AMATH 583 Homework 5

AMATH 583 Course Overview at the University of Washington
Spring Quarter 2025 Section A with Professor Prof. Kenneth J. Roche

Assignment PDF

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Problem 1

exists

$\oplus b$

start z.

Problem 1 $\boxed{\text{tldr: } (\mathbb{Z}, \oplus, \odot) \text{ is ring w/o unity}}$

1. Ring: $\forall x, y \in \mathbb{Z}$, define binary op. $\oplus$ as $x \oplus y = x + y - 1$, and binary operator $\odot$ as $x \odot y = x + y - xy$. $(\mathbb{Z}, \oplus, \odot)$ as a ring.

(a) Additive identity. Find $z \in \mathbb{Z}$, $\ni a \oplus z = z \oplus a = a$, $\forall a \in \mathbb{Z}$

for $a \oplus z = a$:

$x + z - 1 = x$

$z - 1 = 0$

$z = 1 \rightarrow$ the additive identity

(b) Additive inverse: $\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z} \ni a \oplus b = b \oplus a = z$

Find b

$a + b - 1 = 1$

$b = 2 - a \rightarrow$ add. inv. of a is $2 - a$

(c) Show that $\oplus$ commutes

$x \oplus y = y \oplus x$

$x + y - \cancel{1} = y + x - \cancel{1}$ yes? $+$ & $-$ known to commute in $\mathbb{Z}$

$x - x + y - y = -1 + 1$

(d) Multiplicative identity: A ring with unity is a ring with member $u \in \mathbb{Z}$, $u \neq z$, $\ni a \odot u = u \odot a = a$, $\forall a \in \mathbb{Z}$; find u

$a \odot u = u \odot a$ known commutative in $\mathbb{Z}$

$a + u - au = u + a - ua = a + u - au$

$a - a + u - u - au + au = 0 \Rightarrow 0 = 0$ true, misleading

or

$a + u - au = a$

$u(1 - a) = 0$; since $u \neq z = 1$, $\underline{u = 0}$

$\boxed{\text{No unity}}$

Problem 2

2. Recursive Sequence

$T_0 = 2$, $T_1 = 3$, $T_2 = 6$, & for $n \geq 3$,
let $T_n = (n+4)T_{n-1} - 4n\,T_{n-2} + (4n-8)\,T_{n-3}$

(a) Derive a formula for $T_n = A_n + B_n$ where (A_n), (B_n) are well-known sequences

Test initial terms:

linear Exponential

$$\left.\begin{aligned} T_0 &= 2 = A_0 + B_0 \\ T_1 &= 3 = A_1 + B_1 \\ T_2 &= 6 = A_2 + B_2 \end{aligned}\right] \to \text{guess } A_n = 2n,\ B_n = 2^n$$

$$\left.\begin{aligned} T_0 &= 2(0) + 2^0 = 1 \\ T_1 &= 2 + 2^1 = 4 \\ T_2 &= 2 \cdot 2 + 2^2 = 8 \end{aligned}\right\} \text{not quite}$$

combined fibonacci and polynomial?

$T_n = A \cdot F_n + B \cdot G_n$

$T_n = A(2^n) + B(n^2)$

$$\text{set}\left[\begin{aligned} T_0 &= A(2^0) + B(0^2) = 2 \to A = 2 \\ T_1 &= A(2^1) + B(1) = 3 \to B = -1 \end{aligned}\right.$$

$T_2 = A(2^2) + B(2) \overset{?}{=} 8 - 2 \overset{\checkmark}{=} 6$ success

Closed form: $T_n = 2^{n+1} - n^2$

(b) Induction pro...
show that ...

$T_n = 2^{n+1} - n...$

$T_n = (n+4)T...$

$2^{n+1} - n^2 \overset{?}{=} (n+4)(...$
$4n(2^{...}$
$(4n-8)...$

$2(2^n) - n^2 = ...$

$= 2^n(n ...$

$- 4n...$

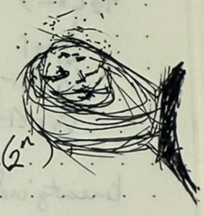
$= 2(2^n...$
$+ n^2(...$
$+ n(...$
$+ (-4...$

$= 2(2^n)...$
L ma...

(b) Inductive proof: assume it works for $n = k, k-1, k-2, \ldots$
show that it also works for $k+1$

$T_n = 2^{n+1} - n^2$ ← closed form

$T_n = (n+4)T_{n-1} - 4n\,T_{n-2} + (4n-8)\,T_{n-3}$



$$2^{n+1} - n^2 \stackrel{?}{=} (n+4)(2^n - (n-1)^2) - 4n(2^{n-1} - (n-2)^2) + (4n-8)(2^{n-2} - (n-3)^2)$$

$$2(2^n) - n^2 = n(2^n) - 4(n-1)^2 - n(n-1)^2 + 4(2^n) - 4n(2^n)(\tfrac{1}{2}) + 4n(n-2)^2 + 4n(2^n)(\tfrac{1}{4}) - 8(2^n)(\tfrac{1}{4}) - 4n(n-3)^2 + 8(n-3)^2$$

$$= 2^n(n+4-2n+n-2) + (-4)(n^2-2n+1) + (-n)(n^2-2n+1) + 4n(n^2-4n+4) - 4n(n^2-6n+9) + 8(n^2-6n+9)$$

$$= 2(2^n) + n^3(-1+4-4) + n^2(-4+2-16+24+8) + n(8-1+16-36-48) + (-4+72)$$

$$= 2(2^n) + (-1)n^3 + 14n^2 - 109n^2 + 68$$

I made a mistake...

Problem 2

Guess: $T_n = A(2^n) + B(n!)$

$T_0 = 2,\ T_1 = 3,\ T_2 = 6$

$n \geq 3 \Rightarrow T_n = (n+4)T_{n-1} - 4n\,T_{n-2} + (4n-8)\,T_{n-3}$

$T_0 = A_0(2^0) + B_0(0!) = 2 \Rightarrow A + B = 2$ $\qquad A, B = 1$

$T_1 = A_1(2^1) + B_1(1) = 3 \Rightarrow 2A + B = 3$ $\qquad 2$

$T_2 = A_2(2^2) + B_2(2!) = 4A + 2B = 6$

Closed form guess: $T_n = 2^n + n!$

$RHS = (n+4)\left(2^{n-1} + (n-1)!\right) - 4n\left(2^{n-2} + (n-2)!\right)$
$\quad + (4n-8)\left(2^{n-3} + (n-3)!\right)$

$RHS = n \cdot \frac{1}{2} \cdot 2^n + n \cdot \frac{n!}{n} + 4 \cdot 2^n \cdot \frac{1}{2} + 4 \cdot \frac{n!}{n}$
$\quad - 4n \cdot 2^n \cdot \frac{1}{4} - 4n \cdot \frac{n!}{n(n-1)}$
$\quad + 4n \cdot 2^n \cdot \frac{1}{8} - 8 \cdot 2^n \frac{1}{8} + 4n \cdot \frac{n!}{n(n-1)(n-2)} - 8 \cdot \frac{n!}{n(n-1)(n-2)}$

$= 2^n\left(\frac{n}{2} + 2 - n + \frac{n}{2} - 1\right)$
$\quad + n!\left(1 + \frac{4}{n} - \frac{4}{(n-1)} + \frac{4}{(n-1)(n-2)} - \frac{8}{n(n-1)(n-2)}\right)$

$n! + 4(n-1)! - 4n(n-2)! + 4n(n-3)! - 8(n-3)!$

Ugh more algebra mis

$E(n) = E_2(n) + E_!($

$E_2(n) = (4n-8)2^{n-3}$
$\quad + (-4n)2^{n-1}$
$\quad + (4+n)2^{n-1} =$

$E_!(n) = (4n-8)(n-3)$

$E_!(n) = \left[(4n-8) + \right.$

$F(n) = \text{bracketed}$
$\quad = 4n - 8 - 4n^2$
$\quad = n^3 - 3n + 2n$

$(n-3)! \cdot (n-1)$

yay!!! $\boxed{T_n =}$

Ugh more algebra mistakes.

$\cancel{(n-1)! = \frac{n!}{n}}$

$\cancel{1 \cdot 2 \cdot 3 \cdot 4}$

$\cancel{(n-2)! = \frac{n!}{n(n-1)}}$

$E(n) = E_2(n) + E_!(n)$

$$E_2(n) = (4n-8)2^{n-3} + (-4n)2^{n-1} + (4+n)2^{n-1} = 2^n$$

already shown

$$E_!(n) = (4n-8)(n-3)! + (-4n)(n-2)! + (n+4)(n-1)!$$

$(-4n)(n-3)!(n-2)$ $\quad$ $(n+4)(n-3)!(n-2)(n-1)$

$$E_!(n) = \left[(4n-8) + (-4n)(n-2) + (n+4)(n-2)(n-1)\right](n-3)!$$

$F(n)$ = bracketed polynomial

$= 4n - 8 - 4n^2 + 8n + n^3 + 2n^2 - 8n - n^2 - 2n + 8$

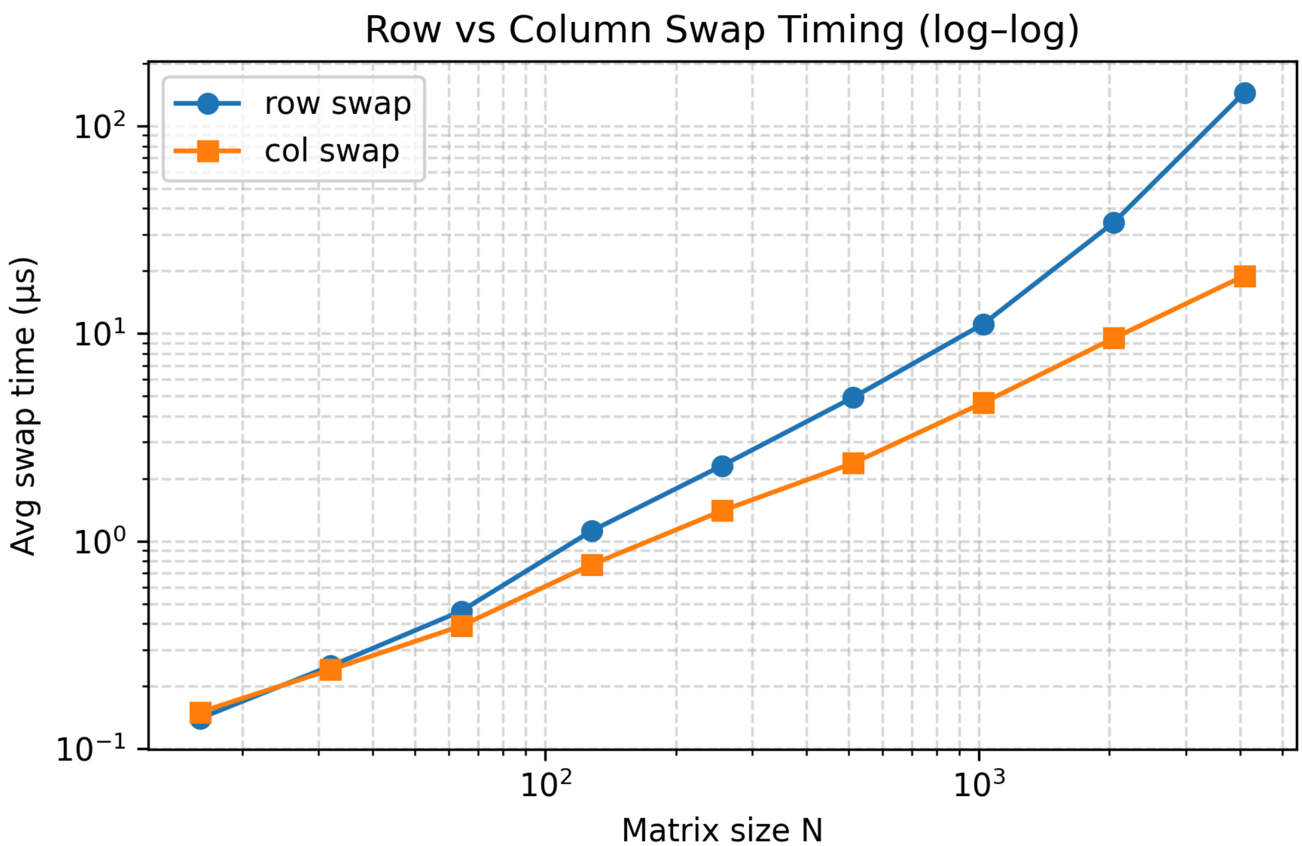
$= n^3 - 3n + 2n = n(n-1)(n-2)$

$(n-3)! \cdot (n-1)(n-2)n = \boxed{n!}$

yay!!! $\boxed{T_n = 2^n + n!}$ strongly induction

Problem 3

mem_swaps.hpp



Problem 4

Problem 4

(a). Basis: ~~Does the given set~~ $\mathcal{S} = \{1, 1-x, (1-x)^2\}$ ~~form a basis for the space~~ of polynomials up to degree 2, $x \in \mathbb{R}$?

must be linearly independent & span the space

$P_2(\mathbb{R})$ is 3D, sufficient to show linear independence

$$v_1 = 1 \qquad 1 + 0x + 0x^2$$
$$v_2 = 1 - x \qquad = 1 - 1x + 0x^2$$
$$v_3 = (1-x)^2 = 1 - 2x + 1x^2$$

$\underset{\text{coeffs}}{\Longrightarrow}$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = A$

linearly independent iff $\det \neq 0$

$\det(A) \Rightarrow$ matrix is upper triangular

$\therefore \det(A) = \prod_{k=1}^{3} A_{k,k} = 1 \times -1 \times 1 = -1$

matrix is non-singular. $\boxed{\checkmark \text{ satisfied } P_2(\mathbb{R}) \text{ basis}}$

(b.) Gram-Schmidt. Given set $S = \{x_1(t), x_2(t), x_3(t)\}$ $\ni x_1(t) = t^2$, $x_2(t) = t$, $x_3(t) = 1$ and $t \in [-1,1] \in \mathbb{R}$

construct by hand an orthonormal basis for S using the inner product $(x, y) = \int_{-1}^{1} x(t)\, y(t)\, dt$

inner product: $\langle f, g \rangle = \int_{-1}^{1} f(t) g(t)\, dt$

Construct orthonormal vectors

$u_1 = x_1 = t^2$, $\|u_1\|^2 = \int_{-1}^{1} t^4\, dt = \left[\frac{t^5}{5}\right]_{-1}^{1} = \frac{2}{5}$

$e_1 = \frac{u_1}{\|u_1\|} = \sqrt{\frac{5}{2}}\, t^2$

$u_2 = x_2 - \text{proj}_{e_1} x_2 = \langle x_2, e_1 \rangle = \int_{-1}^{1} t \sqrt{\frac{5}{2}}\, t^2\, dt = \sqrt{\frac{5}{2}} \left[\frac{t^3}{3}\right]_{-1}^{1} = 0$

$\cancel{= \sqrt{\frac{5}{2}} \cdot \frac{2}{3} = \sqrt{\frac{20}{18}}}$

$u_2 = t$

$\|u_2\|^2 = \int_{-1}^{1} t \cdot t\, dt = \left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{2}{3}$

$e_2 = \frac{u_2}{\|u_2\|} = \left(\frac{2}{3}\right)^{-1/2} t$

$u_3 = x_3 - \langle x_3, \ldots$

$\langle x_3, e_1 \rangle = \int_{-1}^{1} \sqrt{\ldots}$

$\text{proj}_{e_1} \ldots$

$\langle x_3, e_2 \rangle = \sqrt{\frac{3}{2}} \ldots$

$u_3 = 1 - \frac{5}{3} t^2$

$\|u_3\|^2 = \int_{-1}^{1} \left(1 - \frac{5}{3} t \ldots\right.$

orthonormal

$e_1(t) = \ldots$

$e_2(t) = \ldots$

$e_3(t) = \ldots$

$\langle e_i, e_j \rangle = \ldots$

$\sqrt{\frac{3}{2}} \sqrt{\frac{5}{2}} \int_{-1}^{1} t^3\, dt = \ldots$

$u_3 = x_3 - \langle x_3, e_1 \rangle - \langle x_3, e_2 \rangle$

$\langle x_3, e_1 \rangle = \int_{-1}^{1} \sqrt{\frac{5}{2}}\, t^2 \, dt = \sqrt{\frac{5}{2}} \cdot \frac{2}{3}$

$\text{proj}_{e_1} x_3 = \langle x_3, e_1 \rangle e_1 = \frac{2}{3}\sqrt{\frac{5}{2}}\left(\sqrt{\frac{5}{2}}\, t^2\right) = \frac{5}{3} t^2$

$\langle x_3, e_2 \rangle = \sqrt{\frac{3}{2}} \int_{-1}^{1} t \, dt = 0$

$u_3 = 1 - \frac{5}{3} t^2$

$\|u_3\|^2 = \int_{-1}^{1} \left(1 - \frac{5}{3} t^2\right)^2 dt = \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}} t^2$

orthonormal basis vectors:

$$\begin{bmatrix} e_1(t) = \sqrt{\frac{5}{2}}\, t^2, \\ e_2(t) = \sqrt{\frac{3}{2}}\, t, \\ e_3(t) = \frac{3 - 5t^2}{2\sqrt{2}} \end{bmatrix} \quad \text{span } \{t^2, t, 1\} \checkmark$$

satisfied

$\langle e_i, e_j \rangle = \delta_{ij}$?

$\sqrt{\frac{3}{2}}\sqrt{\frac{5}{2}} \int_{-1}^{1} t^3 \, dt = \sqrt{\frac{15}{4}} \left[\frac{t^4}{4}\right]_{-1}^{1} = 0$

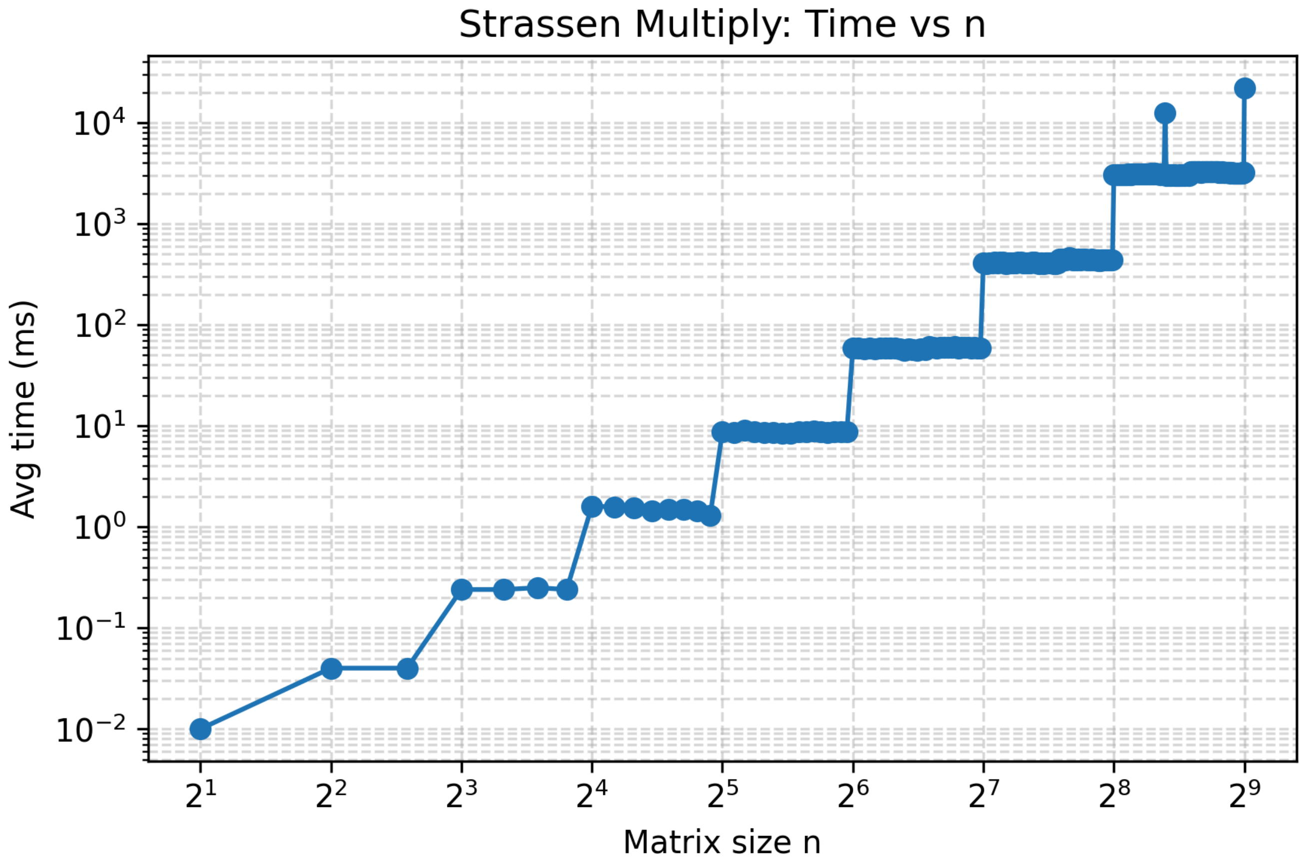
$\vdots$

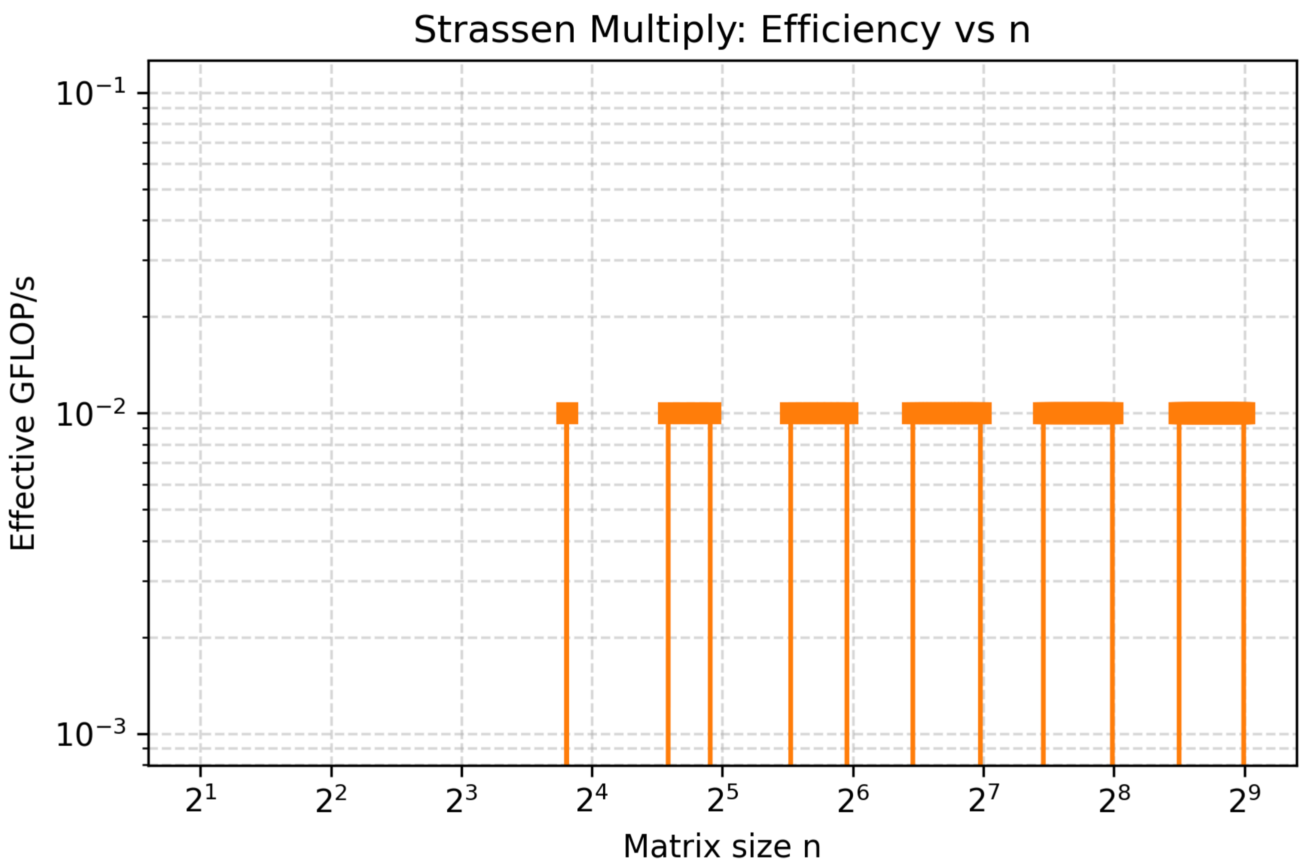
e_3 will cancel ...

e_2 is odd

Problem 5

strassen.cpp


Strassen Multiply: Time vs n
Avg time (ms)
10^4
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
2^1 2^2 2^3 2^4 2^5 2^6 2^7 2^8 2^9
Matrix size n


Strassen Multiply: Efficiency vs n
Effective GFLOP/s
10^{-1}
10^{-2}
10^{-3}
2^1 2^2 2^3 2^4 2^5 2^6 2^7 2^8 2^9
Matrix size n