Matsubara time integrals

April 29, 2019

Abstract

1 MxM

$$
\begin{aligned}
I(\tau) &= \int_0^\beta A(\tau-\tau')B(\tau')d\tau' \\
&= \int_0^\tau A(\tau-\tau')B(\tau')d\tau' + \xi_A \int_\tau^\beta A(\beta+\tau-\tau')B(\tau')d\tau'
\end{aligned}
$$

Define

$$
\begin{aligned}
I^{(1)}(\tau) &= \int_0^\tau A(\tau-\tau')B(\tau')d\tau' \\
I^{(2)}(\tau) &= \xi_A \int_\tau^\beta A(\beta+\tau-\tau')B(\tau')d\tau'
\end{aligned}
$$

In terms of our integration weights this becomes

$$
I^{(1)}_m = \begin{cases} \sum_{k=0}^{p-1}\sum_{l=0}^{p-1} R[m,l,k]A[l]B[k] & ; \quad m=0,...,p-1 \\ \sum_{k=0}^{m} w[m,k]A[m-k]B[k] & ; \quad m=p,...,N-1 \end{cases}
$$

$$
I^{(2)}_m = \begin{cases} \sum_{k=0}^{p-1}\sum_{l=0}^{p-1} R[N-1-m,l,k]A[N-1-l]B[N-1-k] & ; \quad m=N-p,...,N-1 \\ \sum_{k=m}^{N-1} w[N-1-m,k-m]A[k]B[N-1-(k-m)] & ; \quad m=0,...,N-1-p \end{cases}
$$

where the w are Gregory integration weights and R are weights obtained from integrating interpolating polynomials exactly within small time intervals. Under the substitutions $k \to N-1-k$ in the first case and $k \to N-1-(k-m)$ in the second case this becomes:

$$
I^{(2)}_m = \begin{cases} \sum_{k=N-p}^{N-1}\sum_{l=0}^{p-1} R[N-1-m,l,N-1-k]A[N-1-l]B[k] & ; \quad m=N-p,...,N-1 \\ \sum_{k=m}^{N-1} w[N-1-m,N-1-k]A[N-1-(k-m)]B[k] & ; \quad m=0,...,N-1-p \end{cases}
$$

We then define the following coefficients:

$$
C^{(1)}_{m,k} = \begin{cases} \sum_{l=0}^{p-1} R[m,l,k]A[l] & ; \quad m=0,...,p-1 \text{ and } k=0,...,p-1 \\ w[m,k]A[m-k] & ; \quad m=p,...,N-1 \text{ and } k=0,...,m \end{cases}
$$

$$
C^{(2)}_{m,k} = \begin{cases} \sum_{l=0}^{p-1} R[N-1-m,l,N-1-k]A[N-1-l] & ; \quad m=N-p,...,N-1 \text{ and } k=N-p,...,N-1 \\ w[N-1-m,N-1-k]A[N-1-(k-m)]B[k] & ; \quad m=0,...,N-1-p \text{ and } k=m,...,N-1 \end{cases}
$$

And finally,

$$
I_m = \sum_{k=0}^{N-1}\left[C^{(1)}_{m,k} + \xi_A C^{(2)}_{m,k}\right]B[k]
$$

2 RIxM

$$
\begin{aligned}
I(\tau) &= \int_0^\beta A(\tau')B(\tau'-\tau)d\tau' \\
&= \int_\tau^\beta A(\tau')B(\tau'-\tau)d\tau' + \xi_B \int_0^\tau A(\tau')B(\beta+\tau'-\tau)d\tau'
\end{aligned}
$$

Define

$$I^{(1)}(\tau) = \int_\tau^\beta A(\tau')B(\tau'-\tau)d\tau'$$
$$I^{(2)}(\tau) = \int_0^\tau A(\tau')B(\beta+\tau'-\tau)d\tau'$$

In terms of our integration weights this becomes

$$I_m^{(1)} = \begin{cases} \sum_{k=0}^{p-1}\sum_{l=0}^{p-1} R[N-1-m,k,l]A[N-1-k]B[l] & ; \quad m = N-p,...,N-1 \\ \sum_{k=m}^{N-1} w[N-1-m,k-m]A[k]B[k-m] & ; \quad m = 0,...,N-1-p \end{cases}$$

Under the substitutions $N-1-k \to k$ in the first case :

$$I_m^{(1)} = \begin{cases} \sum_{k=N-p}^{N-1}\sum_{l=0}^{p-1} R[N-1-m,N-1-k,l]A[k]B[l] & ; \quad m = N-p,...,N-1 \\ \sum_{k=m}^{N-1} w[N-1-m,k-m]A[k]B[k-m] & ; \quad m = 0,...,N-1-p \end{cases}$$

$$I_m^{(2)} = \begin{cases} \sum_{k=0}^{p-1}\sum_{l=0}^{p-1} R[m,k,l]A[k]B[N-1-l] & ; \quad m = 0,...,p-1 \\ \sum_{k=0}^{m} w[m,k]A[k]B[N-1-m+k] & ; \quad m = p,...,N-1 \end{cases}$$

We then define the following coefficients:

$$C_{k,m}^{(1)} = \begin{cases} \sum_{l=0}^{p-1} R[N-1-m,N-1-k,l]B[l] & ; \quad m = N-p,...,N-1 \ \text{ and } k = N-p,...,N-1 \\ w[N-1-m,k-m]B[k-m] & ; \quad m = 0,...,N-1-p \ \text{ and } k = m,...,N-1 \end{cases}$$

$$C_{k,m}^{(2)} = \begin{cases} \sum_{l=0}^{p-1} R[m,k,l]B[N-1-l] & ; \quad m = 0,...,p-1 \ \text{ and } k = 0,...,p-1 \\ w[m,k]B[N-1-m+k] & ; \quad m = p,...,N-1 \ \text{ and } k = 0,...,m \end{cases}$$

And finally,

$$I_m = \sum_{k=0}^{N-1} A[k]\left[C_{k,m}^{(1)} + \xi_B C_{k,m}^{(2)}\right]$$