Lecture 10 - More on Linear Independence and Characteristic Polynomial

Abel's Theorem:

Suppose $y_1(t), y_2(t), \dots, y_n(t)$ solve the nth order linear homogeneous equation

$$\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + a_{n-2}(t)\frac{d^{n-2}y}{dt^{n-1}} + \dots + a_0(t)y = 0$$

Then the Wronskian $W(y_1, y_2, \dots, y_n)$ solves the first order equation

$$\frac{dW}{dt} + a_{n-1}(t)W = 0$$

or

$$W(t) = Ae^{-\int a_{n-1}(t)dt}$$

ex. $\frac{d^2y}{dt^2} + y = 0$ solutions: $y_1 = \cos t$ $y_2 = \sin t$

$\frac{d^2y}{dt^2} + 0\cdot\frac{dy}{dt} + y = 0$ (general form)

$\frac{dW}{dt} + 0\cdot W = 0$ (Abel's thm)

$\frac{dW}{dt} = 0$ $\therefore$ W constant

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

Also consider $\tilde{y}_1 = \cos t + \sin t$ $\tilde{y}_2 = \cos t - \sin t$ (also solutions)

$$W(\tilde{y}_1, \tilde{y}_2) = \begin{vmatrix} \cos t + \sin t & \cos t - \sin t \\ -\sin t + \cos t & -(\sin t + \cos t) \end{vmatrix} = -(\cos t + \sin t)^2 - (\cos t - \sin t)^2 = -2 \quad \text{(also constant)}$$

ex. $\frac{d^4y}{dt^4} + \frac{d^2y}{dt^2} = 0$ soln: $y_1 = 1$ $y_2 = t$ $y_3 = \cos t$ $y_4 = \sin t$

Abel's: coefficent of y''' (bc $y''' = y^{(4-1)}$)

$\frac{dW}{dt} + 0\cdot W = 0$

$\frac{dW}{dt} = 0$ W constant

$$W = \begin{vmatrix} 1 & t & \cos t & \sin t \\ 0 & 1 & -\sin t & \cos t \\ 0 & 0 & -\cos t & -\sin t \\ 0 & 0 & \sin t & -\cos t \end{vmatrix} = \begin{vmatrix} -\cos t & -\sin t \\ \sin t & -\cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

(by some lin alg shit)

ex. $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 0$ $\quad y_1 = e^t \quad y_2 = e^{3t}$

Abels:

$\frac{dW}{dt} - 4W = 0 \qquad W(t) = Ae^{-\int(-4)dt} = Ae^{4t} \qquad \frac{dW}{dt} = 4Ae^{4t}$

$4Ae^{4t} - Ae^{4t} = 0$

X

$3Ae^{4t} = 0$

$W(t) = \begin{vmatrix} e^t & e^{3t} \\ e^t & 3e^{3t} \end{vmatrix} = 3e^t e^{3t} - e^{3t}e^t = 3e^{4t} - e^{4t} = 2e^{4t} \qquad A = 2$

Corollary for Abel's (Important!):

If $y_1, y_2, \ldots, y_n$ solve

$$\frac{d^ny}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + a_{n-2}(t)\frac{d^{n-2}y}{dt^{n-1}} + \ldots + a_0(t)y = 0$$

and the functions a_i are continuous in some interval (a,b) and the Wronskian is non-zero at any point in (a,b), then it is never zero anywhere in the interval. Important because tells us that (as long as continuous coefficients) if set of solutions satisfy any set of initial conditions at one point, then that solution set satisfies any set of initial conditions at any point.

Characteristic Polynomial:

Given $\frac{d^ny}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + y = 0$

The characteristic polynomial is

$r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0$

Take $\frac{d^ny}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + y = 0$, look for $y = e^{rt}$, $\frac{dy}{dt} = re^{rt}$, $\frac{d^2y}{dt^2} = r^2e^{rt}$

$r^n e^{rt} + a_{n-1}r^{n-1}e^{rt} + \ldots + a_1 re^{rt} + a_0 e^{rt} = 0$

$\underbrace{(r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0)}_{\text{characteristic polynomial}}e^{rt} = 0$

Works when r is a root of the characteristic polynomial

N roots of characteristic polynomial → N solutions (linearly independent) → linear combination is general solution ex. $\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} - 79\frac{dy}{dt} - 220y = 0$

$P(r) = r^3 - 2r^2 - 79r - 220$

roots are $r_1 = -4$, $r_2 = -5$, $r_3 = 11$ $\therefore$ polynomial has 3 solutions:

$y_1 = e^{-4t}$ $\quad y_2 = e^{-5t}$ $\quad y_3 = e^{11t}$

ex. Solve $y''' - 3y'' + 2y' = 0$ $\quad y(0) = 1$ $\quad y'(0) = 0$ $\quad y''(0) = 0$

$P(r) = r^3 - 3r^2 + 2r = 0$

$r(r^2 - 3r + 2) = 0$

$r(r-2)(r-1) = 0$ $\qquad r = 0, 1, 2$

$y_1 = e^{0t} = 1$ $\quad y_2 = e^{1t} = e^t$ $\quad y_3 = e^{2t}$

general solution:

$y = A + Be^t + Ce^{2t}$

$y(0) = A + Be^t + Ce^{2t} = 1$ $\qquad y'(0) = Be^t + 2Ce^{2t} = 0$ $\qquad y''(0) = Be^t + 4Ce^{2t} = 0$

$A + B + C = 1$ $\qquad B + 2C = 0$ $\qquad B + 4C = 0$

$A + 0 + 0 = 1$ $\qquad B = -2C$ $\qquad 2C = 0$

$A = 1$ $\qquad B = 0$ $\qquad C = 0$

$y = 1$ general solution

Synthetic/Polynomial Division:

ex. find three solutions to $y''' - 3y' + 2y = 0$

$P(r) = r^3 - 3r + 2 = 0$

just have to kinda notice this

$r = 1$ is root $(1^3 - 3 + 2 = 0)$

$\therefore (r-1)$ divides $P(r)$

$$\begin{array}{r|l} & r^2 + r - 2 \\ \hline r-1 & r^3 + 0r^2 - 3r + 2 \\ & -(r^3 - r^2) \\ \hline & r^2 - 3r + 2 \\ & -(r^2 - r + 0) \\ \hline & -2r + 2 \\ & -(-2r + 2) \\ \hline & 0 \end{array}$$

$\rightarrow$ $(r-1)(r^2 + r - 2) = 0$

$(r-1)^2(r+2) = 0$

roots are $r = 1, 1, -2$

$y_1 = e^t$ $\quad y_2 = e^{-2t}$ $\quad y_3 = te^t$ from double root $(r-1)^2$, explained in lecture 11