Lecture 11 - Euler Formula

Linear Homogeneous:

- use characteristic polynomial
- n roots give n solutions
- see lecture 10 for more info

ex. $y''' - 7y' + 6y = 0$

$P(r) = r^3 - 7r + 6 = 0$

$r = 1$ root

$$
\begin{array}{r}
r^2 + r - 6 \\
r-1 \overline{\big) r^3 + 0r^2 - 7r + 6} \\
-(r^3 - r^2) \\
\hline
r^2 - 7r + 6 \\
-(r^2 - r \quad\;) \\
\hline
-6r + 6 \\
-(-6r + 6) \\
\hline
0
\end{array}
$$

$\rightarrow (r-1)(r^2 + r - 6) = 0$

$(r-1)(r+3)(r-2) = 0$

$r = -3, 1, 2$

$y_1 = e^{-3t} \quad y_2 = e^t \quad y_3 = e^{2t}$

$y = Ae^{-3t} + Be^t + Ce^{2t}$

ex. $y''' - 3y'' + 2y' = 0 \quad y(0) = 1 \quad y'(0) = 0 \quad y''(0) = -1$

$P(r) = r^3 - 3r^2 + 2r = 0$

$r(r^2 - 3r + 2) = 0$

$r(r-2)(r-1) = 0$

$r = 0, 1, 2$

$y = A + Be^t + Ce^{2t}$

$y(0) = A + B + C = 1 \qquad y'(0) = Be^t + 2Ce^{2t} = 0 \qquad y''(0) = Be^t + 4Ce^{2t} = -1$

$A + 1 - \frac{1}{2} = 1 \qquad B + 2C = 0 \qquad B + 4C = -1$

$A = \frac{1}{2} \qquad B = -2C \qquad 2C = -1$

$B = 1 \qquad C = -\frac{1}{2}$

$y = \frac{1}{2} + e^t - \frac{1}{2}e^{2t}$

Theorem:

The following theorem says that this procedure always works when the roots are distinct:

Theorem

Suppose that $y_1(t) = e^{r_1 t}, y_2(t) = e^{r_2 t}, \ldots y_n(t) = e^{r_n t}$. The Wronskian is given by

$$W(y_1, y_2, \ldots, y_n) = e^{(r_1 + r_2 + \ldots r_n)t} \prod_{1 \leq i < j \leq n} (r_j - r_i)$$

in particular *if the r_i are distinct then the Wronskian is never zero and the y_i are linearly independent.*

The determinant formula is not important here. What is is that if the roots of the characteristic polynomial are distinct then the n solutions are always linearly independent.

ex. $y_1 = e^{t}$ $y_2 = e^{-2t}$ $y_3 = e^{5t}$ matrix left out on purpose cuz it takes hella long to solve

$W(y_1, y_2, y_3) = e^{(1-2+5)}(r_2 - r_1)(r_3 - r_1)(r_3 - r_2) = -84e^{4t}$

Euler Formula:

$e^{it} = \cos t + i\sin t$ $e^{(\alpha + i\beta)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$

ex. $e^{1+i\pi} = e^{1}(\cos\pi + i\sin\pi) = -e$

ex. $e^{2+i\pi/2} = e^{2}\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = ie^{2}$

ex. $e^{i\pi/4} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

Complex roots:

Method 1 - Treat normally (usually not reccomended):

ex. $y'' + y = 0$ $y(0) = 1$ $y'(0) = 1$

$r^2 + 1 = 0$ $r^2 = -1$ $r = \pm i$

$y = Ae^{it} + Be^{-it}$ $y(0) = A + B = 1$

$y' = iAe^{it} - iBe^{it}$ $y'(0) = iA - iB = -1$

$\rightarrow$ $2iB = i - 1$ $B = \frac{i-1}{2i} \cdot \frac{-i}{-i} = \frac{1+i}{2}$ $2iA = i + 1$ $A = \frac{1-i}{2}$ $\rightarrow$ $y = \left(\frac{1-i}{2}\right)e^{it} + \left(\frac{1+i}{2}\right)e^{-it}$

(euler formula)

$y = \left(\frac{1-i}{2}\right)(\cos t + i\sin t) + \left(\frac{1+i}{2}\right)(\cos t - i\sin t) = \cos t + \sin t$ bad method, lots of algebra

Method 2 - Find real linear combinations:

$r_1 = \alpha + i\beta$ $r_2 = \alpha - i\beta$

$y_1 = e^{(\alpha + i\beta)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$

$y_2 = e^{(\alpha - i\beta)t} = e^{\alpha t}(\cos\beta t - i\sin\beta t)$

$w_1 = \frac{y_1 + y_2}{2} = e^{\alpha t}\cos\beta t$ $w_2 = \frac{y_1 - y_2}{2i} = e^{\alpha t}\sin\beta t$

ex. $y'' + y = 0$ $y(0) = 1$ $y'(0) = 1$

$P(r) = r^2 + r = 0$ $r = \pm i$

$r = \alpha \pm \beta i$ $\alpha = 0$ $\beta = 1$

$w_1 = e^{0 \cdot t}\cos(1 \cdot t) = \cos t$ $w_2 = e^{(0 \cdot t)}\sin(1 \cdot t) = \sin t$

$y = A\cos t + B\sin t$

$y(0) = A \cdot 1 + B \cdot 0 = 1$ $y'(0) = -A\sin t + B\cos t = -A \cdot 0 + B \cdot 1 = 1$

$A = 1$ $B = 1$

$y = \cos t + \sin t$

ex. $y''' + 2y'' + 5y' = 0$

$r^3 + 2r^2 + 5r = 0$

$r(r^2 + 2r + 5) = 0$

⋮ (algebra)

$r = 0, -1+2i, -1-2i$ $\alpha = -1$ $\beta = 2$

$y_1 = e^{0 \cdot t} = 1$ $y_2 = e^{-t}\cos 2t$ $y_3 = e^{-t}\sin 2t$

$y = A + Be^{-t}\cos 2t + Ce^{-t}\sin 2t$

Multiple Roots:

ex. $y''' = 0$ $y'' = A$ $y' = At + B$ $y = \frac{At^2}{2} + Bt + C$

$r^3 = 0$ $r = 0$ root w/ multiplicity of 3

just look at this and it should make sense

$y_1 = e^{0 \cdot t} = 1$ $y_2 = t$ $y_3 = t^2$

← linearly independent →