Lecture 5 - Linear Non-homogeneous and Exact Equations

Linear Non-Homogeneous First Order Differential Equations

- Last lecture: homogeneous
- Non-homogeneous: $\frac{dy}{dx} + p(t)y = Q(t)$
- Multiply through by $\mu(t) = e^{\int p(t)dt}$

$$e^{\int p(t)dt}\left(\frac{dy}{dt}\right) + p(t)ye^{\int p(t)dt} = Q(t)e^{\int p(t)dt}$$

$$\int \frac{d}{dt}\left(ye^{\int p(t)dt}\right) = \int Q(t)e^{\int p(t)dt}$$

$$ye^{\int p(t)dt} = \int Q(t)e^{\int p(t)dt}dt + C$$

ex. $\frac{dy}{dt} + \tan t\, y = \sin t$ $\quad \int p(t)dt = \int \tan t\, dt = -\ln(\cos t)$ $\quad \mu(t) = e^{-\ln(\cos(t))} = \frac{1}{\cos t}$

$$\frac{1}{\cos t}\cdot\frac{dy}{dt} + \frac{\sin t}{\cos^2 t}y = \frac{\sin t}{\cos t}$$

$$\frac{d}{dt}\left(\frac{1}{\cos t}y\right) = \frac{\sin t}{\cos t}$$

$$\int \frac{d}{dt}\left(\frac{1}{\cos t}y\right)dt = \int \frac{\sin t}{\cos t}dt$$

$$\frac{1}{\cos t}y = A - \ln|\cos t|$$

$$y = \underbrace{A\cos t}_{\text{homogeneous solution}} \underbrace{- \cos t \ln|\cos t|}_{\text{particular solution}}$$

ex. $t\frac{dy}{dt} + 2y = t^3$

$$\frac{dy}{dt} + \frac{2}{t}y = t^2 \qquad \mu(t) = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$$

$$t^2\frac{dy}{dt} + 2ty = t^4$$

$y \times$ integrating factor (μ)

$$\frac{d}{dt}(t^2y) = t^4$$

$$\int \frac{d}{dt}(t^2y)dt = \int t^4 dt$$

$$t^2y = \frac{t^5}{5} + A$$

$$y = \frac{t^3}{5} + At^{-2}$$

Non-linear Seperable Equations

ex. $\frac{dy}{dt} = y^2$

$$\int \frac{dy}{dt}\left(\frac{1}{y^2}\right)dt = \int 1\, dt$$

$$-\frac{1}{y} = t + A$$

$$y = \frac{-1}{t+A}$$

Exact Equation

$N(x,y)\frac{dy}{dx} + M(x,y) = 0$

$\cdot$ Exact if $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

$N\,dy + M\,dx = 0$

$\vec{v} = (M(x,y), N(x,y))$ Exact: $M_y = N_x$ $M_y - N_x = 0$ $\text{curl}(M,N) = 0$ $\therefore \vec{v}$ conservative/gradient

$\therefore$ Exists $\psi(x,y)$ s.t $(\psi_x, \psi_y) = (M(x,y), N(x,y))$

Can find $\psi(x,y)$ st $\psi_x = M$ $\psi_y = N$

Diff Eq is: $\psi_y\,dy + \psi_x\,dx = 0 = d\psi$

$\psi(x,y) = \text{constant}$

finding ψ:

$\psi_y = N(x,y)$ $\psi_x = M(x,y)$

$\psi_y = N$ integrate wrt y, treat x constant

$\psi = \int N\,dy + c(x)$

$\psi_x = \int N_x\,dy + c'(x)$

ex. $(3x^2 - 3y^2)\frac{dy}{dx} + (3x^2 + 6xy) = 0$

$\frac{d\psi}{dy} = 3x^2 - 3y^2$ $\frac{d\psi}{dx} = 3x^2 + 6xy$

$\psi = 3x^2y - y^3 + c(x)$

$\frac{\partial \psi}{\partial x} = 6xy + c'(x) = 3x^2 + 6xy$

$c'(x) = 3x^2$

$c(x) = x^3$

$\psi(x,y) = 3x^2y - y^3 + x^3$