Signals and linear systems

2 hours

Preface

- “The purpose of computing is insight, not numbers.”- R. W. Hamming, Numerical Methods for Engineers and Scientists, McGraw-Hill, Inc.
- Properly applied, computing can illuminate theory and help students to think, analyze, and reason in meaningful ways.

Analysis, Simulation, and Implementation

- To reduce cost and time, we need analysis and/or simulation first. For example, circuit simulation software (SPICE etc.), flight simulator, etc.
- For digital communication systems, Q(sqrt(SNR)) is **analysis;** sending 1 million bits over AWGN channel and counting the number of bit errors is **simulation;** the HW/SW of the whole system is **implementation.**

Ch. 1 signals and linear systems

- **LTI Systems**
- **Fourier Transform**
- **Sampling Theorem**
- **Frequency-Domain Analysis of LTI Systems**
- **Power and Energy**

Linear Systems

Feature: characterized by impulse response and output can be computed by convolution

It satisfies:

1. Homogeneity
2. Superposition

Homogeneity

$$u(t) \rightarrow y(t)$$

$$ku(t) \rightarrow ky(t)$$

Superposition

$$u_1(t) \to y_1(t)$$
$$u_2(t) \to y_2(t)$$
$$u_1(t) + u_2(t) \to y_1(t) + y_2(t)$$

Fourier Transform

The Fourier transform is the extension of Fourier series to ***non-periodic*** signals.

$$F[x(t)] = X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$
$$F^{-1}[X(f)] = x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

Properties of the Fourier transform

- 1. Linearity
- 2. Duality
- 3. Time shift
- 4. Scaling
- 5. Modulation
- 6. Differentiation
- 7. Convolution
- 8. Parseval’s relation

1. Linearity

A linear combination of two or more signals is the linear combination of the corresponding Fourier transforms:

$$F\left[\alpha x_1(t)+\beta x_2(t)\right]=\alpha F\left[x_1(t)\right]+\beta F\left[x_2(t)\right]$$

2. Duality

- If $X(f) = F[x(t)]$ then

$$F\left[X(t)\right] = x(-f)$$

Rectangular pulse ⇔ sinc

3. Time shift

A shift in the time domain results in a phase shift in the frequency domain.

if $X(f) = F[x(t)]$, then

$$F\left[x(t-t_0)\right] = e^{-j2\pi f t_o} X(f)$$

Time-shifted version ⇔ linear phase
Waveform unchanged (amplifier in Microelectronics course)

4. Scaling

An expansion in the time domain results in a contraction in the frequency domain, and vice versa. If *X(f) = F[x(t)]* then

$$F\left[x(at)\right] = \frac{1}{|a|} X\left(\frac{f}{a}\right), a \neq 0$$

High data rate ⇔ broadband.

5. Modulation

Multiplication by an exponential in the time domain corresponds to a frequency shift in frequency domain. If *X(f) = F[x(t)]* then

$$F\left[e^{j2\pi f_0 t} x(t)\right] = X(f - f_0)$$

$$F\left[x(t)\cos(2\pi f_0 t)\right] = \frac{1}{2}\left[X(f - f_0) + X(f + f_0)\right]$$

6. Differentiation

Differentiation in the time domain corresponds to multiplication by $j2\pi f$ in the frequency domain. If $X(f) = F[x(t)]$ then

$$F\left[x'(t)\right] = j2\pi f X(f)$$

$$F\left[\frac{d^n}{dt^n} x(t)\right] = (j2\pi f)^n X(f)$$

7. Linear Convolution

Convolution in the time domain is equivalent to multiplication in the frequency domain, and vice versa. If $X(f) = F[x(t)]$ and $Y(f) = F[y(t)]$ then

$$F\left[x(t) * y(t)\right] = X(f)Y(f)$$

$$F\left[x(t)y(t)\right] = X(f) * Y(f)$$

Linear convolution ⇔ multiplication in CFT
Circular convolution ⇔ multiplication in DFT
(Thus cyclic prefix (CP) is used in OFDM to approximate circular convolution)

8. Parseval's relation (law of energy conservation)

If $X(f) = F[x(t)]$ and $Y(f) = F[y(t)]$,then

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Fourier transform of periodic signals (Fourier series are special cases of Fourier transform)

For periodic signal x(t), with period T_0,
the Fourier series coefficient are given by x_n

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi nt/T_0}$$

$$X(f) = F[x(t)] = F\left[\sum_{n=-\infty}^{\infty} x_n e^{j2\pi nt/T_0}\right]$$

$$= \sum_{n=-\infty}^{\infty} x_n F\left[e^{j2\pi nt/T_0}\right] = \sum_{n=-\infty}^{\infty} x_n \delta(f - \frac{n}{T_0})$$

Periodic in one domain discrete (sampled) in the other domain

Sampling Theorem

- Basis for relationship between continuous-time signals and discrete-time signals. (fundamental of digital signal processing, DSP)
- A bandlimited signal whose Fourier transform vanishes for |f|> W for some W, can be completely described in terms of its sample values taken at intervals T_s as long as sampling frequency $f_S = 1/T_S > 2W$.

Sample values

$$\{x[n] = x(nT_s)\}_{n=-\infty}^{\infty}$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \operatorname{sin}c(2W(t - nT_s))$$

P.S. non-linear interpolation for channel estimation in OFDMA systems (NWTT7)

Note: rectangle <= *Fourier* => sinc function

(ref next page)

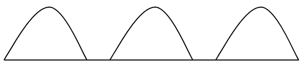
Sampling Theorem

So 1/Ts>2W

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$X_\delta(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(f - \frac{n}{T_s}) \text{ for all f}$$

$$= \frac{1}{T_s} X(f) \text{ for } |f| < W,$$



so low pass filtering (**rectangle**) with a BW W and gain of T_s in the passband will reproduce the original signal

Frequency-Domain Analysis of LTI Systems

- The output of an LTI system with impulse response h(t) when the input signal is x(t) is given by the convolution integral

$$y(t) = x(t) * h(t)$$

$$\therefore Y(f) = X(f)H(f)$$

$$H(f) = F[h(t)] = \int_{-\infty}^{\infty} h(t)e^{-j2\pi tf}dt$$

Can be written in the form

$$\left|Y(f)\right| = \left|X(f)\right|\left|H(f)\right|$$

$$\angle Y(f) = \angle X(f) + \angle H(f)$$

Power and Energy

$$E_X = \int_{-\infty}^{\infty} \left|x(t)\right|^2 dt$$

$$P_X = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left|x(t)\right|^2 dt$$

power = energy per unit time

Energy-type and power-type signal

- A signal with finite energy is called an energy-type signal.
- A signal with positive and finite power is a power type signal. (e.g. noise)

Energy spectral density of an energy-type signal

$$g_X(f) = |X(f)|^2$$

$$(\because \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df)$$

therefore

$$E_X = \int_{-\infty}^{\infty} g_X(f)df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Autocorrelation/spectral density are Fourier transform pair

$$g_X(f) = F[R_X(\tau)] = |X(f)|^2 \ \textbf{spectral density}$$

$$R_X(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt \ \textbf{autocorrelation}$$

$$= x(\tau) * x(-\tau)$$

Time-average autocorrelation function for power-type signal

$$R_X = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t+\tau)dt$$

Power-spectral density

$$S_X(f) = F\left[R_X(\tau)\right]$$

$$P_X = \int_{-\infty}^{\infty} S_X(f)\,df$$