Combinatorics: Operator Counting and Sequence Automata Size

Total orderings of discrete elements may be given two broad classes, strong and weak. *Strong orderings* of elements are counted by the factorial function. *Weak orderings* allow ties between elements, and are counted by *Fubini numbers*. A favored example of weak permutations is the result of a horse race. Studying these sequences under a modulus, the *group of modular integers under multiplication* $(\mathbb{Z}/K\mathbb{Z})^{\times}$ provides a useful bound on the information needed to specify them.

It has been shown by Schreyer [1] that insertion of index shift operators into the generating function of the Stirling numbers of the first kind applied to the vector represented sequence of weak orderings gives a novel and arguably more simple form for the counting of r*-Fubini numbers*, weak permutations where a subset follows strong ordering, than had been previously known. The linearity of this process extends modular periodicity upper bounds from Fubini numbers to r-Fubini numbers easily.

Proposed Efforts

After an initial bout of work summarized above, the study of further questions in a graduate program is proposed. The first set of questions are under operator counting methods. The proposed exploration is on pairs of existing sequences, one providing the generating function interpreted under linear operators, and one as the vector operand. This may be expressed as follows with v, v' a vector represented combinatorial sequence and a new derived sequence respectively, A a linear operator, and F the generating function of another combinatorial sequence.

$$v' = F(A)v \tag{1}$$

Such relations are ripe with possibility to simplify known combinatorial countings.

The other survey based exploratory aspect will be in the facet of sequence modular periodicity. The group $(\mathbb{Z}/K\mathbb{Z})^{\times}$, specifically the work of Carmichael [2], yields a straightforward way to bound the number states needed in a deterministic finite automata (DFA) describing certain sequences of numbers under a modulus. An example such sequence is the Stirling numbers of the second kind, which have the following form as noted by Cheon and Kim [3].

$$S(n,k) = \frac{1}{k!}\sum_{t=0}^{k}(-1)^{k-t}\binom{k}{t}t^n \pmod{K} \tag{2}$$

For fixed k the sequence is only dependent on exponentiations of constant integers to the power n. This form is linearly composed to express r-Fubini numbers under a modulus in previous work [1]. A survey is proposed for other modular integer sequences which may be expressed in their dependence only as exponentiations (and therefore trivially viewable through the lens $(\mathbb{Z}/K\mathbb{Z})^{\times}$). Such a survey would provide exemplary members of a class of modular sequences which have a DFA of size bounded by the exponent of $(\mathbb{Z}/K\mathbb{Z})^{\times}$. Searching for nontrivial properties of a sequence that place it in such a class will occur in parallel to such a survey. This class of sequences envelopes constant linear recurrences which are well studied, but also involves sequences like the Fubini numbers which have no such constant recurrence. Modular periodicity of non-constant linear coefficients defining a sequence appears as a promising condition sufficent to determine $(\mathbb{Z}/K\mathbb{Z})^{\times}$ periodicity for modular sequences.

Of practical use, it is propsed to gather problems whose solution spaces have countings that are directly quantified by the r-Fubini counting or a variation. The most practical expected application is the factory floor scheduling problem. In such scheduling some tasks may be taken at the same time (weak ordering in permutation), while others depend on the completion of a requisite task (strong ordering). The specific strong ordering operator $\binom{E}{r}E^{-r}$ provided by Schreyer [1] could extend the strong-weak dichotomy to factory planning. Programmatic implementations of all novel or unimplemented countings will be written.

To execute such work in a timely manner, collaboration is of utmost importance.The key to not fooling yourself, and doing work that benefits society is to share and discuss mathematics with others. Having funding to attend a graduate program in mathematics or theoretical computer science would allow me to surround myself with critical and constructive peers. Additionally in order to achieve the outline goals I must further my education in discrete mathematics and combinatorics, and continue to teach others in order to master what I learn and share new ideas.

To disseminate this work, papers should be written, which are immediate in survey work, and could stem from generalizations and new formulae that appear. In addition we will work with Professor William Gasarch who inspired this work, to provide students with updated course material to help them understand the simplest cases of results relevant to an advanced complexity and discrete mathematics undergraduate course. Serious effort should also be given to bringing interesting presentation of related material to those outside of the academic sphere or a younger audience. This is not easy, but a starting point is to simply do diagrammatic counting with small sets that have r strongly ordered terms and $n - r$ weakly ordered terms, in short article form, or as a video presentation.

Intellectual Merit

Studying eventually periodic modular integer sequences builds upon the expanding volume of work done in quantifying the information content of a sequence. The base of my work will establish a survey of problems where $(\mathbb{Z}/K\mathbb{Z})^{\times}$ provides a bound for a sequence's DFA size under a modulus with potential for generalizations to be revealed. Simplified expressions for common variations on combinatorial countings will be surveyed. Minimally resulting work can be further used in proofs focused on specific sequences. Connection of combinatorial countings to application will be established by research and survey, and supported by library implementation.

Broader Impacts

Combinatorics is a rich field with application across technical endeavors. In large scale systems across the sciences, the time or space complexity is often linked to some sequence of combinatorial countings. The proposed work building and sharing specific proofs, and tools of proof, is a part in the larger process of reaching for the horizon of combinatorial ideas.

References.

[1] B. Schreyer, Rigged Horse Numbers and their Modular Periodicity, *arXiv preprint* (2024).

[2] P. Erdős, C. Pomerance, E. Schmutz, Carmichael's lambda function, *Acta Arith.* **58** (1991), 363-385.

[3] G. Cheon, J. Kim, Stirling matrix via Pascal matrix, *Linear Algebra Its Appl.* **329** (2001), 49-59.