Lecture 13 - More on Constant Coefficient Equations

Operator Notation:

ex. $y'' + 2y + y = 0$

$r^2 + 2r + 1 = 0$

$(r+1)^2$

$r = -1$ mult. 2

$y_1 = e^{-t} \quad y_2 = te^{-t}$

using operator notation:

$\left(\frac{d^2}{dt^2} + 2\frac{d}{dt} + 1\right)y = 0$

$\left(\frac{d}{dt}+1\right)\left(\frac{d}{dt}+1\right)y = 0$

Let $w = \left(\frac{d}{dt}+1\right)y$

$\left(\frac{d}{dt}+1\right)w = 0$

$\frac{dw}{dt} + w = 0$

$w = Ae^{-t}$

$\frac{dy}{dt} + y = w = Ae^{-t} \qquad \mu(t) = e^{\int dt} = e^t$

$e^t\frac{dy}{dt} + ye^t = A$

$\frac{d}{dt}(ye^t) = A$

$ye^t = At + B$

Non-homogeneous Linear Equations:

$$\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0(t)y = f(t)$$

Theorem

Given an n^{th} order linear nonhomogeneous equation

$$\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0(t)y = f(t)$$

The general solution can be written as follows:

$$y(t) = y_{particular}(t) + A_1y_1(t) + A_2y_2(t) + A_ny_n(t),$$

where $y_{particular}(t)$ is **ANY** *solution to the nonhomogeneous problem and $y_1(t), y_2(t), \ldots, y_n(t)$ solve the homogeneous problem*

$$\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0(t)y = 0$$

ex. $y'' + 2y' + 2y = t \qquad y_p = \frac{t}{2} - \frac{1}{2}$

$y = y_p + Ay_1 + By_2$

$y = \frac{t}{2} - \frac{1}{2} + Ay_1 + By_2 \qquad y_1, y_2$ solve $y'' + 2y' + 2y = 0$

$r^2 + 2r + 2 = 0$

$(r+1)^2+1=0$

$r=-1\pm i \quad \alpha=-1 \quad \beta=1$

$y_1=e^{-t}\cos t \quad y_2=e^{-t}\sin t$

$y=\frac{t}{2}-\frac{1}{2}+Ae^{-t}\cos t+Be^{-t}\sin t$

given initial conditions $y(0)=0 \quad y'(0)=0$
(do some algebra)

$y=\frac{t}{2}-\frac{1}{2}+\frac{1}{2}e^{-t}\cos t$

Alternative Terminology:

You may see the the terminology zero input response and zero state response in some of your engineering classes. These are defined as follows

Definition

Given a n^{th} order linear non-homogeneous equation with initial conditions

$$\frac{d^ny}{dt^n}+a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}}+\ldots+a_0(t)y=f(t)$$
$$y(t_0)=y_0 \quad y'(t_0)=y_0' \quad \ldots \quad y^{(n-1)}(t_0)=y_0^{(n-1)}$$

We define the zero input solution (response) $y_{ZI}(t)$ as follows

$$\frac{d^ny_{ZI}}{dt^n}+a_{n-1}(t)\frac{d^{n-1}y_{ZI}}{dt^{n-1}}+\ldots+a_0(t)y_{ZI}=0$$
$$y_{ZI}(t_0)=y_0 \quad y_{ZI}'(t_0)=y_0' \quad \ldots \quad y_{ZI}^{(n-1)}(t_0)=y_0^{(n-1)}$$

Definition

Given a n^{th} order linear non-homogeneous equation with initial conditions

$$\frac{d^ny}{dt^n}+a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}}+\ldots+a_0(t)y=f(t)$$
$$y(t_0)=y_0 \quad y'(t_0)=y_0' \quad \ldots \quad y^{(n-1)}(t_0)=y_0^{(n-1)}$$

We define the zero state solution (response) $y_{ZS}(t)$ as follows

$$\frac{d^ny_{ZS}}{dt^n}+a_{n-1}(t)\frac{d^{n-1}y_{ZS}}{dt^{n-1}}+\ldots+a_0(t)y_{ZS}=f(t)$$
$$y_{ZI}(t_0)=0 \quad y_{ZI}'(t_0)=0 \quad \ldots \quad y_{ZI}^{(n-1)}(t_0)=0$$

Finding the particular solution:

Method of undetermined coefficients:

Solves $Ly=f(t)$

when 1) L is constant coefficient and 2) $f(t)$ can be built from sums and products of e^{at}, $\sin bt$, $\cos bt$, polynomials in t (eg. t^2+5, $(t^3+11)e^t\cos t+5\sin 4t$, $\cos^3 t\sin t+(58t+19)e^{-t}$ all work, $\frac{\sin t}{\cos t}$, $e^{\cos t}$, $\frac{11t^2+27}{t+5}$ don't work).

"Naive" method of undertermined coefficients

Guess something same form as $f(t)$

polynomial $\rightarrow$ guess polynomial same degree

$\sin bt$ or $\cos bt$ $\rightarrow$ guess $A\sin bt+B\cos bt$

e^{at} $\rightarrow$ guess Ae^{at}

ex. $y'' - 3y' + 2y = t^2+4$

$f(t) = t^2+4$

guess
$$
\begin{aligned}
2\,(y_p &= At^2+Bt+C) \\
-3\,(y_p' &= 2At+B) \\
+\ 1\,(y_p'' &= 2A) \\
\hline
\end{aligned}
$$

$y_p'' - 3y_p' + 2y_p = 2At^2 + (2B-6A)t + 2A - 3B + 2C = t^2 + 4$

$2A = 1 \quad A = \frac{1}{2}$

$2B - 6A = 0 \quad 2B = 3 \quad B = \frac{3}{2}$

$2A - 3B + 2C = 4 \quad 1 - \frac{9}{2} + 2C = 4 \quad 2C = \frac{15}{2} \quad C = \frac{15}{4}$

particular soln.: $y = \frac{1}{2}t^2 + \frac{3}{2}t + \frac{15}{4}$

homogeneous part:

$r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0$

$r = 2, \ r = 1$

$y_1 = e^{2t} \quad y_2 = e^t$

general soln.: $y = \frac{1}{2}t^2 + \frac{3}{2}t + \frac{15}{4} + Ae^{2t} + Be^t$