MA332 Project 1

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1 Introduction

Newton's Method is a numerical root-finding algorithm. To find a root $f(x_\star) = 0$, the algorithm uses f, its derivative f', and some initial value x_0. Starting at x_0 the algorithm iterates, with the $n + 1^{\text{th}}$ approximation given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In most cases Newton's Method

2 Failure to Converge

Depending on the function and starting value, Newton's Method may not converge. There are several circumstances under which this happens.

2.1 Converges to a Cycle

Consider the function $g(x) = x^3 - 2x + 2$, which has one root: $g(-1.769) \approx 0$. Newton's method with $g(x)$ gives the $n + 1^{\text{th}}$ approximation:

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n + 2}{3x_n^2 - 2}$$

Observe that if $x_n = 0$:

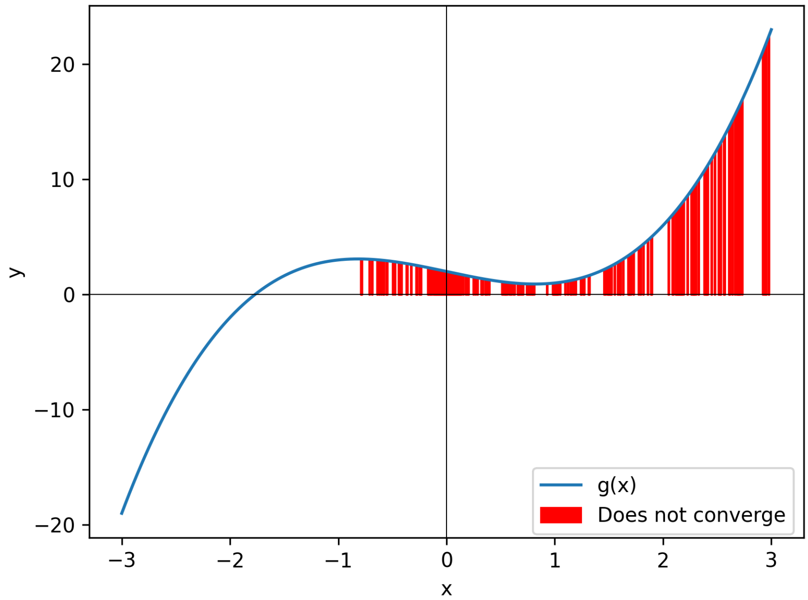
$$x_{n+1} = 0 - \frac{2}{-2} = 1$$

And if $x_n = 1$:

$$x_{n+1} = 1 - \frac{1 - 2 + 2}{3 - 2} = 0$$

If given either 0 or 1 as a starting value, Newton's Method will cycle infinitely. In addition to these two values, there are many other starting points where the series will asymptotically approach the 0-1 cycle

Figure 1: Starting values that do not converge for $g(x) = x^3 - 2x + 2$



2.2 Diverges

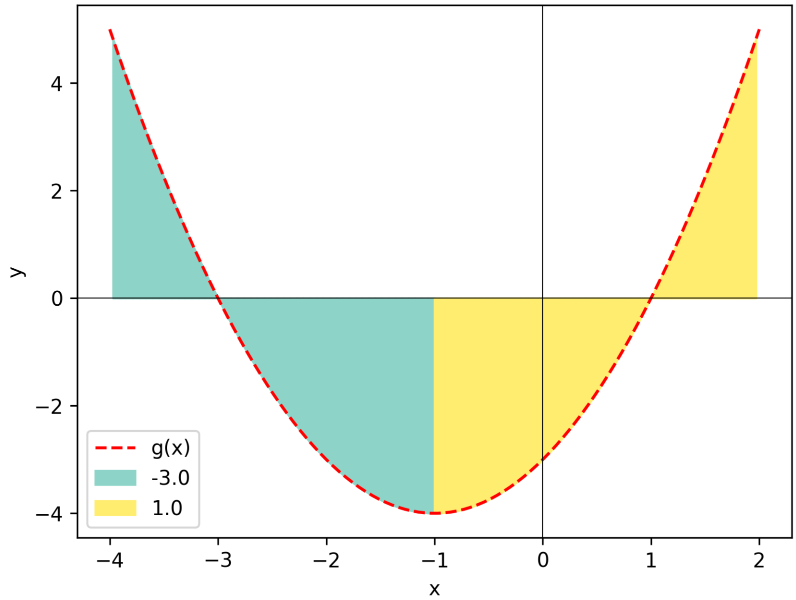
3 Basins of Attraction

For a given root $f(x_\star) = 0$, the *basin of attraction* is the set of starting values x_0 for which Newton's Method will converge to $x_\star$.

3.1 Real-Valued Functions

Consider the function $g(x) = (x - 1)(x + 3)$

Figure 2: Basins of convergence for $g(x) = (x-1)(x+3)$



Consider the function $h(x) = (x-4)(x-1)(x+3)$

Figure 3: Basins of convergence for $h(x) = (x-4)(x-1)(x+3)$

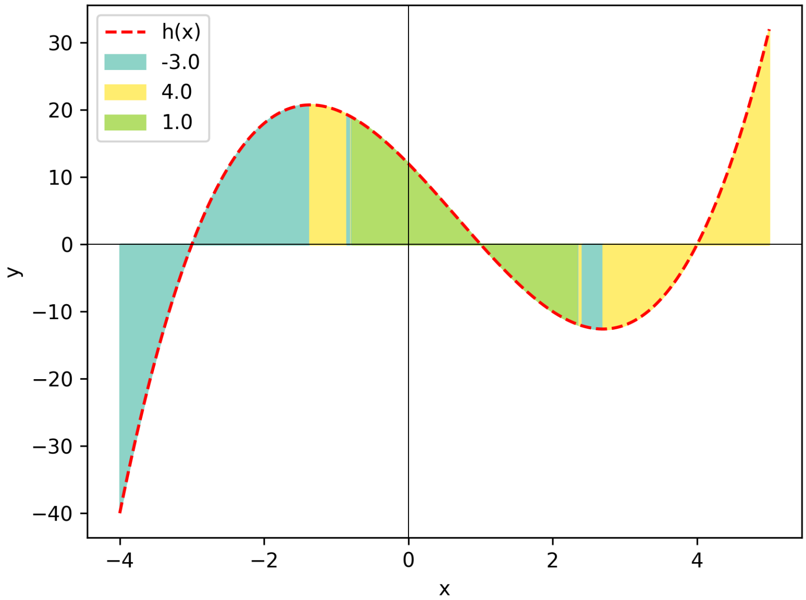
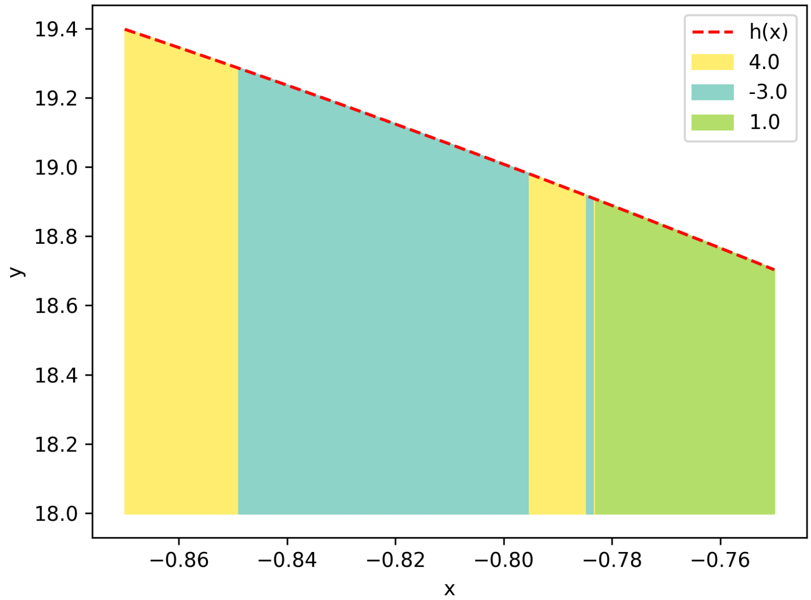


Figure 4: Zooming in shows the fractal pattern of the basins



3.2 Complex-Valued Functions

The fractal behavior of the basins visible in Figure 4 becomes much more intricate when visualized on the complex plane

Figure 5: Basins of convergence for $f(z) = z^3 - 1$ on the complex plane $a + bi$

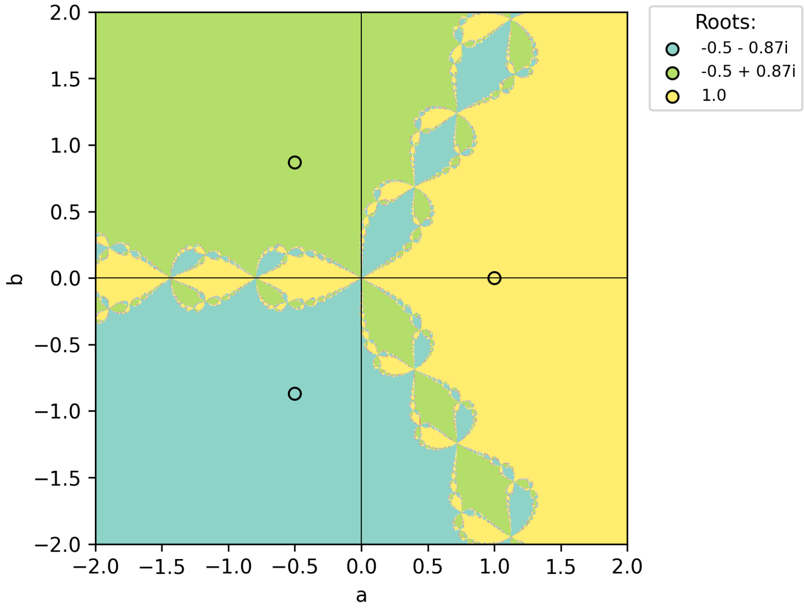
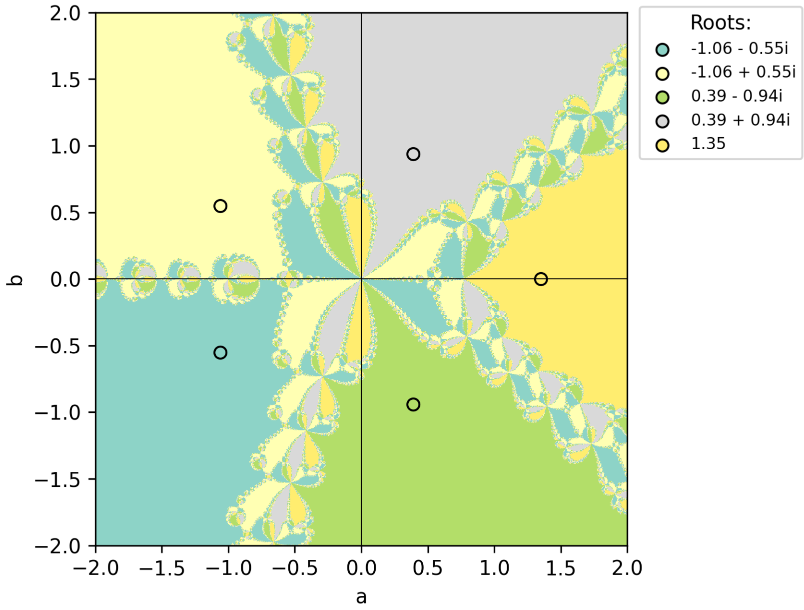


Figure 6: Basins of convergence for $g(z) =$ on the complex plane $a + bi$



4 Discussion