1.) $\frac{dx}{dt} + c(x) = 0$

a.) L = $V(x) = x^2$

1. $V(0) = 0$ since $0^2 = 0$
2. $V(x) > 0 \ \forall x \neq 0$ (positive definite)

   this is true for all x because
   $x^2 > 0 \ \forall x \neq 0$
3. $V'(x) \leq 0 \ \forall x \neq 0$

$$V'(x) = 2x\frac{dx}{dt} = -2xc(x)$$

* since $xc(x) > 0$ (given from problem),

$$-2xc(x) \leq 0$$

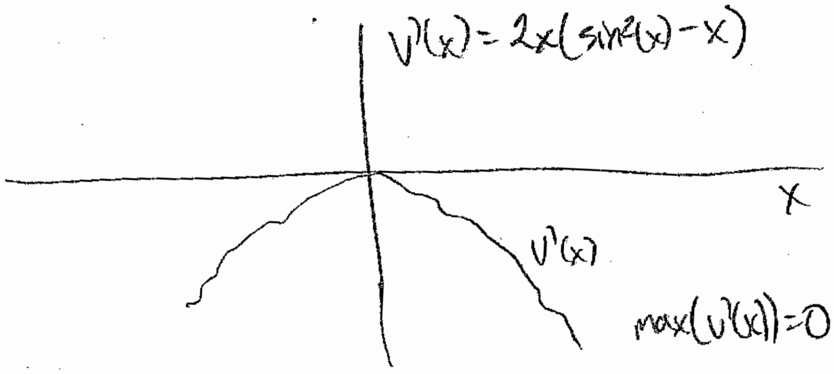
$$\therefore V'(x) = -2xc(x) \leq 0 \ \forall x \neq 0$$

$\boxed{\therefore V \text{ is a Lyapunov function of the system}}$

b.) $\frac{dx}{dt} = \sin^2(x) - x$

In addition to part a), we need to show that $V(x)$ is radially unbounded: $\|x\| \to \infty$ as $|V(x)| \to \infty$. This is true since $\lim_{x \to \infty} V(x) = x^2 = \infty$. Additionally,

$$V'(x) = 2x(\sin^2(x) - x) \leq 0 \ \forall x \neq 0.$$



We showed graphically that $V'(x) < 0 \ \forall x \neq$

$\boxed{\therefore \text{The system is globally asymptotically stable}}$

2.) a.) $f(u) = \frac{1}{(1+e^{-u})}$

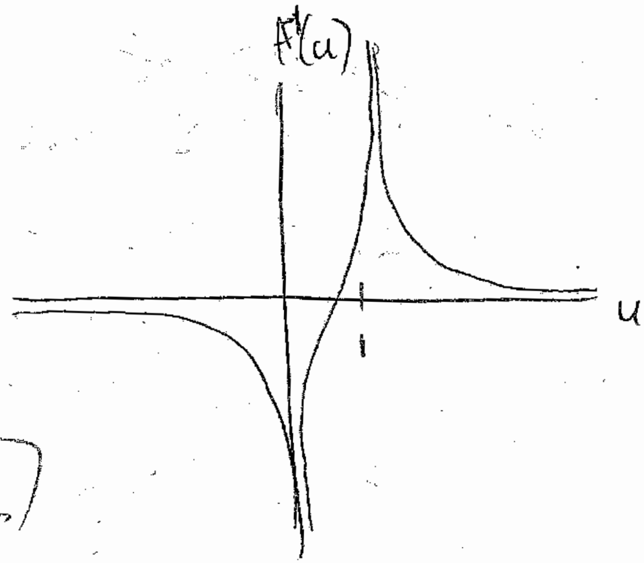
$f^{-1}u = \frac{1}{1+e^{-f(u)}}$

$1+e^{-f^{-1}(u)} = \frac{1}{u}$

$e^{-f^{-1}(u)} = \frac{1}{u} - 1$

$-f^{-1}(u) = \ln(\frac{1}{u} - 1)$

$\boxed{f^{-1}(u) = -\ln(\frac{1}{u} - 1)}$



$F(x) = \int^{x} -\ln(\frac{1}{y} - 1)\, dy$

$= -\ln(\frac{1}{x} - 1)x + \int^{x} \frac{\cancel{y}}{\cancel{y}(y-1)} dy$

$= -\ln(\frac{1}{x} - 1)x + \ln(x-1)$

$f^{-1}(z)$ is defined in: $[(-\infty, 0) \cup (0,1) \cup (1,\infty)]$

b.) From the graph in a), if initially $0<x<1$, then f^{-1} cannot leave $(0,1)$ because $x=0$ and $x=1$ are asymptotes.

c.) $\boxed{L(x) = x\ln(x) + (1-x)\ln(1-x) - 1 - \frac{1}{2}x^T W_x - b^T x}$

3.) $L(x) = \frac{1}{2}x^T(I-W)x - b^T x$

We show 4 cases as in part a)

1. $L(0)=0 \Rightarrow \frac{1}{2}(0) - b^T(0) = 0$

2. $L(x)>0 \quad \forall x \neq 0$

$$\therefore L(x) = \frac{1}{2}x^T \underbrace{(I-W)}_{psd} x - b^T x$$

$(-)\times(+)\times(-)$ or $(+)\times(+)\times(+)$ } in both cases, positive

analyzing signs: ① if $x<0$,

$$L(x) = (-)(+)(-) - (-) \Rightarrow L(x)>0$$

② if $x>0$,

$$L(x) = (+)(+)(+) - (+) \Rightarrow L(x)>0 \text{ since } LHS > RHS$$

$$\frac{1}{2}x^T(I-W)x - b^T x$$

3. $\dot{L}(x) \leq 0 \quad \forall x \neq 0$ consider $f(Wx+b) = Wx+b > 0$

$$\therefore \dot{x} = -x + Wx + b$$
$$= (W-I)x + b$$

$$\dot{L}(x) = \frac{1}{2}\dot{x}^T(I-W)x + \frac{1}{2}x^T(I-W)\dot{x} - b^T\dot{x}$$

in a linear system let $\dot{x} = Ax$

We must check for a pd matrix P such that

$$A^TP + PA < 0$$

In our case, $A = (W-I)$ and $P = (I-W)$

$$\Rightarrow A^TP + PA = (W-I)^T(I-W) + (I-W)(W-I)$$
$$= (+)(-) + (-)(+)$$
$$= (-) < 0$$

$$\boxed{\therefore \dot{L}(x) \leq 0 \quad \forall x}$$

4.) a.) In a circulant matrix, the eigenvectors are

$$V_j = \frac{1}{\sqrt{n}}(1, \omega_j, \omega_j^2, \ldots, \omega_j^{n-1})^T \quad \text{where } j = 0, 1, \ldots, n-1$$

and $\omega_j = e^{\frac{2\pi i j}{n}}$

$\Rightarrow$ the m_{th} component of the α^{th} eigenvector V_α is:

$$(\omega_\alpha)^m = e^{\frac{2\pi i(\alpha-1)m}{n}} = \boxed{e^{ik_\alpha m}, \quad k_\alpha = \frac{2\pi(\alpha-1)}{n}}$$

$$\lambda_j = c_0 + c_{n-1}\omega_j + c_{n-2}\omega_j^2 + \ldots + c_1\omega_j^{n-1}, \quad j = 0, 1, \ldots, n-1$$

$$\Rightarrow \lambda_\alpha = c_0 + c_{n-1}\omega_\alpha + c_{n-2}\omega_\alpha^2 + \ldots + c_1\omega_\alpha^{n-1}$$

If $c_1 = c_{N-1} \equiv c$, and all other $c_i = 0$, then:

$$W = \begin{bmatrix} 0 & c & 0 & \cdots & 0 & c \\ c & 0 & \cdots & & \cdots & 0 \\ 0 & & \ddots & & & \vdots \\ \vdots & & & \ddots & & 0 \\ 0 & & & & 0 & c \\ c & & \cdots & & c & 0 \end{bmatrix}$$