Average Case recurrence for Quicksort:

High level:

$T(n)$ = Expected comparisons for pivot selection +
Exp comp for partitioning +
T(size of left subarray) +
T(size of right subarray)

Random Pivot: $O(1)$

Comp for Partition: $O(n)$

Recurrence Relation:

$T(n) = O(1) + O(n) +$

$T(\text{size of left subarray}) +$

$T(\text{size of right subarray})$.

We shall guess that the recurrence relation solution is: $O(n \log n)$

We need to prove by induction

$T(n) = O(1) + O(n) + T(n/2) + T(n/2)$

$$T(n) \le c\, n \log n, \quad c > 0$$

$$T(n) = 2T(n/2) + n$$

$$T(n) \le 2\left(c(n/2)\log(n/2)\right) + n$$

$$\le c\, n \log(n/2) + n$$

$$= c\, n \log n - c\, n \log 2 + n$$

$$= c\, n \log n - cn + n$$

$$\le c\, n \log n \qquad c \ge 1$$

lets try with Unbalanced sub problems:

$$T(n) = T(9n/10) + T(n/10) + n$$

Assume $O(n \log n)$

$$T(n) = O(n \log n)$$

$$T(n) \leq c n \log n \qquad c \geq 0$$

$$T(n) \leq c \frac{9n}{10} \log \frac{9n}{10} +$$

$$\frac{n}{10} \log \frac{n}{10} + n$$

$$= c\frac{9n}{10}\left(\log n + \log\frac{9}{10}\right) + c\frac{n}{10}\left(\log n + \log\frac{1}{10}\right) + n$$

$$= c\frac{9n}{10}\log n + c\frac{n}{10}\log n + n\left(c\log\frac{9}{10} + c\log\frac{1}{10} + 1\right)$$

$$= cn\log n + n\left(c\log\frac{9}{10} + c\log\frac{1}{10} + 1\right)$$

$$\leq cn\log n + n\left(c\log 1 + 1\right)$$

$$= cn\log n + (c+1) + n$$

$$\leq (c+1)\,n\log n$$

Akra-Baazi

$$T(n) = T\frac{n}{10} + T\frac{9n}{10} + n$$

$$\sum_{i=1}^{K} a_i T(b_i n + h_i(n)) + g(n)$$

$K = 2$

$a_1 = 1$

$b_1(n) = 1/10$

$a_2 = 1$

$b_2(n) = 9/10$

$g(n) = O(n)$

Solve for p:

$$\sum_{i=1}^{K} a_i b_i^{p} = 1$$

$$T(n) = O\left(n^p\left(1 + \int_1^n \frac{g(x)}{x^{p+1}}\,dx\right)\right)$$

$$T(n) = O\left(n^1\left(1 + \int_1^n \frac{x}{x^2}\,dx\right)\right)$$

$$T(n) = O\left(n\left(1 + \int_1^n \frac{1}{x}\,dx\right)\right)$$

<u>Integrate</u>

$$T(n) = O\left(n\left(1 + \int_1^n \frac{1}{x}\,dx\right)\right)$$

Use common integration law

$$\int \frac{1}{x}\,dx = \ln x$$

$T(n) = O(n(1 + \ln x))$

With asymptotic behaviour we can say $\ln x$ and $\log x$ are equivalent.

Remove lower order terms and constents

$T(n) = O(n + n \log n)$

$T(n) = O(n \log n)$