Statistics and data analysis
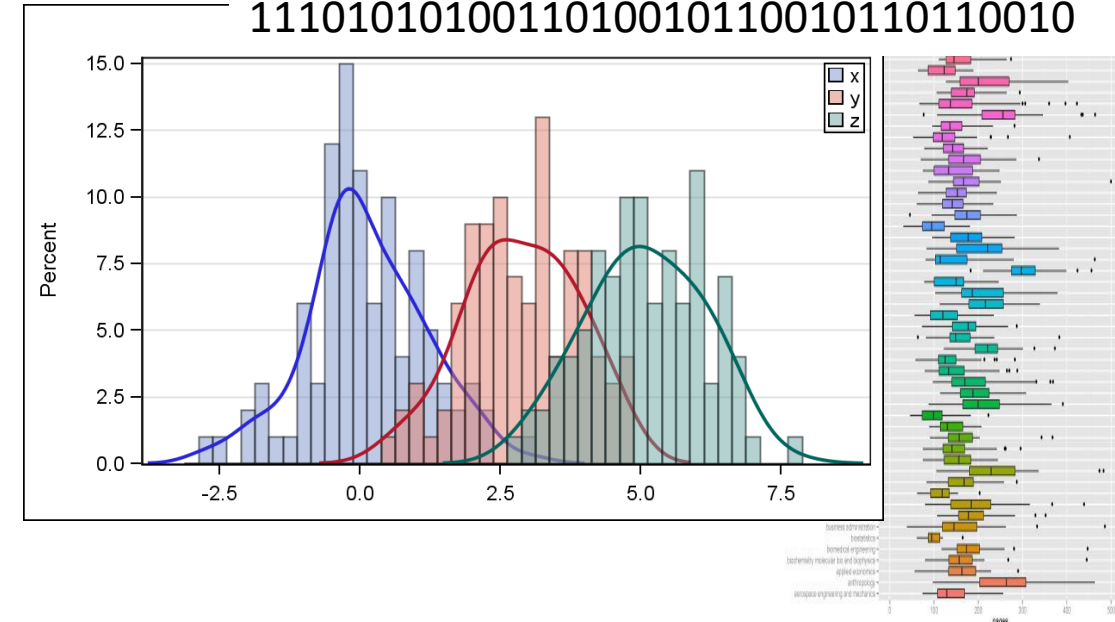
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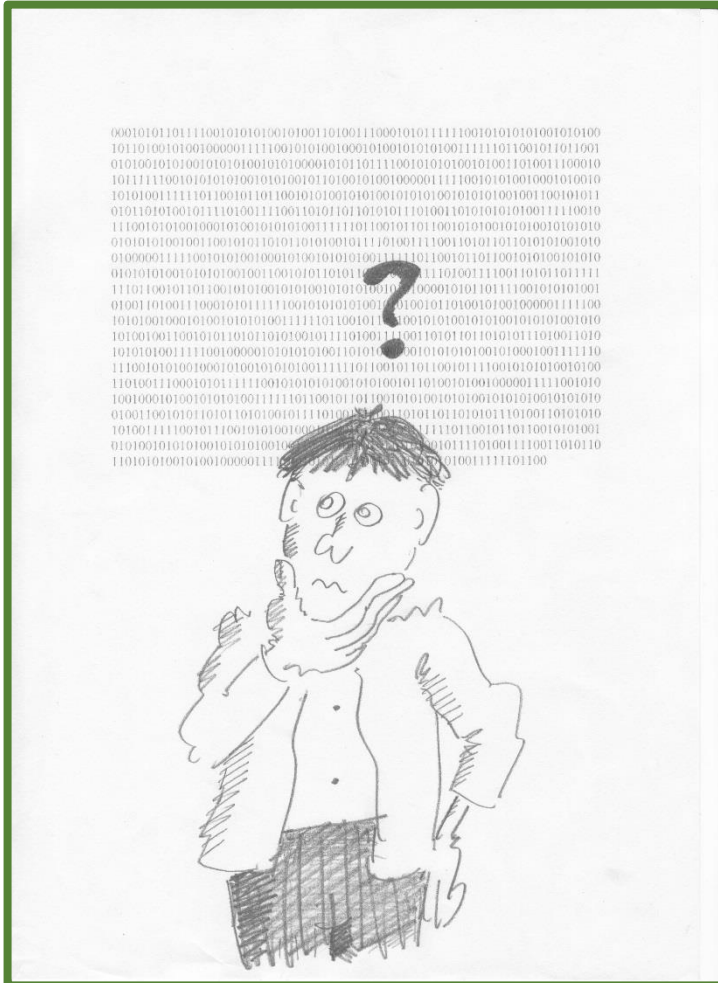
More distributions, independence

Binomial Distribution

- P(Y = k) = $\binom{n}{k} p^k (1-p)^{n-k}$

- $\sum_{k=0}^{n} P(Y=k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + 1 - p)^n = 1$

$\omega \epsilon \Omega$:

The Geometric distribution

$\omega \epsilon \Omega :$

$X(\omega) =$ time of first success

Continue to infinity ...

$X \sim Geom(p)$

$P(X \; = \; k) \; = \; ?$

Geometric Distribution – Expectation and variance

$$E(Y) = \sum_{y=1}^{\infty} y \left[q^{y-1} p \right] = p \sum_{y=1}^{\infty} \frac{dq^y}{dq} = p \frac{d}{dq} \sum_{y=1}^{\infty} q^y = p \frac{d}{dq} \left[q \sum_{y=1}^{\infty} q^{y-1} \right] =$$

$$= p \frac{d}{dq} \left[\frac{q}{1-q} \right] = p \left[\frac{(1-q)(1) - q(-1)}{(1-q)^2} \right] = \frac{p\left((1-q) + q \right)}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$E\left(Y(Y-1) \right) = \sum_{y=1}^{\infty} y(y-1) \left[q^{y-1} p \right] = pq \sum_{y=1}^{\infty} \frac{d^2 q^y}{dq^2} = pq \frac{d^2}{dq^2} \sum_{y=1}^{\infty} q^y = pq \frac{d^2}{dq^2} \left[q \sum_{y=1}^{\infty} q^{y-1} \right] =$$

$$= pq \frac{d^2}{dq^2} \left[\frac{q}{1-q} \right] = pq \frac{d}{dq} \frac{1}{(1-q)^2} = pq \left(-2(1-q)^{-3} (-1) \right) = \frac{2pq}{(1-q)^3} = \frac{2pq}{p^3} = \frac{2q}{p^2}$$

$$\Rightarrow E\left(Y^2 \right) = E\left(Y(Y-1) \right) + E(Y) = \frac{2q}{p^2} + \frac{1}{p} = \frac{2(1-p) + p}{p^2} = \frac{2-p}{p^2}$$

$$\Rightarrow V(Y) = E\left(Y^2 \right) - \left[E(Y) \right]^2 = \frac{2-p}{p^2} - \left[\frac{1}{p} \right]^2 = \frac{2-p-1}{p^2} = \frac{1-p}{p^2} = \frac{q}{p^2}$$

$$\Rightarrow \sigma = \sqrt{\frac{q}{p^2}}$$

Negative Binomial Distribution

- In successive Bernoulli(p) instances, what is the distribution of the number of trials (in some versions – failures) needed until the r th success.
  (the Geometric Distribution is equivalent to $r = 1$)

- For this number to equal k we should have exactly $r - 1$ successes in first $k - 1$ trials, followed by a success

- $P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$

- $E(X) = \dfrac{r}{p}$

- $V(X) = \dfrac{r(1-p)}{p^2}$

Binomial rate vs n

Consider

$$X_1 \sim \mathrm{Binom}(1, \lambda) \text{ and } X_2 \sim \mathrm{Binom}(2, \lambda/2)$$

Which is larger:

$P(X_1 \geq 1)$ or $P(X_2 \geq 1)$?

$E(X_1)$ or $E(X_2)$?

Poisson – a limit of binomials with an increasing n and a fixed mean

Consider repeated coin tossing with increasingly smaller success rates

$$X_1 \sim Binom(1, \lambda)$$

$$X_2 \sim Binom(2, \lambda/2)$$

$$X_3 \sim Binom(3, \lambda/3)$$

$$P(X_1 \geq 1) = P(X_1 = 1) = \lambda$$

$$P(X_2 \geq 1) = 1 - P(X_2 = 0)$$

$$= 1 - \left(1 - \lambda/2\right)^2 = \lambda - \left(\lambda/2\right)^2 < \lambda$$

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Poisson – a limit of binomials with an increasing n and a fixed mean

$$X_n \sim \text{Binom}\left(n, \lambda/n\right)$$

$$P(X_n = k) = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{n^k} \cdot \frac{\lambda^k}{k!} \cdot \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k}$$

as $n \to \infty$

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Poisson – a limit of binomials with an increasing n and a fixed mean

So,

$$\forall k = 0, 1 \dots$$

we have

$$P(X_n = k) \xrightarrow[n \to \infty]{} e^{-\lambda} \frac{\lambda^k}{k!}$$

Poisson Distribution

- X ~ Poisson(λ) if

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

- Example – a website receives visits distributed as Poisson(0.5) per second.
- What is the probability of no visits at a certain second?
- Answer: exp(-0.5)
- What is the probability of no visits in a stretch of 10 seconds?
- Compute in two ways:
  - Poisson(5) yields exp(-5)
  - 10 independent as above yields (exp(-0.5))^10 = exp(-5)

Poisson Distribution

Distribution often used to model the number of incidences in some characteristic unit of time or space:

- Arrivals of customers to a store within one hour
- Numbers of flaws in a roll of fabric of a given length
- Number of visitors to a website in one minute
- Number of calls to a service center in 10 mins

Poisson Distribution – Expectation and Variance

$$f(y) = \frac{e^{-\lambda}\lambda^y}{y!} \qquad y = 0,1,2,\ldots$$

$$E(Y) = \sum_{y=0}^{\infty} y\left[\frac{e^{-\lambda}\lambda^y}{y!}\right] = \sum_{y=1}^{\infty} y\left[\frac{e^{-\lambda}\lambda^y}{y!}\right] = \sum_{y=1}^{\infty}\frac{e^{-\lambda}\lambda^y}{(y-1)!} = \lambda e^{-\lambda}\sum_{y=1}^{\infty}\frac{\lambda^{y-1}}{(y-1)!} = \lambda e^{-\lambda}e^{\lambda} = \lambda$$

$$E\big(Y(Y-1)\big) = \sum_{y=0}^{\infty} y(y-1)\left[\frac{e^{-\lambda}\lambda^y}{y!}\right] = \sum_{y=2}^{\infty} y(y-1)\left[\frac{e^{-\lambda}\lambda^y}{y!}\right] = \sum_{y=2}^{\infty}\frac{e^{-\lambda}\lambda^y}{(y-2)!} =$$

$$= \lambda^2 e^{-\lambda}\sum_{y=2}^{\infty}\frac{\lambda^{y-2}}{(y-2)!} = \lambda^2 e^{-\lambda}e^{\lambda} = \lambda^2$$

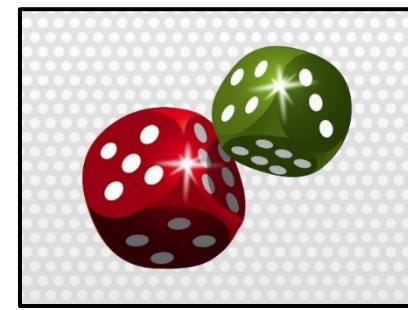$$\Rightarrow E\big(Y^2\big) = E\big(Y(Y-1)\big) + E(Y) = \lambda^2 + \lambda$$

$$\Rightarrow V(Y) = E\big(Y^2\big) - \big[E(Y)\big]^2 = \lambda^2 + \lambda - [\lambda]^2 = \lambda$$

$$\Rightarrow \sigma = \sqrt{\lambda}$$

Statistical independence

Example – Rolling 2 Dice (Red/Green)

Ω = All possible outcomes, that is:

(1,1) (1,2) (1,3) (1,4) (1,5) (1,6)

(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)

(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)

(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)

(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)

(6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

- Assuming that all outcomes have P = 1/36 is based on assuming that the result of one dice DOES NOT AFFECT the rolling of the other in any way.
- What is the probability of G = 3 or 6 and R = 5?
- P(G = 3 or 6) = 1/3
- P(R = 5) = 1/6
- The probability of the JOINT event is, assuming 1/36 in each entry, 1/36+1/36 = 1/18.
- This is just the product of the two probabilities: P(G = 3 or 6 and R = 5) = 1/3 * 1/6 = 1/18
- This is called STATISTICAL INDEPENDENCE.
- When we defined 1/36 in every entry we imply that the two rolls are independent random variables

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Definitions and factoids …

- Two events (subsets of the sample space Ω), A and B, are said to be statistically independent if the occurrence of one doesn't affect the occurrence of the other:

$P(A\,|\,B) \;=\; P(A)$, where $\;\;P(A\,|\,B) \;=\; P(A \cap B)/P(B)\;$ is the conditional probability of A given B.

- Form here we get

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A \cap B)P(B)}{P(A)P(B)}$$

$$= \; P(A|B)\frac{P(B)}{P(A)} = \; P(B)$$

- Show that from here it follows that $P(A\,|\,B) \;=\; P(A\,|\,\neg B)$
- It also clearly follows that $P(A \cap B) \;=\; P(A)P(B)$

Independent random variables

- Two random variables X and Y, defined over the same space Ω have a joint distribution $p(x, y)$.
- They also have marginal distributions
- The same marginal can often be joined (or coupled) in very different ways. The independent copula is only one of them.
- They are called independent if for all numbers x and y we have
$$P(X = x \text{ and } Y = y) = P(X = x) \cdot P(Y = y)$$
- Or – for all x and y as above,
the events $P(X = x)$ and $P(Y = y)$ are independent.
- If X and Y are independent then $E(XY) = E(X) \cdot E(Y)$ (prove this …)
- Is the opposite true?

Linearity of expected values

- $E(X + Y) = E(X) + E(Y)$
- This is true for ANY random variables. They don't have to be independent.
- This generalizes to any sums.

Sample/Coupon collection

- A website is seeking information about users from 100 different cities.
- It needs to observe the action of m users from each city to perform the analysis.
- How many visits will it take if every visit comes from each of the cities with equal probabilities and independent of all previous visits?
- On average?

Poll: 100 countries and m=1

Sample/Coupon collection

At this point we will compute the expected value for the case $m = 1$.

We define random variables X_i, $i = 1 \dots 100$, as follows.
Let X_1 = the number of visits until the first country is in ($X_1 == 1$)
Let X_2 = the number of visits, after the first country is in, until the second country is also in

…

Let X_i = the number of visits, after the first $i - 1$ countries are in, until the i-th country is also in.

Now let

$$T = X_1 + X_2 + X_3 + \dots + X_i + \ \dots + X_{99} + X_{100}$$

Sample/Coupon collection

We are, of course, interested in

$$E(T) = E(X_1 + X_2 + X_3 + \ldots + X_i + \ldots + X_{99} + X_{100}) = \sum_{i=1}^{100} E(X_i)$$

Now note that $X_i \sim Geom(p = (100 - i + 1)/100)$ and we therefore have $E(X_i) = \frac{1}{p} = \frac{100}{100-i+1}$

So:

$$E(T) = 100 \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{100} \right)$$

<u>In general, for n types of coupons, we have $E(T) = nH(n) \sim n \ln n$.</u>

General m and unequal probabilities require a more complex treatment.

Var(X+Y)

Covariance

- Consider X and Y defined on the same sample space Ω

- $Cov(X, Y) = E(\,(X - \mu(X))(Y - \mu(Y)))$

- $Cov(X, Y) = E(XY) - E(X)E(Y)$

- When X and Y are independent, what is $Cov(X, Y)$?

- Is the opposite true?

Binomial Distribution – Variance and S.D.

$$f(y) = \frac{n!}{y!(n-y)!} p^y q^{n-y} \quad y = 0,1,\ldots,n \quad q = 1-p$$

Note : $E(Y^2)$ is difficult (impossibl e?) to get, but $E(Y(Y-1)) = E(Y^2) - E(Y)$ is not :

$$E(Y(Y-1)) = \sum_{y=0}^{n} y(y-1) \left[\frac{n!}{y!(n-y)!} p^y q^{n-y} \right] = \sum_{y=2}^{n} y(y-1) \left[\frac{n!}{y!(n-y)!} p^y q^{n-y} \right]$$

(Summand $= 0$ when $y = 0,1$)

$$\Rightarrow E(Y(Y-1)) = \sum_{y=2}^{n} \frac{n!}{(y-2)!(n-y)!} p^y q^{n-y}$$

Let $y^{**} = y-2 \Rightarrow y = y^{**} + 2$ Note : $y = 2,\ldots,n \Rightarrow y^{**} = 0,\ldots,n-2$

$$\Rightarrow E(Y(Y-1)) = \sum_{y^{**}=0}^{n-2} \frac{n(n-1)(n-2)!}{y^{**}!\left(n-(y^{**}+2)\right)!} p^{y^{**}+2} q^{n-(y^{**}+2)} = n(n-1)p^2 \sum_{y^{**}=0}^{n-2} \frac{(n-2)!}{y^{**}!\left((n-2)-y^*\right)!} p^{y^{**}} q^{(n-2)-y^{**}} =$$

$$= n(n-1)p^2(p+q)^{n-2} = n(n-1)p^2(p+(1-p))^{n-2} = n(n-1)p^2$$

$$\Rightarrow E(Y^2) = E(Y(Y-1)) + E(Y) = n(n-1)p^2 + np = np[(n-1)p+1] = n^2p^2 - np^2 + np = n^2p^2 + np(1-p)$$

$$\Rightarrow V(Y) = E(Y^2) - [E(Y)]^2 = n^2p^2 + np(1-p) - (np)^2 = np(1-p)$$

$$\Rightarrow \sigma = \sqrt{np(1-p)}$$

Or: linearity of variance for independent variables

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Sums of independent random variables

Let X and Y be two independent random variables. Let $Z = X + Y$.
Then

$$P(Z = z) = \sum_{i=-\infty}^{\infty} P(X = i)P(Y = z - i)$$

For continuous random variables, the density function of Z is:

$$h(z) = \int_{-\infty}^{\infty} f(t)g(z - t)dt$$

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Sum of two independent Poissons is Poisson

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

Sum of 2 indpt Poissons

$$P(Y = k) = e^{-\mu} \frac{\mu^k}{k!}$$

X and Y indpt. Let $Z = X + Y$

$$P(Z = k) = \sum_{i=-\infty}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} \cdot e^{-\mu} \frac{\mu^{k-i}}{(k-i)!}$$

Here summands are 0 when either of the denominator factorials are negative

$$= e^{-(\lambda+\mu)} \cdot \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} \lambda^i \mu^{k-i}$$

$$= e^{-(\lambda+\mu)} \frac{(\lambda+\mu)^k}{k!}$$

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Higher moments

The raw kth moment of a random variable X is $E(X^k)$

The central kth moment of a random variable X is $E((X - \mu(X))^k)$

Let $X \sim \text{Binom}(n, p)$. What is the 3rd central moment of X ?

$X = \sum_{i=1}^{n} X_i$, where $X_i \sim \text{Ber}(p)$, independent.

$$\gamma_3 = E\left[\left(\sum_{i=1}^{n}(X_i - p)\right)^3\right] = E\left[\sum_{i,j,k=1 \ldots n}(X_i - p)(X_j - p)(X_k - p)\right]$$

$$= \sum_{i,j,k=1 \ldots n} E((X_i - p)(X_j - p)(X_k - p))$$

The terms of the last summation are all 0 except when $i = j = k$. Therefore:

$$\gamma_3 = nE((X_1 - p)^3) = n(p(1 - p)^3 + (1 - p)(-p)^3).$$

And, after further simplification:
$$\gamma_3 = np(1 - p)(1 - 2p)$$

Mutual independence vs k-wise independence

Summary

- Geometric distribution
- Negative binomials (next week: how to compare them)
- Poisson distribution
- Coupon collector
- Independence and the covariance of two random variables
- Convolution of pdfs (to be continued)
- Higher moments and an example
- Mutual indpce vs lower order indpce

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