Boson-Fermion correspondence and construction of character tables for the symmetric group S n

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Abstract

In these note, we present a simple method to determine the character tables of the symmetric group S_n. Furthermore, specific example for the group S_3 are provided.

Introduction

The determination of the character tables of the symmetric group S_n is not trivial, as it requires an in-depth analysis of symmetries. In these notes. We suggest a concise approach for constructing the character table of the symmetric group. This method is inspired by the work of B. Mohamed [1], and P. Tingley [3], who identified a connection between character tables and Maya diagrams..

Boson-Fermion Correspondence

The bosonic space

$$\mathcal{B} = \mathbb{C}[p_1, \ldots ; z, z^{-1}] = \bigoplus_{l \in \mathbb{Z}} z^l \mathbb{C}[p_1, \ldots]$$

is introduced, with two operators

$$p_m = p_m \cdot \quad \text{and} \quad p_{-m} = m \frac{\partial}{\partial p_m}$$

that act on the space $\mathcal{B}$.

The fermionic space

$$\mathcal{F} := \bigoplus_{l \in \mathbb{Z}} \mathcal{F}^l := \{ v_{i_1} \wedge v_{i_2} \cdots \mid i_1 > i_2 > \cdots , i_k = -k + l + \frac{1}{2} \text{ for } k \gg 0 \}$$

in other words, the basis is indexed by Maya diagrams.

Notation : To shorten notation we will omit below the sign of the wedge product. This space is introduced with two operators

$$\{\psi_n, \psi_n^*\}_{n \in \mathbb{Z} + \frac{1}{2}},$$

that act on the fermionic space as follows:

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$$\psi_n \cdot v_{(\lambda,l)} = v_n v_{(\lambda,l)}$$

$$\psi_n^* \cdot v_{(\lambda,l)} = \begin{cases} 0 & \text{if } v_n \text{ does not appear as a factor of } v_{(\lambda,l)} \\ v_{(\mu,l)} & \text{if } v_{(\lambda,l)} \text{ can be expressed as } v_n v_{(\mu,l)} \text{ for some } v_{(\mu,l)}. \end{cases}$$

Theorem 1 (Boson-Fermion Correspondence)**.** *The bosonic space is isomorphic to the fermionic space, and we have the following correspondences for $m > 0$:*

$$\begin{array}{ccc} \sum_i \psi_{i+m}\psi_i^* & \longleftrightarrow & p_m \\ \sum_i \psi_{i-m}\psi_i^* & \longleftrightarrow & p_{-m} = m\frac{\partial}{\partial p_m} \\ \sum_i :\psi_i\psi_i^*: := \sum_{i<0}\psi_i\psi_i^* - \sum_{i>0}\psi_i^*\psi_i & \longleftrightarrow & l \end{array}$$

From now on, we will work only with the correspondence between $\mathcal{F}^0$ and $\mathcal{B}^0$

Remark 1. *The reader can consult the references [1],[3]*

Example of the Character Table for S_3

First, let us recall a formula derived using Schur-Weyl duality. Let $\lambda = (\lambda_1 > \lambda_2 > \cdots)$ be a partition of an integer n:

Theorem 2. *[2]*

$$p_\lambda = \sum_{\mu \in \mathcal{Y}oung} \chi_\mu(\lambda) s_\mu$$

where $p_\lambda = \prod_i p_{\lambda_i}$ and s_λ is the Schur polynomial associated with the Young diagram λ.

Remark 2. *Since the bosonic space can be generated by Schur polynomials, and by the isomorphism between the bosonic and fermionic spaces, the Schur polynomials s_μ can be identified with $|\mu\rangle$.*

Example 3. *In this example, we need to use the boson-fermion correspondence to identify the p_i via $\sum_i \psi_{i+k}\psi_i^*$, as well as the Maya-Young correspondence to identify Maya diagrams with Young diagrams. These diagrams characterize the irreducible characters of the symmetric group S_n. Then, we use Theorem 2 and Remark 2 to identify the characters $\chi_\mu(\lambda)$. We focus here on the case $n = 3$, but the method presented is valid for any n.*

$$\begin{aligned}
p_{(1)} &= p_1|0\rangle = p_1(\times\times\times|\circ\circ\circ) = \times\times\circ|\times\circ\circ && = \ydiagram{1} \\
p_{(1,1)} &= p_1^2|0\rangle = p_1^2(\times\times\times|\circ\circ\circ) = \times\circ\times|\times\circ\circ + \times\times\times\circ|\circ\times\circ && = \ydiagram{1,1} + \ydiagram{2} \\
p_{(1,1,1)} &= p_1^3|0\rangle = p_1^3(\times\times\times|\circ\circ\circ) = \times\times\circ|\circ\circ\times + 2\times\circ\times|\circ\times\circ + \times\circ\times\times|\times\circ\circ && = \ydiagram{3} + 2\,\ydiagram{2,1} + \ydiagram{1,1,1} \\
p_{(2)} &= p_2|0\rangle = p_2(\times\times\times|\circ\circ\circ) = \times\times\circ|\circ\times\circ - \times\circ\times|\times\circ\circ && = \ydiagram{2} - \ydiagram{1,1} \\
p_{(2,1)} &= p_2p_1|0\rangle = p_2p_1(\times\times\times|\circ\circ\circ) = \times\times\circ|\circ\circ\times - \times\circ\times\times|\times\circ\circ && = \ydiagram{3} - \ydiagram{1,1,1} \\
p_{(3)} &= p_3|0\rangle = p_3(\times\times\times|\circ\circ\circ) = \circ\times\times|\times\circ\circ - \times\circ\times|\circ\times\circ + \times\times\circ|\circ\circ\times && = \ydiagram{1,1,1} - \ydiagram{2,1} + \ydiagram{3}
\end{aligned}$$

From Theorem 2 and the relations we have just calculated, we match both sides and deduce the character table of S_3:

	$\yng(1,1,1)$	$\yng(2,1)$	$\yng(3)$
$\yng(1,1,1)$	*1*	*2*	*1*
$\yng(2,1)$	*1*	*0*	*-1*
$\yng(3)$	*1*	*-1*	*1*

We can also determine the Schur polynomial by inverting this matrix. It is important to note that the inversion of this matrix is made easier by the orthogonality formula of characters.

So, this method determines both the characters of the symmetric group and those of the general linear group.

Now, let us observe the following. We denote by τ_i the transposition $(i\,i{-}1)$ for $1 \leq i \leq n{+}1$. It is well known that the symmetric group S_n can be described by the following relations:

$$\tau_i^2 = 1,$$
$$\tau_i\tau_j = \tau_j\tau_i \quad \text{for } |i-j| > 1,$$
$$\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}.$$

The Hecke algebra is defined in a similar way. Given a parameter q, we define the Hecke algebra with generators T_i, which satisfy the following relations:

$$T_i^2 = (q-1)T_i + q,$$
$$T_iT_j = T_jT_i \quad \text{for } |i-j| > 1,$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}.$$

Question: Can you define the bosonic and fermionic operators p_λ and ψ_i in a similar way, and use this to obtain the boson-fermion correspondence? This would allow us to compute the characters for the Hecke algebra in an easy way.

References

[1] BENSAID Mohamed, Sato tau functions and construction of Somos sequence, 2024, `https://arxiv.org/pdf/2409.05911`

[2] R. Dipper, S. Doty, and J. Hu, Brauer Alegbras, Symplectic Schur Algebras and Schur-Weyl Duality, Trans. Amer. Math. Soc., 360, 189-213 (2008).

[3] Peter Tingley, Notes Fock space, 2023, `https://arxiv.org/pdf/2211.12463`