Lecture 2 - Solving Differential Equations

Solving Differential Equations

- Rule of thumb: N^{th} order equation requires N pieces of data
- Initial value problems: all n pieces of data at same time

ex. $\frac{d^3y}{d\tau^3} + y^2 = 1$ Third order non-linear

$y(0) = 1 \qquad \frac{dy}{dt}(0) = -1 \qquad \frac{d^2y}{dt^2}(0) = 2$

$\frac{d^3y}{d\tau^3}(0) + 1^2 = 1 \qquad \frac{d^3y}{d\tau^3}(0) = 0 \qquad y'''(0) = 0$

what about $y^{(n)}(0)$?

$\frac{d^3y}{d\tau^3} + y^2 = 1 \rightarrow \frac{d^4y}{d\tau^4} + 2y\frac{dy}{d\tau} = 0 \rightarrow \frac{d^4y}{d\tau^4}(0) + 2(1)(-1) = 0 \rightarrow y''''(0) = 2$

- Taylor Series: $y(t) = y(0) + y'(0)t + \frac{y''(0)t^2}{2!} + \frac{y'''(0)t^3}{3!} + \dots$

Ways to Solve Differential Equations

1) Recognize as derivative, integrate using fundamental thmn. of calculus
2) Guess

Fundamental thmn of calc: $\int_a^b f'(t)dt = f(b) - f(a) \qquad \int f'(t)\,dt = f(t) + C$

ex. $F = ma$

$m\frac{d^2y}{dt^2} = -mg$

$\frac{d^2y}{dt^2} = -g$ assume $Y(0) = Y_0$, $\frac{dY}{dt}(0) = V_0$

$\int_0^t \frac{d^2y}{dt^2}dt = \int_0^t -g\,dt$

$y'(t) - y'(0) = -gt$

$y'(t) = -gt + V_0$

$\int_0^t y'(t)\,dt = \int_0^t -gt\,dt + \int_0^t V_0\,dt$

$y(t) - y(0) = \frac{-gt^2}{2} + V_0 t$

$y(t) = -\frac{1}{2}gt^2 + V_0 t + y_0$

ex. Exponential $\underset{(r>0)}{\text{Growth}} / \underset{(r<0)}{\text{Decay}}$

$\frac{dY}{dt} = rY$

$$\left(\frac{1}{y}\right)\frac{dy}{dt} = r$$

$$\int \left(\frac{1}{y}\right)\frac{dy}{dt}\,dt = \int r\,dt$$

$$\int \frac{d}{dt}\left(\ln|y|\right)dt = \int r\,dt$$

$$\ln(y) = rt + C$$

$$y = e^{rt+C}$$

$$y = e^{rt}\cdot e^{C} \qquad \text{let } A = e^{C}$$

$$y = Ae^{rt}$$

Seperable Equations

① $\frac{dy}{dt} = f(y)\,g(t)$

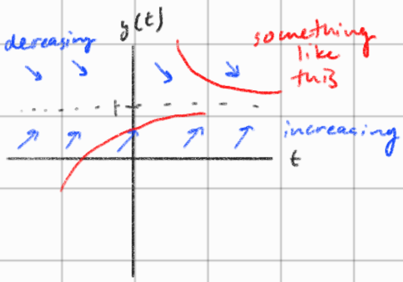
$\left(\frac{1}{f(y)}\right)\frac{dy}{dt} = g(t)$ $\qquad \int \left(\frac{1}{f(y)}\right)\frac{dy}{dt}\,dt = \int g(t)\,dt$

② $\frac{dy}{dt} = 1-y$

what does graph look like?

if $y(t) > 1$ $\frac{dy}{dt} < 0$ (decreasing)

if $y(t) < 1$ $\frac{dy}{dt} > 0$ (increasing)



$\downarrow$

$dt = \frac{dy}{1-y}$

$\int dt = \int \frac{1}{1-y}\,dy$

$t + C = -\ln|1-y|$

$\ln|1-y| = -t-C \qquad |1-y| = e^{-t-C} \qquad \text{let } A = e^{-C}$

$|1-y| = Ae^{-t}$

$y = 1 \pm Ae^{-t} \qquad y \to 1 \text{ as } t \to \infty$