Electrostatic Potential Energy

Benjamin Schreyer

0.1 Potential Energy of a Charge Density

We start by finding the total energy of any charge density as an integral over space.

Let ρ_f be a function of x, y, z with units $\frac{C}{m^3}$ that fully determines the configuration of charge in the system. Consider assembling this charge density from infinity in a manner that allows a single real value $a \in [0, 1]$ to define the progress of arranging the charges, so the intermediate charge density during assembly of the system will be ρ_i where

$$\rho_i = f(a)\rho_f \tag{1}$$

$$f(0) = 0 \tag{2}$$

$$f(1) = 1 \tag{3}$$

In addition assembling charge to a varying voltage will require energy

$$U = \int_0^Q V dq \tag{4}$$

Couloumb's law in voltage charge density form is given by

$$V = \int_{Space} \frac{\rho}{4\pi\epsilon_0 r} d\tau \tag{5}$$

where r is the distance to the point which the voltage is defined and ρ is the charge density.

Starting with equations 4 and 5.

$$U = \int_0^Q \int_{Space} \frac{\rho_i}{4\pi\epsilon_0 r} d\tau dq \tag{6}$$

Then doing a change of variable on the outer integral

$$q = \int_{Space} \rho_i dm = f(a) \int_{Space} \rho_f dm \tag{7}$$

$$dq = (\int_{Space} \rho_f dm) \cdot f'(a) da. \tag{8}$$

Noting that $\rho_f dm$ and equation 5 must be nested together because each $\rho_f dm$ of charge is allocated at a different location and thus potential.

$$= \int_0^1 \int_{Space} f'(a)\rho_f \int_{Space} \frac{\rho_i}{4\pi\epsilon_0 r} d\tau dm da \tag{9}$$

Then substituting equation 1 inside the integral and un-nesting functions of exclusively a

$$= \int_0^1 f'(a)f(a) \int_{Space} \rho_f \int_{Space} \frac{\rho_f}{4\pi\epsilon_0 r} d\tau dm da \tag{10}$$

Notice that

$$\int f'(a)f(a)\, da = \frac{1}{2} f(a)^2 \tag{11}$$

Leading to the factor of $\frac{1}{2}$

$$= \frac{1}{2} \int_{Space} \rho_f \int_{Space} \frac{\rho_f}{4\pi\epsilon_0 r} d\tau dm \tag{12}$$

The innermost integral is just the final voltage of the complete configuration V_f (equation 3) which results in the potential energy of a charge configuration expressed in terms of density:

$$U = \frac{1}{2} \int_{Space} \rho_f V_f dm \tag{13}$$

0.2 Potential Energy of an Electric Field

Start with the potential energy of some electric charge density

$$U = \frac{1}{2} \int_{Space} \rho V d\tau \tag{14}$$

Substituting Maxwell's equation for the divergence of the E-field

$$= \frac{\epsilon_0}{2} \int_{Space} (\nabla \cdot\, E) V d\tau \tag{15}$$

Then using the definition of the potential V

$$= \frac{\epsilon_0}{2} \int_{Space} (\nabla^2\, V) V d\tau \tag{16}$$

Using the first auxiliary identity[1]

$$= -\frac{\epsilon_0}{2} \int_{Space} \frac{\nabla^2\,(V^2)}{2} - \nabla V \bullet \nabla V d\tau \tag{17}$$

Rearranging and using Maxwell's equation for the divergence of the E-Field with the definition of norm

$$= \frac{\epsilon_0}{2} \int_{Space} |E|^2 d\tau - \frac{\epsilon_0}{2} \int_{Space} \frac{\nabla^2\,(V^2)}{2} d\tau \tag{18}$$

[1] See appendix

Rewriting the Laplacian

$$= \frac{\epsilon_0}{2}\int_{Space} |E|^2 d\tau - \frac{\epsilon_0}{2}\int_{Space} \frac{\nabla \bullet \nabla(V^2)}{2} d\tau \tag{19}$$

Using the product rule[2] on $\nabla(V^2)$

$$= \frac{\epsilon_0}{2}\int_{Space} |E|^2 d\tau - \frac{\epsilon_0}{2}\int_{Space} \nabla \bullet (V \cdot \nabla V) d\tau \tag{20}$$

By The Divergence Theorem

$$= \frac{\epsilon_0}{2}\int_{Space} |E|^2 d\tau - \frac{\epsilon_0}{2}\int_{Surface} (V \cdot \nabla V) \bullet da \tag{21}$$

Inspecting the surface integral term in this equation, it is evident that it will approach 0 as we expand the enclosed volume in an otherwise neutral environment. V falls off like $\frac{1}{r}$ and ∇V falls off as $\frac{1}{r^2}$ (∇V is propotional to the E-field). For the limit of an infinitely far surface the surface elements da will grow as r^2 which means the integrand acts like $\frac{1}{r}$. The radius can be expanded to infinity without enclosing any net charge so the integral will be 0. Finally the energy of an E-field is:

$$U = \frac{\epsilon_0}{2}\int_{Space} |E|^2 d\tau \tag{22}$$

Appendix

0.3 Laplacian Identity and Gradient Product Rule

Identity for the Laplacian used in 0.2

$$\nabla^2(f) \cdot f = \frac{\nabla^2(f^2)}{2} - \nabla f \bullet \nabla f \tag{23}$$

Expanding the right side of the equation

$\nabla^2(f) \cdot f = \frac{\nabla^2(f^2)}{2} - (\frac{df}{dx})^2 - (\frac{df}{dy})^2 - (\frac{df}{dz})^2$

$= \frac{1}{2}(\frac{d^2(f^2)}{dx^2} + \frac{d^2(f^2)}{dy^2} + \frac{d^2(f^2)}{dz^2}) - (\frac{df}{dx})^2 - (\frac{df}{dy})^2 - (\frac{df}{dz})^2$

$= \frac{1}{2}(\frac{d}{dx}(2f\frac{df}{dx}) + \frac{d}{dy}(2f\frac{df}{dy}) + \frac{d}{dz}(2f\frac{df}{dz}) - (\frac{df}{dx})^2 - (\frac{df}{dy})^2 - (\frac{df}{dz})^2$

$= f\frac{df^2}{dx^2} + (\frac{df}{dx})^2 + f\frac{df^2}{dy^2} + (\frac{df}{dy})^2 + f\frac{df^2}{dz^2} + (\frac{df}{dz})^2 - (\frac{df}{dx})^2 - (\frac{df}{dy})^2 - (\frac{df}{dz})^2$

$= f\frac{df^2}{dx^2} + f\frac{df^2}{dy^2} + f\frac{df^2}{dz^2}$

[2] Also in the appendix

$= \nabla^2(f) \cdot f$

The Product Rule for The Gradient

$$\nabla(f \cdot g) = \nabla g \cdot f + \nabla f \cdot g \tag{24}$$

Expanding the left side to show equality

$\nabla(f \cdot g) = \frac{d}{dx}(f \cdot g)\hat{i} + \frac{d}{dy}(f \cdot g)\hat{j} + \frac{d}{dz}(f \cdot g)\hat{k}$

$= (g\frac{df}{dx} + f\frac{dg}{dx})\hat{i} + (g\frac{df}{dy} + f\frac{dg}{dy})\hat{j} + (g\frac{df}{dz} + f\frac{dg}{dz})\hat{k}$

$= \nabla g \cdot f + \nabla f \cdot g$