Lecture Notes 6 Introduction to Fourier Optics

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1. 2D FFT IN PYTHON

The numpy library provides basic two dimensional Fast Fourier Transforms *fft2*. As with their one dimensional counterpart, *fft2* requires use of *fftshift* to get physically correct index ordering. Besides taking a 2d grid of values to be transformed, the operation appears to the user identically. The computation time should also be considered. For N samples the FFT is computed in $NlogN$ time. If the grid is square with K samples along each side, the computation time is K^2logK. The 2d FFT approximates the true 2d Fourier transform, and thus Single Fourier Transform Fresnel propagation under similar qualifications to the 1d case.

$$A'(x',y') \propto e^{\frac{i2\pi}{\lambda}L} e^{\frac{i\pi}{\lambda L}(x'^2+y'^2)} FT_{x,y}\{A(x,y)e^{\frac{i\pi}{\lambda L}(x^2+y^2)}\}(\frac{x'}{\lambda L},\frac{y'}{\lambda L}) \tag{1}$$

Goes to the following when discretized:

$$A'_{m,n} = e^{\frac{i2\pi}{\lambda}L} B'_{m,n} \cdot DFT_{j,k}\{B_{j,k}\},\ X'_j = X_j \cdot \lambda L \frac{N}{(2l)^2},\ Y'_j = Y_j \cdot \lambda L \frac{N}{(2l)^2} \tag{2}$$

Where j,k are integer indices. Arrays X,Y,X',Y' are the boundary coordinates and propagated coordinates respectively, $2l$ is the size of the boundary, and $B_{j,k}$ are samples of $A(x,y)e^{\frac{i\pi}{\lambda L}(x^2+y^2)}$ at points $x = \frac{j}{N}2l$, $y = \frac{k}{N}2l$, and $B'_{j,k}$ are samples of $e^{\frac{i\pi}{\lambda L}(x'^2+y'^2)}$ at points $x' = \frac{j\lambda L}{2l}, y' = \frac{k\lambda L}{2l}$.

The following code implements the single Fourier Transform Fresnel propagation in two dimensions, and assumes the amplitude arrays are square in indices and physical extent.

```
#New propagation, Fresnel single Fourier transform
#Almost identical to 1d, except the coordinate squares are summed for x,y
#And fft2, the two dimensional FFT is used
def propagate_fresnel_2d(A, coordinates, photon_lambda, L):

  #Form the sampled function B_j,k

  B = A * np.exp(1.0j * (coordinates[:,:,0]**2 + coordinates[:,:,1]**2) * np.pi / photon_lambda / L)

  #scale the input coordinates correctly, len(coordinates) is the number of samples N
  propagated_coordinates = coordinates * photon_lambda * L / (np.max(coordinates)-np.min(coordinates))**2 * len(coordinates)
  #apply the FFT (fast DFT) to the field to find the diffraction pattern
  #np.fft.fftshift swaps around the indices of the returned fft from np.fft.fft, since the
  #returned indices before fftshift are not in the normal convenient order physicists like to deal with

```

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```python
    #Add on Fresnel phase correction, and phasing due to
    #travel distance L
    Ap = normalize(np.fft.ifftshift(np.fft.fft2(np.fft.fftshift(B)))) * np.exp(1.0j * (
        propagated_coordinates[:,:,0]**2 + propagated_coordinates[:,:,1]**2) * np.pi /
        photon_lambda / L) * np.exp(1.0j * L * 2 * np.pi / photon_lambda)

    #return the calculted propagation
    return Ap, propagated_coordinates
```

One can see little difference between the 1d and 2d implementations. This is for two reasons, first numpy's FFT functionality is wide, and immediately accommodates the 2d FFT. In addition, the problem of 2d propagation is separable, and the product of 1d solutions is a solution.

2. RAY OPTICS OVERVIEW

Now for a short time, ray optics will be considered, so that when the above propagation code is tested, the results may be compared to intuition.

In ray optics, interference and wavelike effects are ignored. This may seem like a large approximation. It is, but there are many cases where ray optics can make predictions some surprising. The cases that will be considered now will be the optics of mirrors and lenses. More complicated predictions of ray optics are accurate too. For example, the shape of a simple laser beam in its transverse profile (Gaussian) can be predicted by treating light as rays. It is only when higher order more structured laser cavities are used that wave optics are required. Such structure can include diffractive gratings. The double slit pattern is not predicted by ray optics for example.

Ray optics starts with the free laser beam. In this case represented by many projectile like paths. This predicts the linear divergence of all beams far enough away except those perfectly aligned with the propagation axis.

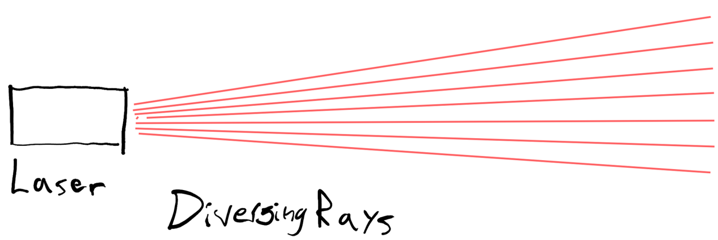

Figure 1. A ray optic laser that produces initially diverging rays.

A laser can also be assumed to produce converging rays, which will also become diverging for large distances.

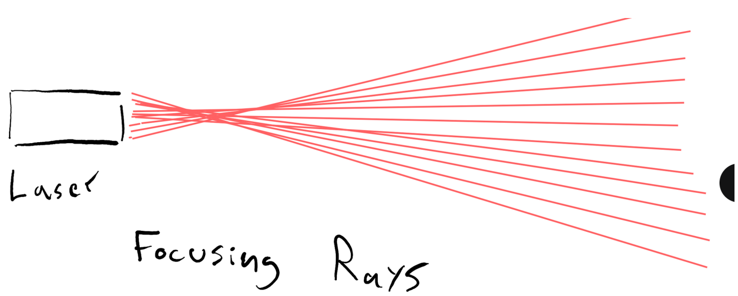

Figure 2. A ray optic laser that produces initially converging rays.

The density of rays may be seen as roughly giving the intensity of the laser. As the laser shoots far into the distance, the density of rays gets lower, and the illumination becomes weaker. Rays can be thought of as conserved, each carrying some power if the laser is a continuous source.

A lens in ray optics is just an element that reorients rays. A common lens is a focusing lens which causes diverging rays to turn around and start converging to a focus. Rays from an idealized laser will reach this convergence at the lenses characteristic focal length f.

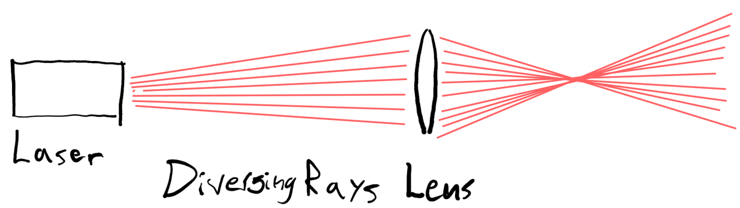

Figure 3. A ray optic laser that produces initially diverging rays intersects a lens, which causes the rays to converge at a point f away from the lens.

The mirror should also be a familiar example.

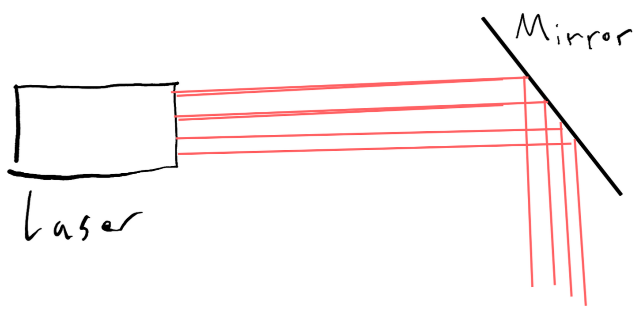

Figure 4. A ray optic laser that produces initially diverging rays intersects a lens, which causes the rays to converge at a point f away from the lens.

This is a short introduction to ray optics. Much more can be done, the next step in complexity would be multi component devices, like a beam expander, or collimator. These components spread out the rays and make them less dense, and straighten the rays to be minimally diverging respectively. For the following examples, it will be ok to test basic single component optical phenomena.

3. COMPARABLE VARIABLES BETWEEN RAY OPTICS AND WAVE OPTICS

The highly general measurable of beam width will be compared between our wave optic simulations, and what we expect from ray optics or pedestrian observations in a lab.

Sometimes, it is useful to define how wide a curve is by its full width half maximum. This is even the case in many optical experiments and phenomena. Generally however, the mathematics of free space optics says that the most convenient way to define a laser's width is the second moment width. This variance width V should be familiar from quantum mechanics. This width has units of square meters, to get meters, taking the square root $\sqrt{V}$ yields a quantity that is more intuitive.

$$V_x = \; < (x - \mu_x)^2 >_A \; = \; < x^2 >_A - < x >_A^2 \tag{3}$$

In our case, ψ is not the wave function of an electron however, it is the complex E-field amplitude A, whose square magnitude is the intensity. This variance width is defined and used for both transverse axes x and y. μ_x is the centroid in the x axis, which is an equally important quantity to the width of laser, μ_x locates the laser approximately in space along the x axis.

$$V_y = \; < (y - \mu_y)^2 >_A \; = \; < y^2 >_A - < y >_A^2 \tag{4}$$

In quantum mechanics expected values as dealt with in terms of probability, the wave function magnitude squared. In the case of wave optics, the probability is taken as proportional to the intensity $|A|^2$. Usually

This width will be seen to follow impressive mathematical structure, and is thus chosen often as a measure of the size of a laser beam or optical pattern. When the density of rays is high, the wave optic width should be small (like the focus of a converging lens). The opposite is true for low density rays.

Let us construct a program to calculate the parameters W_x, W_y the square root of the variance based width (to get units of meters rather than meters squared), and μ_x, μ_y the beam center.

```python
def beam_parameters_2d(coordinates, A):

  An = normalize(A)

  #Calculate the expected values for x**2 and x
  #Normalization is already done so no denominator needed

  #Expected value for x^2
  Ex2 = np.sum(coordinates[:,:,0]**2 *np.abs( np.square(An)))

  #Expected value for x, the centroid in x direction
  Ex = np.sum(coordinates[:,:,0] * np.abs( np.square(An)))

  #Calculate the expected values for y**2 and y
  Ey2 = np.sum(coordinates[:,:,1]**2 *np.abs( np.square(An)))
  Ey = np.sum(coordinates[:,:,1] * np.abs( np.square(An)))

  #Use the variance based width (standard deviation) to report a width, and return the centroid
  return np.sqrt(Ex2 - Ex**2), np.sqrt(Ey2 - Ey**2),Ex,Ey

```

Once the transverse amplitude is normalized, calculating expected values is easy given numpy's great functionality. The value of the coordinate, or its square is multiplied with each probability, then summed. Additional code is then written to produce the following plot, which makes the center and widths of a transverse profile clear graphically using a marker, and a width sized ellipse. Two examples are shown, first the intensity that models an ideal laser, a two dimensional Gaussian.

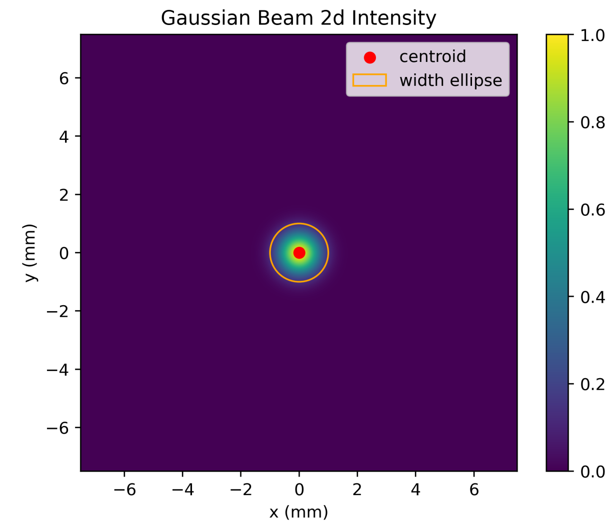

Figure 5. The idealized wave amplitude for a laser: Gaussian, with its four standard deviation diameters $D4\sigma_{x/y} = 4\sqrt{V_{x/y}}$ as an ellipse.

Now, observe the width of a slit diffraction experiment, where the pattern should be much wider in one axis. In a sense this pattern is "lucky" in that the two distincly sized pattern axes are aligned with x and y. If they were not, the axes with which the widths are taken could easily be rotated to maximize the difference between the two transverse widths.

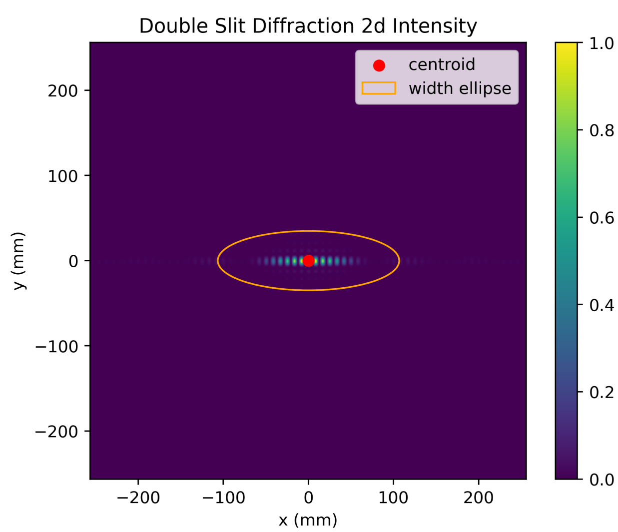

Figure 6. Diffraction of a 2d double slit, with calculated diameter and centroid ellipse.

4. WAVE OPTICS CALCULATIONS FOR SIMPLE LAB SETUPS

4.1 Unimpeded Gaussian Laser

Now that width calculations are implemented, the divergence of an ideal laser beam can be tested with the wave optic Fresnel propagation. First, justifying approximations, and minimizing errors is in order. This is done for the case of a $1mm$ laser beam size with $\lambda = 500nm$ visible light.

The Frensel approximation is $\frac{R^4}{L^3\lambda} << 1$ where R is the widest extent of the amplitude field. Take $R = 1mm$ the laser beams approximate size, and $\lambda = 500nm$. The condition $\frac{R^4}{L^3\lambda} = 1$ will yield a propagation distance where the approximation will begin to fail, in that case $L = O(10mm)$ or L is of order $10mm$. To be safe, the beam will be propagated to $L = 100mm$ and beyond.

Sampling must be high enough such that $A(x, y)e^{\frac{i\pi}{\lambda L}(x^2+y^2)}$ suffers minimal aliasing as well when it is discretely sampled. The highest frequency expected in $e^{\frac{i\pi}{\lambda L}(x^2+y^2)}$ is approximately $\frac{r_{max}}{\lambda L}$ (the derivative of the complex exponentials argument, so called "instantaneous frequency"), which necessitates $N = 2\frac{r_{max}}{\lambda L} \cdot S$ samples (Divide the spatial size by the smallest period , or $2f_{max} \cdot S$), where S is the simulation side length. In this case $N \sim 600$, so 1024 samples are used in each dimension. The frequency content of the Gaussian is of comparable or negligible contribution regarding sampling. This is not always the case, and must be checked. Gaussian Frequency content can generally be determined by the uncertainty relation for width in spectral space, and width in real space. A wide Gaussian is made of low frequencies, a compact Gaussian must be made more significantly from high frequencies because it is wider in frequency space.

With a simple simulation where the outcome is known, it is ok to just play with the sampling and scale parameters until the result physically makes sense. But more strict consideration given above is necessary when the result is not so obvious. For now it is acceptable to take these error preparations for granted, to focus on comparison of wave optic simulation results, to predictions of ray optics.

First, the Gaussian laser beam will undergo Fresnel propagation between $L = 100mm$ and $L = 1000mm$, at each point its width and centroid will be calculated as above. This is coded up using existing propagation and width/centroid calculations, and a simple loop, recording the beam width as each calculation is made.

```
widths_x = []
widths_y = []
Ls = []

for i in range(20):
Ap, propagated_coordinates = propagate_fresnel_2d(gaussian_amplitude, coordinates, 500
    * nanometer, 100 * millimeter + i * 50 * millimeter)

wx, wy, cx, cy = beam_parameters_2d(propagated_coordinates,Ap)

Ls.append(100 * millimeter + i * 50 * millimeter)

widths_x.append(wx)
widths_y.append(wy)

```

Note that propagations are always done from the $L = 0mm$ field *gaussian_amplitude* to successively further distances. Propagations are not done on top of each other. The alternative, propagating each newly found amplitude to the next rather than always from $L = 0$, could work but would introduce needless error build up.

If the width in the x axis is plotted over propagation, the result is as follows. The y axis is ignored for now since the Gaussian beam shown in figure 3 is symmetrical, and the result will be the same.

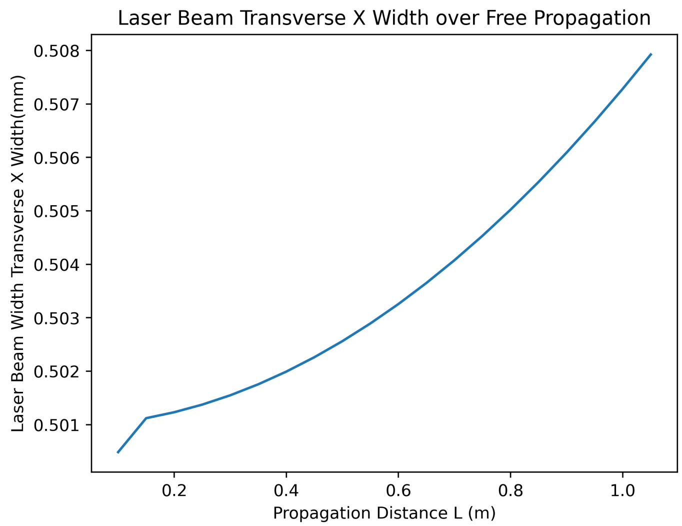

Figure 7. Width of a Fresnel single Fourier Transform propagated ideal laser beam over its extent in space. Error is significant close in, leading to a discontinuity near $L = 0.1m$.

There is clearly an error, sampling was insufficient, or the Fresnel approximation was not actually met, near $L = 0.1m$. This is an example of where it pays to be conservative with error estimation and evaluation of approximations. Besides this error, the beam slowy diverges. As expected the width does not change much over $1m$, only about 2%, which is reasonable for a laser beam. It should be the case that for really far away, the laser beam's width goes linearly with distance, so that the inverse square law holds for radiation. The same plot is made but for L up to $10m$ (the approximations and sampling used in this case are robust for larger distances) as shown in figure 4.1:

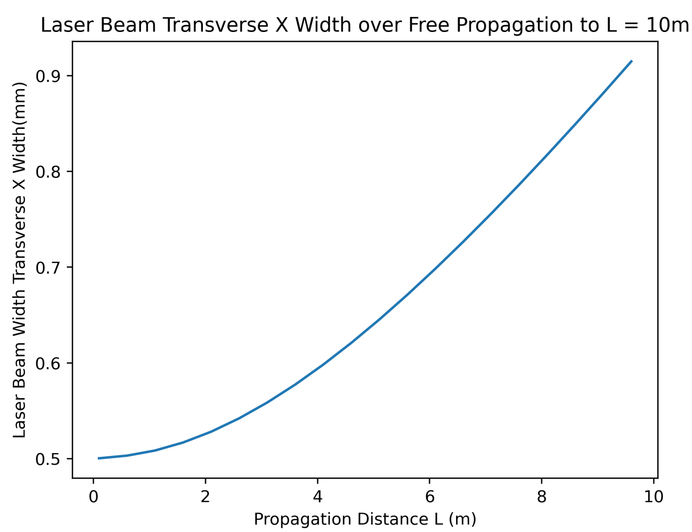

Figure 8. Width of a Fresnel single Fourier Transform propagated ideal laser beam over $10m$ in space.

It does in fact appear that the divergence of the beam width is linear for large distances, in agreement with the conservation of energy and inverse square law. The rate at which the beam width increases can be characterized by the angle the width increase makes with the axis, known as the divergence angle. Higher quality lasers will generally have a smaller divergence angle, although there is a physical limit, the beam must diverge for large distances.

4.2 Lensing of a Laser Beam

The lensing that will be considered at first is convergent lensing, because it has an intuitive explanation as far as any questions of "why" for the parameters involved.

All lenses considered will be thin. This means that the effect of the lens is only to change the phase of the optical amplitude plane that meets it. This is only a good approximation, if the lens is not very thick. If the lens was thick, the light would continue to diffract while in the lens, modifying not only its phase (by increasing optical path length by glass material $n_{glass} \neq 1$) but also its intensity. Figure 4.2 shows a central cross section of a 2d lens, and the simplified calculation of path length under the assumption that diffraction is small.

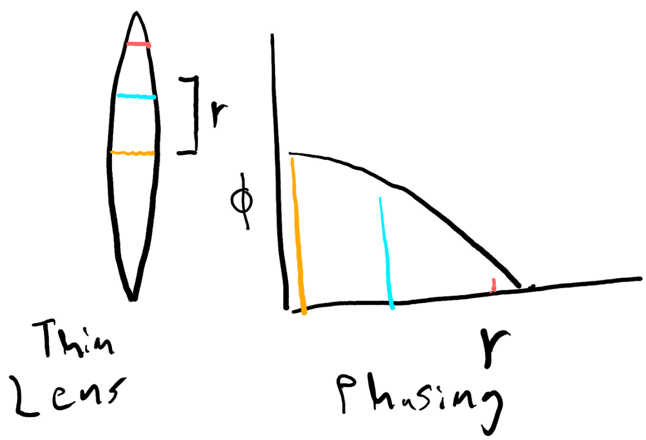

Figure 9. Lens thickness of a thin lens in direct correspondence with phasing caused to the incident beam.

This is a good approximation, that can already be explored a with the tools developed. Consider a propagation calculation for some double slits in the Fresnel (near field) region. Over a short enough distance (that ends up being the thickness of a lens or shorter) the approximation holds. In aid of this approximation, the extent of a lens can be made within one wavelength of light ($[0, 2\pi)$ is the only physically relevant change in the phase in the cases considered), for visible, hundreds of nanometers. The transverse pattern of the light will diffract negligibly over such a scale. Obviously the engineering team will be upset if a lens of thickness $700nm$ is requested, but the point is the thickness of a physical lens is not very limited by physical principles, rather manufacturing ability.

A practical perspective on the thin converging lens is the easiest way to grasp the concept. One wants to organize the light amplitude pattern as best they can such that it converges to a small point. Imagine an unpracticed soccer team, the ball is still on their end of the field, and all the defenders (photons) go for it. In this case like stooges, they all bump heads at the soccer ball, arriving at the same time. In order to arrive at the same time, they must all run at a constant speed proportional to their distance from the ball towards the ball.

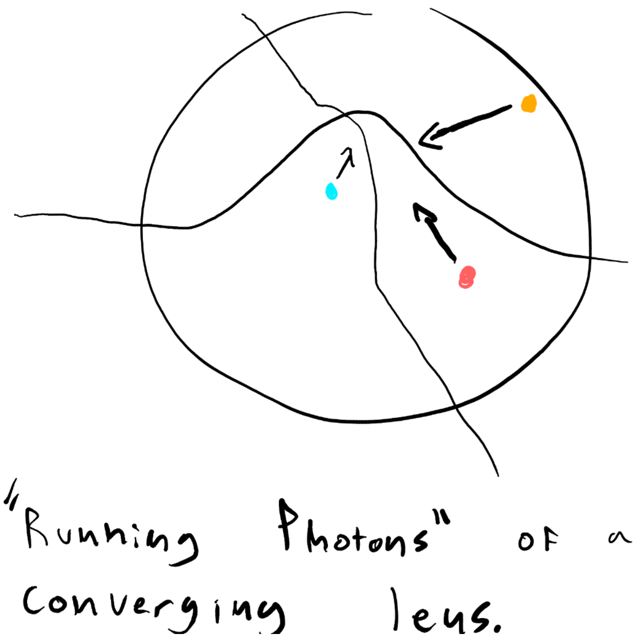

Figure 10. The three stooges if they were photons making up a Gaussian laser beam cross section (suspend your belief in boson counting to accept they may bump heads). Lensing causes the wave amplitude to coincide at a distance of approximately f.

thin-lens-phasing-diagram.png If phasing is the option to make photons "run" inwards and meet at the same time(thin lens), then the phase should be such that it introduces traveling waves in the transverse direction. Traveling waves are clearly admitted by the wave equation. Typically the traveling waves seen are $e^{ikx-\omega t}$. In the case of the lens, this is not going to work. Parts of the light field that are further from the center of the transverse axes need to travel faster to reach the center (and be focused) at the same time as closer in light. The " inward velocity" of light (with quotations because the argument is very heuristic) should increase linearly with distance if all the light is to reach the center at the same time, so. The group velocity of a wave is $\frac{d\omega}{dk}$. The dispersion relation on the free transverse (r waves assuming rotational symmetry) waves is

$$\omega^2 = [k_r^2 + k_z^2]c^2 \tag{5}$$

The group velocity in the r direction is:

$$v_r = -\frac{k_r}{\sqrt{k_r^2 + k_z^2}}c \tag{6}$$

Working with absolute values only, and knowing that we want the velocity to result in convergence in time $\Delta t = \frac{f}{c}$ (so that the focal length f is met) the following holds:

$$v_r \frac{f}{c} = r \tag{7}$$

The approximation that $k_r << k_z$ since the laser or light is assumed to be traveling nearly straight.

$$\frac{k_r}{k_z} c \frac{f}{c} = r \tag{8}$$

Now isolate k_r which is the instantaneous frequency in the transverse direction, and note $k_z \approxeq \frac{2\pi}{\lambda}$ because again the wave is mostly traveling in z:

$$k_r = \frac{d\phi}{dr} = \frac{2\pi r}{f\lambda} \tag{9}$$

Integrating:

$$\phi(r) = \frac{\pi r^2}{\lambda f} \tag{10}$$

Therefore if we have an ideally designed lens to focus a circularly symmetric light pattern, the phase change an ideal lens should cause is expressed in phasor notation as:

$$e^{\pm i \frac{\pi r^2}{\lambda f}} \tag{11}$$

$\pm$ exists so that reversing the sign of the exponent can move the lens between being a focusing lens and a diverging lens.

The geometry

One can see that a few approximations were made in reaching the result, most notably the group velocity being used throughout the entire process. In reality things don't work out perfectly, and there are quantum mechanical limits on the size objects, and the laser or other light will focus down to be much smaller, but will not focus down to an infinitesimal point. The size of the smallest focused spot is called the spotsize, and is often measured with the moment based widths discussed, but also could be measured by full width half maximum.

Another view of a converging lens' phasing hinges on three key facts:

- An ideal laser is a Gaussian amplitude
- Fresnel propagation (usually valid where the focus is) is a Fourier transform, after a multiplication with the exponential $e^{\frac{i\pi}{\lambda L} r^2}$
- A Gaussian is a form of minimum momentum uncertainty

Since a Gaussian is of minimum momentum uncertainty, its Fourier transform is a small as possible. The goal of a lens is to make the intensity profile as small as possible (focused). Therefore a focusing lens should cancel the term $e^{\frac{i\pi}{\lambda L} r^2}$ in the Fresnel propagation Fourier transform, so that the amplitude being Fourier transformed is a Gaussian, and thus its Fourier transform will be as small as possible in width. Taking $L = f$ in the Fresnel inner phasing, the focusing lens (11) phase is recovered.

4.3 Computation: Lensing of a Laser Beam

Now using the Fresnel propagation, the lensing of a laser beam to a focal spot will be computed, as was done in Section 4.1. The lens phase is made into a function as follows, and is the only modification to the $1m$ propagation simulation:

```
def lens_2d(coordinates, f, photon_lambda):
  return np.exp(-1.0j * np.pi / photon_lambda / f * (coordinates[:,:,0]**2+coordinates
    [:,:,1]**2))
```

Plotting the phase diagram of a lens is shown in figure 4.3. The radial waves that causing focusing are clearly, visible, along with some periodic, and aliasing effects further from the center.

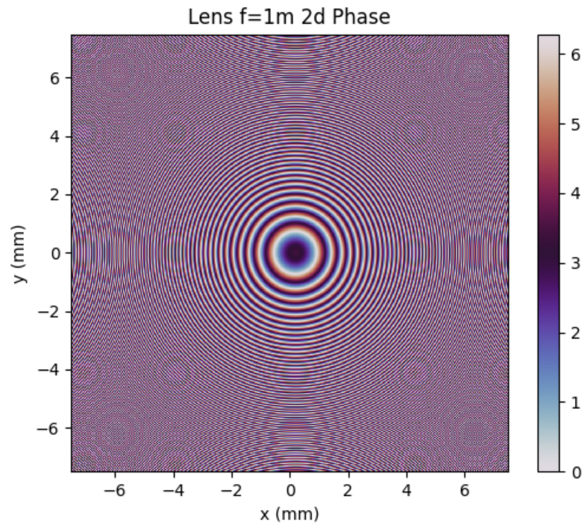

Figure 11. Radial phase waves of a converging lens $f = 1m$. The phase changes much more slowly near the center, reducing the radial inward group velocity closer in.

The resulting width profile is shown for a lens with $f = 10cm$ applied on a Gaussian laser beam before it is propagated $16cm$ in figure 4.3

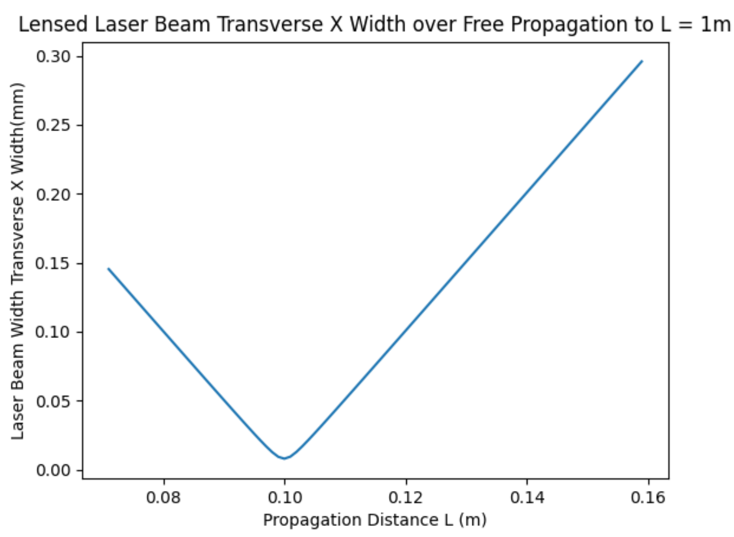

Figure 12. Lensed laser beam ($f = 10cm$) width calculations as it propagates. For the Gaussian laser beam, the minimum width is almost exactly at the focal length $10cm$, for a disturbed laser (say due to air currents varying the index of refraction), the focusing will generally be less ideal (to a larger spot size, and further from f)

In this case for the beam with $D4\sigma$ width of $2mm$, the spot size can be read off the figure as approximately $D4\sigma = 4 \times 8\mu m$. Multiplication by 4 is because the function for width implemented calculates σ. This agrees well with the analytical formula for the spot size of a Gaussian laser beam. Note the beam also for large distances starting diverging in width linearly, as expected. The width will be the same in the y direction, since the Gaussian beam input, and lens phasing were circularly symmetric.

5. CITATIONS/RESOURCES

All the code to generate plots is made available with this PDF.

1. https://en.wikipedia.org/wiki/Thin_lens