Lecture 4 - Integrating Factor Method and Exact Equations

Existence and Uniqueness Theorem (from lecture 3)

Consider the differential equation

$$\frac{dy}{dt} = f(y,t) \qquad y(t_0) = y_0$$

suppose that the function f is continuous in a neighborhood of the point (t_0, y_0) and that the partial derivative $\frac{\partial f}{\partial y}$ is continuous in a neighborhood of the point (t_0, y_0). Then there exists a unique solution to the differential equation for $t \in (t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$.

Picard Iteration

- Construct sequence of functions that converge to actual solution

$$\frac{dy}{dt} = f(y,t) \qquad y(t_0) = y_0$$

$y_0(t) = y_0$ next function: $\frac{dy_1}{dt} = f(y_0, t)$ $\quad y_1(t_0) = y_0$ $\qquad \frac{dy_2}{dt} = f(y_1(t), t)$ $\quad y_2(t_0) = y_0$

$$\ldots \quad \frac{dy_{i+1}}{dt} = f(y_i(t), t) \qquad y_{i+1}(t_0) = y_0$$

ex. $\frac{dy}{dt} = y \quad y(0) = 1$ $\qquad f(y) = y \quad f(y_0) = f(1) = 1$

$y = e^t$ solution just by looking at it

$y_0 = 1 \qquad \frac{dy_1}{dt} = 1 \qquad y_1 = A + t \quad y_1(0) = A = 1$

$y_1 = 1 + t$

$\frac{dy_2}{dt} = y_1 = 1+t \qquad y_2(0) = 1$

$y_2 = t + \frac{t^2}{2} + A \qquad y_2(0) = A = 1$

$y_2 = 1 + t + \frac{t^2}{2}$

$\frac{dy_3}{dt} = y_2 = 1 + t + \frac{t^2}{2}$

$y_3 = t + \frac{t^2}{2} + \frac{t^3}{6} + A \qquad y_3(0) = A = 1$

$y_3 = 1 + t + \frac{t^2}{2} + \frac{t^3}{6}$ $\qquad$ converges to actual solution as $y \to \infty$

ex. $\frac{dy}{dt} = y^2 + t \qquad y(0) = 2$

$y_0(t) = 2 \qquad \frac{dy_1}{dt} = (y_0(t))^2 + t = 4 + t \qquad y_1(0) = 2 \qquad y_1(t) = A + 4t + \frac{t^2}{2} \qquad y_1(t) = 2 + 4t + \frac{t^2}{2}$

$\frac{dy_2}{dt} = (y_1(t))^2 + t = \left(2 + 4t + \frac{t^2}{2}\right)^2 + t \qquad y_2(0) = 2 \qquad \ldots$

First Order Equations

- first order linear: $\frac{dy}{dt} + P(t)y = Q(t)$
- first order linear homogeneous: $\frac{dy}{dt} + P(t)y = 0$

$$\frac{dy}{dt} = -P(t)y$$

$$\int \frac{1}{y}\frac{dy}{dt} = \int -P(t)\,dt$$

$$\ln(y(t)) = C - \int P(t)\,dt \qquad \text{let } A = e^C$$

$$y(t) = e^{C - \int P(t)dt} = Ae^{-\int P(t)dt}$$

general solution to first order linear homogeneous ex. $\frac{dy}{dt}+ry=0$ $\quad P(t)=r$ $\quad \int -P(t)\,dt = \int -r\,dt = -rt$

$y = Ae^{-rt}$

ex. $\cos t \frac{dy}{dt} = \sin t\, y$

$\cos t \frac{dy}{dt} - \sin t\, y = 0$

$\frac{dy}{dt} - \tan t\, y = 0$ $\quad P(t) = -\tan t$ $\quad \int -P(t)\,dt = \int \tan(t)\,dt = -\ln|\cos t|$

$y = Ae^{-\ln|\cos t|} = \frac{A}{\cos t}$