CS 173 - Lecture 12

Summation Review

$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$ if a_i dependent on i

ex. $a_i = i$ $\qquad$ $a_i = 2$

$\sum_{i=1}^{n} a_i = 1+2+3+\ldots+n$ $\qquad$ $\sum_{i=1}^{n} a_i = 2+2+2+2+\ldots+2 = 2n$

Closed forms (succinct formula):

$$\sum_{i=1}^{n} i = 1+2+\ldots+n = \frac{n(n+1)}{2}$$

$$\sum_{k=0}^{n} r^k = r^0 + r^1 + r^2 + \ldots + r^n = \frac{r^{n+1}-1}{r-1}$$

memorize

Induction

- Proof technique to prove claims $P(n)$ for some subset of integers n
- Outline:
  1) state proof by induction, specify inductive variable
  2) prove base case(s)
  3) state inductive hypothesis
  4) prove inductive step

ex. prove $P(n)$ is true for all $n \geq 1$

1) prove by induction on n
2) base case - prove $P(1)$ true
3) IH: suppose $P(n)$ true for $n = 1, 2, \ldots, k-1$
4) IS: show $P(k)$ true

ex. prove by induction that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof: We will prove by induction on n.

Base case: $n=1$, LHS: $\sum_{i=1}^{1} i = 1$, RHS: $\frac{1(1+1)}{2} = 1$, LHS = RHS, so base case true.

IH: suppose $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ is true for $i = 1, 2, \ldots, k-1$

IS: show $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for $n = k$, $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ $= 1+2+3+\ldots+(k-1)+k = \sum_{i=1}^{k-1} i + k$

can plug in formula be assumed true up to $k-1$

$= \frac{(k-1)k}{2} + k$ (by IH) $= \frac{k^2-k}{2} + \frac{2k}{2} = \frac{k(k+1)}{2}$ $\square$

Why this works (recursion theory)

proved: $P(1)$ true

assumed: $P(1) \wedge P(2) \wedge P(3) \ldots P(k-1)$

proved: $P(1) \wedge P(2) \ldots P(k-1) \rightarrow P(k)$

Discussion Problems - 11.1ab, 11.4

11.1 Simple examples

Prove the following formulas using induction:

(a) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ (for all positive integers)

(b) $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ (for all positive integers)

a) Proof: We will prove $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ $\forall$ $\mathbb{Z}^+$ by induction on n. For our base case $(n=1)$, we have $\sum_{i=1}^{1} i^2 = 1^2 = 1$ on the LHS and $\frac{1(1+1)(2+1)}{6} = 1$ on the RHS, so $\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+1)}{6}$ and the base case is true. For our inductive hypothesis, we assume $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ is true for $i = 1, 2, \dots, k-1$. For our inductive step, when $n=k$, we have

$\sum_{i=1}^{k} i^2 = 1^2 + 2^2 + \dots + (k-1)^2 + k^2 = \sum_{i=1}^{k-1} i^2 + k^2 = \frac{(k-1)(k)(2k-1)}{6} + k^2$ (by inductive hypothesis) $= \frac{k(k+1)(2k+1)}{6}$ $\square$

b) We will prove $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ $\forall$ $\mathbb{Z}^+$ by induction on n. For our base case, $n=1$, so for LHS, $\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$, RHS is $\frac{1}{1+1} = \frac{1}{2}$, LHS = RHS so base case is true.

For our inductive hypothesis, we assume $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ is true for $k = 1, 2, \dots, j-1$.

For our inductive step, we have $n=j$, so $\sum_{k=1}^{j} \frac{1}{k(k+1)} = \frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \dots + \frac{1}{(j-1)(j)} + \frac{1}{j(j+1)}$

$= \sum_{k=1}^{j-1} \frac{1}{k(k+1)} + \frac{1}{j(j+1)}$ (by the defn. of summation) $= \frac{j-1}{j} + \frac{1}{j(j+1)}$ (by the inductive hypothesis) $=$

$= \frac{(j-1)(j+1)}{j(j+1)} + \frac{1}{j(j+1)} = \frac{j^2-1+1}{j(j+1)} = \frac{j^2}{j(j+1)} = \frac{j}{j+1}$ $\square$

11.4 A broken induction proof

What's wrong with the following induction "proof?"

Claim: all horses are the same color.

Proof: We'll show that if S is any set of horses, all horses in S have the same color, by induction on the size of S.

Base: The claim is clearly true for a set containing only one horse.

Induction: Suppose that if T is any set of $k-1$ horses, all horses in T have the same color. Let S be a set of k horses. We need to show that all horses in S have the same color.

Suppose S contains horses $H_1, H_2, \dots, H_k$. The set $S' = \{H_2, \dots, H_k\}$ contains only $k-1$ horses, so they must all be the same color by the inductive hypothesis. Similarly, all the horses in the set $S'' = \{H_1, \dots, H_{k-1}\}$ must be the same color. Since S is the union of S' and S'', all the horses in S must have the same color.

Breaks for sets with cardinality of 2 $(n=2)$ bc $S' \cap S'' = \emptyset$. Eg. $S = \{A, B\}$ so $S' = \{B\}$ and $S'' = \{A\}$, but $S' \cap S'' = \emptyset$ so there is no "common" color between S' and S'' that forces A and be to be the same color. For $S = \{A, B, C\}$, $S' = \{B, C\}$ and $S'' = \{A, B\}$, $S' \cap S'' = \{B\}$ which is what makes S' and S'' the same color. Because $P(1) \not\Rightarrow P(2)$, the induction breaks, so proof is invalid.

Takeaway is the inductive step must be generalizable for every case beyond the base case.