Lecture Notes 1 Introduction to Fourier Optics

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Meaning of * I write these notes to be more extensive than what the student learning goals, sections marked with * are overly complex/philosophical and serve only the curious student with extra time to spend. Instead of reading the section, once could equally well just read the starred blurb to see why the section is ignorable, or where to read a more conventional approach.

1. THE SCALAR WAVE EQUATION FROM A QUANTUM MECHANICAL PERSPECTIVE*

*I recommend early sections of chapter 8 of David Morin's *Waves (Draft)* for a more straightforward approach. This approach is mostly "for fun" (fun is subjective), but also shows the utility of complex numbers by shoeing in a basis in quantum mechanics.

As a physics student one is exposed heavily to quantum mechanics. One will be familiar with wave functions for massive particles, which you have seen follow Schrödinger's equation for a massive non-relativistic particle as follows:

$$i\hbar\frac{d}{dt}|\psi(x)\rangle = \hat{H}_{nr}|\psi(x)\rangle \tag{1}$$

$$\hat{H}_{nr} = U - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \tag{2}$$

Note the notation of giving operators a hat, the scalar valued classical Hamiltonian H is converted into $\hat{H}$ by replacing momentum and position with their respective operators $\hat{x}$, $\hat{p}$ proportional to x and $\frac{\partial}{\partial x}$. This is an equation of great utility, but will prevent us from exporing any optical phenomenon in two ways. Photons are massless particles that travel at the speed of light. Taking $m = 0$ in (2), there is immediate issue, the coefficient is undefined, the multiplicative inverse of 0. The second is that the Hamiltonian (2) does not account for relativistic kinetic energies. You may have seen low order relativistic corrections for the spectrum of the Hydrogen atom. Here a non approximate relativistic energy is clearly in order, because a photon will be traveling at the speed of light. As an assumption, we take the relativistic kinetic energy to be:

$$T = \sqrt{m_0^2c^4 + p^2c^2} = m_0c^2 + \frac{p^2}{2m} - \frac{p^4}{8m_0^3c^2} + O(p^6) \tag{3}$$

This formula may equally be expressed in terms of rest mass and velocity, but because the quantum mechanical operator for momentum ($\hat{p} = -i\hbar\frac{d}{dx}$) is easier to work with it is the obvious choice. In free space optics, there is no potential term in the energy so it is appropriate to take $U = 0$ resulting in the following Hamiltonian:

$$H_{photon} = \sqrt{m_0^2c^4 + p^2c^2} \tag{4}$$

Then remember that the rest mass m_0 is zero for photons and take the square root.

$$H_{photon} = \pm pc \tag{5}$$

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The next immediate concern is, how to deal with taking an operator with $\pm$ into a formulation of a quantum mechanical wave. It turns out the plus or minus allows for photons to travel to the left or the right. If one had only $H_{photon} = pc$, photons could only move in one direction through space as time went forward. The easiest solution to this is to square both sides.

$$H_{photon}^2 = p^2 c^2 \tag{6}$$

If this Hamiltonian is to be applied to a quantum mechanical wave, there is clearly a problem. No derivative operator with the time coordinate t is present, so there is no hope to find out the time evolution of our photon wave. The Planck-Einstein relation saves the day:

$$E = \hbar\omega \tag{7}$$

Interpreting the Hamiltonian as the total energy E, and taking the square so it fits (6), we come to the final equation on a scalar Hamiltonian that will be dealt with.

$$\hbar^2\omega^2 = p^2 c^2 \tag{8}$$

It is natural to give ω and p an operator interpretation so the photon wave can be inspected in spatial and time coordinates. The following operator view is given, both stemming from the Fourier transform property that derivatives become scalar operations in frequency space.

$$\hat{w} = i\hbar\frac{\partial}{\partial t} \tag{9}$$

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \tag{10}$$

Making the substitution we find that for photons the following equation on operators holds.

$$\frac{\partial^2}{\partial^2 x} = \frac{1}{c^2}\frac{\partial^2}{\partial^2 t} \tag{11}$$

Then act these operators on a photon wave function $|\psi(x,t)\rangle$

$$\frac{\partial^2|\psi(x,t)\rangle}{\partial^2 x} = \frac{1}{c^2}\frac{\partial^2|\psi(x,t)\rangle}{\partial^2 t} \tag{12}$$

This is called the Klein-Gordon equation. It is a case of the wave equation but specifically falling out of a quantum mechanical model for massless particles. The solutions admitted by such a 1d wave equation are very simple, any function of the form $f(x \pm ct)$ as shown for a specific f in figure 1 for the $+$ and $-$ case of $\pm$. The interesting work to be done is for higher dimensions namely 2 or 3 spatial dimensions. In 3 dimensions the momentum squared is $\hbar^2(k_x^2 + k_y^2 + k_z^2)$, and the spatial representation of this operator is $-\hbar^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})$. One extends to 3 dimensions then by letting $\hat{p}^2 \rightarrow -\hbar^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})$, leaving the following wave equation on $|\psi\rangle$:

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})|\psi(\vec{r},t)\rangle = \frac{1}{c^2}\frac{\partial^2|\psi(\vec{r},t)\rangle}{\partial^2 t} \tag{13}$$

One may ask, why not just derive the wave equation from Maxwell's equations with the plane wave view [1]? For computational purposes this is correct, either derivation works. The derivation of the Klein-Gordon equation explains the following important ideas that just working through Maxwell's equation would not admit:

- Why should complex solutions to the wave equation be happily used? Because quantum mechanical evolution admits complex solutions
- Representations of the differential operators in conjugate variables $p, -i\hbar\frac{\partial}{\partial x}$, and $\omega, i\frac{\partial}{\partial t}$

A lot of freedom of interpretation is left for $|\psi(x,t)\rangle$. For our purposes, it will be assumed that many photons co-propagate, and thus the square of the wave function magnitude will give the relative density of photons.

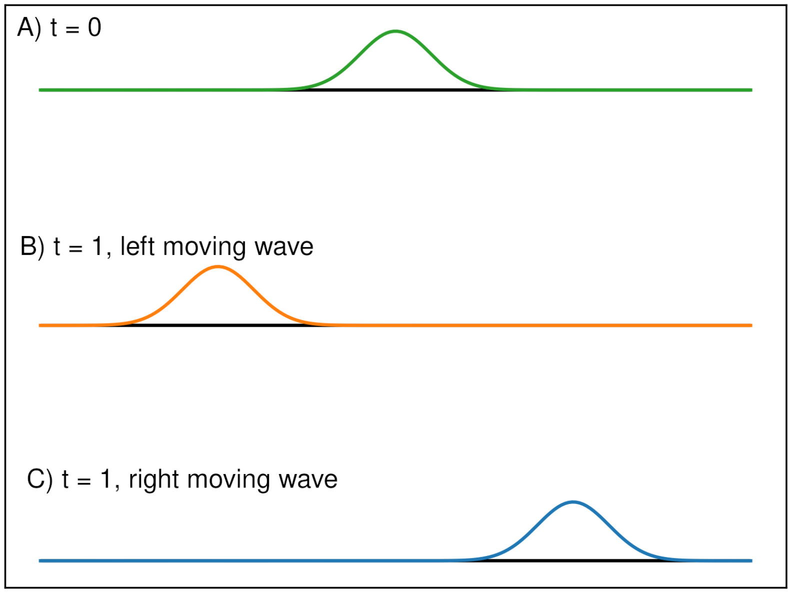

Figure 1. Example of a solution of the scalar wave equation in one dimension, one left moving or the + choice, the other right moving or the − choice.

2. STEADY STATE SOLUTIONS FROM EXPERIMENT: FAR FIELD DIFFRACTION FROM SLITS

The first section discusses a theoretical basis. Diffraction is an example of a laboratory scale phenomenon one should hope to explain with any wave theory of light.

2.1 Experiment

In a diffraction experiment a coherent photon source is shown upon a double slit, and at some distance the illumination pattern is observed. This is illustrated in figure 2.1 where $O(1m)$ expresses the general length scale. The figure shows what is perceived by an observer in the room with experiment, and one can formalize the experiment further by giving a functional form to the pattern of photons observed on the wall. For the experimental case, one may sweep a photo diode across the pattern at the observation distance L, and read the intensity, the results in a function for the intensity $I(x)$ where x is the displacement across the pattern.

2.2 Theoretical approach

The experiment can be seen with varying levels of complexity as will become clear, even within the scope of these lectures. The following view will extract to great accuracy the intensity pattern measured at the observation screen of such a diffraction. Generally regardless of the level of approximation, the input wave amplitude is considered to be some $A(x, y)$ (for the 2d case drop x or y) and the output amplitude is $A'(x', y')$.

Geometry First consider the nature of the slits. Considering the slits to be very wide in one axis, that means many hundreds wavelengths of laser light, and the slit to be very thin in the other axis. As will become apparent later on, the width of the slits in the vertical axis is what allows the pattern to be localized vertically at the observation plane. Intuitively consider that if the slits were not there, this would be equivalent to extremely

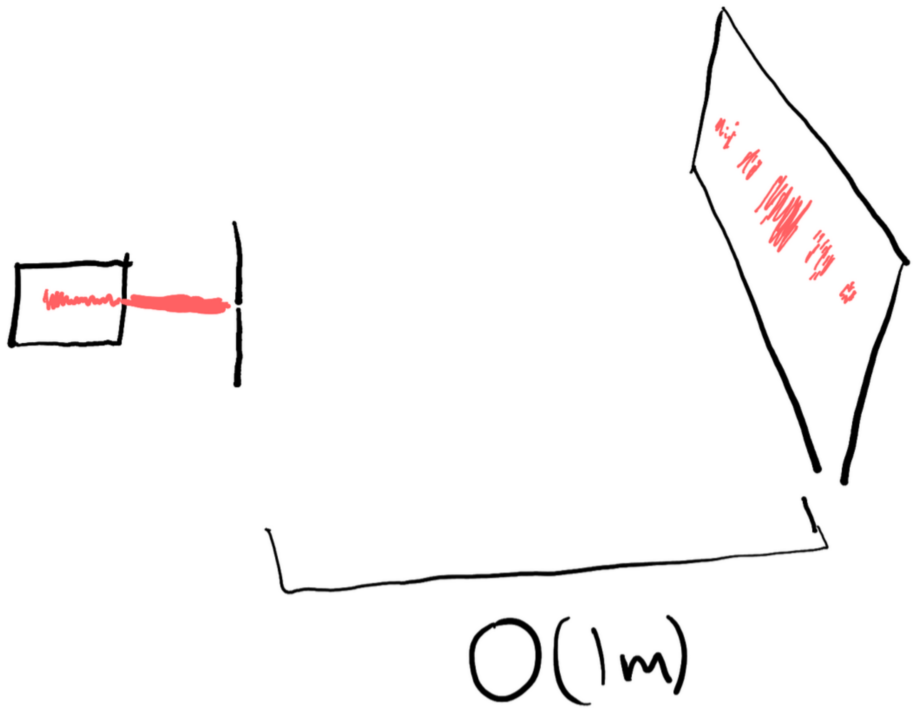

Figure 2. A diffraction experiment with a red laser setup in a small room.

wide slits, such a wide slit would not change the laser propagation as no photons would be incident upon it, and so the laser would act as expected, "pointing" and maintaining a very tight profile as in figure 2.2. This means any interesting intensity profile will be found by analysis along the long axis of the pattern, which is parallel to the thin sides of the slits. The thin axis of the slit is what allows for the slits to be treated as point sources of radiation which have relative phase at the observation plane due to different path lengths causing the interference pattern observed. The functional form of this point source $A_p(r,t) = \frac{e^{i2\pi\frac{r-ct}{\lambda}}}{r}$, where r is the radial distance from the source, will be confirmed further.

Linearity to find the two slit pattern It has been asserted, and will be shown that each of the two slits acts as a point source solution to the 3d wave equation in the axis parallel to the small slit dimension. Then because of the linearity of the wave equation, the sum of two solutions is also a solution, meaning that to find the two slit pattern, one adds up the two spherical point source waves, and calculates their interference pattern at the observation screen. These calculations will be the subject of the first non-coding homework, and just involve some approximations on the geometry of the experiment, and complex phasors. In addition this calculation should have been shown previously in a wave physics or electromagnetism course.

3. SEPARABILITY AND THE FUNCTIONAL FORM OF A RADIATIVE POINT SOURCE, PLANE BOUNDARY CONDITION

In the following analysis the photon wave function $|\psi(\vec{r},t)\rangle$ will be abbreviated as $A(\vec{r},t)$, commonly used notation where A stands for amplitude. A_p will be the amplitude of a point source of radiation. Assuming such a solution will work if all operators are effective only on one variables, so there is no time and space dependent potential, or in the case of free particles, a mixing of a differential on the time coordinate and space coordinate as a single

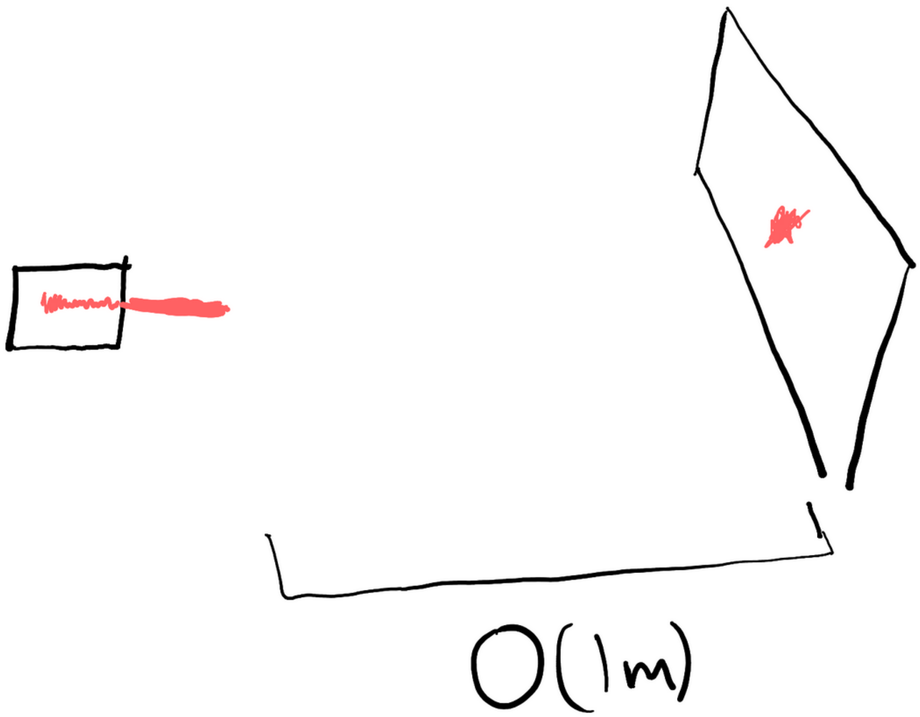

Figure 3. Laser diffraction pattern with no slit.

term.

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})A(\vec{r}, t) = \frac{1}{c^2}\frac{A(\vec{r}, t)}{\partial^2 t} \tag{14}$$

3.1 Separability of the free photon wave equation

As given, solving (14) appears quite daunting. A great and familiar aid in this task is fact that this equation is indeed a separable partial differential equation (PDE). This means we may assume the following form for $A(\vec{r}, t)$ because the PDE governing A is a sum of operators on only one of the coordinates x, y, z, t.

$$A(\vec{r}, t) = T(t)X(x)Y(y)Z(z) \tag{15}$$

This fact will be used indirectly, here in the case of the t coordinate, to simplify the problem, specifically by factoring $e^{i\omega t}$ out of our calculations, the assumption of monochromatic light, which also implies a steady state source.

3.2 Consistency of point sources

A solution for the radiation from an isotropic stationary point source has multiple constraints of symmetry. The first is the conservation of energy. The flux of energy through a surfaces of larger and large radius about the source must be the same, since the only energy available is due to the source. If the source is to remain stationary, then the net momentum of outbound photons from the source must be zero. For a point source the radiation must be spherically symmetric. This necessitates the following for for a point source of constant power:

$$A_p(r, t) = S\frac{e^{i2\pi\frac{r-ct}{\lambda}}}{r} \tag{16}$$

Where λ is the wavelength of the point source, and S the strength and phase. The numerator should be familiar, it looks like a traveling plane wave in the radial axis. The inverse relation to r is such that the radiation follows an inverse square law in intensity, since the intensity at a given radius will be:

$$I_p(r,t)_{radial} = c\hbar\omega |A_p(r,t)|^2 \propto \frac{1}{r^2} \tag{17}$$

$\hbar\omega$ is the energy of one photon, and $|A_p(r,t)|^2$ is proportional to the number density of photons. A factor of c is the radial propagation speed of the waves, which is a familiar factor from the intensity of a plane wave.

3.3 Agreement of the point source with the wave equation

Note that the form of the equation only depends on the variables r, t, so spherical coordinates will be well admitted, since the complexity in the angles θ and ϕ will be seen to immediately drop out. The operator in the left hand side of (14) is known as the Laplacian. The Laplacian in spherical coordinates is:

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r}) + ... \tag{18}$$

Terms contained in ... all have factors of $\frac{\partial A}{\partial \theta}, \frac{\partial A}{\partial \phi}$ in the case that the Laplacian acts on $A(\vec{r}, t)$ and are thus omitted, since the case at hand has only r and t dependence. These coordinates are applied to $A_p(r,t)$ the point source solution.

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})S\frac{e^{i2\pi\frac{r-ct}{\lambda}}}{r} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}S\frac{e^{i2\pi\frac{r-ct}{\lambda}}}{r} \tag{19}$$

$$\frac{S}{r^2}\frac{\partial}{\partial r}(r^2)i\frac{2\pi}{\lambda}\frac{e^{i2\pi\frac{r-ct}{\lambda}}}{r} - \frac{S}{r^2}\frac{\partial}{\partial r}(r^2)\frac{e^{i2\pi\frac{r-ct}{\lambda}}}{r^2} = -(\frac{2\pi}{\lambda})^2(-c)^2\frac{1}{c^2}S\frac{e^{i2\pi\frac{r-ct}{\lambda}}}{r} \tag{20}$$

$$\frac{S}{r^2}\frac{\partial}{\partial r}ir\frac{2\pi}{\lambda}e^{i2\pi\frac{r-ct}{\lambda}} - \frac{S}{r^2}\frac{\partial}{\partial r}e^{i2\pi\frac{r-ct}{\lambda}} = -(\frac{2\pi}{\lambda})^2 S\frac{e^{i2\pi\frac{r-ct}{\lambda}}}{r} \tag{21}$$

$$\frac{S}{r^2}\frac{i2\pi}{\lambda}e^{i2\pi\frac{r-ct}{\lambda}} - (\frac{2\pi}{\lambda})^2\frac{S}{r}e^{i2\pi\frac{r-ct}{\lambda}} - \frac{S}{r^2}(\frac{i2\pi}{\lambda})e^{i2\pi\frac{r-ct}{\lambda}} = -(\frac{2\pi}{\lambda})^2 S\frac{e^{i2\pi\frac{r-ct}{\lambda}}}{r} \tag{22}$$

$$\tag{23}$$

Note the important cancellation on the left hand side.

$$-(\frac{2\pi}{\lambda})^2 S\frac{e^{i2\pi\frac{r-ct}{\lambda}}}{r} = -(\frac{2\pi}{\lambda})^2 S\frac{e^{i2\pi\frac{r-ct}{\lambda}}}{r} \tag{24}$$

The spherical wave obeys the 3d wave equation.

3.4 Plane boundary condition

Care must be taken when applying the result for a point source to a boundary value problem, such as the diffraction double slit experiment that was described. Clearly the plane boundary is the place to start for the double slit experiment, since the laser is shone on a flat plane slit card, imposing a boundary with the geometry of a plane. Here it will be shown to important properties of the point source. A point source's radiation field satisfies the scalar wave equation, or in the specific case the Klein-Gordon equation. Additionally a uniform superposition of point sources matches the boundary condition of an incoming plane wavefront of coherent photons (for example a laser) as shown in 3.5.

This fact will be extrapolated to calculate the propagation of photons for non uniform plane boundary conditions, such as the double slit diffraction, or a laser after it passes through a lens.

3.5 Agreement of the point source with a planewave boundary*

*Spherical waves can be seen by heuristic to act as a Green's function or impulse response for a continuous/steady isotropic source. The intensity of the radiation is infinite at the location of the source point, and finite elsewhere. This means with some trickery/normalization a boundary condition may be recreated by forming a sum of delta function wave sources on the boundary defined by $z = 0$ by convention. Since the system is linear, adding up solutions is allowable to form a new more complicated solution. Below is a first look at the trickery/normalization that shows incident plane waves are equivalent to spherical waves from all points along a plane perpendicular to the wave vector, which is needed to believe a solution formed by sum will be a solution to the intended problem.

Observe integration of the influence all point sources across a plane to one point on a parallel plane displaced z from an initial plane whos points are labeled by r, θ, cylindrical coordinates are chosen as in figure 3.5 to make calculations easiest. Note that the integral is taken to infinite radius, because it is assumed that a long time has passed, and the influence of all point sources may reach the point of observation, although we will weaken this assumption slightly later on.

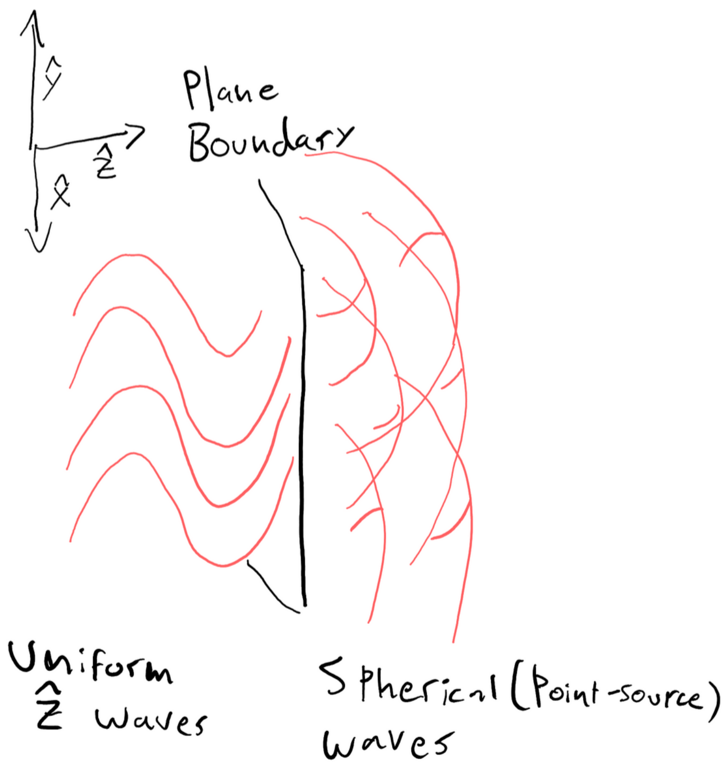

Figure 4. A uniform traveling z plane wave, meets a uniform arrangement of point sources at the boundary $z = 0$

$$\int_0^\infty \int_0^{2\pi} A_p(r,t) = \int_0^\infty \int_0^{2\pi} r d\theta dr \cdot S \frac{e^{i2\pi \frac{\sqrt{r^2+z^2}-ct}{\lambda}}}{\sqrt{r^2+z^2}} \tag{25}$$

Above is the calculated photon amplitude at the cylindrical coordinates of observation for a uniform plane of point sources. This uniform plane of point sources is claimed to match the boundary conditions of a uniform z directed plane wave. Simplifying the expression, (including integrating θ coordinate) gives:

$$\int_0^\infty \int_0^{2\pi} A_p(r,t) = \int_0^\infty 2\pi r dr \cdot S \frac{e^{i2\pi \frac{\sqrt{r^2+z^2}-ct}{\lambda}}}{\sqrt{r^2+z^2}} \tag{26}$$

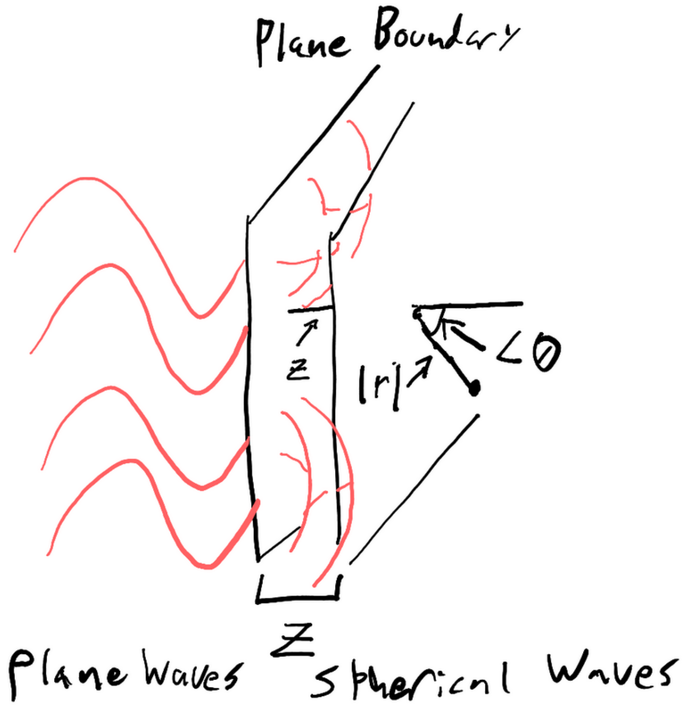

Figure 5. A uniform traveling z plane wave, meets a uniform arrangement of point sources at the boundary $z = 0$, now equipped with cylindrical coordinates about some point source.

Make a substitution $u = \sqrt{r^2 + z^2}, du = \frac{r}{u}dr$

$$\int_0^\infty \int_0^{2\pi} A_p(r,t) = \int_0^\infty 2\pi r dr \cdot S\frac{e^{i2\pi\frac{u-ct}{\lambda}}}{u} \tag{27}$$

dr combined with $\frac{r}{u}$ gives du in the integrand, and the bounds must be changed to the u variable. In addition the dependence on t is factored out, an example of the separability of solutions to the Klein-Gordon equation.

$$\int_0^\infty \int_0^{2\pi} A_p(r,t) = \int_z^\infty 2\pi \cdot Se^{i2\pi\frac{u-ct}{\lambda}} du = e^{-i\omega t}\int_z^\infty 2\pi \cdot Se^{i2\pi\frac{u}{\lambda}} du \tag{28}$$

Now the integral is clearly undefined. A sticky situation, the integral does not converge. Let us cheat by a small ϵ and write the integral as follows, this is called a normalization [2, pg. 3]:

$$\int_0^\infty \int_0^{2\pi} A_p(r,t) = e^{i\omega t}\int_z^\infty 2\pi \cdot Se^{i2\pi\frac{u}{\lambda}-\epsilon u} du = e^{-i\omega t}2\pi S\frac{e^{i2\pi\frac{z}{\lambda}-\epsilon z}}{\frac{i2\pi}{\lambda}-\epsilon} \tag{29}$$

Take ϵ to zero

$$\int_0^\infty \int_0^{2\pi} A_p(r,t) = e^{-i\omega t}(-i\lambda)Se^{i2\pi\frac{z}{\lambda}} \tag{30}$$

This expresses the photon amplitude some distance z from the plane of uniformly distributed point sources. Happily its properties are such that it can be believed to be the solution to the boundary value problem posed in the left side of the plane in figure 3.5.

Let us define the incoming plane wave as follows:

$$A_{pw}(\vec{r},t) = C_0 e^{-i\omega t} e^{i2\pi \frac{z}{\lambda}} \tag{31}$$

This should be a familiar definition, substituting $C_0 = -iS\lambda$ recovers the solution found from a uniform superposition of point sources. Therefore it has been shown that an incident wall of uniform plane waves is equivalent to a uniform plane of point sources emanating spherical waves. Imagine placing a diffraction slit over this array of inbound plane waves, then it is clear that the way to incorporate this into the math is to have a superposition of point sources emitting spherical waves only where the slit allows light through. This is called Huygens' Principle, and seems like a cheat when you first see it, but is compatible with the view taken by the Klein-Gordon or Maxwell's equations view of photon propagation.

4. CITATIONS/RESOURCES

1. David Morin "Waves", Ch. 8
2. Boris Svistunov "Green's Function of the Wave Equation"