Sorbonne Université, Computer Science Master Données, Apprentissage et Connaissances (DAC)
Bayesian Deep Learning

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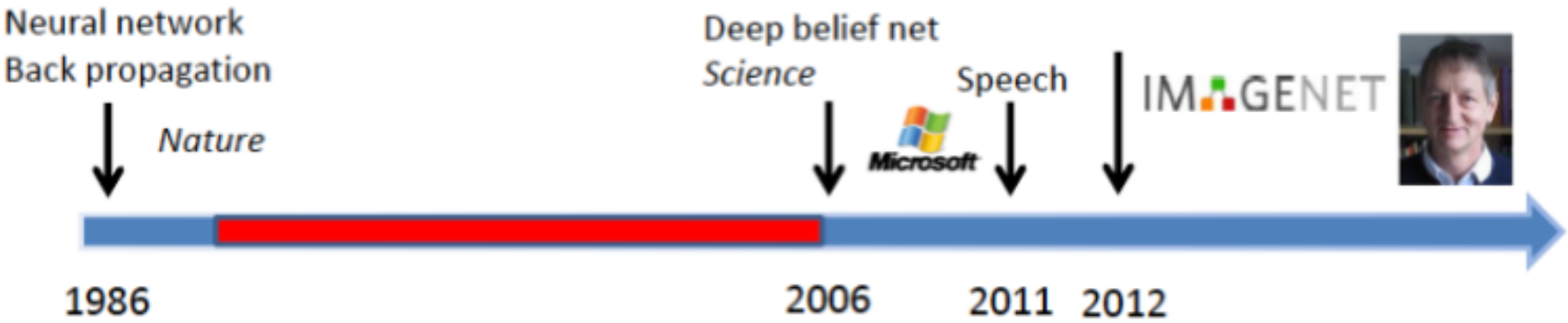
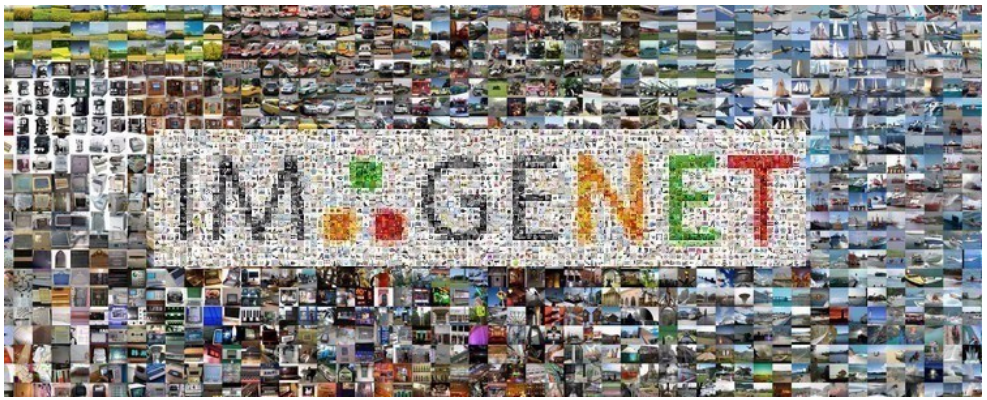
Deep Learning Success since 2010

- 90's / 2000's: difficult to train large deep models on existing databases



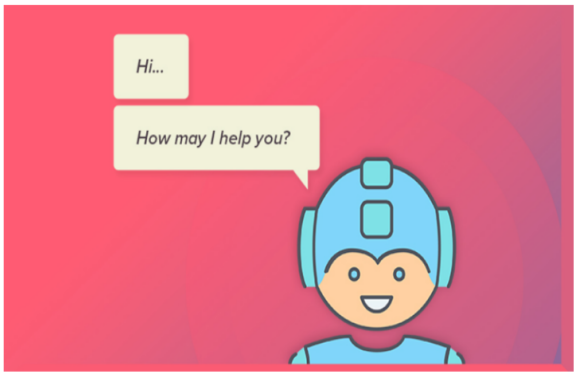
- **ILSVRC'12: the deep revolution**
  ⇒ outstanding success of ConvNets [Krizhevsky et al., 2012]

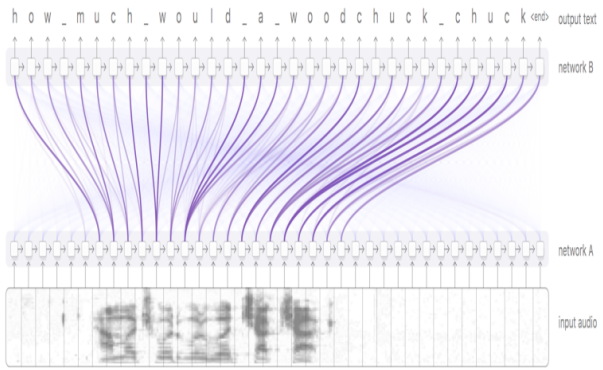


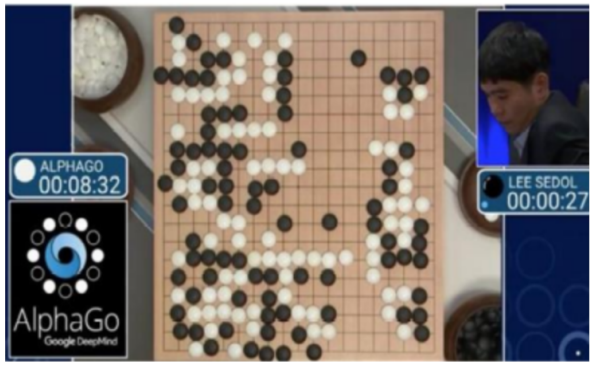
Rank	Name	Error rate	Description
1	**U. Toronto**	0.15315	Deep learning
2	U. Tokyo	0.26172	Hand-crafted features and learning models. Bottleneck.
3	U. Oxford	0.26979	
4	Xerox/INRIA	0.27058	

Deep Learning everywhere since 2012

- Image classification, speech recognition
- chatbots, translation,
- Games, robotics

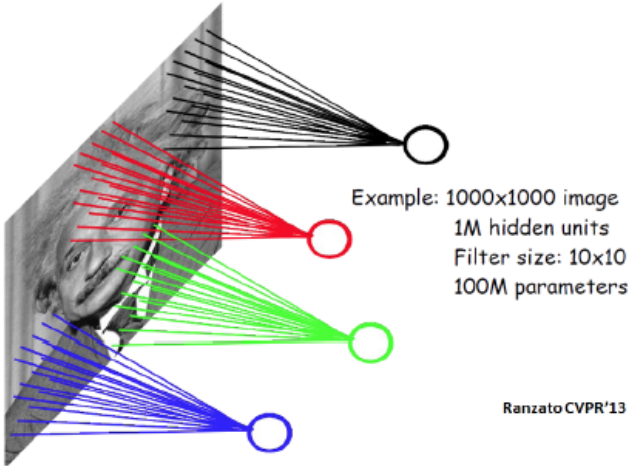


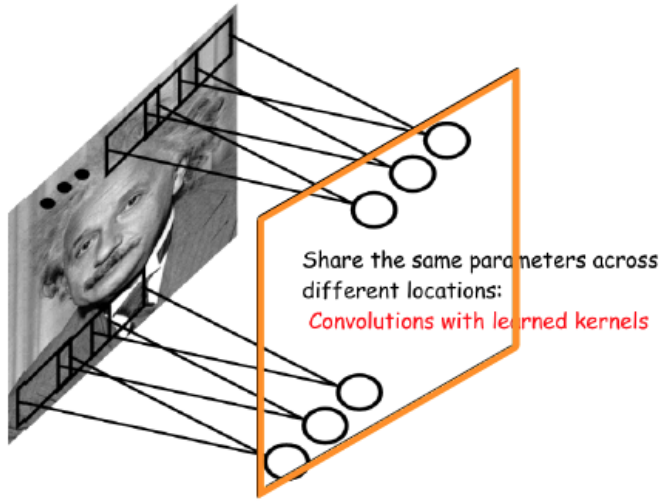




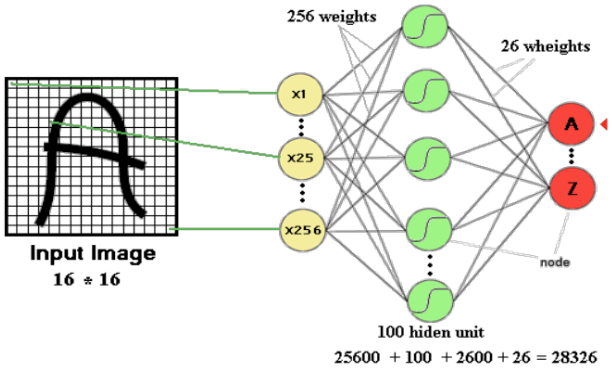
Convolutional Neural Networks (ConvNets)

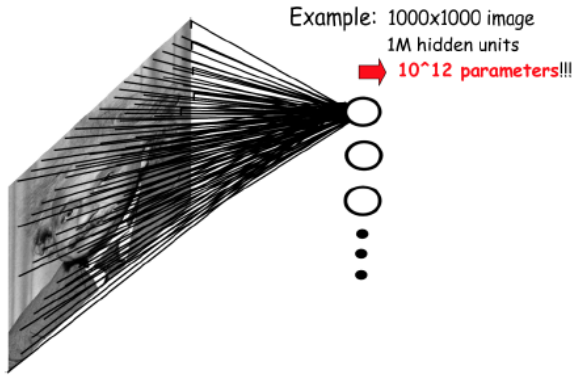
- **ConvNets:** sparse connectivity + shared weights





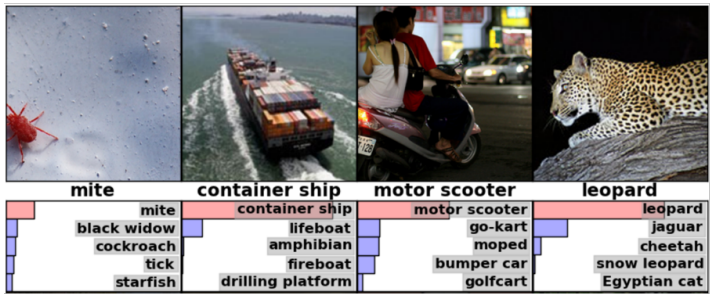
- **Local feature extraction (≠ FCN)**
- Overcome parameter explosion for FCN on images



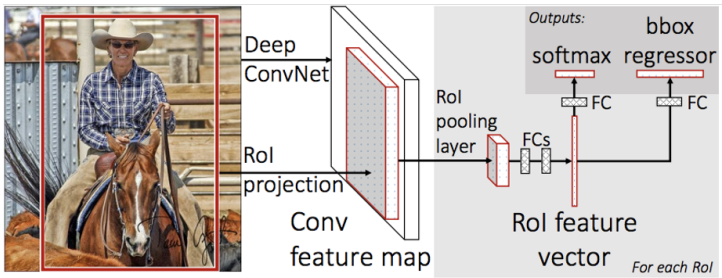


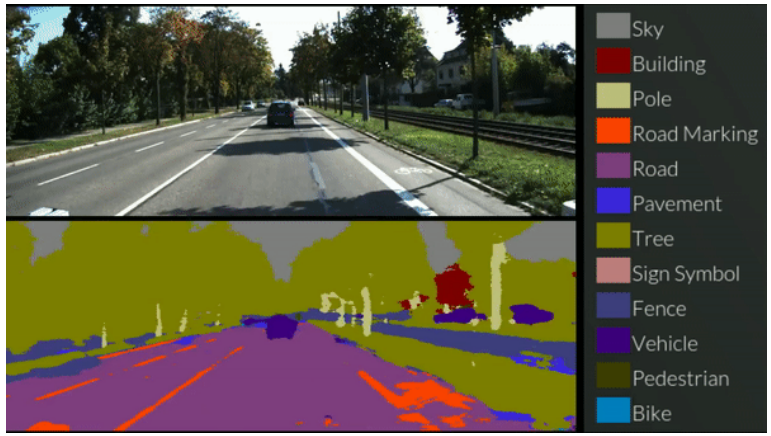
Deep Learning in Computer Vision

[*Krizhevsky*, 2012]



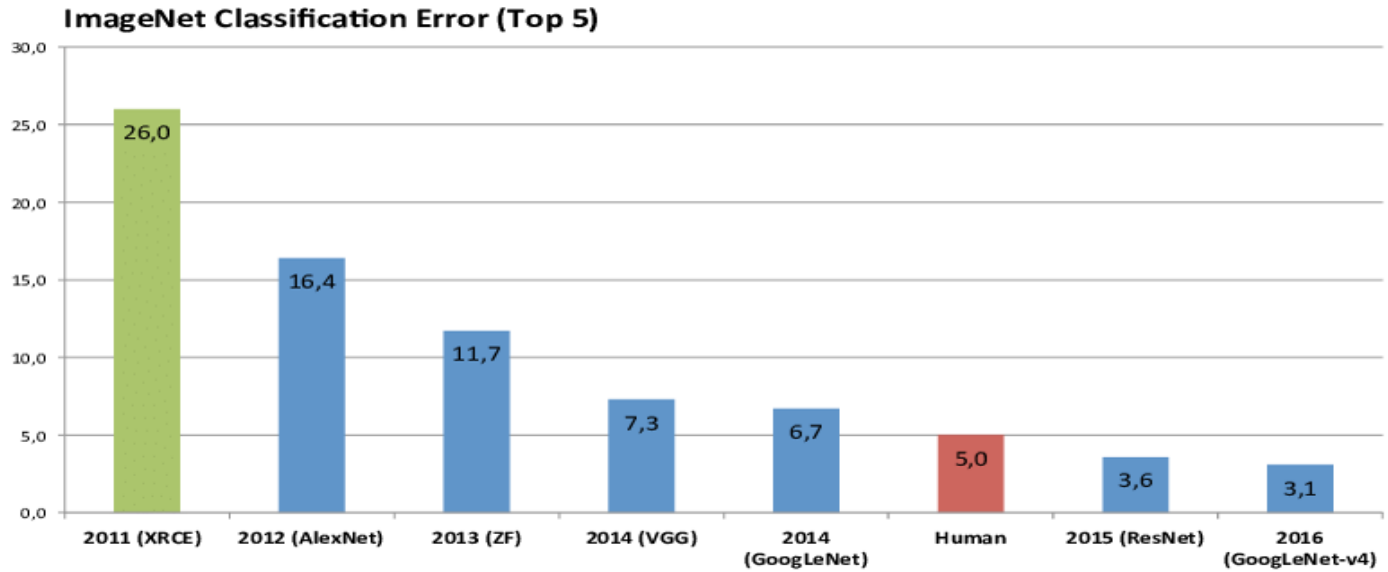
[*Girshick et al*. Fast R-CNN, 2015]





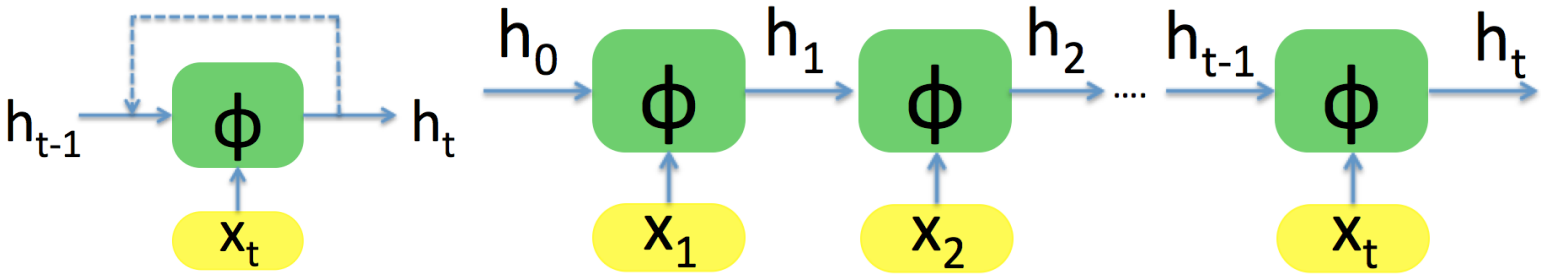
[*Kendall et al*. SegNet, 2015]

Brought significant improvements in multiple vision tasks



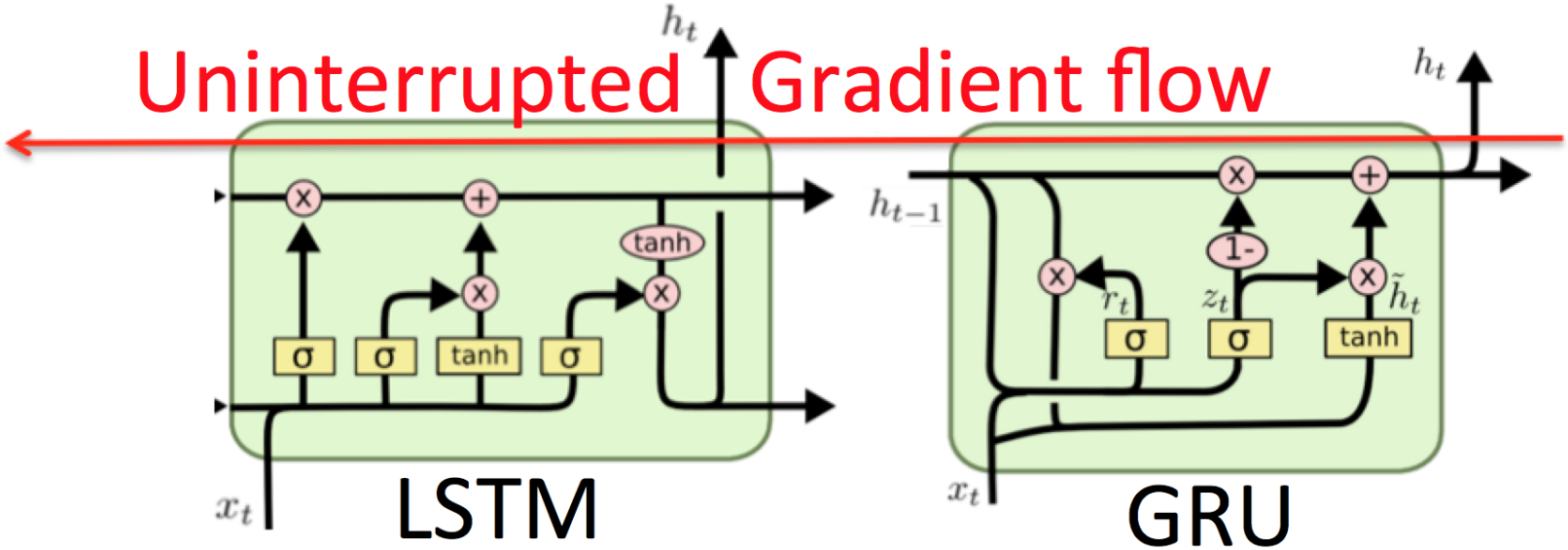
Recurrent Neural Networks (RNNs)

- **RNN Cell:** $h_t = \phi(x_t, h_{t-1}) = f(Ux_t + Wh_{t-1} + b_h)$ [Elman, 1990]
  - h_t: **network memory up to time t $\Rightarrow$ Sequence processing**



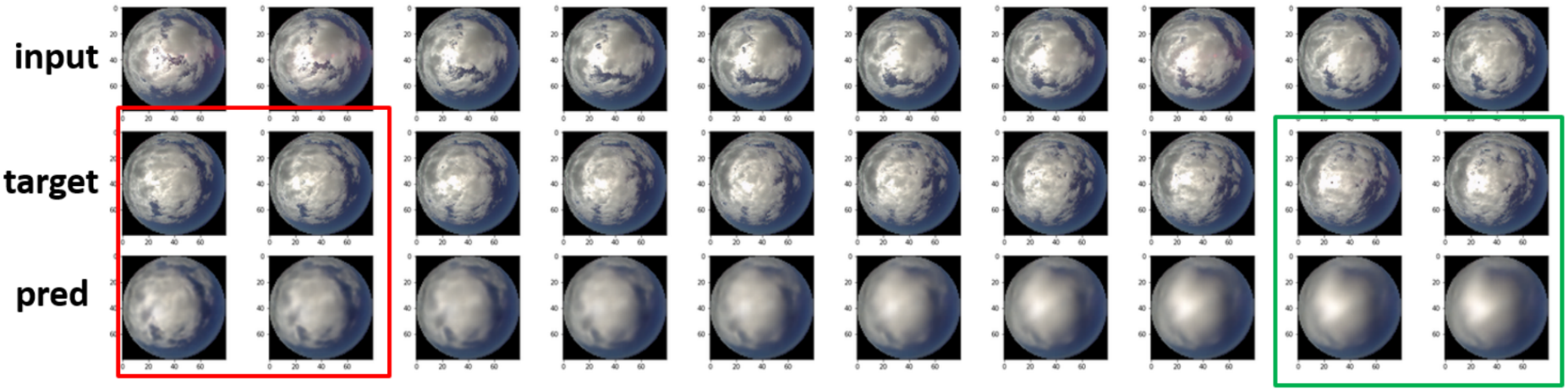
Folded RNN Unfolded RNN

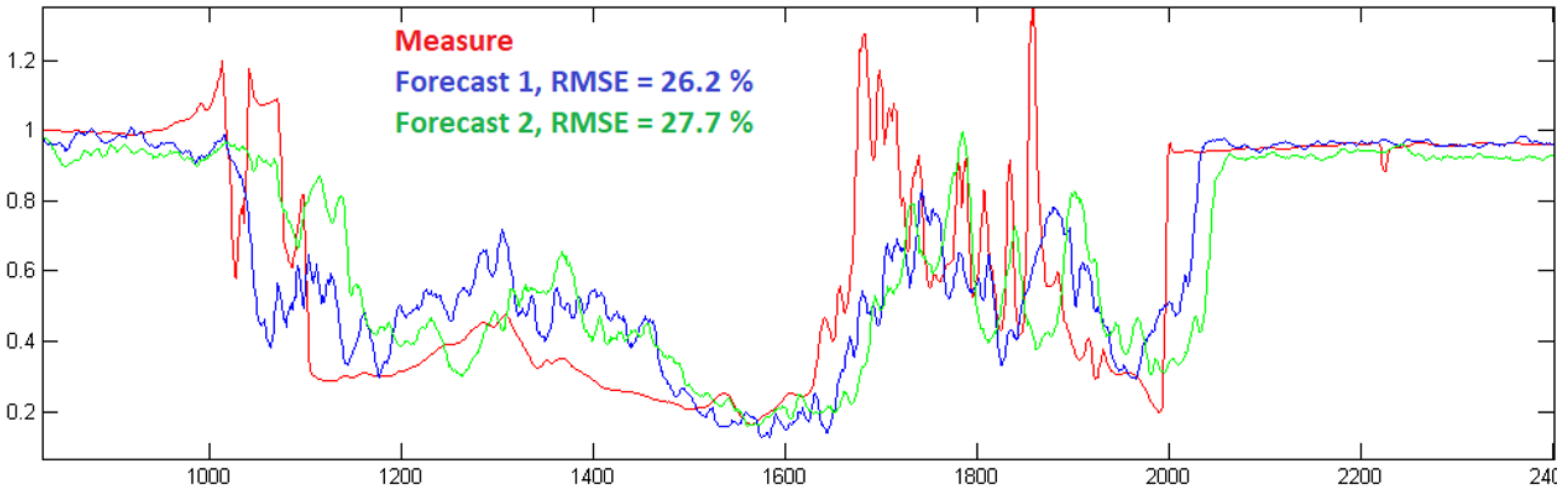
- **Specific architectures for vanishing gradients:**
  LSTM [Hochreiter and Schmidhuber, 1997], GRU [Cho et al., 2014]



Deep Learning for Sequence Processing

- RNNs SOTA for many sequential decision making tasks: speech, translation, text/music generation, times series, etc
- Ex: video forecasting, *i.e.* prediction future frames

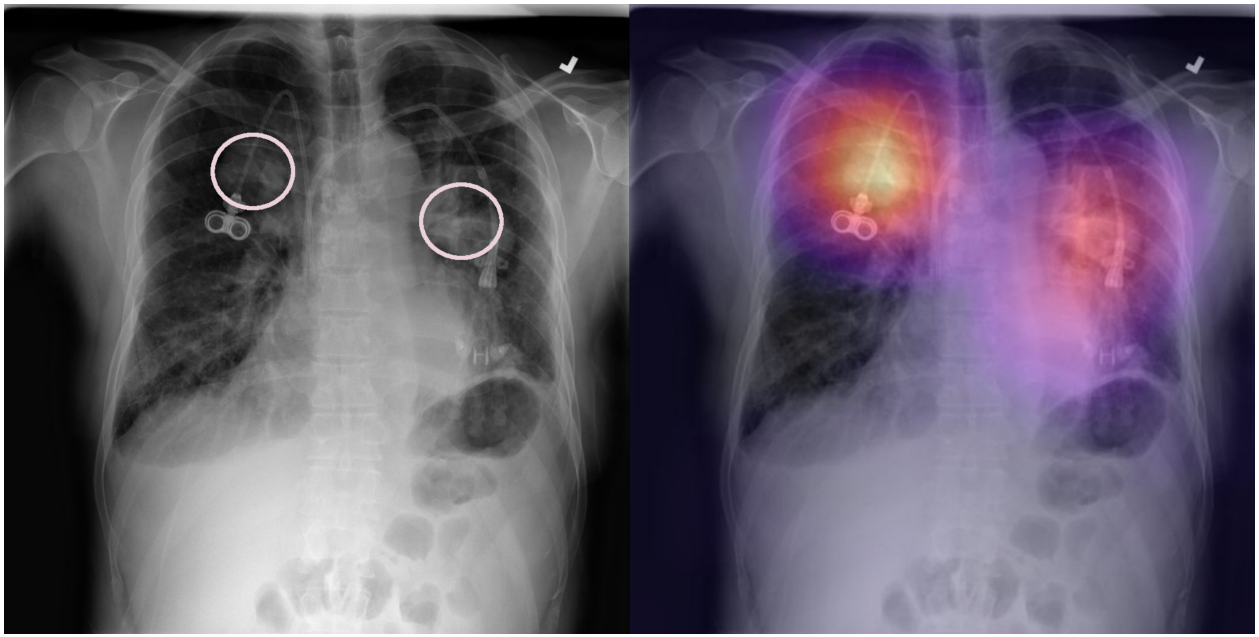


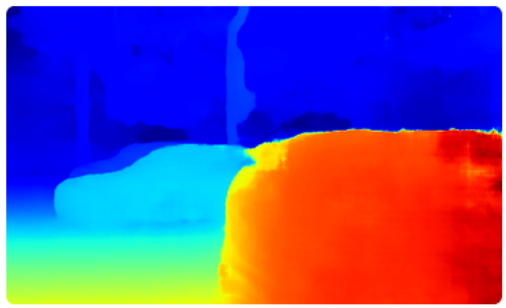


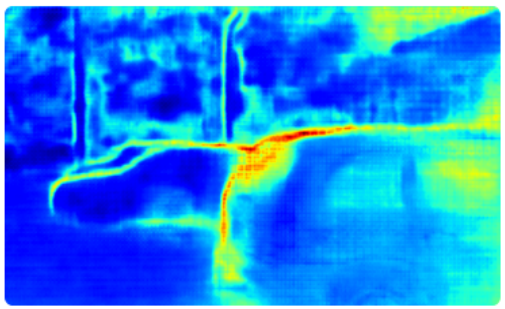
Robustness in Deep Learning

Deep Learning: huge gain in average performance, *e.g.* precision for classification

- Need for **performance certification in safety-critical applications: robustness**
  - Healthcare, autonomous steering, nuclear, defense, *etc*

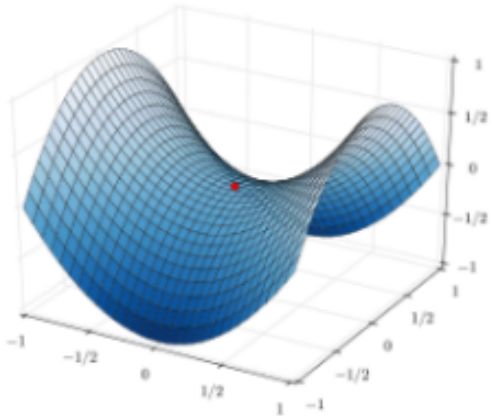






Performance certification

- **Formal understanding of deep learning:**
  - Understanding generalization performances with over-parameterized networks
  - Optimization, landscape loss function



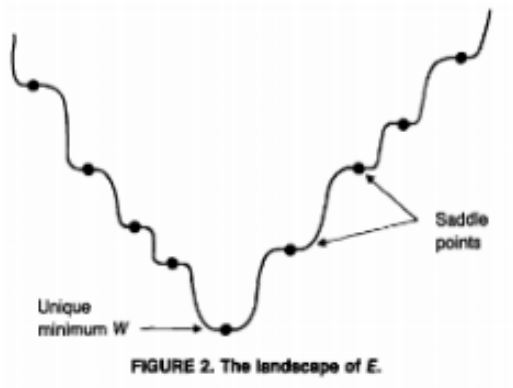
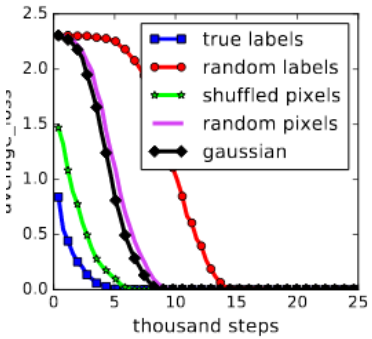
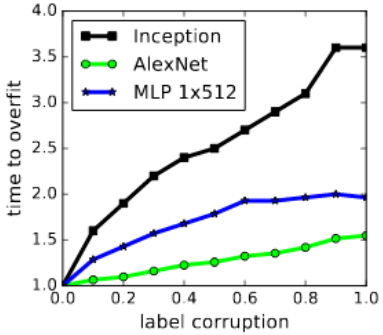


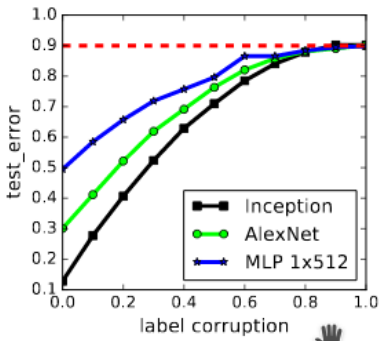
FIGURE 2. The landscape of E.



(a) learning curves



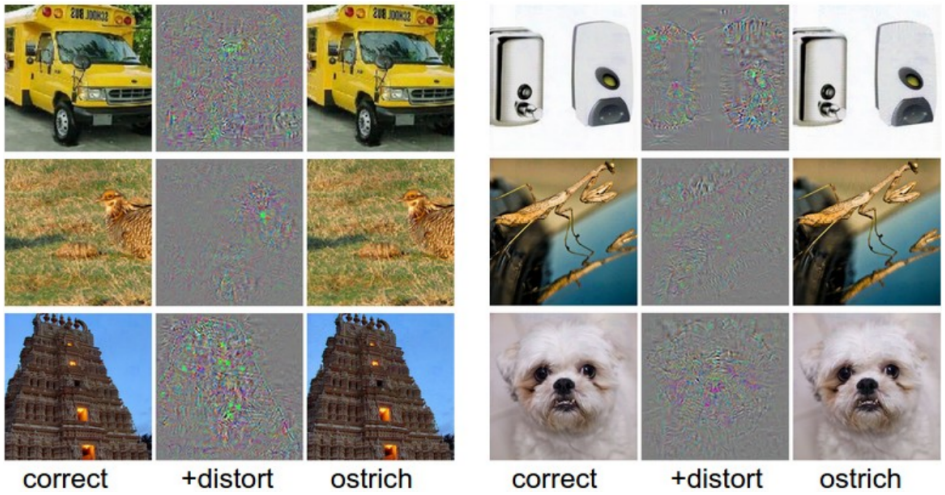
(b) convergence slowdown



(c) generalization error growth

Performance certification

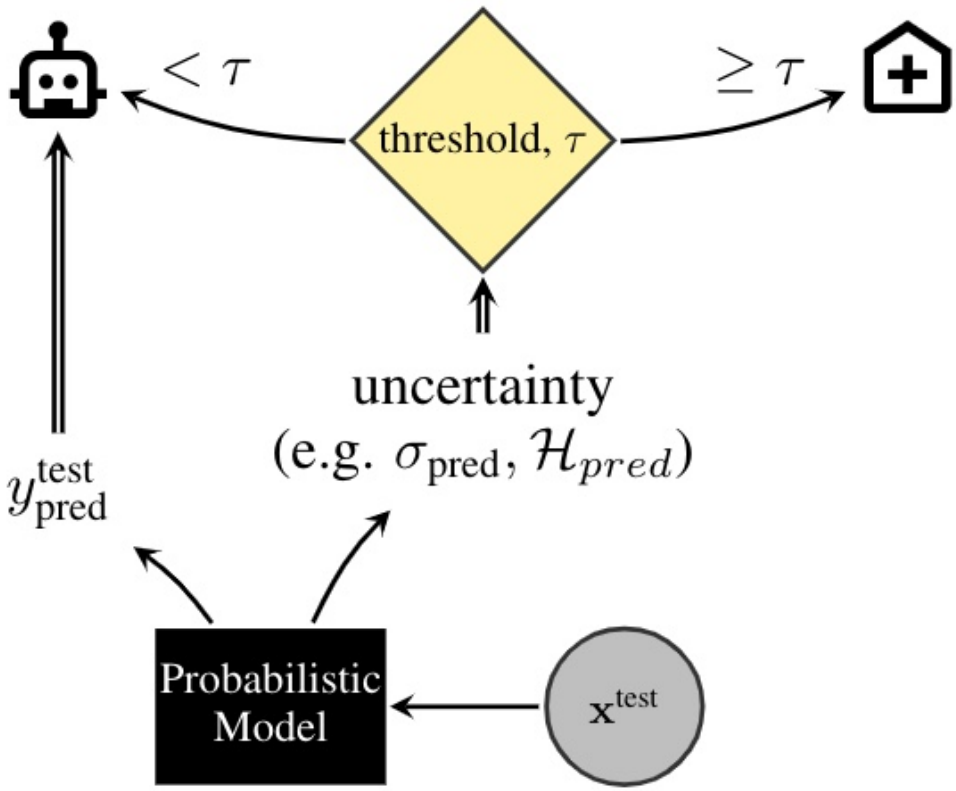
- **Stability of the decision function**, *e.g.* robustness to adversarial attacks



[Evtimov et al., 2017]

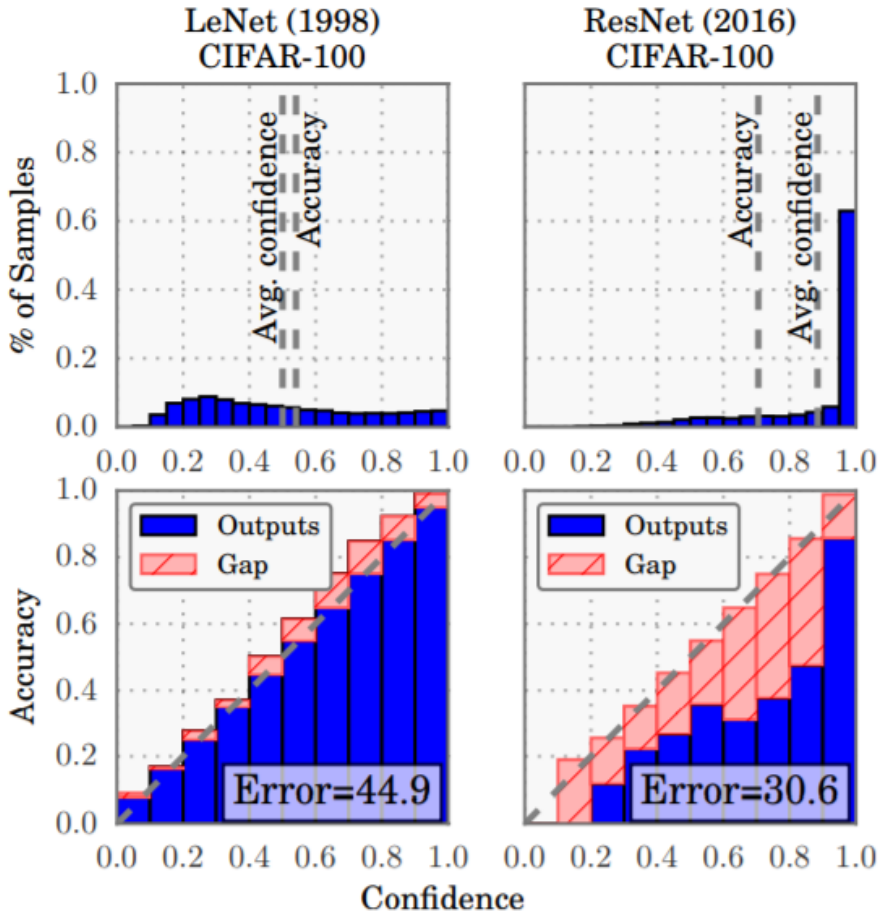
Performance certification

- **Reliable confidence / uncertainly estimation** of the decision process
  - "Know when you do not know"



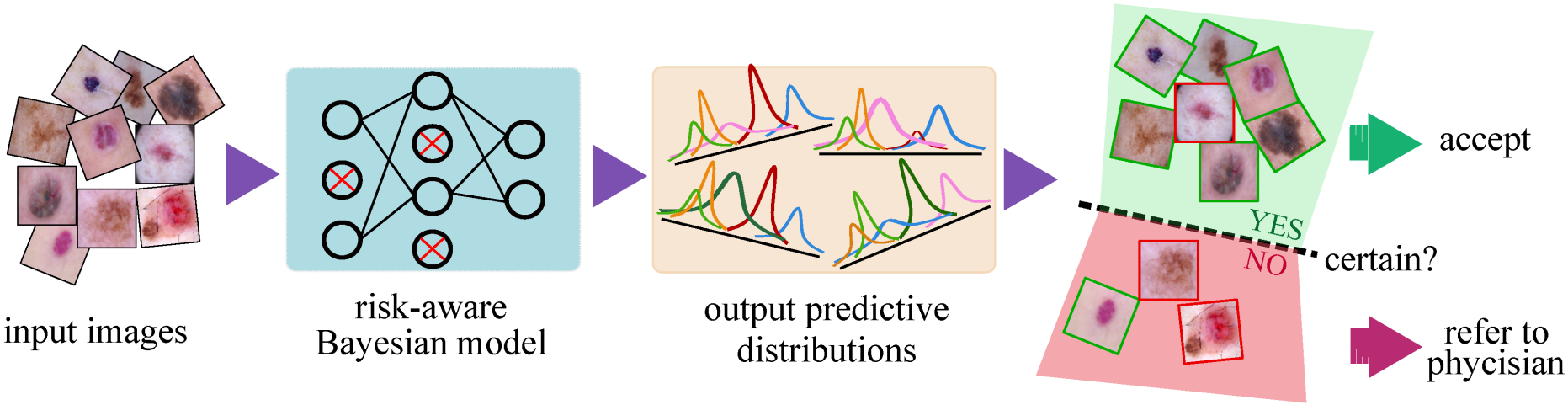
Performance certification

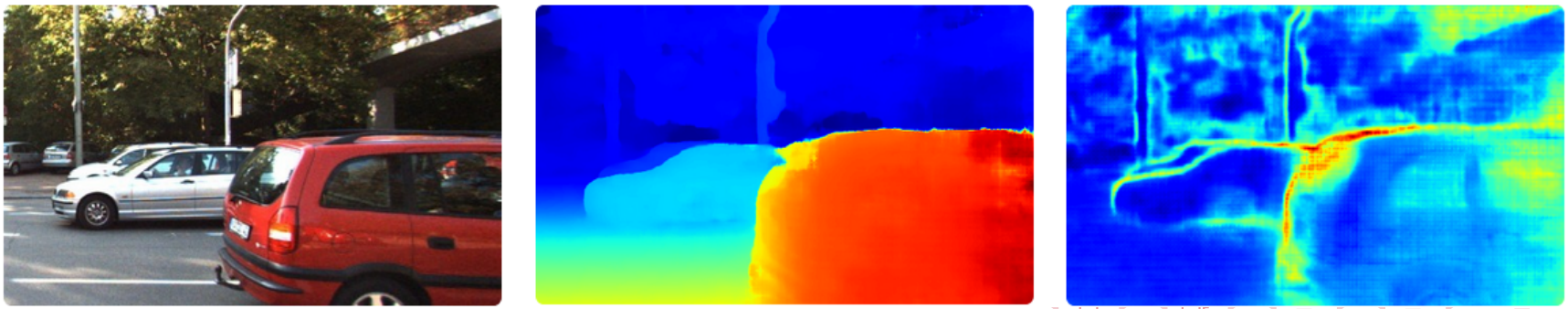
- **Reliable confidence / uncertainly estimation** of the decision process
  - Deep learning models overconfident, poor at this task [Guo et al., 2017]



Performance certification

- **Reliable confidence / uncertainly estimation** of the decision process
  - Crucial for deployment in healthcare / autonomous steering applications





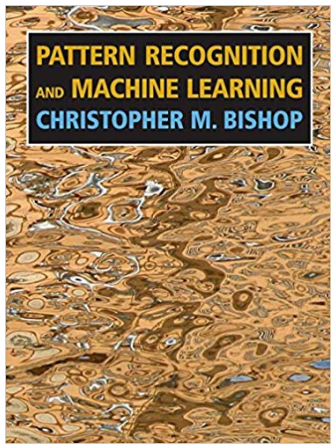
Robustness in Deep Learning

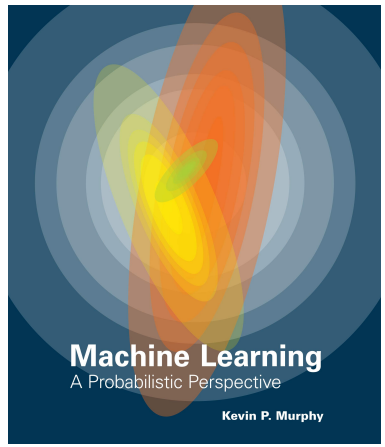
This course: 3 weeks on robust deep learning

`http://cedric.cnam.fr/~thomen/cours/RDFIA.html`

1. Bayesian models and uncertainty estimation
   - Bayesian linear regression and extension to non-linear feature maps
2. Bayesian deep learning
   - Approximate posterior inference, variational approximation
   - Monte Carlo dropout
3. Application of uncertainty, other robustness issues
   - failure prediction and OOD detection

- **References:**
  - Pattern Recognition and Machine Learning [Bishop, 2006]
  - Machine Learning: A Probabilistic Perspective [Murphy, 2012]
- **Eval:** practical session report (20% of RDFIA)





Outline

Type of uncertainties

Aleatoric uncertainty

- *aleator* (lat.) = dice player
- captures noise inherent in the observations
- due to: natural randomness, sensor quality
- inherent stochasticity
- cannot be reduced (need to change sensor), but can be learned
- Two types of aleatoric uncertainty:
  1. **homoscedastic uncertainty**: stays constant for different input values, limited, captures 'average' uncertainty
  2. **heteroscedastic uncertainty**: depends on the input, learned from data

Aleatoric uncertainty

e.g. occlusion, lack of visual features, sun glare, rain drops...

*Rain drops**

Lack of visual features

Glare

*Rain drops**

Lack of visual features

Glare

Type of uncertainties

Epistemic uncertainty

- *episteme* (gr.) = knowledge
- captures our lack of knowledge about the generating process
- can be explained away given enough data
- also refered as model uncertainty
- detects samples far from the training distribution
- use distribution over model's weights

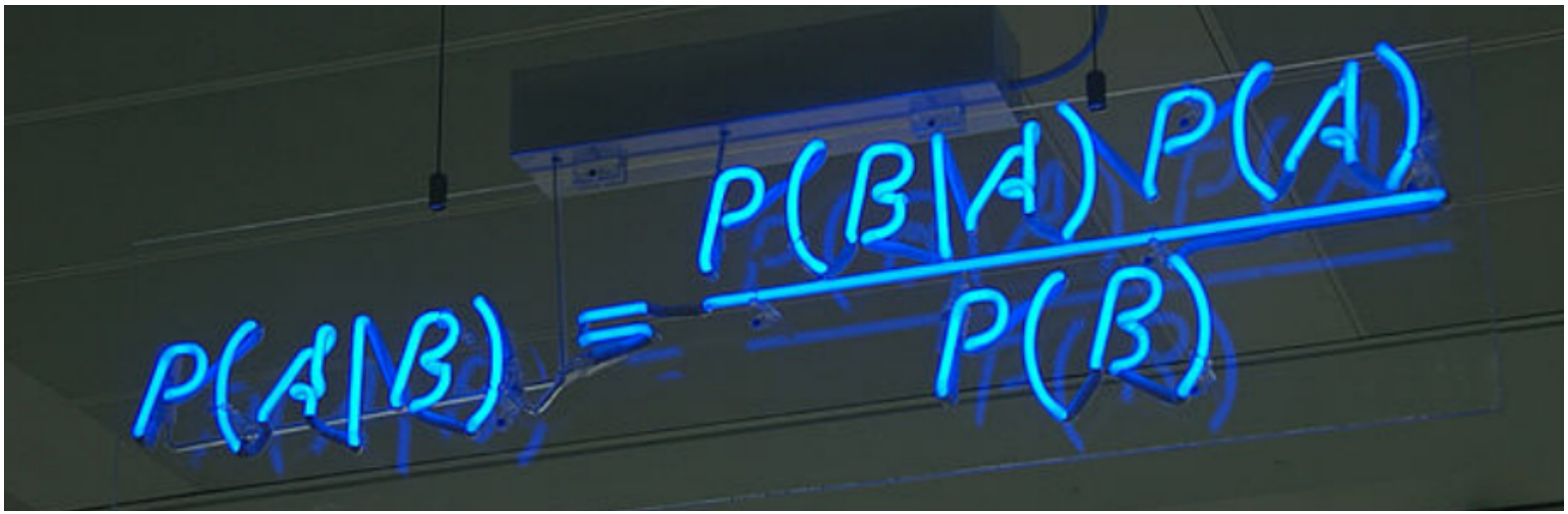


Outline

Bayesian Models

- Observed inputs $\mathrm{X} = \{\mathrm{x}_i\}_{i=1}^N$ and outputs $\mathrm{Y} = \{\mathrm{y}_i^*\}_{i=1}^N$
  - $\mathrm{x}_i \in \mathbb{R}^d$, $\mathrm{y}_i^* \in \mathbb{R}^K$ (classification or regression)
  - Model with parameters w: $\hat{\mathrm{y}}_i = f_{\mathrm{w}}(\mathrm{x}_i)$
- **Bayes rule:** $p(\mathrm{Y}, \mathrm{w}/\mathrm{X}) = p(\mathrm{Y}/\mathrm{X}, \mathrm{w})p(\mathrm{w}) = p(\mathrm{w}/\mathrm{X}, \mathrm{Y})p(\mathrm{Y}/\mathrm{X})$

$$\Rightarrow \boxed{p(\mathrm{w}/\mathrm{X}, \mathrm{Y}) = \frac{p(\mathrm{Y}/\mathrm{X}, \mathrm{w})p(\mathrm{w})}{p(\mathrm{Y}/\mathrm{X})} \propto p(\mathrm{Y}/\mathrm{X}, \mathrm{w})p(\mathrm{w})}$$

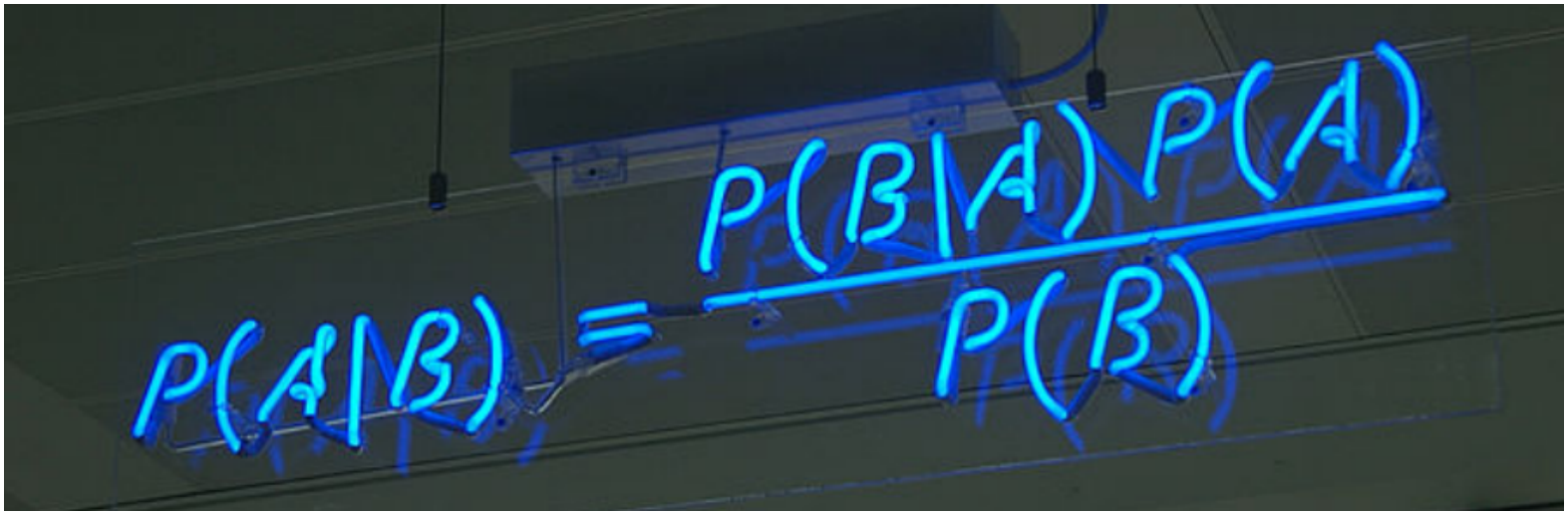


Bayesian Models

- **What we model:** $p(Y/X, w)$: likelihood and $p(w)$: prior knowledge
- **What we compute:** $p(w/X, Y) \propto p(Y/X, w)p(w)$ posterior
  biases prior $p(w)$ once data observed through likelihood $p(Y/w, X)$

How to estimate optimal w**?**

- **Maximum Likelihood (ML):** ignore (or assume uniform) prior
  $\Rightarrow$ find $\hat{w}$ s.t $p(Y/X, w)$ max
- **Maximum A Posteriori (MAP):** use prior $p(w)$
  $\Rightarrow$ find $\hat{w}$ s.t $p(w/X, Y)$ max

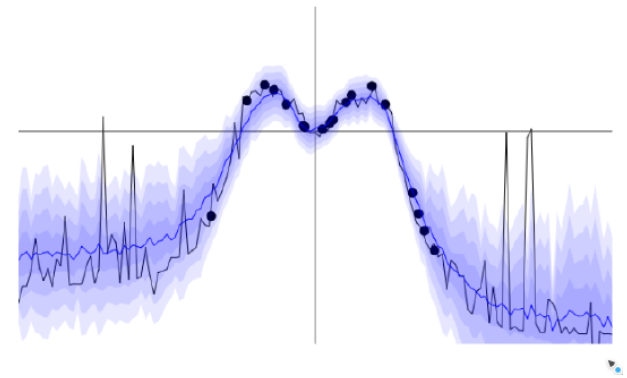


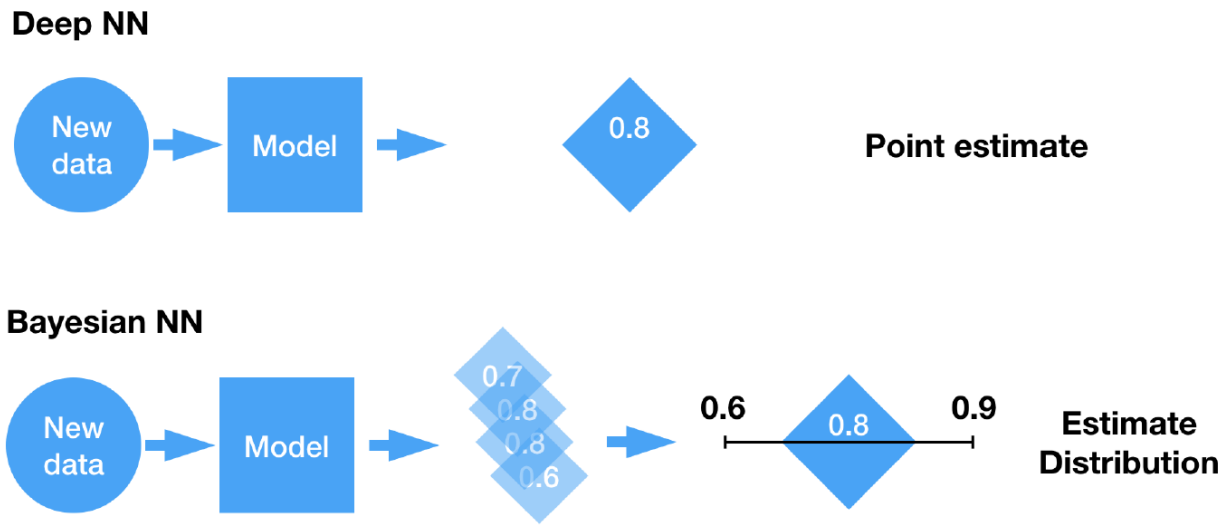
Bayesian Models & Uncertainty

- From posterior $p(w/X, Y) \Rightarrow$ compute **predictive distribution** given new input x^*: $p(y^*/x^*, Y, X) = \int p(y^*, w/x^*, Y, X)dw$

$$\boxed{p(y^*/x^*, Y, X) = \int p(y^*/x^*, w)p(w/X, Y)dw} = \mathbb{E}_{p(w|\mathcal{D})}[p(y^*|x^*, w)]$$

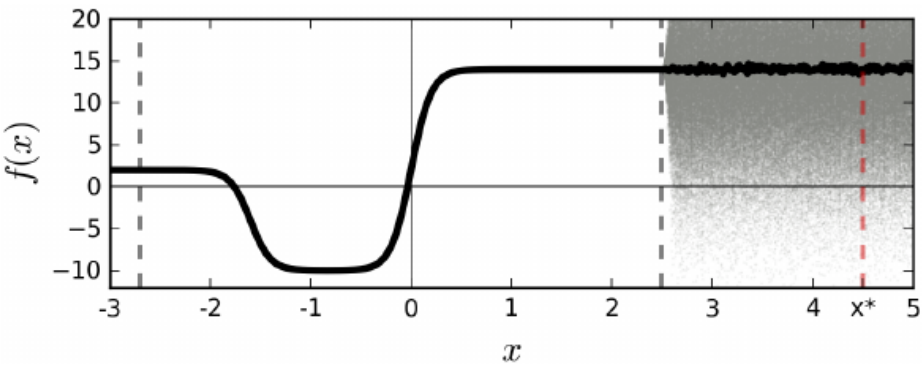
  - Prob distribution of outputs $\neq$ point estimate (ML/MAP)
  - Naturally gives a measure of uncertainty

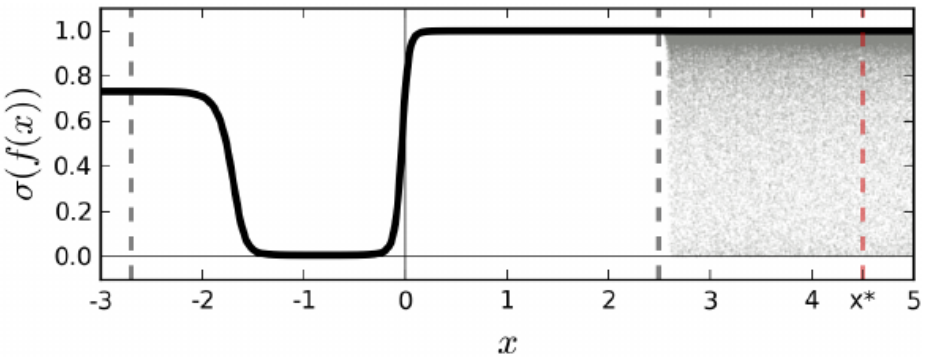




Point-wise vs distribution modeling

In binary classification, training a neural network to learn a function $f^w(x)$ (*e.g.* tanh) from a 1D dataset X=[-3,2], and apply Softmax to obtain probabilities.

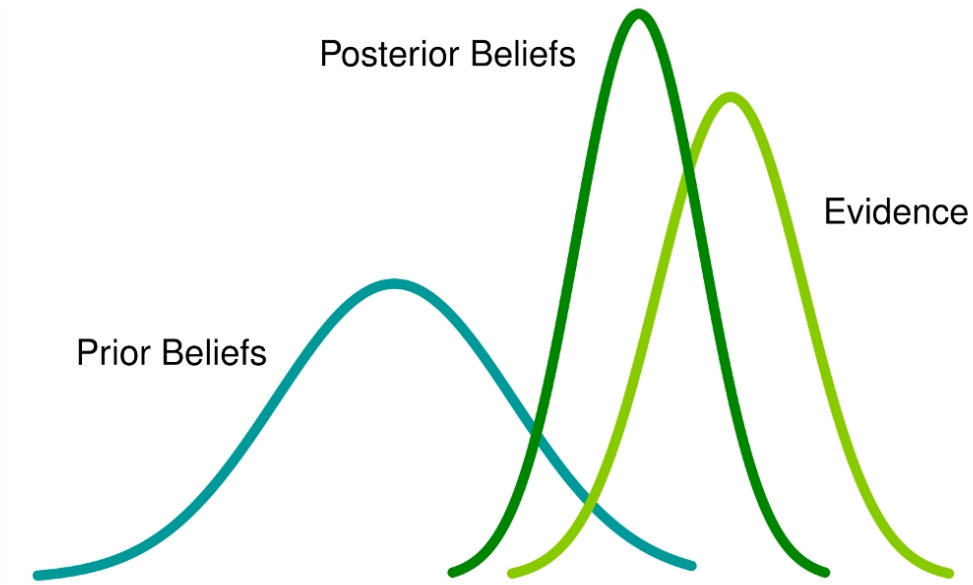




- Point estimate through Softmax
  $\Rightarrow$ unjustified high confidence far from data
- Distribution $\Rightarrow$ better reflects our model uncertainty

Bayesian Models & Uncertainty

- RECAP: for uncertainty estimate with Bayesian models
  1. Define prior $p(w)$ and likelihood $p(Y/X, w)$
  2. Compute posterior distribution $p(w/X, Y)$
  3. Compute predictive distribution $p(y^*/x^*, Y, X)$
- Easy? NO !! $\Rightarrow$ steps 2 and 3 computationally hard in general!
  - Typically no closed form for step 2
  - High-dimensional integration for step 3



Outline

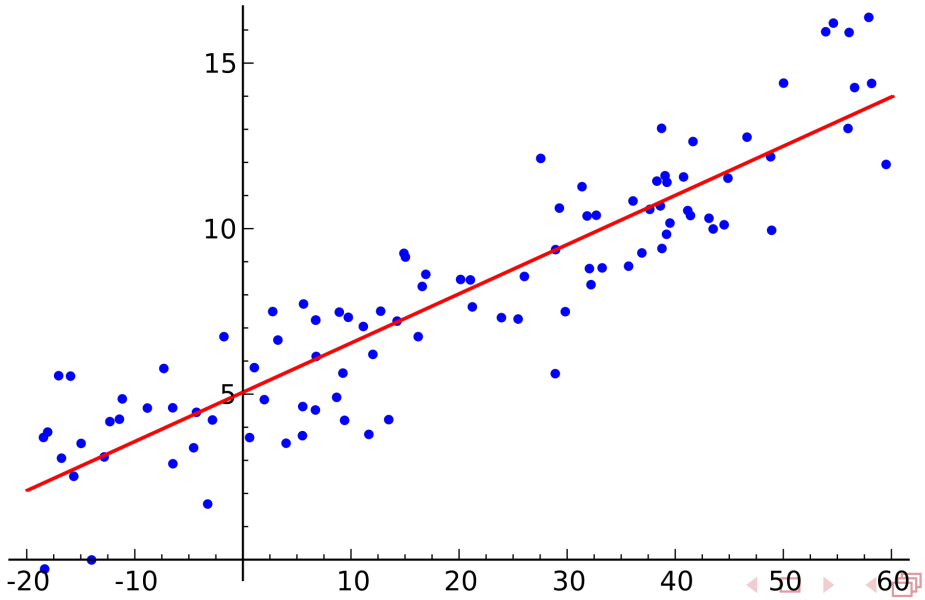
Probabilistic Linear Regression

- N training examples $(x_i, y_i)_{i\in\{1;N\}}$; $x_i \in \mathbb{R}^p$, $y_i \in \mathbb{R}^K$
- Matrix notation including bias in w: Φ of size $N \times (p+1)$

$$\Phi = \begin{pmatrix} 1 & x_{11} & \dots & x_{1p} \\ 1 & x_{21} & \dots & x_{2p} \\ \dots & \dots & \dots & \dots \\ 1 & x_{N1} & \dots & x_{Np} \end{pmatrix}$$

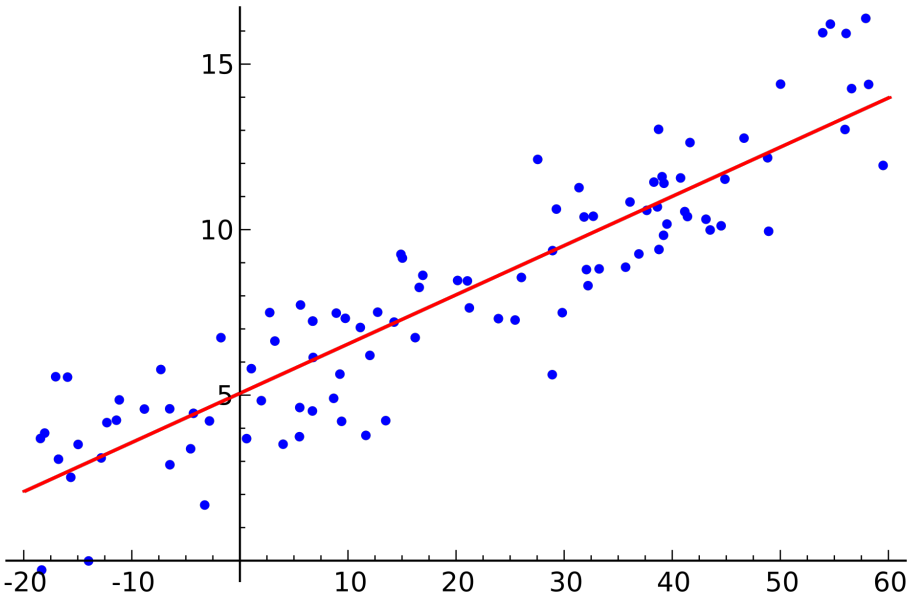
- $\boxed{Y = \Phi w + \varepsilon}$, $\varepsilon \in \mathbb{R}^{N\times K}$, $\varepsilon_{i,k} \sim \mathcal{N}(0, \sigma^2)$
- $Y \in \mathbb{R}^{N\times K}$, $\Phi \in \mathbb{R}^{N\times(p+1)}$, $w \in \mathbb{R}^{(p+1)\times K}$
  - $\Phi_i^T := \begin{pmatrix} 1 & x_{i1} & \dots & x_{ip} \end{pmatrix}^T \in \mathbb{R}^{1\times(p+1)}$
  - $y_i = \Phi_i^T w + \varepsilon_i$, $y_i \in \mathbb{R}^{1\times K}$, $\varepsilon_i \in \mathbb{R}^{1\times K}$

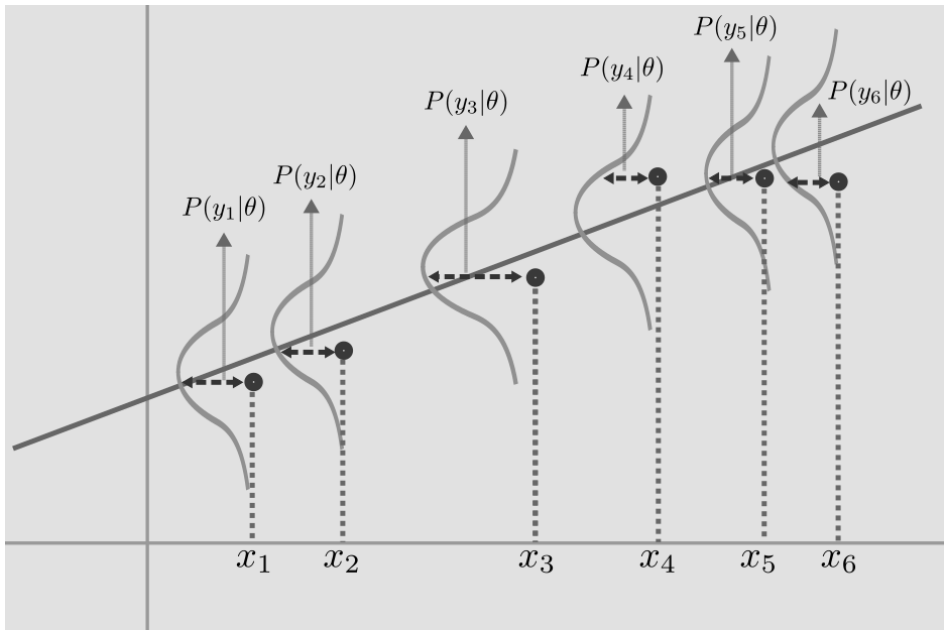
- Ex: scalar inputs and outputs $p = 1$, $K = 1 \Rightarrow \Phi \in \mathbb{R}^{N\times 2}$, $w \in \mathbb{R}^2$:



Probabilistic Linear Regression

- N training examples (x_i, y_i) $y_i = \Phi_i^T w + \varepsilon_i$, with $\epsilon_i \sim \mathcal{N}(0,1)$
- $p(y_i/x_i, w) \sim \mathcal{N}(\Phi_i^T w, \sigma^2)$ or $p(y_i - \Phi_i^T w) \sim \mathcal{N}(0, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(y_i - \Phi_i^T w)^2}{2\sigma^2}}$
  - $p(y_i/x_i, w)$: likelihhod, $\sigma \sim$ aleatoric uncertainty
  - σ independent of x $\Rightarrow$ homoscedastic uncertainty





Probabilistic Linear Regression

- Trained with Maximum Likelihood (ML)
  - Examples assumed to be *i.i.d* (independent and identically distributed)

    $\Rightarrow p(\mathrm{Y}/\mathrm{X}, \mathrm{w}, \sigma) = \prod_{i=1}^{N} p(y_i/x_i, \mathrm{w}, \sigma)$
  - MLE: $(\hat{\mathrm{w}}, \hat{\sigma}) = \arg\max_{(\mathrm{w},\sigma)} p(\mathrm{X}, \mathrm{Y}/\mathrm{w}, \sigma) = \arg\min_{\mathrm{w},\sigma} -\sum_{i=1}^{N} log\left[p(\mathrm{y}_i/\mathrm{x}_i, \mathrm{w})\right]$
- MLE solution w: $\hat{\mathrm{w}} = \arg\min_{\mathrm{w}} C(\mathrm{w}) = \arg\min_{\mathrm{w}} \sum_{i=1}^{N} (\mathrm{y}_i - \Phi_i^T \mathrm{w})^2$

  $\Rightarrow$ Ordinary least square problem (closed form):
  - $C(\mathrm{w}) = ||\mathrm{Y} - \Phi\mathrm{w})||^2 = (\mathrm{Y} - \Phi\mathrm{w})^T(\mathrm{Y} - \Phi\mathrm{w})$, $\nabla_{\mathrm{w}} C = 2\Phi^T(Y - \Phi\mathrm{w})$
  - $\nabla_{\mathrm{w}} C = 0 \Leftrightarrow \Phi^T\Phi\mathrm{w} = \Phi^T\mathrm{Y}$

$$\boxed{\hat{\mathrm{w}} = (\Phi^T\Phi)^{-1}\Phi^T\mathrm{Y}}$$

- ML solution σ: $\arg\min_{\sigma}[\ N\ log(\sigma) + \frac{1}{2\sigma^2} \sum_{i=1}^{N} (\mathrm{y}_i - \Phi_i^T \mathrm{w})^2]$
  - Closed form solution: $\hat{\sigma^2} = \frac{1}{N} \sum_{i=1}^{N} (\mathrm{y}_i - \Phi_i^T \mathrm{w})^2 \Rightarrow$ interpretation: data std

Limits of MLE

- Learning model to predict coin toss with MLE
  - Bernoulli variable X with param p: $P(\mathrm{x}|p) = \prod_{i=1}^{N} P(x_i|p) = \prod_{i=1}^{N} p^{x_i}(1-p)^{1-x_i}$
  - MLE: $\ln P(\mathrm{x}|p) = \sum_{i=1}^{N} [x_i \ln p + (1-x_i)\ln(1-p)] \Rightarrow \boxed{p_{MLE} = \frac{1}{N}\sum_{i=1}^{N} x_i}$
  - MLE: predict $P(X|p_{MLE}) = 1$ for all futures tosses!

- Using prior knowledge on p, *e.g.* $P(p) = 0.5$ or $P(p) = 0.3 \Rightarrow$ MAP

Bayesian Linear Regression

Include prior $p(w) \Rightarrow$ biased posterior $p(w/X, Y)$ (MAP)

- Choose **Prior**, *e.g.* $p(w|\alpha) = \mathcal{N}(w; 0, \alpha^{-1} I)$
- **Likelihood**, as before: $p(y_i/x_i, w) = \mathcal{N}(\Phi_i^T w, \beta^{-1})$ with $\beta = \frac{1}{2\sigma^2}$
  $\Rightarrow$ **Compute posterior** $p(w/x_i, y_i) \propto p(y_i/x_i, w) p(w)$
  - Here, prior same form as likelihood (Gaussian), *i.e.* **Prior conjugate to likelihood**
    $\Rightarrow$ **Posterior as the same form as the prior**
    $\Rightarrow$ **Closed-form for the posterior, which is also Gaussian!**
  - With $p(w|\alpha) = \mathcal{N}(w; 0, \alpha^{-1} I)$ and $p(y_i/x_i, w) = \mathcal{N}(\Phi_i^T w, \beta^{-1})$, we can show that:

$$p(w/X, Y) = \mathcal{N}(w|\boldsymbol{\mu}, \Sigma)$$
$$\Sigma^{-1} = \alpha \boldsymbol{I} + \beta \Phi^T \Phi$$
$$\boldsymbol{\mu} = \beta \Sigma \Phi^T Y$$

- Closed form solution for MAP ($\boldsymbol{\mu}$, median $\leftrightarrow$ mode)
- $\alpha \to 0 \Rightarrow$ recover ML ; $N = 0 \Rightarrow$ recover prior

Bayesian Linear Regression

Come from general results [Bishop, 2006]

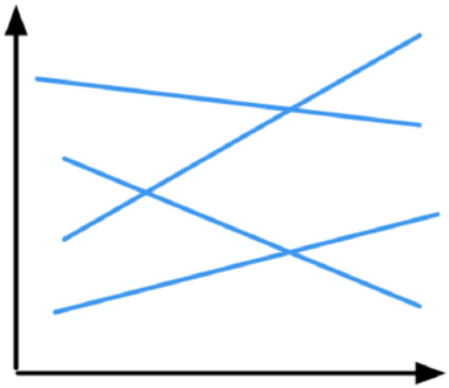
- Assume that:
  - $p(\mathrm{x}) = \mathcal{N}(\mathrm{x}|\boldsymbol{\mu}_x, \Sigma_x)$
  - $p(\mathrm{y}|\mathrm{x}) = \mathcal{N}(\mathrm{y}|A\mathrm{x} + b, \Sigma_y)$
- Then: $p(\mathrm{x}|\mathrm{y}) = \mathcal{N}(\mathrm{x}|\boldsymbol{\mu}_{x|\mathrm{y}}, \Sigma_{x|\mathrm{y}})$, with:

$$\Sigma_{\boldsymbol{x}|\mathrm{y}}{}^{-1} = \Sigma_x^{-1} + \boldsymbol{A}^T \Sigma_y^{-1} \boldsymbol{A}$$

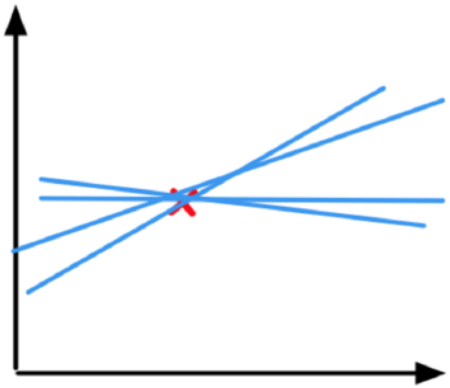
$$\boldsymbol{\mu}_{x|\mathrm{y}} = \Sigma_{\boldsymbol{x}|\mathrm{y}} [\boldsymbol{A}^T \Sigma_y^{-1} (\mathrm{y} - b) + \Sigma_x^{-1} \boldsymbol{\mu}_x]$$

Bayesian Linear Regression: Posterior Sampling

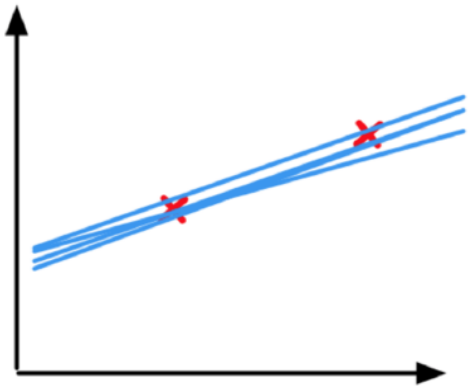
- $p(\mathrm{w}/\mathrm{X},\mathrm{Y}) = \mathcal{N}(\mathrm{w};\boldsymbol{\mu},\Sigma) \Rightarrow$ we can sample from posterior
  - No observation: $p(\mathrm{w}/\mathrm{X},\mathrm{Y}) = p(\mathrm{w})$
  - $N \geq 1$: prior biased by data likelihood



no observations

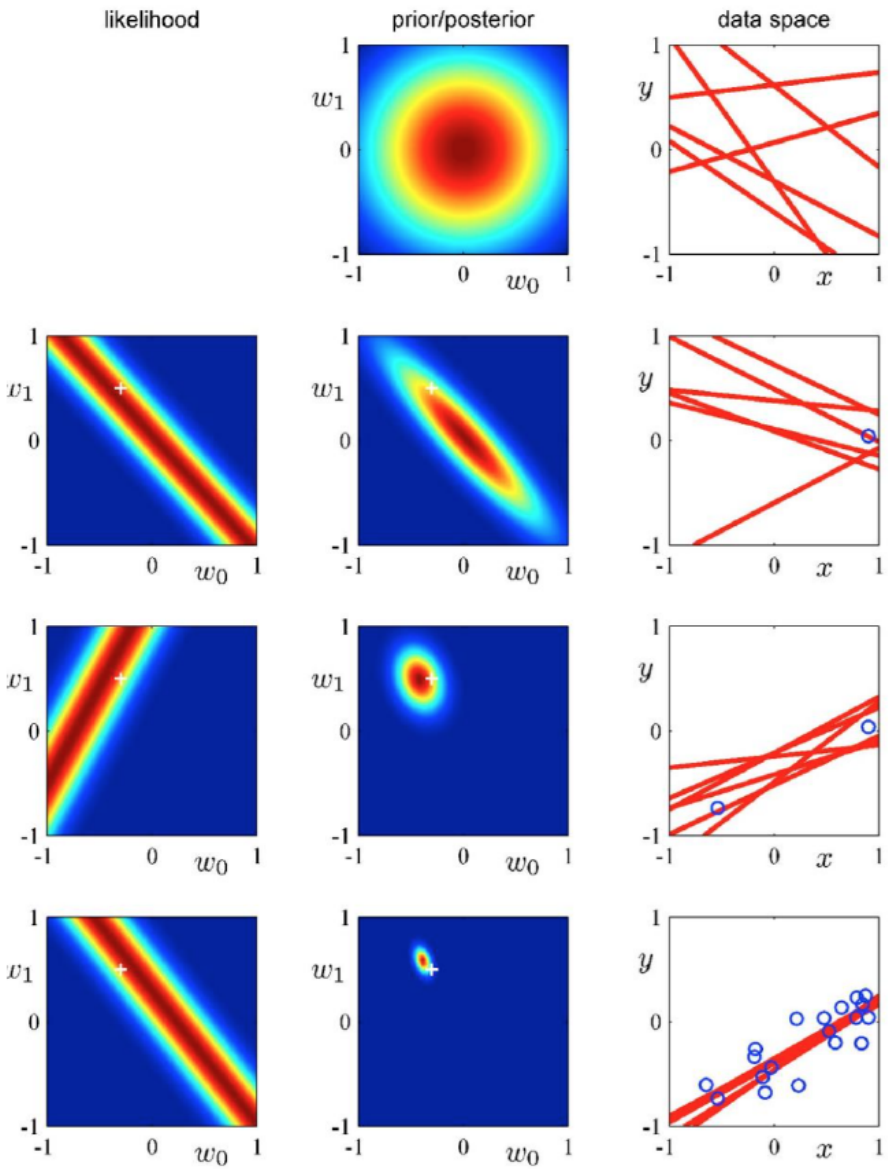


one observation



two observations

Bayesian Linear Regression: Posterior Sampling



0 data points are observed.

1 data point is observed.

2 data points are observed.

20 data points are observed.

Bayesian Linear Regression

- Note on MAP estimate:

$$
\begin{aligned}
\mathrm{w}_{MAP} &= \arg\min_{\mathrm{w}} -\sum_{n=1}^{N} \log p(\mathrm{w}|\mathrm{x}_n, y_n, \beta, \alpha) \\
&= \arg\min_{\mathrm{w}} -\sum_{n=1}^{N} \log p(y_n|\mathrm{x}_n, \mathrm{w}, \beta, \alpha) - \log p(\mathrm{w}|\alpha) \\
&= \arg\min \frac{\beta}{2} \sum_{i=1}^{N} ||y_n - f^{\mathrm{w}}(\mathrm{x}_n)||^2 + \frac{\alpha}{2} \mathrm{w}^T \mathrm{w}
\end{aligned}
$$

$\Rightarrow$ **adding a Gaussian prior with precision on weights α acts like L_2 regularisation (weight decay) with $\lambda = \alpha/\beta$**

Bayesian Linear Regression: Predictive Distribution

$p(\mathrm{w}|\mathcal{D}, \alpha, \beta) \Rightarrow$ compute predictive distribution by marginalizing over w:

- $p(y^*|\mathrm{x}^*, \mathcal{D}, \alpha, \beta) = \int p(y^*|\mathrm{x}^*, \mathrm{w}, \beta) p(\mathrm{w}|\mathcal{D}, \alpha, \beta) dw$
  - $p(y^*|\mathrm{x}^*, \mathrm{w}, \beta) = \mathcal{N}(y^*; \Phi(\mathrm{x}^*)^T \mathrm{w}, \beta^{-1})$: likelihood
  - $p(\mathrm{w}|\mathcal{D}, \alpha, \beta) = \mathcal{N}(\mathrm{w}; \boldsymbol{\mu}, \Sigma)$: w posterior
    - $\Sigma^{-1} = \alpha \boldsymbol{I} + \beta \Phi^T \Phi$
    - $\boldsymbol{\mu} = \beta \Sigma \Phi^T \mathrm{Y}$
- $p(y^*|\mathrm{x}^*, \mathcal{D}, \alpha, \beta)$: convolution of two Gaussians $\Rightarrow$ Gaussian
  - Mean of predictive distribution $\mu^T \Phi(\mathrm{x}^*)$
  - Variance of predictive distribution $\sigma^2_{pred}(\mathrm{x}^*) = \frac{1}{\beta} + \Phi(\mathrm{x}^*)^T \Sigma \Phi(\mathrm{x}^*)$

$$p(y^*|\mathrm{x}^*, \mathcal{D}, \alpha, \beta) = \mathcal{N}(y^*; \mu^T \Phi(\mathrm{x}^*), \frac{1}{\beta} + \Phi(\mathrm{x}^*)^T \Sigma \Phi(\mathrm{x}^*))$$

Bayesian Linear Regression: Predictive Distribution

$$p(y^*|x^*, \mathcal{D}, \alpha, \beta) = \mathcal{N}(y^*; \mu^T \Phi(x^*), \frac{1}{\beta} + \Phi(x^*)^T \Sigma \Phi(x^*))$$

- $\sigma^2_{pred}(x^*) = \frac{1}{\beta} + \Phi(x^*)^T \Sigma \Phi(x^*)$
- β is actually our noise representation (**aleatoric**)
- $\Phi(x^*)^T \Sigma \Phi(x^*)$ is uncertainty over parameters w (**epistemic**)
- $\sigma^2_{pred}(x^*)$ actually depends on N, $\sigma^2_{pred}(x^*, N)$
  - $\sigma^2_{pred}(x^*, N+1) < \sigma^2_{pred}(x^*, N)$
  - $\lim_{N \to \infty} \Phi(x^*)^T \Sigma \Phi(x^*) = 0$: epistemic uncertainty removed by adding data samples

Bayesian Linear Regression: Predictive Distribution

$$p(y^*|x^*, \mathcal{D}, \alpha, \beta) = \mathcal{N}(y^*; \mu^T \Phi(x^*), \sigma^2_{pred}(x^*))$$

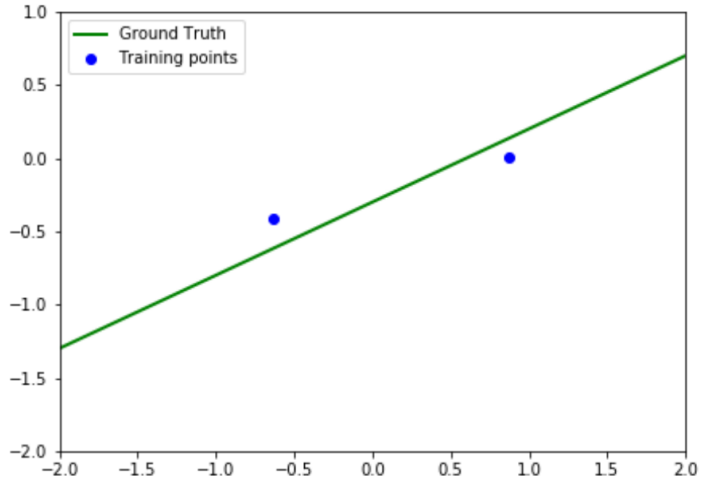
- $\sigma^2_{pred}(x^*) = \beta^{-1} + \Phi(x^*)^T \Sigma \Phi(x^*)$
- $\Phi(x^*)^T \Sigma \Phi(x^*)$ is uncertainty over parameters w (**epistemic**)
- **Practical session:** in case of 1D inputs & outputs, *i.e.* $x_i \in \mathbb{R}$, $X \in \mathbb{R}^{N \times 1}$, $X \in \mathbb{R}^{N \times 1}$, $w \in \mathbb{R}^2$:

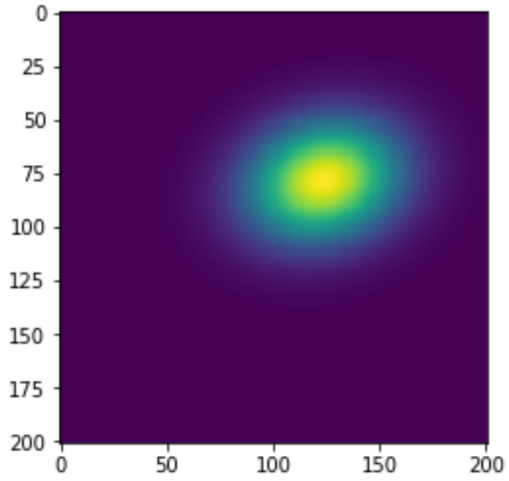
$$\Sigma^{-1} = \begin{pmatrix} \alpha + \beta N & \beta 1^T X \\ \beta 1^T X & \alpha + \beta X^T X \end{pmatrix}$$

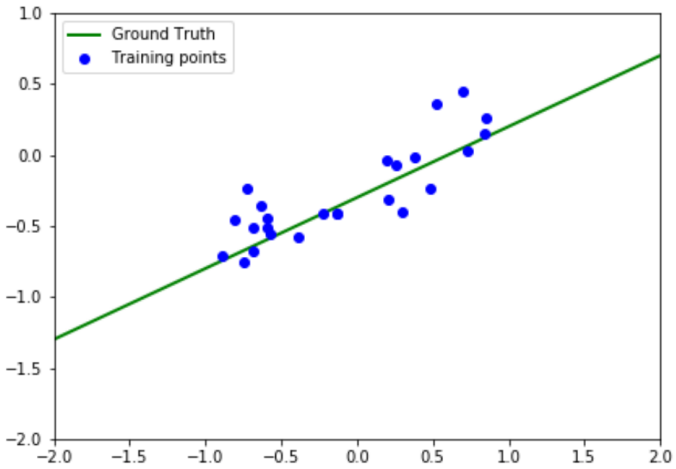
$\Rightarrow \Phi(x^*)^T \Sigma \Phi(x^*)$ **Increases when** x^* **far from training data**

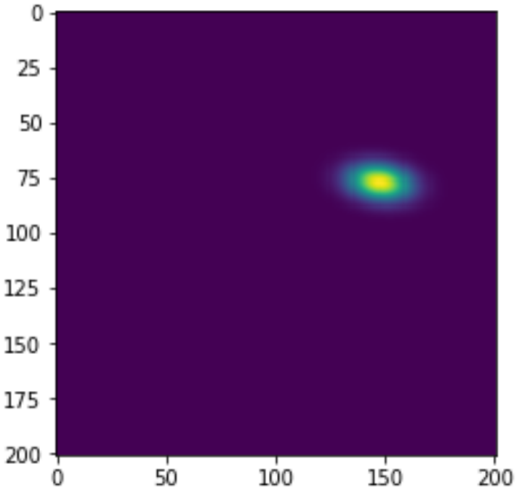
  - $x^*_{min} = \frac{\sum_i x_i}{N + \alpha/\beta}$

Practical session: compute posterior



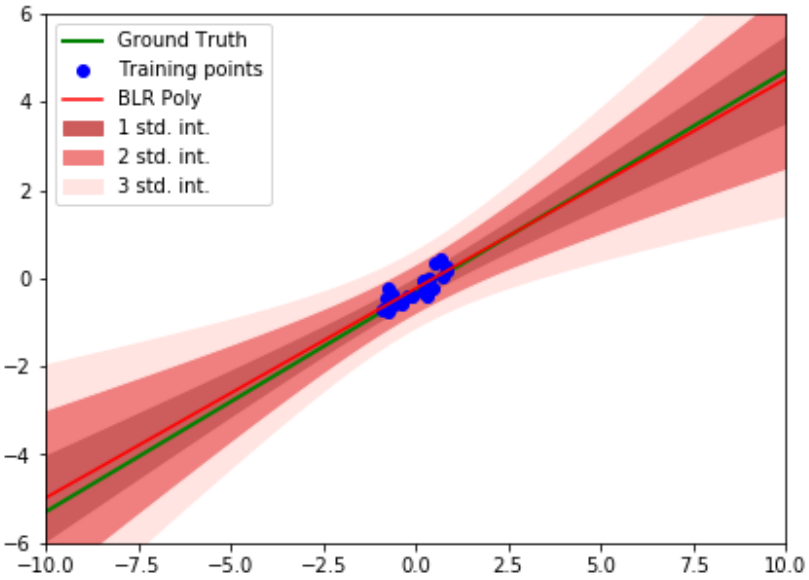


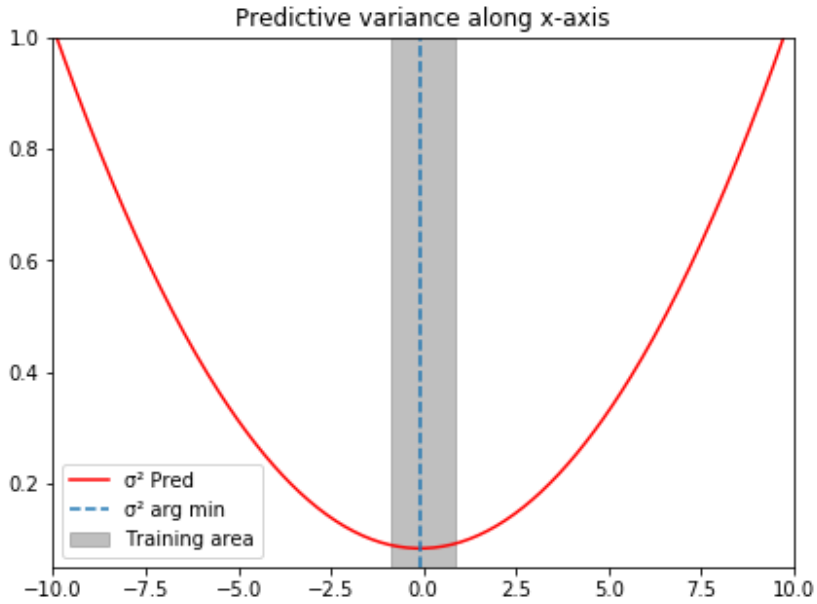




Bayesian Linear Regression

Practical session: predictive distribution and uncertainty

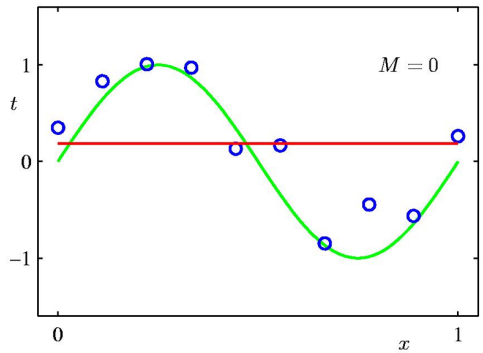


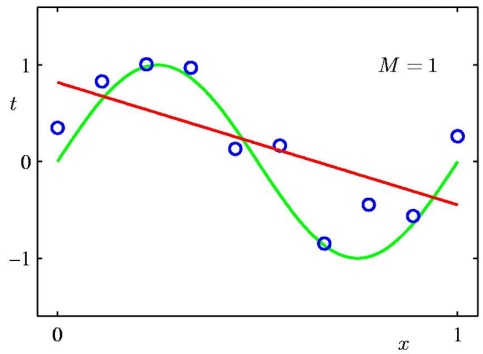


Non-Linear Regression

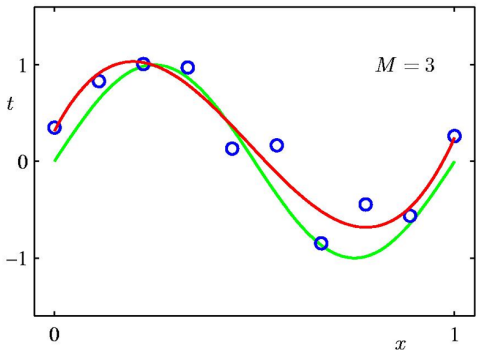
- Linear regression: limited in many datasets
- Non-linear extension by designing explicit non-linear feature maps Φ
  - Ex: Polynomial regression for 1D input, *i.e.* $x_i \in \mathbb{R}$:

$$\Phi = \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^M \\ 1 & x_2 & x_2^2 & \dots & x_2^M \\ \dots & \dots & \dots & \dots & \\ 1 & x_N & x_N^2 & \dots & x_N^M \end{pmatrix} \qquad Y = \Phi w + \varepsilon \qquad \varepsilon = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_N \end{pmatrix}^T, \ \varepsilon_i \sim \mathcal{N}(0, \sigma^2)$$

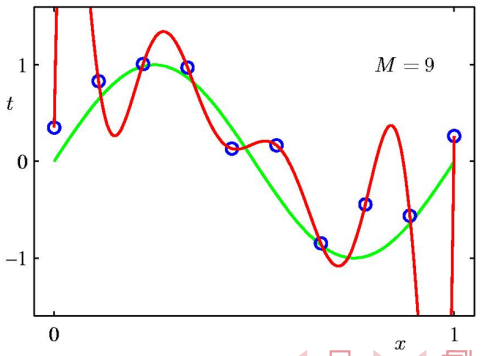




← Poor representations of $\sin(2\pi x)$



← Best Fit to $\sin(2\pi x)$



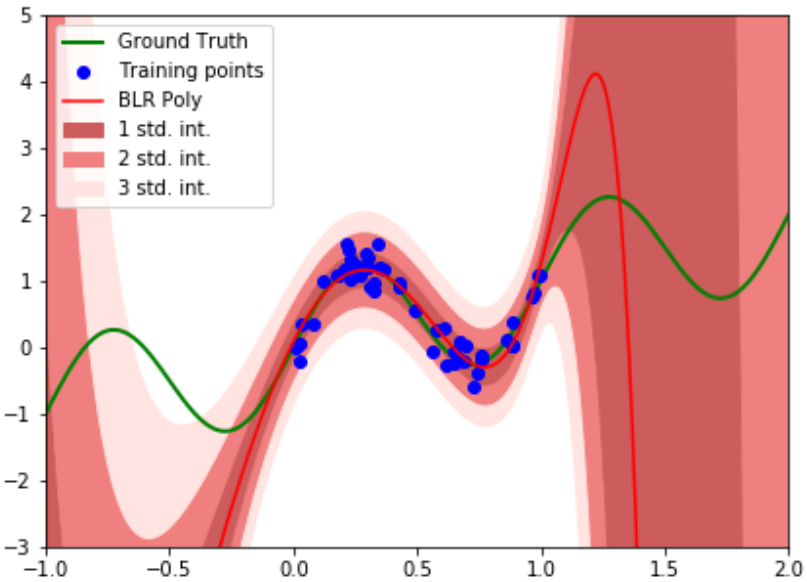
Over Fit Poor representation of $\sin(2\pi x)$

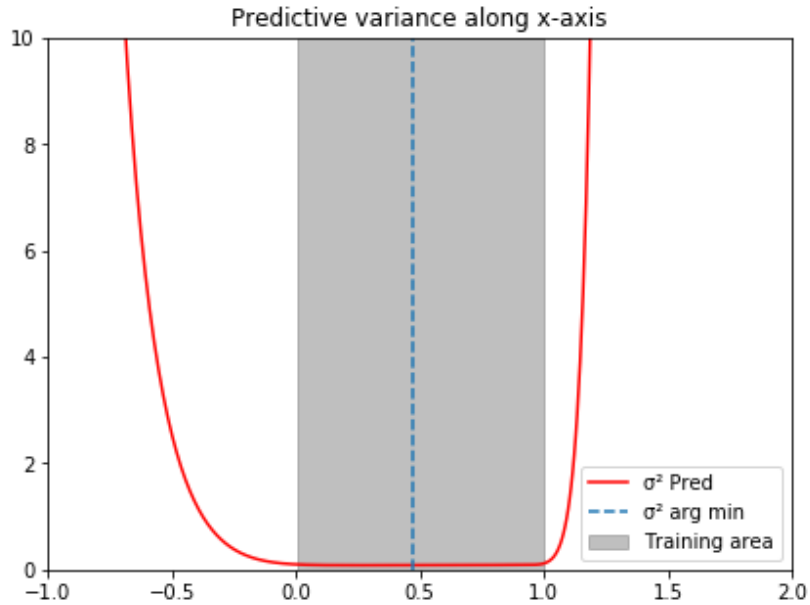
Bayesian Polynomial Regression

- Polynomial regression for 1D input, *i.e.* $x_i \in \mathbb{R}$:

$$\Phi = \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^M \\ 1 & x_2 & x_2^2 & \dots & x_2^M \\ \dots & \dots & \dots & \dots & \\ 1 & x_N & x_N^2 & \dots & x_N^M \end{pmatrix} \qquad \mathsf{Y} = \Phi \mathsf{w} + \boldsymbol{\varepsilon} \qquad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_N \end{pmatrix}^T, \ \varepsilon_i \sim \mathcal{N}(0, \sigma^2)$$

- Apply Bayesian linear regression in non-linear feature space Φ
  - Same closed-form solution for posterior and predictive distribution in Φ
- Practical session: regression for $f(x) = x + sin(2\Pi x)$, $M = 10$, $\alpha = 0.05$



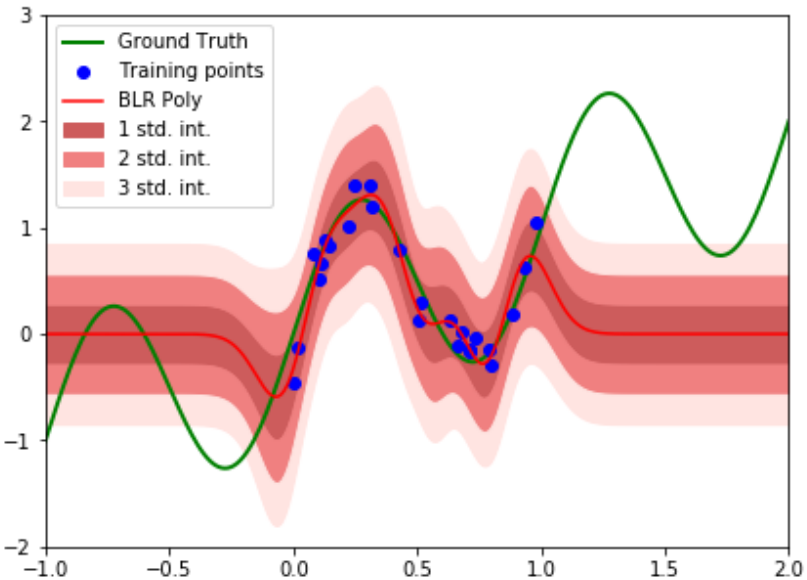


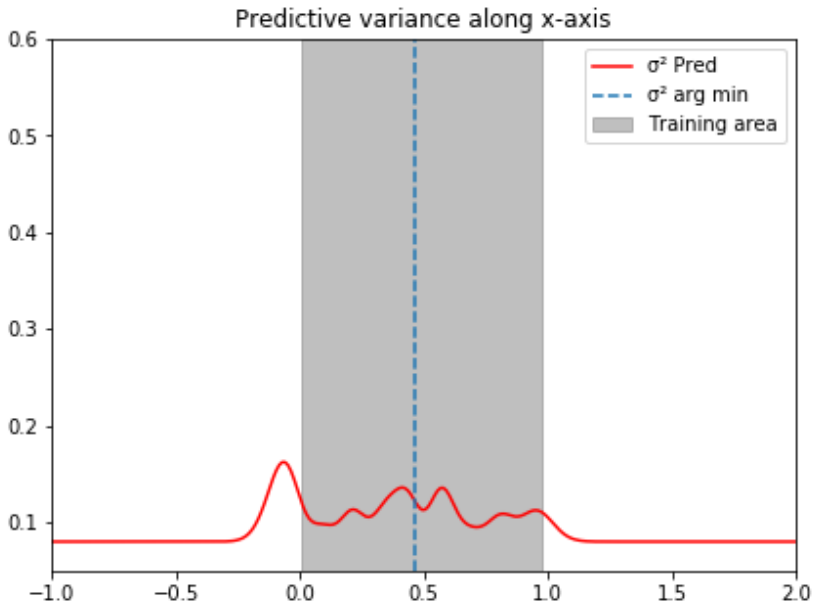
RBF Gaussian Regression

- Gaussian regression for 1D input, *i.e.* $x_i \in \mathbb{R}$, $\Phi_j(x_i) = exp\left(-\frac{(x_i-\mu_j)^2}{2s^2}\right)$:

$$\Phi = \begin{pmatrix} \Phi_1(x_1) & \Phi_2(x_1) & \dots & \Phi_M(x_1) \\ \dots & \dots & \dots & \dots \\ \Phi_1(x_N) & \Phi_2(x_N) & \dots & \Phi_M(x_N) \end{pmatrix} \quad Y = \Phi w + \varepsilon, \ \varepsilon_i \sim \mathcal{N}(0, \sigma^2)$$

- Apply Bayesian linear regression in non-linear feature space Φ





- Practical session: issue with localized features, epistemic uncertainty $\Phi(x^*)^T \Sigma \Phi(x^*) \to 0$ far from trainig data (μ_j)

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