# CS 173 - Lecture 3

## Proof by Cases

ex. Prove using definition of even integers, $\forall$ j, k $\in \mathbb{Z}$, j even $\wedge$ k even $\rightarrow$ jk even

universal     case 1     case 2     concl.

Let j, k be integers. Suppose j even or k even. Then, we have two cases.

Case 1: j even. By definition of even, $j = 2m$ where $m \in \mathbb{Z}$. Then, $jk = 2mk$. We know $mk \in \mathbb{Z}$ since $m, k \in \mathbb{Z}$ so jk follows definition of even, and it is even

Case 2: K is even, ... (similar to case 1)

Since in both cases jk is even, we have proven the statement

## Proof style notes

- use connector words (not all math)
- write math w/ symbols
- declare/define variables even if they're defined in the question
- begin by assuming hypothesis
- end w/ conclusion

## Proof by Contrapositive

- Rephrasing an implication that is logically equivalent to the original statement

$$p \rightarrow q \equiv \neg q \rightarrow \neg p \qquad \forall x,\ P(x) \rightarrow Q(x) \equiv \forall x,\ \neg Q(x) \rightarrow \neg P(x)$$

contrapositives     contrapositives

- Rephrase if contrapositive is easier to prove

ex. $\forall a, b \in \mathbb{Z}$ $a+b \geq 15 \rightarrow (a \geq 8 \vee b \geq 8)$

contrapositive: $\forall a, b \in \mathbb{Z}$ $\neg(a \geq 8 \vee b \geq 8) \rightarrow \neg(a+b \geq 15)$

$\forall a, b \in \mathbb{Z}$ $a < 8 \wedge b < 8 \rightarrow a+b < 15$

proof: Let $a, b \in \mathbb{Z}$ where $a<8$ and $b<8$. Since $a, b \in \mathbb{Z}$, we know $a \leq 7$ and $b \leq 7$. Then, $a+b \leq 14 < 15$. This is what we wanted to prove.

ex. $\forall n \in \mathbb{Z}$, if $n^2$ even $\rightarrow n$ even

contrapositive: $\forall n \in \mathbb{Z}$ $n$ odd $\rightarrow n^2$ odd

proof: Let $n$ be an odd integer. By definition of odd, $n = 2k+1$ where $k \in \mathbb{Z}$. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since $k \in \mathbb{Z}$, $2k^2 + 2k \in \mathbb{Z}$. Let $j = 2k^2 + 2k$, so also $j \in \mathbb{Z}$. Then, $n^2 = 2j + 1$ $\therefore$ $n^2$ is odd by the definition of odd

Discussion Problems

## 1.1 Negations and contrapositives

Negate the following statements, moving all negations (e.g. "not") onto individual predicates/propositions. You may use shorthand to do intermediate work, but your final answer should be in English. Similarly, construct the contrapositive of each statement.

(a) If my plant is dead, then I didn't water it or I left it in the dark.

(b) If vampires exist, then there is a city $c$ such that $c$ is full of vampires and $c$ does not have a blood bank.

(c) For every martian $w$, if $w$ is green, then $w$ is tall or $w$ is ticklish.

(d) For any house $h$, for any dog $d$, $d$ does not live at $h$ or $h$ has a supply of dog food.

(e) For every movie $m$, if $m$ is a fantasy movie and $m$ is popular, then $m$ has a cute lead actor and $m$ has a big special effects budget.

## 1.3 Variations on direct proof

Prove the following claim. Your proof should divide into cases based on the sign of $|x + 7|$.

(a) For any integer $x$, if $|x + 7| > 8$, then $|x| > 1$.

Prove the following claims by contrapositive. Begin your proof by explicitly writing out the contrapositive. Then use direct proof to prove the contrapositive.

(b) For all real numbers $x$ and $y$, if $x + y \geq 2$, then $x \geq 1$ or $y \geq 1$.

(c) For all integers $m$ and $n$, if $mn$ is even, then $m$ is even or $n$ is even.

(d) For all real numbers $x$, if $x^2 - 3x + 2 > 0$, then $x \geq 2$ or $x < 1$.

(e) For any integers m and n, if $7m + 5n = 147$, then $m$ is odd or $n$ is odd.

1.1a) Negation: My plant is dead, and I watered it and left it in light

Contrapositive: If I watered my plant and left it in light, then my plant is alive

1.1c) $\neg(\forall w \in \text{martian}, w \text{ green} \rightarrow w \text{ tall} \vee w \text{ ticklish}) \equiv \exists w \in \text{martian}$ s.t $w$ green and $w$ not tall and $w$ not ticklish

Negation: There exists a martian w such that w is green and not tall and not ticklish

Contrapositive: For every martian w, if w isn't tall and isn't ticklish, then w isn't green

1.3c) $\forall m, n \in \mathbb{Z}$, $mn$ even $\rightarrow m$ even or $n$ even

contrapositive: $\forall m, n \in \mathbb{Z}$, $m$ odd $\wedge$ $n$ odd $\rightarrow mn$ odd

proof: Let $m, n \in \mathbb{Z}$ and odd. By the definition of odd $m = 2j+1$ where $j \in \mathbb{Z}$ and $n = 2k+1$ where $k \in \mathbb{Z}$. Then, $mn = (2j+1)(2k+1) = 4jk + 2k + 2j + 1 = 2(2jk+j+k)+1$. $2jk+j+k \in \mathbb{Z}$ because $j, k \in \mathbb{Z}$. Let $b = 2jk+j+k \in \mathbb{Z}$, then $mn = 2b+1$, which matches the definition of odd, therefore $mn$ is odd when $m$ odd and $n$ odd.

1.3d) $\forall x \in \mathbb{R}$, $x^2 - 3x + 2 > 0 \rightarrow (x \geq 2 \vee x < 1)$

contrapositive: $\forall x \in \mathbb{R}$, $(x < 2 \wedge x \geq 1) \rightarrow x^2 - 3x + 2 \leq 0$

proof: Let $x \in \mathbb{R}$. When $x = [1, 2)$, $x^2 - 3x + 2 \leq 0$. $x^2 - 3x + 2 = (x-2)(x-1)$ and the the coefficient of $x^2$ is positive, therefore between and including the x-intercepts, $x \leq 0$. Therefore, $x^2 - 3x + 2 \leq 0$ on $x = [1, 2)$, and the statement is proven.