GAME THEORY • PRELIM I

I. Assumptions, Rationality, Risk-Aversion, & Types of Games

A **decision** has three features → Actions an agent can choose, possible consequences of these actions (outcomes), and preference relations between these outcomes. ($x_1 \succ x_2$ for strong preference, $x_1 \succsim x_2$ for weak preference).

A payoff function quantifies the utility from an outcome (so if $v(x_1) \geq v(x_2)$, x_1 has $\geq$ payoff function value, and $x_1 \succsim x_2$)

By assumptions of transitivity (If $v(x_1) > v(x_2)$ & $v(x_2) > v(x_3)$, $v(x_1) > v(x_3)$ → see condorcet paradox) and completeness (for any outcome x_n in X, either $x_1 \succsim x_2$, $x_2 \succsim x_1$, or both)

↳ Proposition 1.1: If a set of outcomes X is finite, then any rational preference relation over X can be represented by a payoff function.

↳ Definition 1.2: An agent faced with a decision problem is rational if the agent chooses a^* such that $v(x(a^*)) \geq v(x(a))$ for all possible choices of a ⇒ or, in other words, if this agent **maximizes the payoff function**.

Games can be displayed in forms that include functions, matrices, or decision trees:

$\boxed{1.}$ Cournot Duopoly: Games as functions:

N: $\{1, 2\}$ (2 players)

S_i: $[0, \infty)$ (possible strategies range 0 to ∞)

$v_i(s_i, s_j) = (100 - s_i - s_j)^{+} s_i - s_i^2$ (are the payoffs for these strategies)

*$N / S_i / v_i$ notation is known as normal form

To maximize $(100 - s_i - s_j)^{+} s_i - s_i^2 = 100s_i - 2s_i^2 - s_i s_j$:

$$\frac{d}{ds_i} = 100 - 4s_i - s_j$$

$$\frac{d}{ds_i^2} = -4 < 0 \rightarrow \text{concave down}$$

By symmetry,

$$\frac{d}{ds_j} = 100 - 4s_j - s_i$$

$100 - 4s_i = s_j$

$100 - 4s_j = s_i$

$100 - 4(100 - 4s_i) = s_i$

$100 - 400 + 16s_i = s_i$

$-300 = -15s_i$

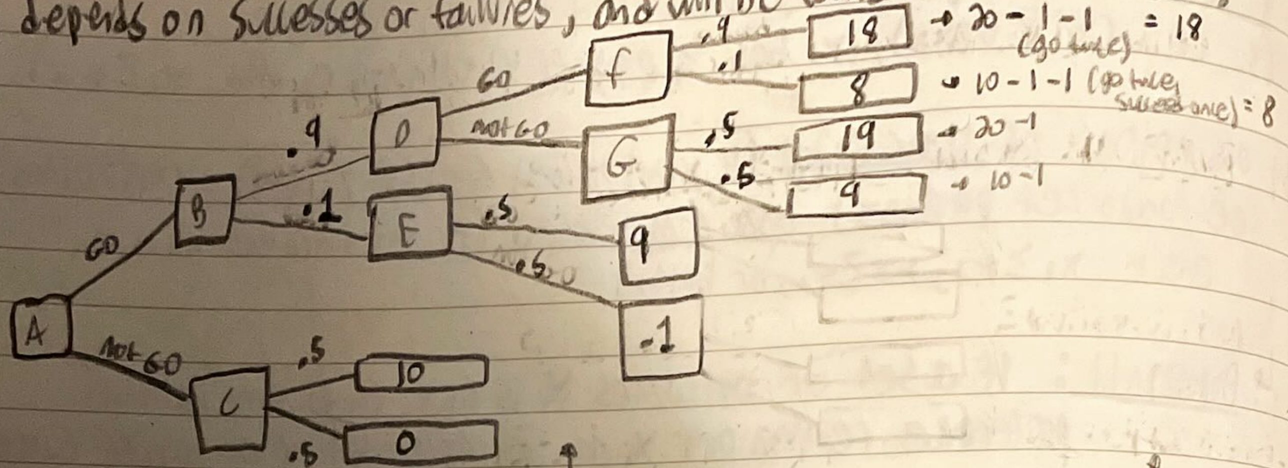
$\boxed{s_i = 20,\ s_j = 20}$

2 R&D Problem: Games as Decision Trees

$N = \{1\}$ (This is more of a decision than a game)

$S_i = \{Go, \text{Not } Go\}$

v_i depends on successes or failures, and will be found from tree below:



Go ahead with initial project, at cost of €1

Going increases P(success) from .5 to .9

If you go ahead with initial project and succeed, you have option for project 2, at cost of €1 (and aim for 2 successes)

A "success" pays 10, a "failure" pays 0

We use <u>expected values</u> and <u>backwards induction</u> to find [A, G]:

$F = 18(.9) + 8(.1) = 17$ $\quad D = \max(F, G) = 17$, by <u>rationality</u>

$G = 19(.5) + 9(.5) = 14$

$E = .5(9) + .5(-1) = 4$

$B = .9(D) + .1(E) = .9(17) + .1(4) = 15.7$

$C = .5(10) = 5$

A (expected payoff) $= 15.7$, found by <u>going</u>.

Note that if going increased P(success) to .6 instead of .9, it would probably be best <u>not</u> to go.

Risk neutral implies $u(x) = x$; this may not be the case in real life, where downside hurts more than upside helps.

To show that most people are risk-averse rather than risk-neutral, consider the following scenario:

A person can accept \$1000, or

$$\left.\begin{array}{l} \$1 \text{ if tails} \\ \$2 \text{ if h/t} \\ \$4 \text{ if h/h/t} \ldots \\ \$\infty \text{ if h} \ldots \infty \ldots \text{t} \end{array}\right\} \text{Expected value is } \sum \underbrace{2^i}_{\text{payoff}} \underbrace{\left(\tfrac{1}{2}\right)^i\left(\tfrac{1}{2}\right)}_{\text{probability}} = \boxed{\$\infty}$$

But few people would take the coin flips; people are risk-averse.

[3] The Prisoner's Dilemma: Games as Matrices

$N = \{1, 2\}$
$S_i = \{\text{confess, stay silent}\}$
$v_i(\text{confess, confess}) = -4$
$v_i(\text{confess, stay silent}) = -1$
$v_i(\text{stay silent, confess}) = -5$
$v_i(\text{stay silent, stay silent}) = -2$

P_1 \ P_2	C	S
C	$(-4,-4)$	$(-1,-5)$
S	$(-5,-1)$	$(-2,-2)$

What should a player do? Strategies will tell us.

II. Strategies for pure games: IESDS, Rationalizability, & Nash Equilibrium

A static game with complete information is a game, like above, where every player chooses simultaneously & independently, and all players know all possible actions of all players, all possible outcomes, actions → outcomes, and payoffs of all players, and pure strategies are deterministic plans of actions a player takes in every "situation of the game."

A strategy is strictly dominated by another strategy if one strategy beats another in all situations:
- ↳ Above, since $-4 > -5$ & $-1 > -2$, P_2's C dominates S
- ↳ same is true for P_1, so $[C, C]$ is the strictly dominant strategy.
- ↳ By Claim 4.1, a player that is rational will never play a strictly dominated strategy

The IESDS method eliminates strictly dominated strategies for example:

$N = \{1, 2\}$

$S_1 = \{U, M, D\}$

$S_2 = \{L, C, R\}$

P2

P1	L	C	R
U	4,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2,8

- <u>For P2</u>: $2>1$, $6>4$, $8>6$, so C is strictly dominated
- <u>For P1</u>: $4>2$ & 3, $6>3$ & 2, so M & D are strictly dominated
- <u>For P2</u>: $3>2$, so R is strictly dominated → $[U, L]$ is the solution by IESDS.

or for Cournot Duopoly...

$\frac{d}{ds_i} = 25 - .25s_i - s_j$, so if $s_i = 0$, $s_j = 25$ $[0, 25]$

$\frac{d}{ds_j} = 25 - .25s_j - s_i$, so if $s_j = 25$, $s_i = 18.75$ $[18.75, 25]$

$25 - .25(18.75) = 20.3125$ → converges to 20

$25 - .25(20.3125) = 19.92...$ $[19.92, 20.3125]$

Note that IESDS may be necessary (rather than solving a system of equations from the derivatives) when convergence is to a <u>range</u>, rather than to a specific value.

However, strong dominance is not the only means of selecting a strategy. Consider a <u>second-price auction</u>, where:

Player v_1 bids b_1, Player v_2 bids b_2, Profit $= v_1 - b_2$

For player v_1, If v_1 (valuation of p1) $= b_2$, profit $= 0$
If $v_1 < b_2$, profit is negative
If $v_1 > b_2$, profit is positive
$\Big\}$ $b_1 > b_2$
If $b_1 < b_2$, profit is 0

So, P_1 loses only when $v_1 < b_2 < b_1$, which cannot happen if $b_1 = v_1$, so $b_1 = v_1$ is <u>weakly dominant</u>. Therefore, it is a <u>best response</u>.

A **best response** is, given all opponent strategies, the best strategy a player can play. <u>BR if $v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i})$ for all s_i.</u>

For the previous game:

$Br_1(L) = U$ $Br_2(U) = L$
$Br_1(C) = D$ $Br_2(M) = R$
$Br_1(R) = U$ $Br_2(D) = R$
↓ ↓
never M never C

so $Br_1(L) = U$ $Br_2(U) = L$
$Br_1(R) = U$ $Br_2(D) = R$
↓
never D
↓
So $Br_1(L) = U$, $Br_2(U) = L$
↳ <u>[U, L] is rationalizable equilibrium</u>

4.3 • Note that a strictly dominated strategy <u>cannot</u> be a best response, because if $v_i(s_i, s_{-i}) < v_i(s_i', s_{-i})$, $v_i(s_i, s_{-i}) \not\geq v_i(s_i', s_{-i})$.

However, rationalizable equilibrium is a <u>subset</u> of IESDS equilibrium:

for:

	L	R
U	(3,0)	(0,0)
M	(0,0)	(3,0)
D	(1,0)	(1,0)

• <u>IESDS</u>

<u>No strong domination</u> →
IESDS still has
[L,U] [R,U]
[L,M] [R,M]
[L,D] [R,D]

• <u>Rationalizable</u>

$Br_1(L) = U$
$Br_1(R) = M$
$Br_2(U) = L$ or R
$Br_2(M) = L$ or R
$Br_2(D) = L$ or R

→ never D →

$Br_1(L) = U$
$Br_1(R) = M$
$Br_2(U) = L$ or R
$Br_2(M) = L$ or R
⇓
[(L,U), (R,M), (R,U), (L,M)]

A **Nash equilibrium** holds when $Br_1(A) = B$, and $Br_2(B) = A$, or more formally, if $s_i^* \in BR_i(s_{-i}^*)$ for all i.

In the above example, $Br_1(L) = U$, $Br_2(U) = L$, $Br_1(R) = M$, $Br_2(M) = R$, so [(L,U), (R,M)] is the Nash Equilibrium.

By proposition 5.1, if a strategy profile is a strictly dominant strategy equilibrium, a unique IESDS survivor, or a unique rationalizable strategy profile, it is N.E.

Unfortunately, rational actors' collective selfishness ends up detracting from the greater good. This is called Tragedy of the Commons.

For example, if there are K units of clean air, each firm uses k_i units, and payoff is $v_i(k_i, k_{-i}) = \ln(k_i) + \ln(K - \sum k_j)$:

$$Br_i(k_{-i}) = \max(\ln(k_i) + \ln(K - \sum k_j)):$$

$$\frac{\partial v_i(k_i, k_{-i})}{\partial k_i} = \frac{1}{k_i} + \frac{1}{K - \sum k_j}(-1) = 0 \rightarrow$$

$$k_1^0 = \frac{1}{2}(K - \sum_{j \neq 1} k_j)$$

$$k_2^0 = \frac{1}{2}(K - \sum k_j)$$

If $n=2$, $k_1 = k_2 = \frac{1}{3}K$ at Nash Equilibrium

A government would want to maximize total welfare, or:

$$\max(\ln(k_1) + \ln(K - k_1 - k_2) + \ln(k_2) + \ln(K - k_1 - k_2))$$

$$\frac{\partial w}{\partial k_1} = \frac{1}{k_1} - \frac{2}{K - k_1 - k_2} = 0 \quad \Bigg| \quad \frac{\partial w}{\partial k_2} = \frac{1}{k_2} - \frac{2}{K - k_1 - k_2} = 0$$

$k_1 = k_2 = \frac{1}{4}K$ at Government optimum

So, N.E. $\neq$ best for total welfare.

III. Mixed Strategies

Some games have no pure strategy Nash Equilibrium. In Rock-Paper-Scissors, for instance, Br(Rock) = Paper, Br(Paper) = Scissors, and Br(Scissors) = Rock.

If $S_i = \{s_1 \dots s_n\}$ is the pure strategy set for player i, mixed strategies are of the form "s_{ik} played with probability $\sigma_i(s_{ik})$", or in the case of rock-paper-scissors:

$$\sigma_i(R) = \frac{1}{3} \qquad \sigma_i(P) = \frac{1}{3} \qquad \sigma_i(S) = \frac{1}{3}$$

(note that $\sum \sigma_i = 1$) $\Rightarrow$ pure strategies are where $\sigma_i(s) = 1$ for an s.

Definitions about mixed strategies:

6.3 Over interval S_i, a mixed strategy is a CDF $F: S_i \to [0,1]$ where a player plays $\le x$ with probability $F(x)$

6.4 A belief for player i is a probability distribution $\pi_i \in \Delta S_i$

6.5 Expected payoff of player i if all players play mixed strategies:

$$v(\sigma_i, \sigma_{-i}) = E(v_i(\sigma_i, \sigma_{-i})) = \sum\sum \sigma_i(s_i)\,\sigma_{-i}(s_i)\,v_i(s_i, s_{-i})$$

For example, in rock-paper-scissors, if Player 2 plays σ_2

$\sigma_2(R) = \sigma_2(P) = \frac{1}{2}$, $\sigma_2(S) = 0$

$v_1(R, \sigma_2) = \frac{1}{2}$(loss - rock loses to paper, -1) $+ \frac{1}{2}(0) = -\frac{1}{2}$

$v_1(P, \sigma_2) = \frac{1}{2}(0) + \frac{1}{2}(1) = \frac{1}{2}$ $\Rightarrow$ so Best response is $\{P\}$

$v_1(S, \sigma_2) = \frac{1}{2}(1) + \frac{1}{2}(-1) = 0$

1 plays, in response to

(where win $= 1$, loss $= -1$)

For Nickel-Dime game:

		P_2: N	P_2: D
P_1	N	5, -5	-5, 10
	D	-10, 5	10, -10

At Nash Equilibrium: Assume P_2 plays σ_2^*, and that $p = P(\text{nickel})$, $1-p = P(\text{dime})$

$v_1(N,N)$

$Br_1(N, \sigma_2^*) = (5)(p) + (-10)(1-p) = 15p - 10$

$Br_1(D, \sigma_2^*) = (-5)(p) + (10)(1-p) = -15p + 10$

$15p - 10 > -15p + 10$
$-20 > -30p$
$2 > 3p$
$\frac{2}{3} > p$, Br_1 is N

And assuming opposite for P_1...

$Br_2(N, \sigma_1) = (-5)(q) + (10)(1-q) = -15q + 10$

$Br_2(D, \sigma_1) = (5)(q) + (-10)(1-q) = 15q - 10$

$-15q + 10 > 15q - 10$
$-30q > -20$
$3q < 2$
$q < \frac{2}{3}$, Br_2 is N

If $p = q = \frac{2}{3}$, players are indifferent N / D

So, for either player:

$$P_1: \quad Br_1((p, 1-p)) \begin{cases} p > \frac{2}{3}, & D \\ p < \frac{2}{3}, & N \\ p = \frac{2}{3}, & N \text{ or } D \end{cases} \qquad P_2: \quad Br_2(q, 1-q) \begin{cases} q > \frac{2}{3}, & N \\ q < \frac{2}{3}, & D \\ q = \frac{2}{3}, & N \text{ or } D \end{cases}$$

Any mixed combination of N or D has equal expectation

Note that Nash Equilibrium here is $\Big(\underbrace{(\overset{N}{\tfrac{2}{3}}, \overset{D}{\tfrac{1}{3}})}_{P_1}, \underbrace{(\overset{N}{\tfrac{1}{3}}, \overset{D}{\tfrac{2}{3}})}_{P_2} \Big)$,

meaning that P_1 should play N $\frac{2}{3}$ and D $\frac{1}{3}$, and P_2 should do the opposite.

By proposition 6.1, if $\sigma^\circ = (\sigma^\circ_1, \sigma^\circ_2, \ldots \sigma^\circ_n)$ is a Nash Equilibrium, and player 1's pure strategies s_i and s_i' as well as player 1's mixed strategy σ_i° which is a combination of s_i & s_i', then $v_i(s_i, \underbrace{\sigma^\circ_{-i}}_{P_2 \text{ mixed strategy}}) = v_i(s_i', \sigma^\circ_{-}) = v_i(\sigma^\circ_i, \sigma^\circ_{-i})$.

If this were not the case - say, $v_i(s_i, \sigma^\circ_{-i}) < v_i(s_i', \sigma^\circ_{-i})$, then it would never be better to play s_i' over s_i. If a player is randomizing between two strategies, the player must be indifferent between them. The reason for playing a mixed strategy is so another player cannot intuit the original player's strategy.

Another example:

	C	R
M	(0,0)	(3,5)
D	(4,4)	(0,3)

By pure strategy,

$Br_1(C) = D$, $Br_2(M) = R$

$Br_1(R) = M$, $Br_2(D) = C$

So [C,D] & [M,R] are pure equilibria

So for a mixed strategy:

$v_1(M, \sigma^\circ_2) = \sigma^\circ_2(C)\, v_1(M, C) + \sigma^\circ_2(R)\, v_1(M, R)$

$v_1(D, \sigma^\circ_2) = \sigma^\circ_2(C)\, v_1(D, C) + \sigma^\circ_2(R)\, v_1(D, R)$

as $v_1(M, C) = 0$ or, $v_1(M, R) = 3 \ldots$

$v_1(M, \sigma^\circ_2) = 3\sigma^\circ_2(R) = 3(1 - \sigma^\circ_2(C)) = 3 - 3\sigma^\circ_2(C)$

$v_1(M, \sigma^\circ_2) = 4\sigma^\circ_2(C)$

$$3 - 3\sigma_2^*(L) > 4\sigma_2^*(L)$$

$$3 > 7\sigma_2^*(L) \rightarrow \sigma_2^* = \left(\tfrac{3}{7}, \tfrac{4}{7}\right)$$

$$\frac{3}{7} > \sigma_2^*(L)$$

Play L Play R.