Period Three Implies Hardcore Chaos

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Outline

Oleksandr Mikolaiovich Sarkovskii

- Ukrainian mathematician born December 7, 1936.
- Paper published in *Ukrainian Mathematical Journal* in 1964 (Vol. 16, pp 61-71) entitled **Coexistence of Cycles of a Continuous Map of a Line into Itself**.
- *Sarkovskii ordering of the natural numbers:* $3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2 \cdot 3 \rhd 2 \cdot 5 \rhd \ldots$ $2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd \cdots \rhd 2^3 \cdot 3 \rhd 2^3 \cdot 5 \rhd \cdots \rhd$ $2^3 \rhd 2^2 \rhd 2 \rhd 1$

Sarkovskii's Theorem

Suppose $F : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function with periodic point of period k ($\exists\, x$ s.t. $F^k(x) = x$ and $F^n(x) \neq x$ for $1 \leq n < k$). Then if $k \rhd l$ in the Sarkovskii ordering, then F also has a periodic point of period l.

Implications of Sarkovskii's Theorem

- The only assumption is that F is a continuous, one-dimensional map.
- If F has a periodic point whose period is not a power of two, it necessarily has infinitely many periodic points.
- Conversely, if F has only finitely many periodic points, then they all necessarily have periods which are powers of 2 (period-doubling route to chaos)
- The "greatest" period in the Sarkovskii ordering is 3, thus the Period Three Theorem is simply a corollary to Sarkovskii's.

Tien-Yien Li and James A. Yorke

- Paper published in *The American Mathematical Monthly* in 1975 (Vol. 82, pp 985-992) entitled **Period Three Implies Chaos**.
- First appearance of the word "chaos" in scientific literature.

- Some time after the publication, Yorke attended a conference in East Berlin, during which Sarkovskii approached him and conveyed that he had proven his results a decade earlier.

yorke1.jpg

Period 3 Implies Chaos Theorem

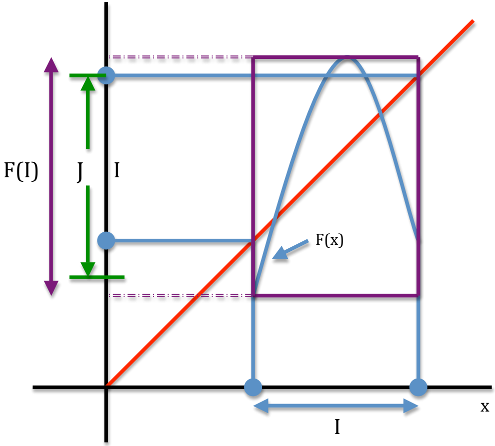
Theorem

Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with periodic point of period three. Then F has periodic points of all other periods.

- Observation 1:
  - Suppose $I = [a, b]$ and $J = [c, d]$ are closed intervals and $I \subset J$. If $F(I) \supset J$, then F has a fixed point in I.
- Observation 2:
  - Suppose I and J are two closed intervals and $F(I) \supset J$. Then there is a closed subinterval $K \subset I$ such that $F(K) = J$.
- **WARNING: This proof is set intensive, bear with me!**

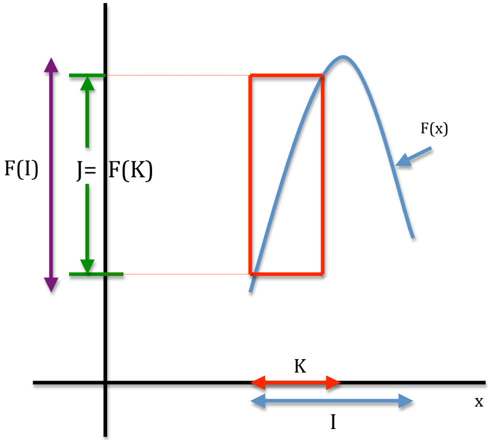
Picture Proof of Observation 1

- Observation 1:
  Suppose $I = [a, b]$ and $J = [c, d]$ are closed intervals and $I \subset J$. If $F(I) \supset J$, then F has a fixed point in I.
  - Follows from the Intermediate Value Theorem.
  - Since $I \subset J$, the graph of F must cross the diagonal



Picture Proof of Observation 2

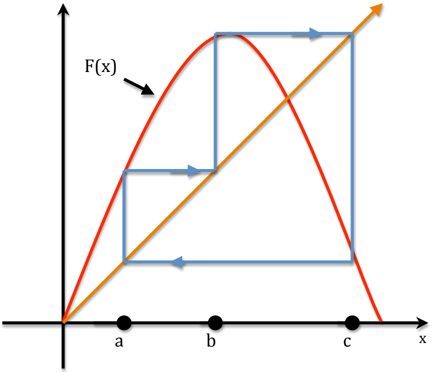
- Observation 2:
  Suppose I and J are two closed intervals and $F(I) \supset J$. Then there is a closed subinterval $K \subset I$ such that $F(K) = J$.



First Steps and Diagram

- Let $a, b, c \in \mathbb{R}$. Suppose $F(a) = b, F(b) = c, F(c) = a$
  $\Rightarrow a \to b \to c \to a$ $\quad F$ has a period three-cycle .
- We will assume a is the leftmost point on the real line. Then we will assume $a < b < c$. The only other case $a < c < b$ is handled similarly.

- Let $I_0 = [a, b]$ and $I_1 = [b, c]$
  By continuity of F:
  1. $F(a) = b, F(b) = c$
     $\Rightarrow \boxed{I_1 \subset F(I_0)}$
  2. $F(b) = c, F(c) = a$
     $\boxed{F(I_1) \supset [a, c] = I_0 \cup I_1}$
- First we will construct periods 1 and 2 and then periods $n > 3$



Cases for n=1 and n=2

Existence of point with period 1

We have just shown that $I_0 \cup I_1 \subset F(I_1)$ thus $I_1 \subset F(I_1)$
Clearly $I_1 \subset I_1$, always, therefore by Observation 1, we know there exists a fixed point of F in I_1.
$\Rightarrow \exists\, x_1 \in I_1$ s.t. $F(x_1) = x_1$ so x_1 has period 1.

Existence of point with period 2

Likewise, we showed that $I_1 \subset F(I_0)$ and we know $I_0 \subset F(I_1)$ since $I_0 \cup I_1 \subset F(I_1)$. Therefore $I_0 \subset F^2(I_0)$.
Again $I_0 \subset I_0$ so by Observation 1, we know there exists a fixed point x_2 of F^2 in I_0 and from diagram of F we can see that this is indeed an orbit of period 2 since $F(x_2) \in I_1$ (more to come on this later).

Periods of length n > 3

- To find a point for each period $n > 3$ we will invoke Observation 2 $n - 1$ times and Observation 1 on the last step.
- Since $F(I_1) \supset I_1$, by Observation 2, there is a subinterval $A_1 \subset I_1$ s.t. $F(A_1) = I_1$
- Again invoking Observation 2, since $F(A_1) \supset A_1$, we can find a closed subinterval $A_2 \subset A_1$ s.t. $F(A_2) = A_1$
  Note that by construction, we now have sets $A_2 \subset A_1 \subset I_1$.
- Continue this process $n - 2$ times. By now we have produced a collection of closed nested intervals A_i , $1 \leq i \leq n - 2$ s.t.

  $$A_{n-2} \subset A_{n-3} \subset \cdots \subset A_2 \subset A_1 \subset I_1$$

  where $F(A_i) = A_{i-1}$ and $F(A_1) = I_1$. Note that $A_{n-2} \subset I_1$ and $F^{n-2}(A_{n-2}) = I_1$

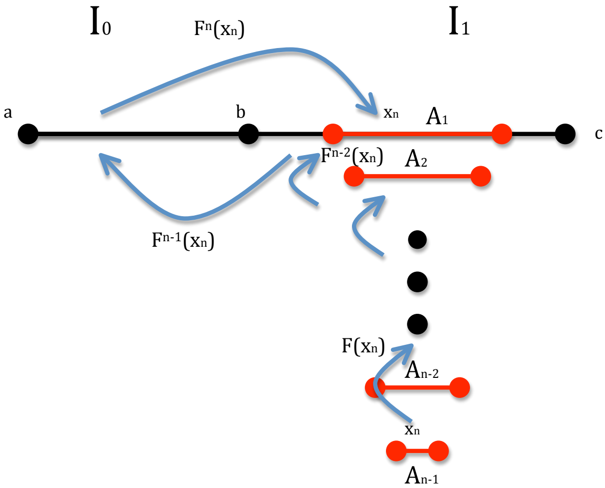
Periods of length n > 3 (cont.)

- Since $I_0 \subset F(I_1)$, equivalently $I_0 \subset F^{n-1}(A_{n-2})$, there exists a closed subinterval (you guessed it, by Obs. 2) $A_{n-1} \subset A_{n-2}$ s.t. $F^{n-1}(A_{n-1}) = I_0$.
- Lastly, since $I_1 \subset F(I_0)$ we have $I_1 \subset F^n(A_{n-1})$. Thus since $A_{n-1} \subset I_1$ by construction, then we invoke Observation 1 to conclude that there exists a point $x_n \in A_{n-1}$ s.t. $F^n(x_n) = x_n$.
- Claim: n is the LEAST period of x_n.
  Note: Suppose $F^{n-1}(x_n)$ lies on the boundary of I_0. Then $F^{n-1}(x_n) = a$ or b (on the period 3 orbit, started from b or c) so $n = 2$ or $n = 3$ necessarily. This contradicts our assumption that $n > 3$, we have already dealt with these cases. Thus $F^{n-1}(x_n)$ lies on the interior of I_0. We know the first $n-2$ iterations of x_n lie in I_1 while $F^{n-1}(x_n) \in I_0$ and $F^n(x_n) = x_n \in I_1$ again. If x_n had period less than n then it would never leave I_1. Thus x_n cannot have period less than n

$$\Rightarrow \boxed{x_n \text{ is a periodic point of } F \text{ with period } n} \qquad \square$$

Diagram of nested interval mappings

- Studying how certain sequences of sets are mapped into or onto each other is common in dynamical systems: ex. Sarkovskii's proof and Smale's horseshoe.



Converse is True!

Converse of Sarkovskii's Theorem

For any natural number n, there exists a continuous function $F : \mathbb{R} \longrightarrow \mathbb{R}$ such that if $k \triangleright n$ in the Sarkovskii ordering, F has a periodic point of period n but none of period k

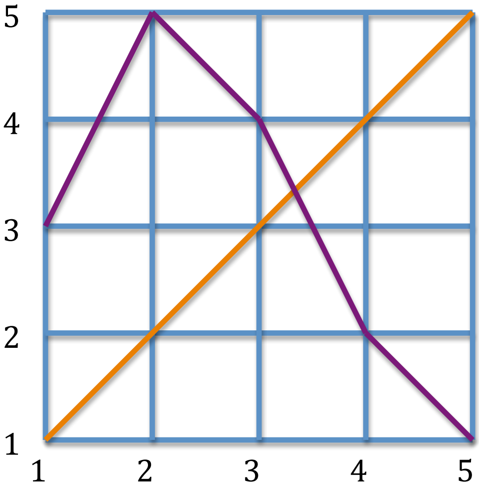
- Of course, this is again a generalization to the same converse result for the Period Three Theorem since 3 precedes all numbers in the Sarkovskii ordering.
- We will show the example of a function F with a period 5 point but no points of period 3 from Li and Yorke's paper.

Converse Example

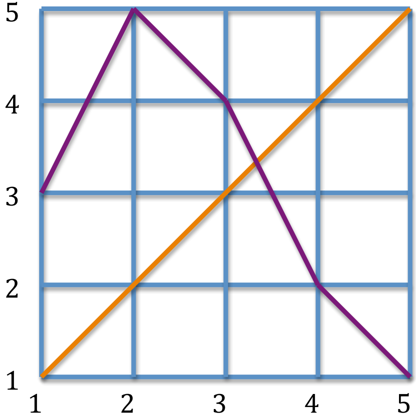
- Consider the piecewise linear map $F : [1,5] \longrightarrow [1,5]$ where

$$1 \to 3 \to 4 \to 2 \to 5 \to 1$$

- That is, $F^5(1) = 1$: F has a 5-cycle.
- We claim F has no 3-cycle



Converse Example (cont.)

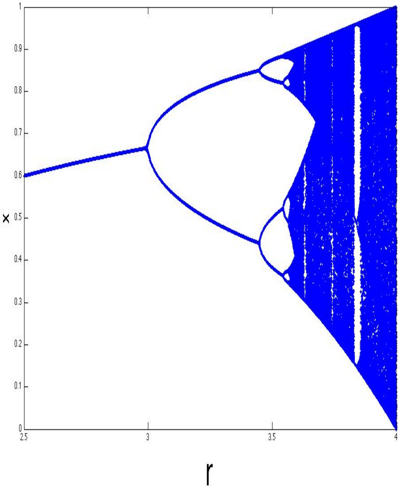


- $F([1,2]) = [3,5], F([3,5]) = [1,4]$, $F([1,4]) = [2,5] \Rightarrow F^3([1,2]) = [2,5]$
- Thus the only fixed point of F^3 in $[1,2]$ could be 2 but 2 has period 5 on the 5-cycle.
- Similarly we can check F^3 has no fixed pts. in $[2,3]$ or $[4,5]$
- F has a fixed pt. on $[3,4]$ and we see that $F([3,4]) = [2,4], F([2,4]) = [2,5]$, $F([2,5]) = [1,5] \Rightarrow F^3$ has a fixed pt. in $[3,4]$
- Claim: This fixed pt. of F^3 is unique and is thus the fixed pt. of F itself (not least period 3).
- F is monotonically decreasing on $[3,4] \to [2,4] \to [2,5] \to [1,5]$ thus F^3 is monotonically decreasing also and the fixed pt. is unique $\to$ F has no 3-cycles

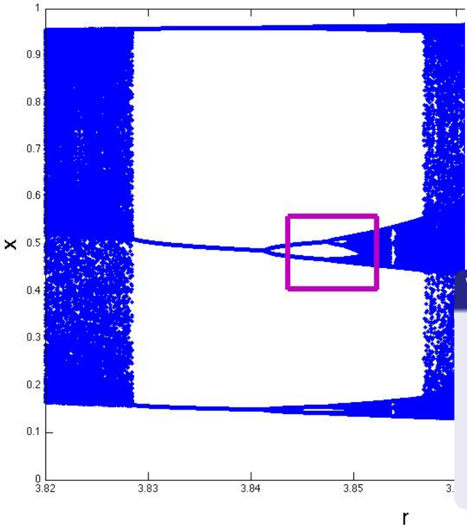
Analysis of Logistic Map

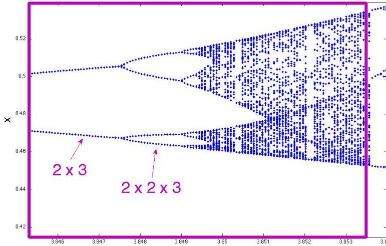
"The snake in the mathematical grass" - Robert May

- $x_{n+1} = r\,x_n(1 - x_n) = F(x_n)$
- NOTE: each r value specifies a DIFFERENT F where Li and Yorke's theorem holds.
- Sarkovskii's results confirmed: If F has only finitely many periodic points, then it necessarily has periods which are powers of 2.



Period 3 window

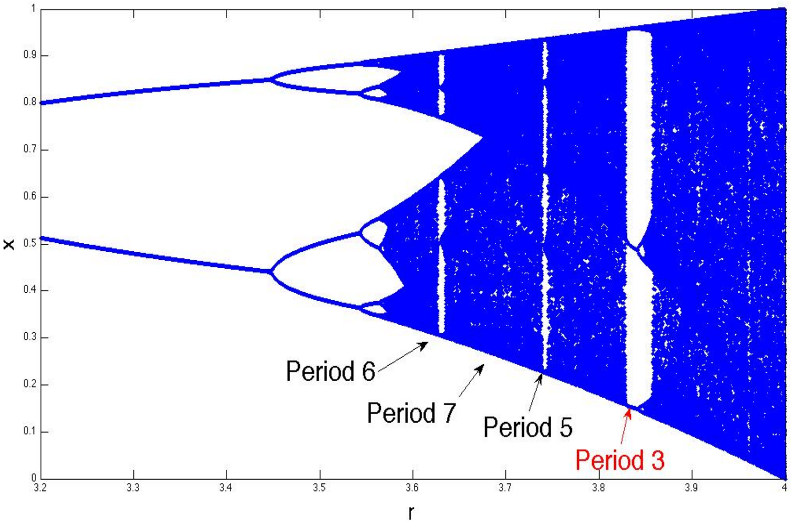




Why only period 3?

- When $1 + \sqrt{8} < r < 3.8415...,\ F$ has a 3-cycle. According to Sarkovskii's theorem, it must also have cycles of any period.
- *Where are these infinite period orbits??*

Logistic map plot



Period 3 analysis

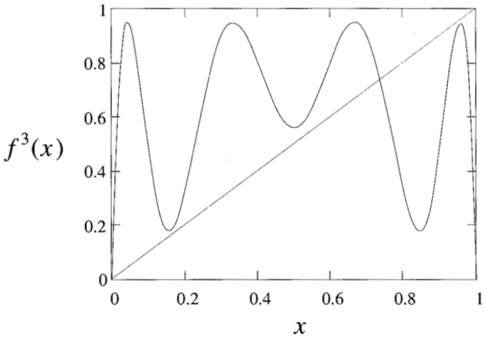


Figure: $r = 3.8$

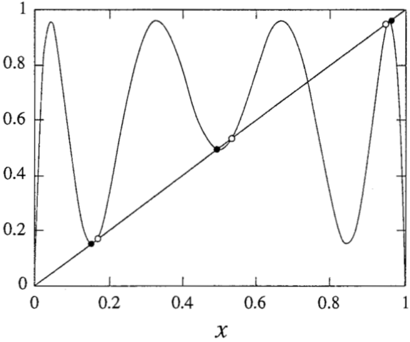


Figure: $r = 3.835$ (See Strogatz)

Logistic map

Logistic map final remarks

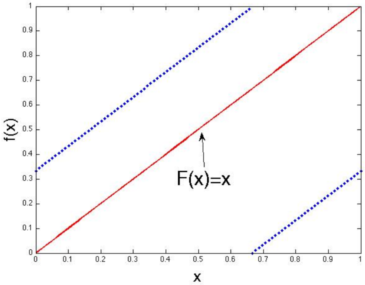
- As r continuously increases from $0 \to 1 + \sqrt{8}$, periodic orbits of periods from bottom up in the Sarkovskii ordering gradually come into existence.
- Before this r value, there are NO period 3 cycles
- Having periods of arbitrary length is a hallmark of chaos, not a necessity, in our mathematical definition.
- Li and Yorke called a system "chaotic" if it had periodic points of arbitrary period length and an uncountable set of aperiodic points.

What if F is not 1-D continuous?

1. Discontinuous Counterexample

$$F(x) = \begin{cases} x + \frac{1}{3} & \text{if } 0 \le x < \frac{2}{3} \\ x - \frac{2}{3} & \text{if } \frac{2}{3} \le x \le 1 \end{cases}$$

It can be shown that every point on $[0, 1)$ has period 3. ex. $\frac{1}{6} \to \frac{1}{2} \to \frac{5}{6} \to \frac{1}{6}$
The value $x = 1$ is eventually period 3.
No other periodic orbits exist for F.



2. Circle Map Counterexample
   - Theorem does not even hold on 1-D system that is not on real line. The map which rotates all points on a circle by 120° makes all points periodic with period 3.

Summary and Discussion points

- *Glimmer of Hope:* Although to theorem does not have a higher-dimensional analogue, we can see typical behavior of period-doubling of cycles and period 3 windows in 3-D continuous time systems like the Rössler system:

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \quad \text{see Strogatz for more detail}\end{aligned}$$

- Possibilities in twist maps (see Laskar et al.) and other higher dimensional systems.
- An elegant result about existence of orbits in 1-D maps with ONLY the assumption that F is continuous!

Herman Melville, *Moby Dick*

"The classification of the constituents of a *chaos*, nothing less here is essayed.“

THANK YOU!

References:

- Devaney, R. L. (2003). An Introduction to Dynamical Systems 2nd ed. . *Advanced Book Program*.
- Gleick, J. (1987). Chaos: Making a New Science *Penguin Books*
- Li, T., Yorke, J. A. (1975). Period Three Implies Chaos. *The American Mathematical Monthly*, **v 82**, 985-992.
- May, R. M. (1976). Simple mathematical models with very complicated dynamics. *Nature*, **v 261**, 459-467.
- Strogatz, S.H. (1994) Nonlinear Dynamics and Chaos. *Da Capo Press*

For your hardcore listening pleasure, check out:

Period Three Implies Chaos
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