CS 173 - Lecture 6

Sets

- Unordered collection of objects

ex. $\{1,2,3\}$, $\{$green, red, blue$\}$, $\{\{1,2\},\{3,7\}\}$

also $\{1,2,3\} = \{3,1,2\} = \ldots$ (order doesn't matter)

also $\{1,2,2\} = \{1,2\}$ (depends on only unique items)

- How do we define sets?

1) curly brackets $\{\ \}$, $\{0,1,2,3,\ldots\}$

↳ only when very obvious pattern

2) Words/verbal descriptions

"set of all integers"

3) Set builder notation

$\{\underbrace{x \in \text{fruits}}_{\text{range}} \mid \underbrace{x \text{ is red}}_{\text{conditions/restrictions}}\}$ ⇒ "set containing all red fruits"

set of all fruits such that they are red

also $\{x \in \text{fruits} : x \text{ is red}\}$ ("$\mid$" = ":")

Cardinality

- Number of unique items in the set A, written as $|A|$
- $|x|$ is abs val (x) if x is num, cardinality if x is set

ex. $|\{1,2,3\}| = 3$

$|\{1,2,2,3,3,3\}| = 3$

$|\{\{1,2\},\{3,3\}\}| = 2$ (bc has two sets)

Subsets

- Let A and B be sets, $A \subseteq B$ iff all elements in A are also in B (if $x \in A \rightarrow x \in B$)

ex. $\underbrace{\{1,2\}}_{A} \subseteq \underbrace{\{3,2,4,1\}}_{B}$ $A \subseteq B$ ✓

ex. $A = \{1,2,3\}$ $B = \{2,0,1,4,5\}$

$A \not\subseteq B$ bc $3 \in A$ and $3 \notin B$

$\subseteq$ = "subset of", A can be same as B

$\subset$ = "proper subset of", A subset of B and not equal to B

similar to $\leq$ and $<$

ex. $A = \{0, 1\}$ $B = \{0, 1, 2\}$ $C = \{0, 1\}$

$A \subseteq B$, $A \subseteq C$, $A \subset B$ $A \not\subset C$

Empty Set

$\{\ \}$ or $\emptyset$ ($\emptyset$ in LaTeX)

$\emptyset \subseteq$ for all sets A

If $x \in \emptyset$, $x \in A$

↳ vaccuously true bc there's nothing in the empty set, so statement always true

$|\emptyset| = 0$

ex. $B = \{\{\emptyset\}\}$

$\emptyset \in B$? False, $\{\emptyset\}$ is only thing in B

$\emptyset \neq \{\emptyset\}$

$|\emptyset| = 0$ but $|\{\emptyset\}| = 1$

Set Operations

$\vee$ = or

$\wedge$ = and

$\cup$ = union $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$\cap$ = intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

$-$ = difference $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

$\times$ = product/cross product/cartesian product $A \times B = \{(x, y) \mid x \in A, y \in B\}$

ex. $A = \{\text{green}, \text{blue}\}$ $B = \{\text{blue}, \text{orange}\}$

$A \cap B = \{\text{blue}\} = B \cap A$

$A \cup B = \{\text{blue}, \text{green}, \text{orange}\} = B \cup A$

$A - B = \{\text{green}\}$

$B - A = \{\text{orange}\}$

$A \times B = \{(\text{green}, \text{blue}), (\text{green}, \text{orange}), (\text{green}, \text{blue}), (\text{blue}, \text{orange})\}$

↳ tuples (ordered, eg. $(1,2) \neq (2,1)$)

$B \times A = \{(\text{blue}, \text{green}), (\text{blue}, \text{blue}), (\text{orange}, \text{green}), (\text{orange}, \text{blue})\}$

Tuples

- Ordered, can have duplicates

ex. vectors

ex. Prove $A \subseteq B$ for $A = \{\lambda(2,3) + (1-\lambda)(7,4) \mid \lambda \in [0,1]\}$ and $B = \{(x,y) \mid x, y \in \mathbb{R}, x \geq 0, y \geq 0\}$

Idea: show arbitrary $(x,y) \in A$ also in B

$\lambda(2,3) = (2\lambda, 3\lambda)$ $\quad (1-\lambda)(7,4) = ((1-\lambda)7, (1-\lambda)4) = (7-7\lambda, 4-4\lambda)$

$(2\lambda, 3\lambda) + (7-7\lambda, 4-4\lambda) = (7-5\lambda, 4-\lambda)$

$A = \{(7-5\lambda, 4-\lambda) \mid \lambda \in \mathbb{R}, \lambda \in [0,1]\}$

pick arbitrary element $(x,y) \in A$ s.t $x = 7-5\lambda$ $\quad y = 4-\lambda$

goal: show $x \geq 0$ $\quad y \geq 0$

$\lambda \in [0,1]$ $\quad -5\lambda \in [-5,0]$ $\quad 7-5\lambda \in [2,7]$ $\quad \therefore\ x = 7-5\lambda \geq 0$

$\lambda \in [0,1]$ $\quad -\lambda \in [-1,0]$ $\quad 4-\lambda \in [3,4]$ $\quad \therefore\ y = 4-\lambda \geq 0$

$\therefore\ x \geq 0,\ y \geq 0,\ (x,y) \in B$ $\quad \therefore\ A \subseteq B$

Discussion Questions - 3.2, 3.3b, 2

3.2 Concrete Subset Proof

Let $A = \{(p,q) \in \mathbb{R}^2 : \; p^2 + q^2 \leq 1\}$ and $B = \{(x,y) \in \mathbb{R}^2 : \; |x| \leq 1, \; |y| \leq 1\}$. Prove that $A \subseteq B$, by choosing a representative element from the smaller set and showing that it is in the larger set.

3.3 Abstract Subset Proofs

Prove the following set containments and show, using a **concrete counterexample**, that the reverse containment does not hold.

(a) $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

(b) $(A - C) - (B - C) \subseteq (A - B)$.

2. Sets warmup

Consider the following sets: $A = \{2\}$, $B = \{A, \{4,5\}\}$, $C = B \cup \emptyset$, $D = B \cup \{\emptyset\}$.

a) Which of the sets have more than two elements?

b) Which of the following are true:

$$2 \in A,\ 2 \in B,\ \{2\} \in A,\ \{2\} \in B,\ \emptyset \in C,\ \emptyset \in D,$$
$$\emptyset \subseteq A,\ \{2\} \subseteq A,\ \{2\} \subseteq B$$

3.2) proof: Let $A = \{(p,q) \in \mathbb{R}^2 : p^2+q^2 \leq 1\}$ and $B = \{(x,y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}$. Let z be an arbitrary element of A. By the definition of A, $z = (m,n) \in \mathbb{R}^2 : m^2+n^2 \leq 1$. Any $\mathbb{R}$ squared must be positive, so $m^2 \geq 0$ and $n^2 \geq 0$. Then, $m^2 \leq 1$ and $n^2 \leq 1$, $|m| \leq 1$ and $|n| \leq 1$. This follows the definition of B, so $(m,n) = z \in B$. z is an abitrary element of B, $\therefore$ every element of A is an element of B and thus $A \subseteq B$

3.3b) $(A-C)-(B-C) \subseteq (A-B)$ Reverse: $(A-B) \subseteq (A-C)-(B-C)$

Proof: Let A, B, and C be sets and let z be an arbitrary element in $(A-C)-(C-B)$. By the definition of difference, $z \in (A-C)$ and $z \notin (B-C)$. Then, $z \in A$ and $z \notin C$. Also, $z \notin B$ or $z \in C$, but $z \notin C$ so $z \notin (B-C)$ is equivalent to $z \notin B$. Because $z \in A$ and $z \notin B$, $z \in (A-B)$. z is an arbitrary element of $(A-C)-(B-C)$ so every element in $(A-C)-(B-C)$ is also in $(A-B)$, $\therefore$ $(A-C)-(B-C) \subseteq (A-B)$

Counter example to reverse:

Let $A = \{1,2,3\}$, $B = \{0,5\}$, $C = \{1\}$

$(A-B) = \{1,2,3\}$

$(A-C)-(B-C) = \{2,3\} - \{0,5\} = \{2,3\}$

$(A-B) \nsubseteq (A-C)-(B-C)$

2) $A = \{2\}$ $B = \{A, \{4,5\}\}$, $C = B \cup \emptyset$ $D = B \cup \{\emptyset\}$

$B = \{\{2\}, \{4,5\}\}$ $C = B = \{\{2\},\{4,5\}\}$ $D = \{\{2\},\{4,5\},\{\emptyset\}\}$

a) D

b) $2 \in A$ ✓, $2 \in B$ ✗, $\{2\} \in B$ ✓, $\emptyset \in C$ ✗, $\emptyset \in D$ ✓, $\emptyset \subseteq A$ ✓, $\{2\} \subseteq A$ ✓, $\{2\} \subseteq B$ ✗

$x \in y \equiv \{x\} \subseteq y$