Lecture 6 - First Order Linear Equations and Autonomous Equations

First Order Linear (from lecture 5)

$$\frac{dy}{dt} + p(t)\,y = Q(t) \qquad \mu(t) = e^{\int p(t)dt}$$

$$\mu(t)\left(\frac{dy}{dt} + p(t)y\right) = Q(t)$$

$$\frac{d}{dt}\left(y e^{\int p(t)dt}\right) = Q(t)e^{\int p(t)dt}$$

ex. $\frac{dy}{dt} + y = t^2 + t$

$p(t) = 1 \quad q(t) = t^2 + t$

$\mu(t) = e^{\int 1 dt} = e^t$

$e^t \frac{dy}{dt} + ye^t = e^t(t^2+t)$

$\frac{d}{dt}(ye^t) = e^t(t^2+t)$

$ye^t = \int (t^2+t)e^t\,dt + c$

$y = \underbrace{e^{-t}\int (t^2+t)e^t\,dt}_{y_p \text{ (particular solution)}} + ce^{-t}$

int. by parts: $Q = t^2+t \quad dP = e^t dt$

$dQ = (2t+1)dt \quad P = e^t$

$\int (t^2+t)e^t dt = (t^2+t)e^t - \int (2t+1)e^t\,dt$

answer in the form: $= (At^2 + Bt + D)e^t$

instead of integrating by parts, guess

$y_p = At^2 + Bt + D$ (particular solution, e^{-t} and e^t cancel out)

$y' + y = t^2 + t$

Guess: $y = At^2 + Bt + D$

$+ \quad y' = 2At + B$

$y' + y = At^2 + (B+2A)t + (B+D) = t^2 + t$

$A = 1 \qquad B + 2A = 1 \qquad B + D = 0$

$B + 2 = 1 \qquad D = 1$

$B = -1$

$\therefore y_p = t^2 - t + 1$

$y = t^2 - t + 1 + ce^{-t}$

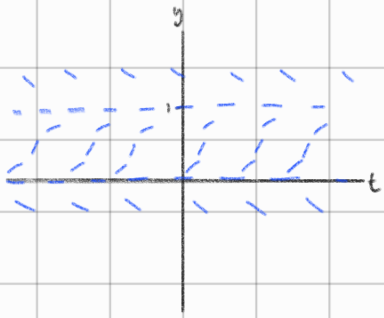
Autonomous Equations

$\frac{dy}{dt} = f(y)$

- always seperable

ex. $\frac{dy}{dt} = y - y^2$

slope field:



horizontal asymptotes at $y=0$ and $y=1$

Equilibria / Fixed Points

$\frac{dy}{dt} = f(y)$ has fixed pts. wherever $f(y) = 0$

ex. $\frac{dy}{dt} = y - y^2$

fixed pts: $y - y^2 = 0$ $\quad y(1-y) = 0$ $\quad y = 0$ or $y = 1$

ex. $\frac{dy}{dt} = \cos(y)$

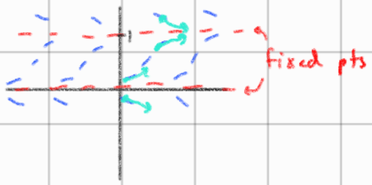
fixed pts. $\cos y = 0$ $\quad y = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

Stability of Fixed Points:

$\frac{dy}{dt} = f(y)$ $\quad$ let y^* fixed pt, $f(y^*) = 0$

- Fixed point is stable if initial conditions near y^* converge to it
- Fixed point is unstable if initial conditions diverge from it

ex. $\frac{dy}{dt} = y - y^2$



$y = 1$ is stable because move towards it

$y = 0$ unstable, move away from it

- Theorem: $\frac{dy}{dt} = f(y)$, y^* fixed bc $f(y^*) = 0$

  y^* stable if $f'(y^*) < 0$

  y^* unstable if $f'(y^*) > 0$

- check example:

  $y^* = 0$ and $y^* = 1$

  $f(y) = y - y^2$

  $f'(y) = 1 - 2y$

  $f'(0) = 1 > 0 \quad \therefore \quad y^* = 0$ is unstable fixed pt

  $f'(1) = 1 - 2 = -1 < 0 \quad \therefore \quad y^* = 1$ is stable fixed pt