Lecture 3 - Verifying Solutions

ex. Show that $\sin(t)e^t$ satisfies $y'' - 2y' + 2y = 0$ just plug it in

$y = e^t \sin t \qquad y' = e^t \sin t + e^t \cos t \qquad y'' = 2e^t \cos t$

$y'' - 2y' + 2y = 0$

$2e^t \cos t - 2(e^t \sin t + e^t \cos t) + e^t \sin t = 0$

$0 = 0$ ✓

ex. find solutions to $y'' - 3y' + 2y = 0$ in the form $y = e^{rt}$

$y = e^{rt} \qquad y' = re^{rt} \qquad y'' = r^2 e^{rt}$

$r^2 e^{rt} - 3re^{rt} + 2e^{rt} = 0$

$e^{rt}(r^2 - 3r + 2) = 0$

$e^{rt}(r-2)(r-1) = 0$

$r = 2, \quad r = 1$

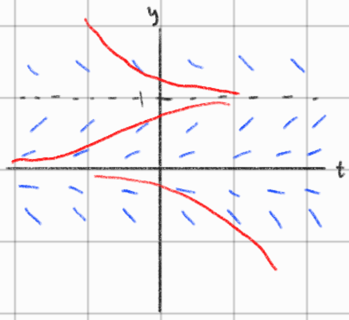
$\boxed{y = e^t, \quad y = e^{2t}}$

Slope fields

$y' = f(y, t)$

ex. $\frac{dy}{dt} = y(1-y)$

t	y	slope
	1	0
	$\frac{1}{2}$	$\frac{1}{4}$
	0	0
	-1	-2



ex. $\frac{dy}{dt} = -\frac{t}{y}$

$y = t \rightarrow$ slope $= -1$

$y = -t \rightarrow$ slope $= 1$

$t = 0 \rightarrow$ slope $= 0$

$y = 0 \rightarrow$ slope $= \infty$

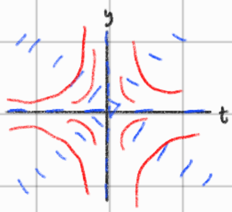
ex. $\frac{dy}{dt} = -\frac{y}{t}$

$y = 0 \rightarrow$ slope $= 0$

$t = 0 \rightarrow$ slope $= \infty$

$y = t \rightarrow$ slope $= -1$

$y = -t \rightarrow$ slope $= 1$



Existance and Uniqueness

1) Is there a solution?

2) Is that solution unique?

ex. $\frac{dy}{dt} = y^{1/3}$ $\quad y(0)=0$

$$\int \left(\frac{1}{y^{1/3}}\right) dy = \int dt$$

second solution: $y(t) = 0$

$\frac{dy}{dt} = 0$

$y^{1/3} = 0^{1/3} = 0$

$$\frac{3}{2} y^{2/3} = t + C$$

initial cond.: $\frac{3}{2}(0) = 0 + C$

$C = 0$

$$\frac{3}{2} y^{2/3} = t \qquad y^{2/3} = \frac{2}{3} t \qquad y = \left(\frac{2}{3} t\right)^{3/2}$$

Theorem:

Consider the differential equation

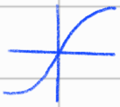
$$\frac{dy}{dt} = f(y,t) \qquad y(t_0) = y_0$$

suppose that the function f is continuous in a neighborhood of the point (t_0, y_0) and that the partial derivative $\frac{\partial f}{\partial y}$ is continuous in a neighborhood of the point (t_0, y_0). Then there exists a unique solution to the differential equation for $t \in (t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$.

(close to t_0)

ex. (from before) $\frac{dy}{dt} = y^{1/3}$ $\quad y(0) = 0$

$f(y,t) = y^{1/3}$ continuous everywhere



$\frac{\partial f}{\partial y} = \frac{1}{3} y^{-2/3}$

$\frac{\partial f}{\partial y}(0) = \frac{1}{3(0)^{2/3}}$ $\therefore$ not continuous at 0

ex. For which of the following equations is existence and uniqueness guaranteed

1. $\frac{dy}{dt} = y^2 + t^2 \quad y(2) = 8$
2. $\frac{dy}{dt} = \frac{t}{y+t} \quad y(2) = -2$
3. $\frac{dy}{dt} = y \ln|y| \quad y(1) = 0$
4. $\frac{dy}{dt} = \frac{y+t-2}{(y-2)^2+t^2} \quad y(0) = 2$
5. $\frac{dy}{dt} = \frac{y+t-2}{(y-2)^2+t^2} \quad y(0) = 0$

1) $\frac{dy}{dt} = y^2 + t^2$ $\quad y(2) = 8$

$f(y,t) = y^2 + t^2$ continuous

$\frac{\partial f}{\partial y} = 2y + t^2$ continuous

unique solution near $(2,8)$

2) f and $\frac{\partial f}{\partial y}$ not continuous near $(2,-2)$

3) $\frac{\partial f}{\partial y}$ not continuous near $(1,0)$

4) f and $\frac{\partial f}{\partial y}$ not continuous near $(0,2)$

5) f and $\frac{\partial f}{\partial y}$ continuous near $(0,2)$