Math 490 Notes: Tuesday September 6, 2016

Definition. A **distance function** on a set X is a nonnegative, real-valued function on pairs of points in X. A **metric** d on a set X is a distance function

$$d\colon X \times X \to [0, \infty)$$

satisfying moreover that

(M1) $d(x, y) = 0$ if and only if $x = y$, (identity of indiscernibles)

(M2) $d(x, y) = d(y, x)$ for all $x, y \in X$, (symmetry)

(M3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. (triangle inequality)

A set X equipped with a metric d is called a **metric space**. We will often denote the resulting metric space by (X, d). If there is no ambiguity we will refer to the metric space as just X.

The most basic example of a metric space is the real line $\mathbb{R}$ with the metric $d_{\mathrm{Euc}}(x, y) = |x - y|$.

Fact. The function $d_{\mathrm{Euc}}\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ given by $d_{\mathrm{Euc}}(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ is a metric, called the **Euclidean metric**, on $\mathbb{R}^n$.[1]

Proof. To show that d_{Euc} is a metric, we must verify properties (M1)–(M3). For convenience, we let $d = d_{\mathrm{Euc}}$ for the proof.

We first notice that $d(\vec{x}, \vec{y}) = 0$ if and only if $\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = 0$ which, since each term is positive, occurs if and only if $(x_i - y_i)^2 = 0$ for all i, which in turn occurs if and only if $x_i = y_i$ for all i which is true if and only if $\vec{x} = \vec{y}$. This establishes property (M1).

If $\vec{x}, \vec{y} \in \mathbb{R}^n$, then

$$d(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} = d(\vec{y}, \vec{x})$$

which establishes property (M2).

In order to establish the triangle inequality (M3) it suffices to show that

$$(d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z}))^2 \geq (d(\vec{x}, \vec{z}))^2$$

for all $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$, since both sides are positive.

It will be easiest to use the language of vectors. Notice that

$$d(\vec{x}, \vec{y}) = \sqrt{(\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y})} = ||\vec{x} - \vec{y}||.$$

We then compute that

$$(d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z}))^2 = (||\vec{x} - \vec{y}|| + ||\vec{y} - \vec{z}||)^2 = ||\vec{x} - \vec{y}||^2 + 2||\vec{x} - \vec{y}||\ ||\vec{y} - \vec{z}|| + ||\vec{y} - \vec{z}||^2,$$

[1] When we discuss $\mathbb{R}^n$ without specifying a metric, we always mean the Euclidean metric.

while

$$
\begin{aligned}
(d(\vec{x},\vec{z}))^2 &= ||\vec{x}-\vec{z}||^2 = (\vec{x}-\vec{z})\cdot(\vec{x}-\vec{z}) \\
&= ((\vec{x}-\vec{y})+(\vec{y}-\vec{z}))\cdot((\vec{x}-\vec{y})+(\vec{y}-\vec{z})) \\
&= (\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})+2(\vec{x}-\vec{y})\cdot(\vec{y}-\vec{z})+(\vec{y}-\vec{z})\cdot(\vec{y}-\vec{z}) \\
&= ||\vec{x}-\vec{y}||^2+2||\vec{x}-\vec{y}||\ ||\vec{y}-\vec{z}||\cos\theta+||\vec{y}-\vec{z}||^2
\end{aligned}
$$

where θ is the angle between the vectors $\vec{x}-\vec{y}$ and $\vec{y}-\vec{z}$. (Recall that if $\vec{a},\vec{b}\in\mathbb{R}^n$, then $\vec{a}\cdot\vec{b}=||a||||b||\cos\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$.)

Since $2||\vec{x}-\vec{y}||\ ||\vec{y}-\vec{z}|| \geq 2||\vec{x}-\vec{y}||\ ||\vec{y}-\vec{z}||\cos\theta$, we conclude that

$$(d(\vec{x},\vec{y})+d(\vec{y},\vec{z}))^2 \geq (d(\vec{x},\vec{z}))^2$$

which completes the proof of (M3) and the proof that d is a metric. $\square$

Math 490 Handout: Tuesday September 6, 2016

In-class exercises. Determine whether the following functions give metrics on the indicated sets. Prove your claim!

- Let X be any set and let $d_0\colon X \times X \to [0,\infty)$ be given by $d_0(x,y) = 0$ if $x = y$ and $d_0(x,y) = 1$ if $x \neq y$.
- The function $d_1\colon \mathbb{R}^2 \times \mathbb{R}^2 \to [0,\infty)$ given by $d_1(\vec{x},\vec{y}) = |x_1 - y_1| + |x_2 - y_2|$.
- Let $X = \{\vec{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$ be the unit circle in $\mathbb{R}^2$. Let $d_2(\vec{x},\vec{y})$ be the length of the shortest arc in the circle joining $\vec{x}$ to $\vec{y}$.
- Let $\mathcal{C}([a,b])$ denote the set of all continuous functions from $[a,b]$ to $\mathbb{R}$. Let $d_{\mathcal{C}}\colon \mathcal{C}([a,b]) \times \mathcal{C}([a,b]) \to [0,\infty)$ be given by
$$d_{\mathcal{C}}(f,g) = \int_a^b |f(x) - g(x)| dx.$$

Explore. Additional questions for in-class.

- What does a circle look like in the d_1 metric?
- Let S be any shape in $\mathbb{R}^2$ containing $\vec{0}$ and let B be the boundary of S. Define a distance d_S by setting $d_S(\vec{x},\vec{x}) = 0$ and, for $\vec{x} \neq \vec{y}$, translate S to be centered at $\vec{x}$ and dilate S by $\lambda > 0$ until $\vec{y} \in \lambda B$.
  - What is the unit ball in the d_S-distance?
  - What conditions on S are necessary for d_S to satisfy (M1), (M2), and (M3)?

Individual Homework. Due Thursday September 8. Students will present solutions to select problems at the beginning of class on Thursday.

Prove (EDIT) or disprove that the following give metric spaces, unless otherwise noted.

1. The function $d_1\colon \mathbb{R} \times \mathbb{R} \to [0,\infty)$ given by $d_1(x,y) = (x-y)^2$.
2. Let $d_\infty\colon \mathbb{R}^2 \times \mathbb{R}^2 \to [0,\infty)$ be given by
$$d_\infty(\vec{x},\vec{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$
Prove that d_∞ is a metric and describe the unit circle in $(\mathbb{R}^2, d_\infty)$.
3. Let $\mathcal{C}([a,b])$ denote the set of all continuous functions from $[a,b]$ to $\mathbb{R}$. Let $d_{\sup}\colon \mathcal{C}([a,b]) \times \mathcal{C}([a,b]) \to [0,\infty)$ be given by
$$d_{\sup}(f,g) = \sup\{|f(x) - g(x)| \mid x \in [a,b]\}.$$
4. Let $X = \{\vec{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$ be the unit circle in $\mathbb{R}^2$. Let $d(\vec{x},\vec{y})$ be the length of the shortest arc in the circle joining $\vec{x}$ to $\vec{y}$.
5. (Optional) Find the distance function $d_{\hexagon}$ on $\mathbb{R}^2$ such that the unit $d_{\hexagon}$-circle is the hexagon with vertices $(1,0)$, $(0,-1)$, $(-1,-1)$, $(-1,0)$, $(0,1)$. Verify that this distance function is a metric.

Math 490 Handout: Thursday September 8, 2016

Definition: If (X, d) is a metric space, $\epsilon > 0$ and $x_0 \in X$, then the **open ball of radius** ϵ about x_0 is defined to be

$$B(x_0, \epsilon) = \{x \in X \mid d(x, x_0) < \epsilon.\}$$

A subset U of X is said to be **open** if for all $x \in U$, there exists $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subset U$.

Fact: The interval $(-1, 1)$ is open in $\mathbb{R}$ (with the Euclidean metric).

Proof. Suppose $x \in (-1, 1)$. Let $\epsilon_x = 1 - |x|$.

We will show that $B(x, \epsilon_x) \subset (-1, 1)$. If $y \in B(x, \epsilon_x)$, then $|y - x| < \epsilon_x$. By the triangle inequality,

$$|y| \leq |x| + |y - x| < |x| + \epsilon_x = |x| + (1 - |x|) = 1,$$

so $y \in (-1, 1)$. Therefore, $B(x, \epsilon_x) \subset (-1, 1)$.

Since there is an open ball of positive radius about every point in $(-1, 1)$ which is contained in $(-1, 1)$, we conclude $(-1, 1)$ is open in $\mathbb{R}$. $\square$

Fact: $[0, 1)$ is **not** an open subset of $\mathbb{R}$ (with the Euclidean metric).

Proof. If $\epsilon > 0$, then $B(0, \epsilon) = (-\epsilon, \epsilon)$ is not contained in $[0, 1)$, since $[0, 1)$ does not contain the point $-\epsilon/2$. Therefore no open ball about 0 is contained in $[0, 1)$ which implies that $[0, 1)$ is not open. $\square$

In-class Exercises:

- If X is a metric space, $x_0 \in X$ and $\epsilon > 0$, then $B(x_0, \epsilon)$ is an open subset of X.
- Are the following subsets of $\mathbb{R}^2$ open with respect to the Euclidean metric?
  - $(0, 1) \times (0, 1) = \{(x, y) \in \mathbb{R}^2 \mid x, y \in (0, 1)\}$.
  - $[0, 1] \times [0, 1] = \{(x, y) \in \mathbb{R}^2 \mid x, y \in [0, 1]\}$.
- If (X, d) is a metric space, then
  - The empty set $\varnothing$ and X are open.
  - If $\{U_\alpha\}_{\alpha \in \Lambda}$ is a collection of open sets, then $\bigcup_{\alpha \in \Lambda} U_\alpha$ is an open set.
  - If $\{U_1, \ldots, U_n\}$ is a finite collection of open sets, then $\bigcap_{i=1}^n U_i$ is open.
- Show that the intersection of an infinite collection of open sets need not be open.
- Every interval of the form (a, b) is open in $\mathbb{R}$ (even if $a = -\infty$ and/or $b = +\infty$).

Math 490 Individual homework: Due Tuesday September 13

1. Prove that if x and y are distinct points in a metric space (X, d), then there exist disjoint open sets U and V in X such that $x \in U$ and $y \in V$.

   (Remark: this property is called Hausdorff, which we will work with more later on.)

2. We say that a subset C of a metric space X is **closed** if its complement $X \setminus C$ is open.

   Prove that if $x_0 \in X$ and $\epsilon \geq 0$, then

   $$D(x_0, \epsilon) = \{x \in X \mid d(x, x_0) \leq \epsilon\}$$

   is a closed set. We call this the closed ball of radius ϵ about x_0.

3. Prove that:

   (a) The empty set $\varnothing$ and X are closed.

   (b) If $\{C_1, \ldots, C_n\}$ is a finite collection of closed sets, then $\bigcup_{i=1}^{n} C_i$ is closed.

   (c) If $\{C_\alpha\}_{\alpha \in \Lambda}$ is a collection of closed sets, then $\bigcap_{\alpha \in \Lambda} C_\alpha$ is a closed set.

   (Hint: You may use of DeMorgan's Laws below. You need not prove DeMorgan's Laws.)

   DeMorgan's Laws: Let X be a set and let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a collection of subsets of X.

   (1) $X \setminus \bigcup_{\alpha \in \Lambda} A_\alpha = \bigcap_{\alpha \in \Lambda} (X \setminus A_\alpha)$

   (2) $X \setminus \bigcap_{\alpha \in \Lambda} A_\alpha = \bigcup_{\alpha \in \Lambda} (X \setminus A_\alpha)$

4. Let $\mathcal{C}([a, b])$ denote the set of all continuous functions from $[a, b]$ to $\mathbb{R}$. Let $d_\infty : \mathcal{C}([a, b]) \times \mathcal{C}([a, b]) \to [0, \infty)$ be given by

   $$d_\infty(f, g) = \sup\{|f(x) - g(x)| \mid x \in [a, b]\}.$$

   Prove that d_∞ is a metric on $\mathcal{C}([a, b])$. (You may assume basic facts from real analysis.)

Math 490 Handout: Tuesday September 13, 2016

Definition: A function $f\colon (X_1, d_1) \to (X_2, d_2)$ is **continuous at** $x \in X_1$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $y \in X_1$ and $d_1(x, y) < \delta$, then $d_2(f(x), f(y)) < \epsilon$.

A function $f\colon X_1 \to X_2$ is said to be **continuous** if it is continuous at every point in X_1.

0. Rewrite the definition of continuity of f at x_1 using metric balls.

1. Show that if $f\colon X_1 \to X_2$ is continuous and $U \subset X_2$ is open in X_2, then $f^{-1}(U)$ is open in X_1.

2. Suppose that $f\colon X_1 \to X_2$ is a function between metric spaces and whenever $U \subset X_2$ is open in X_2, then $f^{-1}(U)$ is open in X_1. Show that f is continuous.

Combining (1) and (2) we have established:

Fact: Let $f\colon X_1 \to X_2$ be a function between metric spaces. Then f is continuous if and only if whenever $U \subset X_2$ is open in X_2, then $f^{-1}(U)$ is open in X_1.

When X_1, X_2 are not metric spaces, but are endowed with a notion of openness, the equivalent characterization of continuity stated above is appropriate.

Definition: A function $f\colon X_1 \to X_2$ is called **open** if $f(U)$ is open in X_2 whenever U is open in X_1.

3. Does there exist a continuous function which is not open?

4. Does there exist an open function which is not continuous?

5. For $f\colon X_1 \to X_2$ such that X_1, X_2 are metric spaces, define openness at $x_1 \in X_1$ using metric balls.

6. Exhibit a function which is neither open nor continuous at any point.

Math 490 Individual homework: Due Thursday September 15, 2016

1. Let (X, d) be a metric space. Show that if $\{U_\alpha\}_{\alpha \in \Lambda}$ is a collection of open subsets of X, then $\bigcup_{\alpha \in \Lambda} U_\alpha$ is open. (You should only use the definition of openness for a metric space, not anything proven in HW or class.)

2. Show that a countably infinite union of closed sets need not be closed with an explicit counterexample.

3. Show that if $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous functions between metric spaces, then $g \circ f\colon X \to Z$ is continuous. (Hint: Use our new characterization of continuity in terms of open sets. Notice how much simpler this is than using the $\delta - \epsilon$ criterion.)

4. Show that if $f\colon X_1 \to X_2$ is continuous and $C \subset X_2$ is closed in X_2, then $f^{-1}(C)$ is closed in X_1.

Math 490 Handout: Thursday September 15, 2016

Definition: A sequence $\{x_n\}$ of points in a metric space (X, d) is said to **converge** to $x_\infty \in X$ if for all $\epsilon > 0$, there exists N such that if $n \geq N$, then $d(x_n, x_\infty) < \epsilon$ for all $n \geq N$. We say that $\lim x_n = x_\infty$.

We may reformulate this definition in the following manner:

A sequence $\{x_n\}$ of points in a metric space (X, d) converges to x_∞ if and only if for all $\epsilon > 0$, there exists N such that if $n \geq N$, then $x_n \in B(x_\infty, \epsilon)$.

In-class Exercises:

1. A sequence $\{x_n\}$ of points in a metric space (X, d) converges to x_∞ if and only if for any open set U in X which contains x_∞, there exists N such that if $n \geq N$, then $x_n \in U$.
2. If $f\colon (X, d_X) \to (Y, d_Y)$ is continuous and $\{x_n\}$ is a sequence in X converging to x_∞, i.e. $\lim x_n = x_\infty$, then $\{f(x_n)\}$ converges to $f(x_\infty)$, i.e. $\lim f(x_n) = f(x_\infty)$.
3. Suppose that $f\colon X \to Y$ is a function between metric spaces which is not continuous at a point $x \in X$. Prove that there exists a sequence $\{x_n\}$ in X which converges to x such that $\{f(x_n)\}$ does not converge to $f(x)$.
4. Explain why we have proven the following criterion for continuity:

Fact: Suppose that $f\colon X \to Y$ is a function between metric spaces. Then f is continuous if and only if whenever $\{x_n\}$ is a convergent sequence in X, then $\{f(x_n)\}$ is convergent in Y and $\lim f(x_n) = f(\lim x_n)$.

More exercises:

- Consider the following statement:

  A sequence $\{x_n\}$ of points in a metric space (X, d) converges to x_∞ if and only if for any open set U in X which contains x_∞, for all N there exists $n \geq N$ such that $x_n \in U$.

  Which (if any) direction(s) of the statement are true? Which (if any) are false? Justify.
- Suppose $f\colon X \to Y$ is a continuous function between metric spaces and $\{x_n\}$ diverges. What can we say about $\{f(x_n)\}$, if anything? Justify with counterexamples or by specifying which of 2. or 3. you are using to make a claim.

Math 490 Team homework: Due Tuesday September 20

1. Suppose that $f\colon X \to Y$ is a function between metric spaces which is not continuous at a point $x \in X$. Prove that there exists a sequence $\{x_n\}$ in X which converges to x such that $\{f(x_n)\}$ does not converge to $f(x)$.

2. Suppose that C is a closed subset of a metric space X and $\{x_n\}$ is a sequence of points converging to $x_\infty \in X$ and $x_n \in C$ for all n. Prove that $x_\infty \in C$.

3. **Definition:** A sequence $\{x_n\}$ in a metric space X is said to be a **Cauchy Sequence** if for all $\epsilon > 0$ there exists N such that $n, m \geq N$ implies that $d(x_n, x_m) < \epsilon$.

   Prove that every convergent sequence in a metric space is a Cauchy sequence.

4. Recall the discrete metric, defined on any set X by $d(x, x) = 0$ and for $x \neq y$, $d(x, y) = 1$. Prove that every subset of X with the discrete metric is open.

5. a) Exhibit a continuous function which is not open. (Prove both your claims)

   b) Exhibit an open function which is not continuous. (Prove both your claims)

   Hint: The discrete metric could be useful here.

Math 490 Handout: Tuesday September 20, 2015

Definition: Suppose $A \subset (X, d)$. We say that $U_A \subset A$ is **open in** A if and only if there exists a set U open in X such that $U_A = U \cap A$.

1. Show that $[a, b)$ is open in $[a, b] \subset (\mathbb{R}, d_{\text{Euc}})$.
2. Suppose $f\colon (X, d_X) \to (Y, d_Y)$ is continuous. Show that the restriction $f\big|_A\colon A \to Y$ is continuous for any subset A of X. (Use the characterization of continuity by preimages of open sets.)
3. Prove that if $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous functions of metric spaces, and f is not surjective, then the composition function $g \circ f\colon X \to Z$ is continuous.

Definition: If x is a point in a metric space X, we say that U is an **open neighborhood** of x in X if U is open in X and $x \in U$. If A is a subset of a metric space X, then $x \in \bar{A}$ if whenever U is an open neighborhood of x, then

$$U \cap A \neq \varnothing.$$

(More informally, $x \in \bar{A}$ if and only if every open neighborhood of x intersects A.)

4. Prove that if A is a subset of X, then $x \in \bar{A}$ if and only if for all $\epsilon > 0$,

$$B(x, \epsilon) \cap A \neq \varnothing.$$

5. Prove that $\bar{A}$ is closed. ($\bar{A}$ is called the **closure** of A.)
6. Suppose that $C \subset X$ is closed in X and $\{x_n\}$ is a sequence of points in C which converges in X. Then $\lim x_n \in C$.
7. Suppose that $A \subset X$ and $\{x_n\}$ is a sequence of points in A which converges in X, then $\lim x_n \in \bar{A}$.

Math 490 Individual HW: Due Thursday September 22

Definition: If A is a subset of a metric space X, we say that $a \in A^0$ if and only if there is an open neighborhood U of a so that $U \subset A$. A^0 is called the **interior** of A.

A sequence $\{x_n\}$ in a metric space X is said to be **bounded** if there exists $y \in X$ and $R > 0$ so that $d(y, x_n) \leq R$ for all $n \in \mathbf{N}$.

1. Prove that if A is a subset of a metric space X, then A^0 is open in X.
2. Prove that if A is a subset of a metric space X, U is open in X and $U \subset A$, then $U \subset A^0$. (Note that Exercises 1. and 2. prove that A^0 is the largest open subset of A.)
3. Prove that $\bar{A}$ is closed.
4. Prove that every convergent sequence in a metric space is bounded.

Math 490 Handout: Thursday September 24, 2012

Recall that $f\colon X \to Y$ is **surjective** if the image of f is the entire codomain, $\mathrm{Im}(f) = Y$.

Remark: Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous. Then to argue that the composition $g \circ f$ is continuous, it suffices to assume f is surjective.

Argument. For any $A \subset Y$ on which we restrict g and any other subset $B \subset Z$, we can write the preimage

$$g^{-1}\big|_A(B) = \{x \in A \mid g(x) \in B\} = \{x \in X \mid g(x) \in B\} \cap A = g^{-1}(B) \cap A$$

Then if B is open in Z then $g^{-1}(B)$ is open in X by continuity, hence $g^{-1}(B) \cap A$ is open in A. We have shown that $g\big|_A$ is continuous if g is continuous.

Now, note that $f\colon X \to Y$ being continuous clearly implies $f\colon X \to \mathrm{Im}(f)$ is continuous. Then $g \circ f$ is continuous if and only if $g\big|_{\mathrm{Im}(f)} \circ f$ is continuous. So we can treat f to be surjective by restricting g to the image of f in Y and the result follows since the restriction of g is still continuous. □

Recall from last class:

Definition: If x is a point in a metric space X, we say that U is an **open neighborhood** of x in X if U is open in X and $x \in U$. If A is a subset of a metric space X, then $x \in \bar{A}$ if whenever U is an open neighborhood of x, then

$$U \cap A \neq \varnothing.$$

(More informally, $x \in \bar{A}$ if and only if every open neighborhood of x intersects A.)

Last class we proved we could equivalently say that $x \in \bar{A}$ if and only if for all $\epsilon > 0$,

$$B(x, \epsilon) \cap A \neq \varnothing.$$

In-class Exercises:

1. Prove that if $X = \mathbb{R}$ with the Euclidean metric and $A = (0, 1)$, then $\bar{A} = [0, 1]$.
2. Suppose that $A \subset X$ and $\{x_n\}$ is a sequence of points in A which converges in X, then $\lim x_n \in \bar{A}$.
3. Prove that if C is closed and $A \subset C$, then $\bar{A} \subset C$. (Note that $\bar{A}$ is the smallest closed set containing A.)
4. What is the closure of the set $\mathbb{Q}$ of rational numbers in the real line? Prove your answer is correct.
5. What is the closure of any set A in X with the discrete metric? Prove it.

Math 490 Team homework: Due Tuesday September 27

1. Suppose that A is a subset of a metric space X.

   (a) Prove that A is closed if and only if $A = \bar{A}$.

   (b) Prove that A is open if and only if $A = A^0$.

2. Show that for all $\{x_n\}$ convergent to x_∞ in (X, d), if $\{x_n\} \subset C$ implies $x_\infty \in C$, then C is closed.

   (Note that, combining with Team HW due 9/20, we now have a characterization of closed sets by limit points of sequences.)

3. Suppose that A is a subset of a metric space X. Prove that $(X \setminus A)^0 = X \setminus \bar{A}$.

4. **Definition:** If A is a subset of a metric space X we define the boundary $\partial(A)$ of A to be

   $$\partial(A) = \bar{A} \cap \overline{X - A}.$$

   Prove that A^0 and $\partial(A)$ are disjoint and that $\bar{A} = A^0 \cup \partial(A)$.

5. Is it always the case that if X is a metric space, $x_0 \in X$ and $R > 0$, then

   $$\overline{B(x_0, R)} = D(x_0, R)?$$

   Either prove that this is the case or give a counterexample. (Recall that

   $$D(x_0, R) = \{x \in X \mid d(x, x_0) \leq R\}$$

   is the closed ball of radius R about x_0.)

Math 490 Handout: Tuesday September 27, 2015

Recall that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in a space X and

$$0 < j_1 < j_2 < \cdots < j_n < j_{n+1} < \cdots$$

is an infinite sequence of increasing integers, then $\{x_{j_n}\}$ is called a **subsequence** of $\{x_n\}$.

Definitions: A metric space X is said to be **sequentially compact** if every sequence in X has a subsequence which converges in X. Moreover, a subset C of a metric space X is said to be **sequentially compact** if every sequence contained in C has a convergent subsequence with limit in C.

Lastly, a subset C of a metric space X is **bounded** if there exists $x_0 \in X$ and $R > 0$ such that $C \subset D(x_0, R)$.

We recall the following important facts from real analysis:

Heine-Borel Theorem: Any closed and bounded interval $[a, b]$ in $\mathbf{R}$ is sequentially compact.

Corollary: Any closed and bounded subset of $\mathbf{R}$ is sequentially compact.

Proof of Corollary: Let C be a closed and bounded subset of $\mathbf{R}$. Since C is bounded it is contained in some closed and bounded interval $[a, b]$. Let $\{x_n\}$ be a sequence in C. Since $C \subset [a, b]$ and $[a, b]$ is sequentially compact, there is a convergent subsequence $\{x_{j_n}\}$ of $\{x_n\}$. Since C is closed and $\{x_{j_n}\}$ is a convergent sequence in C, $\lim x_{j_n} \in C$. Therefore, any sequence in C has a convergent subsequence with limit in C, so C is sequentially compact.

In-class Exercises:

- A closed subset C of a sequentially compact metric space X is sequentially compact.
- If $f\colon X \to Y$ is a continuous map between metric spaces and $C \subset X$ is sequentially compact, then $f(C)$ is sequentially compact.
- A sequentially compact subset C of a metric space X is bounded.

  (Note that in particular, a sequentially compact metric space is bounded.)
- Prove that any finite subset of a metric space is sequentially compact.
- Exhibit a closed and bounded subset of a metric space which is not sequentially compact.
- Show that if X is a sequentially compact metric space, then every Cauchy sequence in X converges.
- Do Cauchy sequences always converge?

Math 490 Individual homework: Due Thursday September 29

1. Prove that if a sequence $\{x_n\}$ in a metric space X converges to $x \in X$ and $\{x_{n_j}\}$ is a subsequence of $\{x_n\}$, then $\{x_{n_j}\}$ converges to x.
2. A sequentially compact subset C of a metric space X is closed in X.
3. Prove that if A and B are sequentially compact subsets of a metric space X, then $A \cap B$ is sequentially compact.
4. Show that if X is a sequentially compact metric space, then every Cauchy sequence in X converges.

Math 490 Handout: Thursday September 29, 2016

In-class Exercises:

- Prove that any finite subset of a metric space is sequentially compact.
- Prove that a sequentially compact subset C of a metric space X is bounded. (Note that in particular, a sequentially compact metric space is bounded.)
- Exhibit a closed and bounded subset of a metric space which is not sequentially compact.
- Prove that $C([0,1],\mathbb{R})$ is not sequentially compact with the metric

$$d_\infty(f,g) = \sup\{|f(x) - g(x)| \mid x \in [0,1]\}.$$

Definition: We say that a function $f\colon (X,d_X) \to (Y,d_Y)$ between metric spaces is a **homeomorphism** if it is a continuous bijection, whose inverse map $f^{-1}\colon (Y,d_Y) \to (X,d_X)$ is also continuous.

We say that two metrics d_1 and d_2 on a set X are **topologically equivalent** if U is open in (X,d_1) if and only if it is open in (X,d_2).

- Prove that $\mathbb{R}$ and $[0,1]$ with the Euclidean metrics are not homeomorphic.
- Prove that $\mathbb{R}$ and $(0,\infty)$ with the Euclidean metrics are homeomorphic.
- Prove that the Euclidean metric and the discrete metric are not topologically equivalent on $\mathbb{R}$.
- Prove that the Euclidean metric and the discrete metric are topologically equivalent on $\mathbb{Z}$.
- Prove that two metrics d_1 and d_2 on a set X are topologically equivalent if and only the identity map $I\colon (X,d_1) \to (X,d_2)$, given by $I(x) = x$ for all $x \in X$, is a homeomorphism.

Math 490 Team Homework due Tuesday October 4

1. Prove that $C([0,1],\mathbb{R})$ is not sequentially compact with the metric

$$d_\infty(f,g) = \sup\{|f(x) - g(x)| \mid x \in [0,1]\}.$$

2. If $f\colon X \to \mathbb{R}$ is continuous and $C \subset X$ is sequentially compact, then there exists $c \in C$ such that $f(c) = \sup f(C)$, i.e. f achieves its supremum on C.

3. **Definition:** A map $f\colon X \to Y$ between metric spaces is said to be **closed** if whenever C is a closed subset of X, then $f(C)$ is a closed subset of Y.

   Prove that if $f\colon X \to Y$ is continuous and X is sequentially compact, then f is a closed map.

4. Give an example of a closed map between metric spaces which is not continuous. Prove your example has the properties you claim.

5. Prove that there does not exist a continuous surjection $f\colon [0,1] \to \mathbb{R}$.

Math 490 Handout: Tuesday October 4, 2016

Definition: A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X with the following properties:

(T1) $\varnothing$ and X are elements of $\mathcal{T}$.

(T2) If $\{U_\alpha\}_{\alpha\in\Lambda}$ is a collection of elements of $\mathcal{T}$, then

$$\bigcup_{\alpha\in\Lambda} U_\alpha \in \mathcal{T}.$$

(T3) If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called the **open sets** in the topological space $(X, \mathcal{T})$. We will often leave the topology $\mathcal{T}$ implicit and simply refer to the topological space X.

We have already established the following fundamental result:

Fact: If (X, d) is a metric space, then the set $\mathcal{T}_d$ of open sets in (X, d) forms a topology on X.

In-class Exercises:

1. Exhibit all possible topologies on the set $X = \{0, 1, 2\}$. How many are there? (You need not prove that each one is a topology.)

2. Let X be a set and $\mathcal{T} = \{X, \varnothing\}$. Prove that $\mathcal{T}$ is a topology on X. This topology is called the **indiscrete topology.**

3. Let X be a set and let $\mathcal{T}$ denote the collection of all subsets of X. Prove that $\mathcal{T}$ is a topology on X. This topology is called the **discrete topology**.

4. Show that if X is a set with topology $\mathcal{T}$, then $V \in \mathcal{T}$ if and only if for all $x \in V$, there exists $U_x \in \mathcal{T}$ such that $x \in U_x \subset V$.

5. Let X be a set and let $\mathcal{T}$ consist of the emptyset $\varnothing$ and all subsets of X whose complement is finite i.e. $U \in \mathcal{T}$ if and only either $U = \varnothing$ or $X \setminus U$ is a finite set. Prove that $\mathcal{T}$ is a topology on X. ($\mathcal{T}$ is called the **finite complement topology** on X.)

6. **Definition:** A topology $\mathcal{T}$ on a set X is said to be **metrizable** if there exists a metric d on X such that $\mathcal{T}$ is the set $\mathcal{T}_d$ of open sets for the metric space (X, d).

   Prove that if X has at least two elements, then the discrete topology is metrizable, but the indiscrete topology is not metrizable. Do you think the finite complement topology is sometimes, always, or never metrizable? Prove your claim.

Math 490 Individual homework: Due Thursday October 6

1. Prove that the Euclidean metric and the discrete metric are topologically equivalent on $\mathbb{Z}$.

2. Prove that two metrics d_1 and d_2 on a set X are topologically equivalent if and only if the identify map $I\colon (X, d_1) \to (X, d_2)$, given by $I(x) = x$ for all $x \in X$, is a homeomorphism.

3. Let X be a set with a topology $\mathcal{T}$. We say that a subset C of X is **closed** if its complement $X \setminus C$ is open, i.e. $X \setminus C \in \mathcal{T}$.

   Prove that:

   (i) The emptyset $\varnothing$ and X are closed.

   (ii) If C and D closed sets, then $C \cup D$ is closed.

   (iii) If $\{C_\alpha\}_{\alpha\in\Lambda}$ is a collection of closed sets, then $\bigcap_{\alpha\in\Lambda} C_\alpha$ is a closed set.

   (Hint: You may make use of DeMorgan's Laws. You need not prove DeMorgan's Laws.)

4. We say that a topological space $(X, \mathcal{T})$ is **Hausdorff** if whenever x and y are distinct points in X, there exist disjoint open sets U and V so that $x \in U$ and $y \in V$.

   Prove:

   (a) If $\mathcal{T}$ is the discrete topology on a set X, then $(X, \mathcal{T})$ is Hausdorff.

   (b) The finite complement topology on $\mathbb{Z}$ is not Hausdorff. (Note that the finite complement topology is therefore not metrizable on $\mathbb{Z}$.)

Math 490 Handout: Thursday October 6, 2016

Recall that a topology $\mathcal{T}$ on a set X is said to be **metrizable** if there exists a metric d on X such that $\mathcal{T}$ is the set $\mathcal{T}_d$ of open sets for the metric space (X, d).

1. Prove that a metrizable topological space is Hausdorff.
2. Explain why an infinite space with the finite complement topology is not metrizable.

Definition: A function $f\colon (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ between topological spaces is said to be **continuous** if whenever $U \in \mathcal{T}_2$, then $f^{-1}(U) \in \mathcal{T}_1$. More informally, we say that a function between topological spaces is continuous if pre-images of open sets are open.

3. If $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are topological spaces and $f\colon (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ is a constant map, then f is continuous.
4. Let $(Y, \mathcal{T}_Y)$ be any topological space and X any set. Put a topology $\mathcal{T}_X$ on X which guarantees every function $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous. Prove your claim.
5. Let $(X, \mathcal{T}_X)$ be any topological space and Y any set. Put a topology $\mathcal{T}_Y$ on Y which guarantees every function $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous. Prove your claim.
6. We define a set to be **closed** in the topological space $(X, \mathcal{T})$ if $X \setminus C \in \mathcal{T}$.

   Prove that a function $f\colon (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ between topological spaces is continuous if whenever C is closed in X_2, then $f^{-1}(C)$ is closed in X_1.

Definition: If $\mathcal{T}$ is a topology on a set X, then $\mathcal{B}$ is a **basis** for $\mathcal{T}$ if $\mathcal{B} \subset \mathcal{T}$ and every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$. (By convention, we regard the empty set $\varnothing$ as the union of an empty collection of elements of $\mathcal{B}$.)

7. Suppose that (X, d) is a metric space. Observe that the set of open balls in X,

   $$\mathcal{B} = \{B(x, \epsilon) \mid x \in X,\ \epsilon > 0\}$$

   is a basis for $\mathcal{T}_d$.
8. Suppose that $f\colon X \to Y$ is a function between topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ and that $\mathcal{B}_Y$ is a basis for $\mathcal{T}_Y$. Prove that f is continuous if and only if whenever B is an element of $\mathcal{B}_Y$, then $f^{-1}(B)$ is open in X (i.e. $f^{-1}(B) \in \mathcal{T}_X$.)
9. Define a topology on $\mathbb{R}$ by letting $\varnothing, \mathbb{R}, [a, b) \in \mathcal{T}$ and letting all countable unions and finite intersections of half-open intervals $[a, b)$ also be in $\mathcal{T}$. Give a basis for this topology. Can you find a countable basis for this topology? What are some open and closed sets? Do you think this topology is Hausdorff? Metrizable?

Math 490 Team homework: Due Tuesday October 11

1. Let (X, d_X) and (Y, d_Y) be metric spaces. We say that a sequence $\{f_n \colon X \to Y\}$ of continuous functions **converges uniformly** to a function $f \colon X \to Y$ if for any $\epsilon > 0$, there exists N such that if $n \geq N$ and $x \in X$, then $d(f_n(x), f(x)) < \epsilon$.

   Prove that if a sequence $\{f_n \colon X \to Y\}$ of continuous functions converges uniformly to $f \colon X \to Y$, then f is continuous.

2. Exhibit all possible topologies on the set $X = \{0, 1, 2\}$. (You need not prove that each one is a topology.)

3. Let $\mathcal{B}$ be the set of open balls with rational radius about rational points in $\mathbb{R}$, i.e. $\mathcal{B} = \{B(x, \epsilon) \mid \epsilon \in \mathbb{Q} \cap (0, \infty), \quad x \in \mathbb{Q}\}$.

   Prove that $\mathcal{B}$ is a basis for the Euclidean topology on $\mathbb{R}$, i.e. the topology induced by the Euclidean metric.

   (You may use, without proof, the density of the rational numbers within the real numbers.)

4. The **lower limit topology** on $\mathbb{R}$ is defined by letting $U \in \mathcal{T}$ if and only if for all $x \in U$, there exists $\epsilon > 0$ such that $[x, x + \epsilon) \subset U$.

   (a) Show that $\mathcal{T}$ is a topology on $\mathbb{R}$.

   (b) For $a < b$, show that $[a, b)$ is both open and closed in the lower limit topology.

   (c) Suppose $f \colon (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \text{Euclidean})$ is continuous when $\mathcal{T}$ is the standard Euclidean topology. Is f necessarily continuous when $\mathcal{T}$ is the lower limit topology? Prove or give a counterexample.

Math 490 Handout: Tuesday October 11, 2016

Definition: If $(X, \mathcal{T})$ is a topological space and $A \subset X$, then the **subspace topology** on A is given by $\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$. More informally, we say that a set V is open in the subspace topology on A if and only if there exists an open set U in X such that $V = A \cap U$.

In-class Exercises:

1. Prove that if $(X, \mathcal{T})$ is a topological space and $A \subset X$, then the subspace topology is a topology on A.
2. Prove that $D \subset A$ is closed in the subspace topology on A if and only if there exists a closed subset C of X such that $D = C \cap A$.
3. Suppose that $(X, \mathcal{T})$ is a topological space, $A \subset X$ and $\mathcal{T}_A$ is the subspace topology on A.

   Prove that if $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\mathcal{B}_A = \{B \cap A \mid B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_A$.
4. Suppose that (X, d) is a metric space and $A \subset X$. Let $d_A = d|_{A \times A}$.

   Prove that d_A is a metric on A and the subspace topology $(\mathcal{T}_d)_A$ on A associated to $\mathcal{T}_d$ is the same as $\mathcal{T}_{d_A}$.

Definition: A topological space $(X, \mathcal{T})$ is said to be T_1 if for any distinct points $x, y \in X$, there is a neighborhood of x which does not contain y and vice versa.

5. Observe that Hausdorff implies T_1. The Hausdorff property is sometimes called T_2.
6. Prove that $(X, \mathcal{T})$ is T_1 if and only if singletons are closed.
7. Prove that Hausdorff is a stronger property than T_1, ie; $T_1 \not\Rightarrow T_2$.
8. Construct a topological space with infinitely many open sets which is not T_1. Verify the construction satisfies the definition of a topology and fails the T_1 property.

Math 490 Individual Homework: due Thursday October 13, 2016

1. Suppose that $(X, \mathcal{T})$ is a topological space and $A \subset X$ is given the subspace topology.
   (a) Prove that if U is open in A and A is open in X, then U is open in X.
   (b) Give an example where U is open in A, but U is not open in X. (Here, A cannot be open in X.)
2. Prove that $D \subset A$ is closed in the subspace topology on A if and only if there exists a closed subset C of X such that $D = C \cap A$.
3. Suppose that $(X, \mathcal{T})$ is a topological space, $A \subset X$ and $\mathcal{T}_A$ is the subspace topology on A.

   Prove that if $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\mathcal{B}_A = \{B \cap A \mid B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_A$.
4. Let X be an infinite set and let $\mathcal{T}$ be the finite-complement topology on X.

   Prove that if A is any finite subset of X, then the subspace topology on A agrees with the discrete topology on A.

Math 490 Handout: Thursday October 13, 2016

Definition: A topological space $(X, \mathcal{T})$ is **disconnected** if there exist disjoint non-empty open subsets A and B of X such that $X = A \cup B$. If X is not disconnected, X is said to be **connected**.

A subset A of X is said to be connected if A is connected in the subspace topology, i.e. $(A, \mathcal{T}_A)$ is connected. Equivalently, a subset A of X is connected if and only if there do not exist open sets U and V in X such that $A \subset U \cup V$, $(U \cap V) \cap A = \varnothing$, and $A \cap U$ and $A \cap V$ are both non-empty.

We recall the Intermediate Value Theorem from Real Analysis:

Intermediate Value Theorem: *If J is an interval in $\mathbb{R}$, $f\colon J \to \mathbb{R}$ is continuous, $a, b \in J$ and d lies between $f(a)$ and $f(b)$ (i.e. either $f(a) < d < f(b)$ or $f(b) < d < f(a)$), then there exists c between a and b so that $f(c) = d$.*

In-class Exercises:

1. Suppose X is a set and $\mathcal{T}$ is the indiscrete topology.

   Prove that $(X, \mathcal{T})$ is connected.

2. Suppose that X is a set with more than one point and $\mathcal{T}$ is the discrete topology.

   Prove that $(X, \mathcal{T})$ is disconnected.

3. Prove that a topological space $(X, \mathcal{T})$ is **disconnected** if and only if there exists a continuous onto function $f\colon X \to \{0, 1\}$ (where $\{0, 1\}$ is given the discrete topology).

4. Prove that any interval in $\mathbb{R}$ is connected. (Hint: You may use the intermediate value theorem.)

5. Prove that any subset of $\mathbb{R}$ which is not an interval is disconnected. (Hint: A subset $A \subset \mathbb{R}$ is an interval if and only if whenever $x, y \in A$ and z lies between x and y, then $z \in A$.)

6. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces.

   Prove that if $f\colon X \to Y$ is continuous and X is connected, then $f(X)$ is connected.

7. Is the lower limit topology on $\mathbb{R}$ connected or disconnected?

8. There are 29 topologies on a set with three elements. Are most of those topologies connected or disconnected?

9. Is the finite completement topology sometimes, always, or never connected? Find necessary and sufficient conditions.

Math 490 Individual Homework: due Thursday October 20, 2016

1. Suppose that (X, d) is a metric space and $A \subset X$. Let $d_A = d|_{A \times A}$, the metric d restricted to pairs of points in A.

   Prove that d_A is a metric on A and the subspace topology $(\mathcal{T}_d)_A$ on A associated to $\mathcal{T}_d$ is the same as $\mathcal{T}_{d_A}$ (i.e.; $(\mathcal{T}_d)_A = \mathcal{T}_{d_A}$).

2. Prove that any subset of $\mathbb{R}$ which is not an interval is disconnected. (Hint: A subset $A \subset \mathbb{R}$ is an interval if and only if whenever $x, y \in A$ and z lies between x and y, then $z \in A$.)

3. Let X be an infinite set and let $\mathcal{T}$ be the finite complement topology.

   Prove that $(X, \mathcal{T})$ is connected.

4. Suppose that $(X, \mathcal{T})$ is a topological space.

   Prove that if A and B are connected subsets of X and $A \cap B \neq \varnothing$, then $A \cup B$ is connected.

5. **Definition:** If $(X, \mathcal{T})$ is a topological space and $x \in X$, then U is an **open neighborhood** of x if U is open in X and $x \in U$. If A is a subset of X, then we say that $x \in \bar{A}$ if and only if every open neighborhood of x intersects A. The set $\bar{A}$ is called the **closure** of A.

   Prove that $A \subset \bar{A}$ and $\bar{A}$ is closed.

Math 490 Handout: Thursday October 20, 2016

Definition: If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces then we say that $W \in \mathcal{T}_{X \times Y}$ if for each $(x, y) \in W$, there exists $U_{(x,y)} \in \mathcal{T}_X$ and $V_{(x,y)} \in \mathcal{T}_Y$ such that $(x, y) \in U_{(x,y)} \times V_{(x,y)} \subset W$.

$\mathcal{T}_{X \times Y}$ is called the **product topology** on $X \times Y$.

In-class Exercises:

1. Prove that if $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, then $\mathcal{T}_{X \times Y}$ is a topology on $X \times Y$ and that it has basis $\mathcal{B}_{X \times Y} = \{U \times V \mid U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y\}$.

2. Suppose that $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$ are topological spaces.

   Prove that if $f : X \to Y$ and $g : X \to Z$ are continuous functions, then the map $h : X \to Y \times Z$ given by $h(x) = (f(x), g(x))$ is continuous if we give $Y \times Z$ the product topology. (Hint: remember our characterization of continuity in terms of elements of the basis of the range.)

3. Suppose that (X, d_X) and (Y, d_Y) are metric spaces.

   a) Prove that $d_\infty : (X \times Y) \times (X \times Y) \to [0, \infty)$ given by $d_\infty\big((x, y), (x', y')\big) = \max\{d_X(x, x'), d_Y(y, y')\}$ is a metric on $X \times Y$.

   b) Prove that the topology $\mathcal{T}_{d_\infty}$ on $X \times Y$ agrees with the product topology associated to the topologies $\mathcal{T}_{d_X}$ and $\mathcal{T}_{d_Y}$ on X and Y.

4. What propoerties does the product topology preserve? In other words, if topological spaces X and Y have property A, does the product topology also have property A?

Math 490 Team homework: Due Tuesday October 25

1. Suppose that $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$ are topological spaces.

   Prove that if $f : X \to Y$ and $g : X \to Z$ are continuous functions, then the map $h : X \to Y \times Z$ given by $h(x) = (f(x), g(x))$ is continuous if we give $Y \times Z$ the product topology. (Hint: remember our characterization of continuity in terms of elements of the basis of the range.)

2. Prove that if $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are Hausdorff topological spaces, then $(X \times Y, \mathcal{T}_{X \times Y})$ is Hausdorff.

3. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces. Define the projection map $p_X : X \times Y \to X$ by $p_X(x, y) = x$.

   Prove that p_X is continuous if we give $X \times Y$ the product topology.

4. **Definition:** A function $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a **homeomorphism** if f is a continuous bijection whose inverse is also continuous.

   Prove that if $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a homeomorphism and $(Y, \mathcal{T}_Y)$ is Hausdorff, then $(X, \mathcal{T}_X)$ is Hausdorff.

Math 490 Handout: Tuesday November 1, 2016

In-class Exercises:

Definition: Suppose that $\{x_n\}$ is a sequence of points in a topological space $(X, \mathcal{T})$. We say that $\{x_n\}$ **converges** to $x \in X$, if given any open neighborhood U of x, there exists N such that $x_n \in U$ if $n \geq N$. (We recall that an open neighborhood of x is an open set containing x.)

1. If $(X, \mathcal{T})$ is Hausdorff, then a sequence in X converges to at most one point in X.
2. Exhibit a sequence in a topological space $(X, \mathcal{T})$ which converges to more than one point.
3. Let C be a closed subset of a topological space $(X, \mathcal{T})$. Suppose that $\{x_n\}$ is a sequence of points in C which converges to $x \in X$. Prove that $x \in C$.
4. Exhibit a continuous, onto function $f : X \to Y$ between topological spaces such that X is Hausdorff, but Y is not Hausdorff. This shows that images of Hausdorff spaces need not be Hausdorff.
5. Suppose that $\{A_\alpha\}_{\alpha \in \Lambda}$ is a collection of connected subsets of a topological space $(X, \mathcal{T})$.

   Prove that if $\bigcap_{\alpha \in \Lambda} A_\alpha$ is non-empty, then $\bigcup_{\alpha \in \Lambda} A_\alpha$ is connected.
6. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are connected topological spaces.

   Prove that if $(a, b) \in X \times Y$, then $(X \times \{b\}) \cup (\{a\} \times Y)$ is a connected subset of $X \times Y$ with the product topology.

Math 490 Individual Homework: due Thursday November 3, 2016

1. If $(X, \mathcal{T})$ is Hausdorff, then a sequence in X converges to at most one point in X.
2. Prove that if $(X, \mathcal{T})$ is Hausdorff then singletons, i.e. sets containing one element, are closed.
3. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces. Prove that the projection map $\pi_X \colon X \times Y \to X$ given by $\pi_X((x, y)) = x$ is open if $X \times Y$ is given the product topology.
4. Prove that if A is connected in a topological space $(X, \mathcal{T})$, then $\bar{A}$ is connected. (Prove this directly, not citing the result that A connected and $A \subset B \subset \bar{A}$ implies B connected).

Math 490 Handout: Thursday November 3, 2016

In-class Exercises:

1. Suppose that (X, d) is a metric space. Prove that if $C \neq \varnothing$ is closed and $x \notin C$, then $r = \inf\{d(x, c) \mid c \in C\}$ exists and is positive.

2. Construct a nontrivial closed set C in a metric space (X, d) so that $\inf\{d(x, c) \mid c \in C\}$ is not realized in X. (Hint: think about choosing X to be a special subset of $\mathbb{R}$ with the Euclidean metric).

3. Suppose that $\{A_\alpha\}_{\alpha \in \Lambda}$ is a collection of connected subsets of a topological space $(X, \mathcal{T})$.

   Prove that if $\bigcap_{\alpha \in \Lambda} A_\alpha$ is non-empty, then $\bigcup_{\alpha \in \Lambda} A_\alpha$ is connected.

4. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Prove that if $(X, \mathcal{T}_X)$ is connected, then $X \times \{b\}$ for any $b \in Y$ is connected in $X \times Y$ with the product topology.

5. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are connected topological spaces.

   Prove that if $(a, b) \in X \times Y$, then $(X \times \{b\}) \cup (\{a\} \times Y)$ is a connected subset of $X \times Y$ with the product topology.

6. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are connected topological spaces. Prove that $X \times Y$ is connected in the product topology.

Math 490 Group Homework: due Tuesday November 8, 2016

1. Let $(X, \mathcal{T})$ be a topological space, $A \subset X$. Prove that
   (a) $\bar{A}$ is the smallest closed set containing A.
   (b) For any sequence $\{x_n\} \subset A$, $\lim x_n \in \bar{A}$.

2. The **cocountable topology** on any set X contains $X, \varnothing$, and all subsets with countable complement.
   (a) Prove that the cocountable topology on $\mathbb{R}$ is not Hausdorff.
   (b) Prove that the only convergent sequences in the cocountable topology are eventually constant sequences.

3. A topological space $(X, \mathcal{T})$ is said to be T_1 if for all distinct $x, y \in X$, there exists an open neighborhood U of x such that $y \notin U$. Note that Hausdorff implies T_1 (the Hausdorff property is sometimes called T_2).

   Prove that T_1 does not imply the Hausdorff property.

4. A topological space $(X, \mathcal{T})$ is said to be **totally disconnected** if the only connected subsets are singletons and the empty set.
   (a) Prove that $\mathbb{R}$ with the lower limit topology is totally disconnected.
   (b) Prove that if $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are totally disconnected, then $X \times Y$ with the product topology is totally disconnected.

Math 490 Handout: Tuesday November 8, 2016

Definition: Let $f\colon X \to \mathbb{R}$ be a function. For any subset $A \subset X$, we say that $\inf\{f(x) \mid x \in A\}$ is **realized** if the infimum exists and there is a point $a \in A$ such that $f(a) = \inf\{f(x) \mid x \in A\}$. (Note that this notion can be made more general if we allow the infimum to be realized outside of A, but let's not worry about that today).

In-class Exercises:

1. Recall that if (X, d) is a metric space and $C \subset X$ closed and nonempty, then for any $x \notin C$, we have that $r = \inf\{d(x, c) \mid c \in C\}$ exists and is positive.

   Find a nonempty closed set C in $\mathbb{R}$ minus a point, and an $x \notin C$ so that $r = \inf\{d(x, c) \mid c \in C\}$ is not realized.

2. Suppose that $\{A_\alpha\}_{\alpha\in\Lambda}$ is a collection of connected subsets of a topological space $(X, \mathcal{T})$.

   Prove that if $\bigcap_{\alpha\in\Lambda} A_\alpha$ is non-empty, then $\bigcup_{\alpha\in\Lambda} A_\alpha$ is connected.

3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Prove that if $(X, \mathcal{T}_X)$ is connected, then $X \times \{b\}$ for any $b \in Y$ is connected in $X \times Y$ with the product topology.

4. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are connected topological spaces.

   Prove that if $(a, b) \in X \times Y$, then $(X \times \{b\}) \cup (\{a\} \times Y)$ is a connected subset of $X \times Y$ with the product topology.

5. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are connected topological spaces. Prove that $X \times Y$ is connected in the product topology.

Math 490 Individual Homework: due Thursday November 10, 2016

PLEASE VOTE!!!! Here is a nice familiar homework so that you will have plenty of time to vote today.

1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Prove that if $(X, \mathcal{T}_X)$ is connected, then $X \times \{b\}$ for any $b \in Y$ is connected in $X \times Y$ with the product topology.

2. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are connected topological spaces.

   Prove that if $(a, b) \in X \times Y$, then $(X \times \{b\}) \cup (\{a\} \times Y)$ is a connected subset of $X \times Y$ with the product topology.

3. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are connected topological spaces. Prove that $X \times Y$ is connected in the product topology.

4. Prove that a topological space $(X, \mathcal{T})$ is T_1 if and only if singleton sets are closed.

Math 490 Handout: Thursday November 10, 2016

Definition: Let $(X, \mathcal{T})$ be a topological space. A collection $\{U_\alpha\}_{\alpha\in\Lambda}$ of open subsets of X is an **open cover** of X if $X = \bigcup_{\alpha\in\Lambda} U_\alpha$, i.e. every point in X lies in some U_α.

A sub-collection $\{U_\alpha\}_{\alpha\in\Lambda_0}$ (where $\Lambda_0 \subset \Lambda$) is a **sub-cover** if $X = \bigcup_{\alpha\in\Lambda_0} U_\alpha$, i.e. every point in X lies in some U_α where $\alpha \in \Lambda_0$.

The space $(X, \mathcal{T})$ is said to be **compact** if every open cover has a finite sub-cover, i.e. if $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover, then there are finitely many elements $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ in $\{U_\alpha\}_{\alpha\in\Lambda}$ which cover X.

For subsets: if A is a subset of a topological space we say that a collection $\{U_\alpha\}_{\alpha\in\Lambda}$ of open subsets of X is an **open cover** of A if $A \subset \bigcup_{\alpha\in\Lambda} U_\alpha$. Moreover, we say that A is **compact** if every open cover of A has a finite subcover, i.e. if $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover of A, then there are finitely many elements $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ in $\{U_\alpha\}_{\alpha\in\Lambda}$ such that $A \subset \bigcup_{i=1}^{n} U_{\alpha_i}$.

In-class Exercises:

1. Let $(X, \mathcal{T})$ be a topological space.

   Prove that if X is finite, then $(X, \mathcal{T})$ is compact.

2. Let X be a set and let $\mathcal{T}$ be the indiscrete topology.

   Prove that $(X, \mathcal{T})$ is compact.

3. Suppose that X is an infinite set and that $\mathcal{T}$ is the discrete topology.

   Prove that $(X, \mathcal{T})$ is not compact.

4. Prove that $\mathbb{R}$ is not compact.

5. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces.

   Prove that if $f: X \to Y$ is continuous and X is compact, then $f(X)$ is a compact subset of Y.

6. Suppose that C and D are compact subsets of a topological space $(X, \mathcal{T})$. Prove that $C \cup D$ is a compact subset of X.

7. Are there any other topologies we know which, on any set X, are compact? Are there any other topologies which are always noncompact?

8. Are compact subsets of topological spaces always closed? Consider some examples and prove your claims.

Math 490 Team homework: Due Tuesday November 15

1. Suppose that (X, d) is a metric space. Prove that if a subset A of X is compact, then A is bounded.

2. Let X be a set and let $\mathcal{T}$ be the finite complement topology.

   Prove that $(X, \mathcal{T})$ is compact.

3. Prove that a closed subset A of a compact topological space $(X, \mathcal{T})$ is compact.

4. Suppose that $(X, \mathcal{T})$ is a Hausdorff topological space.

   Prove that any compact subset A of X is closed.

Math 490 Handout: Tuesday November 15, 2016

Our goal for the week will be to prove the following theorem:

Theorem: *If (X, d) is a metric space, then X is compact if and only if it is sequentially compact.*

In-class Exercises:

1. Prove that a compact subset of $\mathbb{R}$ is sequentially compact.
2. Suppose that $\{x_n\}$ is a sequence in a metric space X which has no convergent subsequence.

   Prove that for all $x \in X$, there exists $\epsilon_x > 0$ such that $B(x, \epsilon_x)$ contains only finitely many elements in the sequence.
3. Prove that a compact metric space X is sequentially compact.
4. **Definition:** A subset A of a metric space X is said to be an **ϵ-net** if $\{B(a, \epsilon) \mid a \in A\}$ is an open cover of X.

   Prove that if $\epsilon > 0$, then every sequentially compact metric space has a finite ϵ-net.
5. **Definition:** If $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ is an open cover of a metric space X, then $\delta > 0$ is a **Lebesgue number** for $\mathcal{U}$ if for all $x \in X$, there exists $\alpha_x \in \Lambda$ such that $B(x, \delta) \subset U_{\alpha_x}$.

   Prove that if $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ is an open cover of a metric space (X, d) which does not have a positive Lebesgue number, then there exists a sequence $\{x_n\}$ in X, such that, for all n, $B(x_n, \frac{1}{n})$ is not contained in U_α for any $\alpha \in \Lambda$.
6. Is there a topological space which is not Hausdorff but for which compact sets are closed?
7. Prove for C, D compact subsets of a topological space $(X, \mathcal{T})$ that $C \cap D$ is not necessarily compact. Find conditions which guarantee that intersections of compact sets are compact.

Math 490 Individual homework: Due Thursday November 17

1. Suppose that C and D are compact subsets of a topological space $(X, \mathcal{T})$. Prove that $C \cup D$ is a compact subset of X.

2. Suppose that $(X, \mathcal{T})$ is a compact topological space.

   Prove that if $f : X \to \mathbb{R}$ is a continuous function, then there exists $z \in X$ such that $f(z) = \sup f(X)$, i.e. f achieves its supremum.

3. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces.

   Prove that if X is compact, Y is Hausdorff and $f : X \to Y$ is continuous, one-to-one and onto, then f is a homeomorphism. (Hint: First prove that f is a closed map, i.e. the image of any closed set is closed.)

4. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, and that $X \times Y$ is given the product topology $\mathcal{T}_{X \times Y}$.

   Prove that if $X \times Y$ is compact, then X is compact.

Math 490 Handout: Thursday November 17, 2016

On Tuesday, we proved that compact metric spaces are sequentially compact. Today we will prove the converse.

In-class Exercises:

1. **Definition:** A subset A of a metric space X is said to be an ϵ**-net** if $\{B(a, \epsilon) \mid a \in A\}$ is an open cover of X.

   Prove that if $\epsilon > 0$, then every sequentially compact metric space has a finite ϵ-net.

2. **Definition:** If $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ is an open cover of a metric space X, then $\delta > 0$ is a **Lebesgue number** for $\mathcal{U}$ if for all $x \in X$, there exists $\alpha_x \in \Lambda$ such that $B(x, \delta) \subset U_{\alpha_x}$.

   Prove that if $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ is an open cover of a metric space (X, d) which does not have a positive Lebesgue number, then there exists a sequence $\{x_n\}$ in X, such that, for all n, $B(x_n, \frac{1}{n})$ is not contained in U_α for any $\alpha \in \Lambda$.

3. Prove that any open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ of a sequentially compact metric space (X, d) has a positive Lebesgue number.

4. Prove that a sequentially compact metric space is compact.

5. We have proven in a Hausdorff space that compact implies closed. Find a topological space which is not Hausdorff but for which all compact sets are closed.

6. We say a topological space $(X, \mathcal{T})$ is **regular** if for all $C \subset X$ closed and $x \notin C$, then there exist disjoint $U, V \in \mathcal{T}$ such that $C \subset U$ and $x \in V$. Construct an example of a Hausdorff topological space which is not regular. (Hint: consider $\mathbb{R}$ with all Euclidean open sets and add sets of the form $(a, b) \setminus K$... choose K wisely.)

7. Prove for C, D compact subsets of a topological space $(X, \mathcal{T})$ that $C \cap D$ is not necessarily compact. Find conditions which guarantee that intersections of compact sets are compact.

Math 490 Team homework: Due Tuesday November 22, 2016

1. Prove that a compact, Hausdorff topological space is regular.

   (i.e.; Prove that if C is a closed subset of X and that $x \in X \setminus C$, then there exist disjoint open subsets U and V of X so that $x \in U$ and $C \subset V$.)

2. Let $(X, \mathcal{T})$ be a Hausdorff topological space.

   Prove that if C and D are compact subsets of X, then $C \cap D$ is compact.

3. Suppose that (X, d) is a compact metric space and $f\colon X \to X$ is an **isometry**, meaning $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

   Prove that f is surjective. (Hint: Suppose that $z \in X - f(X)$. Show that there exists $\epsilon > 0$ so that $d(z, y) \geq \epsilon$ for all $z \in f(X)$. Let $z_n = f^n(z)$ for all n and show that $d(z_n, z_m) \geq \epsilon$ for all $n \neq m$.)

4. Let $(X, \mathcal{T})$ be a compact topological space. Suppose that $\{C_\alpha\}_{\alpha \in \Lambda}$ is a collection of non-empty closed subsets of X. Show that if the intersection $C_{\alpha_1} \cap \cdots \cap C_{\alpha_n}$ of any finite subcollection of $\{C_\alpha\}_{\alpha \in \Lambda}$ is non-empty, then the total intersection $\bigcap_{\alpha \in \Lambda} C_\alpha$ is non-empty.

Math 490 Handout: Tuesday November 22, 2016

Recall: Last Thursday we proved

Lemma: *If (X,d) is a sequentially compact metric space, then for all $\epsilon > 0$, there exists a finite ϵ-net for X.*

Today we complete the proof that for a metric space, compactness and sequential compactness are equivalent.

1. **Definition:** If $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover of a metric space X, then $\delta > 0$ is a **Lebesgue number** for $\mathcal{U}$ if for all $x \in X$, there exists $\alpha_x \in \Lambda$ such that $B(x,\delta) \subset U_{\alpha_x}$.

   Prove that if $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover of a metric space (X,d) which does not have a positive Lebesgue number, then there exists a sequence $\{x_n\}$ in X, such that, for all n, $B(x_n, \frac{1}{n})$ is not contained in U_α for any $\alpha \in \Lambda$.

2. Prove that any open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ of a sequentially compact metric space (X,d) has a positive Lebesgue number.

3. Prove that a sequentially compact metric space is compact.

4. Recall that a topological space $(X,\mathcal{T})$ is **regular** if for all $C \subset X$ closed and $x \notin C$, then there exist disjoint $U, V \in \mathcal{T}$ such that $C \subset U$ and $x \in V$. Construct an example of a Hausdorff topological space which is not regular. (Hint: consider $\mathbb{R}$ with all Euclidean open sets and add sets of the form $(a,b) \setminus K \ldots$ choose K wisely.)

5. We have proven in a Hausdorff space that compact implies closed. Is there a topological space which is not Hausdorff but for which all compact sets are closed?

Math 490 Individual Homework: due Tuesday November 29, 2016

1. Let (X, d_X) and (Y, d_Y) be sequentially compact metric spaces. Prove that $X \times Y$ is sequentially compact with the metric

$$d_\infty((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2))\}.$$

   Conclude that $X \times Y$ is compact in the product topology.

2. **Definition:** Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f\colon (X, d_X) \to (Y, d_Y)$ is said to be **uniformly continuous** if given any $\epsilon > 0$ there exists $\delta > 0$ such that if $x_1, x_2 \in X$ and $d_X(x_1, x_2) < \delta$, then $d_Y(f(x_1), f(x_2)) < \epsilon$. (**Note**: δ depends only on ϵ, and not on the points in X!)

   Prove that if X is compact and $f\colon X \to Y$ is continuous, then f is uniformly continuous.

3. **Definition**: A topological space $(X, \mathcal{T})$ is **normal** if for each pair of disjoint closed sets A, B in X, there exist disjoint open sets U, V such that $A \subset U$ and $B \subset V$.

   Prove that metric spaces are normal.

Math 490 Handout: Tuesday November 29, 2016

1. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are compact topological spaces and that we give $X \times Y$ the product topology.

   Let $x_0 \in X$ and let N be an open set in $X \times Y$ which contains $\{x_0\} \times Y$. Prove that there exists an open neighborhood W of x_0 in X such that $W \times Y \subset N$.

2. Prove that if $\mathcal{U}$ is an open cover of $X \times Y$ and $x_0 \in X$, then there exist finitely many elements in $\mathcal{U}$ which cover $\{x_0\} \times Y$.

3. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are compact topological spaces.

   Prove that $X \times Y$ is compact in the product topology.

4. Suppose that (X, d_X) is a sequentially compact metric space and that $f\colon X \to X$ is a contraction, i.e. there exists $\alpha \in (0, 1)$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$. Prove that there exists a unique point $x_0 \in X$ such that $f(x_0) = x_0$.

   Remark: In fact, it suffices to assume that X is **complete**, i.e. that every Cauchy sequence in X is convergent, and that f is a contraction. This very useful fact is known as the Contraction Mapping Theorem or Banach Fixed Point Theorem. For example, it is used to prove existence and uniqueness theorems for solutions of differential equations.

Math 490 Individual homework: due Thursday December 1, 2016

1. **Definition**: We say that a subset A of a topological space $(X, \mathcal{T})$ is **dense** if $\bar{A} = X$.

   Let $(X, \mathcal{T})$ be an infinite space with the finite complement topology. Prove that any open set is dense.

2. **Definition**: Let $(X, \mathcal{T})$ be a topological space. A subset A of X is **perfect** if every point $a \in A$ is a limit of a sequence $\{a_n\}$ in A such that $a_n \neq a$ for all n.

   Prove that if (X, d) is a perfect metric space, then every open metric ball contains infinitely many points in X.

3. Prove that $\mathbb{R}$ with the Euclidean metric is perfect but $\mathbb{R}$ with the discrete metric is not perfect.

4. Let K be any subset of $\mathbb{R}$. Define a topology $\mathcal{T}_{K^*}$ on $\mathbb{R}$ such that $\mathcal{T}_{\text{Euc}} \subset \mathcal{T}_{K^*}$ and $(a, b) \setminus K \in \mathcal{T}_{K^*}$ for all $a < b \in \mathbb{R}$. (You do not need to prove that $\mathcal{T}_{K^*}$ is a topology.)

   Prove that if $K = \{\frac{1}{n}\}$ then $\mathcal{T}_{K^*}$ is not regular.

Math 490 Handout: Thursday December 1, 2016

Definition: Let $(X, \mathcal{T})$ be a topological space. A **path** from x to y in X is a continuous map $I\colon [a,b] \to X$ such that $I(a) = x$ and $I(b) = y$. Then X is said to be **path-connected** if every pair of points in X can be joined by a path in X. A subset A in X is path-connected if A is path-connected in the subspace topology.

1. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$, and $(Z, \mathcal{T}_Z)$ be topological spaces and $f\colon X \to Y$, $g\colon Y \to Z$ continuous functions. Prove that $g \circ f$ is continuous.
2. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f\colon X \to Y$ continuous. Prove that if X is path-connected then $f(X)$ is path-connected in Y.
3. Show that if there is a path from x to y and there is a path from y to z in $(X, \mathcal{T})$, then there is a path from x to z.
4. Prove that a path-connected topological space is connected.

   (**Remark**: it is NOT true that all connected spaces are path-connected!)
5. Prove that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.
6. If $A \subset X$ is path-connected, is $\bar{A}$ necessarily path-connected?
7. Suppose that (X, d_X) is a sequentially compact metric space and that $f\colon X \to X$ is a contraction, i.e. there exists $\alpha \in (0,1)$ such that $d(f(x), f(y)) \leq \alpha d(x,y)$ for all $x, y \in X$. Prove that there exists a unique point $x_0 \in X$ such that $f(x_0) = x_0$.

   (Hint: Find a candidate fixed point by proving the sequence $\{f^n(x)\}$ converges. Then use that f is continuous.)

Math 490 Team homework: Due Tuesday December 6, 2016

1. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are compact topological spaces.

   Prove that $X \times Y$ is compact in the product topology.

2. Let $p\colon X \to Y$ be a closed continuous surjective function of topological spaces $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$. Prove that if X is normal, then so is Y. (Hint: if U is an open set containing $p^{-1}(\{y\})$, find a neighborhood W of y such that $p^{-1}(W) \subset U$).

3. Let (X, d) be a metric space such that ϵ-balls are path-connected.

   (a) Prove that if U is an open connected subset of X, then U is path-connected. (Hint: Show that for $x_0 \in U$, the set of points that can be joined to x_0 by a path is both open and closed in U.)

   (b) Prove that ϵ-balls are path-connected in $\mathbb{R}^2$.

4. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces. Prove that $X \times Y$ with the product topology is path-connected if and only if X and Y are path-connected.

Countable spaces

Definition: A set A is **countable** if either A is finite or there is a bijection $f\colon A \to \mathbb{N}$ where $\mathbb{N}$ is the set of positive integers.

Facts: Subsets of countable sets are countable. Countable unions of countable sets are countable.

Corollary: $\mathbb{Q}^+ \leftrightarrow \{(p,q) \in \mathbb{N} \times \mathbb{N} \mid \mathrm{GCD}(p,q) = 1\} \subset \bigcup_{p\in\mathbb{N}} \left(\bigcup_{q\in\mathbb{N}}\{(p,q)\}\right)$ is countable so $\mathbb{Q} = \{0\} \cup \mathbb{Q}^+ \cup -\mathbb{Q}^+$ is countable.

Theorem: $\mathbb{R}$ is uncountable.

Proof. We will prove that $[0,1]$ is uncountable. This will suffice because $\mathbb{R}$ is homeomorphic to $(0,1)$ and $[0,1]$ has only 2 more points than $(0,1)$.

Note that it suffices to verify that any fuction $f\colon \mathbb{N} \to [0,1]$ is not surjective. Equivalently, any sequence $x_n = f(n)$ must miss a point in $[0,1]$. We will need the following:

Lemma: Given x in $\mathbb{R}$ and U open and nonempty in $\mathbb{R}$, there exists V open and nonempty such that $V \subset U$ and $x \notin \bar{V}$.

Take $U = (0,1)$, and $x = x_1$, then the lemma gives us a V_1 nonempty and open such that $x_1 \notin \bar{V}_1$ and $V_1 \subset U$. Then for x_2, we can choose $V_2 \subset V_1$ open and nonempty such that $x_2 \notin \bar{V}_2$. Repeat to get a sequence of nonempty open sets

$$V_1 \supset V_2 \supset \cdots \supset V_n \supset V_{n+1} \supset \cdots$$

such that $\{x_1, \ldots, x_n\} \cap \bar{V}_n = \varnothing$.

Note that for any finite subcollection $\bar{V}_{n_1}, \ldots, \bar{V}_{n_k}$ of the collected of closed sets $\{\bar{V}_n\}$, taking $N = \max_{i=1,\ldots,k}\{n_i\}$ we have that

$$\bigcap_{i=1}^{k} \bar{V}_{n_i} = \bar{V}_N \neq \varnothing$$

Since $[0,1]$ is compact, we can conclude $\bigcap_{i=1}^{\infty} V_i \neq \varnothing$ by homework due 11/22, but there is not point in the sequence $\{x_n\}$ which meets this intersection. Thus, f cannot be a bijection. $\square$

Math 490 Handout: Tuesday December 6, 2016

Definition: A topological space $(X, \mathcal{T})$ has a **countable base at** x if there is a **countable** collection $\mathcal{B}$ of neighborhoods of x such that for any open set U containing x, there exists a $B \in \mathcal{B}$ with $B \subset U$. Then X is **first countable** if X has a countable base at every point x.

A topological space is **second countable** if there exists a countable basis for its topology. Note that second countable implies first countable.

Recall: $\mathbb{R}$ is second countable because metric balls centered at rational points with rational radius form a basis (homework due 10/11).

1. Prove that every metric space is first countable.
2. Find a metric space which is not second countable.
3. Suppose A is a subset of a first countable topological space $(X, \mathcal{T})$. If $x \in \bar{A}$ then there exists a sequence of points $\{x_n\}$ in A converging to x.
4. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f\colon X \to Y$. Suppose that whenever a sequence $\{x_n\}$ converges to x in X, we have that $\{f(x_n)\}$ is a convergent sequence with limit $f(x)$. Prove that if X is first countable, then f is continuous.
5. **Definition**: Recall that a subset A of a topological space $(X, \mathcal{T})$ is **dense** if $\bar{A} = X$. Then X is **separable** if there exists a countable dense subset of X.

   Prove that a second countable topological space has a countable dense subset.
6. Explore some of the topological spaces we know. Are these spaces first countable? Second countable? Separable?

Math 490 Individual homework: Due Thursday December 8, 2016

1. Let $(X, \mathcal{T})$ be a Hausdorff topological space such that singletons are not open. Prove that for all $x \in X$ and U open and nonempty, there exists an open, nonempty set V contained in U such that $x \notin \bar{V}$.

2. Suppose $f\colon X \to Y$ is continuous and $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ are topological spaces. Prove that if X has a countable dense subset, then so does $f(X)$.

3. Prove that a second countable topological space has a countable dense subset.

Math 490 Handout: Thursday December 8, 2016

1. Suppose A is a subset of a first countable topological space $(X, \mathcal{T})$. If $x \in \bar{A}$ then there exists a sequence of points $\{x_n\}$ in A converging to x.

2. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f\colon X \to Y$. Suppose that whenever a sequence $\{x_n\}$ converges to x in X, we have that $\{f(x_n)\}$ is a convergent sequence with limit $f(x)$. Prove that if X is first countable, then f is continuous.

3. **Definition**: Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces. A continuous surjective map $q\colon X \to Y$ is a **quotient map** if for all $U \subset Y$, we have $q^{-1}(U)$ is open in X if and only if U is open in Y.

   Prove that if a continuous surjection is either open or closed, then it is a quotient map. (Note that projection maps are quotient maps.)

4. Let $A = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ or } y = 0\}$. Let $q\colon A \to \mathbb{R}$ be defined by $q(x, y) = x$. Prove that q is a quotient map, but q is neither open nor closed.

5. **Definition**: If $(X, \mathcal{T})$ is a topological space and $A \subset X$ is given the subspace topology, then a continuous map $r\colon X \to A$ is a **retraction** if $r(a) = a$ for all $a \in A$.

   Prove that a retraction is a quotient map.

6. Let $X = [1, 2] \cup [2, 3] \subset \mathbb{R}$ and $Y = [1, 2] \subset \mathbb{R}$. Let $q\colon X \to Y$ be defined by

$$q(x) = \begin{cases} x & \text{if } x \in [1, 2] \\ x - 1 & \text{if } x \in [2, 3] \end{cases}$$

   Let $A = [1, 2] \cup (2, 3) \subset X$. Prove that q is a quotient map, but that $q\big|_A$ is a continuous, surjective map which is not a quotient map.

Math 490 Team homework: due Tuesday December 13, 2016.

1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $q\colon X \to Y$ a quotient map. Prove that if Y is connected and $p^{-1}(\{y\})$ is connected for all $y \in Y$, then X is connected.

2. Prove that (X, d) is a separable metric space, then X is second countable. (Hint: use density of the rationals in $\mathbb{R}$ as in real analysis: that is, between any two distinct real numbers, there exists a rational number.)