Lecture Notes 4 Introduction to Fourier Optics

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1. SAMPLING

In the cases of forward propagation and reverse propagation, sampling is the most obvious constraint. If the slit pattern or generally amplitude boundary features slits or other features of size w, in order to ensure these features are resolved in propagation, the pattern must be sampled at points separated by $\frac{w}{2}$ or less. This can also be stated in terems of frequency, if the frequency components of an optical pattern (or signal generally) are known to be up to a frequency f, then to discern this signal sampling must be done with frequency at least $2f$.

The minimum frequency to avoid aliasing is known as the Nyquist limit. If the limit is not overtaken, a phenomena known as aliasing will occur. This is when signals or optical patterns of one frequency appear as another frequency under sampling. The simplest example of such aliasing is shown in figure 1, where a sin function appears as a constant function because it is sampled at a rate lower than its frequency. Figure 1 also shows the same function, but sampled twice as fast, to meet the Nyquist limit. The result is that the function's true frequency is not aliased. A proof of Nyquist's theorem is omitted, but one is provided as a link, however its utility is immediate.

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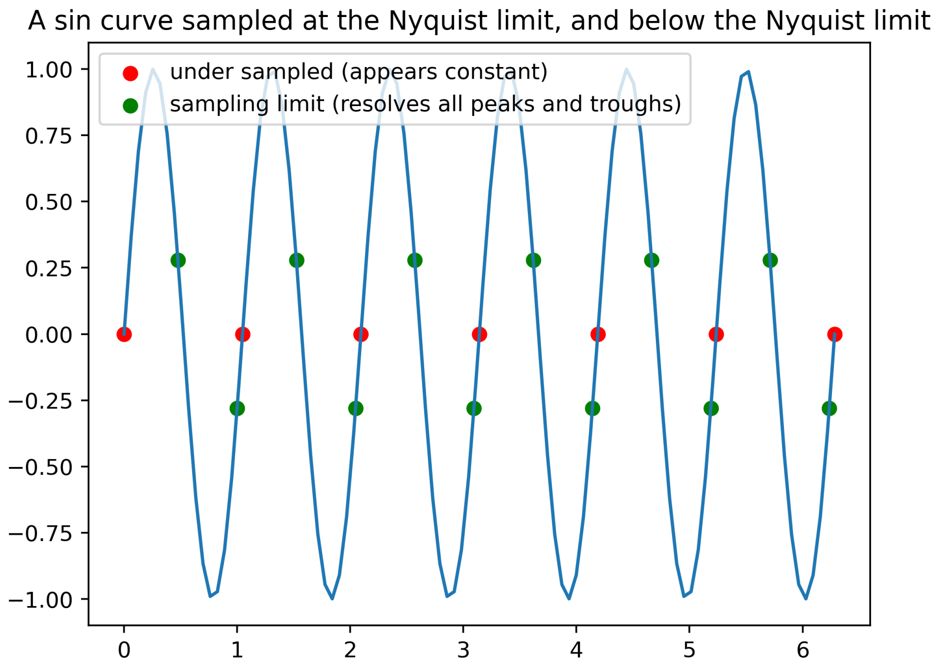

Figure 1. Sampling of a sinusoidal signal in two cases. The signal amplitude is still misrepresented sampling at the Nyquist limit.

If such aliasing enters any optical propagation algorithm, a correct answer is not guaranteed. The optical patterns that have been dealt with are tophat or boxcar shaped, but obey the sample sampling limits, because they may easily be decomposed into a sum of sinusoids by Fourier analysis.

The example given in figure 1 is dramatic, a clearly varying underlying function appears constant after insufficiently fine grained sampling. This is a coincidence of the example, in reality, a function that has frequency past the Nyquist limit set by the sampling frequency will more generally just be jumbled, ie some frequency f will be aliased and appear as a different frequency f', which is equally good at causing failures of optical propagation algorithms as is being sent to the constant function $f' = 0$. Note even in the example given, which the frequency signal is resolved, its true amplitude is lost, this occurs more generally, and so the Nyquist limit is a useful lower bound, but is not always sufficient, because two frequencies may be attenuated by sampling at significantly different rates, causing the signal or optical pattern to be incorrectly characterized. Figures 1 and 1 show how two signals of acceptable frequency will have different attenuation.

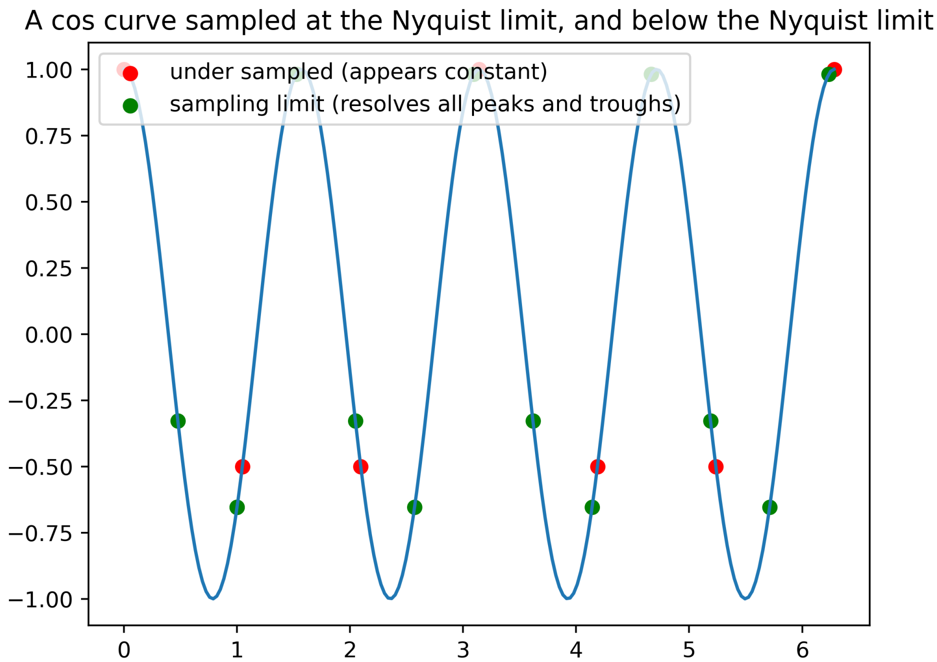

Figure 2. Sampling of a sinusoidal signal in two cases. The signal amplitude is correctly represented sampling at the Nyquist limit. Intuitively this is because sample points occur at the maxima or minima in this case.

2. PERIODICITY ASSUMPTION IN CIRCULAR CONVOLUTIONS

When taking the convolution, it was previously discussed that there are two slightly different definitions, circular and linear. Linear convolution is usually the convolution that best represents physical systems, but circular convolution is much faster to compute. The solution to this offered informally was to have the functions convolved have enough of the boundary of each functions sampled array be zero, such that boundary overflow effects due to the circular indexing would come out to zero.

To state this formally the support enumerated in indices of the two sampled functions being convolved, S_1 and S_2, must total to less than the total number of samples (assuming sample spacing and number is equal between both functions) [1], which in the last lecture were a diffraction pattern and a boxcar function, sampled into discrete space. This is written formally as:

$$S_1 + S_2 < N \tag{1}$$

The support of a discretely sampled function is the length of the shortest sub array that contains all nonzero values in the array. In other words it is the size of the region where the sampled function is nonzero. Figure 2 shows a sampled function with its support marked in green. Notice the support spans regions that are zero, because it must envelope all nonzero values contiguously. This condition ensures that any time the indices overflow the array and loop around circularly, at least one of the two factors in the convolution is zero, giving zero contribution, which matches with the linear convolution, which ignores these indices. An example of such loop around is shown in figure 2, the two functions shown are superimposed as would be the case in a circular convolution, since the support is large enough to wrap around from the boundary to a nonzero value, the circular convolution will not equal the linear convolution.

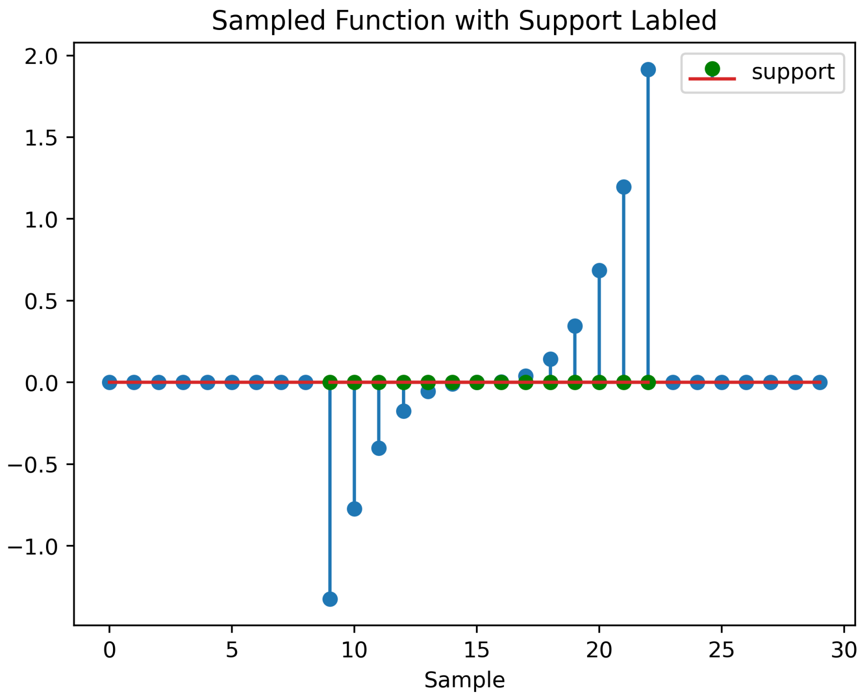

Figure 3. Sampled function with support highlighted in green.

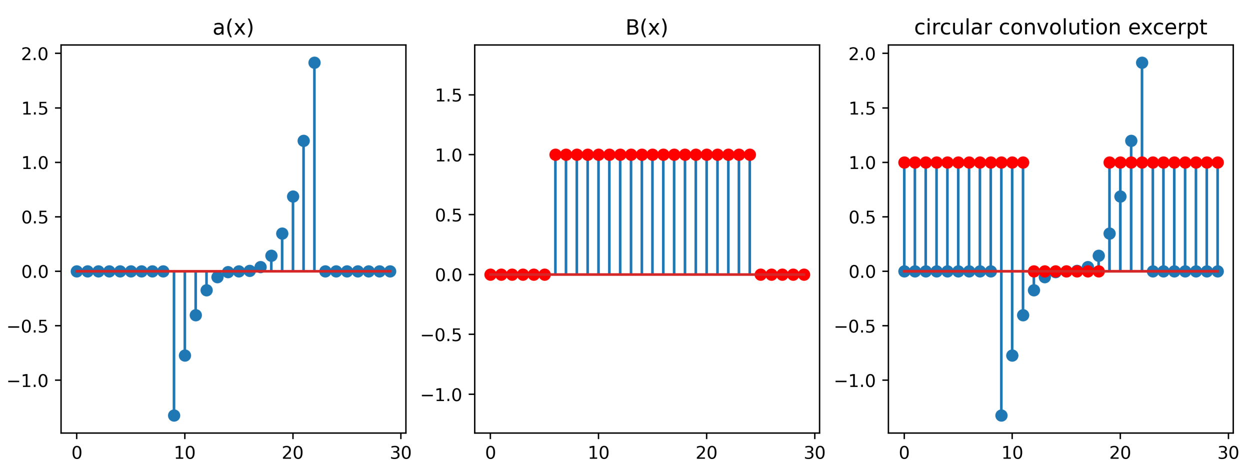

Figure 4. Excerpt from a circular convolution calculation, showing a value of the convolved results argument (namely 15) as it would be depicted graphically. Because the support of both functions is sufficiently large, their sum is greater than the number of samples, so the circular nature of the convolution differentiates it from a linear convolution.

3. INCREASING SAMPLING DENSITY AT THE OUTPUT

The scaling factor on the coordinates of the single Fourier transform far field propagation is:

$$\alpha = \lambda L \frac{N}{a^2} \tag{2}$$

Where a is the sampled region size to be propagated, and L is the distance propagated. A common problem is that one wishes to increase the sample density at the plane of observation. Most generally, the plane of observation has a desired extent b, and has the same number of samples as is in the input N, the sample density is $\frac{N}{b}$. Taking the ratio of a and b which defines α the desired parameters are found.

$$\frac{b}{a} = \lambda L \frac{N}{a^2} \tag{3}$$

Since b and N are target values, the only unknown is a, the sampled region size must be:

$$a = \lambda L \frac{N}{b} \tag{4}$$

4. FRESNEL APPROXIMATION, CLOSER IN COORDINATES

Having wrapped up the first look at far field diffraction, ($\frac{a^2}{L} << \lambda$ where a is the size of the diffraction slit pattern) a weaker approximation will be shown, due to Fresnel. By taking a weaker approximation, one my either view the diffraction of a larger pattern at the same distance, than with Fraunhofer propagation, or the same sized pattern's diffraction closer to the slit boundary.

The approximation of Fresnel is simpler to derive in terms of the distance approximations made, but introduces a term inside of the integral, complicating the diffraction pattern to be more than just the Fourier transform.

Start by considering a 1 dimensional diffraction problem, exactly as previously, with a 1d plane boundary, usually occupied by some slits. Exactly as previously, the plane wave amplitude at the slit boundary must be integrated to find the resulting amplitude at the opposite boundary as a superposition of point sources placed whenever the diffraction slit pattern is open. Recall that the wave equation which models the light is linear, so the sum of solutions is always a solution (matching boundary conditions of course). The steady state case is considered, so propagation speed of light may be seen as infinite, and so the only time dependence is a global phase ($e^{i\frac{2\pi c}{\lambda}t}$). Note $2l = a$ as a useful shorthand, and $R(y, y')$ is the distance between two points on the opposing boundaries, with respective coordinates y, y'. Proportionality is introduced to save writing down normalization (either to total intensity 1.0 in the probability view, or total true intensity).

$$A'(y') \propto \int_{-l}^{l} A(y) \frac{1}{R(y, y')} e^{\frac{i2\pi R(y,y)}{\lambda}} dy \tag{5}$$

Explicitly through simple euclidean geometry, the path length between two points on diffraction planes (y, y'),as shown in figure 4:

$$R(y, y') = \sqrt{L^2 + (y - y')^2} \tag{6}$$

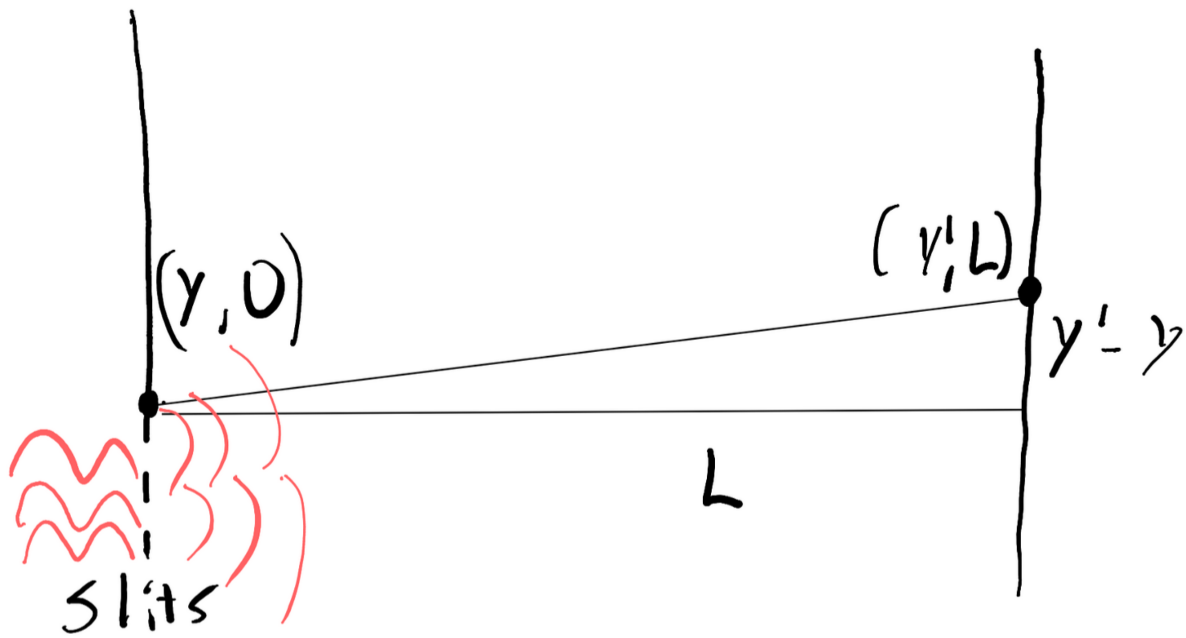

Figure 5. Coordinate labeling for the general 1d diffraction problem.

Where L is the propagation distance between the diffraction planes (slits and observation). It should be clear that if this fully unapproximated R is placed into the integral, there is no possiblity to recover something resembling a Fourier transform, as is the case for the Fraunhofer far field diffraction. To specify what "resembles" a Fourier transform, there should be some factor $e^{-i\alpha yy'}$ in the integral, such that is resembles the Fourier transform integral between coordinates y and y'. In addition all other factors inside the integral should only depend on y. The approximation of Fresnel will be seen to accomplish both of these goals, which being weaker than that of Fraunhofer, and thus applicable over a wider range of parameters.

$$FT_{modified}\{A(y)\} = \int_{-l}^{l} A(y) \cdot B(y) e^{-i2\pi\alpha yy'} dy \tag{7}$$

It has been shown how to deal with the scale factor α of the coordinates, as the Fraunhofer approximation of light propagation also required $\alpha = \frac{1}{L\lambda}$. Arbitrary function $B(y)$ is introduced, as it costs us realtively little computationally to multiply the boundary amplitude $A(y)$ by some arbitrary function, so long as the following Fourier transform is done as a discrete approximation.

Making the approximation of Fresnel will end up simplifying (10) enough to make the most general 1d scalar diffraction (5) appear as a Fourier transform of form (7). From there discretization will be taken, to arrive at a less approximate version of the first single Fourier transform numerical diffraction method that was discussed. This method is less approximate than using Fraunhofer coordinates in the sense that it keeps terms of the Taylor series of the path length (as illustrated in figure 4) to a higher order, and is valid for a wider range of parameters.

To get underway with Fresnel's approximation on the diffraction path length, start will the full non-approximate path length R which determines the phase difference $\Delta\phi$ for a source at point y propagating to point y' in the steady state.

$$R(y,y') = \sqrt{L^2 + (y-y')^2} \tag{8}$$

$$\Delta\phi = 2\pi \frac{R(y,y')}{\lambda} \tag{9}$$

λ is a constant so the equation of concern for approximation will be $(y-y')$, the approximation is when $(y-y')$ is small, so for convenience take $\Delta y = (y-y')$. The equation reads:

$$R(\Delta y) = \sqrt{L^2 + \Delta y^2} \tag{10}$$

Fresnel's approximation is to take this expression for R to second order in Δy and use that as an approximate path length. To do error analysis, the third and fourth order must also be calculated (the third order is zero (distance is an even function about minima)).

$$R(\Delta y) = \sqrt{L^2 + \Delta y^2} \tag{11}$$

$$\frac{dR}{d\Delta y} = \frac{\Delta y}{\sqrt{L^2 + \Delta y^2}} \tag{12}$$

$$\frac{d^2R}{d\Delta y^2} = \frac{L^2}{\sqrt{L^2 + \Delta y^2}(L^2 + \Delta y^2)} \tag{13}$$

$$\frac{d^3R}{d\Delta y^3} = -\frac{3L^2\Delta y}{(L^2 + \Delta y^2)^{\frac{5}{2}}} \tag{14}$$

$$\frac{d^4R}{d\Delta y^4} = -\frac{3L^2(L^2 - 4\Delta y^2)}{(L^2 + \Delta y^2)^{\frac{7}{2}}} \tag{15}$$

Now to expand around zero set $\Delta y = 0$ in the derivatives for use in a Taylor series.

$$R \approxeq L + \frac{1}{2L}\Delta y^2 - \frac{1}{8L^3}\Delta y^4 \tag{16}$$

Fresnel's approximation is that the fourth order term of R the path length is much smaller then a wavelength

$$\frac{\Delta y^4}{L^3} << \lambda \tag{17}$$

In the case of (17) the path length is as follows. (17) essentially constrains the propagation to be between diffraction patterns and slit patterns of limited size, based on the parameters L and λ. Compared to the Fraunhofer condition of previous lectures, the small parameter Δy in the numerator and the large parameter L in the denominator are to higher powers, make the inequality easier to satisfy.

$$R \approxeq L + \frac{1}{2L}\Delta y^2 \tag{18}$$

It is now useful to express Δy again in terms of the coordinates y, y' since the Fourier transform form that is sought should involve the variables y, y'

$$R \approxeq L + \frac{1}{2L}(y - y')^2 \tag{19}$$

Now as promised, making this approximation on (7) will yield a single Fourier transform propagation between photon amplitudes in the plane described by y, and the plane described by y'. Note that in (7) there is a factor of $\frac{1}{R}$ scaling. This is ignored as being approximately a constant of proportionality, which does require the additional assumption, that $|\Delta \frac{1}{R}| << 1$ which is $\frac{|\Delta y|}{R^2} << 1$ to first order, where $\Delta \frac{1}{R}$ is the change in the scaling factor due to some Δy. This condition is usually satisfied if the Fresnel approximation is being used, but it never hurts to check. Especially in the case of a laser beam propagating in the lab, given the diameter of the laser beam is always very small $\frac{\Delta y}{L}$ is already small.

$$A'(y') \propto \int_{-l}^{l} A(y) e^{\frac{i2\pi}{\lambda}(L + \frac{1}{2L}(y-y')^2)} dy \tag{20}$$

$$\tag{21}$$

A constant phase between A' and A should be apparent $e^{\frac{-i2\pi}{\lambda}L}$. Usually such a global phase is ignored, but can be important in calculations where the light beam will come back on itself (like an interferometer) because it is the relative phase due to propagation distance. This phase is factored out, along with the expansion of $(y - y')^2$.

$$A'(y') \propto e^{\frac{i2\pi}{\lambda}L} \int_{-l}^{l} A(y) e^{\frac{i2\pi}{\lambda}(\frac{1}{2L}(y^2 - 2yy' + y'^2))} dy \tag{22}$$

$$\tag{23}$$

Now all that is left to be done is to break up the exponent proportional to $y^2 + 2yy' + y'^2$, and bring the factor dependent only on y'^2 outside of the integral.

$$A'(y') \propto e^{\frac{i2\pi}{\lambda}L} e^{\frac{i\pi}{2\lambda L}y'^2} \int_{-l}^{l} A(y) e^{\frac{i\pi}{\lambda L}y^2} e^{\frac{-i2\pi}{\lambda L}yy'} dy \tag{24}$$

$$\tag{25}$$

For many applications where a simple propagation is used, the phases outside of the integral may be ignored (the classic case being a slit diffraction pattern experiment where only intensities matter).

The inside of the integral is also where the numerical computations are interesting, the phases outside can be calculated, in continuous variables or in a discretized approximation as just multiplication. On the otherhand the inside of the integral has a phase factor that should be familiar, namely $e^{\frac{-i2\pi}{\lambda L}yy'}$ which is associated with the Fourier transform from the variable y to a new variable y', since the integral is with y. The other factors inside the integral $A(y)e^{\frac{i\pi}{\lambda L}y^2}$ may be treated as the function being Fourier transformed, since they only depend on y and constants. Therefore the integral is rewritten as:

$$A'(y') \propto e^{\frac{i2\pi}{\lambda}L} e^{\frac{i\pi}{\lambda L}y'^2} FT\{A(y)e^{\frac{i\pi}{\lambda L}y^2}\}(\frac{y'}{\lambda L}) \tag{26}$$

This resembles the integral of the Fraunhofer approximated propagation extremely closely:

$$A'(y') \propto e^{\frac{i2\pi}{\lambda}L} FT\{A(y)\}(\frac{y'}{\lambda L}) \tag{27}$$

Only additional phasing from the y^2 and y'^2 terms that must be introduced if the weaker Fresnel approximation is to be used. Note the derivation for the Fraunhofer propagation using a single Fourier transform did not involve the global phase due to the propagation distance L, but it was in fact there, just ignored because the slit diffraction experiment does not require this global phasing to predict all possible measurements.

This similarity leads to the exact same approach as for Fraunhofer propagation. Sample the function $A(y)e^{\frac{i2\pi}{\lambda L}y^2}$ at discrete points, then apply the discrete Fourier transform, scaling the new coordinates in the exact same way as for the Fraunhofer diffraction. There are similar error considerations as those that existed for the Fraunhofer diffraction. Additionally the requirement to sample $e^{\frac{i\pi}{\lambda L}y^2}$ and $A(y)$ fast enough to accommodate the Fourier transform, not just $A(y)$. For now the discretized approximation of the single Fourier transform Fraunhofer propagation is stated, and further its error considerations will be made rigorous.

$$A'_m = e^{\frac{i2\pi}{\lambda}L} B'_m \cdot DFT\{B_j\},\ Y'_j = Y_j \cdot \lambda L \frac{N}{(2l)^2} \tag{28}$$

Where j is an integer index. Arrays Y, Y' are the boundary coordinates and propagated coordinates respectively, $2l$ is the size of the boundary, and B_j are samples of $A(y)e^{\frac{i\pi}{\lambda L}y^2}$ at points $y = \frac{j}{N}2l$, and B'_j are samples of $e^{\frac{i\pi}{\lambda L}y'^2}$ at points $y' = \frac{j\lambda L}{2l}$.

Implementing this propagation (after sampling) is as follows:

```
def propagate_fresnel(A, coordinates, photon_lambda, L):

  #Form the sampled function B_j

  B = A * np.exp(1.0j * (coordinates**2) * np.pi / photon_lambda / L)

  #scale the input coordinates correctly, len(coordinates) is the number of samples N
  propagated_coordinates = coordinates * photon_lambda * L / (np.max(coordinates)-np.min(coordinates))**2 * len(coordinates)
  #apply the FFT (fast DFT) to the field to find the diffraction pattern
  #np.fft.fftshift swaps around the indices of the returned fft from np.fft.fft, since the
  #returned indices before fftshift are not in the normal convenient order physicists like to deal with

  #Add on Fresnel phase correction, and phasing due to
  #travel distance L
  Ap = normalize(np.fft.ifftshift(np.fft.fft(np.fft.fftshift(B)))) * np.exp(1.0j * (propagated_coordinates**2) * np.pi / photon_lambda / L) * np.exp(1.0j * L * 2 * np.pi / photon_lambda)

  #return the calculted propagation
  return Bp, propagated_coordinates
```

The propagation under the Fresnel approximation should match that of the Fraunhofer propagation for large enough distances (where both approximations are valid). The two propagations are applied a boxcar aperture, or single slit of width $0.1mm$. In the far field ($0.3m$) the propagation with the Fresnel approximation and with the Fraunhofer approximation match, as shown in figure 4. Figure 4 shows a disagreement when the light is propagated a shorter distance of $0.0003m$. While intuition about what nearfield propagation should look like will not be immediate, it is clear that in the limit of an infinitesimal propagation, the light pattern should be of two slits. The Fresnel propagation in the near field exhibits this, by featuring two prominences in the amplitude envelope, which if the Fresnel approximation could calculate even nearer fields, would eventually become the double slits. In comparison the Fraunhofer approximation has failed, and cannot predict appearance of the two slits in the light pattern for any near field, including the example given.

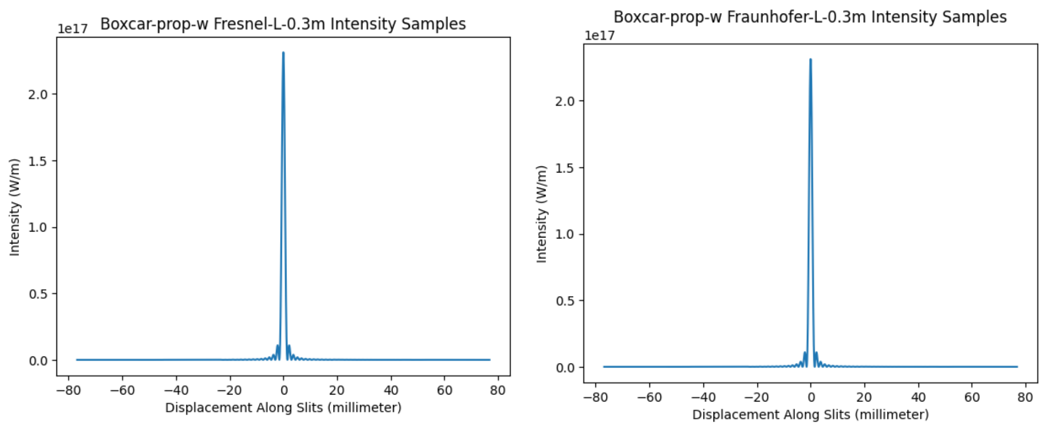

Figure 6. Comparison of the Fresnel and Fraunhofer propagation to the far field. Agreement is clear.

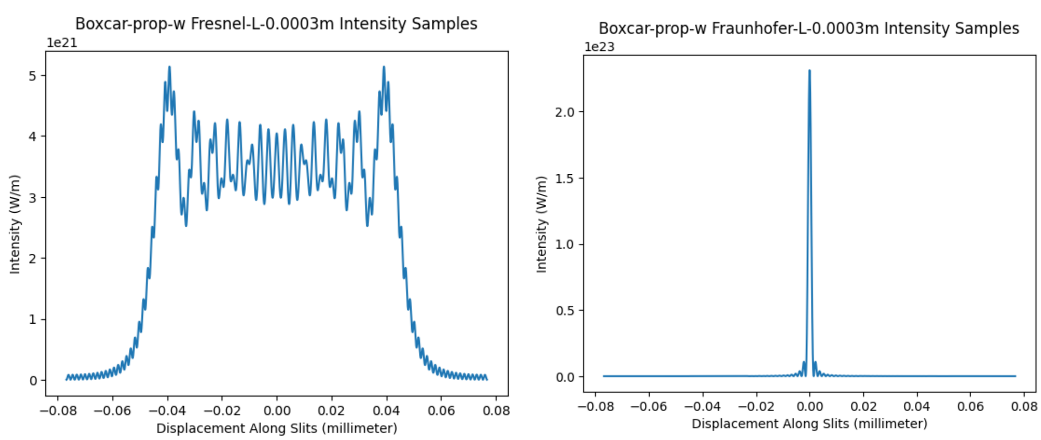

Figure 7. Only the Fresnel propagation shows the amplitude pattern approach that of the initial condition. The Fraunhofer calculation is unsurprisingly static, and incorrect as the propagation is reduced into the near field.

5. CITATIONS/RESOURCES

All the code to generate plots is made available with this PDF.

1. Computational Fourier Optics: A Matlab Tutorial, D. Voelz, Page 25.
2. nyquist proof https://brianmcfee.net/dstbook-site/content/ch02-sampling/Nyquist.html
3. Fresnel approximation derivation https://qiweb.tudelft.nl/aoi/wavefielddiffraction/wavefielddiffraction.html