

It was guessed that the General restricted horse numbers (strong ordering on variables $x_1, x_2, x_3 \dots x_k$) is the form $B'(k) = s(k, k)H(n) + s(k, k-1)H(n-1) + s(k, k-1)H(n-2) \dots s(k, 1)H(n-k+1)$ where $s(a, b)$ are the signed Stirling numbers.

This may be proven by assuming the following:

The translation operator $\exp(d/dn)$, $\exp(d/dn)H(n) = H(n+1)$

Then build the problem as a counting problem, first remove the k variables that are strongly ordered

$$e^{-(k-1)d/dn}H(n)$$

Then add them back one at a time, which is just increasing the number of variables, but also, need to subtract out any choice where the added variable is equal to one of the k variables, which if m variables are already added back, is just an additive constant $-m$ multiplied on the possible number of configurations without that variable,

$$(e^{d/dn} - 1) e^{-(k-1)d/dn}H(n) [k = 2]$$

$$(e^{d/dn} - 2)(e^{d/dn} - 1) e^{-(k-1)d/dn}H(n) [k = 3]$$

$$(e^{d/dn} - 3)(e^{d/dn} - 2)(e^{d/dn} - 1) e^{-(k-1)d/dn}H(n) [k = 4]$$

The form of the polynomials $(x-1)(x-2)(x-3) \dots (x-m)$ is the generating polynomial for the Stirling Numbers of the first kind, so letting $x^K \sim H(n+k)$ [via translation operator], the formula

$$B'(k, n) = s(k, k)H(n) + s(k, k-1)H(n-1) + s(k, k-1)H(n-2) \dots s(k, 1)H(n-k+1)$$

Is recovered, if one wants a specific ordering, say $x_1 < x_3 < x_4 \dots < x_k$

Just divide by $k!$ Or Bill number $k! * B(k, n) = B'(k, n)$. Note the primed B !