Differential equations

ex. $\frac{dY}{dT} = -Y$ $\quad \frac{d^2Y}{dT} + \frac{dY}{dT} + 16Y = \cos T$ $\quad \frac{d^3Y}{dT^3} + Y\frac{dY}{dT} - Y = 1$

In real life:

- Newton's law of cooling: $\frac{dT}{dt} = -k(T - T_0)$
- Newton's second law: $m\frac{d^2y}{dt^2} = F\left(\frac{dy}{dt}, y, t\right)$
- Hooke's Law: $m\frac{d^2y}{dt^2} = -kY$

Linear vs Non-Linear

Eqn. of line:

$$a_1 y_1 + a_2 y_2 + a_3 y_3 + \dots + a_n y_n = b$$

Linear diff. eqn.:

$$a_n(t)\frac{d^n y}{dt} + a_{n-1}(t)\frac{d^{n-1}y}{dt} + \dots + a_1(t)\frac{dy}{dt} + a_0(t)y = f(t)$$

anything not in this form is non-linear

ex. $\frac{dY}{dT} = -Y \rightarrow \frac{dY}{dT} + y = 0$ linear

$\frac{d^2Y}{dT^2} + \frac{dY}{dT} + 16Y = \cos t$ linear

$\frac{d^3Y}{dT^3} + \boxed{Y\frac{dY}{dT}} - Y = 1$ Non-linear

makes it non-linear

☆ Only dependance on $Y, Y', Y'', \dots Y^{(n)}$ matters

ex. $\frac{d^2y}{dt^2} + tY^2 = 0$ non-linear $\quad \frac{d^2Y}{dt^2} + t^2Y = 0$ linear

Order of a Differential Equation

- Order = Order of highest derivative

ex. $\frac{dY}{dT} = -Y \rightarrow \frac{dY}{dT} + y = 0$ first order

$\frac{d^2Y}{dT^2} + \frac{dY}{dT} + 16Y = \cos t$ second order

$\frac{d^3Y}{dT^3} + Y\frac{dY}{dT} - Y = 1$ third order

- tells us what data we need to solve the problem

Lecture 2 - Solving Differential Equations

Solving Differential Equations

• Rule of thumb: N^{th} order equation requires N pieces of data

• Initial value problems: all n pieces of data at same time

ex. $\frac{d^3y}{d\tau^3} + y^2 = 1$ Third order non-linear

$y(0) = 1$ $\quad \frac{dy}{dt}(0) = -1$ $\quad \frac{d^2y}{dt^2}(0) = 2$

$\frac{d^3y}{d\tau^3}(0) + 1^2 = 1$ $\quad \frac{d^3y}{d\tau^3}(0) = 0$ $\quad y'''(0) = 0$

what about $y^{(n)}(0)$?

$\frac{d^3y}{d\tau^3} + y^2 = 1 \rightarrow \frac{d^4y}{d\tau^4} + 2y\frac{dy}{d\tau} = 0 \rightarrow \frac{d^4y}{d\tau^4}(0) + 2(1)(-1) = 0 \rightarrow y''''(0) = 2$

• Taylor Series: $y(t) = y(0) + y'(0)t + \frac{y''(0)t^2}{2!} + \frac{y'''(0)t^3}{3!} + \ldots$

Ways to Solve Differential Equations

1) Recognize as derivative, integrate using fundamental thmn. of calculus

2) Guess

Fundamental thmn of calc: $\int_a^b f'(t)dt = f(b) - f(a)$ $\quad \int f'(t)dt = f(t) + C$

ex. $F = ma$

$m\frac{d^2y}{dt^2} = -mg$

$\frac{d^2y}{dt^2} = -g$ $\quad$ assume $y(0) = y_0$, $\frac{dy}{dt}(0) = V_0$

$\int_0^t \frac{d^2y}{dt^2}dt = \int_0^t -g\,dt$

$y'(t) - y'(0) = -gt$

$y'(t) = -gt + V_0$

$\int_0^t y'(t)dt = \int_0^t -gt\,dt + \int_0^t V_0\,dt$

$y(t) - y(0) = \frac{-gt^2}{2} + V_0t$

$y(t) = -\frac{1}{2}gt^2 + V_0t + y_0$

ex. Exponential $\overset{(r>0)}{\text{Growth}} / \overset{(r<0)}{\text{Decay}}$

$\frac{dy}{dt} = ry$

$$\left(\frac{1}{y}\right)\frac{dy}{dt} = r$$

$$\int\left(\frac{1}{y}\right)\frac{dy}{dt}\,dt = \int r\,dt$$

$$\int\frac{d}{dt}\left(\ln|y|\right)dt = \int r\,dt$$

$$\ln(y) = rt + C$$

$$y = e^{rt+C}$$

$$y = e^{rt}\cdot e^{C} \qquad \text{let } A = e^{C}$$

$$y = Ae^{rt}$$

Seperable Equations

① $\frac{dy}{dt} = f(y)\,g(t)$

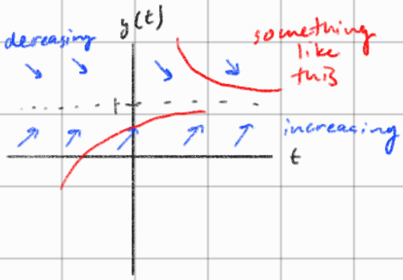
$$\left(\frac{1}{f(y)}\right)\frac{dy}{dt} = g(t) \qquad \int\left(\frac{1}{f(y)}\right)\frac{dy}{dt}\,dt = \int g(t)\,dt$$

② $\frac{dy}{dt} = 1 - y$

what does graph look like?

if $y(t) > 1$ $\frac{dy}{dt} < 0$ (decreasing)

if $y(t) < 1$ $\frac{dy}{dt} > 0$ (increasing)



$$dt = \frac{dy}{1-y}$$

$$\int dt = \int\frac{1}{1-y}\,dy$$

$$t + C = -\ln|1-y|$$

$$\ln|1-y| = -t - C \qquad |1-y| = e^{-t-C} \qquad \text{let } A = e^{-C}$$

$$|1-y| = Ae^{-t}$$

$$y = 1 \pm Ae^{-t} \qquad y \to 1 \text{ as } t \to \infty$$

Lecture 3 - Verifying Solutions

ex. Show that $\sin(t)e^t$ satisfies $y'' - 2y' + 2y = 0$ just plug it in

$y = e^t \sin t \qquad y' = e^t \sin t + e^t \cos t \qquad y'' = 2e^t \cos t$

$y'' - 2y' + 2y = 0$

$2e^t \cos t - 2(e^t \sin t + e^t \cos t) + e^t \sin t = 0$

$0 = 0$ ✓

ex. find solutions to $y'' - 3y' + 2y = 0$ in the form $y = e^{rt}$

$y = e^{rt} \qquad y' = re^{rt} \qquad y'' = r^2 e^{rt}$

$r^2 e^{rt} - 3re^{rt} + 2e^{rt} = 0$

$e^{rt}(r^2 - 3r + 2) = 0$

$e^{rt}(r-2)(r-1) = 0$

$r = 2, \quad r = 1$

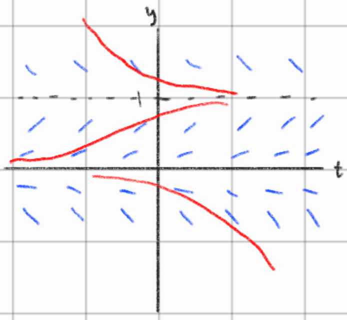
$\boxed{y = e^t, \quad y = e^{2t}}$

Slope fields

$y' = f(y, t)$

ex. $\frac{dy}{dt} = y(1-y)$

t	y	slope
	1	0
	$\frac{1}{2}$	$\frac{1}{4}$
	0	0
	-1	-2



ex. $\frac{dy}{dt} = -\frac{t}{y}$

$y = t \to$ slope $= -1$

$y = -t \to$ slope $= 1$

$t = 0 \to$ slope $= 0$

$y = 0 \to$ slope $= \infty$

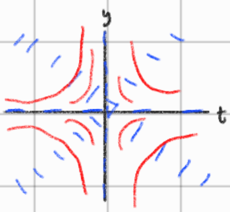
ex. $\frac{dy}{dt} = -\frac{y}{t}$

$y = 0 \to$ slope $= 0$

$t = 0 \to$ slope $= \infty$

$y = t \to$ slope $= -1$

$y = -t \to$ slope $= 1$



Existance and Uniqueness

1) Is there a solution?

2) Is that solution unique?

ex. $\frac{dy}{dt} = y^{1/3} \qquad y(0)=0$

$$\int \left(\frac{1}{y^{1/3}}\right) dy = \int dt$$

$$\frac{3}{2} y^{2/3} = t + C$$

initial cond.: $\frac{3}{2}(0) = 0 + C$

$C = 0$

$$\frac{3}{2} y^{2/3} = t \qquad y^{2/3} = \frac{2}{3} t \qquad y = \left(\frac{2}{3} t\right)^{3/2}$$

second solution: $y(t) = 0$

$\frac{dy}{dt} = 0$

$y^{1/3} = 0^{1/3} = 0$

Theorem:

Consider the differential equation

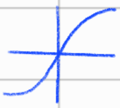
$$\frac{dy}{dt} = f(y,t) \qquad y(t_0) = y_0$$

suppose that the function f is continuous in a neighborhood of the point (t_0, y_0) and that the partial derivative $\frac{\partial f}{\partial y}$ is continuous in a neighborhood of the point (t_0, y_0). Then there exists a unique solution to the differential equation for $t \in (t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$.

(close to t_0)

ex. (from before) $\frac{dy}{dt} = y^{1/3} \qquad y(0) = 0$

$f(y,t) = y^{1/3}$ continuous everywhere



$\frac{\partial f}{\partial y} = \frac{1}{3} y^{-2/3}$

$\frac{\partial f}{\partial y}(0) = \frac{1}{3(0)^{2/3}}$ $\therefore$ not continuous at 0

ex. For which of the following equations is existence and uniqueness guaranteed

1 • $\frac{dy}{dt} = y^2 + t^2 \quad y(2) = 8$

2 • $\frac{dy}{dt} = \frac{t}{y+t} \quad y(2) = -2$

3 • $\frac{dy}{dt} = y \ln|y| \quad y(1) = 0$

4 • $\frac{dy}{dt} = \frac{y+t-2}{(y-2)^2+t^2} \quad y(0) = 2$

5 • $\frac{dy}{dt} = \frac{y+t-2}{(y-2)^2+t^2} \quad y(0) = 0$

1) $\frac{dy}{dt} = y^2 + t^2 \qquad y(2) = 8$

$f(y,t) = y^2 + t^2$ continuous

$\frac{\partial f}{\partial y} = 2y + t^2$ continuous

unique solution near $(2,8)$

2) f and $\frac{\partial f}{\partial y}$ not continuous near $(2,-2)$

3) $\frac{\partial f}{\partial y}$ not continuous near $(1,0)$

4) f and $\frac{\partial f}{\partial y}$ not continuous near $(0,2)$

5) f and $\frac{\partial f}{\partial y}$ continuous near $(0,2)$