Lecture

Math Review

$\mathbb{N}$: natural numbers (incl. 0)

$\{0, 1, 2, \ldots\}$

$\mathbb{Z}$: integers

$\{0, -1, 1, -2, 2, \ldots\}$

$\mathbb{Z}^+$: positive integers (excl. 0)

$\{1, 2, 3, \ldots\}$

$\mathbb{R}$: Real numbers (incl. irrational)

$\{-2, 2.5, \pi, \ldots\}$

$\mathbb{Q}$: Rational numbers, written as m/n where m, n integers, $n \neq 0$

$\mathbb{C}$: complex numbers

$\{i, 2i+1, \pi, \ldots\}$

Notation

$x \in \mathbb{R}$ → x is an element of reals

is an element of

$y \in (0, 5]$ for $y \in \mathbb{Z}$ → $y \in \{1, 2, 3, 4, 5\}$

excluded included

$(a, b) \in \mathbb{Z}^2$ → $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, so (a, b) is an ordered pair of integers.

(NOT integer squared ie 2, 4, 9, ...)

$(x, y, z) \in \mathbb{R}^3$

Exponents

(basic exponent stuff)

change of base: $\log_b x = \log_a x \cdot \log_b a$

Propositional Logic

Proposition − T or F statement

Complex Propositions − one or more propositions combined w/ logic operators

Operators:

$\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies), $\leftrightarrow$ (bi-direction implies), $\neg$ (not)

Truth Tables

p	$\neg p$
T	F
F	T

} entirely defines operator

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

F → only false if p true and q false

vacuously true

vacuous truth example: If it's raining when I wake up, then I will bring an umbrella to campus

If raining and no umbrella, I lied

If not raining and umbrella or no umbrella, no lie

Negation

logical equivalence (same truth tables)

$\neg(\neg p) \equiv p$

DeMorgan's Law

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) = \neg p \wedge \neg q$

Negation of Implication

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

Discussion Problems

1.5 Logic operators

The late 19th century philosopher Charles Peirce (rhymes with 'hearse,' not 'fierce') wrote about a set of logically dual operators and, in his writings, coined the term 'Ampheck' to describe them. The two most common Ampheck operators, the Peirce arrow (written $\downarrow$ or $\perp$ or ∇ by different people) and the Sheffer stroke (written $\uparrow$ or $|$ or $\barwedge$ by different people), are defined by the following truth table:

p	q	$p \uparrow q$	$p \downarrow q$
T	T	F	F
T	F	T	F
F	T	T	F
F	F	T	T

1. (4 points) The set of operators $\{\wedge, \vee, \neg\}$ is *functionally complete*, which means that *every* logical statement can be expressed using only these three operators. Is the smaller set of operators $\{\vee, \neg\}$ also functionally complete? Explain why or why not.
2. (4 points) Express $\neg p$ using only the Sheffer stroke operation $\uparrow$.
3. (5 points) Express $p \vee q$ using only the Sheffer stroke operation $\uparrow$. Justify your answer (e.g. using a truth table).
4. (3 points) Explain why the set of operators $\{\uparrow\}$ is functionally complete.
5. (4 points) Express the Sheffer stroke operation $p \uparrow q$ using only the Peirce arrow $\downarrow$ operation. Explain why the set of operators $\{\downarrow\}$ is functionally complete.

1. $p \wedge q \equiv \neg(\neg p \vee \neg q)$, $\therefore$ $\{\wedge, \vee, \neg\}$ can be described using $\{\vee, \neg\}$. $\{\wedge, \vee, \neg\}$ is functionally complete $\therefore$ $\{\vee, \neg\}$ must also be functionally complete

2.

p	$\neg p$	$p \uparrow p$
T	F	F
F	T	T

$\neg p \equiv p \uparrow p$

3.

p	q	$p \vee q$	$p \uparrow q$	$\neg p$	$\neg q$	$\neg p \uparrow \neg q$
T	T	T	F	F	F	T
T	F	T	T	F	T	T
F	T	T	T	T	F	T
F	F	F	T	T	T	F

$p \vee q \equiv \neg p \uparrow \neg q$

4. $\{\uparrow\}$ functionally complete because $\{\wedge, \vee, \neg\}$ can be expressed using $\{\uparrow\}$
$(\neg p \equiv p \uparrow p,\ p \vee q \equiv \neg p \uparrow \neg q,\ p \wedge q \equiv \neg(p \uparrow q))$

5.

p	q	$\neg p$	$\neg q$	$p \uparrow q$	$p \downarrow q$	$\neg(\neg p \uparrow \neg q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

$\{\downarrow\}$ is functionally complete because it can be expressed using $\{\uparrow, \neg\}$, which is functionally complete ($\{\uparrow\}$ is functionally complete $\therefore$ $\{\uparrow, \neg\}$ must be functionally complete)

1. $p \wedge q \equiv \neg(\neg p \vee \neg q)$ $\therefore$ $\{\wedge, \vee, \neg\}$ can be described using $\{\vee, \neg\}$. $\{\vee, \wedge, \neg\}$ is functionally complete $\therefore$ $\{\vee, \neg\}$ must also be functionally complete.
2. $\neg p \equiv p \uparrow p$
3. $p \vee q \equiv \neg p \uparrow \neg q$
4. $\{\uparrow\}$ is functionally complete because $\{\wedge, \vee, \neg\}$ can be expressed using $\{\uparrow\}$. $\neg p \equiv p \uparrow p, p \vee q \equiv \neg p \uparrow \neg q, p \wedge q \equiv \neg(p \uparrow q)$.
5. $\{\downarrow\}$ is functionally complete because it can be expressed using the functionally complete set $\{\uparrow, \neg\}$. $\{\uparrow, \neg\}$ is functionally complete because $\{\uparrow\}$ is functionally complete $\therefore$ $\{\uparrow, \neg\}$ must also be functionally complete.