Statistics and data analysis

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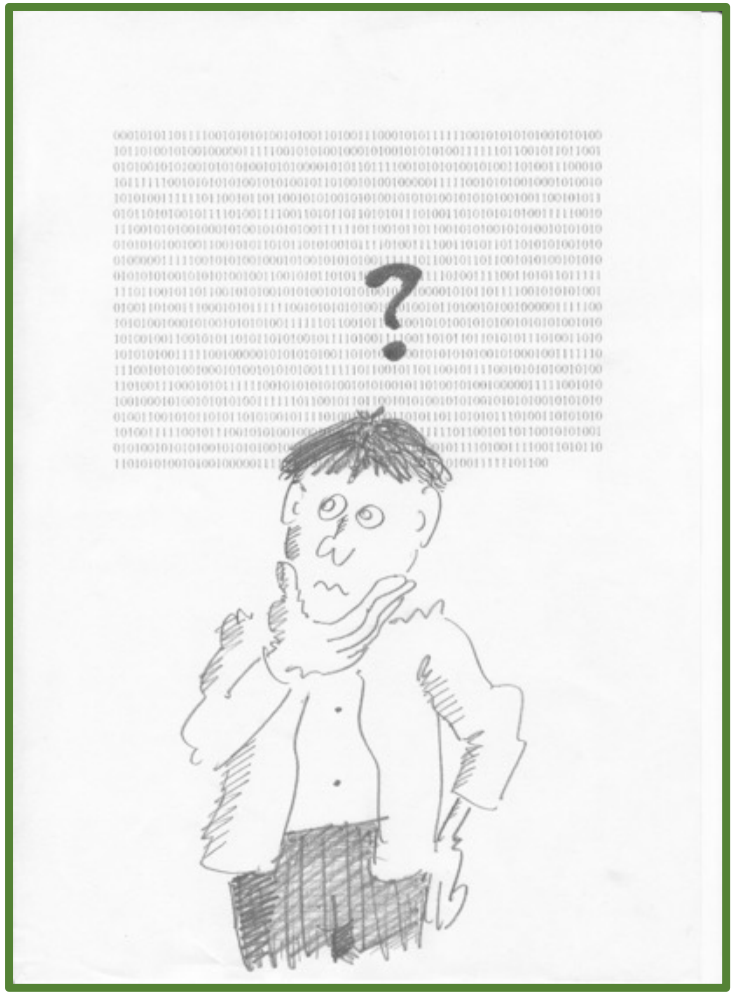
IDC, Herzeliya

Independence and variations, convolution, computer age statistics



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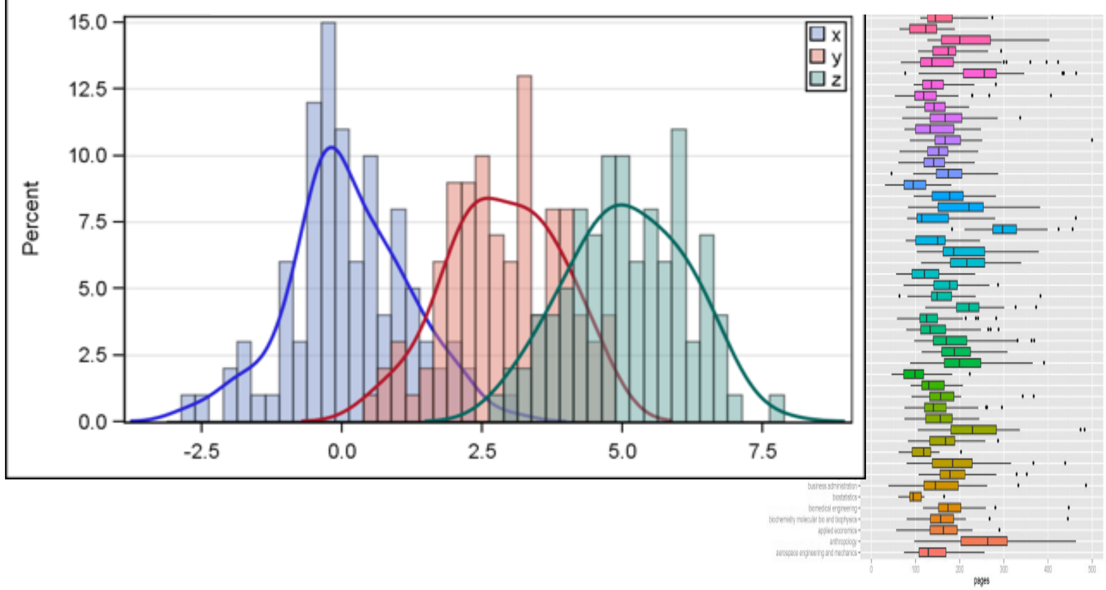


Negative Binomial Distribution

- In successive Bernoulli(p) instances, what is the distribution of the number of trials (in some versions – failures) needed until the r^{th} success.
  (the Geometric Distribution is equivalent to $r = 1$)
- For this number to equal y we should have exactly $r - 1$ successes in first $y - 1$ trials, followed by a success

$$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r} \qquad y = r, r+1, \ldots$$

$$E(Y) = \frac{r}{p}$$

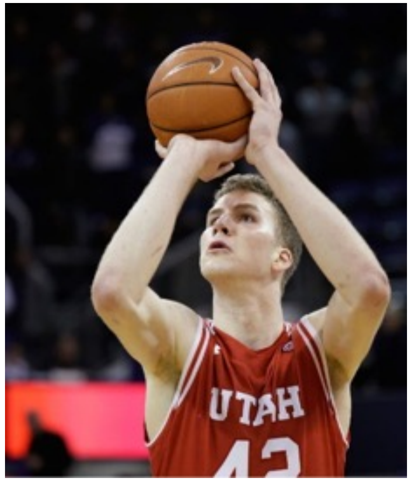
$$V(Y) = \frac{r(1-p)}{p^2}$$

Randomistan basketball, again

Players shoot synchronously.

Player 1:
Probability of scoring = $p < 1/2$
Shoots until he has r successes.
X_1 is the attempt when that happened.

Player 2:
Probability of scoring = mp for some integer $1 < m$ so that $mp < 1$
Shoots until she has mr successes.
X_2 is the attempt when that happened.



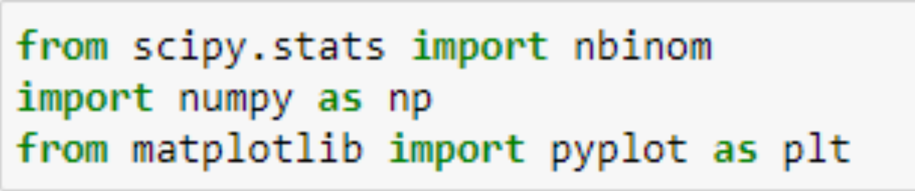
- Which is higher $\mathrm{E}(X_1)$ or $\mathrm{E}(X_2)$?
- Which is higher V(X_1) or V(X_2)?
- Placing a bet on $X_1 > X_2$? (Player 2 is better)

$$\mathrm{E}(X_1) = \frac{r}{p} = \frac{mr}{mp} = E(X_2)$$

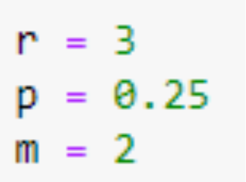
$$\mathrm{V}(X_1) = \frac{r(1-p)}{p^2} \; ? \; \frac{mr(1-mp)}{(mp)^2} = V(X_2)$$

From: DeGroot, Probability and Statistics

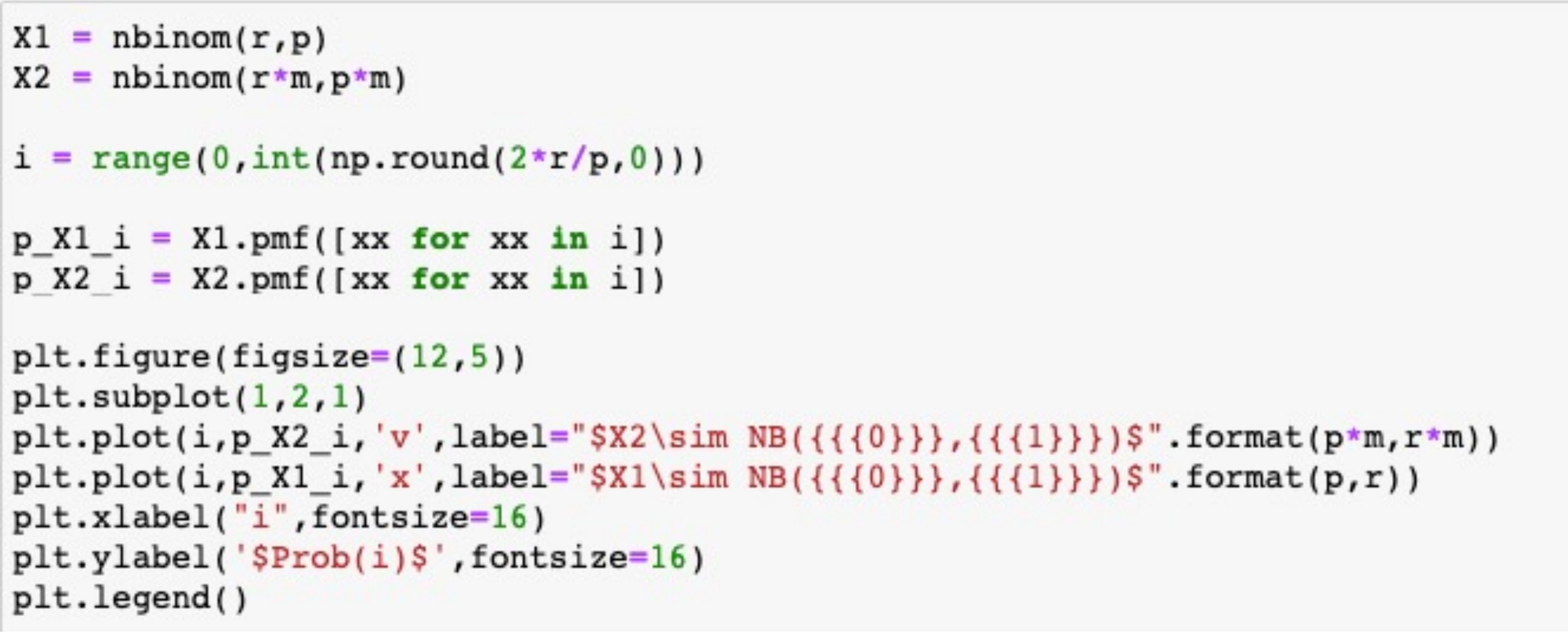
scipy.stats.nbinom



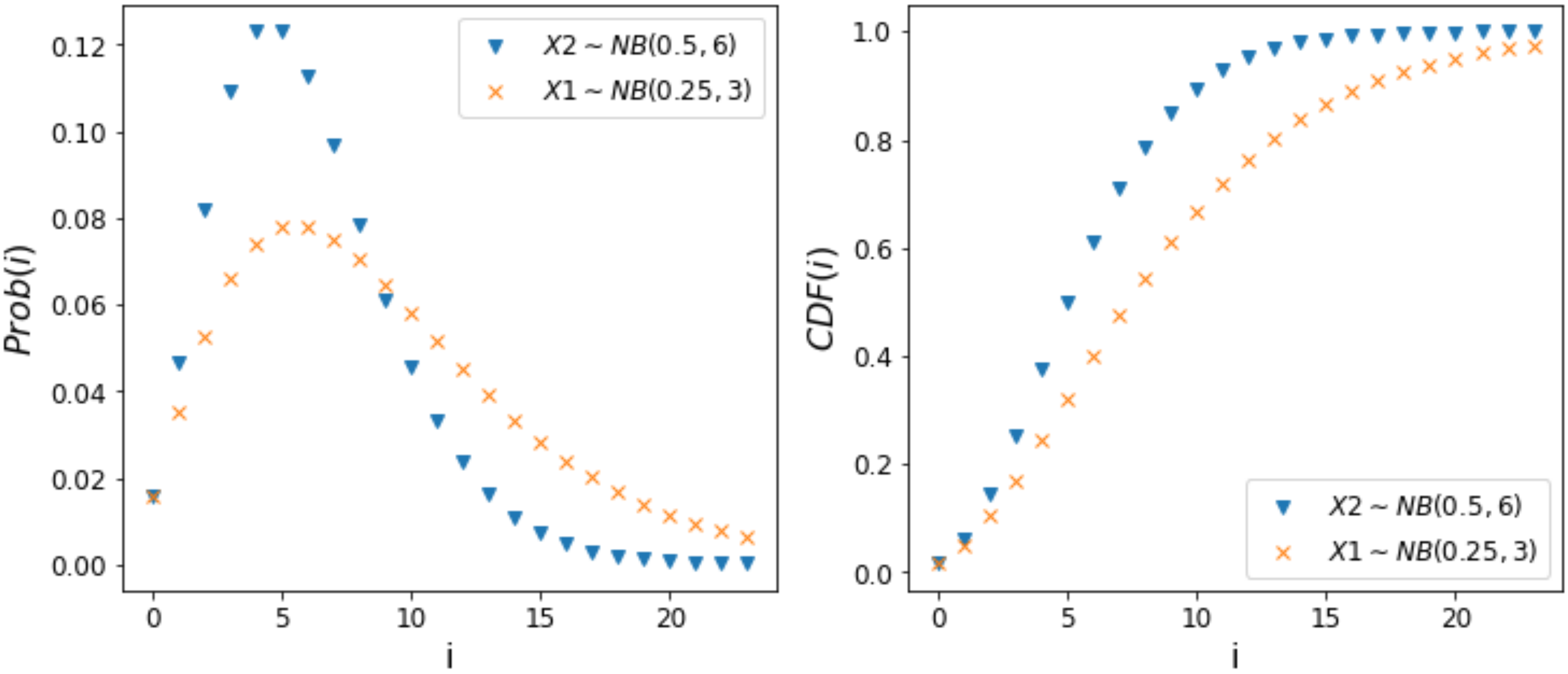
```python
from scipy.stats import nbinom
import numpy as np
from matplotlib import pyplot as plt
```

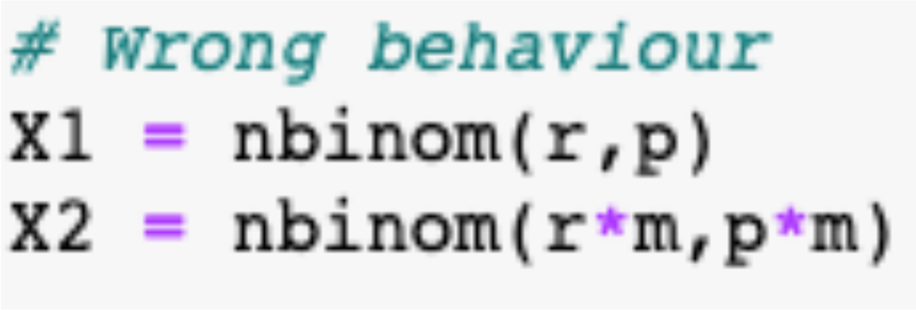


```python
r = 3
p = 0.25
m = 2
```

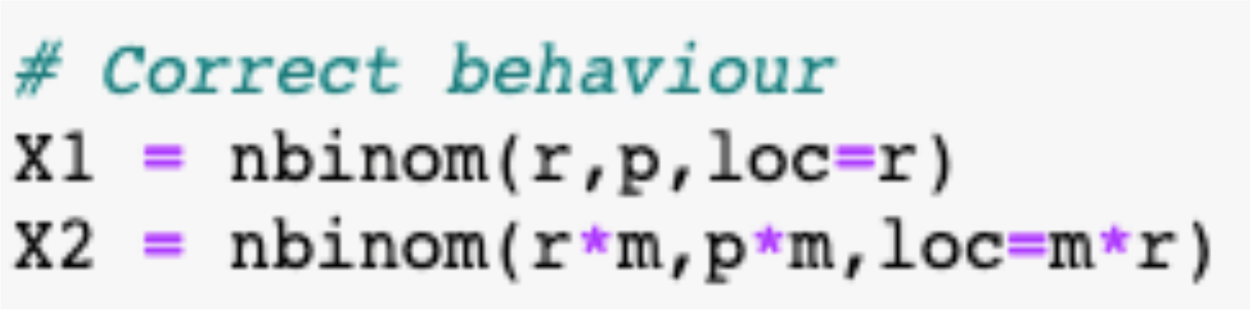


```python
X1 = nbinom(r,p)
X2 = nbinom(r*m,p*m)

i = range(0,int(np.round(2*r/p,0)))

p_X1_i = X1.pmf([xx for xx in i])
p_X2_i = X2.pmf([xx for xx in i])

plt.figure(figsize=(12,5))
plt.subplot(1,2,1)
plt.plot(i,p_X2_i,'v',label="$X2\sim NB({{{0}}},{{{1}}})$".format(p*m,r*m))
plt.plot(i,p_X1_i,'x',label="$X1\sim NB({{{0}}},{{{1}}})$".format(p,r))
plt.xlabel("i",fontsize=16)
plt.ylabel('$Prob(i)$',fontsize=16)
plt.legend()
```

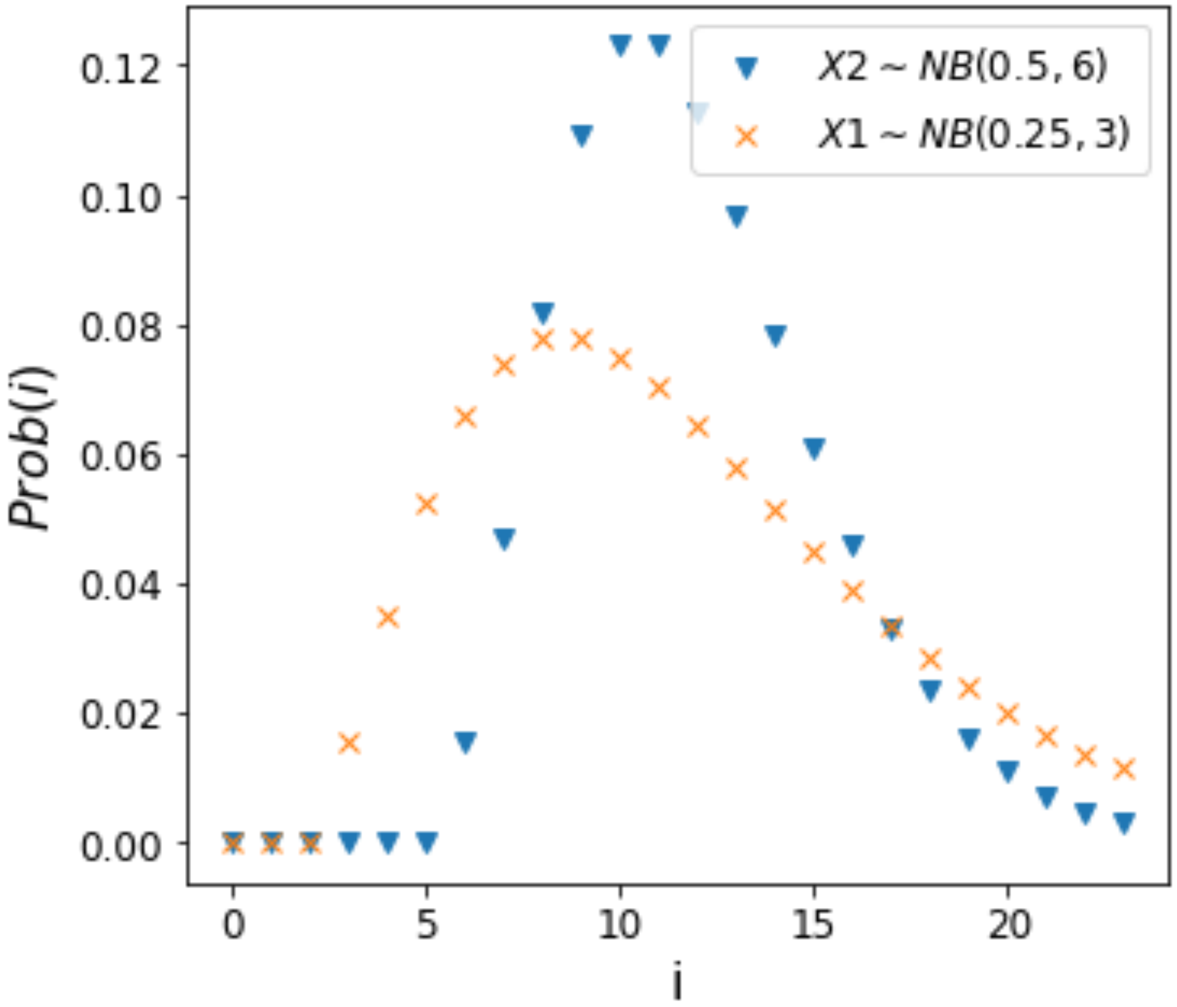

Prob(i)
0.12
0.10
0.08
0.06
0.04
0.02
0.00
0
5
10
15
20
i
▼ $X2 \sim NB(0.5, 6)$
× $X1 \sim NB(0.25, 3)$
CDF(i)
1.0
0.8
0.6
0.4
0.2
0.0
0
5
10
15
20
i
▼ $X2 \sim NB(0.5, 6)$
× $X1 \sim NB(0.25, 3)$

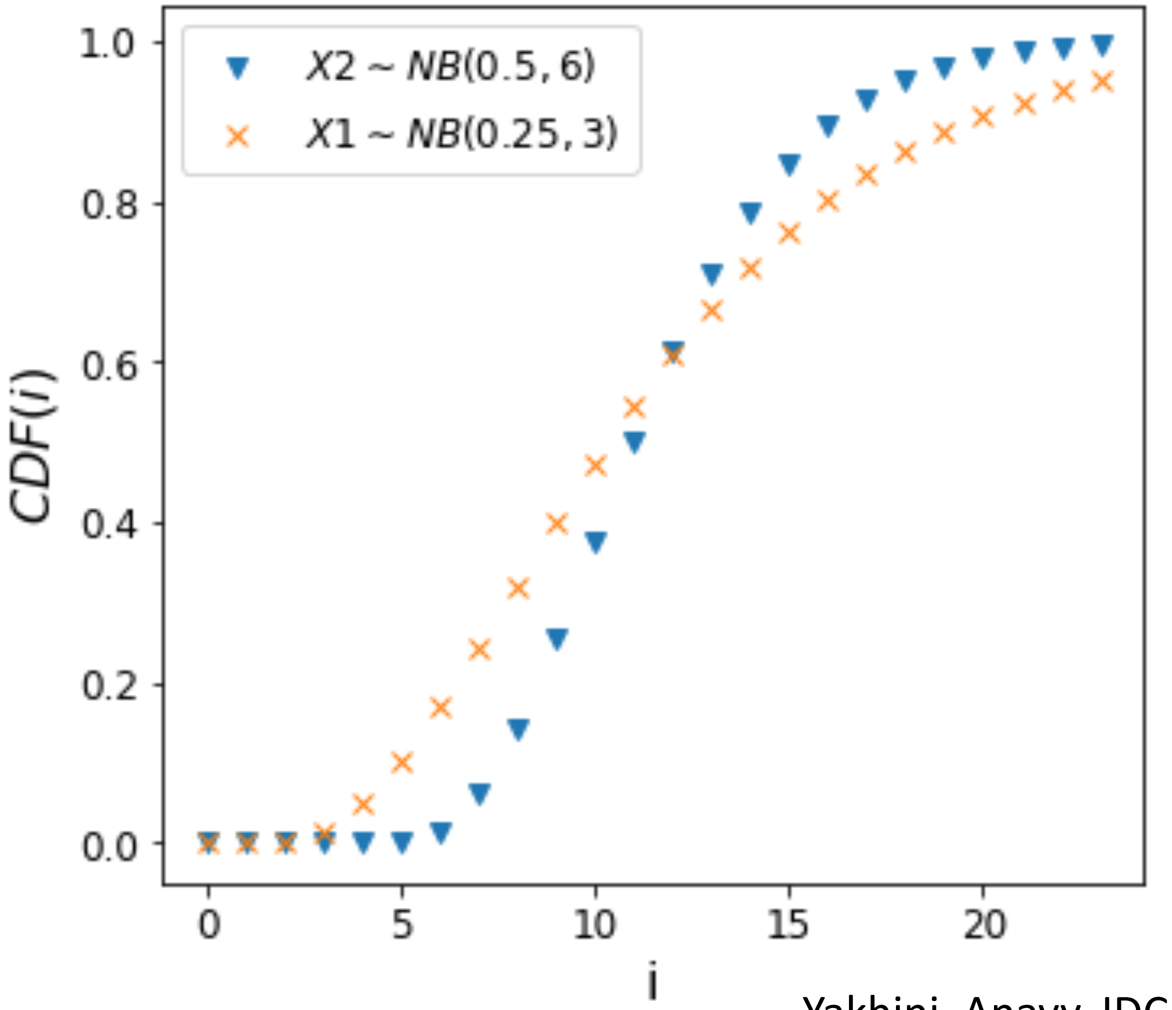


```
# Wrong behaviour
X1 = nbinom(r,p)
X2 = nbinom(r*m,p*m)
```



```
# Correct behaviour
X1 = nbinom(r,p,loc=r)
X2 = nbinom(r*m,p*m,loc=m*r)
```



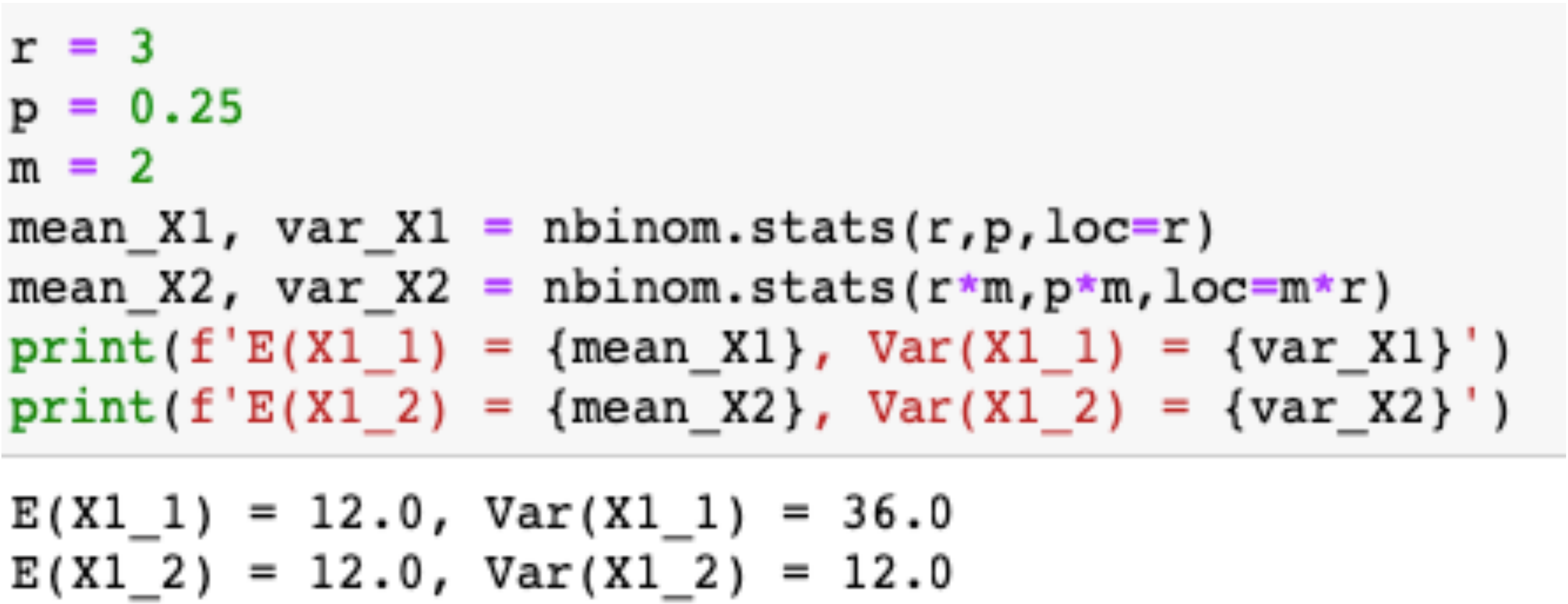


Note: E vs mod of a distribution

Yakhini, Anavy, IDC

Randomistan basketball story

- Which is higher $\mathrm{E}(X_1)$ or $\mathrm{E}(X_2)$?
- Which is higher V(X_1) or V(X_2)?
- Placing a bet on $X_1 > X_2$? (Player 2 is better)



```
r = 3
p = 0.25
m = 2
mean_X1, var_X1 = nbinom.stats(r,p,loc=r)
mean_X2, var_X2 = nbinom.stats(r*m,p*m,loc=m*r)
print(f'E(X1_1) = {mean_X1}, Var(X1_1) = {var_X1}')
print(f'E(X1_2) = {mean_X2}, Var(X1_2) = {var_X2}')
```

```
E(X1_1) = 12.0, Var(X1_1) = 36.0
E(X1_2) = 12.0, Var(X1_2) = 12.0
```

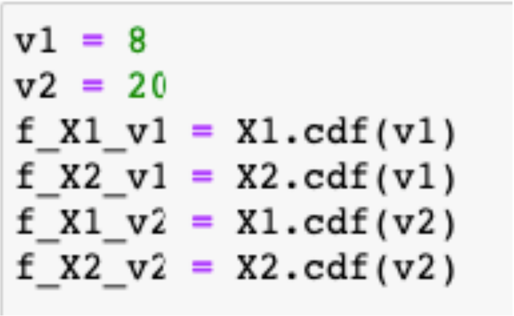
How to assess betting on the players?

- Placing a bet on $X_1 > X_2$? (Player 2 is better)

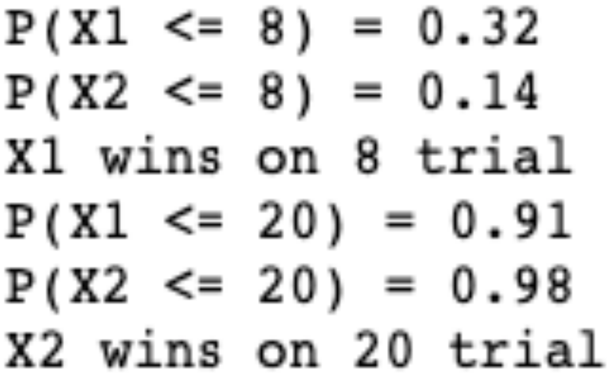
We can choose a player and bet on whether they succeed before 8, before 20. Which player should we prefer?

Calculate $P(X_1 \leq 8)$ and $P(X_2 \leq 8)$

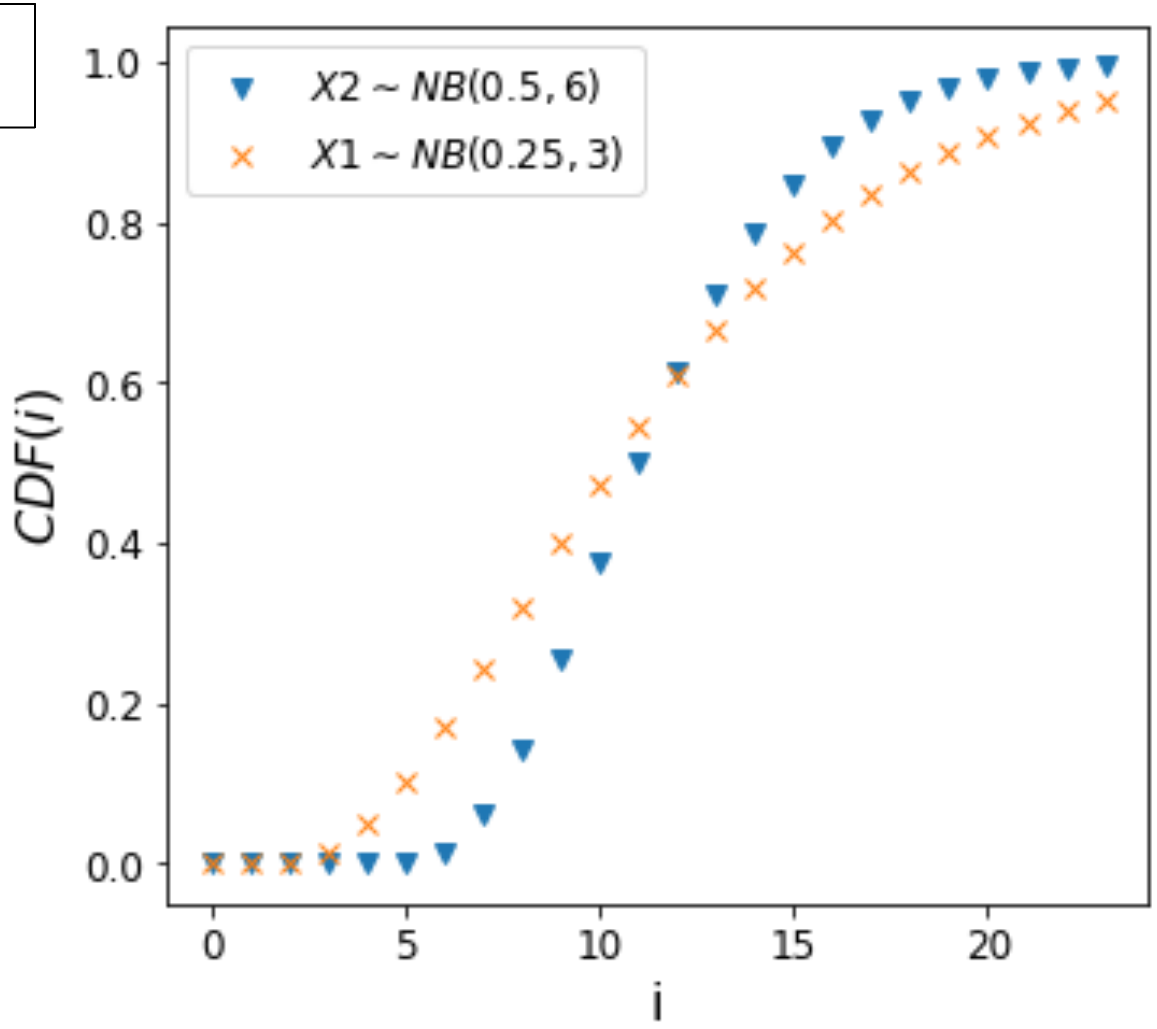
Calculate $P(X_1 \leq 20)$ and $P(X_2 \leq 20)$



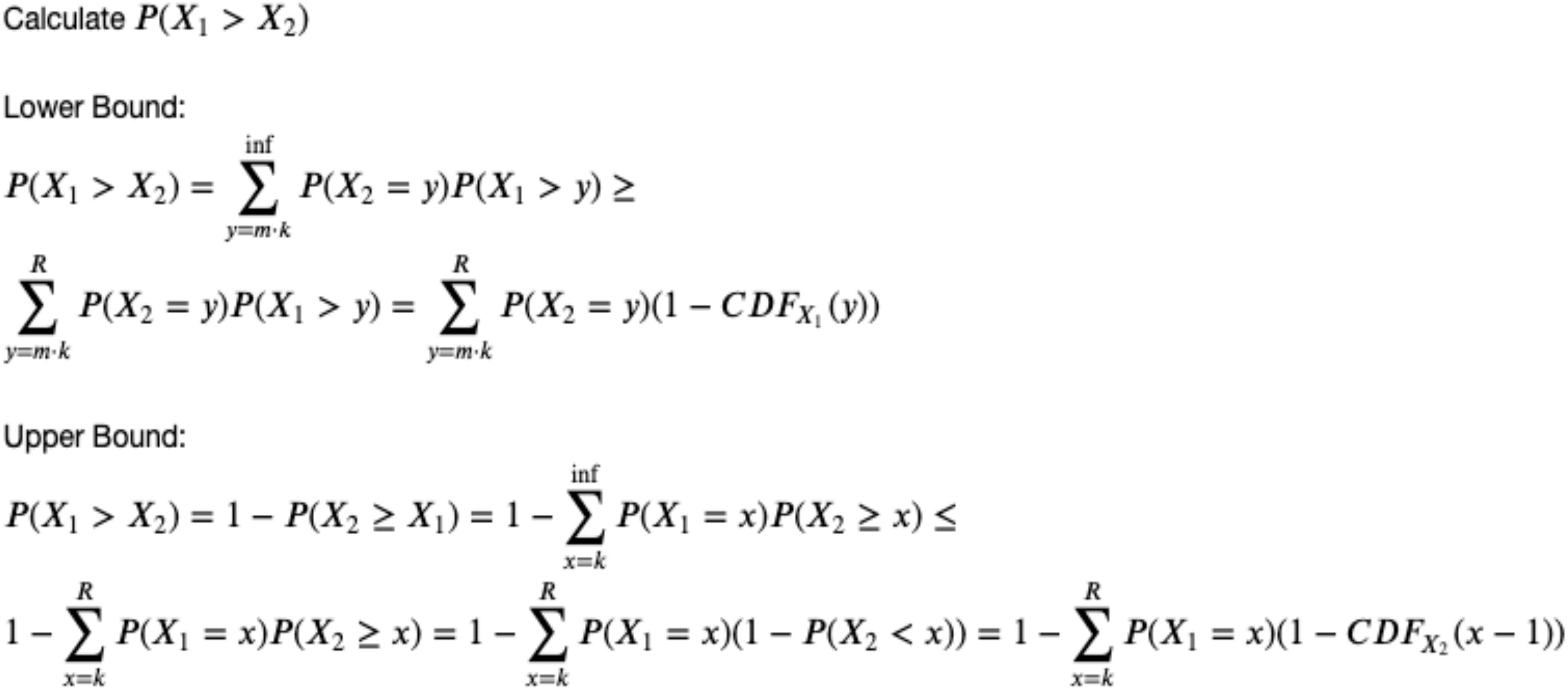
```
v1 = 8
v2 = 20
f_X1_v1 = X1.cdf(v1)
f_X2_v1 = X2.cdf(v1)
f_X1_v2 = X1.cdf(v2)
f_X2_v2 = X2.cdf(v2)
```



```
P(X1 <= 8) = 0.32
P(X2 <= 8) = 0.14
X1 wins on 8 trial
P(X1 <= 20) = 0.91
P(X2 <= 20) = 0.98
X2 wins on 20 trial
```



Who completes the task earlier?
Computer age statistics



Calculate $P(X_1 > X_2)$

Lower Bound:

$$P(X_1 > X_2) = \sum_{y=m\cdot k}^{\text{inf}} P(X_2 = y)P(X_1 > y) \geq$$

$$\sum_{y=m\cdot k}^{R} P(X_2 = y)P(X_1 > y) = \sum_{y=m\cdot k}^{R} P(X_2 = y)(1 - CDF_{X_1}(y))$$

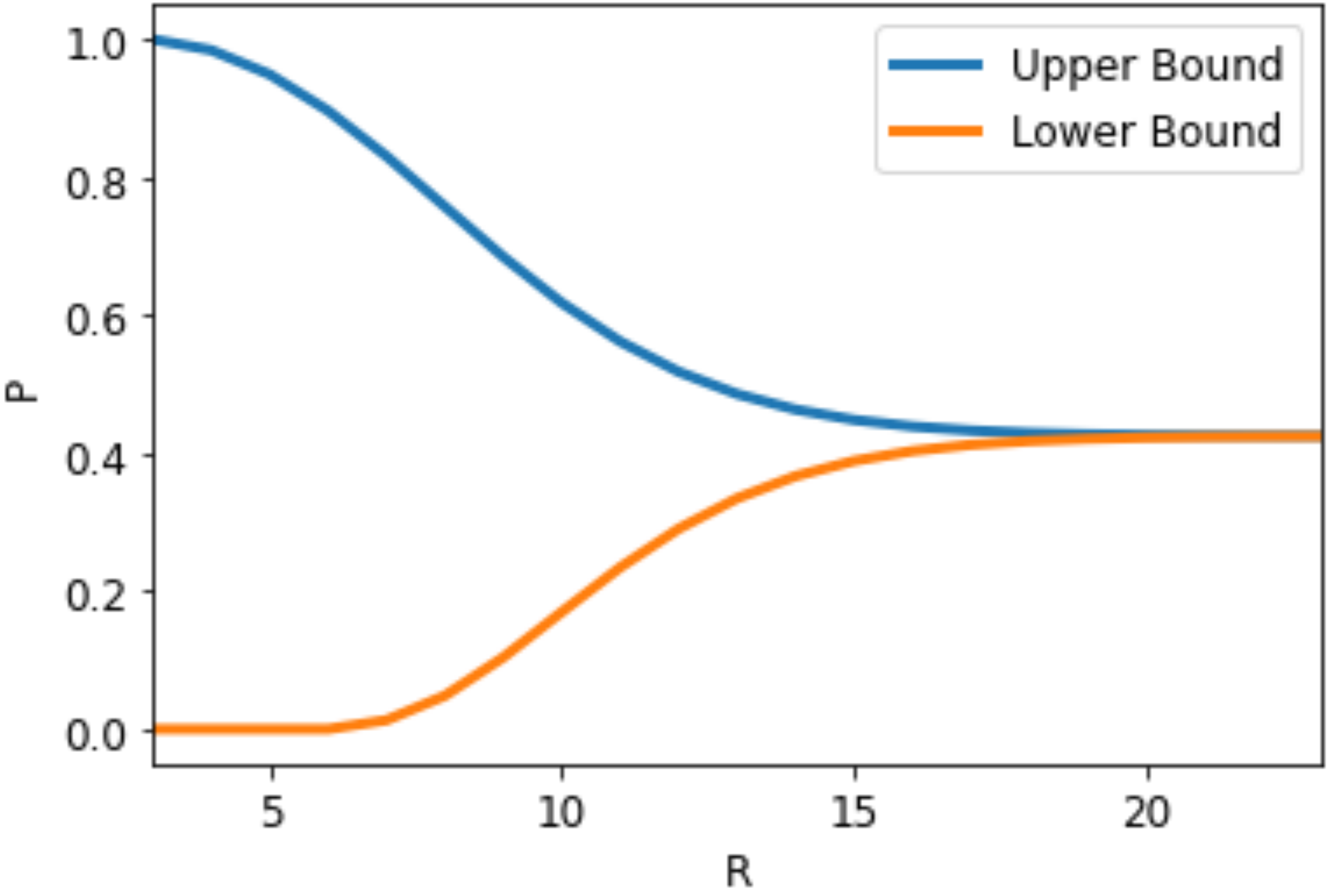
Upper Bound:

$$P(X_1 > X_2) = 1 - P(X_2 \geq X_1) = 1 - \sum_{x=k}^{\text{inf}} P(X_1 = x)P(X_2 \geq x) \leq$$

$$1 - \sum_{x=k}^{R} P(X_1 = x)P(X_2 \geq x) = 1 - \sum_{x=k}^{R} P(X_1 = x)(1 - P(X_2 < x)) = 1 - \sum_{x=k}^{R} P(X_1 = x)(1 - CDF_{X_2}(x - 1))$$

$$P(X_1 > X_2)$$

Calculate $P(X > Y)$



$$\mathrm{P}(\mathrm{X}_1 > X_2) \in [0.4246, 0.4251]$$

Sample/Coupon collection

The RV T counts the number of observations required to see at least m=1 users from each country of the n=100. How many visits will it take if every visit comes from each of the countries with equal probabilities and independent of all previous visits?

$$T \;= X_1 + X_2 + X_3 + \ldots + X_i + \;\ldots + X_{99} + X_{100}$$

Where the random variable X_i counts the number of visits, after the first $i-1$ countries are in, until the i-th country is also in.

Sample/Coupon collection

We saw

$$E(T) = E(X_1 + X_2 + X_3 + \dots + X_i + \ \dots + X_{99} + X_{100}) = \sum_{i=1}^{100} E(X_i)$$

Note that $X_i \sim Geom\left(p_i = \frac{100-i+1}{100}\right)$ and we therefore have $E(X_i) = \frac{1}{p_i} = \frac{100}{100-i+1}$

So:

$$E(T) = 100\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ \dots + \frac{1}{100}\right)$$

$$\underline{E(T) = nH(n) \sim n \ln n.}$$

Sample/Coupon collection

How can we use Chebyshev's inequality to get a bound on:

$$P(T > nH(n) + cn) \text{ ?}$$

$$P(|X - \mu| \geq \lambda) \leq \frac{\mathrm{V(X)}}{\lambda^2}$$

Sample/Coupon collection

$$P(|X-\mu| \geq \lambda) \leq \frac{\mathrm{V(X)}}{\lambda^2}$$

$$P(|T-E(T)| \geq \lambda) = P(T \leq E(T) - \lambda) + P(T \geq E(T) + \lambda)$$

$$\geq P(T \geq E(T) + \lambda)$$

$$P(T \geq nH(n) + cn) \leq \frac{\mathrm{V(T)}}{c^2 n^2}$$

Sample/Coupon collection

What about the Variance of T?

$$Var(T) = Var(X_1 + X_2 + X_3 + \ldots + X_i + \ \ldots + X_{99} + X_{100}) = \sum_{i=1}^{100} Var(X_i)$$

$X_i \sim Geom\left(p_i = \frac{100-i+1}{100}\right)$ and we therefore have $Var(X_i) = \frac{1-p_i}{p_i^2}$

So:

$$Var(T) = \sum_{i=1}^{100} \frac{1-p_i}{p_i^2} = \sum_{i=1}^{100} \frac{1}{p_i^2} - \sum_{i=1}^{100} \frac{1}{p_i} = 100^2 \sum_{i=1}^{100} \frac{1}{i^2} - 100 \sum_{i=1}^{100} \frac{1}{i}$$

$$Var(T) = n^2 \sum_{i=1}^{n} \frac{1}{i^2} - nH(n)$$

Sample/Coupon collection

$$Var(T) = n^2 \sum_{i=1}^{n} \frac{1}{i^2} - nH(n) < n^2 \sum_{i=1}^{n} \frac{1}{i^2} < n^2 \frac{\pi^2}{6}$$

$$P(T \geq nH(n) + cn) \leq \frac{\mathrm{V(T)}}{c^2 n^2} < \frac{\pi^2}{6c^2}$$

Taking $n = 100$:

$c = 2$: $P(T \geq 518 + 200) < \frac{\pi^2}{24} = 0.4112$

$c = 3$: $P(T \geq 518 + 300) < \frac{\pi^2}{54} = 0.1828$

$c = 4$: $P(T \geq 518 + 400) < \frac{\pi^2}{96} = 0.1028$

Computer age statistics

We can calculate the true value of the variance and use this for the Chebyshev bound:

$$Var(T) = n^2 \sum_{i=1}^{n} \frac{1}{i^2} - nH(n)$$

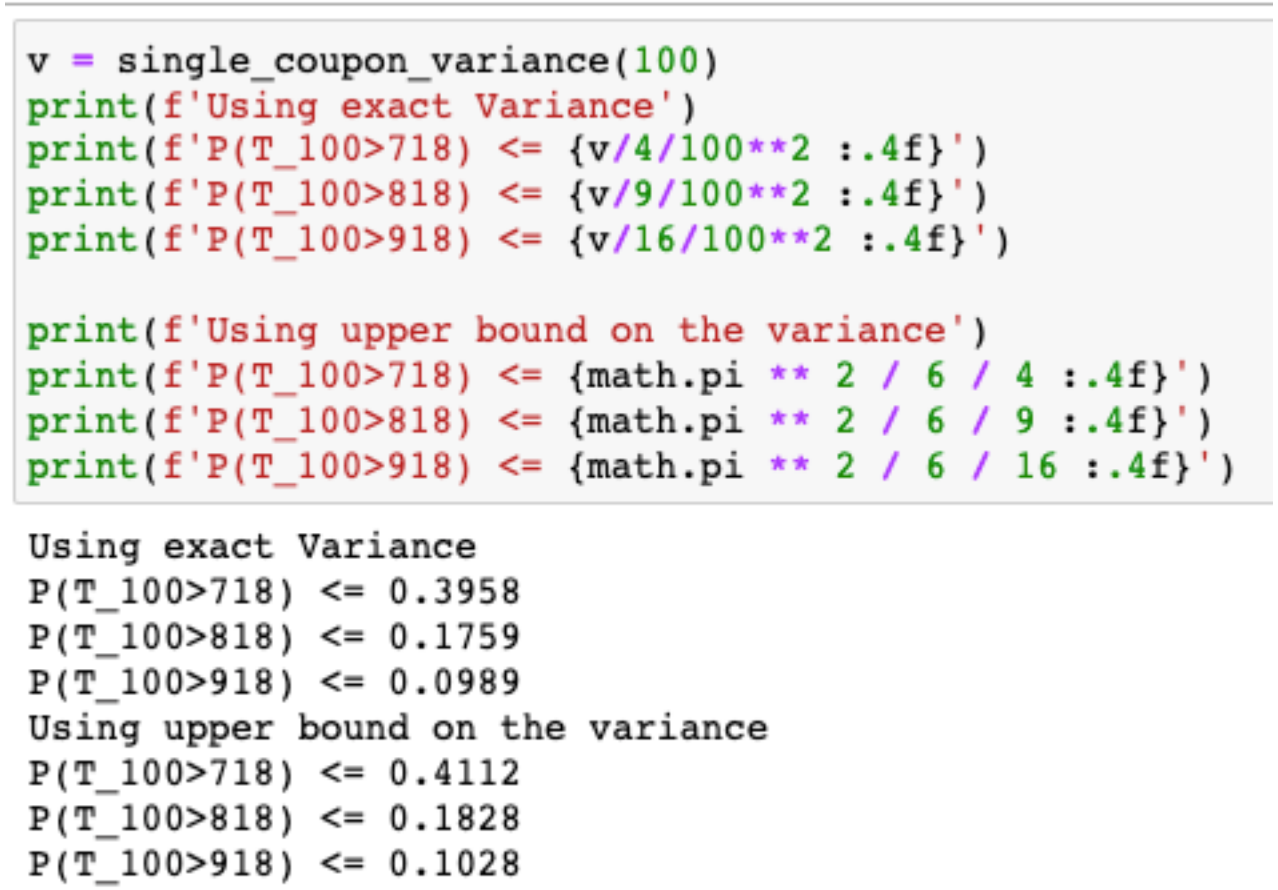
$$P(T \geq nH(n) + cn) \leq \frac{\mathrm{V(T)}}{c^2 n^2}$$

Taking $n = 100$: $Var(T) = 16449$

$c = 2$: $P(T \geq 518 + 200) \leq 0.3958$

$c = 3$: $P(T \geq 518 + 300) \leq 0.1759$

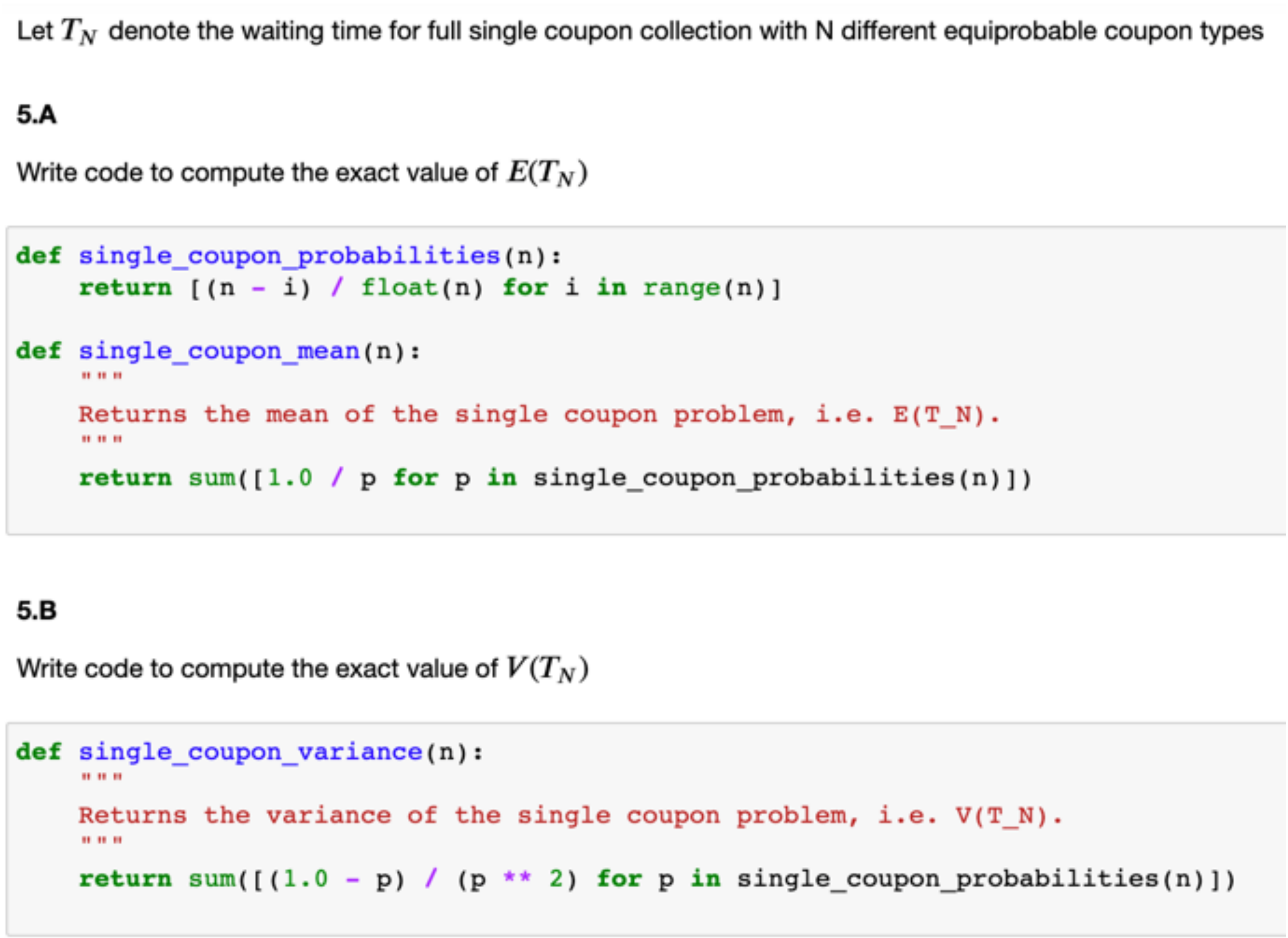
$c = 4$: $P(T \geq 518 + 400) \leq 0.0989$



```
v = single_coupon_variance(100)
print(f'Using exact Variance')
print(f'P(T_100>718) <= {v/4/100**2 :.4f}')
print(f'P(T_100>818) <= {v/9/100**2 :.4f}')
print(f'P(T_100>918) <= {v/16/100**2 :.4f}')

print(f'Using upper bound on the variance')
print(f'P(T_100>718) <= {math.pi ** 2 / 6 / 4 :.4f}')
print(f'P(T_100>818) <= {math.pi ** 2 / 6 / 9 :.4f}')
print(f'P(T_100>918) <= {math.pi ** 2 / 6 / 16 :.4f}')
```

```
Using exact Variance
P(T_100>718) <= 0.3958
P(T_100>818) <= 0.1759
P(T_100>918) <= 0.0989
Using upper bound on the variance
P(T_100>718) <= 0.4112
P(T_100>818) <= 0.1828
P(T_100>918) <= 0.1028
```

Computer age statistics



Let T_N denote the waiting time for full single coupon collection with N different equiprobable coupon types

5.A

Write code to compute the exact value of $E(T_N)$

```python
def single_coupon_probabilities(n):
    return [(n - i) / float(n) for i in range(n)]

def single_coupon_mean(n):
    """
    Returns the mean of the single coupon problem, i.e. E(T_N).
    """
    return sum([1.0 / p for p in single_coupon_probabilities(n)])
```

5.B

Write code to compute the exact value of $V(T_N)$

```python
def single_coupon_variance(n):
    """
    Returns the variance of the single coupon problem, i.e. V(T_N).
    """
    return sum([(1.0 - p) / (p ** 2) for p in single_coupon_probabilities(n)])
```

Taking $n = 100$:

$c = 2$: $P(T \geq 518 + 200) \leq 0.3958$

$c = 3$: $P(T \geq 518 + 300) \leq 0.1759$

$c = 4$: $P(T \geq 518 + 400) \leq 0.0989$

Sample/Coupon collection

How can we calculate the probability directly:

$$P(T > nH(n) + cn)\ ?$$

$$T \ = X_1 + X_2 + X_3 + \ldots + X_i + \ \ldots \ + X_{99} + X_{100}$$

Sums of independent random variables

Let $X$and Y be two independent random variables. Let $Z = X + Y$. Then

$$P(Z = z) = \sum_{i=-\infty}^{\infty} P(X = i)P(Y = z - i)$$

For continuous random variables, the density function of Z is:

$$h(z) = \int_{-\infty}^{\infty} f(t)g(z - t)dt$$

Computer age statistics

We can use convolutions to compute the actual FULL (or rather – the interesting part) distribution of T_N :

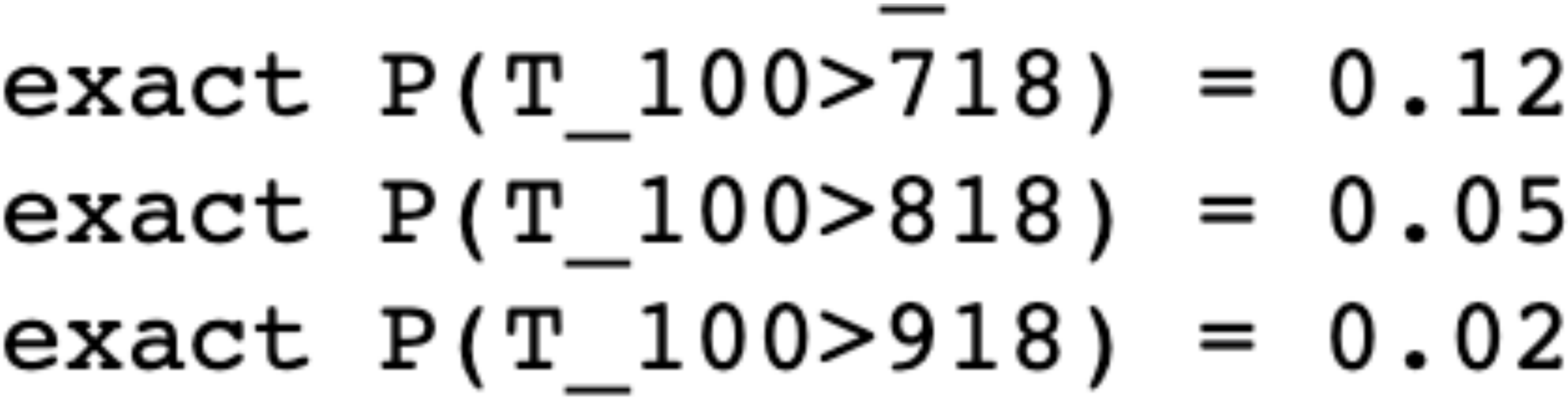
$$P(T_N = k) = \sum_{i=-\infty}^{\infty} P(G_N = i)P(T_{N-1} = k - i)$$

$$= \sum_{i=1}^{k-1} P(G_N = i)P(T_{N-1} = k - i)$$

where $G_s \sim Geo(\, p = \frac{N-s+1}{N})$.

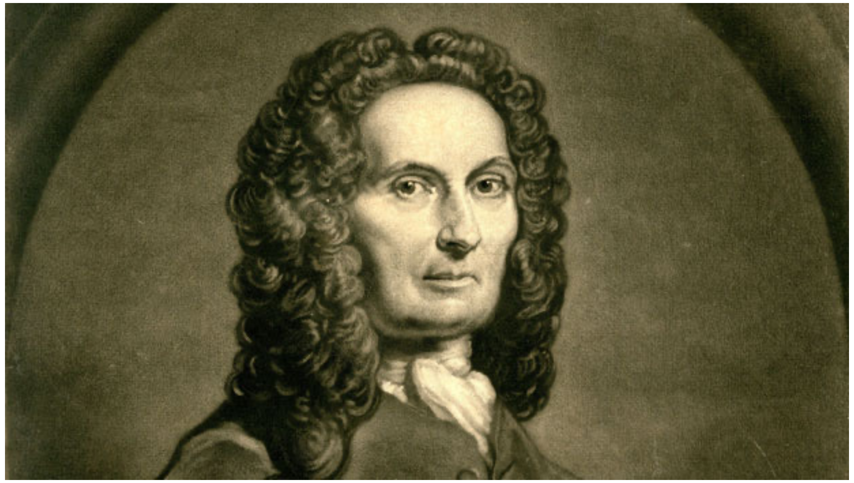
We need to initialize this with $P(T_1 = 1) = 1$ and 0 for all other values.

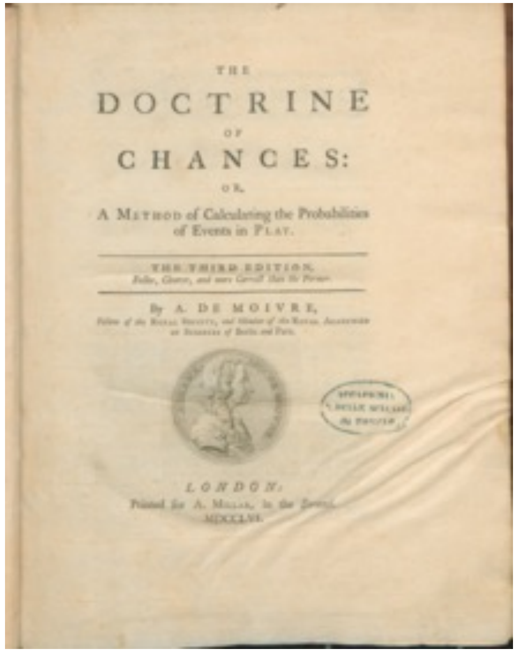
Exact coupon collector waiting time



```
exact P(T_100>718) = 0.12
exact P(T_100>818) = 0.05
exact P(T_100>918) = 0.02
```

Sample/coupon collector and computers: Historical notes





THE

DOCTRINE

OF

CHANCES:

OR,

A METHOD of Calculating the Probabilities
of Events in PLAY.

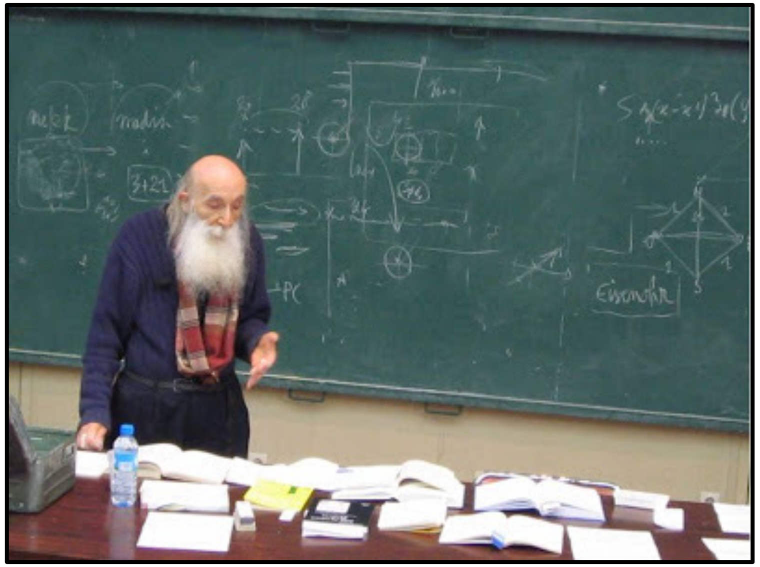
THE THIRD EDITION,
Fuller, Clearer, and more Correct than the Former.

By A. DE MOIVRE,
Fellow of the ROYAL SOCIETY, and Member of the ROYAL ACADEMIES
OF SCIENCES of Berlin and Paris.

LONDON:
Printed for A. MILLAR, in the Strand.
MDCCLVI.

- The T_N discussion dates back to Abraham de Moivre 1667 (France) – 1754 (England)
- Rigorously treated by William Feller in the 1940s
- Variants are still being studied as an active field of research

Jean Paul Benzecri, French statistician (1932-2019):
"It is unthinkable to use methods conceived before the invention of the computer. Statistics will have to be completely rewritten!"
Stated in 1965.



Pairwise independence

A set of random variables $(X_1, X_2, \dots, X_n)$ is said to be pairwise independent if any two random variables X_i and X_j are independent.

Recall – a set of random variables as above is called (collectively or mutually) independent if

$$\forall(x_1, x_2, \dots, x_n)$$

$$P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = \prod_{i=1}^{n} P(X_i = x_i)$$

Equivalence?

- Does collective independence imply pairwise independence?
- Does pairwise independence imply collective independence?

Var of a sum?

Pairwise independence is sufficient for the linearity of variances.

Let X and Y two independent Bernoulli w $p = \frac{1}{2}$.
Let $Z = XOR(X, Y)$.
We work in $\Omega = \{0,1\}^3$.

We have the following joint probability mass function:

X	Y	Z	P
0	0	0	0.25
0	1	1	0.25
1	0	1	0.25
1	1	0	0.25

$X + Y + Z$ vs $Binom(0.5,3)$

$$E(X + Y + Z) = ?$$
$$V(X + Y + Z) = ?$$

X	Y	Z	P
0	0	0	0.25
0	1	1	0.25
1	0	1	0.25
1	1	0	0.25

Multinomial Distribution

Roll a die n times, Y counts the number of 5's

- What is the distribution of Y?
- Can you treat this as a coin? What is p?
- What is $\mathrm{E}(Y)$?
- What is $\mathrm{V}(Y_i)$?

Multinomial Distribution

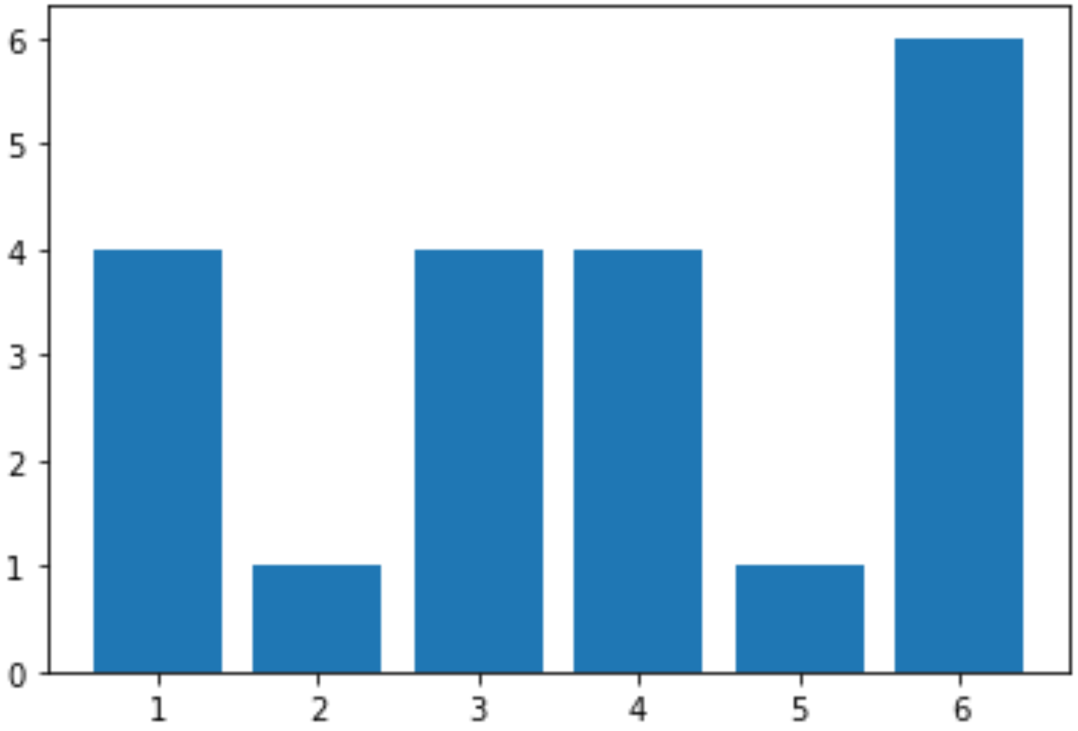
Now roll a die n times and define $X \ = (X_1, X_2\ , \dots , X_6)$ where X_i counts the number of i’s

$$X \ = (X_1, X_2\ , \dots , X_d) \sim \text{Multinomial}(n, p)$$

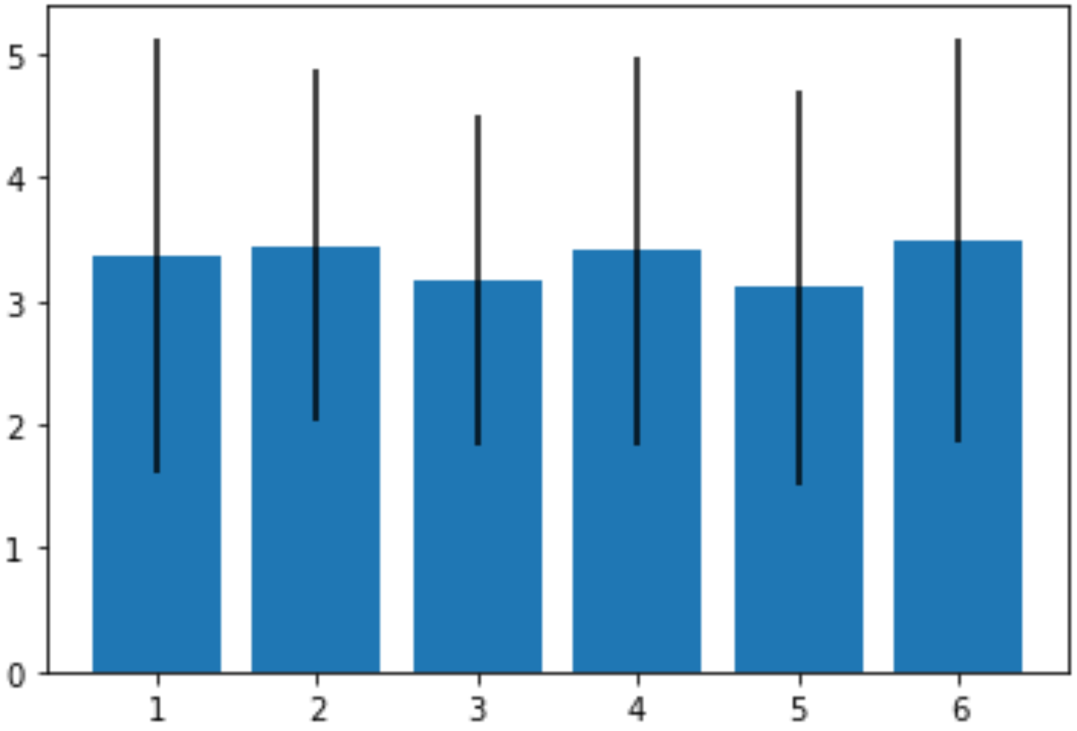
- What is d?
- What is p?
- What is $\mathrm{E}(X_i)$?
- What is $\mathrm{V}(X_i)$?

Multinomial Distribution

$$X = (X_1, X_2, \dots, X_6) \sim \text{Multinomial}\left(20, \left(\frac{1}{6}, \dots, \frac{1}{6}\right)\right)$$



One experiment



100 repeated experiments, with avergaging

Multinomial Distribution

Now roll a die n times and define $X \ = (X_1, X_2 \ , \ldots, X_6)$ where X_i counts the number of i's

$$X \ = (X_1, X_2 \ , \ldots, X_d) \sim \text{Multinomial}(n, p)$$

- Are these random variables collectively independent?
- Pairwise independent?

Multinomial Distribution - covariances

Let $X \sim \mathrm{MNom}(N, P)$, $X = (X_1, X_2, \dots, X_d)$.

$$Var\left(X_i + X_j\right) = V(X_i) + 2Cov\left(X_i, X_j\right) + V\left(X_j\right)$$

Now observe that $X_i + X_j \sim \mathrm{Binom}(N, p_i + p_j)$ and therefore, from the above identity we get:

$$2Cov\left(X_i, X_j\right) = Var\left(X_i + X_j\right) - V(X_i) - V\left(X_j\right) =$$

$$= N\left[\left(p_i + p_j\right)\left(1 - p_i - p_j\right) - p_i(1 - p_i) - p_j(1 - p_j)\right]$$

$$= -2Np_ip_j$$

Multinomials, example

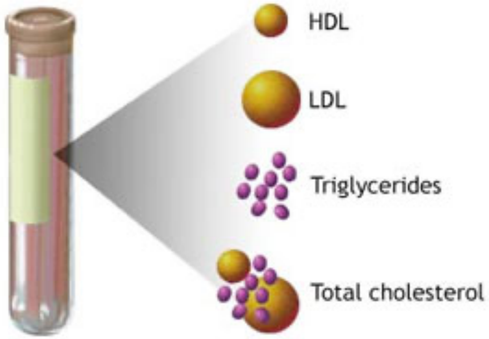
Let $X \sim \mathrm{MNom}(N, P)$, $X = (X_1, X_2 , \ldots , X_{10})$

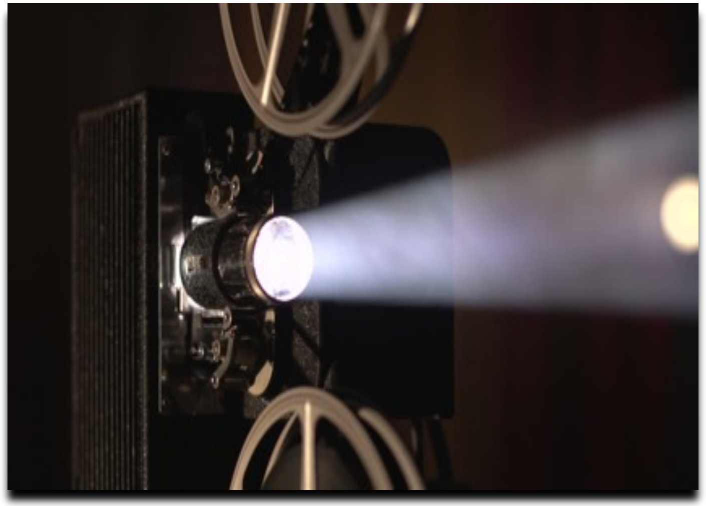
Also assume that P is uniform $\frac{1}{10}$

What is $Cov(X_1 + X_2, X_7 + X_8)$?

Conditional independence

- Is the blood cholesterol level of a person independent of the number of movies watched by that person so far?
- No – they are both related to the age of the person.
- But – they are conditionally independent given the age.
- Presumably …, socioeconomic and behavioral factors ignored …
- Notation: $X \perp Y \mid C$





Conditional Independence - Definition

Two random variables X and Y are conditionally independent given a third rv C if <u>for all</u> pairs (x, y) AND <u>for all</u> possible values c of C, we have:

$$\boxed{P\big((X = x \ \wedge \ Y = y)|C = c\big) = P\big((X = x)|C = c\big) \cdot P\big((Y = y)|C = c\big)}$$

Conditionally independent but not independent?

Independent but not conditionally independent?

Summary

- Computer age statistics:
  - + Comparing negative binomials
  - + Coupon collector – exact calculations
  - + Independence and convolutions
- Mutual indpce vs lower order indpce
- Multinomials
- Conditional independence