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Social Networks and the Identification of Peer Effects

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There is a large and growing literature on peer effects in economics. In the current article, we focus on a Manski-type linear-in-means model that has proved to be popular in empirical work. We critically examine some aspects of the statistical model that may be restrictive in empirical analyses. Specifically, we focus on three aspects. First, we examine the endogeneity of the network or peer groups. Second, we investigate simultaneously alternative definitions of links and the possibility of peer effects arising through multiple networks. Third, we highlight the representation of the traditional linear-in-means model as an autoregressive model, and contrast it with an alternative moving-average model, where the correlation between unconnected individuals who are indirectly connected is limited. Using data on friendship networks from the Add Health dataset, we illustrate the empirical relevance of these ideas.

KEY WORDS: Endogeneity; Linear-in-means model; Network formation.

1. INTRODUCTION

There is a large and growing literature on peer effects in economics. This article focuses on a Manski-type linear-in-means model (Manski 1993) that has proved to be popular in empirical work. We present our ideas in the context of data from the National Longitudinal Study of Adolescent Health (Add Health), where the substantive focus is on peer effects through friendship networks on high-school grades. We examine some aspects of the linear-in-means statistical model that may be restrictive or problematic in empirical analyses. Specifically, we focus on three aspects of the model and suggest generalizations of the basic model to address potential concerns. In all three cases, we augment the peer-effect model with a simple model for network formation to illustrate these concerns.

First, we explore the possible endogeneity of the network. This issue has been raised before in the econometric literature and is often mentioned as a potential concern in the interpretation of peer-effect estimates in empirical work. Although endogeneity of the network can be a concern even in cases where peer groups are assigned centrally, such as classrooms or military squadrons, it is a particular concern where networks are formed by individuals making deliberate choices to establish links. The specific concern is often that individuals have unobserved characteristics that are associated with their outcomes, and that these characteristics are also associated with the decision to establish links. The precise mechanism that is often articulated assumes that individuals exhibit homophily in these unobserved characteristics, making it more likely that individuals with similar characteristic values form links. If these characteristics are also correlated with the outcomes, researchers will find that individuals who are connected have correlated outcomes even without true peer effects. We examine evidence for the presence of endogeneity by developing a specific model for network formation that can incorporate this specific omitted-variable form of endogeneity.

Second, the Manski linear-in-means model assumes that being a peer is a zero/one property of a relationship; individuals are either peers (are linked) or not, and we observe this without error. This embodies the restriction that peers are exchangeable, so that the effect of one peer is identical to that of another peer, as well as the restriction that individuals who are not peers have no direct influence over each other. These assumptions are restrictive, especially in settings where the peer groups are self-reported and the result of choices by the individuals, and where they do not form a partition of the population. For example, the self-reporting may lead to measurement error in the links, or links may be of different strengths. We explore the implications of these restrictions and we find that outcomes for individuals who are not reported to be in one's peer group may still have direct effects on an individual's outcome through a generalization of the linear-in-means model. Moreover, within peer groups, some individuals appear to have stronger influences than others.

Finally, the linear-in-means model implies that while individuals who are not part of an individual's peer group have no direct effect on that individual's outcomes, there are indirect effects through common friends. This ultimately implies that there are positive correlations in outcomes for individuals with no direct connections, as long as they are connected through chains of friends, with the magnitude of these correlations decreasing toward zero as a function of the distance between individuals.

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We investigate whether there is evidence on the correlations between outcomes for individuals only indirectly connected, as implied by the linear-in-means model. This issue can partly be understood in both the context of networks and peer effects, and within the time series and spatial statistics literature. In the spatial statistics literature, observations on units are located in some space, with a distance associated with each pair of units. In the peer-effects literature, the distance between pairs of units can be thought of as the number of links one needs to travel to find a connection between two units. In the spatial and time series literatures, researchers have used both autoregressive type models, where correlations decrease at an exponential rate with the distance without completely disappearing, and moving-average type models where the correlations vanish at some finite distance. The traditional linear-in-means model fits in with the autoregressive structure. We explore moving-average type correlation structures and compare them to the autoregressive linear-in-means model using the Add Health data.

There are other aspects of the linear-in-means model we do not consider. Important in practice is the presence of missing data on links and nodes (see Chandrasekhar 2012, for a recent discussion).

In our empirical analyses, we use a Bayesian approach for estimation. We take this approach for three reasons. The first reason is a principled one. Bayesian methods have well-known optimality properties. In many simple settings, the inferences derived from a Bayesian approach are similar to those based on repeated sampling (frequentist) approaches (through the Bernstein-Von Mises Theorem), but that is not necessarily the case here, and so one has to make a more principled choice. A second argument is that there are no general frequentist results for the properties of maximum likelihood estimators in the type of models we consider. Finally, the motivation is one of computational convenience. Computing maximum likelihood estimates in the models we consider is challenging, and in many cases not feasible.

2. FRIENDSHIP NETWORKS AND HIGH-SCHOOL GRADES

To illustrate the methods and models discussed in this article, we use data from the Add Health survey. Here we use information on $N = 534$ individuals from a single high school, having dropped the 21 individuals who reported no friends within this school. We observe for each of these 534 students their grade-point average at two points in time, as well as their friendship links at the same two points in time. Other covariates are available, but to focus on conceptual issues we restrict the analyses to a single covariate, initial grade-point average. Friendships are interpreted in our discussion as symmetric relationships: the pair of students i and j are coded as friends if either i lists j as a friend, or j lists i as a friend, or both. Table 1 presents some summary statistics for this sample. On average, students have 6.0 friends in their school in the first wave, and 5.2 in the second.

Table 1. Summary statistics for Add Health sample ($N = 534$)

	Average	Standard deviation	Min.	Max.
GPA_1 (Y_i)	2.5	(1.0)	0.0	4.0
GPA_0 (X_i)	2.6	(0.8)	0.0	4.0
Number of friends, Wave 1	6.0	(3.2)	0	19
Number of friends, Wave 2	5.2	(3.0)	1	18

Friendship patterns change substantially over time. Table 2 shows the frequency of friendship patterns over time. Out of the 1604 pairs of students who reported to be friends in the initial period, only 662 (41%) reported to still be friends in the second period. Interestingly, friendships do not appear to be more stable if we define them based on both individuals claiming the other as a friend. In that case, 171 out of 434 (39%) of the friendships are maintained through the second period (see Table 3).

Table 2. Dynamic friendship patterns: all friendships

		Wave 1			
		Not friends		Friends	
Wave 2	Not friends	139,987	(98.37%)	942	(0.66%)
	Friends	720	(0.51%)	662	(0.47%)

Table 4 gives the distribution of the degree of separation. With 534 students, there are 142,311 pairs of students. Out of this set, there are 1382 pairs of friends (0.97%), and 5856 (4.11%) pairs of students who are not friends but who do have friends in common. For 8892 (6.25%) pairs of students, there are no links that connect them.

Table 3. Dynamic friendship patterns: two-way friendships

		Wave 1			
		Not friends		Friends	
Wave 2	Not friends	141,666	(99.55%)	263	(0.18%)
	Friends	211	(0.15%)	171	(0.12%)

3. SETUP AND NOTATION

In this section, we discuss the general setup and introduce the notation. We also outline the general type of identification questions we address.

3.1 Notation

The outcome of interest is denoted by Y_i. In our application, this is the grade-point average (gpa) for a student in the second period. The N-component vector of outcomes is denoted by $\mathbf{Y}$ with typical element Y_i. For each individual we also observe a $K-$vector of exogenous covariates, X_i, with $\mathbf{X}$ the $N \times K$ matrix with ith row equal to X_i'. In our application X_i is a scalar, initial grade-point average. More generally one could also include fixed characteristics such as sex, ethnicity, and descriptions of initial choices such as out-of-school activities. The network in the second (final) period is captured by the symmetric adjacency matrix $\mathbf{D}$, with typical element D_{ij} equal to one if i and j are friends and zero otherwise. The links are not directed, so $D_{ij} = D_{ji}$. The diagonal elements of $\mathbf{D}$ are zero. We

Table 4. Distribution of degree of separation (number of pairs 142,311)

Degree of separation	Number of pairs
1	1374
2	5215
3	15,573
4	32,019
5	39,310
6	25,852
7	10,283
8	2696
9	487
10	70
11	7
12	0
∞	8892

also observe the network in the previous period, with adjacency matrix $\mathbf{D}_0$ and (i, j)th element equal to $D_{0,ij}$.

For individual i the number of friends in the second period is $M_i = \sum_{j=1}^{N} D_{ij}$, with $\mathbf{M}$ the N-component vector with ith element equal to M_i. We have dropped from the original sample all individuals with no friends within the school, so that $M_i > 0$ for all $i = 1, \dots, N$. It is also convenient to have a notation for the row-normalized adjacency matrix:

$$\mathbf{G} = \mathrm{diag}(\mathbf{M})^{-1}\mathbf{D}, \qquad \text{with } G_{ij} = \begin{cases} D_{ij}/M_i & \text{if } M_i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that although $\mathbf{D}$ is symmetric, the row-normalized $\mathbf{G}$ is not in general symmetric. Finally, define $F_{0,ij}$ to be the indicator for the event that students i and j have one or more friends in common in the initial period:

$$F_{0,ij} = \begin{cases} 1 & \text{if } \sum_{k=1}^{N} D_{0,ik} \cdot D_{0,jk} > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and let $\mathbf{F}_0$ be the corresponding matrix with (i, j)th element equal to $F_{0,ij}$.

An important special case is when the peer groups partition the sample. Manski (1993) studied this case (see Sacerdote 2001, for a very interesting application). In this case, it is useful to have some additional notation.

Definition 1. If, for all triples (i, j, k), with $i \neq j$, $j \neq k$, and $i \neq k$,

$$D_{ij} = D_{jk} = 1 \implies D_{ik} = 1,$$

then the network forms a partition of the sample.

If the network forms a partition of the sample, we can characterize the network through a vector of group or cluster indicators. Specifically, we can define an N-vector $\mathbf{C}$ with ith element $C_i \in \{1, 2, \dots\}$, indexed by the label for the unit, such that for all $i \neq j$

$$C_i = C_j \iff D_{ij} = 1.$$

As pointed out by Bramoullé, Djebbari, and Fortin (2009), in the case of a network that partitions the sample, the row-normalized adjacency matrix $\mathbf{G}$ satisfies

$$\mathbf{GG} = \mathbf{G}.$$

3.2 Identification

We observe the vector of outcomes $\mathbf{Y}$, the covariate matrix $\mathbf{X}$, and the networks $\mathbf{D}$ and $\mathbf{D}_0$ for a sample of size N. We assume these can be viewed as a realization of a draw from a joint distribution $f(\mathbf{Y}, \mathbf{X}, \mathbf{D}, \mathbf{D}_0)$. For some of the identification questions we explore, we assume that we know this joint distribution. This is not as straightforward as it is in other settings where identification questions are studied. In many settings it is assumed that unit-level responses can be viewed as independent and identically distributed random variables, so that in a large sample one can precisely estimate the joint distribution of these variables, and thus is assumed to be known for identification purpose. Then we can simply refer to large sample arguments to argue that it is reasonable to focus on the case where the joint distribution of the variables is known (e.g., Wooldridge 2002). Here this is not the case because outcomes for different units may be correlated. In special cases, such as the Manski setting where the network partitions the sample, we can still appeal to such arguments because there is independence between clusters or groups. For the general case, the issue is more complicated. Without giving formal arguments here, the motivation for focusing on identification questions that take the joint distribution of $(\mathbf{Y}, \mathbf{X}, \mathbf{D}, \mathbf{D}_0)$ as given can take two different forms.

The simplest case is observations on a large number of exchangeable networks. In that case, the argument is as before. We can estimate the joint distribution of $(\mathbf{Y}, \mathbf{X}, \mathbf{D}, \mathbf{D}_0)$ from a sufficiently large sample of networks with arbitrary precision, and for identification purposes, it is reasonable to proceed by assuming this joint distribution is known. Outside of the partitioned case, it is difficult, however, to envision cases where we observe a large number of independent networks. In the current case that would entail observing full networks on a large number of schools, including characteristics of these schools that would make them exchangeable.

The more complicated case is data from a single, possibly connected, network. We need to impose some structure that limits the dependencies between observations, so that in a large sample we can estimate the joint distribution of all variables. Formal conditions for this have not been worked out for general cases. One approach taken by Goldsmith-Pinkham and Imbens (2012) was to use mixing conditions similar to those used in spatial statistics and time series analyses. The idea was to find conditions such that one can construct subsamples of increasing size that are approximately independent. More specifically, suppose there is a covariate X_i such that if X_i and X_j are far apart, then the probability of a link between individuals i and j is very small. This could arise if the individuals exhibit homophily in this variable X_i, and the support of this covariate is large. Then, if we think of large samples where the distribution of X_i is indexed by i, with the location of that distribution increasing in i, it may be possible, under certain conditions, to view blocks of observations as essentially independent. That in turn will allow us to view the data as similar to data based on samples of networks rather than as a single network. This is similar in spirit to domain-increasing asymptotics in time series analyses, where large sample arguments are based on increasing the sample size by adding observations further and further away in time, as opposed to infill asymptotics where the large sample arguments are based on increasing the number of observations within the time range of the current sample. The same issues arise in spatial statistics, where this choice between infill and domain-increasing asymptotics is also important (see, e.g., Schabenberger and Gotway 2004; Gelfand et al. 2010). Formal conditions that lead to consistency for estimators in such settings have not been established, although in many cases researchers proceed as if consistency and asymptotic normality holds (see Kolaczyk 2009; Kline 2011; Goldsmith-Pinkham and Imbens 2012, for more details).

4. THE MANSKI LINEAR-IN-MEANS MODEL FOR PEER EFFECTS UNDER EXOGENEITY OF THE NETWORK

In this section, we discuss the linear-in-means model under the assumption that the network is exogenous.

4.1 The MLIM Model

The starting point is a linear-in-means model of the type studied in depth by Manski (1993), which we will hereafter refer to as the Manski Linear-In-Means (MLIM) model. We focus initially on the specification

$$Y_i = \beta_0 + \beta_x X_i + \beta_{\overline{y}} \overline{Y}_{(i)} + \beta_{\overline{x}} \overline{X}_{(i)} + \eta_i, \tag{4.1}$$

also used by Bramoullé, Djebbari, and Fortin (2009) who allowed for more general network structures (see also Manski 2000; Brock and Durlauf 2001; Blume et al. 2011). Here the covariates are defined as averages over the peer groups, excluding the own outcome or covariate:

$$\overline{Y}_{(i)} = \frac{1}{M_i} \sum_{j=1}^{N} D_{ij} Y_j = \sum_{j=1}^{N} G_{ij} Y_j,$$

$$\overline{X}_{(i)} = \frac{1}{M_i} \sum_{j=1}^{N} D_{ij} X_j = \sum_{j=1}^{N} G_{ij} X_j.$$

The main object of interest is the effect of peer's outcomes on own outcomes, $\beta_{\overline{y}}$, the endogenous peer effect in Manski's terminology. Also of interest is the exogenous peer effect $\beta_{\overline{x}}$. Here we interpret the endogenous effect as the average change we would see in an individual's outcome if we changed their peer's outcomes directly (see the discussion by Kline and Tamer for an interesting interpretation of the coefficients as the parameters of a utility function in a game-theoretical setup). In some cases, this may be difficult to envision. In our example, we will think of this as something along the lines of the direct causal effect of providing special tutoring to one's peers on one's own outcome (see also Manski 2013). The exogenous effect is interpreted as the causal effect of changing the peer's covariate values. For some covariates, this thought experiment may be difficult, but for others, especially lagged values of choices, it may be feasible to consider interventions that would change those values for the peers.

It is useful to write the MLIM model in matrix notation. First, using the definition of the row-normalized adjacency matrix $\mathbf{G}$, we have

$$\mathbf{GY} = \begin{pmatrix} \overline{Y}_{(1)} \\ \vdots \\ \overline{Y}_{(N)} \end{pmatrix}, \qquad \mathbf{GX} = \begin{pmatrix} \overline{X}_{(1)} \\ \vdots \\ \overline{X}_{(N)} \end{pmatrix},$$

so that Equation (4.1) can be written in matrix form as

$$\mathbf{Y} = \beta_0 \iota_N + \beta_x \mathbf{X} + \beta_{\overline{y}} \mathbf{GY} + \beta_{\overline{x}} \mathbf{GX} + \eta. \tag{4.2}$$

4.2 Identification Under Exogeneity

Initially we assume that η are independent of the exogenous covariates and the peer groups:

Assumption 1. (Exogeneity)

$$\eta \perp \mathbf{X}, \mathbf{D}.$$

Because $\mathbf{G}$ is a deterministic function of $\mathbf{D}$, it follows that also $\eta \perp (\mathbf{X}, \mathbf{D}, \mathbf{G})$. We also assume normality:

Assumption 2. (Normality)

$$\eta | \mathbf{X}, \mathbf{D} \sim \mathcal{N}(0, \sigma^2 I_N).$$

The normality assumption is mainly for convenience and will be used for estimation and inference. It can be relaxed for identification questions. The independence assumption, Assumption 1, is critical for identification. Manski (1993) raised the concern that the residual is correlated with the network. In his setting, where peer groups partition the sample, one can define a group indicator C_i associated with each individual, so that having a link between i and j, or $D_{ij} = 1$, is equivalent to the condition that $C_i = C_j$. Manski formulated his concern with the independence assumption as coming from potential dependence of $\mathbb{E}[\eta_i | C_i = c]$ on c. In the next section, we return to this issue, where in a more general context we will view this as a concern with the exogeneity of the network.

Although under Assumption 1 the residual η is independent of $\mathbf{G}$ and $\mathbf{X}$, it is *not* independent or uncorrelated with $\mathbf{Y}$ and therefore not independent of $\mathbf{GY}$. Hence, we cannot simply regress $\mathbf{Y}$ on a constant, $\mathbf{X}$, $\mathbf{GY}$, and $\mathbf{GX}$ to get unbiased estimators for β. Manski (1993) and Bramoullé, Djebbari, and Fortin (2009) studied identification in essentially this setting. Manski focuses on the case where the peer groups partition the population, and as a result, $\mathbf{GG} = \mathbf{G}$, which creates a particular set of identification problems. Bramoullé, Djebbari, and Fortin (2009) focused on the identifying power of peer groups that do not partition the population, and in particular on the assumption that $\mathbf{GG} \neq \mathbf{G}$.

To study identification of the MLIM model in the Bramoullé, Djebbari, and Fortin (2009) case, it is useful to look at the conditional distribution of $\mathbf{Y}$ given $\mathbf{X}$ and $\mathbf{D}$. We can rewrite Equation (4.2) as

$$\mathbf{Y} = (\mathbf{I} - \beta_{\overline{y}} \mathbf{G})^{-1} \beta_0 \iota_N + (\mathbf{I} - \beta_{\overline{y}} \mathbf{G})^{-1} (\beta_x + \beta_{\overline{x}} \mathbf{G}) \mathbf{X} + (\mathbf{I} - \beta_{\overline{y}} \mathbf{G})^{-1} \eta,$$

where ι_N is the N-vector with all elements equal to one. Now, under the normality assumption on η, it follows that

$$\mathbf{Y} | \mathbf{X}, \mathbf{G} \sim \mathcal{N}(\mu_Y, \Sigma_Y),$$

where

$$\mu_Y = (\mathbf{I} - \beta_{\overline{y}}\mathbf{G})^{-1}\beta_0\iota_N + (\mathbf{I} - \beta_{\overline{y}}\mathbf{G})^{-1}(\beta_x + \beta_{\overline{x}}\mathbf{G})\mathbf{X},$$

and

$$\Sigma_Y = \sigma^2(\mathbf{I} - \beta_{\overline{y}}\mathbf{G})^{-1}(\mathbf{I} - \beta_{\overline{y}}\mathbf{G}')^{-1}.$$

Manski (1993) showed that identification of $\beta_{\overline{y}}$ is difficult in settings where the the peer groups partition the population. Under conditions described by Bramoullé, Djebbari, and Fortin (2009), we can identify the parameters β_0, β_x, $\beta_{\overline{x}}$, and $\beta_{\overline{y}}$ from the conditional distribution of $\mathbf{Y}$ given $\mathbf{X}$ and $\mathbf{G}$ even without normality. A key condition is that the network $\mathbf{D}$ does not partition the population, and therefore $\mathbf{GG} \neq \mathbf{G}$. In that case, at least for some individuals, friends' friends are not their friends. Alternatively, identification can be achieved by having different-sized peer groups (on this issue, see also Graham 2008).

4.3 Estimation and Inference in the MLIM Model

We use Bayesian methods to estimate the models. There are two main reasons. First, the posterior distributions given the model and given the prior distributions have clear interpretations. In contrast, maximum likelihood estimates are difficult to interpret. There is no well-developed theory for the properties of maximum likelihood estimates, even in large samples. Although there is evidence that the logarithm of the likelihood is approximately quadratic around its maximum, there have been no formal properties established for these estimators and for confidence intervals based on maximum likelihood estimators and the information matrix. Second, for some of the models we consider, maximum likelihood estimators are difficult to calculate. In contrast, obtaining draws from the posterior distribution is relatively straightforward, although computationally intensive in many cases. These advantages of Bayesian methods have been noted before, and many researchers use Bayesian computational methods, often suggesting that classical repeated sampling (frequentist) interpretations of the resulting estimators may be appropriate, without formal justification. Here we follow a fully Bayesian approach and focus on the posterior distributions.

First, we estimate the model assuming exogeneity of the network. We use normal independent prior distributions for the parameters β_0, β_x, $\beta_{\overline{y}}$, and $\beta_{\overline{x}}$, and an inverse chi-squared distribution for σ^2. Some summary statistics for posterior distributions are reported in the first panel of Table 5. The model for the outcomes exhibits evidence of substantial peer effects. The posterior mean for the endogenous peer effect is 0.16, and the posterior mean for the exogenous effect is 0.11. The posterior standard deviations are 0.05 and 0.07, respectively. Having current friends with good past academic performance, or friends with good current academic performance is associated with better performance for the individual.

Table 5. Summary statistics for posterior distribution: exogenous network

Outcome equation			Network model		
	Posterior			Posterior	
	Mean	Stand dev.		Mean	Stand dev.
β_0	−0.13	(0.12)	α_0	−2.56	(0.04)
β_x	0.74	(0.04)	α_x	−0.20	(0.03)
$\beta_{\overline{y}}$	0.16	(0.05)	α_d	2.52	(0.05)
$\beta_{\overline{x}}$	0.11	(0.07)	α_f	1.20	(0.04)

5. AN EXOGENOUS NETWORK FORMATION MODEL

To assess potential endogeneity of the network, it is useful to combine the model for peer effects with a formal model for network formation.

5.1 Indirect Peer Effects

In the setting where the peer groups are nonoverlapping, a key issue is how the networks are formed, and how they might have been different. As a result, we may need to worry about peer effects not only through the direct effect of the peer groups' characteristics and outcomes on an individual's outcomes, but also on the effects of outcomes and covariates on the peer group itself.

Currently, $\beta_{\overline{x}}$ measures the effect of changing the average characteristics of one's peer group on an individual's outcome, keeping fixed the peer group itself. It is possible that changing these covariates may change the peer group, even under the exogeneity condition $\eta \perp (\mathbf{D}, \mathbf{X})$. Suppose that X_i is the grade-point average of student i at the beginning of the period. Now suppose that we could have changed this by giving some of the students special tutoring. The effect of such a change would be to raise X_j for the affected students and would affect their peers' gpa in the final period. That in turn would affect their friends' friends, and so on. However, a more subtle effect is that the tutoring may also affect whom the tutored students form friendships with as a result of preferential network formation. To find the total effect of the change in the covariates, we need to model the effect of the covariates on the outcome as well as the effect of the covariates on the network itself (see Carrell, Sacerdote, and West 2012, for a fascinating study of the effect of an intervention on peer effects and the possibility of indirect peer effects through changes in the network).

5.2 Strategic Network Formation Models

We start by modeling the network formation process. The first assumption we make is that the decision to form a link is the result of two choices. Both individuals need to agree to form the link, and will do so if they view the net utility from the link as positive. Formally,

$$D_{ij} = \mathbf{1}_{U_i(j)>0} \cdot \mathbf{1}_{U_j(i)>0},$$

where $U_i(j)$ is the utility for individual i of forming a link with individual j. Following Jackson (2008), we refer to this type of model as a Strategic Network Formation Model (see also Jackson 2003, 2006, 2011). In the sociological literature, these models are also referred to as Network Evolution Models (Toivonen et al. 2009), or Actor Based Models (Snijders 2001; Snijders, Koskinen, and Schweinberger 2010). The specific models studied in the literature differ by the utility that the agents associate with links, given the characteristics of the other agents and the current state of the network, as well as the opportunities the agents have for establishing or changing the status of their links (for theoretical discussions, see Jackson and Wolinsky 1996; Jackson and Rogers 2007; for recent empirical examples, see Calvó-Armengol and Jackson 2004; Christakis et al. 2010; Currarini, Jackson, and Pin 2010; De Melo 2011).

5.3 A Dynamic Network Formation Model

In this discussion, we focus on a particular version of a strategic network formation model, exploiting the presence of data on the network at two points in time. At the same point in time, each pair of agents evaluates the utility of a link between them. This utility depends on the characteristics of the two agents, and on the status of the network at the beginning of the period. Unlike, for example, Christakis et al. (2010), the utility of a link does not depend on the decisions of the other pairs of agents. Although this is a strong assumption, because the decision to form a link is based on the network in the previous period, the model can accommodate important geometric features of the network. Conditional on the network at the beginning of the period, $\mathbf{D}_0$, the utility for i of forming a link with j depends on the distance between the units in the covariate space, $|X_i - X_j|$, on whether the two were friends in the previous period, $D_{0,ij}$, and whether they had friends in common in the previous period, $F_{0,ij}$:

$$U_i(j) = \alpha_0 + \alpha_x |X_i - X_j| + \alpha_d D_{0ij} + \alpha_f F_{0,ij} + \epsilon_{ij}. \quad (5.1)$$

Initially, let us assume that ϵ_{ij} are independent across all i and j (and in particular that ϵ_{ij} is independent of ϵ_{ji}) and that ϵ_{ij} have a logistic distribution. The covariates enter in a specific way, reflecting homophily: the utility of a friendship goes down with the distance in covariate space. This implies that the probability of a link between i and j, given the previous version of the network, and given the covariates, is

$$\mathrm{pr}(D_{ij} = 1 | \mathbf{D}_0, \mathbf{X}) = p_{ij} \cdot p_{ji},$$

where the probability that i values the link with j positively is the same as the probability that j values the link with i positively:

$$p_{ij} = p_{ji} = \frac{\exp(\alpha_0 + \alpha_x |X_i - X_j| + \alpha_d D_{0ij} + \alpha_f F_{0,ij})}{1 + \exp(\alpha_0 + \alpha_x |X_i - X_j| + \alpha_d D_{0ij} + \alpha_f F_{0,ij})}.$$

More general models are possible here. Such models need not have the implication that $p_{ij} = p_{ji}$. For example, the utility associated with a friendship link may depend on the number of friends the potential friend had in the initial period, or on the level of the attribute X_j for the potential friend rather than solely on the difference $|X_i - X_j|$.

5.4 Estimation and Inference for a Dynamic Network Formation Model

Again we use Bayesian methods for inference. The prior distributions for α_0, α_x, α_d, and α_f are independent normals. The normal prior distributions are centered at zero with prior standard deviation equal to one. Results are not sensitive to these assumptions about the prior distributions. Some summary statistics for posterior distributions are reported in the second panel of Table 5. The network model suggests that there is substantial sorting on academic performance, with the utility of a friendship link decreasing in the difference in past academic performance. The utility increases with the presence of past friendships, and with having friends in common in the past.

6. AN ENDOGENOUS NETWORK FORMATION MODEL

A major concern with a causal interpretation of estimates of the parameters in Equation (4.1) is that the peer groups themselves may be endogenous. Correlations in outcomes between peers need not be the (causal) effect of peers. Instead, because peers are partly the result of individual choices, these correlations may reflect prior similarities between individuals (what Manski, 1993, called correlated effects). More specifically, individuals who are peers may be similar in terms of unobserved characteristics that also affect the outcomes, generating correlation in outcomes between peers as a result. Conceptually, this is different from the endogenous peer effect in Manski's terminology that stems from a simultaneity problem. Rather, it is a correlated effect arising from omitted variable bias. In econometrics, such omitted variable problems are traditionally also referred to as endogeneity problems (e.g., the ability bias in regression estimates of the returns to education).

6.1 Endogeneous Network Formation

As described before, when peer groups partition the sample, we can assign each individual a cluster or group indicator C_i, so that $D_{ij} = 1$ if $C_i = C_j$. In this setting, we can conceptualize the correlated effect (as defined by Manski) as $\mathbb{E}[\eta_i | C_i = c] = \delta_c$, with δ_c varying with the group c. In the case we study here, with the peer groups varying by individual, we use a different notion. What we wish to capture is that the N-vector of unobserved components of the outcome, η, is not independent of the $N \times N$ adjacency matrix $\mathbf{D}$ and the matrix of covariates $\mathbf{X}$. We model this potential dependence through the presence of unobserved individual characteristics. Let ξ_i be a individual-specific unobserved component that enters the outcome equation. We generalize the outcome equation to

$$Y_i = \beta_0 + \beta_x X_i + \beta_{\overline{y}} \overline{Y}_{(i)} + \beta_{\overline{x}} \overline{X}_{(i)} + \beta_\xi \xi_i + \eta_i. \quad (6.1)$$

Note that we do not allow for a peer effect of the unobserved covariate. Doing so would complicate the analyses.

We also modify the network formation process by generalizing the network formation model. The utility associated with a link between individuals i and j now depends also on the distance between these two individuals in terms of the unobserved characteristic ξ_i:

$$U_i(j) = \alpha_0 + \alpha_x |X_i - X_j| + \alpha_\xi |\xi_i - \xi_j| + \alpha_d D_{0ij} + \alpha_f F_{0,ij} + \epsilon_{ij}. \quad (6.2)$$

Individuals with similar values for ξ_i are more likely to form links (if α_ξ is negative), and if β_ξ differs from zero, this covariate has a direct effect on the outcome.

The special case that Manski studies would correspond to the unobserved individual characteristic ξ_i being equal to the group indicator, or $\xi_i = C_i$, combined with α_ξ being negative and very large in absolute value, so that all units in the same cluster are linked.

6.2 Testable Implications of Endogeneity of the Network Formation: Heterogeneity of Peer Effects

Before attempting to estimate models that allow for endogeneity of the network, we wish to assess whether exogeneity is testable given knowledge of the joint distribution of $\mathbf{X}$, $\mathbf{D}$, $\mathbf{D}_0$, and $\mathbf{Y}$. We proceed using the fact that endogeneity, within the model with exchangeability of peers, has some testable implications, and thus there is motivation for estimating models with network endogeneity. This does not answer the question of what models with endogenous networks are identified, but implies that exogeneity itself is testable. Given the model without endogeneity, Equations (4.1) and (5.1), suppose we have the parameter values for β and α. Then we can calculate for the pair of individuals i and j the probability of being friends,

$$q_{ij} = \text{pr}(U_i(j) > 0, U_j(i) > 0|\mathbf{X}) = p_{ij} \cdot p_{ji},$$

where

$$p_{ij} = \frac{\exp(\alpha_0 + \alpha_x|X_i - X_j| + \alpha_d D_{0ij} + \alpha_f F_{0,ij})}{1 + \exp(\alpha_0 + \alpha_x|X_i - X_j| + \alpha_d D_{0ij} + \alpha_f F_{0,ij})}.$$

We can also calculate the residuals η_i from the outcome equation:

$$\eta_i = Y_i - \beta_0 - \beta_x X_i - \beta_{\overline{y}}\overline{Y}_{(i)} - \beta_{\overline{x}}\overline{X}_{(i)}.$$

Under the exogeneity assumption, it follows that

$$\eta_i \perp q_{ij}.$$

However, if ξ_i are nondegenerate and both β_ξ and α_ξ differ from zero, this no longer holds. For example, if $\alpha_\xi < 0$ (homophily in the unobserved characteristic) and $\beta_\xi > 0$ (the outcome increases in the unobserved characteristic), then the absolute value of the difference in residuals, $|\eta_i - \eta_j|$, is, in expectation, increasing in the ex ante probability of the link. Formally,

$$\mathbb{E}\left[|\eta_i - \eta_j| \,\middle|\, q_{ij} = q, D_{ij} = d\right]$$

is increasing in q for $d = 0, 1$.

The intuition goes as follows. Among pairs (i, j) for whom $D_{ij} = 1$ (i.e., friend pairs), a low value of q_{ij} implies that $|\xi_i - \xi_j|$ must be relatively close to zero. As a result, the absolute value of the difference $|\eta_i - \eta_j|$ is expected to be relatively low. High values of q_{ij} do not contain information regarding $|\xi_i - \xi_j|$, and thus do not predict the value of $|\eta_i - \eta_j|$. A similar argument goes through for nonfriend pairs. If q_{ij} is high with observed $D_{ij} = 0$, one would expect that $|\xi_i - \xi_j|$ is relatively large, and hence $|\eta_i - \eta_j|$ is relatively large.

We can also look at this directly in terms of covariates.

$$\mathbb{E}\left[|\xi_i - \xi_j| \,\middle|\, |X_i - X_j| = x, D_{ij} = 1\right]$$

is decreasing in x. Hence, under endogeneity,

$$\mathbb{E}\left[|\eta_i - \eta_j| \,\middle|\, |X_i - X_j| = x, D_{ij} = 1\right]$$

is decreasing in x. As a result, we can look among friend pairs at the correlation between the absolute value of the difference in the covariates, x, and compare that to the absolute value of the difference in residuals $|\eta_i - \eta_j|$, in the outcome equations.

One concern with this comparison is that it relies on equal influence among peers. If peers have differential effects on an individual depending on their similarity, one might find that the correlation in residuals η_i is higher for friends with similar characteristics. Although this is not captured in the MLIM model, one could generalize the model by modifying the weights in the row-normalized adjacency matrix $\mathbf{G}$. For example, one might model

$$G_{ij} = \frac{D_{ij} \cdot |X_i - X_j|}{\sum_{k \neq i} D_{ik} \cdot |X_i - X_k|}.$$

A second approach to assessing evidence of exogeneity does not rely on the equality of peer effects among peers. Instead, we can look at correlations in outcomes for nonfriends. Within this set of links, we compare the correlation between the absolute value of the difference in residuals η_i and η_j and the ex ante probability of a link. Pairs of individual who are not friends, but who had a high probability of a link, have, in expectation, a larger difference in the absolute value of the difference ξ_i and ξ_j, and thus a larger expected value for the difference in absolute values of the residuals η_i and η_j. Here the key assumption is that, under the null model, there is no correlation in residuals for pairs of individuals who are not friends.

6.3 An Empirical Network Formation Model With Endogeneity

We now turn to an estimable version of the model with network endogeneity. We use the model in Equations (6.1) and (6.2), with an unobserved scalar component that appears in the outcome equation and the utility associated with links:

$$\begin{aligned} Y_i &= \beta_0 + \beta_x X_i + \beta_{\overline{y}}\overline{Y}_{(i)} + \beta_{\overline{x}}\overline{X}_{(i)} + \beta_\xi \xi_i + \eta_i, \\ U_i(j) &= \alpha_0 + \alpha_x|X_i - X_j| + \alpha_\xi|\xi_i - \xi_j| + \alpha_d D_{0,ij} \\ &\quad + \alpha_f F_{0,ij} + \epsilon_{ij}. \end{aligned}$$

We make the distributional assumptions

$$\eta|\xi, \mathbf{X}, \mathbf{D} \sim \mathcal{N}(0, \sigma^2 I_N),$$

and a logistic distribution for ϵ_{ij}, independent of η_i and ξ_i, and with all ϵ_{ij} independent. Finally, we assume that the unobserved type ξ_i is binary:

$$\text{pr}(\xi_i = 1|\mathbf{X}, \mathbf{D}_0) = 1 - \text{pr}(\xi_i = 0|\mathbf{X}, \mathbf{D}_0) = p.$$

In the empirical analysis we fix $p = 1/2$. This leads to a parametric distribution for $(\mathbf{Y}, \mathbf{G})$ given $(\mathbf{X}, \mathbf{D}_0)$. First, define the probability of a link conditional on observed and unobserved covariates:

$$\begin{aligned} &p(x_1, x_2, d_0, f_0, \xi_1, \xi_2; \alpha_0, \alpha_x, \alpha_\xi, \alpha_d, \alpha_f) \\ &\quad = \frac{\exp(\alpha_0 + \alpha_x|x_1 - x_2| + \alpha_\xi|\xi_1 - \xi_2| + \alpha_d d_0 + \alpha_f f_0)}{1 + \exp(\alpha_0 + \alpha_x|x_1 - x_2| + \alpha_\xi|\xi_1 - \xi_2| + \alpha_d d_0 + \alpha_f f_0)}. \end{aligned}$$

Then, conditional on ξ, $\mathbf{X}$, and $\mathbf{D}_0$, we have the likelihood function for the network,

$$\mathcal{L}_{\text{network}}(\alpha|\mathbf{G};\xi,\mathbf{X},\mathbf{D}_0) = \prod_{i\neq j}(p(X_i,X_j,D_{0,ij},F_{0,ij},\xi_i,\xi_j;\alpha_0,\alpha_x,\alpha_\xi,\alpha_d,\alpha_f) \times p(X_j,X_i,D_{0,ji},F_{0,ji},\xi_j,\xi_i;\alpha_0,\alpha_x,\alpha_\xi,\alpha_d,\alpha_f))^{D_{ij}} \times (1-p(X_i,X_j,D_{0,ij},F_{0,ij},\xi_i,\xi_j;\alpha_0,\alpha_x,\alpha_\xi,\alpha_d,\alpha_f) \times p(X_j,X_i,D_{0,ji},F_{0,ji},\xi_j,\xi_i;\alpha_0,\alpha_x,\alpha_\xi,\alpha_d,\alpha_f))^{1-D_{ij}}.$$

The likelihood function for the outcome is

$$\mathcal{L}_{\text{outcome}}(\beta,\sigma^2|\mathbf{Y};\mathbf{D},\mathbf{X},\xi) = \frac{1}{(2\pi)^{N/2}|\Sigma_Y|}\exp\left(-(\mathbf{Y}-\mu_Y)\,\Sigma_Y^{-1}\,(\mathbf{Y}-\mu_Y)/2\right),$$

where

$$\Sigma_Y = \sigma^2(\mathbf{I}-\beta_{\overline{y}}\mathbf{G})^{-1}(\mathbf{I}-\beta_{\overline{y}}\mathbf{G}')^{-1},$$

and

$$\mu_Y = (\mathbf{I}-\beta_{\overline{y}}\mathbf{G})^{-1}\beta_0\iota_N + (\mathbf{I}-\beta_{\overline{y}}\mathbf{G})^{-1}(\beta_x+\beta_{\overline{x}}\mathbf{G}) + (\mathbf{I}-\beta_{\overline{y}}\mathbf{G})^{-1}\beta_\xi\xi.$$

Finally,

$$\mathcal{L}(\beta,\alpha,\sigma^2,p|\mathbf{Y},\mathbf{G},\xi;\mathbf{X},\mathbf{D}_0) = \mathcal{L}_{\text{outcome}}(\beta,\sigma^2|\mathbf{Y};\mathbf{D},\mathbf{X},\xi)\times\mathcal{L}_{\text{network}}(\alpha|\mathbf{G};\xi,\mathbf{X},\mathbf{D}_0),$$

is the conditional likelihood function. Finally, to get the likelihood function in terms of the observed data $(\mathbf{Y},\mathbf{G};\mathbf{X},\mathbf{D}_0)$, we need to integrate out ξ_i over the conditional distribution given the observed data: integrating out ξ_i we have

$$\mathcal{L}(\beta,\alpha,\sigma^2,p|\mathbf{Y},\mathbf{G};\mathbf{X},\mathbf{D}_0) = \sum_{\xi}\mathcal{L}(\beta,\alpha,\sigma^2,p|\mathbf{Y},\mathbf{G},\xi;\mathbf{X},\mathbf{D}_0) \times p(\xi|\mathbf{Y},\mathbf{G},\mathbf{X},\mathbf{D}_0;(\beta,\alpha,\sigma^2,p).$$

We could use alternative parametric distributions for the unobserved component ξ_i. Our choice is motivated by the flexibility of the distribution and the computational tractability of the resulting model.

6.4 Endogeneity in the Add Health Friendship Network

Maximum likelihood estimation of this likelihood is particularly computationally demanding because of the difficulty of integrating out the unobserved ξ_i. Moreover, there are no results for the repeated sampling properties of the maximum likelihood estimators for this type of model, even in large samples. We therefore again focus on Bayesian methods. We specify a prior distribution for the parameter $\theta = (\beta,\sigma^2,\alpha,p)$, and use MCMC methods to obtain draws from the posterior distribution of θ given the data $(\mathbf{D},\mathbf{D}_0,\mathbf{X},\mathbf{Y})$. This Bayesian approach allows us to treat ξ_i as unobserved random variables, and exploit the fact that with ξ_i known, we have exogeneity of the network and can obtain draws from the posterior distribution.

Table 6. Summary statistics for posterior distribution: endogenous network

Outcome equation			Network model		
	Posterior			Posterior	
	Mean	Stand dev.		Mean	Stand dev.
β_0	−0.10	(0.13)	α_0	−2.26	(0.04)
β_x	0.73	(0.04)	α_x	−0.21	(0.04)
$\beta_{\overline{y}}$	0.15	(0.05)	α_ξ	−1.06	(0.07)
$\beta_{\overline{x}}$	0.11	(0.06)	α_d	2.63	(0.03)
β_ξ	−0.01	(0.10)	α_f	1.22	(0.04)
σ^2	0.61	(0.02)			

The Appendix contains details of the MCMC algorithm for the specific case. We use Metropolis-Hastings steps separately for α, β, and ξ, and a Gibbs sampler for the conditional posterior for σ^2 given the other parameters and the data. The results are reported in the second part of Table 6. The unobserved component ξ matters substantially for the network estimation, with the coefficient on the difference in the unobserved characteristics large and precisely estimated. This appears to allow the formation model to fit the observed clustering of friendships better than the model without the unobserved component. However, the unobserved component does not appear to matter much for the outcome. Its coefficient is close to zero with the 95% posterior probability interval comfortably including zero. The posterior distribution for the endogenous peer effect is not much affected by this, with the posterior mean equal to 0.15, and the posterior standard deviation equal to 0.05, compared to 0.16 and 0.05 in the exogenous network model.

7. HETEROGENEITY IN PEER EFFECTS

The MLIM model assumes that the effect of all peers are equal, that nonpeers have no direct effects on outcomes, and that peer status is measured without error. In practice, none of these assumptions is likely to hold exactly. Friendships are not an all or nothing notion, and it is likely that there is measurement error. Here we discuss alternative assumptions, and assess whether the violations of the MLIM model assumptions are likely to affect the magnitude of the estimates of the peer effects in the context of the Add Health friendship network. We propose a particular technical generalization of the MLIM model involving multiple networks to assess these issues.

7.1 Multiple Networks

We study the implications of uncertainty in the definition or measurement of peers by estimating a generalization of the MLIM model where we consider the presence of two networks simultaneously, each with their associated peer effects. Let $\mathbf{D}_A$ denote the adjacency matrix for the first network, and $\mathbf{D}_B$ the adjacency matrix for the second network. Both are symmetric matrices with elements equal to zero or one, and zeros on the diagonal. Let the averages of outcomes and covariates be

$$\overline{Y}_{A,(i)} = \frac{1}{M_{A,i}} \sum_{j=1}^{N} D_{A,ij} Y_j, \qquad \overline{X}_{A,(i)} = \frac{1}{M_{A,i}} \sum_{j=1}^{N} D_{A,ij} X_j,$$

and similarly for $\overline{Y}_{B,(i)}$ and $\overline{X}_{B,(i)}$. Then we estimate a linear-in-means model where both networks have exogenous and endogenous peer effects:

$$Y_i = \beta_0 + \beta_x X_i + \beta_{\overline{y},A} \overline{Y}_{A,(i)} + \beta_{\overline{x},A} \overline{X}_{A,(i)} + \beta_{\overline{y},B} \overline{Y}_{B,(i)} + \beta_{\overline{x},B} \overline{X}_{B,(i)} + \eta_i. \tag{7.1}$$

We generalize Assumption 1 to assume that the residuals are independent of both networks:

Assumption 3. (Exogeneity)

$$\eta \perp \mathbf{X}, \mathbf{D}_A, \mathbf{D}_B.$$

We estimate three different versions of this model, corresponding to different definitions of the second network.

7.2 Measurement Error

There are many mechanisms through which the links may be mismeasured. We exploit the presence of observations on the network in an earlier period to assess the presence of measurement error. We use the multiple network setup from the previous section, where the first network measures the current state of the link, and the second network measures links that were present in the past, but not in the current network. Formally,

$$D_{A,ij} = D_{ij}, \qquad D_{B,ij} = (1 - D_{ij}) \cdot D_{0,ij}.$$

This model nests the previous model as the special case with $\beta_{\overline{x},B} = \beta_{\overline{y},B} = 0$. The idea is that if a link is present currently, it is likely that it was also present in the previous period. Therefore, if measurement error is random unconditionally, then conditional on no link being reported in the current period, it is more likely that an actual link is present if a link was reported in the previous period. Therefore, there should be correlations between outcomes for individuals who were linked in the past but who are not linked in the present. There are other models that would imply such correlations, but measurement error in the links is one interpretation of such correlations.

In the first two columns of Table 7, we report posterior means and standard deviations for this model under exogeneity of the network formation (β_0, β_x, $\beta_{\overline{y},A}$, $\beta_{\overline{x},A}$, $\beta_{\overline{y},B}$, $\beta_{\overline{x},B}$) using this definition of the network. We find that the peer effects of former friends are substantial (posterior mean 0.13, posterior standard deviation 0.05), almost the same as the peer effects for current friends (posterior mean 0.15, posterior standard deviation 0.05). This casts substantial doubt on the notion that relying on self-reported friendship links captures all the connections that matter for correlations in outcomes.

7.3 Effects of One-Way Friendships

A second approach to assessing heterogeneity of peer effects is by considering an alternative definition of friendships. We implement this by setting the first network to be that of one-way or mutual friends in the second period, and the second one to be the network where links are defined as corresponding to mutual friendships. Formally, define $\mathbf{D}_M$ to be the adjacency matrix with $D_{M,ij}$ an indicator that is equal to one if both i lists j as a friend *and* j lists i as a friend, so that $D_{M,ij} \leq D_{ij}$. Then we define:

$$D_{A,ij} = D_{ij}, \qquad D_{B,ij} = D_{M,ij}.$$

We report estimates for this model in the second pair of columns in Table 7. There appears to be little evidence that the definition of friendships based on a single side reporting the friendship is not adequate.

Table 7. Summary statistics for posterior distribution: exogenous network with second network effects

	Lagged links Posterior		Mutual links Posterior		Friends of friends Posterior	
	Mean	Stand dev.	Mean	Stand dev.	Mean	Stand dev.
β_0	−0.16	(0.13)	−0.11	(0.12)	−0.17	(0.16)
β_x	0.73	(0.04)	0.73	(0.04)	0.73	(0.04)
$\beta_{\overline{y},A}$	0.14	(0.06)	0.14	(0.06)	0.13	(0.06)
$\beta_{\overline{x},A}$	0.09	(0.07)	0.11	(0.08)	0.09	(0.07)
$\beta_{\overline{y},B}$	0.13	(0.05)	0.03	(0.05)	0.19	(0.10)
$\beta_{\overline{x},B}$	−0.07	(0.06)	0.00	(0.05)	−0.10	(0.12)
σ	0.63	(0.04)	0.62	(0.02)	0.66	(0.08)

7.4 Effects of Friends One-Degree Removed

Finally, we consider the two-network peer-effect model where we define the second network in terms of friends of friends. Formally, the second network has a link between i and j if i and j have at least one friend in common but are not friends themselves:

$$D_{A,ij} = D_{ij}, \qquad D_{B,ij} = F_{M,ij} \cdot (1 - D_{ij}).$$

In the third pair of two columns of Table 7, we report posterior means and standard deviations for this model. There is substantial evidence of direct effects of the outcomes of friends-of-friends on own outcomes, with posterior mean of $\beta_{\overline{y},B}$ of 0.19 and posterior standard deviation of 0.10. The posterior probability that the coefficient is negative is 0.03.

8. INDIRECT PEER EFFECTS: AUTOREGRESSIVE VERSUS MOVING-AVERAGE REPRESENTATIONS

The MLIM model has strong implications for the correlations in outcomes between individuals. Peer effects are mediated through friends' outcomes, but ultimately outcomes for nonfriends can still affect an individual's outcomes, albeit indirectly. In this section, we investigate some implications of this and interpret some features of the model in terms of the differences between autoregressive and moving-average representations in time series and spatial analyses. We show that the MLIM model has similarities to autoregressive models, and develop alternatives that are more like moving-average models with finite dependencies.

8.1 Autoregressive Models

Taking the matrix version of the linear-in-means model,

$$\mathbf{Y} = \beta_0 \iota_N + \beta_x \mathbf{X} + \beta_{\overline{y}} \mathbf{G}\mathbf{Y} + \beta_{\overline{x}} \mathbf{G}\mathbf{X} + \eta,$$

we can write the outcome in terms of the exogenous variables as

$$\mathbf{Y} = (\mathbf{I} - \beta_{\overline{y}}\mathbf{G})^{-1}\beta_0\iota_N + (\mathbf{I} - \beta_{\overline{y}}\mathbf{G})^{-1}(\beta_x + \beta_{\overline{x}}\mathbf{G})\mathbf{X} + (\mathbf{I} - \beta_{\overline{y}}\mathbf{G})^{-1}\eta.$$

The conditional expectation of $\mathbf{Y}$ given $\mathbf{D}$ and $\mathbf{X}$ is therefore under exogeneity (Assumption 1), equal to

$$\mathbb{E}[\mathbf{Y}|\mathbf{X}, \mathbf{G}] = (\mathbf{I} - \beta_{\overline{y}}\mathbf{G})^{-1}\beta_0\iota_N + (\mathbf{I} - \beta_{\overline{y}}\mathbf{G})^{-1}(\beta_x + \beta_{\overline{x}}\mathbf{G})\mathbf{X}.$$

Following the arguments by Bramoullé, Djebbari, and Fortin (2009), we can reparameterize this as

$$\mathbb{E}[\mathbf{Y}|\mathbf{X}, \mathbf{G}] = \gamma_0\iota + \gamma_x\mathbf{X} + \sum_{k=1}^{\infty}\gamma_k\mathbf{G}^k\mathbf{X},$$

where, for $k \geq 1$, $\gamma_k = (\beta_x\beta_{\overline{y}} + \beta_{\overline{x}})\beta_{\overline{y}}^{k-1}(k-1)!$. Here we estimate an approximation to this conditional expectation based on the first two terms:

$$\mathbb{E}[\mathbf{Y}|\mathbf{X}, \mathbf{G}] \approx \gamma_0\iota_N + \gamma_x\mathbf{X} + \gamma_1\mathbf{G}\mathbf{X} + \gamma_2\mathbf{G}\mathbf{G}\mathbf{X}.$$

The linear-in-means model suggest that the terms on the friends-in-common average covariates should be, ignoring higher-order terms,

$$\gamma_1 = \beta_x\beta_{\overline{y}} + \beta_{\overline{x}}, \qquad \text{and} \quad \gamma_2 = (\beta_x\beta_{\overline{y}} + \beta_{\overline{x}})\beta_{\overline{y}}.$$

Two implications emerge. First, if the own covariate effect β_x, and the exogenous and endogenous peer effects, $\beta_{\overline{x}}$ and $\beta_{\overline{y}}$, are all positive, the second coefficient in this expansion should be positive. Second, the ratio of the first and second coefficients in this expansion should equal $\beta_{\overline{y}}$.

Table 8 reports estimation results for this model. We find that the posterior 95% probability interval of the effect of the second-friends' average includes zero. The posterior variance is large, so in fact the data are consistent with wide range of values. As a result, we cannot conclusively establish that there is any indirect peer effect of friends of friends.

8.2 Moving-Average Models

The second attempt in establishing whether there are indirect effects for individuals who are not friends takes a different approach. It adds a moving-average type component to the linear-in-means model so that it allows for the possibility that there is no correlation between outcomes for individuals who are not friends, while allowing for correlations in outcomes between friends. The starting point is again the MLIM model

$$\mathbf{Y} = \beta_0 \iota_N + \beta_x \mathbf{X} + \beta_{\overline{y}} \mathbf{G}\mathbf{Y} + \beta_{\overline{x}} \mathbf{G}\mathbf{X} + \eta. \tag{8.1}$$

Now instead of modeling the unobserved component η_i as independent across units, we model it as having a clustering-type structure:

$$\eta_i = \nu_i + \sum_{j=1}^{N} D_{ij}\epsilon_{ij},$$

with the ϵ_{ij} independent across i and j, and independent of ν, $\mathbf{D}$, and $\mathbf{X}$, normally distributed with mean zero and variance σ_ϵ^2. The ν_i have a normal distribution with mean zero and variance σ_ν^2. This leads to the following covariance matrix for η:

$$\mathbb{E}[\eta\eta'|\mathbf{X}, \mathbf{D}] = \Omega,$$

$$\text{with} \quad \Omega_{ij} = \begin{cases} \sigma_\nu^2 + M_i \cdot \sigma_\epsilon^2, & \text{if } i = j, \\ \sigma_\epsilon^2 & \text{if } i \neq j, D_{ij} = 1, \\ 0 & \text{if } i \neq j, D_{ij} = 0. \end{cases}$$

In the Manski case, where the network partitions the sample,

$$\Omega_{ij} = \begin{cases} \sigma_\nu^2 + M_i \cdot \sigma_\epsilon^2, & \text{if } i = j, \\ \sigma_\epsilon^2 & \text{if } i \neq j, C_i = C_j, \\ 0 & \text{if } C_i \neq C_j. \end{cases}$$

This covariance structure is closely related to that generated by Manski's correlated effects, where each group has its own unobserved component. In model (8.1), if $\sigma_\epsilon^2 > 0$ and there are no peer effects, $\beta_{\overline{y}} = 0$, there would be no spillovers beyond friends, but there would be positive correlation between outcomes for friends.

Table 9 presents summary statistics for the posterior distribution for this model. The posterior mean for the endogenous peer effects, $\beta_{\overline{y}}$, goes down from 0.15 in the baseline model to 0.09 in the model with the additional flexibility in the error-covariance structure. The posterior standard deviation is 0.06, so there is now considerable uncertainty about the sign of this effect. The posterior mean for the variance component σ_ϵ^2 is 0.10^2, large enough to create a substantial correlation between outcomes for friends.

Both exercises carried out in this section show that the evidence for indirect peer effects, that is, effects not from friends but from friends-of-friends, is limited. One explanation may be

Table 8. Summary statistics for posterior distribution: exogenous network with no endogenous peer effects

	Posterior	
	Mean	Stand dev.
Intercept	−0.25	(0.18)
β_x	0.73	(0.04)
$\beta_{\overline{x}}$	0.21	(0.08)
$\beta_{\mathrm{FIC},\overline{x}}$	0.10	(0.11)

Table 9. Summary statistics for posterior distribution: exogenous network with no endogenous peer effects

	Posterior	
	Mean	Stand dev.
Intercept	−0.08	(0.12)
β_x	0.74	(0.04)
$\beta_{\overline{y}}$	0.09	(0.06)
$\beta_{\overline{x}}$	0.16	(0.07)
σ^2	0.58^2	
σ_ν^2	0.10^2	

that the indirect peer effects we find in these model are relatively modest. It may be the case that the direct peer effects are simply too small to generate substantial indirect effects, or that the indirect effects are not there for academic outcomes such as grades.

9. CONCLUSION

In this article, we explore extensions of the Manski-type linear-in-means model for identifying and estimating peer effects. We study possible evidence for endogeneity of the network, and develop models that allow for endogeneity.

In our application to friendship networks in the Add Health data, we find that one's friends's grades are correlated with one's own grades. Whether these are causal peer effects is more difficult to establish. We find there is limited evidence that the friendships are endogenous with grades, but there is evidence that current friendships are not sufficient for capturing all correlations in outcomes. Correlations in grades with former friends, as well as those with friends-of-friends, are almost as strong as those with current friends, casting doubt on causal interpretations of the correlations between current friends and current grades. We also explore the direct evidence for indirect peer effects, that is, the effects of the characteristics of friends-of-friends. We find that the data are inconclusive regarding the presence of such effects, and that the estimated effects in models that allow for them are largely driven by functional form assumptions that tie these friends-of-friends effects to those of friends.

Many more aspects of peer-effect models remain unexplored in this discussion, although some are touched upon in the discussions. The area remains a fruitful one for future research.

APPENDIX: APPROXIMATING THE POSTERIOR DISTRIBUTION

Here we provide some details on the evaluation of the posterior distribution. We focus on the model with endogeneity. The model with exogenous network formation is simpler to estimate. In particular, we can separately analyze the network formation model and the model for the primary outcome. We use the results from that model to provide starting values for the model with endogeneity.

The model with endogenous network formation has four components. The first describes the conditional distribution of $\mathbf{Y}$ given $\mathbf{X}$, $\mathbf{G}$, $\mathbf{G}_0$, and the unobserved ξ:

$$\begin{aligned}&\mathbf{Y}|\mathbf{X}, \mathbf{G}, \mathbf{G}_0, \xi \\ &\quad\sim \mathcal{N}\big((\mathbf{I}-\beta_{\overline{y}}\mathbf{G})^{-1}\beta_0\iota_N + (\mathbf{I}-\beta_{\overline{y}}\mathbf{G})^{-1}(\beta_x+\beta_{\overline{x}}\mathbf{G})\mathbf{X} \\ &\qquad + (\mathbf{I}-\beta_{\overline{y}}\mathbf{G})^{-1}\beta_\xi\xi, \sigma^2(\mathbf{I}-\beta_{\overline{y}}\mathbf{G})^{-1}(\mathbf{I}-\beta_{\overline{y}}\mathbf{G}')^{-1}\big).\end{aligned}$$

The second part describes the network formation given $\mathbf{X}$ and the unobserved ξ:

$$\begin{aligned}D_{ij} &= \mathbf{1}_{U_i(j)>0}\cdot\mathbf{1}_{U_j(i)>0},\\ U_i(j) &= \alpha_0 + \alpha_x|X_i - X_j| + \alpha_d D_{0,ij} + \alpha_f F_{0,ij}\\ &\quad + +\alpha_\xi|\xi_i - \xi_j| + \epsilon_{ij}. \qquad \text{(A.1)}\end{aligned}$$

The third component gives the conditional distribution of ξ and ϵ_{ij} given $\mathbf{X}$:

$$\epsilon_{ij} \sim \frac{\exp(-x)}{(1+\exp(-x))^2},$$
$$\mathrm{pr}(\xi_i = 1) = 1/2, \qquad \xi_i \perp \xi_j \text{ for } i \neq j.$$

This model leads to a likelihood function in terms of $\theta = (\beta, \alpha, \sigma^2)$. The prior distributions for all parameters are independent. The prior distributions for α, β_x are normal with mean zero and variance equal to one. The prior distribution for σ^2 is inverse χ^2 with 10 degrees of freedom.

We pick starting values for θ based on the model with exogeneity for all parameters other than β_ξ and α_ξ. We take the starting value for α_ξ from a normal distribution centered at -1 and a variance equal to 0.01. The starting values for β_ξ are drawn from a normal distribution centered at zero with variance 0.01. Then we draw ξ_i from a binomial distribution with mean 1/2, for $i = 2, \ldots, N$. The first value ξ_1 is set equal to 1. Before changing the values for θ, we repeatedly update ξ to obtain values more in line with the data. We have found that this leads to faster convergence. We update ξ_i sequentially, given the starting values for θ. Each time we update a single ξ_i. We cycle through the full set of ξ_i 100 times without changing any of the values for θ.

Next, we start cycling through the other parameter values. We divide the updating into three parts. First we update β given σ^2, α, and ξ. We use a Metropolis-Hastings step here, using a candidate normal distribution centered at the current values, with covariance matrix equal to the covariance matrix estimated from the exogenous model multiplied by 1/16.

The second step updates σ^2. Here we have the exact posterior distribution given values for the other parameters and given ξ and hence use a Gibbs sampler.

The third step involves updating α given ξ. Here we use a Metropolis-Hastings step, with the candidate distribution centered at the current values, and the covariance matrix estimated on the model with exogenous network.

In the fourth step we update ξ given all the parameter values.

We use five starting values. For each of the five chains, we drop the first 2000 iterations. We then compute the average value of the elements of θ, and the overall average. We monitor convergence by comparing for each of the elements of θ the ratio of the overall variance and the average of the within-chain variances. Following the suggestion by Gelman and Rubin (1992), we aim for ratios below 1.1. After 30,000 iterations, the convergence criteria for all elements of θ were below 1.1, with most below 1.05.

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Comment

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In their article, Goldsmith-Pinkham and Imbens tackle three important issues in the study of peer effects in social networks: network endogeneity, network heterogeneity, and the structure of the error term. The endogenous nature of social links is probably the most important and the most vexing of these three issues. In this note, I discuss first how network endogeneity is addressed. I then make a comment on network heterogeneity, a clarification on group interactions and a proposition of further research.

1. CORRECTING FOR NETWORK ENDOGENEITY

Goldsmith-Pinkham and Imbens' approach can be viewed as a network version of the classical Heckman correction for sample selection. By studying network formation and outcome determination jointly, we may recover enough information on the error term of the outcome equation to solve the underlying endogeneity problem. Goldsmith-Pinkham and Imbens propose one of the first convincing applications of this idea to peer effects and networks, see Conti et al. (2012) and Hsieh and Lee (2012) for two recent studies in the same vein. The way identification works, however, is not very clear. With classical selection problems, the empirical importance of exclusion restrictions is well known. Without exclusion restrictions, identification relies on nonlinearities and may not be very robust. While peer effects in networks constitute a new setup, the question of the strength of identification remains key. In what follows, I highlight some properties and limitations of the model whose study may help clarify this issue.

Let us examine, first, the role played by exclusion restrictions in the analysis. Current links D_{ij} are affected by past network features and by covariates (Equation (6.2)). Past network features D_{0ij} and F_{0ij} do not affect outcome Y_i (Equation (6.1)). However, if unobserved characteristics are persistent, past

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