PHYS3990 HW1

August 30, 2024

1. Spherical waves Show that the spherical wave ($S(\vec{r},t) = \frac{e^{i(k|\vec{r}|-\omega t)}}{|\vec{r}|}$) is a solution to the 3d wave equation by direct substitution. The following expression in spherical coordinates of the wave equation I find easiest to use. The large linear operator in brackets is the Laplacian ∇^2 expressed in spherical coordinates.

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\phi}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\frac{\partial}{\partial\phi})\right]A(r,\theta,\phi,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}A(r,\theta,\phi,t) \tag{1}$$

Do not forget that it is assumed $\frac{\omega^2}{k^2} = c^2$.

Comments (not to be answered) Since the 3d scalar wave equation is shift invariant (we can move the origin and be left with the same PDE), one can also conclude that spherical waves centered at any finite location $\vec{r_0}$ are equally good solutions.

$$S(\vec{r},t) = \frac{e^{i(k|\vec{r}-\vec{r_0}|-\omega t)}}{|\vec{r}-\vec{r_0}|} \tag{2}$$

2. Intensity for highly directional scalar waves The intensity (a vector because it has direction) of electromagnetic plane waves is $I = \frac{1}{\mu_0}\vec{E} \times \vec{B}$. Under the assumption of plane wave propagation ($|\vec{B}| = \frac{1}{c}|\vec{E}|$, $\vec{B} \perp \vec{E}$), $|I| = \sqrt{\frac{\epsilon_0}{\mu_0}}|\vec{E}|^2$. By choosing a primary direction of propagation, and selecting a polarization, one arbitrary axis of the electric field may be studied in a single scalar wave equation (1). With the assumption of monochromaticity:

$$|I| = \sqrt{\frac{\epsilon_0}{\mu_0}}|Re(e^{-i\omega t}A_{\vec{r}}(\vec{r}))|^2 \tag{3}$$

Where the intensity is always oriented in the z direction given a nearly fixed propagation direction.

Question Show the useful identity $< |Re(e^{-i\omega t}A_{\vec{r}}(\vec{r}))|^2 >_t = \frac{1}{2}|A_{\vec{r}}(\vec{r})|^2$, where $<>_t$ is the time average over a period $\frac{1}{T}\int_a^{a+T} dt$. To do this fix $\vec{r}$ constant, and remember both factors $e^{-i\omega t}$ and $A_{\vec{r}}(\vec{r})$ are complex numbers, so the real part is nontrivial.

Useful

$$\int_0^T \sin^2(\frac{2\pi t}{T} + \phi)dt = \frac{T}{2} \tag{4}$$

Where ϕ is a free parameter.