Predicate logic

- Define functions that produce logical statements
- ex. define $P(x)$ as "x is purple"
  predicate
  - say x is "grape", P(grape) means grape is purple
    true (assuming purple grape)

Quantifiers

$\exists$: There exists

$\forall$: For all

ex. $\exists\ x \in$ fruits such that $P(x)$ = there exists one or more purple fruits   True
quantifier; $x \in$ fruits: is an element of, x must be a fruit; $P(x)$: predicate; "one or more" ↳ $\exists$

ex. $\forall\ x \in$ fruits, $P(x)$ = all fruits are purple   false
quantifier; predicate

ex. $\exists\ x, y \in$ fruits such that $P(x) \wedge \neg P(y)$ = there exists some purple fruit and some non-purple fruit   True

Negating a statement with a quantifier

- Define $Q(x)$ to mean "x is sweet"
- ex. $\neg(\forall\ x \in \text{fruits}, P(x)) \rightarrow \exists\ x \in \text{fruits}, \neg P(x)$
  not all fruits are purple → there exists a non purple fruit
- ex. $\neg(\exists\ x \in \text{fruits such that } P(x) \wedge Q(x))$ there does not exist a sweet and purple fruit
  $\forall\ x \in \text{fruits}, \neg(P(x) \wedge Q(x))$
  $\forall\ x \in \text{fruits}, \neg P(x) \vee \neg Q(x)$
  all fruits are either not purple or not sweet
  false

Proof Techniques

	Prove	Disprove
universal ($\forall$)	direct proof	concrete counter-example
existential ($\exists$)	concrete example	same as proving universal (direct proof)

Proving a Universal

For any integer, if k is odd, then k^2 is odd. Prove this statement.
"k is odd": assume; "k^2 is odd": prove this

☆ don't assume conclusion is true

Let k be odd integer. Then, $k = 2n+1$ where $n \in \mathbb{Z}$.

declare — assume hypothesis of implication is true — type — definition of odd — declare new variables

Then, $k^2 = (2n+1)^2 = 4n^2 + 4n + 2$. We know $k^2 = 2(2n^2+2n)+1$, and $2n^2+2n \in \mathbb{Z}$

show this is odd

since $n \in \mathbb{Z}$. Call $2n^2 + 2n = j \in \mathbb{Z}$. So $k^2 = 2j+1$, and by the definition of odd numbers, k^2 is odd

Proving an existential

Prove there exists an integer k such that $k^2 = 0$

Let $k=0$, $0^2 = 0$, and $0 \in \mathbb{Z}$

Disprove a universal

Every rational number q has a multiplicative inverse. Disprove this.

We define multiplicative inverse of q as $r \in \mathbb{Q}$ such that $r \cdot q = 1$

$q = 0 \in \mathbb{Q}$ does not have a multiplicative inverse because $r \cdot 0 = 0 \neq 1$ regardless of our choice of r

Disprove an existential

There exists a $k \in \mathbb{Z}$ such that $k^2+2k+1 < 0$. Disprove this.

$\rightarrow$ It is not true that there exists $k \in \mathbb{Z}$ s.t $k^2+2k+1 < 0$

$\rightarrow \neg(\exists\, k \in \mathbb{Z}$ s.t $k^2+2k+1<0)$

$\forall\, k \in \mathbb{Z} \ \neg(k^2+2k+1<0)$

$\forall\, k \in \mathbb{Z} \ k^2+2k+1 \geq 0$

(review proof by cases in textbook)

3.10 - Proof by Cases

- Do proof twice, like if you have an abs val.
- ex. $\forall \mathbb{Z}$ $j \wedge k$, if j even or k is even, then jk is even

Case 1: j even

then $j = 2m$ where $m \in \mathbb{Z}$, $jk = 2mk$, $mk \in \mathbb{Z}$ $\therefore$ jk must be even

Case 2: k even

then $k = 2n$ where $n \in \mathbb{Z}$, $kj = 2nj$, $nj \in \mathbb{Z}$ $\therefore$ jk must be even

even in case 1 and 2

- you can have 2+ cases with overlap, as long as all posibilities covered

3.11 - Rephrasing Claims

Given a claim that's not in a good form for direct proof

ex. No integer k s.t k odd and k^2 even

$=$ $\forall k$, not the case that k odd and k^2 even

$=$ $\forall k \in \mathbb{Z}$, k not odd and k^2 not even

$=$ $\forall k \in \mathbb{Z}$, k not odd and k^2 odd

($\neg p \vee q$ same as $p \to q$)

$\forall k \in \mathbb{Z}$, k odd $\to$ k^2 odd

3.12 - Proof by Contrapositive

$\forall x, P(x) \to Q(x)$

contrapositive: $\forall x, \neg Q(x) \to \neg P(x)$ (logically equivalent)

ex. $\forall a, b \in \mathbb{Z}$, $a+b \geq 15 \to (a \geq 8 \vee b \geq 8)$

contrapositive: $\forall a, b \in \mathbb{Z}$ $\neg(a \geq 8 \vee b \geq 8) \to \neg(a+b \geq 15)$

demorgans: $\forall a, b \in \mathbb{Z}$ $\neg a \geq 8 \wedge \neg b \geq 8 \to \neg(a+b \geq 15)$

simplify: $\forall a, b \in \mathbb{Z}$ $a < 8 \wedge b < 8 \to a+b < 15$

Include rewritten claim in proof

Proof: We'll prove the contrapositive of this statement. That is, for any integers a and b, if $a < 8$ and $b < 8$, then $a + b < 15$.

So, suppose that a and b are integers such that $a < 8$ and $b < 8$. Since they are integers (not e.g. real numbers), this implies that $a \leq 7$ and $b \leq 7$. Adding these two equations together, we find that $a + b \leq 14$. But this implies that $a + b < 15$. □

ex. For any integer k, if $3k+1$ is even, then k is odd

$\forall k \in \mathbb{Z}$ $3k+1 = 2n$ $n \in \mathbb{Z}$ $\to$ $k = 2m+1$ $m \in \mathbb{Z}$

contrapositive: $\forall k \in \mathbb{Z}$ k even $\rightarrow$ $3k+1$ odd

Proof: We will prove the contrapositive of this statement: $\forall k \in \mathbb{Z}$ k even $\rightarrow$ $3k+1$ odd.

k even $\therefore$ $k = 2n$ $n \in \mathbb{Z}$ $3k+1 = 3(2n)+1 = 6n+1$ which is odd, so $3k+1$ must be odd $\forall$ k even.

Discussion Problems

Discussion Problem

CS 173: Discrete Structures

Problem 1. Logical Reasoning

Suppose we are given the following facts:

1. All chestnut-eating animals are fun-loving
2. No penguin eats mulberries
3. Some well-dressed animals are uncomfortable
4. At least one penguin is uncomfortable
5. All animals eat mulberries or chestnuts
6. No uncomfortable animal eats mulberries

Which of the following statements can be proved (inferred) from the above facts?

You may assume that "not comfortable" is the same as "uncomfortable". You may also assume that penguins are known to be animals.

Select one or more:

(a) Every comfortable penguin eats mulberries

(b) At least one penguin is well-dressed

(c) No penguins are fun-loving

(d) There is at least one fun-loving well-dressed animal

(e) All penguins are fun-loving

(f) All penguins are uncomfortable

1.2 Direct proof and disproof

For each claim, prove it using direct proof or disprove it using a concrete counterexample.

(a) For any integer k, if k is odd, then k^3 is odd.

(b) For any integers p and q, $(p+q)^2 = p^2 + q^2$.

(c) For any real numbers w, x, y, and z, if $w < x$ and $y < z$ then $wy < xz$.

(d) For all real numbers x and y, where $x \neq 0$, if x and $\frac{y+1}{3}$ are rational, then $\frac{1}{x} + y$ is rational.

1.3 Variations on direct proof

Prove the following claim. Your proof should divide into cases based on the sign of $|x+7|$.

(a) For any integer x, if $|x+7| > 8$, then $|x| > 1$.

Prove the following claims by contrapositive. Begin your proof by explicitly writing out the contrapositive. Then use direct proof to prove the contrapositive.

(b) For all real numbers x and y, if $x + y \geq 2$, then $x \geq 1$ or $y \geq 1$.

(c) For all integers m and n, if mn is even, then m is even or n is even.

(d) For all real numbers x, if $x^2 - 3x + 2 > 0$, then $x \geq 2$ or $x < 1$.

(e) For any integers m and n, if $7m + 5n = 147$, then m is odd or n is odd.

1.2b) For any integers p and q, $(p+q)^2 = p^2+q^2$

$p = 1 \in \mathbb{Z}$ and $q = 2 \in \mathbb{Z}$, $(p+q)^2 = (1+2)^2 = 9 \neq p^2+q^2 = 1^2+2^2 = 5$

1.2c) For any real numbers w, x, y, and z, if $w<x$ and $y<z$, then $wy<xz$

$w = -2 \in \mathbb{R}$, $y = -3 \in \mathbb{R}$, $x = 1 \in \mathbb{R}$, $z = 2 \in \mathbb{R}$, $w<x$ because $-2<1$ and $y<z$ because $-3<2$, but $wy \not< xz$ because $-2(-3) \not< 1(2)$

1.3a) For any integer x, if $|x+7|>8$, then $|x|>1$

statement: $\forall x \in \mathbb{Z}\ |x+7|>8 \to |x|>1$

contrapositive: $\forall x \in \mathbb{Z}\ |x| \leq 1 \to |x+7| \leq 8$

Proof: We will prove the claim by using the contrapositive: $\forall x \in \mathbb{Z}\ |x| \leq 1 \to |x+7| \leq 8$

$x \in \mathbb{Z} \wedge |x| \leq 1 \therefore x = \{-1, 0, 1\}$. We can do three cases for all possible values of x.

Case 1: $x = -1$

$|x+7| = |-1+7| = |6| = 6 \leq 8$

Case 2: $x=0$

$|x+7| = |0+7| = |7| = 7 \leq 8$

Case 3: $x=1$

$|x+7| = |1+7| = |8| = 8 \leq 8$

The claim is valid for all cases, and thus it is valid for all possible values of x.

Problem 1)

a) no b) no c) no d) yes e) yes f) no

Option a is not provable because statement 2 says no penguins eat mulberries.

Option b is not provable because there are no statements that prove that there is a well dressed penguin. Statement 4 says that at least one penguin is uncomfortable and statement 3 says that some well-dressed animals are uncomfortable, but that doesn't necessarily mean that the uncomfortable penguin(s) are well-dressed.

Option c is not provable because statement 4 says that at least one penguin is uncomfortable, statement 6 says says that no uncomfortable animal eats mulberries, and statement 5 says that all animals eat mulberries or chestnuts. Therefore, there is at least 1 penguin that eats chestnuts. Statement 1 says all chestnut-eating animals are fun-loving, therefore there is at least 1 penguin that is fun-loving.

Option d is provable because statement 3 says that some well-dressed animals are uncomfortable, so there is at least one well -dressed uncomfortable animal. Statement 6 says that no uncomfortable animal eats mulberries and all animals eat mulberries or chestnuts (statement 5), so there is at least one well-dressed, uncomfortable, chestnut-eating animal. Statement 1 says that all chestnut-eating animals are fun-loving, therefore there is at least one fun-loving, well-dressed animal.

Option e is provable because no penguin eats mulberries (statement 2) and all animals eat mulberries or chestnuts (statement 5), therefore all penguins eat chestnuts. Statement 1 says all chestnut-eating animals are fun-loving, therefore all penguins are fun-loving.

Option f is not provable because although all penguins eat chestnuts and don't eat mulberries (statement 2 and 5), and no uncomfortable animal eats mulberries (statement 6), that doesn't necessarily mean that all penguins are uncomfortable.