Mixed Effects Models

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Introduction

Mixed effects models are often used in cases where we want to take repeated measurements or nuisance variables in to account without treating them as a fixed variable in the final model. The model will generally look something like:

$$Y_i = \mu + \alpha Z_{j(i)} + \beta X_i + \epsilon_i, i \in 1, ..., n, j \in 1, ..., m.$$

$$\epsilon_i \sim N(0, \sigma^2)$$

In this case, we have n total individual observations which are each in one of m groups of a variable in the data. The x_i are called fixed effects and are the variables that we want to measure the effect of in our final model. The $Z_{j(i)}$ are the random effects that we want to account for but do not wish to do inference on; each individual i is a member of one of the groups j. μ is the intercept term that we normally include in regression models.

This will probably make more sense with the example below. We will analyze the `nlschools` data set from the `MASS` package, which contains data from students in schools in the Netherlands. The variables in the data:

- `lang`: language test score.
- `IQ`: verbal IQ score.
- `class`: class ID.
- `GS`: class size.
- `SES`: socioeconomic status.
- `COMB`: multi-grade class?

We will use the `lme4` package to fit our mixed effects models.

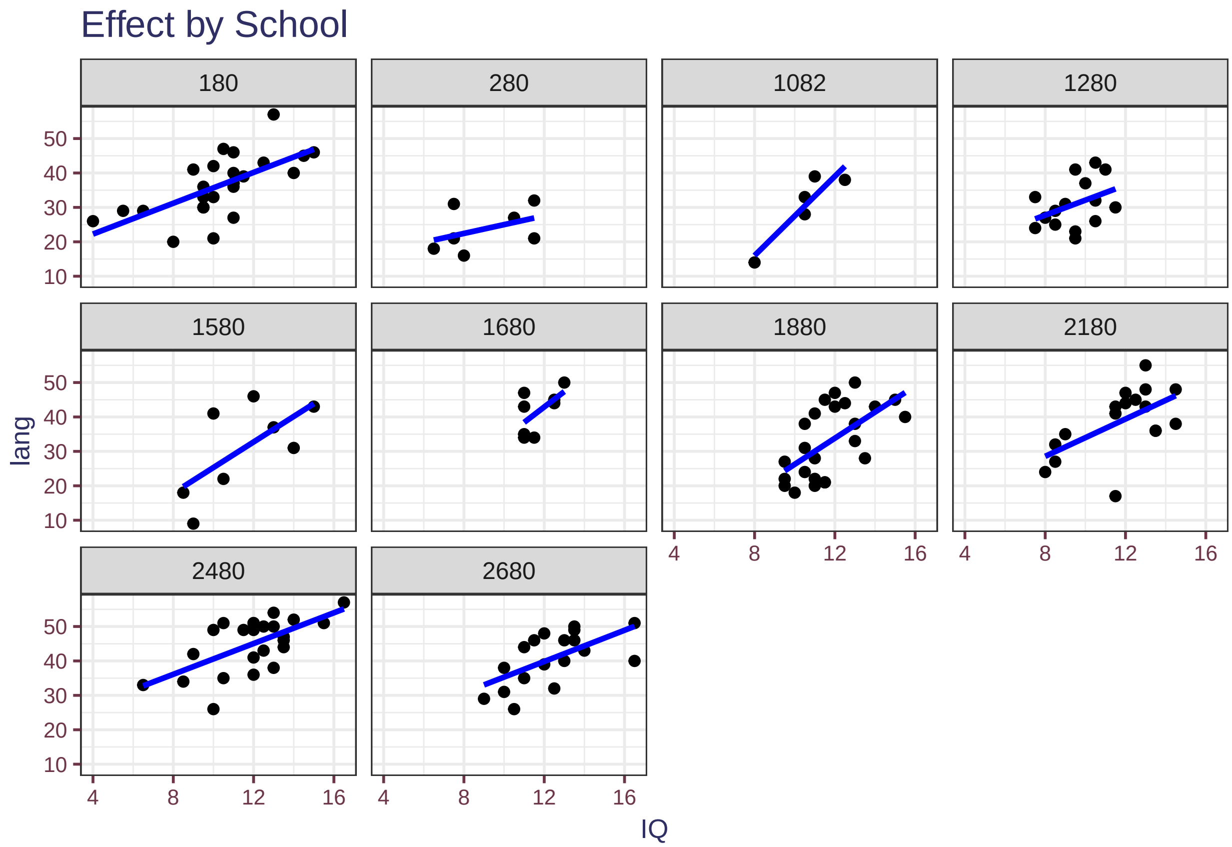
```r
library(MASS)
#?nlschools
data(nlschools)
head(nlschools)
```

```
  lang   IQ class GS SES COMB
1   46 15.0   180 29  23    0
2   45 14.5   180 29  10    0
3   33  9.5   180 29  15    0
4   46 11.0   180 29  23    0
5   20  8.0   180 29  10    0
6   30  9.5   180 29  10    0
```

Intuition

Say for the `nlschools` data we want to predict a student's language test score based on their verbal IQ score, controlling for the other variables in the data set. However, it seems that students were sampled in blocks from individiual classes. Thus, we have repeated measures on individuals from the same class. This would be a case where we would include class ID as a random effect. We certainly want to take the class ID in to account in our model, as there may be class-level effects on the language test score that are not directly measured in the other variables in the data; perhaps the teachers for certain classes are better or worse than others, or perhaps some classes are honors-level courses while others are remedial level. Indeed, the plots below appear to show that the relationship between a student's language test score and their verbal IQ score appears to be different for the different classes However, because of block sampling scheme, we can't just include this as a fixed effect since these class IDs do not represent all possible classrooms in the Netherlands. In particular, if we wanted to predict for a student in a new class not in the data, a model with class ID as a fixed effect could not be used. Thus, we include class ID as a random effect.

```r
ggplot(data = nlschools[1:151, ]) +
  geom_point(aes(x = IQ, y = lang)) +
  geom_smooth(aes(x = IQ, y = lang), se = FALSE,
              method = "lm", color = "blue") +
  facet_wrap(~ class) +
  labs(title = "Effect by School") +
  theme1
```



How are these random effects accounted for?

Random effects are most often included in a mixed effects model as a parametric distribution around a "global" underlying average effect. One might think of this as a sampling distribution for the random effect. For example, we might assume that the classroom level effect on the relationship between a student's language test score and their verbal IQ score is normally distributed with mean 0 and variance τ^2 around the true underlying parameter values.

Random Intercept

The first model example we will look at is a random intercept model. This model assumes that the class-level effect changes the average language score of the individual students. Mathematically, this is:

$$Y_i = \mu + \alpha_{j(i)} + \Lambda W_i + \beta x_i + \epsilon_i, i \in 1, ..., n, j \in 1, ..., m.$$
$$\epsilon_i \sim N(0, \sigma^2), \alpha_j \sim N(0, \tau^2).$$

- μ is the global intercept - α is the class-level random intercept effect - x_i is the fixed effect of interest - W_i are the individual level covariates - ϵ_i is the individual level error

The results are shown below. Note the syntax of the formula - we specify a random intercept by adding `(1|class)`. We can see the estimates for the coefficients for `IQ` as well as the covariates under "Fixed effects:", and we can see the estimates for τ^2 and σ^2 under "Random effects:".

```r
library(lme4)

model1 <- lmer(lang ~ IQ + GS + SES + COMB + (1|class), data = nlschools)
summary(model1)
```

```
Linear mixed model fit by REML ['lmerMod']
Formula: lang ~ IQ + GS + SES + COMB + (1 | class)
   Data: nlschools

REML criterion at convergence: 15132.9

Scaled residuals:
    Min      1Q  Median      3Q     Max
-3.9588 -0.6488  0.0486  0.7204  3.0765

Random effects:
 Groups   Name        Variance Std.Dev.
 class    (Intercept)  8.507   2.917
 Residual             39.998   6.324
Number of obs: 2287, groups:  class, 133

Fixed effects:
            Estimate Std. Error t value
(Intercept) 10.63524    1.50012   7.090
IQ           2.24696    0.07133  31.502
GS          -0.01562    0.04770  -0.327
SES          0.16538    0.01479  11.185
COMB1       -2.02161    0.60055  -3.366

Correlation of Fixed Effects:
      (Intr) IQ     GS     SES
IQ    -0.484
GS    -0.798 -0.002
SES   -0.062 -0.292 -0.057
COMB1 -0.168  0.033 -0.006  0.017
```

Random Slope

The first model example we will look at is a random intercept model. This model assumes that the class-level effect changes the linear relationship between the language score and verbal IQ of the individual students. Mathematically, this is:

$$Y_i = \mu + \alpha_{j(i)} x_i + \Lambda W_i + \beta x_i + \epsilon_i, i \in 1, ..., n, j \in 1, ..., m.$$

$$\epsilon_i \sim N(0, \sigma^2), \alpha_j \sim N(0, \tau^2).$$

- μ is the global intercept - α is the class-level random slope effect - x_i is the fixed effect of interest - W_i are the individual level covariates - ϵ_i is the individual level error

The results are shown below. We specify a random slope WITHOUT a random intercept by adding `(IQ - 1|class)`. Again, we can see the estimates for the coefficients for `IQ` as well as the covariates under "Fixed effects:", and we can see the estimates for τ^2 and σ^2 under "Random effects:".

```r
library(lme4)

model2 <- lmer(lang ~ IQ + GS + SES + COMB + (IQ - 1|class), data = nlschools)
summary(model2)
```

```
Linear mixed model fit by REML ['lmerMod']
Formula: lang ~ IQ + GS + SES + COMB + (IQ - 1 | class)
   Data: nlschools

REML criterion at convergence: 15157.4

Scaled residuals:
    Min      1Q  Median      3Q     Max
-3.9450 -0.6453  0.0590  0.7063  3.4214

Random effects:
 Groups   Name Variance Std.Dev.
 class    IQ    0.0561  0.2369
 Residual      40.5701  6.3695
Number of obs: 2287, groups:  class, 133

Fixed effects:
            Estimate Std. Error t value
(Intercept) 10.70203    1.43466   7.460
IQ           2.26638    0.07481  30.295
GS          -0.02471    0.04466  -0.553
SES          0.16820    0.01484  11.334
COMB1       -2.45972    0.56308  -4.368

Correlation of Fixed Effects:
      (Intr) IQ     GS     SES
IQ    -0.508
GS    -0.805  0.037
SES   -0.063 -0.284 -0.060
COMB1 -0.140 -0.031  0.009  0.020
```

Extensions

Obviously, we can add more random effects to the model. In the particular example above, we could add both a random slope and intercept at the class level using `(IQ|class)`. We could have multiple levels of hierarchies (e.g., students within classes within schools within districts…) We could also use different distributions or parameterizations for the random effects. The possibilities are endless!

Model Comparison & Inference

We should not use the t-values returned by `lmer` to do statistical inference. Period. Instead, we should use some sort of likelihood ratio testing between different potential models. There are a couple of methods to do this:

ANOVA

ANOVA can be used in situations where we want to test for the statistical significance of individual or sets of variables. This is done by fitting 2 models: one with the variable(s) of interest and one without. These are then compared formally using an F-test, one of the classic likelihood ratio tests. The key in this case is that one model is nested in the other. meaning that the variables included in the smaller model must be a subset of the variables included in the larger model. Below, we fit a model without `IQ` before we run the ANOVA

```
model3 <- lmer(lang ~ GS + SES + COMB + (1|class), data = nlschools)
anova(model3, model1)
```

```
Data: nlschools
Models:
model3: lang ~ GS + SES + COMB + (1 | class)
model1: lang ~ IQ + GS + SES + COMB + (1 | class)
       Df   AIC   BIC  logLik deviance  Chisq Chi Df Pr(>Chisq)
model3  6 15956 15990 -7971.9    15944
model1  7 15133 15173 -7559.3    15119 825.09      1  < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The results are shown above. In this example, the model with `IQ` is significantly better than the one without, based on the p-value from the ANOVA output.

AIC/BIC

Infromation criterion tests are used in the case where we do not have nested models. The upside is, of course, that we can more flexibly compare two models. The downside is that there is no formal statistical test or standard for these types of tests to show that one model is definitively better than the other. We can say in general that a smaller AIC or BIC is better; by most standards, a difference of 6 is generally considered enough evidence that one model is better than another.

```
AIC(model3, model1)
```

```
       df      AIC
model3  6 15964.70
model1  7 15146.86
```

```
BIC(model3, model1)
```

```
       df      BIC
model3  6 15999.11
model1  7 15187.00
```

For both of these criteria, the AIC and BIC of the model without `IQ` is much larger than the respective measures on the model with `IQ`. There is almost certainly enough evidence to say that the model with `IQ` is better.