Stochastic Variance Reduced Gradient Methods

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References for this class

Section 6.3:

Sébastien Bubeck (2015)
Foundations and Trends
Convex Optimization: Algorithms and Complexity

M. Schmidt, N. Le Roux, F. Bach (2016), Mathematical Programming **Minimizing Finite Sums with the Stochastic Average Gradient.**

How to transform convergence results into iteration complexity

Section 1.3.5, R.M. Gower, Ph.d thesis: Sketch and Project: Randomized Iterative Methods for Linear Systems and Inverting Matrices University of Edinburgh, 2016

Solving the Finite Sum Training Problem

Optimization Sum of Terms

A Datum Function

$$f_i(w) := \ell\left(h_w(x^i), y^i\right) + \lambda R(w)$$

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} \ell\left(h_w(x^i), y^i\right) + \lambda R(w) &= \frac{1}{n}\sum_{i=1}^{n} \left(\ell\left(h_w(x^i), y^i\right) + \lambda R(w)\right) \\ &= \frac{1}{n}\sum_{i=1}^{n} f_i(w) \end{aligned}$$

Finite Sum Training Problem

$$\min_{w \in \mathbf{R}^d} \frac{1}{n}\sum_{i=1}^{n} f_i(w) =: f(w)$$

SGD shrinking stepsize

SGD 1.0: Descreasing stepsize

Set $w^0 = 0$, choose $\alpha > 0$, $\alpha_t = \frac{\alpha}{\sqrt{t+1}}$,

for $t = 0, 1, 2, \ldots, T-1$

sample $j \in \{1, \ldots, n\}$

$w^{t+1} = w^t - \alpha_t \nabla f_j(w^t)$

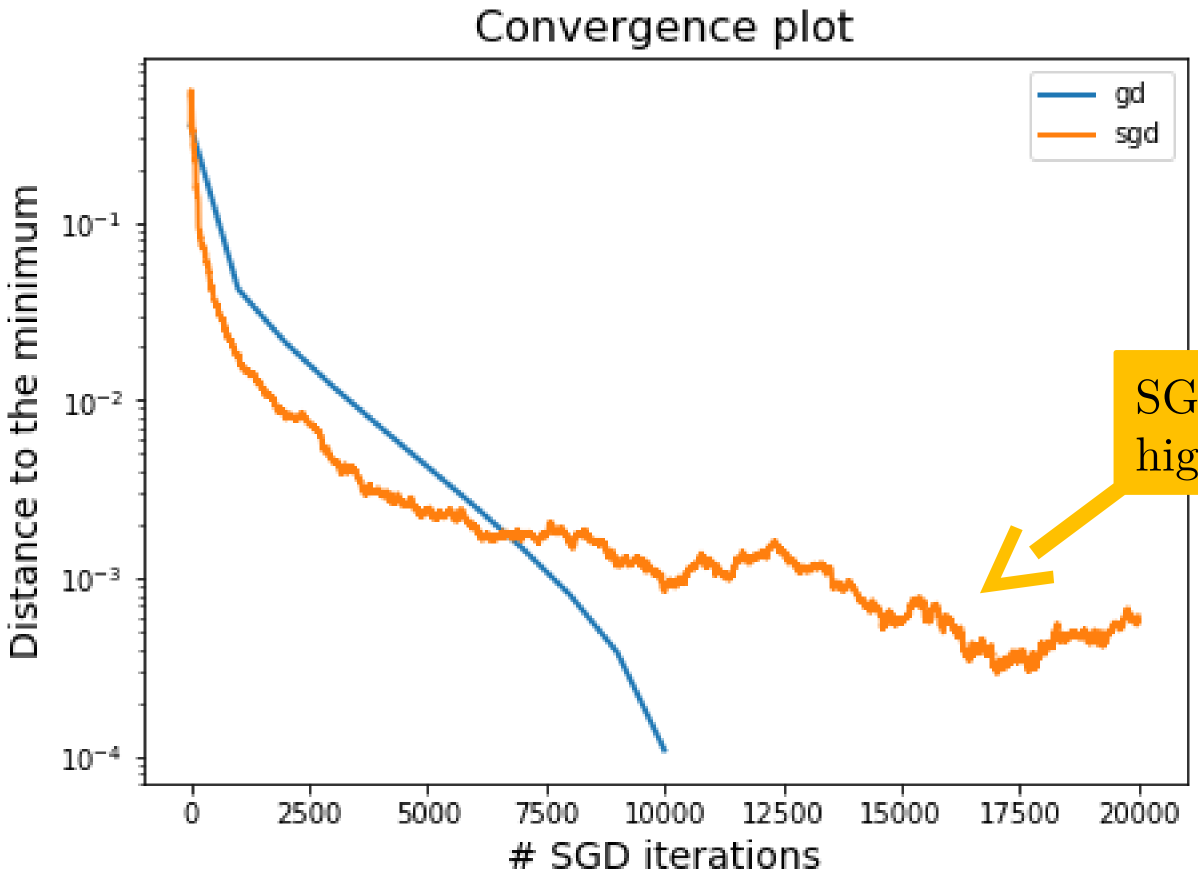
Output w^T

Convergence for Strongly Convex

- $f(w)$ is λ - strongly convex
- Subgradients bounded

$$\alpha_t = O\left(\frac{1}{\lambda t}\right) \quad \Rightarrow \quad \mathbb{E}[f(w^T)] - f(w^*) = O\left(\frac{1}{\lambda T}\right)$$

SGD initially fast, slow later



SGD suffers from high variance

Variance reduced methods through Sketching

Build an Estimate of the Gradient

Instead of using directly $\nabla f_j(w^t) \approx \nabla f(w^t)$
Use $\nabla f_j(w^t)$ to update estimate $g_t \approx \nabla f(w^t)$

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$$w^{t+1} = w^t - \alpha g^t$$

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$$w^{t+1} = w^t - \alpha g^t$$

We would like gradient estimate such that:

Unbiased

$$\mathbb{E}[g^t] = \nabla f(w^t)$$

Converges in *L2*

$$\mathbb{E}||g^t||_2^2 \underset{w^t \rightarrow w^*}{\longrightarrow} 0$$

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Use $\nabla f_j(w^t)$ to update estimate $g_t \approx \nabla f(w^t)$

$$w^{t+1} = w^t - \alpha g^t$$

We would like gradient estimate such that:

Unbiased

$$\mathbb{E}[g^t] = \nabla f(w^t)$$

Solves problem of
$||\nabla f_j(w)||_2^2 \leq B^2$

Converges in $L2$

$$\mathbb{E}||g^t||_2^2 \underset{w^t \to w^*}{\longrightarrow} 0$$

Covariates

Let x and z be random variables. We say that x and z are covariates if:

$$\mathrm{cov}(x, z) \geq 0$$

Variance Reduced Estimate:

$$x_z = x - z + \mathbb{E}[z]$$

Covariates

$$\text{cov}(x, z) := \mathbb{E}[(x - \mathbb{E}[x])(z - \mathbb{E}[z])]$$

Let x and z be random variables. We say that x and z are covariates if:

$$\text{cov}(x, z) \geq 0$$

Variance Reduced Estimate:

$$x_z = x - z + \mathbb{E}[z]$$

EXE:

1. Show that $\mathbb{E}[x_z] = \mathbb{E}[x]$
2. $\mathbb{VAR}[x_z] = \mathbb{E}[(x_z - \mathbb{E}[x_z])^2] =?$
3. When is $\mathbb{VAR}[x_z] \leq \mathbb{VAR}[x]$

Covariates

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Variance Reduced Estimate:

$$x_z = x - z + \mathbb{E}[z]$$

EXE:

1. Show that $\mathbb{E}[x_z] = \mathbb{E}[x]$
2. $\mathbb{VAR}[x_z] = \mathbb{E}[(x_z - \mathbb{E}[x_z])^2] = ?$
3. When is $\mathbb{VAR}[x_z] \leq \mathbb{VAR}[x]$

$$\begin{aligned}
\mathbb{E}[(x_z - \mathbb{E}[x_z])^2] &= \mathbb{E}[(x-\mathbb{E}[x] - (z-\mathbb{E}[z]))^2] \\
&= \mathbb{E}[(x-\mathbb{E}[x])^2] - 2\mathbb{E}[(x-\mathbb{E}[x])(z-\mathbb{E}[z])] \\
&\quad +\mathbb{E}[(z-\mathbb{E}[z])^2] \\
&= \mathbb{VAR}[x] - 2\text{cov}(x,z) + \mathbb{VAR}[z]
\end{aligned}$$

SVRG: Stochastic Variance Reduced Gradients

$$w^{t+1} = w^t - \alpha g^t$$

Reference point	$\tilde{w} \in \mathbb{R}^d$
Sample	$\nabla f_i(w^t), \quad i \in \{1, \ldots, n\}$ uniformly
grad estimate	$g^t = \nabla f_i(w^t) - \nabla f_i(\tilde{w}) + \nabla f(\tilde{w})$

$$x_z = \quad x \quad - \quad z \quad + \quad \mathbb{E}[z]$$

SVRG: Stochastic Variance Reduced Gradients

Set $w^0 = 0$, choose $\alpha > 0, m \in \mathbb{N}$

$\tilde{w}^0 = w^0$

for $t = 0, 1, 2, \ldots, T-1$

  calculate $\nabla f(\tilde{w}^t)$

  $w^0 = \tilde{w}^t$

  for $k = 0, 1, 2, \ldots, m-1$

    sample $j \in \{1, \ldots, n\}$

    $g^k = \nabla f_j(w^k) - \nabla f_j(\tilde{w}^t) + \nabla f(\tilde{w}^t)$

    $w^{k+1} = w^k - \alpha g^k$

  Option I: $\tilde{w}^{t+1} = w^m$

  Option II: $\tilde{w}^{t+1} = \frac{1}{m}\sum_{i=0}^{m-1} w^i$

Output $\tilde{w}^T$

Freeze reference point for m iterations

SAGA: Stochastic Average Gradient

$$w^{t+1} = w^t - \alpha g^t$$

Sample

$$\nabla f_i(w^t), \quad i \in \{1, \ldots, n\} \text{ uniformly}$$

Reference points

if i is sampled store $w^{t_i} = w^t$

grad estimate

$$g^t = \nabla f_i(w^t) - \nabla f_i(w^{t_i}) + \frac{1}{n}\sum_{j=1}^{n} \nabla f_j(w^{t_j})$$

$$x_z = \quad x \quad - \quad z \quad + \quad \mathbb{E}[z]$$

SVRG: Stochastic Variance Reduced Gradients

Set $w^0 = 0$, choose $\alpha > 0, m \in \mathbb{N}$
$\tilde{w}^0 = w^0$
for $t = 0, 1, 2, \ldots, T-1$
    calculate $\nabla f(\tilde{w}^t)$
    $w^0 = \tilde{w}^t$
    for $k = 0, 1, 2, \ldots, m-1$
        sample $j \in \{1, \ldots, n\}$
        $g^k = \nabla f_j(w^k) - \nabla f_j(\tilde{w}^t) + \nabla f(\tilde{w}^t)$
        $w^{k+1} = w^k - \alpha g^k$
    Option I: $\tilde{w}^{t+1} = w^m$
    Option II: $\tilde{w}^{t+1} = \frac{1}{m}\sum_{i=0}^{m-1} w^i$
Output $\tilde{w}^T$

Freeze reference point for m iterations

SAG: Stochastic Average Gradient (Biased version)

$$w^{t+1} = w^t - \alpha g^t$$

Sample

$\nabla f_i(w^t), \quad i \in \{1, \ldots, n\}$ uniformly

Reference points

if i is sampled store $w^{t_i} = w^t$

grad estimate

$$g^t = \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w^{t_j})$$

$$\mathbb{E}[g^t] \neq \nabla f(w^t)$$

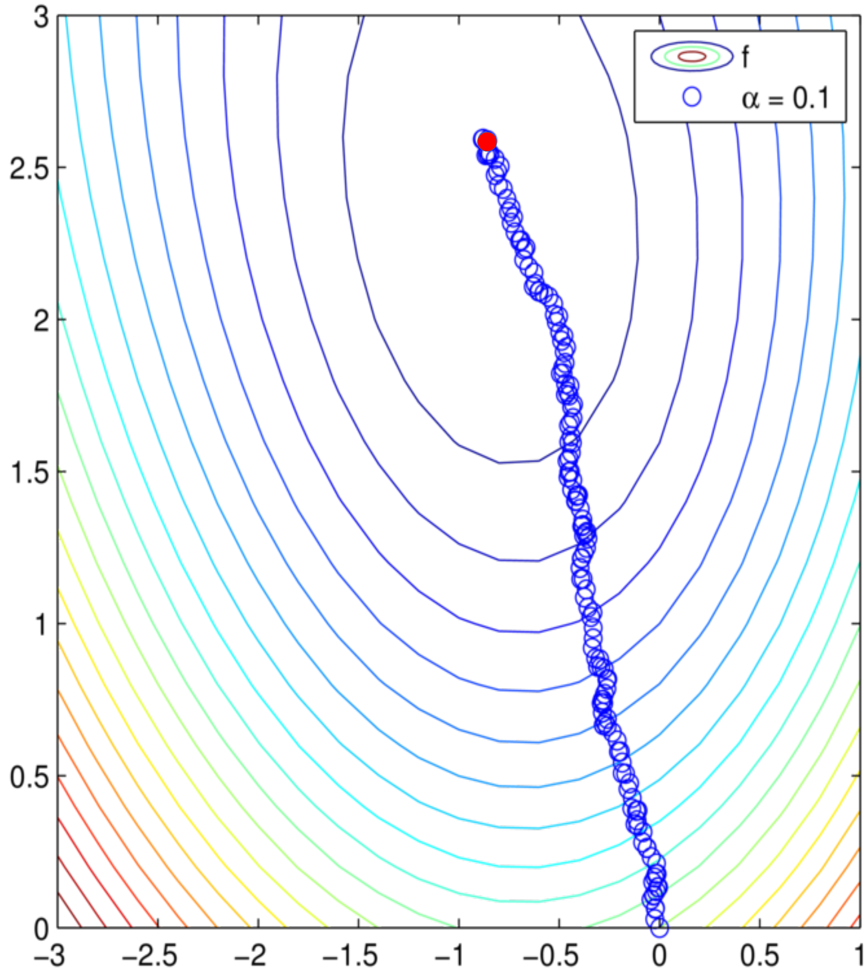
$$\cancel{x_z = x - z + \mathbb{E}[z]}$$

SAGA: Stochastic Average Gradient

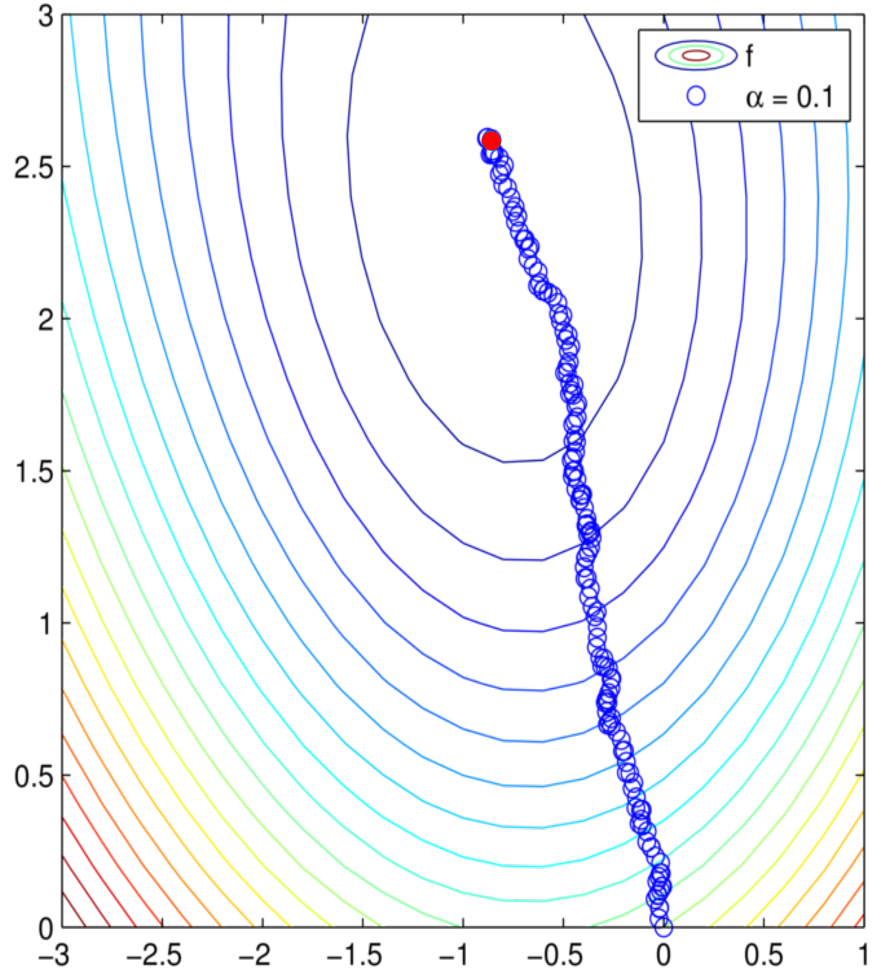
Set $w^0 = 0$, choose $\alpha > 0, m \in \mathbb{N}$
$z_i = \nabla f_i(w^0)$, for $i = 1, \ldots, n$
for $t = 0, 1, 2, \ldots, T-1$
    sample $j \in \{1, \ldots, n\}$
    $g^t = \nabla f_j(w^t) - z_j + \frac{1}{n}\sum_{i=1}^n z_i$
    $w^{t+1} = w^t - \alpha g^t$
    $z_j = \nabla f_j(w^t)$
Output w^T

Store all n vectors $z_i \in \mathbb{R}^d$

The Stochastic Average Gradient



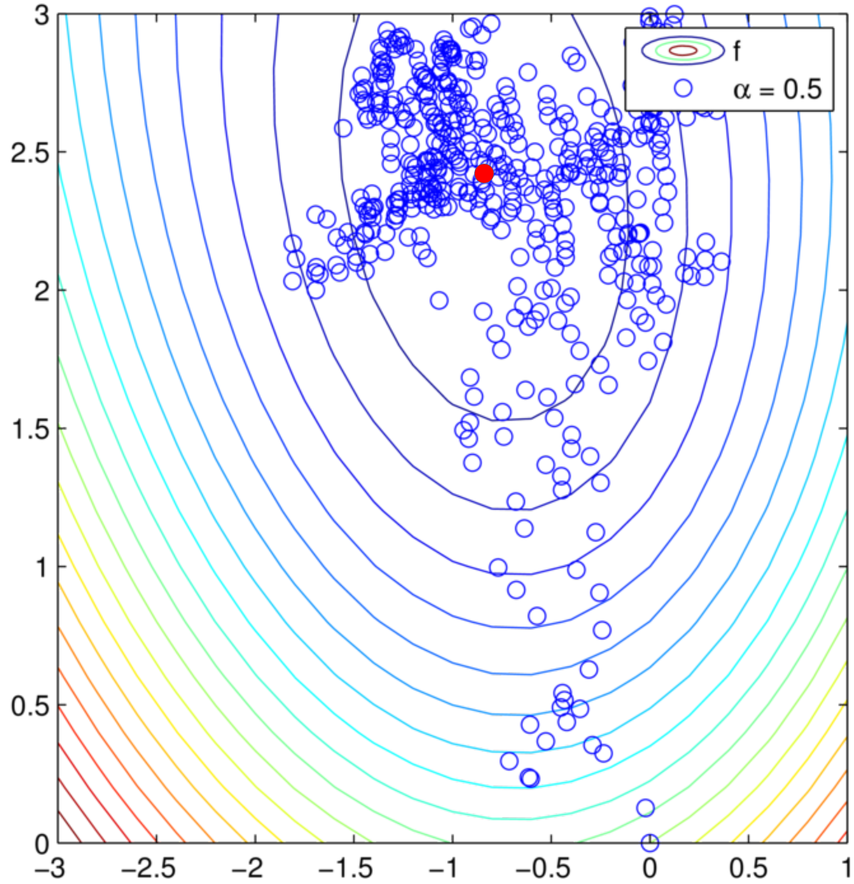
The Stochastic Average Gradient



How to prove this converges? Is this the only option?

Stochastic Gradient Descent

$\alpha = 0.5$



Proving Convergence

Assumptions for Convergence

Strong Convexity

$$f(w) \geq f(y) + \langle \nabla f(y), w - y \rangle + \frac{\lambda}{2}||w - y||_2^2$$

Smoothness

$$f_i(w) \leq f_i(y) + \langle \nabla f_i(y), w - y \rangle + \frac{L_i}{2}||w - y||_2^2, \quad \text{for } i = 1, \ldots, n$$

EXE: Calculate L_i and $L_{\max} := \max_{i=1,\ldots,n} L_i$ for

1. $f(w) = \frac{1}{2}||Aw - y||_2^2 + \frac{\lambda}{2}||w||_2^2, \quad \text{where } A \in \mathbb{R}^{n \times d}$
2. $f(w) = \frac{1}{n}\sum_{i=1}^{n} \ln(1 + e^{-y_i \langle w, a_i \rangle}) + \frac{\lambda}{2}||w||_2^2$

Assumptions for Convergence

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$$= \frac{1}{n}\sum_{i=1}^{n} f_i(w)$$

Assumptions for Convergence

EXE: Calculate L_i for

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$$= \frac{1}{n}\sum_{i=1}^{n} f_i(w)$$

$$\nabla^2 f_i(w) = nA_{i:}A_{i:}^\top + \lambda \quad \preceq \quad (n||A_{i:}||_2^2 + \lambda)I \quad = \quad L_i \ I$$

Assumptions for Convergence

EXE: Calculate L_{i} for

$$2. \quad f(w) = \frac{1}{n}\sum_{i=1}^{n} \ln(1+e^{-y_i\langle w, a_i\rangle}) + \frac{\lambda}{2}||w||_2^2$$

Assumptions for Convergence

EXE: Calculate L_i for

$$2. \quad f(w) = \frac{1}{n}\sum_{i=1}^{n} \ln(1+e^{-y_i\langle w, a_i\rangle}) + \frac{\lambda}{2}||w||_2^2$$

$$2. \quad f_i(w) = \ln(1+e^{-y_i\langle w, a_i\rangle}) + \frac{\lambda}{2}||w||_2^2,$$

Assumptions for Convergence

EXE: Calculate L_i for

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$$2. \quad f_i(w) = \ln(1+e^{-y_i\langle w,a_i\rangle}) + \frac{\lambda}{2}||w||_2^2,$$

$$\nabla f_i(w) = \frac{-y_i a_i e^{-y_i\langle w,a_i\rangle}}{1+e^{-y_i\langle w,a_i\rangle}} + \lambda w$$

$$\nabla^2 f_i(w) = a_i a_i^\top \left(\frac{(1+e^{-y_i\langle w,a_i\rangle})e^{-y_i\langle w,a_i\rangle}}{(1+e^{-y_i\langle w,a_i\rangle})^2} - \frac{e^{-2y_i\langle w,a_i\rangle}}{(1+e^{-y_i\langle w,a_i\rangle})^2} \right) + \lambda I$$

$$= a_i a_i^\top \frac{e^{-y_i\langle w,a_i\rangle}}{(1+e^{-y_i\langle w,a_i\rangle})^2} + \lambda I \quad \preceq \quad \left(\frac{||a_i||_2^2}{4} + \lambda \right) I = L_i \ I$$

Assumptions for Convergence

EXE: Let $f(w)$ be L–smooth and $f_i(w)$ be L_i–smooth for $i = 1, \ldots, n$.

Show that

$$L \quad \leq \quad \frac{1}{n}\sum_{i=1}^{n} L_i \quad \leq \quad L_{\max} := \max_{i=1,\ldots,n} L_i$$

Proof: From definition of $f_i(w)$ smoothness

Assumptions for Convergence

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Proof: From definition of $f_i(w)$ smoothness

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n f_i(w) &\leq \frac{1}{n}\sum_{i=1}^n f_i(y) + \langle \frac{1}{n}\sum_{i=1}^n \nabla f_i(y), x-y\rangle + \frac{1}{2n}\sum_{i=1}^n L_i ||w-y||_2^2 \\ &= f(y) + \langle \nabla f(y), x-y\rangle + \frac{1}{2n}\sum_{i=1}^n L_i ||w-y||_2^2\end{aligned}$$

Convergence SVRG

Theorem

If $\alpha = 1/10L_{\max}$ and $m = 20L_{\max}/\lambda$ then

$$\mathbb{E}[f(\tilde{w}^t)] - f(w^*) \quad \leq \quad 0.9^t(f(\tilde{w}^0) - f(w^*))$$

Need $O(L_{\max}/\lambda)$ inner iterations to have linear convergence

In practice use $\alpha = 1/L_{\max}, \; m = n$

Johnson, R. & Zhang, T. **Accelerating Stochastic Gradient Descent using Predictive Variance Reduction,** NIPS 2013

Proof:

$$\begin{aligned} ||w^{k+1} - w^*||_2^2 &= ||w^k - w^* - \alpha g^k)||_2^2 \\ &= ||w^k - w^*||_2^2 - 2\alpha\langle g^k, w^k - w^*\rangle + \alpha^2 ||g^k||_2^2. \end{aligned}$$

Taking expectation with respect to j

Unbiased estimator

$$\begin{aligned} \mathbb{E}_j\left[||w^{k+1} - w^*||_2^2\right] &= ||w^k - w^*||_2^2 - 2\alpha\langle \nabla f(w^k), w^k - w^*\rangle + \alpha^2 \mathbb{E}_j\left[||g^k||_2^2\right] \\ &\overset{\text{conv.}}{\leq} ||w^k - w^*||_2^2 - 2\alpha(f(w^k) - f(w^*)) + \alpha^2 \mathbb{E}_j\left[||g^k||_2^2\right] \end{aligned}$$

Must control this!

$$\boxed{\mathbb{E}_j\left[||g^k||_2^2\right]}$$

Smoothness Consequences I

Smoothness

$$f(w) \leq f(y) + \langle \nabla f(y), w - y \rangle + \frac{L}{2}||w - y||_2^2, \quad \text{for } i = 1, \ldots, n$$

EXE: Lemma 1

$$f(y - \frac{1}{L}\nabla f(y)) - f(y) \leq -\frac{1}{2L}\nabla f(y), \quad \forall y.$$

Proof:

Substituting $w = y - \dfrac{1}{L}\nabla f(y)$ into the smoothness inequality gives

$$\begin{aligned} f(y - \tfrac{1}{L}\nabla f(y)) - f(y) &\leq \langle \nabla f(y), -\frac{1}{L}\nabla f(y) \rangle + \frac{L}{2}|| - \frac{1}{L}\nabla f(y)||_2^2 \\ &= -\tfrac{1}{2L}\nabla f(y). \quad \blacksquare \end{aligned}$$

Smoothness Consequences II

Smoothness

$$f_i(w) \leq f_i(y) + \langle \nabla f_i(y), w - y \rangle + \frac{L_i}{2}||w - y||_2^2, \quad \text{for } i = 1, \ldots, n$$

EXE: Lemma 2

$$\mathbb{E}[||\nabla f_i(w) - \nabla f_i(w^*)||_2^2] \leq 2L_{\max}(f(w) - f(w^*))$$

Proof: Let $g_i(w) = f_i(w) - f_i(w^*) - \langle \nabla f_i(w^*), w - w^* \rangle$ which is L_i–smooth.

Smoothness Consequences II

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$$f_i(w) \leq f_i(y) + \langle \nabla f_i(y), w - y \rangle + \frac{L_i}{2}||w-y||_2^2, \quad \text{for } i = 1, \ldots, n$$

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$$\mathbb{E}[||\nabla f_i(w) - \nabla f_i(w^*)||_2^2] \leq 2L_{\max}(f(w) - f(w^*))$$

Proof: Let $g_i(w) = f_i(w) - f_i(w^*) - \langle \nabla f_i(w^*), w - w^* \rangle$ which is L_i–smooth.

Convexity of $f_i(w) \Rightarrow g_i(w) \geq 0$ for all w. From Lemma 1 we have

$$g_i(w) \geq g_i(w) - g_i(w - \frac{1}{L_i}\nabla g_i(w)) \underset{\text{Lemma 1}}{\geq} \frac{1}{2L_i}||\nabla g_i(w)||_2^2 \geq \frac{1}{2L_{\max}}||\nabla g_i(w)||_2^2$$

Inserting definition of $g_i(w)$ we have

$$\frac{1}{2L_{\max}}||\nabla f_i(w) - \nabla f_i(w^*)||_2^2 \leq f_i(w) - f_i(w^*) - \langle \nabla f_i(w^*), w - w^* \rangle$$

Result follows by taking expectation of i. ■

Bounding gradient estimate

EXE: Lemma 3

$$\mathbb{E}[||g^k||_2^2] \leq 4L_{\max}(f(w^k) - f(w^*)) + 4L_{\max}(f(\tilde{w}^t) - f(w^*))$$

Proof: Hint: use $||a+b||_2^2 \leq 2||a||_2^2 + 2||b||_2^2$ and Lemma 2

Where we used in the first inequality that $\mathbb{E}[||X - \mathbb{E}X||_2^2] \leq \mathbb{E}[||X||_2^2]$ with $X = \nabla f_i(w^*) - \nabla f_i(\tilde{w}^t)$ thus $\mathbb{E}[X] = -\nabla f(\tilde{w}^t)$

Bounding gradient estimate

EXE: Lemma 3

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with $X = \nabla f_i(w^*) - \nabla f_i(\tilde{w}^t)$ thus $\mathbb{E}[X] = -\nabla f(\tilde{w}^t)$

Bounding gradient estimate

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Proof: Hint: use $||a+b||_2^2 \leq 2||a||_2^2 + 2||b||_2^2$ and Lemma 2

$$\begin{aligned}
\mathbb{E}_j[||g^k||_2^2] &= \mathbb{E}_j[||\nabla f_i(w^k) - \nabla f_i(w^*) + \nabla f_i(w^*) - \nabla f_i(\tilde{w}^t) + \nabla f(\tilde{w}^t)||_2^2] \\
&\leq 2\mathbb{E}_j[||\nabla f_i(w^k) - \nabla f_i(w^*)||_2^2] + 2\mathbb{E}_j[||\nabla f_i(w^*) - \nabla f_i(\tilde{w}^t) + \nabla f(\tilde{w}^t)||_2^2] \\
&\leq 2\mathbb{E}_j[||\nabla f_i(w^k) - \nabla f_i(w^*)||_2^2] + 2\mathbb{E}_j[||\nabla f_i(w^*) - \nabla f_i(\tilde{w}^t)||_2^2] \\
&\underset{\text{Lemma 2}}{=} 4L_{\max}\left(f(w^k) - f(w^*) + f(\tilde{w}^t) - f(w^*)\right) \qquad \blacksquare
\end{aligned}$$

Where we used in the first inequality that $\mathbb{E}[||X - \mathbb{E}X||_2^2] \leq \mathbb{E}[||X||_2^2]$ with $X = \nabla f_i(w^*) - \nabla f_i(\tilde{w}^t)$ thus $\mathbb{E}[X] = -\nabla f(\tilde{w}^t)$

Proof:

$$\begin{aligned} ||w^{k+1} - w^*||_2^2 &= ||w^k - w^* - \alpha g^k)||_2^2 \\ &= ||w^k - w^*||_2^2 - 2\alpha\langle g^k, w^k - w^*\rangle + \alpha^2 ||g^k||_2^2. \end{aligned}$$

Taking expectation with respect to j

Unbiased estimator

$$\begin{aligned} \mathbb{E}_j\left[||w^{k+1} - w^*||_2^2\right] &= ||w^k - w^*||_2^2 - 2\alpha\langle \nabla f(w^k), w^k - w^*\rangle + \alpha^2 \mathbb{E}_j\left[||g^k||_2^2\right] \\ &\overset{\text{conv.}}{\leq} ||w^k - w^*||_2^2 - 2\alpha(f(w^k) - f(w^*)) + \alpha^2 \mathbb{E}_j\left[||g^k||_2^2\right] \end{aligned}$$

Must control this!

$$\boxed{\mathbb{E}_j\left[||g^k||_2^2\right]}$$

$$\mathbb{E}[||g^k||_2^2] \leq 4L_{\max}(f(w^k) - f(w^*)) + 4L_{\max}(f(\tilde{w}^t) - f(w^*))$$

Proof (continued I):

$$\begin{aligned} ||w^{k+1} - w^*||_2^2 &= ||w^k - w^* - \alpha g^k)||_2^2 \\ &= ||w^k - w^*||_2^2 - 2\alpha\langle g^k, w^k - w^*\rangle + \alpha^2 ||g^k||_2^2. \end{aligned}$$

Taking expectation with respect to j

Unbiased estimator

$$\begin{aligned} \mathbb{E}_j\left[||w^{k+1} - w^*||_2^2\right] &= ||w^k - w^*||_2^2 - 2\alpha\langle \nabla f(w^k), w^k - w^*\rangle + \alpha^2 \mathbb{E}_j\left[||g^k||_2^2\right] \\ &\overset{\text{conv.}}{\leq} ||w^k - w^*||_2^2 - 2\alpha(f(w^k) - f(w^*)) + \alpha^2 \mathbb{E}_j\left[||g^k||_2^2\right] \\ &\leq ||w^k - w^*||_2^2 - 2\alpha(1 - 2\alpha L_{\max})(f(w^k) - f(w^*)) \\ &\quad +4\alpha^2 L_{\max}(f(\tilde{w}^t) - f(w^*)) \end{aligned}$$

Proof (continued I):

$$
\begin{aligned}
||w^{k+1} - w^*||_2^2 &= ||w^k - w^* - \alpha g^k)||_2^2 \\
&= ||w^k - w^*||_2^2 - 2\alpha\langle g^k, w^k - w^*\rangle + \alpha^2 ||g^k||_2^2.
\end{aligned}
$$

Taking expectation with respect to j

Unbiased estimator

$$
\begin{aligned}
\mathbb{E}_j\left[||w^{k+1} - w^*||_2^2\right] &= ||w^k - w^*||_2^2 - 2\alpha\langle \nabla f(w^k), w^k - w^*\rangle + \alpha^2 \mathbb{E}_j\left[||g^k||_2^2\right] \\
&\overset{\text{conv.}}{\leq} ||w^k - w^*||_2^2 - 2\alpha(f(w^k) - f(w^*)) + \alpha^2 \mathbb{E}_j\left[||g^k||_2^2\right] \\
&\leq ||w^k - w^*||_2^2 - 2\alpha(1 - 2\alpha L_{\max})(f(w^k) - f(w^*)) \\
&\quad +4\alpha^2 L_{\max}(f(\tilde{w}^t) - f(w^*))
\end{aligned}
$$

Taking expectation and summing from $k = 0, \ldots, m-1$ gives

$$
\begin{aligned}
\mathbb{E}\left[||w^m - w^*||_2^2\right] &\leq \mathbb{E}\left[||w^0 - w^*||_2^2\right] - 2\alpha(1 - 2\alpha L_{\max})\mathbb{E}[\textstyle\sum_{k=0}^{m-1}(f(w^k) - f(w^*))] \\
&\quad +4m\alpha^2 L_{\max}\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right]
\end{aligned}
$$

Proof (continued II):

$$\begin{aligned}\mathbb{E}\left[||w^m - w^*||_2^2\right] \quad &\leq \quad \mathbb{E}\left[||w^0 - w^*||_2^2\right] - 2\alpha(1 - 2\alpha L_{\max})\mathbb{E}[\textstyle\sum_{k=0}^{m-1}(f(w^k) - f(w^*))] \\ &\quad + 4m\alpha^2 L_{\max}\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right]\end{aligned}$$

$$\begin{aligned}2\alpha(1 - 2\alpha L_{\max})\mathbb{E}[\textstyle\sum_{k=0}^{m-1}(f(w^k) - f(w^*))] \quad &\leq \quad \mathbb{E}\left[||w^0 - w^*||_2^2\right] - \mathbb{E}\left[||w^m - w^*||_2^2\right] \\ &\quad + 4m\alpha^2 L_{\max}\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right] \\ &\leq \quad 2(2m\alpha^2 L_{\max} - \lambda^{-1})\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right]\end{aligned}$$

Proof (continued II):

$$\mathbb{E}\left[||w^m - w^*||_2^2\right] \leq \mathbb{E}\left[||w^0 - w^*||_2^2\right] - 2\alpha(1 - 2\alpha L_{\max})\mathbb{E}[\textstyle\sum_{k=0}^{m-1}(f(w^k) - f(w^*))]$$

$$+4m\alpha^2 L_{\max}\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right]$$

Re-arranging and using strong convexity $f(\tilde{w}^t) - f(w^*) \geq \frac{\lambda}{2}||\tilde{w}^t - w^*||_2^2$

$$2\alpha(1 - 2\alpha L_{\max})\mathbb{E}[\textstyle\sum_{k=0}^{m-1}(f(w^k) - f(w^*))] \leq \mathbb{E}\left[||w^0 - w^*||_2^2\right] - \mathbb{E}\left[||w^m - w^*||_2^2\right]$$

$w^0 = \tilde{w}^t$

$$+4m\alpha^2 L_{\max}\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right]$$

$$\leq 2(2m\alpha^2 L_{\max} - \lambda^{-1})\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right]$$

Proof (continued II):

$$\mathbb{E}\left[||w^m - w^*||_2^2\right] \leq \mathbb{E}\left[||w^0 - w^*||_2^2\right] - 2\alpha(1-2\alpha L_{\max})\mathbb{E}[\textstyle\sum_{k=0}^{m-1}(f(w^k) - f(w^*))]$$
$$+4m\alpha^2 L_{\max}\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right]$$

Re-arranging and using strong convexity $f(\tilde{w}^t) - f(w^*) \geq \frac{\lambda}{2}||\tilde{w}^t - w^*||_2^2$

$$2\alpha(1-2\alpha L_{\max})\mathbb{E}[\textstyle\sum_{k=0}^{m-1}(f(w^k) - f(w^*))] \leq \mathbb{E}\left[||w^0 - w^*||_2^2\right] - \mathbb{E}\left[||w^m - w^*||_2^2\right]$$
$$+4m\alpha^2 L_{\max}\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right]$$
$$\leq 2(2m\alpha^2 L_{\max} - \lambda^{-1})\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right]$$

$w^0 = \tilde{w}^t$

Re-arranging again

$$\mathbb{E}[(f(\sum_{k=0}^{m-1}\frac{w^k}{m}) - f(w^*))] \leq \mathbb{E}[\frac{1}{m}\sum_{k=0}^{m-1}(f(w^k) - f(w^*))]$$

Jensen's inequality

$$\leq \left(\frac{2\alpha L_{\max}}{1-2\alpha L_{\max}} + \frac{1}{\lambda\alpha(1-2\alpha L_{\max})m}\right)\mathbb{E}\left[f(\tilde{w}^t) - f(w^*)\right]$$

Now plug in values $\alpha = 1/(10L_{\max})$ and $m = 20L_{\max}/\lambda$ $\blacksquare$

Convergence SAGA

Theorem SAGA

If $\alpha = 1/3L_{\max}$ then

$$\mathbb{E}\left[||w^t - w^*||_2^2\right] \leq \left(1 - \min\left\{\frac{1}{4n}, \frac{\lambda}{3L_{\max}}\right\}\right)^t ||w^0 - w^*||_2^2$$

A practical convergence result!

M. Schmidt, N. Le Roux, F. Bach (2016)
Mathematical Programming
Minimizing Finite Sums with the Stochastic Average Gradient.

Comparisons in complexity for strongly convex

Approximate solution

$$\mathbb{E}[f(w^T)] - f(w^*) \leq \epsilon$$

SGD

$$O\left(\frac{1}{\lambda\epsilon}\right)$$

Gradient descent

$$O\left(\frac{nL}{\lambda}\log\left(\frac{1}{\epsilon}\right)\right)$$

SVRG/SAGA

$$O\left(\left(n + \frac{L_{\max}}{\lambda}\right)\log\left(\frac{1}{\epsilon}\right)\right)$$

Variance reduction faster than GD when $L \geq \lambda + L_{\max}/n$

How did I get these complexity results from the convergence results?

Section 1.3.5, R.M. Gower, Ph.d thesis: Sketch and Project: Randomized Iterative Methods for Linear Systems and Inverting Matrices University of Edinburgh, 2016

Take for home Variance Reduction

- Variance reduced methods use only **one stochastic gradient per iteration** and converge linearly on strongly convex functions
- Choice of **fixed stepsize** possible
- **SAGA** only needs to know the smoothness parameter to work, but requires storing n past stochastic gradients
- **SVRG** only has $O(d)$ stored, but requires full gradient computations every so often