MA332 Project 1

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1 Introduction

Newton's Method is a numerical root-finding algorithm. To find a root $f(x_\star) = 0$, the algorithm uses f, its derivative f', and some initial value x_0. Starting at x_0 the algorithm iterates, with the $n+1^{\text{th}}$ approximation given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In most cases, the approximation generated by Newton's Method will be quite accurate within a few iterations.

2 Failure to Converge

Depending on the function and starting value, Newton's Method may not converge. There are several circumstances under which this happens.

2.1 Converges to a Cycle

In the case of certain functions which have a local minimum or maximum but no root on an interval, Newton's Method can get stuck in a cycle around the minimum/maximum.

Consider the function $g(x) = x^3 - 2x + 2$, which has one root: $g(-1.769) \approx 0$. Newton's method with $g(x)$ gives the $n+1^{\text{th}}$ approximation:

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n + 2}{3x_n^2 - 2}$$

Observe that if $x_n = 0$:

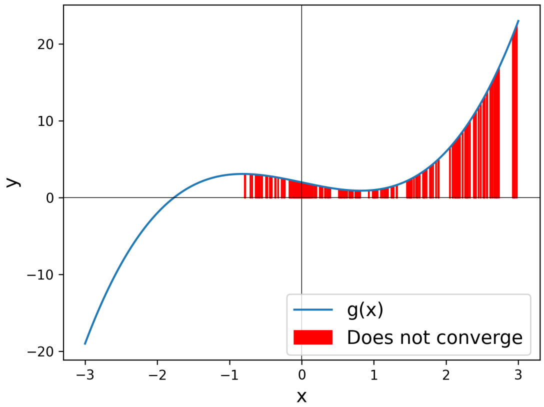
$$x_{n+1} = 0 - \frac{2}{-2} = 1$$

And if $x_n = 1$:

$$x_{n+1} = 1 - \frac{1 - 2 + 2}{3 - 2} = 0$$

If given either 0 or 1 as a starting value, Newton's Method will cycle infinitely. These are not the only starting points for which Newton's Method fails to converge. Figure 1 shows the sub-intervals on [-3, 3] where Newton's Method does not find a root.

Figure 1: Starting values that do not converge for $g(x) = x^3 - 2x + 2$



To see what happens for non-convergent starting values besides 0 and 1, look at the series of approximations generated by one of these starting points ($x = 0.1$):

Table 1: Newton's Method on $g(x)$ after n iterations starting at 0.1

n	x_n
0	0.1
1	1.014213
2	0.079656
3	1.009099
4	0.052227
5	1.003965
6	0.023329
7	1.000804
8	0.004807
9	1.000035

Table 1 shows that the series of approximate roots for non-convergent starting points besides 0 and 1 asymptotically approaches the 0-1 cycle.

2.2 Diverges

Newton's Method will also encounter issues if the derivative does not exist at the root. Consider the function $h(x) = x^{1/5}$. Newton's Method gives the $n+1^{\text{th}}$ approximation of the root:

$$x_{n+1} = x_n - 5\frac{x_n^{1/5}}{x_n^{-4/5}} = x_n - 5x_n = -4x_n$$

Each x_{n+1} will have the opposite sign to x_n and be four times further from the root. The output of the Newton's Method implementation used for this project confirms this:

Table 2: Newton's Method on $h(x) = x^{1/5}$ after n iterations starting at 1

n	x_n
0	1
1	-4
2	16
3	-64
4	256
5	-1024
6	4096

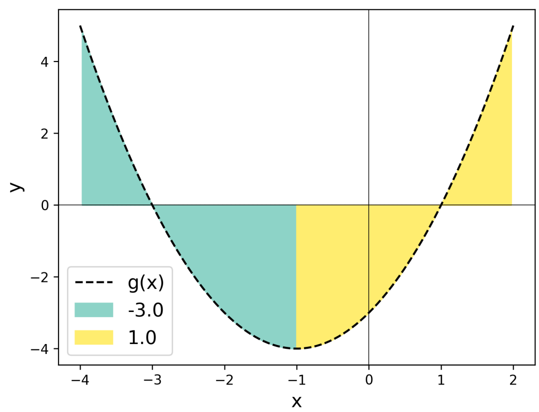
3 Basins of Attraction

For a given root $f(x_\star) = 0$, the *basin of attraction* is the set of starting values x_0 for which Newton's Method on f will converge to $x_\star$.

3.1 Real-Valued Functions

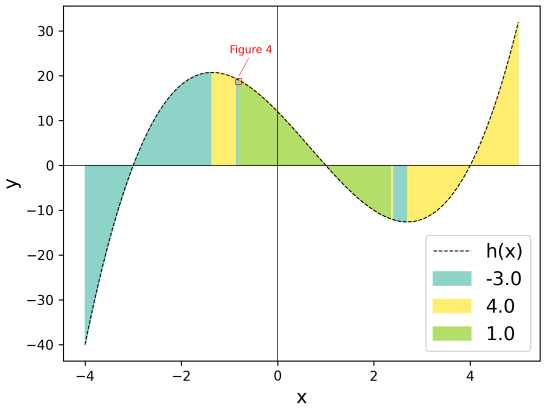
Consider the function $g(x) = (x-1)(x+3)$

Figure 2: Basins of convergence for $g(x) = (x-1)(x+3)$



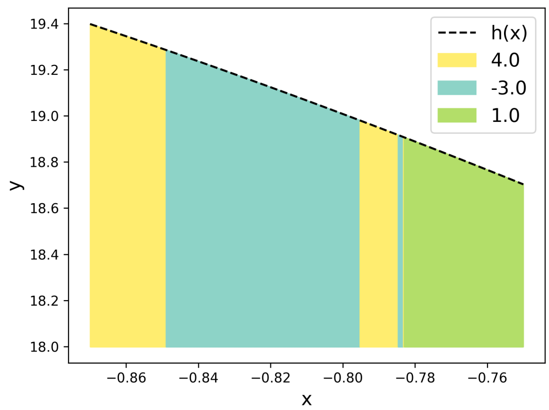
For a second-degree polynomial with two roots the basins are very simple, meeting at the minimum/maximum of the function and extending out to infinity. Introducing a third root has an interesting effect. Consider the function $h(x) = (x-4)(x-1)(x+3)$. Figure 3 shows the basins of convergence for the three roots:

Figure 3: Basins of convergence for $h(x) = (x-4)(x-1)(x+3)$



Two of the three basins are no longer contiguous. When the derivative is small compared to the value of the function (around $x = -1.5$ and $x = 2.5$ in Figure 2), the approximation will change by a large amount. This means Newton's Method can jump over the middle basin and find a root far from where it started. A subset of those jumps will land in regions which jump back again (and a further subset will jump a third time, etc.)

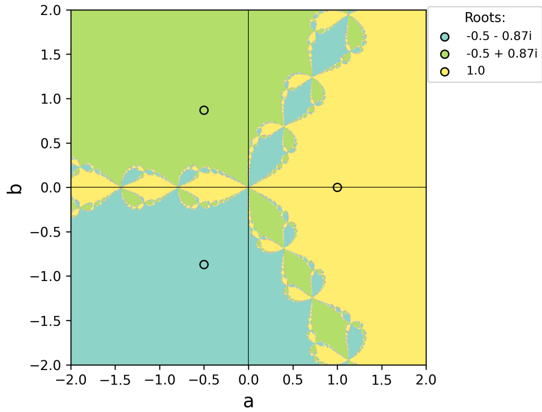
Figure 4: Zooming in shows the fractal pattern of the basins



3.2 Complex-Valued Functions

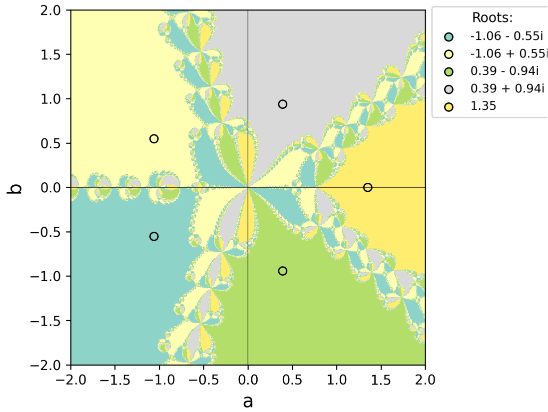
Extending this to complex-valued functions adds another "dimension" along which basins can interface with each other. The one-dimensional fractal behavior visible in Figure 4 becomes much more intricate when visualized on the complex plane.

Figure 5: Basins of convergence for $f(z) = z^3 - 1$ on the complex plane $a + bi$



$f(z) = z^3 - 1$ is not unique, $g(z) = z^5 - z^3 - 2$ produces a similar (though less symmetrical) image:

Figure 6: Basins of convergence for $g(z) = z^5 - z^3 - 2$ on the complex plane $a + bi$



The roots do not need to be complex, as long as there are at least three the basins will form a fractal. Consider again the function $h(x) = (x-4)(x-1)(x+3)$, with real basins of convergence shown in Figure 3. The basins are shown again in Figure 7, this time in the complex plane.

Figure 7: Basins of convergence for $h(x) = (x-4)(x-1)(x+3)$ on the complex plane $a + bi$

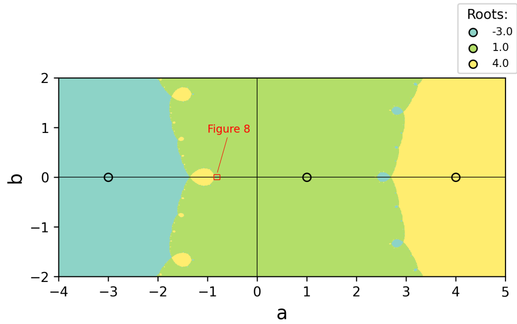
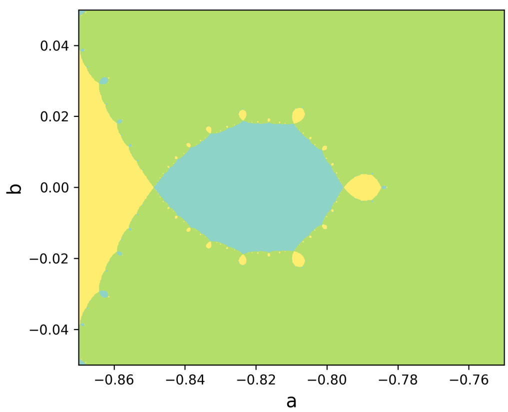


Figure 8: Zoom in on Figure 7 to show the fractal



4 Discussion

In this project I examined the convergence of Newton's Method. First I looked at two functions for which Newton's Method does not converge. Then I visualized the basins of convergence for real- and complex-valued functions created by Newton's Method. For a function with at least three roots, the basins form fractals. I was somewhat surprised to find that the roots don't need to be complex, even three real roots will create fractal basins.