Shift Symmetry in Physics and Combinatorics

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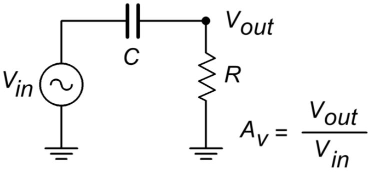
Presentation Overview

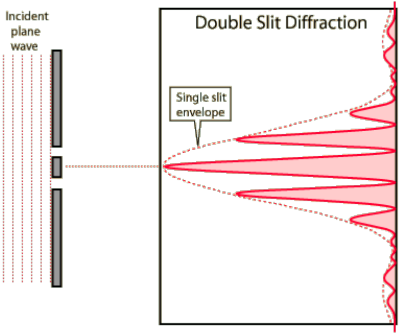
1. Physical Systems
   - Shift invariance
   - Shift operator
   - Sequences
2. Combinatorics
   - Basics
   - Operator counting
   - Operator counting example
   - Modular periodicity
   - Further work

Examples of Shift Invariance

- The system's response for a set of independent coordinates does has no dependence except by translation (time or space typically).
- With the addition of linearity, solving such systems becomes trivial.



(a) Basic electrical components are linear and time invariant



(b) Free space optics is linear and invariant in all coordinates

Shift Operator Formalism

- $X_\Delta f(x,y,z) = f(x+\Delta, y, z)$. Importantly X_Δ acts as the identity on constant functions.
- Commutativity between governing operators of a partial differential equation (PDE) and shift operations is shift invariance.

Consider the case of the 1-d heat equation:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)u(x,t) = 0 \tag{1}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)u(x+\Delta_x, t+\Delta_t)? = 0 \tag{2}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)X_{\Delta_x} T_{\Delta_t} u(x,t)? = 0 \tag{3}$$

If the order of operators $(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2})$ and $X_{\Delta_x} T_{\Delta_t}$ may be swapped, then $u(x+\Delta_x, t+\Delta_t)$ is a solution.

Commutation

Partial derivatives with any independent variable commute, ie $\frac{\partial}{\partial t}\frac{\partial}{\partial z} = \frac{\partial}{\partial z}\frac{\partial}{\partial t}$.

For analytic functions, shift operators may be expressed as a sum of derivatives, $X_\Delta = e^{\Delta \frac{\partial}{\partial x}}$.

The expansion of the shift operator has terms that all commute with partial derivatives.

$$e^{\Delta \frac{\partial}{\partial x}} = \sum_{n=0}^{\infty} \frac{1}{n!}(\Delta \frac{\partial}{\partial x})^n$$

Try at home:

$$e^{\frac{\partial}{\partial x}} x^2 ? = (x+1)^2$$

Sequences

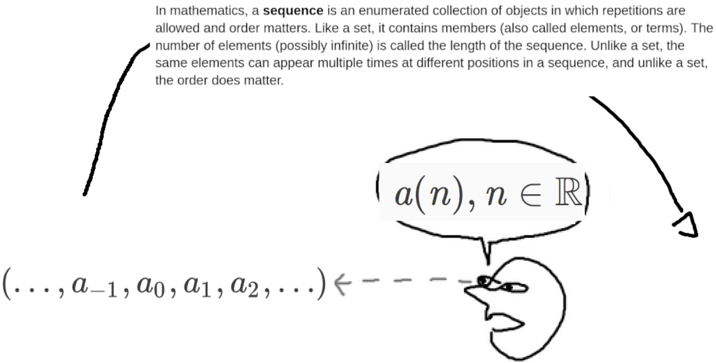


Figure: Sequences.

Sequences in Physics

Coupled oscillators equations of motion (a sequence of equations of motion):

$$m\frac{dx_n^2}{dt^2} = k(x_{n-1} - 2x_n + x_{n+1})$$

This problem is bothersome, usually involves matrix methods. A simple approach is to rewrite it as a PDE.

$$m\frac{d^2x(n,t)}{dt^2} = k(e^{\frac{d}{dn}} - 2 + e^{\frac{-d}{dn}})x(n,t) \tag{4}$$

This is a shift invariant PDE (linear combinations of derivative operators only), solutions are exponentials.

$$e^{\pm i\omega t \pm ik_n n} \to \sin(\pm\omega t \pm k_n n + \phi) \tag{5}$$

Ansatz yields the dispersion relation between k_n, ω.

$$k_n = \cos^{-1}(1 - \frac{k}{2m}\omega^2) \tag{6}$$

Operators in Combinatorics

Combinatorics is all about making countings of permutations or arrangements. In a sense a permutation is a symmetry of an ordering property.

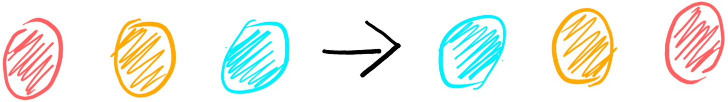


Figure: Transposition $(1\ 3) \equiv S_{1,3}$

For example all bijections (general permutations) may be written as linear operators with $A^T = A^{-1}$.

Combinatorial Sequences

If a counting may be classified by specifying one or more integers, it its a combinatorial sequence.

(a) Weak permutations (with ties) counted by *horse numbers* $F(n)$. Also counts possible expressions for an integral on n variables.
$1, 3, 13, 75, 541, \ldots$

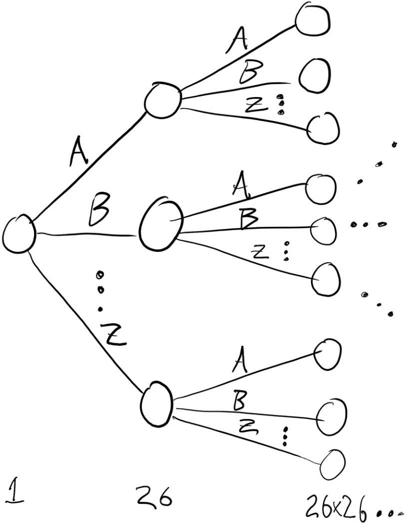
(b) Strong permutations (without ties), $f(n) = n!$.
$1, 2, 6, 24, 120, \ldots$

If I label my presidents x_1, x_2, x_3, x_4, x_5 I can write their ranking as $x_1 < x_3 < x_4 < x_5 < x_2$ or $x_3 < x_5 < x_1 < x_4 < x_2$, and so on.

Enumeration

General approaches exist to counting problems.

(a) Addition principle



(b) Multiplication principle

Extending the Multiplication Principle

Well known single index enumeration function, factorial:
$n! = (n)(n-1)(n-2)\cdots(3)(2)(\mathbf{1})$
For a multi-index combinatorial sequence, the multiplication principle may be interpreted as an *operator multiplication principle*.

$$C'(n,r) = \hat{A}_0\hat{A}_1\hat{A}_2\cdots\hat{A}_r\boldsymbol{C(n)}$$

Form a new counting in the r parameter by extending an existing counting sequence. Analogous to integer multiplication counting on unity.

Constrained Weak Ordering

A previous example was given where some elements $x_1, x_2, \ldots, x_n$ are ordered, with ties. Consider additional constraint $x_1 < x_2 < \cdots < x_r, r \leq n$.

Figure: parallelized factory floor (ties)

Weak Orderings

A weak ordering (with ties) on $x_1, x_2, \ldots, x_n$. [] denote an equivalence class (total mutual ties)
Examples:

$$x_4[x_1x_5]x_3x_7[x_9x_6]\cdots$$

Shorthand for

$$x_4 < (x_1 = x_5) < x_3 < x_7 < (x_9 = x_6)\cdots$$

$$[x_1x_2x_3]x_4x_7[x_9x_6x_5x_{10}]x_{11}x_{12}\cdots$$

Shorthand for

$$(x_1 = x_2 = x_3) < x_4 < x_7 < (x_9 = x_6 = x_5 = x_{10}) < x_{11} < x_{12}\cdots$$

Intuitive first steps

Take the basic case $x_1, x_2, \ldots, x_n, x_1 < x_2$. Start with the original weak ordering $F(n)$. Let us remove any case where $x_1 = x_2$

$$F(n) - F(n-1)$$

Examples ($[x_1 x_2]$ acts as 1 symbol, removed):

$$\cancel{x_4[x_1 x_2]x_3 \cdots}; \cancel{[x_4 x_3]x_9[x_1 x_2] \cdots}; \cancel{[x_4 x_6]x_7 x_{12}[x_1 x_2]x_{10} \cdots}$$

$$x_3[x_1 x_4 x_6]x_2[x_9 x_{12}]x_{10} \cdots ; \cancel{[x_1 x_2 \cdots]}; x_{11} x_8 [x_3 x_2] x_4 x_1 \cdots$$

$\neg(x_1 = x_2) \equiv x_1 < x_2 \vee x_1 > x_2$. There is a degeneracy due to ranking x_1, x_2, we only want $x_1 < x_2$, divide it out.

$$H_2(n) = \frac{F(n) - F(n-1)}{2!} = \frac{(N_1 - 1)N_{-1}F(n)}{2!}$$

Generalized r

$x_1, x_2, \ldots, x_n, x_1 < x_2 < \cdots < x_r$, use (7).
Do this by expanding the shift operator form of the intuitive case:

$$[(N_1 - (r-1))N_{-1}] \cdots [(N_1 - 3)N_{-1}][(N_1 - 2)N_{-1}][(N_1 - 1)N_{-1}]F(n)$$

Now the degeneracy is of ordering r elements to get $x_1 < x_2 < \cdots < x_r$, so divide by $r!$.

$$H_r(n) = \frac{1}{r!}\sum_{j=0}^{r} s(r, r-j)F(n-j) = \left[\binom{N_1}{r} N_{-r}\right] F(n)$$

Analogues to previously known formula that applies to unordered groupings (partitions) derived via chromatic polynomials with

$$\begin{aligned} \neg(x_1 = x_2 \vee x_1 = x_2 \vee \cdots \vee x_1 = x_r \vee x_2 = x_3 \\ \vee x_2 = x_4 \vee \cdots \vee x_2 = x_r \ldots x_{r-1} = x_r) \end{aligned} \tag{7}$$

Modular Periodicity

On a computer, the size of a number is limited by the size of a memory unit. This can be expressed as a modulus. A sequence that was never to repeat can become periodic under a modulus.

Figure: The 8-bit register is a modular integer 0-255, like a clock (0-11)

Modular Periodicity

F(n): 1, 1, 3, 13, 75, 541, 4683, 47293, 545835, 7087261, 102247563, 1622632573, 28091567595, 526858348381, 10641342970443, 230283190977853, 5315654681981355, 130370767029135901, 3385534663256845323, 92801587319328411133, 2677687796244384203115

F(n) (mod 11): 1, 1, 3, 2, 9, 2, 8, 4, 4, 5, 0, 1, 3, 2, 9, 2, 8, 4, 4, 5, 0

Where Next?

- What other enumerations are well suited by the operator multiplication principle?
- What nontrivial properties makes a sequence behave exponentially under a modulus?
- Spectral methods?