Lecture 12 - Fundamental Theorem of Algebra

Solving constant coefficient linear homogeneous (see lecture 11):

root	solution
r	e^{rt}
$\alpha \pm i\beta$	$e^{\alpha t}\cos\beta t,\ e^{\alpha t}\sin\beta t$
r w/ multiplicity k	$e^{rt},\ te^{rt},\ t^2e^{rt},\ \ldots,\ t^{(k-1)}e^{rt}$

ex. $y'' - 4y' + 5y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$

$r^2 - 4r + 5 = 0$

$r = 2 \pm i \qquad \alpha = 2 \quad \beta = 1$

$y_1 = e^{2t}\cos t \qquad y_2 = e^{2t}\sin t$

$y = Ae^{2t}\cos t + Be^{2t}\sin t \qquad y(0) = A\cdot 1 + B\cdot 0 = 1$

$A = 1$

$y' = 2Ae^{2t}\cos t - Ae^{2t}\sin t + 2Be^{2t}\sin t + Be^{2t}\cos t$

$y'(0) = 2A + B = 0$

$B = -2A = -2$

$y = e^{2t}\cos t - 2e^{2t}\sin t$

Fundamental Theorem of Algebra:

Polynomial $P(r)$ of degree n has n roots counted by multiplicity

If r_1 has multiplicity k, $(r - r_1)^k$ is a factor of $P(r)$

ex. $P(r) = (r-2)(r-5)^3(r+7)^4$

degree 8

$r = 2$ (simple) multiplicity 1

$r = 5$ multiplicity 3

$r = -7$ multiplicity 4

8

ex. $y''' + 3y'' + 3y' + y = 0$

$r^3 + 3r^2 + 3r + r = 0$

$(r+1)^3 = 0 \qquad r = 0$ w/ multiplicity 3

$y_1 = e^{-t} \qquad y_2 = te^{-t} \qquad y_3 = t^2e^{-t}$

ex. $y''' - 3y' + 2y = 0$ $\quad y(0)=1 \quad y'(0)=0 \quad y''(0)=1$

$r^3 - 3r + 2 = 0$

$r=1$ is root

see prev lecture for factoring

$(r-1)^2(r+2) = 0$

$r=1$ (multiplicity 2) $\quad r=-2$

$y_1 = e^t \quad y_2 = te^t \quad y_3 = e^{-2t}$

$y = Ae^t + Bte^t + Ce^{-2t}$

do a bunch of algebra w/ initial conditions to get

$y = \frac{7}{9}e^t - \frac{1}{3}te^t + \frac{2}{9}e^{-2t}$

ex. $y^{(4)} + 4y''' + 8y'' + 8y' + 4y = 0 \qquad p(r) = (r^2+2r+2)^2$

$r = 1+i$ (multiplicity 2), $r = 1-i$ (multiplicity 2),

$y = Ae^t\cos t + Be^t\sin t + Cte^t\cos t + Dte^t\sin t$

Operators:

Acts on function, returns another function (derivatives, integrals, multiplying by a function, etc.)

ex. $L_1 = 2\frac{d}{dt} \qquad L_2 = t$

$L_1 \sin t = 2\cos t \qquad L_2 \sin t = t\sin t$

$L_1 e^{2t} = 4e^{2t} \qquad L_2 e^{2t} = te^{2t}$

$L_1 t = 2 \qquad L_2 t = t^2$

Don't commute, order usually matters

ex. $L_1 = 2\frac{d}{dt} \qquad L_2 = t$

$L_1 L_2 \sin t = L_1(t\sin t) = 2(\sin t + t\cos t)$

$L_2 L_1 \sin t = 2t\cos t$

Constant Coefficient Differential Operator

$$L = \frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \dots + a_1\frac{d}{dt} + a_0$$

$$y = \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_1\frac{dy}{dt} + a_0 y = 0$$

$Ly = 0$