mathfrak{A} \text{ and } G_{\sigma(t)\sigma(t')} = \{1\}.\end{array} \qquad (*)$$

The last implication follows from the fact that the sets $\rho(e)$ are guarded, so that $\sigma(t) \cap \sigma(t') = \emptyset$ if $\{a_t, a_{t'}\}$ is not guarded.

We claim that, for nonempty J, $(g_t)_{t\in J}$ would be a nontrivial colored cycle in G, colored by the natural restriction of σ. For this, we verify that any violation of condition (iii) in Definition 2.8 would allow us to eliminate one of the remaining jumps, contradicting the minimality of $|J|$.

Consider next neighbors $t' < t$ in J along this cycle. Since there are no jumps between t' and t, $\hat{a}_{t'}$ and $\hat{a}_t$ are both represented in layer $\mathfrak{A} \times \{h_t\}$, which by $(*)$ implies $\sigma(t) \cap \sigma(t') = \emptyset$ and $G_{\sigma(t)\sigma(t')} = \{1\}$ unless $t = t' + 1$.

Assume then that $t' < t < t''$ are next neighbors in J and that—contrary to condition (iii) in Definition 2.8—we had $g_t \in G_{\sigma(t')\sigma(t)} \circ G_{\sigma(t)\sigma(t'')}$, or equivalently by Observation 2.9 , that $h_t G_{\sigma(t')\sigma(t)} \cap h_{t''} G_{\sigma(t)\sigma(t'')} \neq \emptyset$.

Clearly this implies that $t' = t-1$ or $t'' = t+1$, since otherwise $G_{\sigma(t')\sigma(t)} = G_{\sigma(t)\sigma(t'')} = \{1\}$ while $g_t \neq 1$.

Suppose first that $t' = t - 1$, but $t'' \neq t + 1$ so that $G_{\sigma(t)\sigma(t'')}$ is trivial. Then, $g_t \in G_{\sigma(t')\sigma(t)} \circ G_{\sigma(t)\sigma(t'')} = G_{\sigma(t')\sigma(t)} \subseteq G_{\sigma(t')}$ implies that $\hat{a}_{t'}$ is also represented in layer $\mathfrak{A} \times \{h_{t''}\}$: $\hat{a}_{t'}$ is represented in layer $\mathfrak{A} \times \{h_t\}$ by assumption and can also be represented in layer $\mathfrak{A} \times \{h_{t''}\}$ if the transition from layer $\mathfrak{A} \times \{h_t\}$ to $\mathfrak{A} \times \{h_{t''}\}$, which is affected by g_t, preserves $\hat{a}_t$. But this contradicts minimality of $|J|$.

The case of $t' \neq t - 1$ but $t'' = t + 1$ is symmetric.

If $t' = t-1$ and $t'' = t+1$, then we may use $h \in h_t G_{\sigma(t-1)\sigma(t)} \cap h_{t+1} G_{\sigma(t)\sigma(t+1)}$ to represent all three vertices, $\hat{a}_{t-1}, \hat{a}_t$ and $\hat{a}_{t+1}$ in the common layer $\mathfrak{A} \times \{h\}$, thus again reducing the number of jumps by 1 and contradicting the minimality of $|J|$. To see that all three vertices are contained in layer $\mathfrak{A} \times \{h\}$, we observe that

—$h \in h_t G_{\sigma(t-1)\sigma(t)}$ implies $(h_t)^{-1} \circ h \in G_{\sigma(t-1)}$ and $(h_t)^{-1} \circ h \in G_{\sigma(t)}$, so that

$$\begin{aligned} \hat{a}_{t-1} &= [a_{t-1}, h_t] = [a_{t-1}, h], \\ \hat{a}_t &= [a_t, h_t] \quad = [a_t, h]; \end{aligned}$$

—and that $h \in h_{t+1} G_{\sigma(t)\sigma(t+1)}$ similarly implies $(h_{t+1})^{-1} \circ h \in G_{\sigma(t+1)}$, whence also

$$\hat{a}_{t+1} = [a_{t+1}, h_{t+1}] = [a_{t+1}, h].$$

Towards (ii), let n be minimal such that $\hat{\mathfrak{A}} := \mathfrak{A} \times_\rho G$ has a clique of size n *not* contained in a single layer. Let $(\hat{a}_t)_{t\in\mathbb{Z}_n}$ be such a clique. Put $a_t := \pi(\hat{a}_t)$ and $\mathbf{a} = \{a_t : t \in \mathbb{Z}_n\}$ (clearly $\mathbf{a}$ is a clique in $\mathfrak{A}$). By minimality of n, we have that every subset of up to $n-1$ elements among the $\hat{a}_t$ is represented within a single layer of the stack. In particular, for every $t \in \mathbb{Z}_n$, there is some $h_t \in G$ such that $\hat{a}_s = [a_s, h_t]$ for all $s \neq t$. Consider then the group elements

$$g_t := h_t^{-1} \circ h_{t+1} \quad \text{for } t \in \mathbb{Z}_n.$$

Clearly, $\prod_{t\in\mathbb{Z}_n} g_t = 1$. By our assumptions, all $\hat{a}_s$ for $s \neq t, t+1$ are represented in layers h_t and $h_{t=1}$, whence

$$g_t \in G_{\alpha(t)} \text{ for } \alpha_t := \{e \in E \colon \mathbf{a} \setminus \{a_t, a_{t+1}\} \subseteq \rho(e)\}.$$

Therefore, $\sigma(t) := \alpha_t$ is a coloring of the cycle $(g_t)_{t\in\mathbb{Z}_n}$. We claim that, since $(\hat{a}_t)_{t\in\mathbb{Z}_n}$ is not contained in any single layer, this coloring is nontrivial in the sense of Definition 2.8 and Observation 2.9, whence $n > N$ follows.

According to Observation 2.9, we need to verify that

$$h_t G_{\alpha_{t-1}\alpha_t} \cap h_{t+1} G_{\alpha_t\alpha_{t+1}} = \emptyset.$$

Note that $G_{\alpha_{t-1}\alpha_t} = G_{\alpha_{t-1}} \cap G_{\alpha_t}$ is generated by $\alpha_{t-1} \cap \alpha_t = \{e \in E \colon \mathbf{a} \setminus a_t \subseteq \rho(e)\}$; similarly $G_{\alpha_t\alpha_{t+1}}$ is generated by $\{e \in E \colon \mathbf{a} \setminus a_{t+1} \subseteq \rho(e)\}$.

Suppose to the contrary that $h \in h_t G_{\alpha_{t-1}\alpha_t} \cap h_{t+1} G_{\alpha_t\alpha_{t+1}}$. We claim that then the whole clique $(\hat{a}_t)_{t \in \mathbb{Z}_n}$ would be represented in layer $\mathfrak{A} \times \{h\}$, contrary to our assumptions.

For $s \neq t$:
$\hat{a}_s = [a_s, h_t] = [a_s, h]$, because $h \in h_t G_{\alpha_{t-1}\alpha_t}$ so that $(h_t)^{-1} \circ h \in G_{\alpha_{t-1}\alpha_t} \subseteq G_{a_s}$.

For $s \neq t+1$:
$\hat{a}_s = [a_s, h_{t+1}] = [a_s, h]$, because $h \in h_{t+1} G_{\alpha_t\alpha_{t+1}}$, $(h_{t+1})^{-1} \circ h \in G_{\alpha_t\alpha_{t+1}} \subseteq G_{a_s}$.

For the conformality claim, finally, assume that $\mathfrak{A}$ is conformal and of width less than N. Then, any clique in $\hat{\mathfrak{A}}$ of size up to N is contained in a single layer and therefore guarded within that layer by conformality of $\mathfrak{A}$; but by conformality of $\mathfrak{A}$, the single layer cannot have any cliques of size $N > w$, so that $\hat{\mathfrak{A}}$ cannot have any cliques of size N or larger. $\square$

3.4. Local Covers

In this section, we want to use N-acyclic covers of width less than w to obtain N-acyclic hypergraphs that cover hypergraphs of width w at least locally (disregarding defects near the rim, far from the center). In Section 3.5, we shall see how such local covers can be stacked and glued (and defects mended in the process) so as to obtain full N-acyclic covers. Overall the construction of these covers will therefore be by induction with respect to hypergraph width.

It may also be instructive to consider possibly infinite full N-acyclic covers that are *locally finite* in the sense that all 1-neighborhoods in the cover are finite. For instance, a process of local unfolding of the tetrahedron hypergraph (cf. Figure 1) would result in some locally finite amalgam of infinitely many $3n$-cartwheel hypergraphs. Generally, the existence of locally finite N-acyclic covers of hypergraphs of width $w+1$ implies the existence of finite N-acyclic covers of width w hypergraphs—and thus also points to an inductive approach as just outlined.

OBSERVATION 3.10. *Let $\mathfrak{A}'$ be obtained from $\mathfrak{A} = (A, S)$ by adding one new vertex* 0 *that is also adjoined to every hyperedge $s \in S$. Let $\pi : \hat{\mathfrak{A}}' \to \mathfrak{A}'$ be a conformal N-acyclic cover and let $\pi(\hat{0}) = 0$. Then, the restriction of π to the* 1*-neighborhood of $\hat{0}$ induces a conformal N-acyclic cover of $\mathfrak{A}$. If $\hat{\mathfrak{A}}'$ is locally finite, then this induced cover of $\mathfrak{A}$ is finite.*

We write $N^\ell(a)$ for the Gaifman neighborhood of radius ℓ of a, consisting of nodes at distance up to ℓ in the Gaifman graph.

Definition 3.11. Let $L \in \mathbb{N}$. A homomorphism $\pi : \mathfrak{B} \to \mathfrak{A}$ between hypergraphs is called an *L-local cover* at $a \in \mathfrak{A}$ if for some $b \in \pi^{-1}(a)$, π satisfies the *back*-condition for bisimilar covers as far as extensions at hyperedges in $N^{L-1}(b)$ are concerned:

if $\hat{s} \subseteq N^{L-1}(b)$ and $s = \pi(\hat{s})$ and $s' \in S$ are such that $s \cap s' \neq \emptyset$,
then there is some $\hat{s}' \in \hat{S}$ such that $\pi(\hat{s}') = s'$ and $\pi(\hat{s} \cap \hat{s}') = s \cap s'$.

In this situation, we speak of an L-local cover $\pi : \mathfrak{B}, b \to \mathfrak{A}, a$.

Example 3.12. Consider a connected graph $\mathfrak{A} = (A, E)$ with a distinguished central vertex a. Let $\mathfrak{A}^*_a$ be the usual tree unfolding of $\mathfrak{A}$ with root a. Then, the depth L truncation $\mathfrak{B} := \mathfrak{A}^*_a \restriction N^L(a)$ provides an L-local cover $\pi : \mathfrak{B}, a \to \mathfrak{A}, a$ at a.

As previously discussed, the construction of local covers at higher width will rely on the availability of (full rather than local) conformal and N-acyclic covers of hypergraphs of smaller width. The basic step in the construction is reflected in the following simple observations. For technical reasons, we here assume, without loss of generality, that the set of hyperedges is closed under subsets.

Consider a node a in a hypergraph $\mathfrak{A} = (A, S)$. The *localization* of $\mathfrak{A}$ at a is the hypergraph $\mathfrak{A} \restriction N^1_*(a)$ induced by S on the subset $N^1_*(a) := N^1(a) \setminus \{a\}$. Its hyperedges are the intersections of hyperedges $s \in S$ with $N^1_*(a)$. Note that for conformal $\mathfrak{A}$, every $s \cap N^1_*(a)$ is contained in some hyperedge s' with $a \in s'$. For conformal $\mathfrak{A}$, therefore, the width of $\mathfrak{A} \restriction N^1_*(a)$ is strictly less than that of $\mathfrak{A}$.

Example 3.13. The localization of the width 3 cartwheel hypergraph from Example 1.1 (cf. Figure 1) at its pivot vertex is its perimeter cycle (width 2). An N-acyclic cover of an n-cycle is obtained as a k-fold cover for any $k > N/n$. If we extend this kn-cycle to a width 3 hypergraph again by putting a pivot vertex, we obtain an N-acyclic (and even fully conformal) hypergraph cover of the cartwheel hypergraph.

OBSERVATION 3.14. *Let $a \in \mathfrak{A} = (A, S)$ be conformal and consider a cover $\pi : \mathfrak{B}_0 \to \mathfrak{A} \restriction N^1_*(a)$ with $\mathfrak{B}_0 = (B_0, T_0)$. For a new element $b \notin B_0$, let $B := B_0 \cup \{b\}$ and extend π by $\pi(b) := a$. Then, the hypergraph $\mathfrak{B} := (B, T)$ with*

$$T := \{t \subseteq B : t \setminus \{b\} \in T_0, \pi(t) \in S\}$$

provides a cover of $\mathfrak{A} \restriction N^1(a)$ at a. Moreover,

(i) if $\mathfrak{B}_0$ is (N-)conformal, then so is $\mathfrak{B}$.
(ii) if $\mathfrak{B}_0$ is N-chordal, then so is $\mathfrak{B}$.

PROOF. For (i), consider cliques in $\mathfrak{B}$. If the clique is contained in B_0, (N-)conformality of $\mathfrak{B}_0$ settles this. A clique including $b \in B$ must be of the form $t \cup \{b\}$ for a clique $t \subseteq B_0$ which therefore is a hyperedge $t \in T_0$; but then $\pi(t) \cup \{a\}$ is a clique in $\mathfrak{A}$ and thus in $S \restriction N^1(a)$, and hence $t \cup \{b\}$ was turned into a hyperedge of $\mathfrak{B}$.

For (ii), similarly, the case of cycles with nodes just from B_0 is settled in $\mathfrak{B}_0$; and any cycle involving $b \in B$ of length greater than 3 is chordal as any node of $\mathfrak{B}$ is linked to b by a hyperedge. □

In order to enlarge the radius of local covers based on this idea, we first discuss a simple gluing mechanism that preserves acyclicity and conformality. In effect, the following lemma allows us to extend a given homomorphism $\pi_0 : \mathfrak{B}_0 \to \mathfrak{A}$, which provides an incomplete cover, to a full cover by means of gluing isomorphic copies of some given full cover $\rho : \mathfrak{C} \to \mathfrak{A}$ to mend defects of the cover π_0. The point is that the given incomplete cover $\mathfrak{B}_0$ is retained in the resulting cover and that no new chordless cycles or unguarded cliques are produced.

LEMMA 3.15. *Let $\pi_0 : \mathfrak{B}_0 \to \mathfrak{A}$ a homomorphism that bijectively maps hyperedges of $\mathfrak{B}_0$ onto hyperedges of $\mathfrak{A}$, and let $\rho : \mathfrak{C} \to \mathfrak{A}$ be a cover. Then there is a cover $\pi : \mathfrak{B} \to \mathfrak{A}$ extending π_0 in the sense that $\mathfrak{B} \supseteq \mathfrak{B}_0$ and $\pi_0 = \pi \restriction B_0$. Moreover:*

(i) if $\mathfrak{B}_0$ and $\mathfrak{C}$ are (N-)conformal, then so is $\mathfrak{B}$.
(ii) if $\mathfrak{B}_0$ and $\mathfrak{C}$ are N-chordal, then so is $\mathfrak{B}$.

PROOF. $\mathfrak{B}$ is obtained by gluing one new disjoint isomorphic copy of $\mathfrak{C}$ onto each individual hyperedge of $\mathfrak{B}_0$.

Consider a hyperedge t of $\mathfrak{B}_0$ with image $s = \pi_0(t)$ in $\mathfrak{A}$. Let $\rho^{(t)} : \mathfrak{C}^{(t)} \to \mathfrak{A}$ be a fresh isomorphic copy of the cover $\rho : \mathfrak{C} \to \mathfrak{A}$. In $\mathfrak{C}^{(t)}$ choose a hyperedge $t' \subseteq C^{(t)}$ above $\pi_0(t) = s$. Let $f^{(t)} : t' \to t$ be the bijection between $t' \subseteq C^{(t)}$ and $t \subseteq B_0$, which is induced by $\rho^{(t)}$ and π_0, that is, such that $\pi_0 \circ f^{(t)} = \rho^{(t)} \restriction t'$.

We let $\mathfrak{B}$ be the hypergraph obtained by gluing $\mathfrak{B}_0$ and all the disjoint $\mathfrak{C}^{(t)}$, where each $\mathfrak{C}^{(t)}$ is glued via the corresponding $f^{(t)}$ so as to identify just the chosen $t' \subseteq C^{(t)}$ and $t \subseteq B_0$.

It is clear that $\mathfrak{B}_0 \subseteq \mathfrak{B}$ and that $\pi : \mathfrak{B} \to \mathfrak{A}$ is a cover of the required kind. Moreover, (N-)conformality and N-chordality are preserved in this gluing:

(i) every clique in $\mathfrak{B}$ is fully contained in B_0 or in one of the $\mathfrak{C}^{(t)}$.
(ii) every chordless cycle is fully contained in B_0 or in one of the $\mathfrak{C}^{(t)}$.

For the second claim, consider a cycle linking nodes in $B_0 \setminus t$ to nodes in $C^{(t)} \setminus t'$; as the identification of t with t' is the only bridge between these two parts, the cycle would have to pass through this common patch at least twice; as this common part is a hyperedge of $\mathfrak{B}$, this induces a chord and the cycle cannot be chordless. □

LEMMA 3.16. *Suppose that N-acyclic, conformal covers are available for all width w hypergraphs. Then, there is, for every conformal hypergraph $\mathfrak{A}$ of width $w+1$, every element $a \in A$ and every $L \in \mathbb{N}$, an L-local cover $\pi : \mathfrak{B}, b \to \mathfrak{A}, a$ at a by an N-acyclic and conformal hypergraph $\mathfrak{B}$.*

PROOF. The construction of $\pi : \mathfrak{B}, b \to \mathfrak{A}, a$ is by induction on the radius L, starting from a cover of the localization $\mathfrak{A} \restriction N^1_*(a)$ and of $\mathfrak{A} \restriction N^1(a)$ (as in Observation 3.14). We successively extend incomplete 1-neighborhoods of points $b' \in N^{L-1}(b)$ to 1-neighborhoods that provide covers for $\mathfrak{A} \restriction N^1(\pi(b'))$. Let $\mathfrak{B}_0$ be the current, incomplete N-acyclic cover, b' on the boundary in the sense that $\mathfrak{B}_0 \restriction N^1(b')$ is not yet a full cover of $\mathfrak{A} \restriction N^1(\pi(b'))$.

The extension step is performed at the level of width w hypergraphs:

—we extend the partial cover of $\mathfrak{A} \restriction N^1_*(\pi(b'))$ provided by $\mathfrak{B}_0 \restriction N^1_*(b')$ to a full cover of $\mathfrak{A} \restriction N^1_*(\pi(b'))$ according to Lemma 3.15,
—we fill in b' according to the trick in Observation 3.14, to obtain a full cover $\mathfrak{B}^1$ of $\mathfrak{A} \restriction N^1(\pi(b'))$ that has $\mathfrak{B}_0 \restriction N^1(b')$ as a substructure, and
—we glue this cover $\mathfrak{B}^1$ to $\mathfrak{B}_0$ in $\mathfrak{B}_0 \restriction N^1(b')$ (this part is common to both hypergraphs, which are taken to be otherwise disjoint).

(N-)conformality and N-chordality are preserved in this gluing as well.

This is clear for conformality: any clique in the resulting structure must be contained in either of the two parts, as no new (hyper-)edges are introduced.

For N-chordality consider a chordless cycle in the resulting structure that is not fully contained in either of the two parts, $\mathfrak{B}_0$ or $\mathfrak{B}^1$. Since the cycle is not contained in $\mathfrak{B}^1$, it must have at least two nodes at distance greater than 1 in $\mathfrak{B}_0 \restriction N^1_*(b')$ that are linked by a segment of the cycle that is fully within $\mathfrak{B}_0 \setminus N^1(b')$. If $b_1, b_2 \in \mathfrak{B}_0 \restriction N^1_*(b')$ are such, then we may close this segment to form a new cycle by filling in b' between b_1 and b_2. This cycle would be chordless in $\mathfrak{B}_0$, because it cannot have intermediate nodes in $N^1(b')$, and can only be shorter than the given one; hence, the given one had length greater than N. □

Finite conformal hypergraph covers are available for arbitrary finite hypergraphs as shown in Hodkinson and Otto [2003], see Theorem 3.6. Since covers can naturally be composed we may, without loss of generality, assume that the hypergraph $\mathfrak{A}$ to be covered is itself conformal (as was assumed for the last lemma).

Availability of conformal N-acyclic covers for width 2 hypergraphs follows from Otto [2004], see Proposition 3.7. Width 2 hypergraphs are graphs $\mathfrak{A} = (A, E)$ and N-acyclic (or even N-locally acyclic) covers can be obtained as products $\mathfrak{A} \otimes G$ with Cayley groups G with generator set E of sufficiently large girth, as discussed in connection with Proposition 3.7. Cayley groups of large girth are obtained as $G = \mathrm{sym}(H)$ from E-graphs; in this case, from regularly E-colored trees of depth N, as indicated in Section 2.1.

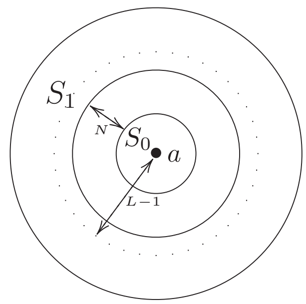

Fig. 6. Regions in local cover.

This settles the base case for the inductive application of the lemma to the construction of conformal N-acyclic covers of finite hypergraphs of any width.

Towards the induction step, we see in the following section how the local covers, whose existence is guaranteed by the last lemma, can be stacked and glued (and defects mended in the process) so as to obtain full N-acyclic covers.

3.5. From Local to Global Covers

Suppose $\mathfrak{A} = (A, S)$ and $a \in \mathfrak{A}$ and $S_0, S_1 \subseteq S$ are such that

$$\bigcup S_0 \subseteq A \setminus \bigcup S_1 \subseteq N^{L-1}(a)$$

and $d(\bigcup S_0, \bigcup S_1) > N$, cf. Figure 6.

Think of S_0 as the core region of some L-local cover of a given hypergraph that is such that every hyperedge of that original hypergraph is covered by some $s \in S_0$; the set S_1, on the other hand, comprises all those hyperedges in the periphery of this local cover, which may still be lacking responses to *back*-requirements. Missing hyperedge neighbours of peripheral hyperedges are to be supplied through gluing with hyperedges in the core region of new copies of $\mathfrak{A}$. For this we need a surplus of core hyperedges compared to the demands created by the peripheral hyperedges. It is to this end that stacking is used: to create many layers of copies of core hyperedges without unduly increasing the number of peripheral ones.

In the given situation, the gluing of isomorphic copies of $\mathfrak{A}$ is achieved with $\hat{\mathfrak{A}} = \mathfrak{A} \times_\rho G$, where $E = \{1, \dots, K\} \times S_1$ and $\rho\colon (i, s) \mapsto s \subseteq A$. As before, we let G be a group with generator set E, reflecting intersections and without nontrivial colored cycles of length up to N.

$\hat{\mathfrak{A}} = (\hat{A}, \hat{S})$ is a cover of $\mathfrak{A}$ with respect to the natural projection $\pi\colon \hat{\mathfrak{A}} \to \mathfrak{A}$, such that all the copies of $s \in S_0$ in the different layers of $\hat{\mathfrak{A}}$ are far from each other and far from the copies of elements $s \in S_1$. Moreover, the multiplicity ratio between center and boundary is improved at least by a factor of K. On one hand, $\hat{\mathfrak{A}}$ has $|G|$ many disjoint isomorphic copies of $\mathfrak{A} \restriction \bigcup S_0 \subseteq \mathfrak{A} \restriction (A \setminus \bigcup S_1)$, because these regions are far from any gluing sites. For $s \in S_1$, on the other hand, the number of distinct covers $[s, g]$ above s is at most $|G|/|G_s|$, where G_s is the subgroup generated by $\{(i, s)\colon 1 \leqslant i \leqslant K\}$ and therefore has at least K elements. This is because $[s, g] = [s, g']$ whenever $g^{-1} \circ g \in G_s$.

Choosing $K > |S_1|$, there is an injection κ from hyperedges $\hat{s}$ of $\hat{\mathfrak{A}}$ above S_1 into layers of $\hat{\mathfrak{A}}$: the number of such hyperedges $\hat{s}$ is bounded by $|S_1||G|/K < |G|$.

Let now $\pi_0\colon \mathfrak{A}, a \to \mathfrak{A}_0, \pi_0(a)$ be an L-local cover of $\mathfrak{A}_0$ at $a_0 = \pi_0(a)$ by some conformal and N-acyclic $\mathfrak{A}$; let $S_0, S_1 \subseteq S$ be as before and such that, for every $s \in S_1$, there is some $s' \in S_0$ such that $\pi_0(s) = \pi_0(s')$—we fix such a selection of s' for every $s \in S_1$. Let further $\pi\colon \hat{\mathfrak{A}} = \mathfrak{A} \times_\rho G \to \mathfrak{A}$ be constructed for $S_0, S_1 \subseteq S$ as discussed, with $K > |S_1|$

and an injection κ from $\pi^{-1}(S_1)$ into G. Clearly, $\hat{\pi}: \hat{\mathfrak{A}} \rightarrow \mathfrak{A}_0$, $\hat{\pi} := \pi_0 \circ \pi$, is an L-local cover by a conformal and N-acyclic hypergraph. We may then construct a full conformal and N-acyclic cover $\tilde{\pi}: \tilde{\mathfrak{A}} \rightarrow \mathfrak{A}_0$ as follows.

The hypergraph $\tilde{\mathfrak{A}}$ is obtained from $\hat{\mathfrak{A}}$ simply by identifying $\hat{s} \in \pi^{-1}(S_1)$ with $[s', \kappa(\hat{s})] \in \hat{S}_0$. As this identification is compatible with $\hat{\pi}$, we can choose $\tilde{\pi}$ to be the natural projection induced by $\hat{\pi}$. It is obvious that $\tilde{\pi}: \tilde{\mathfrak{A}} \rightarrow \mathfrak{A}_0$ is a full cover since all defects in $\pi_0: \mathfrak{A} \rightarrow \mathfrak{A}_0$ have been healed through the gluing of peripheral with central hyperedges. It is also not hard to see that the identifications between $\hat{s} \in \pi^{-1}(S_1)$ with $[s', \kappa(\hat{s})] \in \hat{S}_0$ do not violate conformality or N-chordality: any connected configuration of up to N vertices in $\tilde{\mathfrak{A}}$ is isomorphic to some configuration in a hypergraph $\tilde{\mathfrak{A}}'$ obtained by gluing disjoint isomorphic copies of $\mathfrak{A} \restriction N^{L-1}(a)$ in peripheral hyperedges of $\hat{\mathfrak{A}}$; as these constituents are conformal and N-chordal, so is $\tilde{\mathfrak{A}}$ (compare the arguments in the proof of Lemma 3.15).

We are ready to prove our main theorem on finite hypergraph covers, Theorem 3.8.

Proof of Theorem 3.8. Let $\mathfrak{A}_0$ be the given finite hypergraph to be covered. Without loss of generality, we may assume that $\mathfrak{A}_0$ is connected. Replacing $\mathfrak{A}_0$ by a finite conformal cover of $\mathfrak{A}_0$ according to Theorem 3.6 if necessary, we may further assume that $\mathfrak{A}_0$ is conformal. We may also assume inductively that finite conformal N-acyclic covers are available for every finite hypergraph of smaller width, so that Lemma 3.16 guarantees the existence of finite, conformal and N-acyclic L-local covers $\mathfrak{A}$ for $\mathfrak{A}_0$, for any desired value of L. Let $\pi_0: \mathfrak{A}, a \rightarrow \mathfrak{A}_0, a_0$ be such an L-local cover at $a_0 = \pi_0(a) \in \mathfrak{A}_0$. If L is large enough in relation to N and to the diameter of $\mathfrak{A}_0$, then some collection S_0 of hyperedges of $\mathfrak{A}$ contained in $N^{L-(N+2)}(a)$ provides at least one covering hyperedge for each hyperedge of $\mathfrak{A}_0$. At the same time, any hyperedge of $\mathfrak{A}$ that may have some defect with respect to the *back*-property must be disjoint from $N^{L-2}(a)$, since by definition the L-local cover π_0 satisfies all *back*-requirements at hyperedges that are fully contained in $N^{L-1}(a)$. The collection S_1 of all hyperedges in $\mathfrak{A}$ with defects is therefore contained in the complement of $N^{L-1}(a)$ and thus has distance greater than N from S_0 (cf. Figure 6).

The construction of a suitable $\tilde{\pi}: \tilde{\mathfrak{A}} \rightarrow \mathfrak{A}_0$ from $\pi: \mathfrak{A} \rightarrow \mathfrak{A}_0$ as outlined in this section, then provides a full conformal and N-acyclic cover for $\mathfrak{A}_0$ as desired. □

3.6. Richer Covers: Freeness

The N-acyclic covers obtained previously realize in finite covers degrees of acyclicity that in full can only be realized in infinite covers. Unqualified acyclicity is the key property of bisimilar tree unfoldings. Another property that can easily be achieved in infinite unfoldings is that of unbounded branching, a richness property. Just like ordinary bisimulations, hypergraph bisimulations (or guarded bisimulations between relational structures) cannot control multiplicities, whence the branching degree in covers can essentially be varied freely, cf. Remark 3.5. This section shows that also the feature of unbounded branching admits qualified approximations in finite hypergraph covers. In fact, the stacking of layers in millefeuilles of hypergraphs can be used for this purpose, too.

Definition 3.17. Let $\mathfrak{A} = (A, S)$ be a hypergraph.

(i) For $s \in S$, $B \subseteq A$ and $t \subseteq s \cap B$, let $d_t(s, B)$ be the usual distance between $s \setminus t$ and $B \setminus t$ in the induced subhypergraph $\mathfrak{A} \restriction (A \setminus t)$ (obtained by removing all vertices in t from the universe and from every hyperedge).
(ii) For $s \in S$ and $B \subseteq A$, we say that s and B are *n-free* if $d_t(s, B) > n$ for $t = s \cap B$.

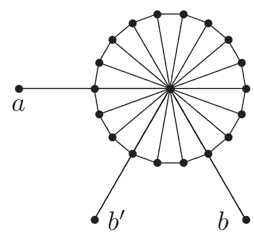

Fig. 7. Freeness.

(iii) A cover $\pi : \hat{\mathfrak{A}} \to \mathfrak{A}$ is called *(n, K)-free* if, for all $\hat{s} \in \hat{S}$ and $\hat{B} \subseteq \hat{A}$ of size $|\hat{B}| \leqslant K$ and $\hat{t} \subseteq \hat{s} \cap \hat{B}$, there is some $\hat{s}' \in \hat{S}$ such that $\pi(\hat{s}') = \pi(\hat{s})$, $\hat{s}' \cap \hat{B} = \hat{t}$ and $\hat{s}'$ and $\hat{B}$ are n-free.[5]

Consider the configuration of Figure 7, which could appear as a subgraph of the Gaifman graph of a width 3 hypergraph, for instance of a 17-acyclic cover of the tetrahedron. While the spokes pointing to a and to b' are not 3-free, those pointing to a and b are.

LEMMA 3.18. *Let $n, K \in \mathbb{N}$ and let $\mathfrak{A} = (A, S)$ be a hypergraph. For sufficiently large M and N, consider $E = \{0, \ldots, M\} \times \{t \subseteq s : s \in S\}$ with the association $\rho : (i, t) \mapsto t \subseteq A$ and let G be an N-acyclic group with generator set E. Then, the hypergraph $\hat{\mathfrak{A}} := \mathfrak{A} \times_\rho G$ is an (n, K)-free cover of $\mathfrak{A}$ with respect to the natural projection.*

PROOF. We use the terminology and notation from Section 3.3.

Let $\hat{s}$, $\hat{B}$, $\hat{t}$ be given as in (iii) of Definition 3.17. We seek to find a suitable $\hat{s}'$ that is n-free from $\hat{B}$.

Let $\hat{s} = [s, h]$, $\hat{t} = [t, h]$.

Let $\alpha := \{(i, t) : 0 \leqslant i \leqslant M\}$. Then, the natural candidates $\hat{s}' \sim \hat{s}$ are of the form $\hat{s}' = [s, h \circ k]$ for $k \in G_\alpha$. The key to the argument is that sufficiently large G_α (sufficiently large M) will allow us to avoid close links between $\hat{s}' \setminus \hat{t}$ and $\hat{B} \setminus \hat{t}$ for some choice of $k \in G_\alpha$.

First, we avoid intersections with $\hat{B} \setminus \hat{t}$. If $\hat{B} \setminus \hat{t}$ intersects $\hat{s}' = [s, k]$, then $\hat{b} = [b, g] = [a, h \circ k]$ for some $a \in s \setminus t$ implies that $b = a$ and $g^{-1} \circ h \circ k \in G_a$. Note that $a \not\in t$ implies that $G_a \cap G_\alpha = \{1\}$; therefore, restriction of the last equation to the α-component (cf. Lemma 2.4) yields $k = (h^{-1} \circ g) \restriction \alpha$. To avoid direct intersection with $\hat{B} \setminus \hat{t}$, therefore, k just needs to be distinct from at most K many elements of G_α (at most one for each $\hat{b} \in \hat{B} \setminus \hat{t}$).

Second, we want to avoid short chordless links outside $\hat{t}$ between $\hat{B} \setminus \hat{t}$ and $\hat{s}' \setminus \hat{t}$. For this, it will suffice to show that no two elements from distinct candidates $\hat{s}' = [s, h \circ k_1]$ and $\hat{s}'' = [s, h \circ k_2]$ can be linked by a chordless path of length up to $2n$ running outside $\hat{t}$. For then, each $\hat{b} \in \hat{B} \setminus \hat{t}$ can again only rule out one further element $k \in G_\alpha$ for which it could possibly have a short link outside $\hat{t}$ to some $[a, h \circ k]$: if one $\hat{b}$ had short paths of this kind to two distinct layers, then the concatenation of these two paths at $\hat{b}$ would yield a short connection outside $\hat{t}$ between two distinct layers $\hat{s}' = [s, h \circ k_1]$ and $\hat{s}'' = [s, h \circ k_2]$ (which after contraction along chords could be made chordless, and of length bounded by $2n$). In the following, we may also assume $2n < N$.

[5]There is also a natural intrinsic notion of freeness of a hypergraph, rather than of a hypergraph cover; we shall consider just the analogue for relational structures, see Definition 4.4.

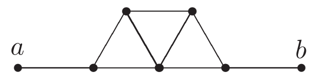

Fig. 8. Short chordless paths.

So assume towards a contradiction that for some $a', a'' \in s \setminus t$ and $k_1 \neq k_2 \in G_\alpha$ there were a short chordless path from $\hat{a}' = [a', h \circ k_1]$ to $\hat{a}'' = [a'', h \circ k_2]$ that does not meet $\hat{t}$ (here *short* means of length up to $2n$). Let this path be $\hat{a}' = \hat{a}_0, \ldots, \hat{a}_{m-1} = \hat{a}''$. Let $\hat{s}_i = [s_i, h_i]$ for $0 < i < m$ be the linking hyperedges along this path such that $\hat{a}_i \in \hat{s}_i \cap \hat{s}_{i+1}$, where these hyperedges and their representatives are chosen so as to minimize the number of jumps between layers h_i and $h_{i+1} \neq h_i$. We also put $h_0 := h \circ k_1$ and $h_m := h \circ k_2$, and let

$$g_i := h_i^{-1} \circ h_{i+1} \text{ for } i \in \mathbb{Z}_{m+1}.$$

We claim that, after elimination of factors $g_i = 1$, the cyclic tuple $(g_i)_{i \in \mathbb{Z}_{m+1}}$ gives rise to a nontrivial colored cycle in G based on the coloring $\sigma(i) := \alpha_{a_i} = \{e \in E \colon a_i \in \rho(e)\}$ for $0 \leqslant i < m$ and $\sigma(m) = \alpha$. With respect to the elimination of trivial factors and the verification of condition (iii) from Definition 2.8 for nontrivial colored cycles, we may reason in close analogy with the proof of Proposition 3.9.

Note that, since $a_0, a_{m-1} \notin t$, $\sigma(m) \cap \sigma(m \pm 1) = \emptyset$ so that $G_{\sigma(m)\sigma(m\pm 1)} = \{1\}$. It is important for the argument, though, that this disjointness extends beyond a number of factors u_i flanking $g_m = k_2^{-1} \circ k_1 \in G_\alpha$ that might happen to be trivial ($g_0 = \cdots = g_j = 1$ or $g_\ell = \cdots = g_{m-1} = 1$). Here, we critically use the condition that the given path does not meet $\hat{t}$. If, for example, a number of initial factors $g_0, \ldots, g_j$ are all equal to 1, then a corresponding initial segment of the path is represented in the same layer as $\hat{a}_0$ and, since it must not run into $\hat{t}$, stays outside t even in projection. Hence, $\hat{a}_j \notin \hat{t}$ implies $a_j \notin t$, which further implies $\sigma(j) \cap \sigma(m) = \emptyset$.

Hence, for sufficiently large N, such chordless paths cannot exist. And this implies that for sufficiently large M (e.g., $M > 2K$ certainly suffices) there must be $k \in G_\alpha$ such that $\hat{s}' := [s, h \circ k]$ is n-free from $\hat{B}$ as desired. □

Combining Lemma 3.18 with Proposition 3.9, we obtain the following strengthening of Theorem 3.8.

COROLLARY 3.19. *For all $N, K, n \in \mathbb{N}$, every finite hypergraph admits a bisimilar (n, K)-free cover by some finite conformal and N-acyclic hypergraph.*

3.7. Bounded Convexity in *N*-Acyclic Hypergraphs

This section is devoted to some basic structure theory of N-acyclic hypergraphs in general. We are particularly interested in small subconfigurations, which are acyclic by N-acyclicity. In order to understand how certain small acyclic subconfiguration are embedded into the ambient hypergraph we study a notion of closure, which is reminiscent of convex hulls. Instead of closure under all shortest connecting paths, we consider only connecting paths up to a certain length n (this is a restriction); but instead of just shortest connecting paths we admit connecting paths that are minimal in the sense of having no chords (this is a relaxation).

A *chordless path* in $\mathfrak{A}$ is a chordless path in the Gaifman graph of $\mathfrak{A}$. Shortest paths are chordless, but there may be (short) chordless that are not of minimal length: see Figure 8 for examples of short but not necessarily shortest chordless paths from a to b.

In the following, we sketch a corresponding analysis in N-acyclic hypergraphs where N can always be assumed to be greater than the width of the hypergraph. So the N-acyclic hypergraphs under consideration will not only be N-conformal but outright conformal, cf. Observation 3.2. In particular, we shall often refer to *sufficiently acyclic* hypergraphs or structures to appeal to some not necessarily explicitly specified bound N such that corresponding constructions go through for all (conformal and) N-acyclic hypergraphs. Uniformity of a suitable bound N in explicitly specified parameters is always understood.

Besides the parameter N specifying the global acyclicity requirements for all $\mathfrak{A}$ under consideration, we often deal with a locality parameter n to say which Gaifman distances and path lengths are currently considered as *short*. In typical game arguments, for instance, n will be shrinking from round to round, with a dependency like $n_i = 2n_{i+1}+1$ in round i of an m-round game. With a choice for n set, we often refer to *short* paths when we mean paths of length up to n.

An interesting feature of sufficiently acyclic $\mathfrak{A}$ is that the number of all nodes on shortest paths between two given nodes a and a' at distance $d(a, a') \leqslant n$ can be bounded in terms of the width w of $\mathfrak{A}$. More precisely, if N is sufficiently large in relation to n and w, then the set $D(a, a')$ of nodes on shortest paths between a and a' satisfies $|D(a, a')| \leqslant nw$. Similarly, even the number of nodes on any short chordless paths between two nodes at short distance can be bounded—and this is strengthened even further to yield a corresponding bounded closure operator.

To simplify notation, we freely switch between tuples and their sets of components; instead of finite subsets we sometimes work with tuples enumerating them, and apply set operations also to tuples, as in $B \setminus \mathbf{a}$ or $B \cup \mathbf{a}$, where the meaning is clear.

Definition 3.20.

(i) A subset $B \subseteq \mathfrak{A}$ is *n-closed* if any chordless path of length up to n between nodes $a, a' \in B$ is fully contained in B.
(ii) For $n \in \mathbb{N}$, the convex n-closure of a tuple $\mathbf{a}$ in $\mathfrak{A}$ is

$$\mathrm{cl}_n(\mathbf{a}) := \bigcap \{B \subseteq \mathfrak{A} \colon \mathbf{a} \subseteq B \ n\text{-closed}\}.$$

Consider the (induced subgraphs of) the Gaifman graph of some width 3 hypergraph in Figures 7 and 8. The entire subgraph in Figure 8 is part of $\mathrm{cl}_n(a, b)$ for $n \geqslant 4$; the 4-closure of $\{a, b\}$ in Figure 7 consists of just the vertices on the shortest connecting path, while the shorter of the two connecting perimeter arcs is part of the 4-closure of $\{a, b'\}$.

Example 3.21. For some simple generic examples, note that arbitrary cliques are n-closed, since elements linked by an edge cannot by connected by a chordless path (of any length). The 1-neighborhood of a single node, $N^1(b) \subseteq \mathfrak{A}$, is n-closed provided $\mathfrak{A}$ is at least $(n+1)$-chordal. In fact the beginning of the proof of Lemma 3.22 shows that the 1-neighborhood of any connected subset of $\mathfrak{A}$ is n-closed provided that $\mathfrak{A}$ is sufficiently chordal in relation to the diameter of this subset.

We turn to bounds on the size of n-closures. For $B \subseteq \mathfrak{A}$ and $a \in \mathfrak{A}$, let $D(a, B)$ be the set of precisely those nodes that are on shortest paths between a and B. It is not hard to see that the size of the set $D(a, B) \setminus B$ is bounded by the product of $d(a, B)$ and the width w of $\mathfrak{A}$, provided $\mathfrak{A}$ is sufficiently acyclic in relation to $d(a, B)$, w and the diameter of B, $\mathrm{diam}(B)$. In fact $3n$-chordality implies that the subset D_k of elements at distance k from B in D must be a clique for $1 \leqslant k \leqslant d(a, B)$ if $d(a, B), \mathrm{diam}(B) \leqslant n$; hence, by conformality, each D_k is contained in a hyperedge and its size bounded by w. It is considerably harder to show that also $\mathrm{cl}_n(B)$ is uniformly size bounded (in terms of $|B|$, w, n) in all sufficiently acyclic $\mathfrak{A}$. For this we establish (by induction on w) the existence of *some* size-bounded n-closed superset of B. We shall directly only need the following.

LEMMA 3.22. *For $n \in \mathbb{N}$, there is a function $f_n(w, k)$ such that, for all sufficiently acyclic $\mathfrak{A}$ of width w, every $\mathbf{a} \in A^k$ is contained in some n-closed subset $B(\mathbf{a})$ of size $\leqslant f_n(w, k)$. Hence $|\mathrm{cl}_n(\mathbf{a})| \leqslant f_n(w, k)$.*

PROOF. Let $B^{(0)} \supseteq \mathbf{a}$ be such that any two distinct connected components of $B^{(0)}$ have distance greater than $n+2$. Such $B^{(0)}$ can always be found of size $2k(n+1)$. This bound is based on the following. Starting with the set of components of the tuple $\mathbf{a}$, we keep joining any two distinct connected components of the current set that are at distance up to $n+2$ in $\mathfrak{A}$ by a connecting path of that length (iteratively and in any order); as the number of connected components decreases in the process, it is trivially bounded by k; as components are joined by the addition of at most $n+1$ new nodes, at most $(k-1)(n+1)$ nodes are added to the original k overall.

For $w = 2$ and in sufficiently acyclic *graphs* $\mathfrak{A}$, all short chordless paths are shortest paths. In this case, any set $B^{(0)}$ with these properties is already n-closed and hence contains $\mathrm{cl}_n(\mathbf{a})$. The base case for the construction of the desired set $B(\mathbf{a})$ with respect to induction on w is thus established.

Consider now $\mathfrak{A}$ of width $w > 2$ and $\mathbf{a} \in \mathfrak{A}$. We assume that $\mathfrak{A}$ is sufficiently acyclic to guarantee acyclicity of the chosen $B^{(0)}$ and some of its size-bounded extensions that arise in the construction.

We claim that the 1-neighborhood of $B^{(0)}$ is closed under short chordless paths. The size of this set cannot a priori be bounded but it will serve as an envelope for the desired B.

Towards the closure claim, suppose to the contrary that $c = c_0, \ldots, c_\ell = c'$, $\ell \leqslant n$, were a short chordless path in $\mathfrak{A}$ between $c, c' \in N^1(B^{(0)})$ with $c_1, \ldots, c_{\ell-1} \notin N^1(B^{(0)})$. Let $d(c, b), d(c', b') \leqslant 1$ for suitable $b, b' \in B^{(0)}$. By choice of $B^{(0)}$, b and b' are linked by a chordless path within $B^{(0)}$ (note that their distance is at most $n+2$). We obtain a cycle by joining the disjoint chordless paths between b and b' and between c and c' by the edges (b, c) and (b', c'). This cycle must be chordal. Any triangulation must join every node c_i by an edge to at least one node on the connecting path in $B^{(0)}$, whence $c_i \in N^1(B^{(0)})$.

To cut down from $N^1(B^{(0)})$ to the desired $B(\mathbf{a})$, we focus on the sets $\bigcap_{b \in \mathbf{b}} N^1(b)$ for cliques $\mathbf{b} \in B^{(0)}$. To ease notation, let us write $N^1_*(\mathbf{b})$ for $\bigcap_{b \in \mathbf{b}} N^1(b) \setminus \mathbf{b}$. Consider the induced hypergraph

$$\begin{aligned} \mathfrak{A}[\mathbf{b}] \; &:= \; \big(N^1_*(\mathbf{b}), S[\mathbf{b}]\big) \\ \text{where} \qquad & S[\mathbf{b}] = \{s \setminus \mathbf{b} \colon s \in S, \mathbf{b} \subseteq s\}. \end{aligned}$$

This *localization* of $\mathfrak{A}$ at $\mathbf{b}$ has width $w - |\mathbf{b}| < w$. Hence, the induction hypothesis applies. Note also that any nodes $a, a' \in N^1_*(\mathbf{b}) \subseteq A$ that are linked by an edge in $\mathfrak{A}$ are also linked by an edge in $\mathfrak{A}[\mathbf{b}]$: as $\mathbf{b}a$ and $\mathbf{b}a'$ are cliques, an edge between a and a' implies that $\mathbf{b}aa'$ is a clique, which gives rise to a hyperedge of $\mathfrak{A}[\mathbf{b}]$ that links a to a'.

As parameters in $\mathfrak{A}[\mathbf{b}]$, we collect all nodes in $N^1_*(\mathbf{b})$ from $B^{(0)}$ and all those in any $N^1(b)$ for any $b \in B^{(0)}$ such that $\mathbf{b}b$ is *not* a clique (these are the $b \in B^{(0)} \setminus (\mathbf{b} \cup N^1_*(\mathbf{b}))$):

$$\begin{aligned} C(\mathbf{b}) := \; & \big(B^{(0)} \cap N^1_*(\mathbf{b})\big) \\ & \cup \; \bigcup \big\{ N^1(b) \cap N^1_*(\mathbf{b}) \colon b \in B^{(0)} \setminus (\mathbf{b} \cup N^1_*(\mathbf{b})) \big\}. \end{aligned}$$

The size of this parameter set can be bounded uniformly in the size of $B^{(0)}$ and w: the contributions from $N^1_*(\mathbf{b}) \cap N^1(b)$ for the relevant b are contained in intersections $N^1(b_i) \cap N^1(b)$ for some $b_i \in \mathbf{b}$ with $d(b_i, b) = 2$; as $\mathfrak{A}$ is sufficiently acyclic and conformal, any such intersection is a clique and hence bounded by w.

We apply the induction hypothesis to the set $C(\mathbf{b})$ in $\mathfrak{A}[\mathbf{b}]$ to obtain a subset $B(\mathbf{b}) \subseteq \mathfrak{A}[\mathbf{b}]$ with $C(\mathbf{b}) \subseteq B(\mathbf{b})$ that is n-closed in $\mathfrak{A}[\mathbf{b}]$. We claim that

$$B := B(\mathbf{a}) := B^{(0)} \cup \bigcup \{B(\mathbf{b}) \colon \mathbf{b} \text{ a clique in } B^{(0)}\}$$

is n-closed in $\mathfrak{A}$. It is obvious that the size of this set can be bounded in terms of the size of $B^{(0)}$, the size of the $B(\mathbf{b})$ and w.

Let $b_0, \ldots, b_\ell$ ($2 \leqslant \ell \leqslant n$) be a short chordless path in $\mathfrak{A}$ between b_0 and $b_\ell \in B$. We already know that this path stays within $N^1(B^{(0)})$; it remains to show that it also stays within B. Suppose $b_j \notin B$ for some $0 < j < \ell$. As $b_j \in N^1(B^{(0)}) \setminus B^{(0)}$, there must be some $b \in B^{(0)}$ such that $b_j \in N^1(b) \setminus b$. Let $\mathbf{b}$ be a clique in $B^{(0)}$ that is maximal with the property that $b_j \in N^1_*(\mathbf{b})$.

Clearly, $b_j \notin N^1(b)$ for any $b \in B^{(0)}$ such that $\mathbf{b}b$ is not a clique, as otherwise $b_j \in C(\mathbf{b}) \subseteq B(\mathbf{b}) \subseteq B$. As the given path is chordless and $b_j \in N^1_*(\mathbf{b})$, the path cannot intersect $\mathbf{b}$.

Let $[b_i, \ldots, b_j, \ldots, b_m]$ be a maximal segment of the path that stays within $N^1_*(\mathbf{b})$.

We want to find a chordless path of the form $b, b_{i'}, \ldots, b_j, \ldots, b_{m'}, b'$ with $b, b' \in C(\mathbf{b})$. Here, the path segment $[b_{i'}, \ldots, b_j, \ldots, b_{m'}] \subseteq [b_i, \ldots, b_j, \ldots, b_m]$ is chordlessly extended by suitable $b, b' \in C(\mathbf{b})$; these may be found within $[b_i, \ldots, b_j, \ldots, b_m]$ if nodes to the left or right of b_j happen to be in $C(\mathbf{b})$, or else will be found as new nodes in $C(\mathbf{b})$. That $b, b' \in C(\mathbf{b})$ then implies that the whole path, and therefore b_j is contained in $B(\mathbf{b}) \subseteq B$.

If $[b_i, b_j)$ contains some node in some $N^1(b)$ such that $\mathbf{b}b$ is not a clique, then this node is in $C(\mathbf{b})$. If not, and if $b_i \neq b_0$, then b_i must also be linked to some $b \in B^{(0)} \setminus \mathbf{b}$, but such that $\mathbf{b}b$ is a clique, whence $b \in N^1_*(\mathbf{b})$ and hence in $C(\mathbf{b})$. Now (b, b_j) is not an edge because $\mathbf{b}$ was a maximal clique with $b_j \in N^1_*(\mathbf{b})$. Let $b_{i'}$ be the last node along the path segment $[b_i, \ldots, b_j)$ that has an edge to b. Then $b, b_{i'}, \ldots, b_m$ is chordless, because the first edge from b into the segment $(b_j, \ldots, b_m]$ would otherwise create a chordless cycle of length greater than 3.

The same reasoning on the other side of b_j yields either a node in $(b_j, b_m] \cap C(\mathbf{b})$ or a segment $[b_i, b_{m'})$ and an element $b' \in C(\mathbf{b})$ such that $b_i, \ldots, b_{m'}, b'$ is chordless.

In all these cases, the choice of $B(\mathbf{b})$ guarantees that some chordless segment containing b_j is contained in $B(\mathbf{b})$.

It remains to deal with the cases that $[b_0, b_j]$, or $[b_j, b_\ell]$, or both, are fully contained in $N^1_{\mathbf{b}}$ while the corresponding segment is disjoint from every $N^1(b)$ for which $\mathbf{b}b$ is not a clique. Consider b_0. If $b_0 \in B(\mathbf{b})$, we reason as before. If $b_0 \in B \setminus B(\mathbf{b})$, then $b_0 \in N^1(b)$ for some b for which $\mathbf{b}b$ is a clique, and again we may reason as before. $\square$

The following two lemmas will be useful towards understanding how the addition of new elements affects closures—a process of importance for the application to back & forth games. The first lemma treats the addition of a clique of new elements to an n-closed set; we shall eventually use this for the addition of a single new element, but state the slight generalization to be able to use it in an inductive proof of the second lemma.

LEMMA 3.23. *Let $\mathfrak{A}$ be sufficiently acyclic, $B \subseteq \mathfrak{A}$ n-closed, $n > 1$, $\mathbf{a}$ a clique with $1 \leqslant d(B, \mathbf{a}) \leqslant n$. Let $\hat{B} := \mathrm{cl}_n(B \cup \mathbf{a})$ and consider the region in which this extended closure attaches to B:*

$$D := B \cap N^1(\hat{B} \setminus B).$$

Then

(i) $\hat{B} \setminus B$ is connected.
(ii) D separates $\hat{B} \setminus B$ from $B \setminus D$, whence $\hat{B} = B \cup \mathrm{cl}_n(D \cup \mathbf{a})$.

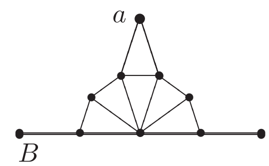

Fig. 9. Extending B to $\mathrm{cl}_n(B \cup a)$ for $n = 2$.

PROOF. For (i), it suffices to observe that the union of B with the connected component of $\mathbf{a}$ in $\hat{B} \setminus B$ is closed under chordless paths of length up to n—hence contains $\mathrm{cl}_n(B \cup \mathbf{a})$. Any short chordless path visiting another connected component and running between two nodes from outside that component would—as far as its passage through this component is concerned—be a chordless path between nodes of B, hence running within B.

For (ii), $d(\hat{B} \setminus B, B \setminus D) > 1$ is obvious from the definition of D: if $b' \in \hat{B} \setminus B$ is directly linked to some $b \in B$, then $b \in D$. □

We want to show that the contact region D is in fact a clique, provided that B is even $(2n + 1)$-closed. We first show that D is connected if B is $(2n + 1)$-closed.

Example 3.24. The example of a line graph of length $2n + 1$ with B containing just the end points and $\mathbf{a}$ being the central edge shows that $2n$-closure of B would not be sufficient for this claim. In the example of Figure 9, the bottom line of five nodes forms a subset B that is 4-closed in this acyclic hypergraph of width 3. The 2-closure of B together with a comprises the whole set of nodes; the contact region D consists of the three central nodes of B and is connected, but not a clique. The 5-closure of B, however, would itself comprise the half circle around the central node of B; its contact region with the 2-closure of this set together with a then consists of the single horizontal edge above B, which is a clique.[6]

For connectedness of D, consider any shortest path p from B to $\mathbf{a}$ with footpoint $b \in D \subseteq B$. We identify p with its trace $p \subseteq \mathfrak{A}$ and show that

(a) $\hat{B} \subseteq B \cup N^1(p)$;
(b) $b \in D \subseteq N^1(b)$.

For claim (a) observe that $N^1(p)$ itself is n-closed (compare the corresponding argument in the proof of Lemma 3.22 with $p \subseteq N^1(p)$ in the role of the initial set $B^{(0)}$); and that any chordless path linking some $b' \in B \setminus N^1(p)$ to an element in $N^1(p) \setminus B$ of length up to n would give rise to a chordless path of length up to $2n + 1$ from b' to b, which would have to stay in B as B is $(2n + 1)$-closed.

Claim (b) is a consequence of (a). Any $d \in D$ is directly linked to some $c \in \hat{B} \setminus B$, which by (a) must be linked to b or to one of the next two elements along the path p (p is a shortest path from B). For $d \neq b$, d and b cannot be linked by a chordless path of length up to 4 that leaves B, since B is $(2n + 1)$-closed. It follows that d must be directly linked to b, whence $d \in N^1(b)$ as claimed.

The proof of the claim that D is even a clique is by induction on the width w of $\mathfrak{A}$. For this induction, we may restrict attention to the situation of the previous lemma in cases where the clique $\mathbf{a}$ is either disjoint from or fully contained in $N^1(B)$. In either

[6] I wrongly stated $2n$-closure of B as a sufficient condition in the context of Lemma 3.25 in Otto [2010].

case, we analyze the connectivity of $(\hat{B} \setminus B) \cap N^1(B)$ with the rest of $\hat{B}$. The goal is to find a suitable separator of $\hat{B}$ that is a clique in $N^1(b)$ for some $b \in B$.

Let $C := (\hat{B} \setminus B) \cap N^1(B)$ be the contact region with B in $\hat{B} \setminus B$, and let C' be the connected component of $\mathbf{a}$ in $\hat{B} \setminus (B \cup C)$ (this set is empty if $\mathbf{a} \subseteq N^1(B)$). Let $C_0 \subseteq C$ consist of just $\mathbf{a} \subseteq C$ in the case that $\mathbf{a} \subseteq N^1(B)$; otherwise, the set

$$C_0 = C \cap N^1(C')$$

of those nodes in C with direct links into C'. Then

$(*)$ C_0 is a clique and separates the connected component of $\mathbf{a}$ in $\hat{B} \setminus (B \cup C)$ from the rest, whence the n-closure of $B \cup \mathbf{a}$ also decomposes according to $\hat{B} = \mathrm{cl}_n(B \cup C_0) \cup \mathrm{cl}_n(C_0 \cup \mathbf{a})$.

To see that C_0 is a clique, consider the nontrivial case in which C' is nonempty. Then, any two distinct elements of C_0 are linked by a path running outside $B \cup C$. If there were no direct edge between them, these nodes would therefore be joined by a nontrivial chordless path through $\hat{B} \setminus (B \cup C)$. As elements of C, they are also directly linked to D and hence, as D is connected, connected by a path running in $D \subseteq B$. Assuming that $\mathfrak{A}$ is sufficiently chordal, the resulting cycle would have to have a chord from a node in B to some node in $\hat{B} \setminus (B \cup C)$, contradicting the definition of C. The separation claim follows from the observation that C separates B from $\hat{B} \setminus (B \cup C)$ and that edges from $C' \subseteq \hat{B} \setminus (B \cup C)$ cannot go to B, and also not to $C \setminus C'$ by the definition of C'.

With this preparation, we are ready to prove the main claim.

LEMMA 3.25. *For $B \subseteq \mathfrak{A}$, $\mathbf{a}$, $\hat{B} = \mathrm{cl}_n(B \cup \mathbf{a})$ as previously mentioned, with $\mathbf{a} \subseteq N^1(B)$ or disjoint from $N^1(B)$: if $B \subseteq \mathfrak{A}$ is even $(2n+1)$-closed, then $D = B \cap N^1(\hat{B} \setminus B)$ is a clique.*

PROOF. The proof is by induction on the width w of $\mathfrak{A}$. For $w = 2$ and a sufficiently acyclic graph $\mathfrak{A}$ (which also means that $\mathbf{a}$ can at most have two components), the claim is easily verified. This is the base case for the induction; the induction step uses a localization at the footpoint of some shortest connecting path from B to $\mathbf{a}$ in order to reduce the width.

We show that, for C and C_0 as previously defined, there is a choice of a footpoint b of a shortest connecting path p from B to $\mathbf{a}$ for which $C_0 \subseteq N^1(b)$. In the nontrivial case that $\mathbf{a} \cap N^1(B) = \emptyset$, this can be inferred from the analysis of the family of subsets $(C_0 \cap N^1(d))_{d \in D}$. First, every $c \in C_0 \subseteq C$ is contained in at least one of these sets by the definition of C and D; second, no two of these sets can be incomparable by inclusion, as this situation would imply the existence of a chordless path of length 4 between the two footpoints in D (impossible since B is 4-closed) or of a chordless 4-cycle in $\mathfrak{A}$. But then any inclusion-maximal element in this family stems from a footpoint d that is directly linked to every $c \in C_0$ and therefore can serve as a footpoint b of a shortest path as desired.

Together with $(*)$, this latter condition can be used to analyze the situation in restriction to the localization $\mathfrak{A} \restriction N^1_*(b)$, which is of smaller width. Since $N^1(b)$ is itself n-closed (cf. Example 3.21), $\mathrm{cl}_n(B \cup C_0) = \mathrm{cl}_n(D \cup C_0) = \mathrm{cl}_n((B \cup C_0) \cap N^1(b)) \subseteq N^1(b)$. The passage to the localization $\mathfrak{A} \restriction N^1_*(b)$ is compatible with n-closures, and it suffices to show that $D \setminus b$ is a clique in the n-closure of $(B \cup C_0) \cap N^1_*(b)$ in $\mathfrak{A} \restriction N^1_*(b)$, which follows from the induction hypothesis, that is, from the claim of the lemma for hypergraphs of width $w - 1$. $\square$

That the contact region D is a clique means that the extension of closures necessitated by the inclusion of an additional node can be made compatible with the tree structure of tree-decompositions: in the situation of the lemma, any tree decompositions of B and of $\hat{B} \setminus B$ can be joined in a node representing the clique D. Indeed, in light of Lemma 3.23(ii), any node representing the clique D in a tree decomposition of B can serve as a port to glue a tree decomposition of $(\hat{B} \setminus B) \cup D$ rooted at D; the resulting merged tree represents all induced hyperedges on $\hat{B}$ since none can bridge D.

4. TWO APPLICATIONS TO THE GUARDED FRAGMENT

We deal with the guarded fragment GF $\subseteq$ FO as introduced in Andréka et al. [1998]. For motivation, we assume some familiarity with its role as a versatile analogue of modal logic in the much richer setting of arbitrary relational structures; for background and key results in its model theory we refer in particular to Grädel [1999]. Some basic notions are reviewed in the following.

4.1. Guardedness

Definition 4.1. Let $\mathfrak{A} = (A, (R^{\mathfrak{A}})_{R\in\tau})$ be a relational structure, τ finite and relational. The *hypergraph of guarded subsets* associated with $\mathfrak{A}$ is the hypergraph $(A, S[\mathfrak{A}])$ whose hyperedges are precisely all singleton sets together with all subsets of the sets $\{a : a \in \mathbf{a}\}$ for $\mathbf{a} \in R^{\mathfrak{A}}$, $R \in \tau$.

A subset $s \subseteq A$ is *guarded* if $s \in S[\mathfrak{A}]$; a *guarded tuple* in $\mathfrak{A}$ is a tuple whose set of components is guarded.

Clearly, the width of this induced hypergraph is bounded by the width of the signature τ (the maximal arity in τ).

The Gaifman graph of the relational structure $\mathfrak{A}$ also is the Gaifman graph associated with the hypergraph $(A, S[\mathfrak{A}])$ of guarded sets of $\mathfrak{A}$. We note that closure under subsets and inclusion of all singleton sets, which are built into the definition, have no effect on the associated Gaifman graph or on acyclicity.

Via $(A, S[\mathfrak{A}])$, hypergraph theoretic notions like conformality and N-acyclicity transfer naturally to relational structures $\mathfrak{A}$, and notions like n-neighborhoods or n-closures can be applied interchangeably to the Gaifman graph of $\mathfrak{A}$ or of $(A, S[\mathfrak{A}])$.

The key feature of the guarded fragment GF is its relativized quantification pattern, which only allows quantification over guarded tuples. Instead of general FO quantification, GF admits *guarded quantification* of the form

$$\exists \mathbf{y}.\alpha(\mathbf{x})\varphi(\mathbf{x}) \quad \text{and, dually} \quad \forall \mathbf{y}.\alpha(\mathbf{x})\varphi(\mathbf{x})$$

where $\mathbf{y} \subseteq \mathbf{x}$ is a tuple of variables among those that occur in the *guard* $\alpha(\mathbf{x})$, which is an atom in which all the free variables of φ *must* occur.[7] Here, we use the shorthand $\exists \mathbf{y}.\alpha\varphi$ for $\exists \mathbf{y}(\alpha \wedge \varphi)$ and $\forall \mathbf{y}.\alpha\varphi$ for $\forall \mathbf{y}(\alpha \rightarrow \varphi)$.

The natural game-based back & forth equivalence that captures the restricted nature of guarded quantification is *guarded bisimulation equivalence*. We briefly review the underlying back & forth game, which is played over two τ-structures $\mathfrak{A}$ and $\mathfrak{A}'$. The positions in the game are correspondences $(\mathbf{a}, \mathbf{a}')$ between locally isomorphic guarded tuples in $\mathfrak{A}$ and $\mathfrak{A}'$. We may think of pebbles marking the components of a guarded tuple $\mathbf{a}$ in $\mathfrak{A}$ and a guarded tuple $\mathbf{a}'$ in $\mathfrak{A}'$ in such a manner that the correspondence between the pebbles induces an isomorphism between the induced substructures $\mathfrak{A}{\restriction}\mathbf{a}$ and $\mathfrak{A}'{\restriction}\mathbf{a}'$.

The typical round in the game consists of a challenge-response exchange between the two players. The first player picks up some of the pebbles from one of the marked tuples and relocates them (possibly together with currently unused pebbles) freely apart from the constraint that the resulting tuple must again be guarded; the second player has

[7] Semantically vacuous equality atoms $x = x$ may serve as guards for singletons.

to respond likewise in the opposite structure by relocating the corresponding pebbles there. As usual, the second player loses if no such response is available, and the notions of a winning strategy in the ℓ-round game and in the unbounded game are defined as usual. In these terms, *guarded bisimulation equivalence* $\mathfrak{A}, \mathbf{a} \sim_{\mathsf{g}} \mathfrak{A}', \mathbf{a}'$ is defined by the condition that the second player has a winning strategy in the unbounded game starting from position $(\mathbf{a}, \mathbf{a}')$. A matching notion of $\mathfrak{A}, s \sim_{\mathsf{g}} \mathfrak{A}', s'$ for guarded subsets $s \in S[\mathfrak{A}]$ and $s' \in S[\mathfrak{A}']$ is defined in terms of $\mathfrak{A}, \mathbf{a} \sim_{\mathsf{g}} \mathfrak{A}', \mathbf{a}'$ for suitable guarded tuples $\mathbf{a}, \mathbf{a}'$ that enumerate s and s'.

Finite approximations $\sim_{\mathsf{g}}^{\ell}$ are similarly induced by ℓ-round games, for $\ell \in \mathbb{N}$.

The guarded variant of the classical Ehrenfeucht–Fraïssé correspondence then associates $\sim_{\mathsf{g}}^{\ell}$, that is, the existence of a winning strategy for the second player in the ℓ-round guarded bisimulation game, with GF-equivalence up to nesting depth ℓ. See, for example, Grädel [1999] and Otto [2011] for expositions. The analysis shows, in particular, that any $\varphi \in \mathrm{GF}$ is preserved under $\sim_{\mathsf{g}}$; and, more specifically, that any $\varphi \in \mathrm{GF}$ of nesting depth ℓ is preserved under $\sim_{\mathsf{g}}^{\ell}$. It also shows that for finite relational signatures τ and any class of τ-structures $\mathcal{C}$, $\mathcal{C}_0 \subseteq \mathcal{C}$ is definable by a GF-sentence of nesting depth ℓ within $\mathcal{C}$ if, and only if, $\mathcal{C}_0$ is closed under $\sim_{\mathsf{g}}^{\ell}$ within $\mathcal{C}$.

The following characterization of GF as the guarded bisimulation invariant fragment of FO is due to Andréka et al. [1998].

THEOREM 4.2. *In the sense of classical model theory,* $\mathrm{GF} \subseteq \mathrm{FO}$ *is the* $\sim_{\mathsf{g}}$*-invariant fragment of* FO, $\mathrm{GF} \equiv \mathrm{FO}/{\sim_{\mathsf{g}}}$. *that is, the following are equivalent for any sentence* $\varphi \in \mathrm{FO}(\tau)$*:*

(i) φ *is preserved under guarded bisimulation equivalence (*$\sim_{\mathsf{g}}$*-invariant):*
$\mathfrak{A} \sim_{\mathsf{g}} \mathfrak{A}' \;\Rightarrow\; \big(\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{A}' \models \varphi\big)$.
(ii) φ *is logically equivalent to some* $\varphi' \in \mathrm{GF}(\tau)$*.*

We stress that this is a statement of classical model theory, whose classical proof involves the use of compactness and a detour through infinite structures, so that $\sim_{\mathsf{g}}$-invariance over just finite structures would not be good enough to support the argument. For trivial examples of first-order formulae that are $\sim_{\mathsf{g}}$-invariant over all finite structures but not over all infinite structures, it suffices to look at conjunctions of sentences that are not $\sim_{\mathsf{g}}$-invariant (over some infinite structures) with suitable sentences that have only infinite models. In Section 4.3, we shall prove the analogue of Theorem 4.2 for finite model theory.

We may think of guarded bisimulations between τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ as hypergraph bisimulations between the hypergraphs $(A, S[\mathfrak{A}])$ and $(B, S[\mathfrak{B}])$, in which we additionally require the local bijections between hyperedges (guarded sets) to be local isomorphisms of the relational structures. Correspondingly, we define guarded covers $\pi : \hat{\mathfrak{A}} \to \mathfrak{A}$ as follows.

Definition 4.3. A *guarded cover* is a homomorphism $\pi : \hat{\mathfrak{A}} \to \mathfrak{A}$ between τ-structures that induces a hypergraph cover between the hypergraphs of guarded subsets $(\hat{A}, S[\hat{\mathfrak{A}}])$ and $(A, S[\mathfrak{A}])$. In other words, π is a surjective relational homomorphism such that

(i) its restrictions to guarded subsets of $\hat{\mathfrak{A}}$ are partial isomorphisms;
(ii) the *back*-property with respect to guarded subsets is satisfied:
for every guarded subset $s \subseteq \mathfrak{A}$, there is some guarded subset $\hat{s} \subseteq \hat{\mathfrak{A}}$ such that $\pi(\hat{s}) = s$, and for every guarded $\hat{s} \subseteq \hat{\mathfrak{A}}$ such that $\pi(\hat{s}) = s$ and for every guarded $s' \subseteq \mathfrak{A}$, there is some guarded $\hat{s}' \subseteq \hat{\mathfrak{A}}$ for which $\pi(\hat{s}') = s'$ and $\pi(\hat{s} \cap \hat{s}') = s \cap s'$.

Importantly, any hypergraph cover $\pi : (\hat{A}, \hat{S}) \to (A, S[\mathfrak{A}])$ induces a unique relational structure $\hat{\mathfrak{A}}$ on universe $\hat{A}$ that turns π into a guarded cover. For this, we observe that the hypergraph homomorphism $\pi : (\hat{A}, \hat{S}) \to (A, S[\mathfrak{A}])$ can be used to pull back the relational interpretation from $\mathfrak{A} \restriction s$ to every $\hat{s} \in \hat{S}$ with $\pi(\hat{s}) = s$ in a unique and well-defined manner. This process turns $(\hat{A}, \hat{S})$ into the hypergraph of guarded subsets of the relational structure $\hat{\mathfrak{A}}$ thus obtained.

Moreover, guarded covers that are derived from (n, K)-free hypergraph covers satisfy the following freeness condition. Compared to the freeness condition for covers of Definition 3.17, this new one has the advantage of applying to the structures themselves.

Definition 4.4. A relational structure $\mathfrak{A}$ with associated hypergraph $(A, S[\mathfrak{A}])$ of guarded sets is called (n, K)*-free* if, for all guarded $s \subseteq \mathfrak{A}$ and arbitrary subsets $B \subseteq A$ of size $|B| \leqslant K$ and all $t \subseteq s \cap B$, there is some guarded s' in $\mathfrak{A}$ with $\mathfrak{A}, s' \sim_{\mathsf{g}} \mathfrak{A}, s$ and $s' \cap B = t$ such that s' and B are n-free.

We therefore obtain the following as a direct corollary to Theorem 3.8 and Corollary 3.19.

COROLLARY 4.5. *For every $N \in \mathbb{N}$, every finite relational structure admits a guarded bisimilar cover by some finite conformal and N-acyclic structure.*

Moreover, such covers can additionally be chosen (n, K)-free, for any choice of the parameters $n, K \in \mathbb{N}$.

In particular, the class of finite N-acyclic τ-structures is fully representative of the class of all finite τ-structures up to guarded bisimulation equivalence. The finite model theory of GF thus reduces to the model theory of GF over finite N-acyclic structures—just as the (classical) model theory of ML reduces to the model theory of tree structures, or its finite model theory to the model theory of locally acyclic transition systems.

4.2. A Strong Finite Model Property for GF

The following generalizes the finite model property of GF [Grädel 1999] and its strengthening in Bárány et al. [2010].

We note that any satisfiable $\varphi \in$ GF has an acyclic model obtained as a guarded bisimilar unfolding of an arbitrary model; but even if the given model is finite, its unfolding typically is not. Indeed, φ may have only infinite acyclic models: a simple example is given by the sentence saying that R is irreflexive and antisymmetric and that every vertex has an outgoing R-edge.

COROLLARY 4.6. GF *has the finite model property in restriction to every class of relational structures that is defined in terms of finitely many forbidden cyclic configurations.*

PROOF. By the finite model property for GF, (see Grädel [1999]), any satisfiable $\varphi \in$ GF also has a finite model $\mathfrak{A} \models \varphi$. Replacing $\mathfrak{A}$ by a finite bisimilar cover $\hat{\mathfrak{A}} \sim_{\mathsf{g}} \mathfrak{A}$ that is N-acyclic, we obtain a finite model that does not have any cyclic configurations of size up to N. In fact, we may also make $\hat{\mathfrak{A}}$ conformal rather than just N-conformal. □

That this cannot be strengthened to arbitrary choices of finitely many forbidden configurations follows for instance from the undecidability of GF with functionality constraints [Grädel 1999]. The reduction would enforce functionality of the irreflexive binary relation R (i.e., $\forall x \forall y \forall y'((Rxy \wedge Rxy') \to y = y')$, which is not expressible in GF) by ruling out induced substructures whose R-reduct is isomorphic to the 3-vertex R-structure $Y_R := (\{x, y, y'\}, \{(x, y), (x, y')\})$. Suppose GF had the finite model property in restriction to the class of Y_R-free relational structures. Then, GF with a functionality constraint on an irreflexive relation R would also have the finite model property. As a

$$\begin{tikzcd} \mathfrak{A}_0 \arrow[r, dash, "\sim_{\mathrm{g}}^{\ell}"] \arrow[d, "\sim_{\mathrm{g}}"'] & \mathfrak{A}'_0 \arrow[d, "\sim_{\mathrm{g}}"] & \text{from arbitrary finite} \arrow[d, Rightarrow, "\text{suitable covers}"] \\ \mathfrak{A} \arrow[r, dash, "\equiv_q", "\varphi"'] & \mathfrak{A}' & \text{to sufficiently acyclic and free} \end{tikzcd}$$

Fig. 10. Upgrading in expressive completeness argument.

fragment of FO with the finite model property it would be decidable, contradicting the undecidability result in Grädel [1999].

4.3. The FMT Characterization Theorem

We prove the following finite model theory version of the classical characterization of GF as the $\sim_{\mathrm{g}}$-invariant fragment of first-order logic by Andréka–van Benthem–Németi [1998], cf. Theorem 4.2.

THEOREM 4.7. GF *precisely captures the guarded bisimulation invariant fragment of* FO *also in restriction to just finite relational structures:* $\mathrm{FO}/{\sim_{\mathrm{g}}} \equiv_{\mathrm{fin}} \mathrm{GF}$. *That is, the following are equivalent for any sentence* $\varphi \in \mathrm{FO}(\tau)$*:*

(i) φ *is preserved under guarded bisimulation between finite* τ*-structures.*
(ii) φ *is logically equivalent over finite* τ*-structures to some* $\varphi' \in \mathrm{GF}(\tau)$*.*

The proof of the crucial expressive completeness assertion over finite structures uses sufficiently acyclic and sufficiently free finite structures to show that any first-order φ that is $\sim_{\mathrm{g}}$-invariant over finite structures is in fact $\sim_{\mathrm{g}}^{\ell}$-invariant for a suitable finite level ℓ of guarded bisimulation equivalence. By the guarded variant of the Ehrenfeucht–Fraïssé theorem, this implies that φ is logically equivalent over all finite structures to some GF formula of nesting depth ℓ.

The argument for $\sim_{\mathrm{g}}^{\ell}$-invariance of φ involves an upgrading of $\mathfrak{A} \sim_{\mathrm{g}}^{\ell} \mathfrak{A}'$ for suitable ℓ to $\mathfrak{A} \equiv_q \mathfrak{A}'$, where q is the quantifier rank of φ; cf. Figure 10. The latter equivalence manifests itself as q-isomorphism, that is, back & forth equivalence in the classical q-round first-order Ehrenfeucht–Fraïssé game. It is obvious that $\sim_{\mathrm{g}}^{\ell}$-equivalence will imply $\equiv_q$-equivalence only for very special structures $\mathfrak{A}$ and $\mathfrak{A}'$. The crux of the upgrading argument therefore is the isolation of a class of (finite) structures

(a) which is fully representative up to $\sim_{\mathrm{g}}$ of all (finite) τ-structures, and
(b) over which $\mathfrak{A} \sim_{\mathrm{g}}^{\ell} \mathfrak{A}'$ (for suitable ℓ) indeed implies $\mathfrak{A} \equiv_q \mathfrak{A}'$.

N-acyclic, conformal structures that also satisfy the richness condition of (n, K)-freeness (for suitable parameters N, n, K) can serve this purpose.

Then any given finite pair of structures $\mathfrak{A}_0 \sim_{\mathrm{g}}^{\ell} \mathfrak{A}'_0$ may be replaced by $\sim_{\mathrm{g}}$-equivalent companion structures $\mathfrak{A} \sim_{\mathrm{g}} \mathfrak{A}_0$ and $\mathfrak{A}' \sim_{\mathrm{g}} \mathfrak{A}'_0$ from that class, for which therefore $\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{A}' \models \varphi$. It follows that $\mathfrak{A}_0 \models \varphi \Leftrightarrow \mathfrak{A}'_0 \models \varphi$ so that $\sim_{\mathrm{g}}^{\ell}$-invariance of φ is proved. Sections 4.3.1 and 4.3.2 establish the suitability of sufficiently acyclic and sufficiently free structures for this upgrading according to the diagram in Figure 10.

4.3.1. Free Realizations of Small Convex Configurations. Towards the upgrading argument indicated in Figure 10, we want to establish the class of finite conformal, N-acyclic and (n, K)-free τ-structures as a class of structures for which $\mathfrak{A} \sim_{\mathrm{g}}^{\ell} \mathfrak{A}'$ implies $\mathfrak{A} \equiv_q \mathfrak{A}'$.

(Suitable values of N, K, n and ℓ need to be determined in relation to q and the width of τ.)

N-acyclicity, for sufficiently large N, is one useful requirement, because short chordless cycles or small unguarded cliques are FO-definable; richness in the sense of (n, K)-freeness for sufficiently large n and K is useful, for instance, because small branching degree for certain extensions is FO-definable, too. As none of these features is controlled by the GF-type of a configuration—and yet GF-types need to control FO-types in the upgrading—the right class of structures must avoid these obstructions outright.

It may be useful again to compare the graph case. Locally acyclic covers can there be used to guarantee that configurations in the q-round classical Ehrenfeucht–Fraïssé game can locally be analyzed in terms of tree structures that span the components of the pebbled configurations. Vertex colors in the vicinity of any tree node are controlled by the bisimulation type of that node, but multiplicities are not. The easiest remedy in that case is to boost all multiplicities by a factor of q, so that remaining differences in multiplicities between $\mathfrak{A}$ and $\mathfrak{A}'$ are compatible with $\equiv_q$.

For GF over relational structures, we may similarly use sufficiently acyclic structures in order to ensure that some n-closures of pebbled configurations break up into small local components, which are acyclic and therefore tree-decomposable. In an extension move that goes close to one of these components, the tree-decomposition needs to be extended to encompass an extended closure that includes the new element. The analysis of n-closures in sufficiently acyclic hypergraphs in Section 3.7 indeed supports this kind of extension argument.

The richness condition of (n, K)-freeness from Section 3.6, on the other hand, will serve to guarantee the existence of a corresponding extension of the matching configuration in the opposite structure. To this end, we need to link GF-types in sufficiently acyclic and free structures to the existence of suitably embedded small, and therefore tree-decomposable, configurations.

In more precise terms, Lemma 4.8 guarantees that in such structures, the guarded bisimulation type determines the extension properties of small acyclic configurations as desired.

Let $B \subseteq \mathfrak{A}$ be connected and n-closed, that is, such that $\mathrm{cl}_n(B) = B$, in a sufficiently free and acyclic structure $\mathfrak{A}$, where in particular $|B|$ is small enough to guarantee acyclicity of $\mathfrak{A}\restriction B$. Then, $\mathfrak{A}\restriction B$ admits a tree decomposition by guarded subsets, $\mathcal{T} = (T, \delta)$ where $\delta\colon v \mapsto \delta(v) \in S[\mathfrak{A}]$. Let the guarded tuple $\mathbf{b} \in B$ be represented in the designated root λ of T, $\delta(\lambda) = \mathbf{b}$.

With $\mathcal{T}$, we associate a GF-formula $\varphi_{\mathcal{T}}(\mathbf{x}) := \varphi_{\mathcal{T},\lambda}$ describing the existential GF-type of $\big(\mathfrak{A}\restriction B, \mathbf{b}\big)$. Formulae $\varphi_{\mathcal{T},v}$ are defined by induction with respect to the depth of $v \in T$. For leaves v, $\varphi_{\mathcal{T},v}$ is a quantifier-free description of the atomic type of $\delta(v)$ in $\mathfrak{A}$. From formulae $\varphi_{\mathcal{T},v_i}(\mathbf{x}^{(i)})$ for the children v_i of $v \in T$ we obtain $\varphi_{\mathcal{T},v}$ in the obvious manner as a formula of the form

$$\chi(\mathbf{x}) \wedge \bigwedge_i \exists \mathbf{x}^{(i)}.\alpha^{(i)}\varphi_{\mathcal{T},v_i}\big(\mathbf{x}^{(i)}\big)$$

with guards $\alpha^{(i)}$ abstracted from v_i and a description $\chi(\mathbf{x})$ of the atomic type of $\delta(v)$ in $\mathfrak{A}$. Note that the variable tuples $\mathbf{x}^{(i)}$ for the elements of the child nodes v_i need to be chosen consistently and in agreement with the new variable tuple $\mathbf{x}$ for the elements at v so as to impose the required identifications for the overlaps.

LEMMA 4.8. *For $\mathfrak{A}$, $\mathcal{T}$, $\mathbf{b}$, $\varphi_{\mathcal{T}}(\mathbf{x})$ as just discussed: if $\mathfrak{A}'$ is sufficiently acyclic and free, then $\mathfrak{A}', \mathbf{b}' \models \varphi_{\mathcal{T}}(\mathbf{x})$ implies that there is an n-closed subset $B' \subseteq \mathfrak{A}'$ such that*

$$\mathfrak{A}\restriction B, \mathbf{b} \simeq \mathfrak{A}'\restriction B', \mathbf{b}'.$$

PROOF. We find the desired B' in stages corresponding to an induction with respect to the height in the underlying tree decomposition $\mathcal{T} = (T, \delta)$. Along with B', we produce a tree decomposition $\mathcal{T}' = (T, \delta')$ isomorphic to the decomposition $\mathcal{T}$ of B and make sure that $\mathfrak{A}', \delta'(v) \models \varphi_{\mathcal{T},v}$ for all $v \in \mathcal{T}$. Enumerate the nodes of T in breadth-first fashion as $(v_i)_{i \leqslant M}$, starting with the root $v_0 = \lambda$. Let $B_i := \bigcup\{\delta(v_j) \colon j \leqslant i\}$ so that $B_0 = \mathbf{b}$. We obtain B' as the union of sets B'_i for which $\mathfrak{A}' \restriction B'_i, \mathbf{b}' \simeq \mathfrak{A} \restriction B_i, \mathbf{b}$, starting with $B'_0 := \mathbf{b}'$.

The extension from B'_{i-1} to B'_i corresponds to a choice of $\delta'(v)$ in $(\mathfrak{A}', B'_i)$ matching $\delta(v)$ in $(\mathfrak{A}, B_i)$, for $v = v_i$. Let $u = v_j$ for some $j < i$ be the immediate predecessor u of v in T; let $\mathbf{c} \in B_i$ and its match $\mathbf{c}' \in B'_i$ be the guarded tuples represented at u so that $\mathfrak{A}, \mathbf{c}$ and $\mathfrak{A}', \mathbf{c}'$ satisfy $\varphi_{\mathcal{T},u}$. According to $\varphi_{\mathcal{T},u}$, $\mathbf{c}'$ overlaps with some tuple $\mathbf{d}'$, similar to the manner in which $\mathbf{c}$ overlaps with $\delta(v) = \mathbf{d}$, such that $\mathfrak{A}', \mathbf{d}' \models \varphi_{\mathcal{T},v}$. Note, however, that the overlap between $\mathbf{d}'$ and $\mathbf{c}'$ could be strictly larger than that between $\mathbf{d}$ and $\mathbf{c}$—more generally, modified distances $d_t(\cdot,\cdot) > n$ with respect to subsets of overlaps in B need not be reproduced automatically. In order to remedy this, we need to replace our ad-hoc first choice for $\mathbf{d}'$ by a free realization $\mathbf{d}''$ over the image $t' \subseteq B'_{i-1}$ of $t := \delta(v) \cap \delta(u) \subseteq B_{i-1}$ (just as $\mathbf{d} = \delta(v)$ is n-free over B_{i-1}, due to the n-closed nature of B). We set $\delta'(v) := \mathbf{d}''$.

We claim that the resulting subset $B' = \bigcup B'_i$ is n-closed in $\mathfrak{A}'$ and that $\mathfrak{A}' \restriction B', \mathbf{b}'$ and $\mathfrak{A} \restriction B, \mathbf{b}$ are isomorphic. Clearly, distances in $\mathfrak{A}' \restriction B'$ can only be shorter than distances in $\mathfrak{A} \restriction B$, as overlaps in B are analogously enforced in B' through the formulae $\varphi_{\mathcal{T},v}$.

That short distances cannot actually shrink and that long distances cannot become short follow from the n-free nature of the choices in the assembly of B'.

It suffices to show by induction on i that distances between elements of B'_i in $\mathfrak{A}$ match those in $\mathfrak{A} \restriction B_i$, if we regard all distances $> n$ as equivalent. This is clear for $i = 0$. For the passage from $i-1$ to i, consider any new element in $\delta(v_i)$, that is, in $\delta(v_i) \setminus \delta(u)$ where u is the immediate predecessor of v_i in T. With our n-free choice of $\delta'(v_i)$, we made sure that $d_{t'}(b', B_{i-1}) > n$ for every $b' \in \delta'(v_i) \setminus \delta'(u)$, and for $t' = \delta'(v_i) \cap \delta'(u)$. As far as distances in $\mathfrak{A}'$ rather than in $\mathfrak{A}' \setminus t'$ are concerned, these can only be smaller if the shortest path necessarily passes through t'; but then the corresponding distance preservation follows from the induction hypothesis.

The reasoning for any short chordless path from a new element $b' \in \delta'(v_i)$ to some element in B'_{i-1} is similar: there cannot be any short chordless paths avoiding t', since $d_{t'}(b', B'_{i-1}) > n$. Any short chordless paths from b' through t', on the other hand, consists of a single edge from b' into t' and a continuation within B'_{i-1} that is taken care of by the induction hypothesis.

It follows that B' is n-closed, and that the natural association between the elements of B and B' is a bijection that preserves distances up to n exactly, and leaves distances larger than n larger than n. It follows in particular that the local isomorphisms guaranteed by the $\varphi_{\mathcal{T},v}$ combine to yield $\mathfrak{A}' \restriction B', \mathbf{b}' \simeq \mathfrak{A} \restriction B, \mathbf{b}$. □

Remark 4.9. The proof shows that for any subtree $T_0 \subseteq T$ such that $B_0 := \bigcup\{\delta(v) \colon v \in T_0\}$ is n-closed, any isomorphism $\mathfrak{A} \restriction B_0 \simeq \mathfrak{A}' \restriction B'_0$ with some n-closed $B'_0 \subseteq \mathfrak{A}'$ extends to an isomorphism $\mathfrak{A} \restriction B \simeq \mathfrak{A}' \restriction B'$ with n-closed B'.

4.3.2. Back & Forth in Free and Acyclic Models. The crux in Theorem 4.7 is the proof of expressive completeness of GF for $\sim_{\mathrm{g}}$-invariant FO-sentences. As discussed at the beginning of Section 4, this is achieved with an upgrading of suitable levels $\sim_{\mathrm{g}}^{\ell}$ of guarded bisimulation equivalence to levels $\equiv_q$ of elementary equivalence—as indicated diagrammatically in Figure 10. The target structures in this upgrading are such that $\sim_{\mathrm{g}}^{\ell}$ implies $\equiv_q$, that is, such that back & forth extensions can be guaranteed in the classical q-round Ehrenfeucht–Fraïssé game for first-order logic, based on just their $\sim_{\mathrm{g}}^{\ell}$ equivalence. This upgrading then shows that any φ as in the theorem is actually preserved under some $\sim_{\mathrm{g}}^{\ell}$. Referring to the diagram: if $\mathfrak{A}_0 \models \varphi$, then $\mathfrak{A} \models \varphi$ because $\mathfrak{A} \sim_{\mathrm{g}} \mathfrak{A}_0$ and because φ is preserved under $\sim_{\mathrm{g}}$ between finite structures; then $\mathfrak{A}' \models \varphi$ simply because $\mathfrak{A} \equiv_q \mathfrak{A}'$ and $\mathrm{qr}(\varphi) \leqslant q$; finally, again by preservation under $\sim_{\mathrm{g}}$, $\mathfrak{A}'_0 \models \varphi$ follows.

The usual Ehrenfeucht–Fraïssé techniques imply that φ is equivalent to the disjunction of those GF-sentences that characterize the $\sim_{\mathrm{g}}^{\ell}$-types of models of φ; this yields the desired $\varphi' \in \mathrm{GF}$ since there are only finitely many $\sim_{\mathrm{g}}^{\ell}$-types.

For the desired upgrading, we provide an extension lemma, which will cover the crucial back & forth requirements of a single round in the first-order Ehrenfeucht–Fraïssé game. We rely on sufficient levels of acyclicity and freeness to make sure we can maintain the appropriate closure conditions.

For a local isomorphism ρ between two τ-structures $\mathfrak{A}$ and $\mathfrak{A}'$ with domain $B \subseteq \mathfrak{A}$ and image $B' \subseteq \mathfrak{A}'$, we use the notation

$$\rho\colon B \longmapsto_{\mathrm{g}}^{\ell} B'$$

to indicate that ρ is compatible with GF-equivalence up to nesting depth ℓ, or with $\sim_{\mathrm{g}}^{\ell}$ in the sense that $\mathfrak{A}, \mathbf{b} \sim_{\mathrm{g}}^{\ell} \mathfrak{A}', \mathbf{b}'$ for all guarded tuples $\mathbf{b} \subseteq B$ and $\mathbf{b}' = \rho(\mathbf{b}) \subseteq B'$.

LEMMA 4.10. *Let $L \geqslant \ell + f_n(w, w+1)$, where f_n is the bound on sizes of n-closures from Lemma 3.22 and w the width of τ. Let $\mathfrak{A}$ and $\mathfrak{A}'$ be sufficiently free and acyclic,*

$$\rho\colon B \longmapsto_{\mathrm{g}}^{L} B'$$

a local isomorphism between subsets $B = \mathrm{dom}(\rho) \subseteq \mathfrak{A}$ and $B' = \mathrm{im}(\rho) \subseteq \mathfrak{A}'$ that are $(2n+1)$-closed. Then, there is, for every $a \in \mathfrak{A}$, an extension to a local isomorphism $\hat{\rho} \supseteq \rho$,

$$\hat{\rho}\colon \hat{B} \longmapsto_{\mathrm{g}}^{\ell} \hat{B}'$$

with $a \in \mathrm{dom}(\hat{\rho})$ and such that $\hat{B} = \mathrm{dom}(\hat{\rho})$ and $\hat{B}' = \mathrm{im}(\hat{\rho})$ are n-closed.

PROOF. Let us work in the expansions $\mathfrak{A}_\ell$ and $\mathfrak{A}'_\ell$ of $\mathfrak{A}$ and $\mathfrak{A}'$ by predicates that mark the $\sim_{\mathrm{g}}^{\ell}$-types of guarded tuples. In effect, this means that ρ is compatible with $\sim_{\mathrm{g}}^{L-\ell}$ over the expansions, and we need $\hat{\rho}$ to be just a local isomorphism with respect to these expansions.

If $d(a, B) > n$, pick any $a' \in \mathfrak{A}'$ such that $\mathfrak{A}_\ell, a \sim_{\mathrm{g}}^{0} \mathfrak{A}'_\ell, a'$ and $d(a', B') > n$ too. This is possible in sufficiently free and acyclic $\mathfrak{A}'$, since $\mathfrak{A}'_\ell \sim_{\mathrm{g}}^{1} \mathfrak{A}_\ell$ and an n-free realization of the appropriate type can be found according to Lemma 4.8.

If $d(a, B) \leqslant n$, we apply Lemmas 3.23 and 3.25 to the analysis of $\hat{B} := \mathrm{cl}_n(B \cup a)$. We locate the clique (guarded tuple) $\mathbf{d} \subseteq B$ in which $\hat{B} \setminus B$ is linked to B, and find its counterpart $\mathbf{d}' := \rho(\mathbf{d}) \subseteq B'$ in $\mathfrak{A}'$. As $\mathrm{cl}_n(B \cup a) = \mathrm{cl}_n(B) \cup \mathrm{cl}_n(\mathbf{d}a)$, $|\hat{B} \setminus B| \leqslant |\mathrm{cl}_n(\mathbf{d}a)| \leqslant f_n(w, w+1)$. It follows that there is a tree decomposition $\mathcal{T}$ of $\hat{B}$, in which $\mathbf{d}$ is represented at a node $v \in T$ such that the subtree $T_v \subseteq T$ that represents $\mathrm{cl}_n(\mathbf{d}a)$ has depth at most $f_n(w, w+1) \leqslant L - \ell$.

Now $\mathbf{d}' = \rho(\mathbf{d})$ satisfies $\varphi_{\mathcal{T},v}$ in $\mathfrak{A}'_\ell$, since the nesting depth of $\varphi_{\mathcal{T},v}$ is bounded by $L - \ell$.

We therefore find, according to Lemma 4.8 and Remark 4.9, an extension of B' to an n-closed subset $\hat{B}'$ such that $\mathfrak{A}_\ell, \hat{B} \simeq \mathfrak{A}'_\ell, \hat{B}'$, which implies that the corresponding extension $\hat{\rho}$, as a local isomorphism over the expansions, is compatible with $\sim_{\mathrm{g}}^{\ell}$ over $\mathfrak{A}$ and $\mathfrak{A}'$, as required. □

COROLLARY 4.11. *For sufficiently large ℓ and sufficiently free and acyclic $\mathfrak{A}$ and $\mathfrak{A}'$, $\mathfrak{A} \sim_{\mathrm{g}}^{\ell} \mathfrak{A}'$ implies $\mathfrak{A} \equiv_q \mathfrak{A}'$.*

PROOF. We consider the following collections of partial isomorphisms

$$I_k := \left\{ \rho \in \mathrm{Part}(\mathfrak{A}, \mathfrak{A}') \colon \begin{array}{l} \rho \colon \mathrm{dom}(\rho) \mapsto_g^{\ell_k} \mathrm{im}(\rho), \\ \mathrm{dom}(\rho), \mathrm{im}(\rho)\; n_k\text{-closed} \end{array} \right\}$$

for suitable parameters ℓ_k and n_k. Let $(\ell_k)_{k \leqslant q}$ and $(n_k)_{n \leqslant q}$ be chosen such that $n_{k+1} \geqslant 2n_k + 1$ and $\ell_{k+1} \geqslant \ell_k + f_{n_k}(w, w+1)$ (with f_n the bound on n-closures from Lemma 3.22, w the width of τ). Then, Lemma 4.10 shows that $(I_k)_{k \leqslant q}$ forms a back & forth system, provided $\mathfrak{A}$ and $\mathfrak{A}'$ are sufficiently acyclic and free. If $\mathfrak{A} \sim_{\mathsf{g}}^{\ell_q} \mathfrak{A}'$, then $\emptyset \in I_q \neq \emptyset$ serves to show that

$$(I_k)_{k \leqslant q} \colon \mathfrak{A} \simeq_q \mathfrak{A}' \;\Rightarrow\; \mathfrak{A} \equiv_q \mathfrak{A}',$$

providing the desired upgrading. □

As discussed previously, the upgrading according to Figure 10 establishes the expressive completeness claim of Theorem 4.7. For sufficiently large ℓ and arbitrary $\mathfrak{A}_0 \sim_{\mathsf{g}}^{\ell} \mathfrak{A}_0'$, we invoke Corollary 4.5 to find $\mathfrak{A}$ and $\mathfrak{A}'$ that are conformal, N-acyclic and (n, K)-free for our preferred choice of parameters N, n, k. For some such suitable choice, Corollary 4.11 implies that $\mathfrak{A} \equiv_q \mathfrak{A}'$, and $\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{A}' \models \varphi$ follows.

5. OUTLOOK

We have introduced a new construction of finite hypergraph covers and guarded covers that seems to achieve the highest possible degree of acyclicity that can generally be guaranteed—viz., N-acyclicity for some N, or acyclicity in substructures of bounded size. The rudimentary study of N-acyclic structures has revealed some striking features, for example, in connection with the closure operation $\mathrm{cl}_n(\,\cdot\,)$. Also an investigation into potential algorithmic benefits of N-acyclicity, akin in spirit maybe to that of local tree decomposability as in Grohe [2008], may be interesting. N-acyclic hypergraphs and relational structures and the underlying model constructions may also prove useful in the further study of extended modal and guarded logics.

The key combinatorial construction of highly acyclic Cayley groups and graphs in Section 2.1 is very uniform and the result seems natural and canonical (cf. Observation 3.3). Its application to the cover construction is far less so, due to the local-to-global construction and due to the arbitrariness of gluing sites in the final steps towards the completion of the cover. Unlike the results in the graph case [Otto 2004], or the conformal covers in Hodkinson and Otto [2003], or the new results in Bárány et al. [2010], the more highly acyclic covers obtained here are neither canonical nor homogeneous, nor naturally compatible with automorphisms of the base structure. It remains to be seen whether this can be improved—or whether there are systematic obstacles that prevent some good features of graph covers from being lifted to hypergraphs. To mention but one obvious such phenomenon, the cartwheel hypergraphs from Figure 1 show that finite N-acyclic hypergraph covers can in general not be faithful with respect to incidence degrees between hyperedges—unlike the canonical graph covers obtained from products with suitable Cayley groups (Proposition 3.7), which are degree-preserving.

It is also interesting to compare the two recent breakthroughs concerning finite hypergraph covers that display certain qualified degrees of acyclicity, viz. *weak N-acyclicity* in Bárány et al. [2010], and *N-acyclicity* here. While the weakly N-acyclic covers are more regular and, above all, constructible within reasonable size and complexity bounds, the current construction of N-acyclic covers is far less concrete and does not seem to offer good complexity bounds, but it offers what seems to be the maximal achievable degree of acyclicity.

ACKNOWLEDGMENTS

I am grateful to the anonymous referees for valuable suggestions.

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Received December 2010; revised September 2011; accepted October 2011