27 th internet seminar: harmonic analysis techniques for elliptic operators
Solutions for exercise sheet 7

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Exercise 1 (Characterization of m-accretivity)**.** Provide a proof of Lemma 7.11.

Proof. ($\Longrightarrow$) For $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) > 0$ it follows by assumption that the operator $(\lambda + L)^{-1}$ is bounded. Thus we have

$$\|(\lambda + L)u\|^2 \geq \mathrm{Re}(\lambda)^2 \|u\|^2, \tag{1}$$

for all $u \in \mathrm{dom}(L)$.

For the first statement pick $\lambda > 0$ and $u \in \mathrm{dom}(L)$. Using equation (1) and the identity

$$\|(\lambda + L)u\|^2 = \langle(\lambda + L)u, (\lambda + L)u\rangle = \|Lu\|^2 + 2\mathrm{Re}(\lambda \langle Lu, u\rangle) + |\lambda|^2 \|u\|^2$$

we get

$$\begin{aligned}
\|Lu\|^2 + 2\mathrm{Re}(\lambda\langle Lu, u\rangle) + |\lambda|^2 \|u\|^2 &\geq \mathrm{Re}(\lambda)^2 \|u\|^2 \\
\|Lu\|^2 + 2\lambda\mathrm{Re}(\langle Lu, u\rangle) + \lambda^2 \|u\|^2 &\geq \lambda^2 \|u\|^2 \\
\|Lu\|^2 + 2\lambda\mathrm{Re}(\langle Lu, u\rangle) &\geq 0.
\end{aligned}$$

Since λ was arbitrary, it follows that $\mathrm{Re}(\langle Lu, u\rangle) \geq 0$ for all $u \in \mathrm{dom}(L)$.

For the second statement pick any $\lambda > 0$. Since L is m-accretive, $-\lambda$ will be in the resolvent set of L and thus the operator $(\lambda + L)^{-1}$ will be bounded, which means that

$$H = \mathrm{dom}((\lambda + L)^{-1}) = \mathrm{ran}(\lambda + L).$$

($\Longleftarrow$) Using the Cauchy-Schwarz inequality we get

$$\begin{aligned}
\|(\lambda + L)u\| \|u\| &\geq |\langle(\lambda + L)u, u\rangle| \\
&\geq \mathrm{Re}(\langle(\lambda + L)u, u\rangle) \\
&= \mathrm{Re}(\lambda) \|u\|^2 + \mathrm{Re}(\langle Lu, u\rangle) \\
&\geq \mathrm{Re}(\lambda) \|u\|^2, \qquad (2)
\end{aligned}$$

for every $u \in \mathrm{dom}(L)$ and $\lambda \in \mathbb{C}$. Here we used the fact that $\mathrm{Re}(\langle Lu, u\rangle) \geq 0$. By assumption there exists some $\lambda_0 > 0$ such that $\mathrm{ran}(\lambda_0 + L)$ is dense in H. Because of equation (2) the operator $(\lambda_0 + L)^{-1}$ is continuous on $\mathrm{ran}(\lambda_0 + L)$ and thus its domain is closed. This implies that $(\lambda_0 + L)^{-1}$ is a bounded operator and that $-\lambda_0 \in \rho(L)$. From (2) it also follows that for any $-\nu \in \rho(L)$ we have

$$\|\mathrm{R}(-\nu, L)\| = \left\|(\nu + L)^{-1}\right\| \leq \frac{1}{\mathrm{Re}(\nu)}. \tag{3}$$

From Proposition 1.15 and $\mathrm{Re}(-\lambda_0) \leq \|\mathrm{R}(-\nu, L)\|^{-1}$ it follows that for every $\mu \in \mathbb{C}$ if $|\lambda_0 - \mu| < \mathrm{Re}(\lambda_0)$ then $-\mu \in \rho(L)$. In particular, since $\lambda_0 > 0$ and

$$\left|\lambda_0 - \frac{1}{2}\lambda_0\right| = \frac{1}{2}\lambda_0 < \lambda_0 = \mathrm{Re}(\lambda_0)$$

it follows $-\frac{3}{2}\lambda_0 \in \rho(L)$. By equation (3) applied with $\nu = \frac{3}{2}\lambda_0$ we see that all complex numbers whose distance from $-\frac{3}{2}\lambda_0$ is at most $\frac{3}{2}\mathrm{Re}(\lambda_0)$ are in the resolvent set, in particular $-\frac{9}{4}\lambda_0$. We continue this reasoning inductively and find that for any $n \in \mathbb{N}$ the complex numbers whose distance from $-\left(\frac{3}{2}\right)^n \lambda_0$ is at most $\left(\frac{3}{2}\right)^n \mathrm{Re}(\lambda_0)$ are in the resolvent set. In particular this shows that all negative real numbers are in the resolvent set.

Now let $\lambda \in \mathbb{C}$ be a complex number such that $\mathrm{Re}(\lambda) > 0$. We have shown that $-\mathrm{Re}(\lambda) \in \rho(L)$. From equation (3) it follows that the set

$$\{z \in \mathbb{C} \mid \mathrm{Re}(z) = -\mathrm{Re}(\lambda), |\mathrm{Im}(z)| < \mathrm{Re}(\lambda)\}$$

is contained in $\rho(L)$. Using the same reasoning as above we obtain, via Proposition 1.15, that the set

$$\left\{z \in \mathbb{C} \mid \mathrm{Re}(z) = -\mathrm{Re}(\lambda), |\mathrm{Im}(z)| < \frac{3}{2}\mathrm{Re}(\lambda)\right\}$$

is contained in $\rho(L)$. Again we continue inductively and for any $n \in \mathbb{N}$ find that the set

$$\left\{z \in \mathbb{C} \mid \mathrm{Re}(z) = -\mathrm{Re}(\lambda), |\mathrm{Im}(z)| < \left(\frac{3}{2}\right)^n \mathrm{Re}(\lambda)\right\}$$

is contained in $\rho(L)$. In particular any complex number z with $\mathrm{Re}(z) = -\mathrm{Re}(\lambda)$ is in the resolvent set. Applying the equation (3) completes the proof.

In the case where L is bounded, the spectrum of L will be compact. Thus for large enough $\lambda > 0$ the number $-\lambda$ will be in the resolvent set of L which implies

$$\mathrm{ran}(L + \lambda) = H.$$

$\square$

Exercise 2 (m-accretive fractional powers)**.** Let L be an m-accretive operator in H and let $\alpha \in (0, 1)$. Prove L^α is m-accretive-

Proof. We use Lemma 7.11 to prove the statement. By Proposition 6.18 we have the representation

$$L^\alpha u = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty t^{\alpha-1} L(t + L)^{-1} u \, dt$$

and thus

$$\mathrm{Re}\left(\langle L^\alpha u, u\rangle\right) = \mathrm{Re}\left(\left\langle \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty t^{\alpha-1} L(t + L)^{-1} u \, dt, u \right\rangle\right).$$

Using Proposition A.13 we can calculate

$$\begin{aligned}
&\mathrm{Re}\left(\left\langle \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty t^{\alpha-1} L(t + L)^{-1} u \, dt, u \right\rangle\right) \\
&= \frac{\sin(\alpha\pi)}{\pi} \mathrm{Re}\left(\left\langle \int_0^\infty t^{\alpha-1} L(t + L)^{-1} u \, dt, u \right\rangle\right) \\
&= \frac{\sin(\alpha\pi)}{\pi} \mathrm{Re}\left(\int_0^\infty t^{\alpha-1} \left\langle L(t + L)^{-1} u, u \right\rangle \, dt\right) \\
&= \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty t^{\alpha-1} \mathrm{Re}\left(\left\langle L(t + L)^{-1} u, u \right\rangle\right) \, dt.
\end{aligned}$$

Since $(t+L)^{-1}$ is a bijection, we can write $u = (t+L)v$. Thus we get

$$
\begin{aligned}
&\frac{\sin(\alpha\pi)}{\pi}\int_0^\infty t^{\alpha-1}\mathrm{Re}\left(\left\langle L(t+L)^{-1}u, u\right\rangle\right)\,dt \\
&= \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty t^{\alpha-1}\mathrm{Re}\left(\langle Lv, (t+L)v\rangle\right)\,dt \\
&= \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty t^{\alpha-1}\mathrm{Re}\left(t\langle Lv, v\rangle + \|Lv\|^2\right)\,dt \\
&= \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty t^{\alpha-1}\left(t\mathrm{Re}\left(\langle Lv, v\rangle\right) + \|Lv\|^2\right)\,dt.
\end{aligned}
$$

Since L is m-accretive the last line is positive, which implies that $\mathrm{Re}(\langle L^\alpha u, u\rangle) \geq 0$.

It remains to show that there exists some $\lambda > 0$ such that $\mathrm{ran}(\lambda + L^\alpha)$ is dense in H. But this follows from exercise 6.5 which states that L^α is sectorial of angle less that π. This implies that $-1 \in \rho(L^\alpha)$ and thus $L^\alpha + 1$ is invertible and so $\mathrm{ran}(L^\alpha + 1) = H$. So we may pick $\lambda = 1$ to complete the proof. □

Exercise 3. Fill in the details left out in the discussion of the H^∞-calculus for multiplication operators in example 7.7.

Exercise 4 (An automatic bound for H^∞-calculus.)**.** Let L be an injective sectorial operator in H. Let $\phi \in (\phi_L, \pi)$ and suppose that we have $f(L) \in \mathcal{L}(H)$ for every $f \in H^\infty(S_\phi)$. Prove that these is a constant $C \geq 0$ such that

$$\|f(L)\|_{\mathcal{L}(H)} \leq C\,\|f\|_{\infty,\phi}$$

holds for every such f.

Proof. □

Exercise 5. (The injective part) Dolgo navodilo.