Recursive Induction

ex. $f: \mathbb{N} \to \mathbb{N}$ st

$f(0) = 2$

$f(1) = 3$

$f(n) = 3f(n-1) - 2f(n-2) \quad n \geq 2$

Claim: $\forall n \in \mathbb{N}, f(n) = 2^n + 1$

Proof: We will prove by induction on n. We have two base cases: $n=0$ and $n=1$. For $n=0$, $f(0) = 2^0+1 \Rightarrow 2 = 1+1 \Rightarrow 2=2$, so the base case holds for $n=0$. For $n=1$, $f(1) = 2^1+1 \Rightarrow 3=3$, so the base case holds for $n=1$. For our inductive hypothesis, suppose $f(n) = 2^n+1$ is true for $n = 0, 1, 2, \ldots, k-1$. For our inductive step, $n=k$ and our goal is to show $f(k) = 2^k+1$.
$f(k) = 3f(k-1) - 2f(k-2) = 3(2^{k-1}+1) - 2(2^{k-2}+1)$ (by the inductive hypothesis) $= 3 \cdot 2^{k-1} + 3 - 2 \cdot 2^{k-2} - 2$
$= 3 \cdot 2^{k-1} - 2^{k-1} + 1 = 2 \cdot 2^{k-1} + 1 = 2^k + 1$ $\square$

ex. Fibonacci

$F_0 = 0 \quad F_1 = 1 \quad F_n = F_{n-1} + F_{n-2}$

Claim: $\forall n \in \mathbb{N}$, F_{3n} is even $(F_0, F_3, F_6, F_9, \ldots)$

Proof: We will prove by induction on n. For our base case, we have $n=0$ and $n=1$. $F_0 = 0$ which is even so base case holds for $n=0$. $F_3 = F_2 + F_1 = F_1 + F_0 + F_1 = 2$ which is even, so base case holds for $n=1$. For our inductive hypothesis, suppose F_{3n} is even for $n = 0, 1, 2, \ldots, k-1$. For our inductive step, $n=k$ and we're trying to show F_{3k} is even. $F_{3k} = F_{3k-1} + F_{3k-2}$ (by defn. of fibonacci)
$= F_{3k-2} + F_{3k-3} + F_{3k-2} = 2F_{3k-2} + F_{3(k-1)}$. $F_{3k-2} \in \mathbb{N}$, so $2 \cdot F_{3k-2}$ must be even by the defn. of even. $F_{3(k-1)}$ must be even by the inductive hypothesis. The sum of two even numbers is also even, so F_{3k} is even $\square$

ex. $g: \mathbb{N} \to \mathbb{N}$

$g(0) = 2$

$g(1) = 7$

$g(k) = g(k-1) + 2g(k-2) \quad k \geq 2$

claim: $\forall n \in \mathbb{N}$, $g(n) = 3 \cdot 2^n + (-1)^{n+1}$

proof: We will prove by induction on n. Base cases are $k=0$ and $k=1$. $g(0) = 3 \cdot 2^0 + (-1)^{0+1} \rightarrow 2 = 3-1 \rightarrow 2=2$, so base case holds for $k=0$. $g(1) = 3 \cdot 2^1 + (-1)^{1+1} \rightarrow 7 = 6+1 \rightarrow 7=7$ so base case holds for $k=1$. For our inductive hypothesis, suppose $g(n) = 3 \cdot 2^n + (-1)^{n+1}$ is true for

$n = 0, 1, 2, \ldots, k-1$. For our inductive step, $n = k$ and we're trying to show $g(k) = 3 \cdot 2^k + (-1)^{k+1}$.

$g(k) = g(k-1) + 2g(k-2)$ (by function defn.) $= 3 \cdot 2^{k-1} + (-1)^k + 2\left(3 \cdot 2^{k-2} + (-1)^{k-1}\right)$

$= 2 \cdot 3 \cdot 2^{k-1} + (-1)^k + 2(-1)^{k-1} = 3 \cdot 2^k + (-1)^k + (-1)^{k-1} + (-1)^{k-1} = 3 \cdot 2^k + (-1)^{k+1}$ $\square$

Discussion Problems – 12.1bd

12.1 Induction on recursive definition

For each of the following functions, compute the first few values of the function and then prove the closed form is correct.

(a) Define a function $g : \mathbb{Z}^+ \to \mathbb{Z}$ by

$g(1) = 1$
$g(n) = g(n-1) + 6n - 6$ (for all integers $n >= 2$)

Closed form: $g(n) = 3n^2 - 3n + 1$

(b) Define a function $g : \mathbb{N} \to \mathbb{N}$ by

$g(0) = 0$
$g(n) = n + 3g(n-1)$ for all integers $n \geq 1$

Closed form: $g(n) = \frac{3^{n+1}-2n-3}{4}$

(c) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by

$f(1) = 3,\ f(2) = 5$
$f(n) = 3f(n-1) - 2f(n-2)$ for all $n \geq 3$.

Closed form: $f(n) = 2^n + 1$

(d) Define a sequence of values x_n as follows:

$x_1 = 1,\ x_2 = 7$
$x_{n+1} = 7x_n - 12x_{n-1}$ for $n \geq 2$

Closed form: $x_n = 4^n - 3^n$

b) $g(0) = 0 = \frac{3^1 - 3}{4}$ $\quad g(1) = 1 = \frac{3^2-2-3}{4}$ $\quad g(2) = 2+3 = 5 = \frac{3^3-4-3}{4} = \frac{20}{4} = 5$

Proof: We will prove by induction on n. For our base case, we have $n=0$. $\frac{3^{0+1}-2(0)-3}{4} = g(0) = 0$ so the base case holds. For our inductive hypothesis, suppose $g(n) = \frac{3^{n+1}-2n-3}{4}$ is true for $n = 0, 1, 2, \dots, k-1$. For our inductive step, $n=k$, and we're trying to show $g(k) = \frac{3^{k+1}-2k-3}{4}$.

$g(k) = k + 3g(k-1)$ (by function defn.) $= k + 3\left(\frac{3^k - 2(k-1) - 3}{4}\right) = \frac{4k + 3 \cdot 3^k - 6(k-1) - 9}{4} = \frac{4k + 3^{k+1} - 6k + 6 - 9}{4}$

$= \frac{3^{k+1} - 2k - 3}{4}$, so we've proven the inductive step and our proof is complete.

d) $x_1 = 1$ $\quad x_2 = 7$ $\quad x_3 = 4^3 - 3^3 = 37$ $\quad x_4 = 4^4 - 3^4 = 175$

proof: We will prove by induction on n. For our base cases, $n=1$ and $n=2$. For $n=1$, $x_1 = 4^1 - 3^1 \to 1 = 1$ so base case holds for $n=1$. For $n=2$, $x_2 = 4^2 - 3^2 \to 7 = 7$ so base case holds for $n=2$. For inductive hypothesis, suppose $x_n = 4^n + 3^n$ for $n = 0, 1, 2, \dots, k-1$. For inductive step, $n=k$ and we're trying to show $x_k = 4^k - 3^k$.

$x_k = 7x_{k-1} - 12x_{k-2} = 7(4^{k-1} - 3^{k-1}) - 12(4^{k-2} - 3^{k-2}) = 7 \cdot 4^{k-1} - 3 \cdot 4 \cdot 4^{k-2} - 7 \cdot 3^{k-1} + 4 \cdot 3 \cdot 3^{k-2}$

$= 7 \cdot 4^{k-1} - 3 \cdot 4^{k-1} - 7 \cdot 3^{k-1} + 4 \cdot 3^{k-1} = 4 \cdot 4^{k-1} - 3 \cdot 3^{k-1} = 4^{k-1} - 3^{k-1}$ $\square$