Lecture 9 - Linear Independence

Higher order linear eqns:

- depends linearly on y and derivatives of y
- general nth order linear homogeneous:

$$\frac{d^n y}{dt^n} + \sum_{k=0}^{n-1} a_k(t) \frac{d^k y}{dt^k} = 0$$

Superposition (Lecture 8)

ex. $\frac{d^2y}{dt^2} + y = 0$

$y_1 = \sin t$ first soln.

$y_1' = \cos t$

$y_1'' = -\sin t$

$y_1'' + y_1 = \sin t - \sin t = 0$

$y_2 = 2\sin t$ linearly dependent on y_1

redundant

To see:

$\frac{d^2y}{dt^2} + y = 0$

$y(0) = 1$

$y'(0) = 2$

$y = A\sin t + 2B\sin t$

$y(0) = A\sin(0) + 2B\sin(0) = 0 \neq 1?$ ✗

$y'(0) = A\cos(0) + 2B\cos(0) = A + 2B = 2$ ✓

linear combination doesn't solve initial condition $y(0)=1$, so must be another linear combo

Linear Independence:

Defn: Given $y_1(t), y_2(t), \ldots, y_{(n)}(t)$ linearly independent if

$$A_1 y_1(t) + A_2 y_2(t) + \ldots + A_n y_n(t) = 0$$

Then

$$A_1 = A_2 = \ldots = A_n = 0$$

Linear Dependence:

Defn: linearly dependent if

$$A_1 y_1(t) + A_2 y_2(t) + \ldots + A_n y_n(t) = 0$$

with some $A_i \neq 0$

Redundancy

ex. $y_1 = \sin t$ $\quad y_2 = 2\sin t$

linearly dependent

$2Y_1 = 2\sin t = Y_2$ $\qquad 2Y_1 + Y_2 = 0$

ex. $y_1 = 1$ $y_2 = 1 + t + t^2$ $y_3 = 1 - t - t^2$

linearly dependent

$y_2 + y_3 = 2 = 2y_1$

$-2y_1 + y_2 + y_3 = 0$

ex. $y_1(t) = 1$ $y_2(t) = t$

$A_1 \cdot 1 + A_2 \cdot t = 0$

at $t=0$: $A_1 \cdot 1 = 0$, $A_1 = 0$

at $t=1$: $A_1 \cdot 1 + A_2 \cdot 1 = 0 \rightarrow A_2 + A_1 = 0 \rightarrow A_2 = -A_1$ $A_1 = 0$ $A_2 = 0$ $\therefore$ linearly independent

ex. $y_1 = \sin t$ $y_2 = \cos t$

$A_1 \sin t + A_2 \cos t = 0$

at $t=0$: $A_1 \cdot 0 + A_2 = 0 \rightarrow A_2 = 0$

at $t = \frac{\pi}{2}$: $A_1 + A_2 \cdot 0 = 0 \rightarrow A_1 = 0$

ex. $\frac{d^2y}{dt^2} + y = 0$ $y(t_0) = y_0$ $y'(t_0) = y_0'$

$y_1 = \sin t$

$y_2 = \cos t$ (linearly independent)

$y = A_1 \sin t + A_2 \cos t$ (soln. by superposition)

$y(t_0) = A_1 \sin(t_0) + A_2 \cos(t_0) = y_0$

$y'(t) = A_1 \cos t - A_2 \sin t$

$y'(t_0) = A_1 \cos(t_0) - A_2 \sin(t_0) = y_0'$

$A_1 \sin t_0 + A_2 \cos t_0 = y_0$

$A_1 \cos t_0 - A_2 \sin t_0 = y_0'$ $\rightarrow$ $\begin{bmatrix} \sin t_0 & \cos t_0 \\ \cos t_0 & -\sin t_0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$

$M\vec{x} = \vec{b}$

$M_{2\times 2}$

for unique soln, need $\det(M) \neq 0$

$\det(M) = -\sin^2 t_0 - \cos^2 t_0 = -1 \neq 0$ $\therefore$ unique solution

$\frac{d^2y}{dt^2} + y = 0$ $y(t_0) = y_0$ $y'(t_0) = y_0'$ $y = A_1 \sin t + A_2 \cos t$

found a unique soln. with this form, $\therefore$ every solution is of this form

Wronskian:

Given $y_1(t), y_2(t), \dots, y_n(t)$

$$W(y_1, y_2, \dots, y_n) = \begin{bmatrix} y_1(t) & y_2(t) & \dots & y_n(t) \\ y_1'(t) & y_2'(t) & \dots & y_n'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \dots & y_n^{(n-1)}(t) \end{bmatrix}$$

$y(t) = A_1 y_1(t) + \dots + A_n y_n(t)$

$y(t_0) = A_1 y_1(t_0) + \dots + A_n y_n(t_0)$

$y'(t_0) = A_1 y_1'(t_0) + \dots + A_n y_n'(t_0)$

$\vdots$

$y^{(n-1)}(t_0) = A_1 y^{(n-1)}(t_0) + \dots A_n y^{(n-1)}(t_0)$

$\downarrow$

$$\begin{bmatrix} y_1(t) & y_2(t) & \dots & y_n(t) \\ y_1'(t) & y_2'(t) & \dots & y_n'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \dots & y_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

unique soln if $W(y_1, y_2, \dots, y_n) \neq 0$

ex. find $W(y_1, y_2)$ $y_1 = e^t$ $y_2 = e^{-t}$

$$W(e^t, e^{-t}) = \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix} = e^t e^{-t} - e^t e^{-t} = -2$$

If $y_1, y_2 \dots y_n$ are linearly dependent, then the wronskian is zero