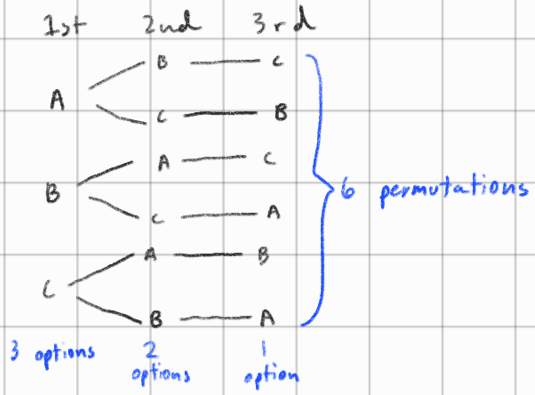
CS 173 - Lecture 8

Permutations

- Different ways to pick elements with ordering

ex. $\{1, 2, 3\}$



$3 \times 2 \times 1 = 3! = 6$

- \# permutations for n objects $= n!$

ex. permutations for $\{A, B, C, D\} = 4! = 24$

ex. \# ways to pick and order 3 elements of $\{1,2,3,4,5\}$?

1st — 2nd — 3rd

1 — 2,3,4,5 — ...

2 — 1,3,4,5

3 ⋮

4

5

5 options — 4 options — 3 options

$5 \times 4 \times 3 = 60$ permutations $= \dfrac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \boxed{\dfrac{n!}{(n-k)!}}$ if permuting k elements from total of n elements

$= \dfrac{5!}{(5-3)!}$

Permutations with Repetitions

ex. "COLLEGE" How many different ways to reorder?

1) Treat everything as unique $\left(\frac{n!}{(n-k)!}\right)$

2) Count ways to permute repeated items

2 L's		2 E's	
L_1	L_2	E_1	E_2
L_2	L_1	E_2	E_1

$$\left.\begin{array}{l} COL_1L_2E_1GE_2 \\ COL_2L_1E_1GE_2 \\ COL_1L_2E_2GE_1 \\ COL_2L_1E_2GE_1 \end{array}\right\} 4 \text{ duplicates}$$

3) $\dfrac{\frac{n!}{(n-k)!}}{\#\text{ ways to arrange repeated elements}}$ $\qquad \dfrac{7!}{4}$ $\qquad (n=7,\ k=7)$

Combinations

- Permutations ordered, combinations not

ex. 5 elements $\{1,2,3,4,5\}$, # of combinations of 3 elements?

$1,2,3 = 3,2,1 = 1,3,2 = \dots$

1) Count permutations

$$\frac{n!}{(n-k)!} = \frac{5!}{(5-3)!} = 5\times4\times3 = 60$$

2) Count duplicates for each combination

1,2,3   1,3,2   2,1,3   2,3,1   3,1,2   3,2,1

6 permutations $\Rightarrow$ 1 combination

3) $$\frac{\frac{n!}{(n-k)!}}{\text{permutations/combination}} = \frac{n!}{(n-k)!\,k!} = \frac{5\times4\times3}{6} = 10 \text{ combinations}$$

Combinations with Repetitions

ex. 3 piles of colored balls red, green, blue. Pick 6 balls (w/ repetitions) out of 3 colors.

RRGBB, GGGGGG, RGRRBB, ...

1) Group by types

RGRBBB $\Rightarrow$ RRGBBB

2) Insert divider

RRGBBB $\Rightarrow$ RR|G|BBB

3) Fix order (red $\to$ green $\to$ blue)

$$\left.\begin{array}{lll} RR|G|BBB & \Rightarrow & **|*|*** \\ R|GGG|BB & \Rightarrow & *|***|** \\ BBBBBB & \Rightarrow & |\;|****** \end{array}\right\} \to 8 \text{ elements } (6\ *\text{'s},\ 2 \text{ bars})$$

Insert 2 bars / 6 stars into 8 spaces $\to$ Ways to insert?

Inserting 2 bars: $\frac{8!}{(8-2)!} = 8\times7 = 56$ $\frac{56}{2} = 28$

2 ways to get same result
(eg. $**|_1**|_2** = **|_2**|_1**$)

Viewing as permutation w/ repetition:

$$\frac{n!}{(n-k)!\,k!} = n \text{ choose } k$$

In general, if x types to form combination of n elements:

$$\#\text{ ways} = \frac{(n+x-1)!}{n!\,(x-1)!}$$

Notation

- Permutations
  - $P(n,k)$
  - # ways to permute k items from n total items
- Combinations
  - $C(n,k)$
  - $\binom{n}{k}$ (binomial coefficent)
  - n choose k
  - # ways to choose k elements from n elements
  - $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n} y^n$

Discussion Problems - 17.5ade

17.5 Counting and Combinations

Solve the following word problems. Providing brief explanations to justify your answers. You do not need to numerically compute all of the factorials. For instance, $\frac{10!}{6!}$ is an acceptable final answer.

(a) You need to form a battle group of 11 made up of orcs, elves, and goblins. In how many ways can you choose the composition of your battle group?

(b) How many ways can you construct a string of 20 decimal digits that contains exactly 3 zeros, no two of which are consecutive? (Hint: set up the other 17 digits with spaces between them and at the ends. Pick three of these spaces to put the zero's in.)

(c) How many bit strings of length 100 have exactly 10 zeros?

(d) Your latest cheapo cell phone keyboard only includes the uppercase alphabet (26 characters total). How many 12-character strings can you type that start with ST and contain no more than three T's? (Hint: you will need to consider cases of zero T's, one T, two T's, and three T's)

(e) Suppose a set S has 10 elements, how many subsets of S have an odd number of elements?

a) Combination w/ repetition

formula: $\frac{(n+x-1)!}{(x-1)!\,n!}$ $\quad n = 11 \quad x = 3$

$\#\text{ ways} = \frac{13!}{2!\,11!} = \frac{13 \times 12}{2} = 13 \times 6 = 78$ ways

d) permutation

case 1: zero T's

impossible cuz starts w/ ST

case 2: one T

ST _ _ _ _ _ _ _ _ _ _ $\quad 25^{10}$ because each slot has 25 choices

case 3: two T's

ST _ _ _ ← T → _ _ _ _ $\quad 25^{9} \times 10$, 9 free slots, 10 slots for T to go

case 4: three T's

ST _ ← T → _ _ _ ← T → _ $\quad 25^{8} \times \binom{10}{2} = 25^{8} \times 45$, 8 free slots, 45 ways to place 2 T's

$\#\text{ways} = 25^{10} + 25^{9} \times 10 + 25^{8} \times 45$

e) 1 element:

10 subsets

3 elements:

combination bc not ordered (ie $\{1,2,3\} = \{3,2,1\}$

$\binom{10}{3} = \frac{10!}{7!\,3!} = \frac{10 \times 9 \times 8}{6} = 120$ subsets

5 elements:

$\binom{10}{5} = \frac{10!}{5!\,5!} = 252$ subsets

7 elements:

$\binom{10}{7} = 120$ subsets

9 elements:

$\binom{10}{9} = 10$ subsets

total subsets $= 10 + 120 + 252 + 120 + 10 = 512$