Homework 2

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[130]:
```
from numpy import abs, sqrt, linspace, pi, sin, nan, min, max, unique, empty, exp
import matplotlib
import matplotlib.pyplot as plt
```

1 Algorithms

1.1 Fixed Point Iteration (FPI):

[159]:
```
def fpi(f, x, iter=1000, tol=1e-2):
    ''' Finds the fixed point x* = f(x*) of function f starting at x using fixed␣
 ↪point iteration'''
    x_n = []  # list to hold guesses

    for n in range(iter):
        x_n.append(x)

        if abs(f(x) - x) < tol:  # check stopping condition
            return x_n

        elif x > 1e10:  # other stopping condition
            return x_n

        x = f(x)  # update approximation
    return x_n
```

1.2 Bisection Method:

[3]:
```
def bisection(f, a, b, iter=100, tol=1e-2):
    ''' Finds the root of function f on [a,b] using the bisection method'''

    if f(a)*f(b) > 0: raise 'No root on [a, b]'

    x_n = []  # list to hold guesses

    for n in range(iter):
```

```python
        mid = (a + b)/2  # compute approximate root (midpoint)
        x_n.append(mid)

        if abs(f(mid)) < tol: return x_n  # check stopping condition

        elif f(a)*f(mid) > 0: a = mid  # check which bound to update

        else: b = mid
    return x_n
```

1.3 Newton's Method:

[13]:
```python
def newton(f, df, x, iter=1000, tol=1e-2):
    ''' Finds the root of function f with derivative df using Newton's method␣
↪starting at x'''

    x_n = []  # list to hold guesses

    for i in range(iter):
        x_n.append(x)

        if abs(f(x)) < tol: return x_n  # check stopping condition

        x = x - f(x)/df(x)  # update approximation
    return x_n
```

2 Problem 1

Starting with:

$$x^3 + x - 2 = 0$$

Add cx to both sides:

$$x^3 + x - 2 + cx = cx$$

Divide the equation by c to get a function $g(x)$ with fixed point $g(1) = 1$:

$$g(x) = \frac{1}{c}x^3 + \frac{c+1}{c}x - \frac{2}{c} = x$$

For fixed point problem $g(p) = p$, fixed point iteration will converge linearly with rate s, if s satisfies:

$$s \equiv |g'(p)| < 1$$

Compute the derivative of g:

$$g'(x) = \frac{3}{c}x^2 + \frac{c+1}{c}$$

Which gives s:

$$s = |g'(1)| = |\frac{3}{c} + \frac{c+1}{c}| = |1 + \frac{4}{c}|$$

To satisfy $|1 + \frac{4}{c}| < 1$

Error on the $(n+1)^{\text{th}}$ iteration is related to s by:

$$E_{n+1} \approx sE_n$$

3 Problem 2

4 Problem 3

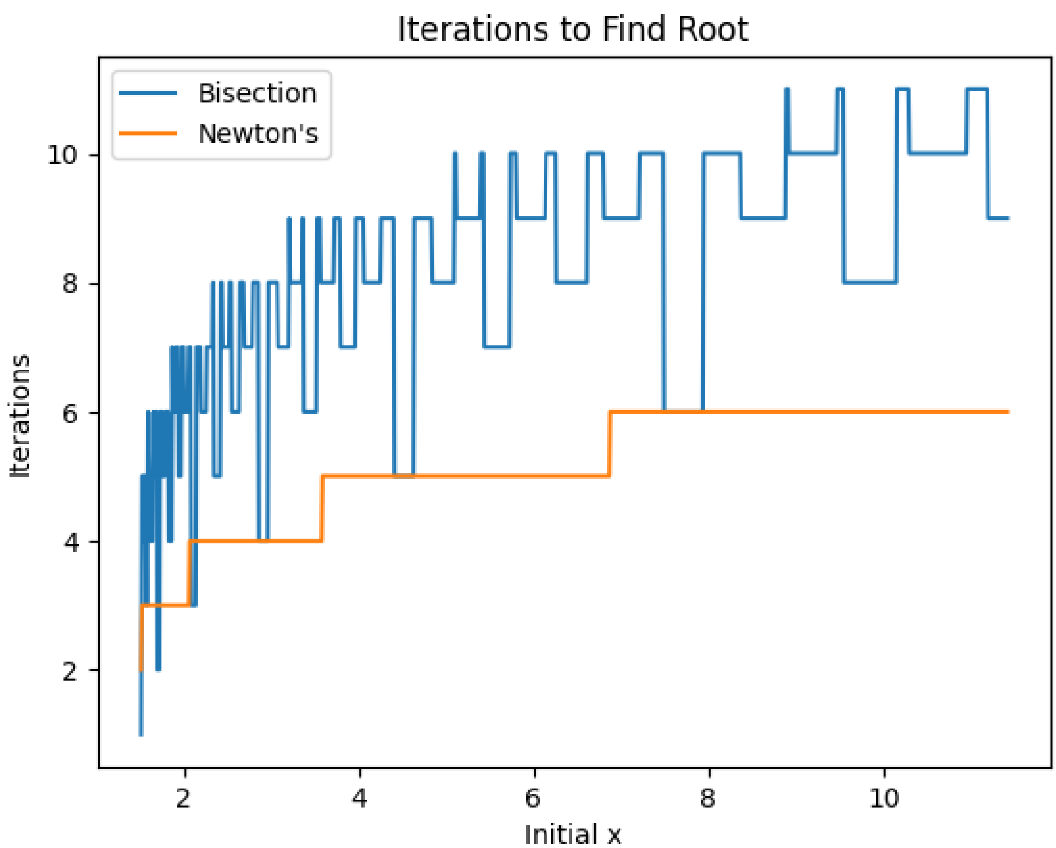
First define $f(x) = x^2 - 2$ and $\frac{df(x)}{dx}$:

[170]:
```
def f(x): return x**2 - 2
def df(x): return 2*x
```

Create an array of test x values around the root $f(\sqrt{2}) = 0$ to compare the convergence of the bisection method and Newton's method

[201]:
```
sqrt_2 = sqrt(2)
test_x = linspace(sqrt_2 + 0.1, sqrt_2 + 10, 1000)
```

[203]:
```
bisection_iter = []
newtons_iter = []

for i in range(1000):
    bisection_iter.append(len(bisection(f, sqrt_2-0.1, test_x[i])))
    newtons_iter.append(len(newton(f, df, test_x[i])))

plt.plot(test_x, bisection_iter, label='Bisection')
plt.plot(test_x, newtons_iter, label="Newton's")
plt.legend()
plt.title('Iterations to Find Root')
plt.xlabel('Initial x')
plt.ylabel('Iterations')
plt.show()
```



5 Problem 4

5.1 (a)

By the Fixed Point Theorem, $g(x)$ will have a unique fixed point $p \in [a, b]$ if:

- $g \in C[a, b]$
- $g(x) \in [a, b]$
- **There exists a k such that $k < 1$ and $|g'(x)| \leq k$ on $[a, b]$**

$g(x)$ satisfies the first two conditions, to check the third find $\frac{dg(x)}{dx}$:

$$\frac{dg(x)}{dx} = \frac{1}{8}\sin(\frac{x}{2})$$

Clearly one can choose k such that the third condition is satisfied, therefore $g(x)$ must have a unique fixed point on $[a, b]$

5.2 (b)

Define $g(x) = \pi + \frac{1}{2}\sin(\frac{x}{2})$

[124]:
```
def g(x): return pi + sin(x/2)/2
```

Find the fixed point on $[0, 2\pi]$

[163]:
```
approx = fpi(g, 1)
print(approx[-1])
```

```
3.626269950250724
```

5.3 (c)

Using the corollary, write:

$$E_n = |x_n - x| \leq k^n \textbf{max}\{x_0 - a, b - x_0\}$$

$$E_n = 0.01 \leq 0.13^n(2\pi - 1)$$

Which gives:

$$\frac{0.01}{(2\pi - 1)} \leq 0.13^n$$
$$\frac{\ln\left(\frac{0.01}{2\pi - 1}\right)}{\ln(0.13)} \leq n$$
$$n \geq 3.073$$

Compare to the number of steps taken by the algorithm:

[164]:
```
print(len(approx))
```

```
4
```

6 Problem 5

6.1 (a)

To find the maximum of $c(t)$**, find where** $\frac{dc(t)}{dt} = 0$ **using Newton's method. This requires computing the first and second derivative of** $c(t)$**:**

$$c(t) = Ate^{-t/3}$$

$$\frac{dc(t)}{dt} = -\frac{A(t-3)}{3}e^{-t/3}$$
$$\frac{d^2c(t)}{dt^2} = \frac{A(t-6)}{3}e^{-t/3}$$

Define $c(t)$, $\frac{dc(t)}{dt}$, **and** $\frac{d^2c(t)}{dt^2}$**:**

[156]:
```
def c(t, A): return A*t*exp(-t/3)
def dc(t): return -(t-3)*exp(-t/3)/3
```

```
def d2c(t): return (t-6)*exp(-t/3)/9
```

Find where $\frac{dc(t)}{dt} = 0$ using Newton's method:

[157]:
```
print('c(t) = 0 at t = ' + str(round(newton(dc, d2c, 1)[-1], 2)))
```

```
c(t) = 0 at t = 2.99
```

Evaluate $c(t)$ at $t_{\textbf{max}}$:

[158]:
```
print(round(c(3, 1), 4))
```

```
1.1036
```

For $A = 1$, $c(3) = 1.1036$. So to make $c(3) = 1$, let $A = \frac{1}{1.1036}$

6.2 (b)

To find when $c(t) = 0.25$, rewrite the equation as a root finding problem:

$$f(t) = c(t) - 0.25 = 0$$

Define $f(t)$ and $\frac{df(t)}{dt}$:

[148]:
```
def f(t): return (1/1.1036)*t*exp(-t/3) - 0.25
def df(t): return -(1/1.1036)*(t-3)*exp(-t/3)/3
```

Using Newton's method, find the root of $f(t)$:

[154]:
```
root = round(newton(f, df, 1)[-1], 2)
print('f(t) = 0 at t = ' + str(root))
print('c(' + str(root) + ') = ' + str(c(root, 1/1.1036)))
```

```
f(t) = 0 at t = 0.31
c(0.31) = 0.2532990464138039
```

7 Problem 6

First define $g(x)$ and $\frac{dg(x)}{dx}$:

[8]:
```
def g(x): return (x - 1)*(x + 3)

def dg(x): return (x - 1) + (x + 3)
```

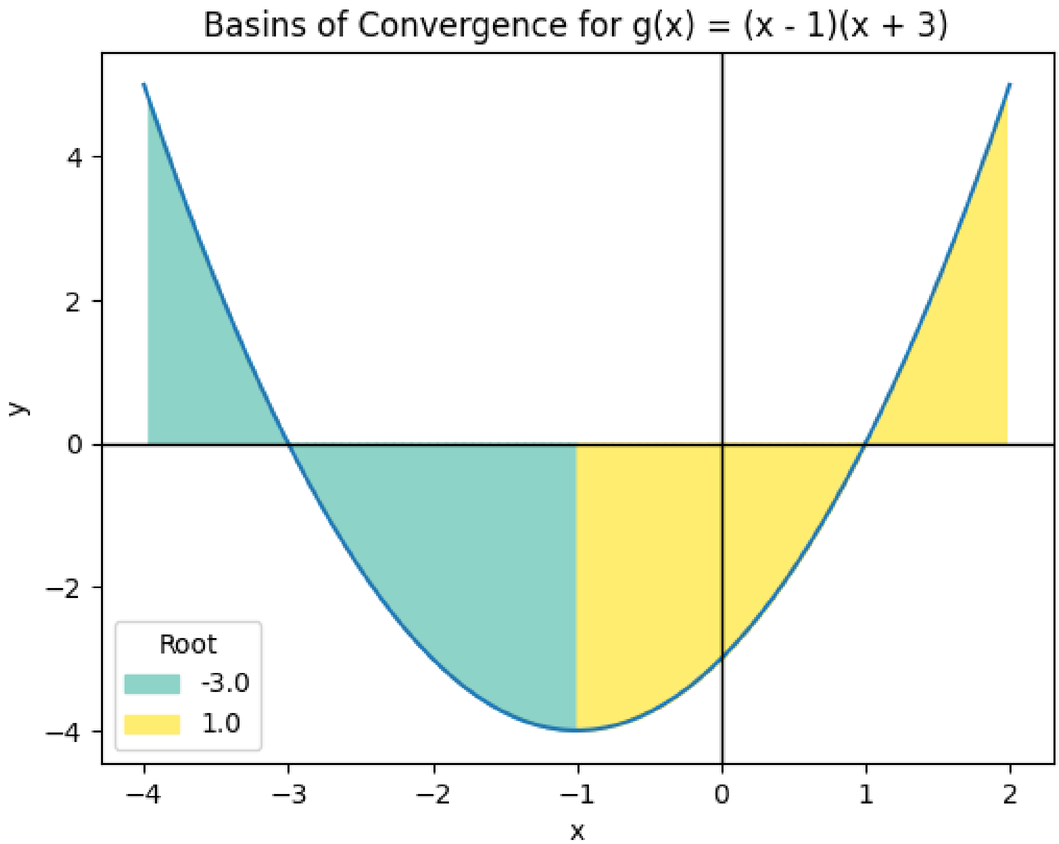
Create an array of x samples to use as starting points for Newton's Method:

[ ]:
```
x = linspace(-4, 2, 100)
y = g(x)
```

Find roots starting at each point:

[65]:
```
roots  = []
for i in range(100):
    approx = newton(g, dg, x[i])
    roots.append(round(approx[-1], 2))

# copy of roots array with first and last values removed
trimmed = empty(98)
trimmed = roots[1:-1]
```

[74]:
```
cmap=matplotlib.colormaps['Set3']
colors = cmap(roots)

# find the first x value that led to each root
_, root_indices = unique(trimmed, return_index=True)

# correct for using trimmed roots
root_indices += 1

fig, axes = plt.subplots()
axes.plot(x, y)

# for each starting x (except endpoints)
for i in range(1, len(roots)-1):

    fill_x = [(x[i-1] + x[i])/2, (x[i] + x[i+1])/2]
    fill_y = [(y[i-1] + y[i])/2, (y[i] + y[i+1])/2]

    axes.fill_between(fill_x, fill_y, 0, color=colors[i], label=roots[i] if i in␣
 ↪root_indices else "")

# draw x and y axis
axes.axvline(color='Black', lw=1)
axes.axhline(color='Black', lw=1)

axes.set_title('Basins of Convergence for g(x) = (x - 1)(x + 3)')
axes.set_xlabel('x')
axes.set_ylabel('y')
axes.legend(title='Root')
plt.show()
```


Basins of Convergence for g(x) = (x - 1)(x + 3)
y
x
Root
-3.0
1.0
4
2
0
−2
−4
−4
−3
−2
−1
0
1
2