

Intro to linear programming

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Overview

- ▶ What is linear programming?
- ▶ Example
- ▶ Motivation for general solution

What is linear programming?

NOT “programming” in the typical sense

- ▶ More related to scheduling (like TV programming)

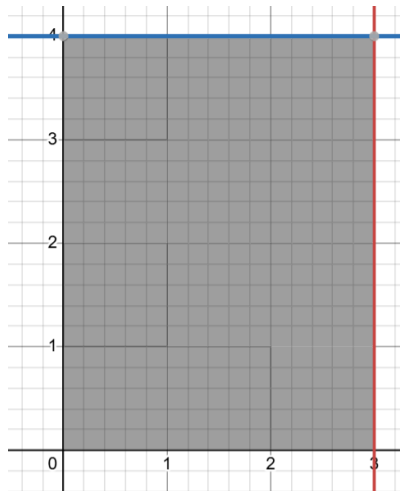
A way to model problems as linear inequalities - Efficiently solvable by many tools: - Python packages, many backend solvers in C++, even MS Excel

Example

$$\begin{array}{ll}\max & x_1 + x_2 \\ \text{s.t.} & x_1 \leq 3 \\ & x_2 \leq 4\end{array}$$

Example (cont.)

$$\begin{array}{ll}\max & x_1 + x_2 \\ \text{s.t.} & x_1 \leq 3 \\ & x_2 \leq 4 \\ & x_1, x_2 \geq 0\end{array}$$



More complexity

Not all linear programs are this easy to solve:

$$\max \quad x_1 + x_2 + \cdots + x_n$$

$$s.t. \quad x_1 \leq 3$$

$$x_2 + x_3 \leq 4$$

$$x_2 + x_3 - x_5 \leq 4$$

$$x_5 + x_1 - x_9 \geq 50$$

$$\vdots$$

$$x_1, x_2, \dots, x_n \geq 0$$

Real-world usecases for linear programming

- ▶ Airline scheduling
- ▶ Efficient redistribution of citibikes
- ▶ Portfolio optimization
- ▶ Petroleum blending
- ▶ Food ingredient mixing
- ▶ Reach maximization for advertisements
- ▶ Minimizing waste in factory settings
- ▶ Routing delivery vehicles (e.g., UberEats or Doordash)

These problems are not so simple!

Example: Meal delivery routing problem (2024, Kancharla, S.R., Van Woensel, T., Waller, S.T. et al.)

$$\max \sum_{i \in T_d} p_i - \sum_{i,j \in T} c_{ij} - \alpha \left(\sum_{i \in T_d} \mu_i - E[Q(x, \xi)] \right)$$

$$p_i \leq w_i \sum_{j \in N} \sum_{k \in V} x_{ji}^k \quad \forall i \in T_d$$

$$c_{ij} \geq \sum_{k \in V} m_k x_{ij}^k t_{ij} \quad \forall i, j \in T, i \neq j$$

$$\sum_{k \in V} \left(\sum_{j \in T \setminus \{i\}} x_{ji}^k \right) \leq 1 \quad \forall i \in T$$

$$\sum_{k \in V} \left(\sum_{j \in T \setminus \{i\}} x_{ij}^k \right) = \sum_{k \in V} \left(\sum_{j \in T \setminus \{i\}} x_{ji}^k \right) \quad \forall i \in T$$

Average LP in a published paper...

$$\sum_{j \in T} \sum_{i \in T_p} x_{ij}^k = \sum_{j \in T} \sum_{i \in T_d} x_{ji}^k \quad \forall k \in V, i \neq j$$

$$\sum_{j \in N \setminus \{i\}} \sum_{k \in V} a_{ji}^k \leq \sum_{j \in N \setminus \{i\}} \sum_{k \in V} a_{ij}^k - (s_i + t_{ij}) x_{ij}^k \quad \forall i \in T$$

$$e_i x_{ij}^k \leq a_{ij}^k \quad \forall k \in V, i \in T, j \in T \setminus \{i\}$$

$$e_k x_{ij}^k \leq a_{ij}^k \leq l_k x_{ij}^k \quad \forall k \in V, i \in N, j \in N \setminus \{i\}$$

$$\sum_{j \in N \setminus \{i\}} a_{ji}^k \leq \sum_{j \in T \setminus \{i+n\}} a_{ji+n}^k \quad \forall k \in V, i \in P$$

$$\sum_{k \in V} \sum_{j \in N \setminus \{i\}} y_{ij}^k - d_i x_{ij}^k = \sum_{k \in V} \sum_{j \in N \setminus \{i\}} y_{ji}^k \quad \forall i \in T$$

$$y_{ij}^k \leq \beta_k x_{ij}^k \quad \forall k \in V, i \in N, j \in N \setminus \{i\}$$

$$\mu_i + l_i \geq \sum_{k \in V} \sum_{j \in N} a_{ji}^k + s_i x_{ij}^k \quad \forall i \in T_d$$

$$x_{ij}^k \in \{0, 1\} \quad \forall k \in V, i \in N, j \in N$$

$$y_{ij}^k \geq 0, a_{ij}^k \geq 0, p_i \geq 0, c_{ij} \geq 0 \quad \forall k \in V, i \in N, j \in N$$

Canonical form

Need to make things simpler to reason about them

Canonical form:

$$\begin{array}{ll}\max & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0\end{array}$$

- ▶ A is a matrix
- ▶ x is now a vector of decision variables
- ▶ c is a (row) vector (so c^T is a column vector)
- ▶ b is a (column) vector

References

Kancharla, S.R., Van Woensel, T., Waller, S.T. et al. Meal Delivery Routing Problem with Stochastic Meal Preparation Times and Customer Locations. *Netw Spat Econ* 24, 997–1020 (2024).
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