CS 173 - Lecture 9

Functions

- Assume A, B sets, function f from A to B is assignment of exactly one element of B to each element of A

$$f : A \to B$$

name of function; domain; co-domain

ex. $A = \{Ben, Pegg\}$

$B = \{Blue, orange, purple\}$

$C : A \to B$

$C(Ben) = Orange$, $C(Pegg) = Blue$ valid

$C(Ben) = Orange$, $C(Ben) = Purple$, $C(Pegg) = blue$ invalid

$C(Ben) = Pink$ invalid (not in co-domain)

- Image of function $f : A \to B$ is set of values we actually get when applied to A (can be different from B)

Onto

- function onto iff image equals co-domain
- for function $f : A \to B$, $\forall y \in B, \exists x \in A$ s.t $f(x) = y$

(quantifier order matters, swapping gives different definition)

☆ ex. $g : \mathbb{Z} \to \mathbb{Z}$ $g(x) = x + 2$ Prove g is onto (show g can output any integer)

Strat: define arbitrary element in co-domain, show it must have a pre-image in domain

Proof: Let $y \in \mathbb{Z}$, $\underbrace{a = y - 2}_{\text{inverse}}$ where $a \in \mathbb{Z}$. Since $y \in \mathbb{Z}$, a is an element of the domain. Then, $g(a) = g(y-2) = y - 2 + 2 = y$. We found pre-image in the domain of our arbitrary element y of the co-domain, $\therefore$ g is onto.

One-to-one

- Pre-image: for $f : A \to B$, if $y \in B$ and $x \in A$ and $f(x) = y$, then x is a pre-image of y
  - Can have multiple pre-images (ie $f(x) = y$, $f(z) = y$, $f(a) = y$, etc)
- One-to-one: iff no element of co-domain has more than one pre-image
  - $f : A \to B$, $\underbrace{\forall x, y \in A,\ x \neq y}_{\text{for different elements in domain}} \to \underbrace{f(x) \neq f(y)}_{\text{we get diff outputs}}$

$f : A \to B$, $\forall x, y \in A$, $f(x) = f(y) \to x = y$ (contrapositive)

(ie for two same outputs, inputs were same, f one-to-one) ex. $A = \{Ben, Pegg\}$ $B = \{orange, blue, purple\}$ $C: A \to B$

$C(Ben) = Orange$ $C(Pegg) = Orange$ yes function, not one-to-one

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ex. Let $f: \mathbb{Z} \to \mathbb{Z}$ and $f(x) = 2x+1$. Prove f is one-to-one.

Strat: use contrapositive, assume $f(x) = f(y)$, show $x = y$

Proof: Let $x, y \in \mathbb{Z}$ and suppose $f(x) = f(y)$. By definition of f, $2x+1 = 2y+1$, $2x = 2y$, $x = y$. Since $x = y$, f is one-to-one.

<u>Extra Stuff</u>

- Bijection: $f: A \to B$ that is one-to-one and onto
  - $f^{-1}: B \to A$ exists and is also a bijection
  - $|A| = |B|$
- Composition: $(f \circ g)(x) = f(g(x))$

Discussion Problems - 5.1bc, 5.3b, 7.1b, 7.3ab

5.1 Nested Quantifiers

Explain why each of the following propositions is true or false:

(a) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R}, y^3 \leq x$

(b) $\exists x \in \mathbb{N},\ \forall y \in \mathbb{N}, xy = x$

(c) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{Q}, |x - y| \leq 0.01$

(d) $\exists y \in \mathbb{Q},\ \forall x \in \mathbb{R}, |x - y| \leq 0.01$

(e) $\exists x \in \mathbb{N},\ \forall y \in \mathbb{N}, \mathrm{GCD}(x, y) = 1$

(f) $\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z}, x = y^2$

(g) $\forall x \in \mathbb{Q},\ \exists (w, y) \in \mathbb{Z}^2, x = \frac{w}{y}$

(h) $\exists (w, y) \in \mathbb{Z}^2,\ \forall x \in \mathbb{Q}, x = \frac{w}{y}$

a) True. When $x<0$, choose a $y \geq \sqrt[3]{x}$ and $y<0$. When $x=0$, choose a $y \leq 0$. When $x>0$, choose a $y \leq \sqrt[3]{x}$

b) True, $x=0$. $0 \in \mathbb{N}$ and $\forall y \in \mathbb{N}$, $xy = x$, $0y = 0$, $0=0$ ✓

c) True

5.3 Concrete Onto Proofs

Prove that the following functions are onto:

(a) $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 17 - 2x$

(b) $f : \mathbb{Z}^2 \to \mathbb{Z}$ by $f(x, y) = xy + 27$

b) Let $f: \mathbb{Z}^2 \to \mathbb{Z}$ by $f(x,y) = xy+27$. Let $m \in \mathbb{Z}$, an arbitrary element in the co-domain. Let a,b be the pre-image of m such that $a=1$ and $b=m-27$. $a,b \in \mathbb{Z}^2$ because $a=1 \in \mathbb{Z}$ and $b=m-27$, $m \in \mathbb{Z}$, $27 \in \mathbb{Z}$, therefore $b \in \mathbb{Z}$ too. Then, $f(a,b) = f(1, m-27) = 1(m-27)+27 = m$. Because m is an abitrary element of the co-domain and we found a pre-image to m in the domain of f, f is onto.

$17-2y=x$ $\quad \frac{17-x}{2}=y$

a) Let $g: \mathbb{R} \to \mathbb{R}$ by $g(x) = 17-2x$. Let $z \in \mathbb{R}$, an arbitrary element in the co-domain of g. Let m be the pre-image of z such that $m = \frac{17-z}{2}$. Then, $m \in \mathbb{R}$ because $z \in \mathbb{R}$. $g(m) = g\left(\frac{17-z}{2}\right) = 17 - 2\left(\frac{17-z}{2}\right) = z$. $z \in \mathbb{R}$, therefore m is in the domain of g and is the pre-image of z. Because z is an abitrary element of the co-domain of g, g must be onto

① define arbitrary element z in co-domain

② define element $m = f^{-1}(z)$ (not always inverse tho), show $m \in$ domain

③ show $f(m) = z$

④ show onto be found pre-image to arbitrary in the domain

7.1 Concrete One-to-one Proofs

Prove that each of the following functions is one-to-one.

(a) $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 2^{x+1}$

(b) $h : \mathbb{N} \to \mathbb{Z}$ by $h(x) = x^2 + 27$

b) Let $h: \mathbb{N} \to \mathbb{Z}$ by $h(x) = x^2 + 27$. Let $a, b \in \mathbb{N}$ and assume $f(a) = f(b)$. Then, $a^2 + 27 = b^2 + 27$, $a^2 = b^2$, and $a \geq 0$ and $b \geq 0$, so $a = b$. Thus, $h(x)$ is one-to-one.

7.3 Abstract proof using Composition

(a) Suppose that A,B, and C are sets and $f : B \to C$ and $g : A \to B$ are functions. Prove that if $f \circ g$ is onto and f is one-to-one, then g is onto.

(b) Give a concrete counter-example (involving small sets!) showing why the assumption that f is one-to-one is necessary in (a).

a) Let $f: B \to C$ and $g: A \to B$, assume $f \circ g$ is onto and f is one-to-one. Let $m \in B$, $z = f(m)$.
Since $f \circ g$ is onto, $\exists$ $k \in A$ st. $f(g(k)) = z$. Since f is one-to-one, $f(g(k)) = z = f(m)$,
then $g(k) = m$. We found the pre-image of m in A, $\therefore$ g is onto.

b) Let $A = \{1, 2\}$, $B = \{3, 4\}$, $C = \{5, 6\}$. Assume $g(1) = 3$ and $f(3) = 4$ ...