Lecture 8 - Linear Differential Equations and Superposition Principle

Linear Equations

- Depends linearly on y and derivatives of y
- General form:

$$\frac{d^n y}{dt^n} + \sum_{k=0}^{n-1} a_k(t) \frac{d^k y}{dt^k} = f(t)$$

- In standard form, coefficent of highest order is 1 (might need to divide to get standard)
- Coefficents $a_k(t)$ can depend arbitrarily on t but not y
- $f(t)$ called inhomogenous term / forcing term
- When $f(t) = 0$, homogeneous linear equation

ex. $\frac{d^2y}{dt^2} + 2(t^2+1)\frac{dy}{dt} - e^t y = \cos(t^2)$

second order linear non-homogeneous

ex. $(1-t^2)\frac{d^3y}{dt^3} + 5e^{t^2}\frac{d^2y}{dt^2} - e^t\frac{dy}{dt} + \cos(t^3)y = 0$

third order linear homogeneous

ex. $\frac{d^4y}{dt^4} + 2(y^2-1)\frac{dy}{dt} - e^t y = 0$

fourth order non-linear

ex. $\frac{d^5y}{dt^5} + 4\frac{d^2y}{dt^2} - ty = 0$

fifth order linear homogeneous

ex. $\frac{d^2y}{dt^2} + 4y = \cos(t)$

second order linear non-homogeneous

Existence and Uniqueness Theorem for Linear Equations

Suppose that the linear differential equation

$$\frac{d^n y}{dt^n} + \sum_{k=0}^{n-1} a_k(t)\frac{d^k y}{dt^k} = f(t)$$
$$y(t_0) = y_0$$
$$\frac{dy}{dt}(t_0) = y_0'$$
$$\vdots$$
$$\frac{d^{n-1}y}{dt^{n-1}}(t_0) = y_0^{(n-1)}$$

has coefficients $\{a_k(t)\}_{k=0}^{n-1}, f(t)$ that are continuous in an interval (a,b) containing the initial point t_0. Then there exists a unique solution $y(t)$ for $t \in (a,b)$

ex. Linear: $\frac{dy}{dt} = t^2 y \quad y(0) = 5$

continuous $\forall\, t \in (-\infty, \infty)$ $\therefore$ guaranteed unique solution $\forall$ vals of t $\left(y = 5e^{\frac{1}{3}t^3}\right)$

$\frac{1}{y}\frac{dy}{dt} = t^2$

$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int t^2\,dt$

$\ln(y) = \frac{1}{3}t^3 + c$

$\ln(5) = c$

$\ln(y) = \frac{1}{3}t^3 + \ln(5) \qquad y = 5e^{\frac{1}{3}t^3}$

ex. Non-linear: $\frac{dy}{dt} = ty^2 \qquad y(0)=5$

$\frac{1}{y^2}\frac{dy}{dt} = t$

$\int \frac{1}{y^2}\frac{dy}{dt}\,dt = \int t\,dt$

$-\frac{1}{y} = \frac{1}{2}t^2 + c$

$c = -\frac{1}{5}$

$-\frac{1}{y} = \frac{t^2}{2} - \frac{1}{5}$

$y = \left(\frac{1}{5} - \frac{t^2}{2}\right)^{-1} = \frac{5}{1 - \frac{5t^2}{2}}$ not continuous at $t = \pm\sqrt{\frac{2}{5}}$, needed to solve to find discontinuities

ex. $\frac{d^2y}{dt^2} + y = \cos(t) \qquad y(2) = 3$

guaranteed unique solution $\forall\ t \in (-\infty, \infty)$

ex. $\frac{dy}{dt} + \frac{y}{t^2+t-2} = \cos(t) \qquad y(\frac{1}{2}) = 4$

$t^2+t-2=0 \qquad (t+2)(t-1)=0$

continuous on $(-\infty,-2) \cup (-2,1) \cup (1,\infty)$

$t_0 = \frac{1}{2}$, so use $(-2,1)$

guaranteed unique solution $\forall\ t \in (-2,1)$

ex. $\frac{dy}{dt} + \frac{y}{t^2+t-2} = \cos(t) \qquad y(3) = 2$

guaranteed unique solution $\forall\ t \in (1,\infty)$

ex. $\frac{d^3y}{dt^3} + \ln|1-t^2|\frac{dy}{dt} + \frac{e^t}{4t-1}y = 0 \qquad y(0)=0$

$1-t^2=0 \qquad t=\pm 1 \qquad t=\frac{1}{4}$

continuous on $(-\infty,-1) \cup (-1,\frac{1}{4}) \cup (\frac{1}{4},1) \cup (1,\infty)$

guaranteed unique soln. $\forall\ t \in (-1, \frac{1}{4})$

ex. $(t^2+t-\frac{1}{2})\frac{dy}{dt} + \frac{y}{1+t^2} = e^t \qquad y(\frac{1}{3}) = 3$

need to divide to get standard form

$\frac{dy}{dt} + \frac{y}{(t^2+t-\frac{1}{2})(1+t^2)} = \frac{e^t}{t^2+t-\frac{1}{2}}$

$t = \frac{-1\pm\sqrt{3}}{2}$ continuous on $(-\infty, \frac{-1-\sqrt{3}}{2}) \cup (\frac{-1-\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2}) \cup (\frac{-1+\sqrt{3}}{2}, \infty)$

guaranteed unique soln. $\forall\ t \in \left(\frac{-1-\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2}\right)$

Homogeneous Equations

General Equation:

$$\frac{d^n y}{dt^n} + \sum_{k=0}^{n-1} a_k(t) \frac{d^k y}{dt^k} = 0$$

Superposition Principle:

Suppose that $y_1(t)$ and $y_2(t)$ are solutions to a linear homogeneous differential equation. A linear combination $y(t) = Ay_1(t) + By_2(t)$ (where A, B are constants) is also a solution. More generally if $y_1(t), y_2(t) \dots y_n(t)$ are solutions to a linear homogeneous differential equation then so is an arbitrary linear combination $y = A_1y_1(t) + A_2y_2(t) + \dots A_ny_n(t)$.

ex. $\frac{d^2y}{dt^2} + y = 0$, suppose $y_1(t)$, $y_2(t)$ solve eqn.

$y_1'' + y_1 = 0 \qquad y_2'' + y_2 = 0$

Then, $y = A_1 y_1(t) + A_2 y_2(t)$ also solves eqn.

$$\frac{d^2}{dt^2}\left(A_1 y_1(t) + A_2 y_2(t)\right) = A_1 y_1'' + A_2 y_2''$$

$$\frac{d^2}{dt^2}\left(A_1 y_1 + A_2 y_2\right) = A_1 y_1'' + A_2 y_2'' + A_1 y_1 + A_2 y_2$$

$$= A_1 \underbrace{(y_1'' + y_1)}_{0} + A_2 \underbrace{(y_2'' + y_2)}_{0}$$

ex. $t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} - 4y = 0$

$y_1 = t^2 \qquad y_2 = t^{-2}$

$y = At^2 + Bt^{-2}$ also solves eqn.

if given $y(1) = 2 \qquad y'(1) = 0$

$t^2 y'' + ty' - 4y = 0 \qquad y = At^2 + Bt^{-2} \qquad y(1) = A + B = 2$

$y' = 2At - 2Bt^{-3} \qquad y'(1) = 2A - 2B = 0$

$A = B \qquad A = 1 \quad B = 1$

$y = t^2 + t^{-2}$

ex. Solve $y'' + 3y' + 2y = 0 \qquad y(0) = y_0 \qquad y'(0) = y_0'$

solutions are $y = e^{-t}$, $y = e^{-2t}$

$y = Ae^{-t} + Be^{-2t}$

$y(0) = A + B = y_0$

$y'(0) = -A - 2B = y_0'$

$-B = y_0 + y_0' \qquad A = 2y_0 + y_0'$

$y = (2y_0 + y_0')e^{-t} - (y_0 + y_0')e^{-2t}$

Superposition is generally not true for nonlinear equations. For instance $y_1(t) = \frac{-1}{t}$ and $y_2(t) = \frac{-1}{t-1}$ are both solutions to

$$\frac{dy}{dt} = y^2$$

However the function $y = -\frac{A}{t} - \frac{B}{t-1}$ is generally not a solution. In fact a little algebra shows that this is a solution only when $A = 0, B = 1$ or $A = 1, B = 0$

Quiz 9