Rigged Horse Numbers and Modular Periodicity

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Abstract

The permutations of horse racing, where ties are possible, are counted by the *Fubini numbers*, also called the *horse numbers*. The *r-Fubini numbers* are a counting of such horse race finishes, where some subset of r horses agree to fix their relative ordering in the race rankings. The r-Fubini numbers for fixed r are expressed as a sum of r index shifted sequences of Fubini numbers weighted with the signed Stirling numbers of the first kind. Then eventual modular periodicity of r-Fubini numbers is shown and their maximum period is determined to be the Carmichael function of the modulus. The maximum is attained in the case of an odd modulus.

1 Introduction

1.1 Contributions

A new expression is found for the r-Fubini numbers. For fixed r, r-Fubini numbers are a constant coefficient sum of the Fubini numbers under index shifts. Elegant proofs for the modular periodicity of Fubini and r-Fubini numbers are formulated, which give an upper bound for their modular eventual period. The upper bound is the Carmichael function $\lambda(K)$ where K is the modulus. When K is odd it is shown $\lambda(K)$ is the exact period.

1.2 Orderings weak and strong

Definition 1. *Fubini numbers* are denoted $H(n)$, which count weak orderings of n elements.

In the case of a horse race, the ordering is weak, equality determines ties, and $<$ and $>$ determine clear succession. The number of such orderings are named the Fubini numbers, horse numbers, or ordered Bell numbers denoted $H(n)$. Without possibility of ties, such orderings are regular permutations (strong).

1.3 The r-Fubini numbers or rigged weak orderings

Definition 2. The *r-Fubini numbers* $H_r(n)$ count weak orderings such that r elements of the finite set of cardinality n are distinguished, and constrained to follow a specific strong ordering.

Here this counting is exemplified. Consider arbitrary elements under total weak ordering $x_1, x_2, \ldots, x_n$. A strong ordering of a subset of distinguished elements inducing r-Fubini counting could be $x_1 < x_2 < \cdots < x_r$. In this text these permutations are indexed by denoting the number of total elements to be ordered n, and size of the distinguished subset that follow a fixed strong ordering r.

Others choose to denote the number of distinguished strongly ordered elements, and the number of undistinguished elements. These numbers have been studied by Rácz [1] who derived an expression in terms of the r-Lah numbers and factorials for $H_r(n)$. Asgari and Jahangiri [2] proved the eventual periodicity of the r-Fubini numbers modulo any natural number, which will also be shown here more briefly. Asgari and Jahangiri also gave calculations for the period.

1.4 Stirling numbers of the first and second kind

Definition 3. *Signed Stirling numbers of the first kind* $s(n, k)$ count partitions of n elements into k cycles (the sign gives the parity of permutation). Such two index sequences may naturally be arranged into a matrix with rows n and columns k. Let M_s denote the matrix of $s(n, k)$.

Definition 4. *Stirling numbers of the second kind*, $S(n, k)$ count ways to partition a set into unordered groups. The matrix of $S(n, k)$ is labeled M_S.

The Stirling numbers are here introduced with the addendum of three useful properties. Stated in *Advanced combinatorics* [3].

Proposition 1. *M_s and M_S are inverses of each other. This applies even to the infinite matrices, where $n, k \geq 0$:*

$$M_s M_S = M_S M_s = I \tag{1}$$

These matrices are additionally both lower triangular.

The second important property is listed in *Concrete Mathematics* [4].

Proposition 2. *The Stirling numbers of the first kind give the coefficient for fixed powers of the argument of the falling factorial. The notation* $(x)_n$ *is the falling factorial of* x *with* n *multiplicative terms.*

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k \tag{2}$$

Before the last property is introduced, a further definition is needed.

Definition 5. Eventual modular periodicity for a sequence $f(n)$ means for sufficiently large n the following holds with fixed $T \in \mathbb{Z}$, T being the eventual period.

$$f(n) \;= f(n+T) \tag{3}$$

The third important property, specific to $S(n,k)$, is their eventual periodicity in n modulo $K \in \mathbb{N}$ for fixed k.

This may be shown using the following formula lifted from the paper *Stirling matrix via Pascal matrix* [5] which is re-expressed under multiplication by unity, $\frac{k}{k}$.

Lemma 3.

$$S(n,k) = \frac{1}{k!}\sum_{t=0}^{k}(-1)^{k-t}\binom{k}{t}t^n \tag{4}$$

The finite period of modular exponentiation determines eventual periodicity for $S(n,k)$ for fixed k.

1.5 The Carmichael function

Definition 6. $(\mathbb{Z}/K\mathbb{Z})^\times$ is the group of integers coprime to $K \in \mathbb{N}$ under multiplication modulo K.

Definition 7. $\lambda(K)$ is the *Carmichael function*, which gives the exponent of integers under multiplication, often in the context of the group $(\mathbb{Z}/K\mathbb{Z})^\times$.

The previously stated equation for $S(n,k)$ has dependence on n exclusively as a sum of exponentiations of integers (4) by n. Importantly under a modulus K, $\lambda(K)$ is the maximum eventual period of exponentiation of integers.

Two properties of the Carmichael function are leveraged.

Proposition 4. *For any* $a \in \{0,1,\ldots,K-1\}$*, the following holds.*

$$a^R = a^{\lambda(K)+R} \pmod{K} \tag{5}$$

Where R *is the greatest exponent in the factorization of* K *into unique prime powers.*

For coprime elements to K a stronger statement may be made.

Proposition 5. *For $b \in (\mathbb{Z}/K\mathbb{Z})^{\times}$:*

$$b^{\lambda(K)} = 1 \pmod{K} \tag{6}$$

1.6 Summation, shifting, and scaling of eventually periodic sequences

Simple proofs are given for important operations that preserve eventual modular periodicity in the appendix A. These operations are scaling by an integer, addition of eventually periodic sequences, and index shifting. Upper bounds for eventual period are preserved for scaling and shifting. For addition, the least common multiple must be considered.

1.7 Shift operators

Shift operators are used to formally show r-Fubini numbers may be expressed using the signed Stirling numbers of the first kind.

Definition 8. $T_{\pm}$ are the right and left shift operators respectively. Often multiple single shift operations are abbreviated T_m, $m \in \mathbb{Z}$.

Computation of the r-Fubini numbers uses the left and right shift operators T_+, T_- on the sequence $H(0), H(1), \ldots, H(n+r)$. Shift operators applied on a sequence $F(n)$ are linear operators, defined such that $T_+F(n) = F(n+1)$ and $T_-F(n) = F(n-1)$. Sequences also distribute over addition of shift operators so $(AT_a + BT_b)F(n) = AF(n+a) + BF(n+b)$. Importantly $T_+T_- = T_-T_+ = I$ (I being the identity operation or zero shift), so any product of shift operators may be abbreviated T_a, $a \in \mathbb{Z}$, and $T_aF(n) = F(n+a)$ so long as $n+a \in \mathbb{N} \cup \{0\}$. In their use here, the shift operators will not leave the sequence with an index outside of $\mathbb{N} \cup \{0\}$.

2 General rigged orderings of $r \leq n$ elements

the following lemma is needed in the proof of the novel expression for $H_r(n)$ the r-Fubini numbers.

Lemma 6. *Including an additional element in the ordered set $\{x_1, x_2, \ldots x_n\}$ so that the new element x' satisfies $x' \notin \{x_1, x_2, \ldots, x_m\}$ is counted by $F(n+1) - mF(n)$, where $F(n)$ is the number of permutations the set had before the element was added.*

Proof. Consider adding the new element, with no restriction. The new number of orderings is $F(n+1)$, since no new element was distinguished, the number of elements is simply increased.

To distinguish x' such that $x' \notin \{x_1, x_2, \ldots, x_m\}$, note that for each ordering counted by $F(n)$, when x' is introduced, it may be set equal to one of $x_1, x_2, \ldots, x_m$ to form a unique disallowed permutation. By the multiplication rule and exclusion of these orderings the lemma follows. $\square$

Theorem 7.

$$H_r(n) = \frac{1}{r!} \sum_{j=0}^{r} s(r, r-j) H(n-j) \tag{7}$$

Proof. The proof first counts the case where the subset $\{x_1, x_2, x_3, \ldots, x_r\}$ are strongly ordered, then gives them a single ordering by dividing by $r!$.

As elements of the distinguished subset are added back to the counted set, there are more possible mutual equalities that must be subtracted out. This results in subtraction of ascending integers in accordance with the above lemma 6. The subtractions yield the desired strongly ordered subset counting. To begin counting first remove or ignore the counting of the r distinguished elements which is $T_{-r}H(n)$.

- Reintroduce the element x_1 by increasing the argument to $H(n-r+1)$, then subtract any case where $x_1 \in \emptyset$. That is $T_+T_{-r}H(n)$. The subtraction is redundant (hence $\emptyset$), for the first step, since no elements in the strongly ordered subset exist in the remaining elements.
- Reintroduce the element x_2 by increasing the argument, then subtract any case where $x_2 \in \{x_1\}$. That is $(T_+ - 1I)T_+T_{-r}H(n)$.
- Reintroduce the element x_3 by increasing the argument, then subtract any case where $x_3 \in \{x_1, x_2\}$. That is $(T_+ - 2I)(T_+ - 1I)T_+T_{-r}H(n)$.
- $\cdots$
- Reintroduce the element x_r by increasing the argument, then subtract any case where $x_k \in \{x_1, x_2, \ldots, x_{r-1}\}$. The total is $(T_+ - (r-1)I) \cdots (T_+ - 2I)(T_+ - 1I)T_+T_{-r}H(n)$.

Now all elements are included with their respective ordering, with those from $x_1, x_2, \ldots, x_r$ without mutual equalities, such that any counted ordering has $x_1, x_2, \ldots, x_r$ strongly ordered. The falling factorial appears with argument T_+ and r terms.

$$T_{-r}(T_+)_r H(n) \tag{8}$$

The count is now expressed as follows via the expansion previously introduced for falling factorials (2).

$$T_{-r}[\sum_{j=0}^{r} s(r, j) T_j] H(n) \tag{9}$$

The formula applies because repetition of T_+ may be treated as would multiplication of a polynomial variable. The effect of the shift operators is now trivial upon $H(n)$.

$$\sum_{j=0}^{r} s(r,j)H(n-r+j) \tag{10}$$

By re-indexing the sum.

$$\sum_{j=0}^{r} s(r,r-j)H(n-j) \tag{11}$$

The number of arrangements of $x_1, x_2, \ldots, x_n$ where $x_1, x_2, \ldots, x_r$ are strongly ordered have been counted. Given the strongly ordered subset count it is straightforward to determine $H_r(n)$ by dividing by $r!$.

$$H_r(n) = \frac{1}{r!}\sum_{j=0}^{r} s(r,r-j)H(n-j)$$

□

An interesting notation is apparent.

$$H_r(n) = \frac{T_+(T_+-1)\cdots(T_+-r+1)}{r!}H(n-r) \tag{12}$$

$$H_r(n) = \binom{T_+}{r}H(n-r) \tag{13}$$

It appears that the fully removed operator $\binom{T_+}{r}T_{-r}$ could be used in repeated application to create counting for multiple disjoint strongly ordered subsets of different sizes. Additionally different subsets could be counted relative to T_+ via subtraction such as using a binomial coefficient or other count instead of successive integers.

2.1 A useful alternating recurrence

Corollary 8.

$$H(n) = n! - \sum_{j=1}^{n} s(n,n-j)H(n-j) \tag{14}$$

Proof. $H_n(n) = 1$, if the set is $\{x_1, x_2, \dots, x_n\}$, and $x_1 < x_2 < \cdots < x_n$ the number of arrangements is 1. It follows from the Theorem 7:

$$1 = \frac{1}{n!} \sum_{j=0}^{n} s(n, n-j) H(n-j) \tag{15}$$

Which may be rearranged after the substitution $s(n, n) = 1$ to the result.

$$H(n) = n! - \sum_{j=1}^{n} s(n, n-j) H(n-j)$$

□

3 Linear transformation between strong and weak orderings

Corollary 9. *The infinite vectors f_n and H_n with entries $n!$ and $H(n)$ respectively obey the following relation with matrices M_s and M_S as defined in the introduction 1.4:*

$$M_S f_n = H_n \tag{16}$$
$$M_s H_n = f_n \tag{17}$$

Proof. The above equation (15) may be written as a matrix equation by multiplication by $n!$ and reindexing.

$$\sum_{j=0}^{n} s(n, j) H(j) = n! \tag{18}$$

The infinite lower triangular matrix $s(n, k)$ multiplied on a vector may be expressed as exactly the sum derived.

$$M_s H_n = f_n$$

The first relation then immediately follows given the previously stated inverse of M_s being M_S (1).

$$\begin{aligned} M_S M_s H_n &= M_S f_n \\ H_n &= M_S f_n \end{aligned} \tag{19}$$

□

The signed Stirling numbers of the first kind act naturally on H_n as a matrix. The Stirling numbers of the second kind correspondingly on f_n.

4 Modular periodicity

Asgari and Jahangiri [2] give eventual modular periodicity to $H_r(n)$ and an explicit calculation for the eventual period. Here the modular periodicity is determined elegantly via the perspective of exponentially generated sequences. In addition an upper bound for the period is proven to be the Carmichael function $\lambda(K)$ of the modulus. Two conditions for $\lambda(K)$ to be the exact modular period follow.

4.1 Fubini numbers modulo K

Theorem 10. *The Fubini numbers $H(n)$ are eventually periodic modulo $K \in \mathbb{N}$, with maximum possible modular period $\lambda(K)$, the Carmichael function of K.*

Proof. Consider the existing relation (16).

$$H_n = M_S f_n$$

The entries of f_n are simply $n!$. Modulo K the vector entries f_n are certainly zero for $n \geq K$.

The relation stands in a simplified form:

$$H(n) = \sum_{k=0}^{K-1} S(n,k)k! \pmod{K} \tag{20}$$

$H(n) \pmod{K}$ is written as a finite sum of $S(n,k)$ with coefficients independent of n. Each $S(n,k) \pmod{K}$ for $0 \leq k \leq K-1$ contributes sums of exponential dependence in n for fixed k by the explicit form for the Stirling numbers of the second kind (4). The finite sum of such eventually periodic exponentially generated sequences scaled by constants is also eventually periodic 1.6. Due to Carmichael all such exponentiations of any $j \in \mathbb{Z}$, $0 \leq j \leq K$ follow:

$$j^{R+\lambda(K)} = j^R \pmod{K} \tag{21}$$

The onset of eventual periodicity must occur after $H(R)$, $R = \max(R_1, R_2, \ldots)$ increments of the argument, where $K = p_1^{R_1} p_2^{R_2} \ldots$, where p_i are unique and prime. All the exponentials each with fixed coefficients will have entered periodicity after R increments. The longest possible eventual period given $H(n) \pmod{K}$ is a weighted sum of integer powers j^n is clearly $\lambda(K)$.

□

4.2 Exact Carmichael periodicity

Let C_i be the unique cyclic groups in the decomposition by direct product of the group $(\mathbb{Z}/K\mathbb{Z})^\times$ into cycles. An element of $(\mathbb{Z}/K\mathbb{Z})^\times$ may be written as a tuple $(b_1, b_2, b_3, \ldots)$ where b_i are residues in each cycle. Let a_i denote a generator which exists for each C_i such that $\langle a_i \rangle = C_i$.

Theorem 11. *The Carmichael function $\lambda(K)$ is exactly the period of $H(n) \pmod K$ if $\forall C_i$, a_i is part of the cyclic decomposition of a base exponentiated to the n'th in the sum for $H(n) \pmod K$.*

Proof. The definition of the exponent as the least common multiple of finite orders of group elements is extended to the sum of exponentiations, since the modular period of a sum of sequences is their least common multiple. So if all a_i are exponentiated in the sum, the period is $\operatorname{lcm}(|C_1|, |C_2|, \ldots) = \lambda(K)$ □

4.3 Extension to r-Fubini numbers

Corollary 12. *$H_r(n)$ the r-Fubini numbers are eventually periodic in n modulo K for fixed r with maximum period $\lambda(K)$ and periodicity onset after at latest $r - 1 + R$ for $r > 0$, where R is the maximum exponent in the unique prime power decomposition of K.*

Proof. $H(n)$ is eventually periodic in accordance with the above proposition 10. Next note the operation $\binom{T_+}{r}T_{-r}$ on $H(n)$ to generate $H_r(n) = \binom{T_+}{r}T_{-r}H(n)$ per Theorem 7 preserves the structure of $H(n)$ as a weighted sum of exponentials of n, which again have maximum period modulo K of $\lambda(K)$. The periodic onset may be delayed by shift operations, hence the addition of $r - 1$ to the onset index. □

4.4 The odd case

Theorem 13. *For $H(n) \pmod K$ if K is odd then the eventual period of the sequence is $\lambda(K)$.*

Proof. This can be seen immediately in the last theorem in Asgari and Jahangiri's paper [2], noting that *Euler's totient* $\varphi(n)$ is equal to the Carmichael function for odd prime power arguments.

The eventual period for this case is $\operatorname{lcm}(\varphi(p_1^{k_1}), \varphi(p_2^{k_2}), \ldots)$ where $p_i^{k_i}$ are the unique prime powers composing K. When the argument of $\lambda(n)$ is an odd prime power, it is equal to $\varphi(n)$. This allows the following expression for the eventual period T:

$$T = \operatorname{lcm}(\lambda(p_1^{k_1}), \lambda(p_2^{k_2}), \ldots) \tag{22}$$

$$T = \lambda(K) \tag{23}$$

Where the recurrence for $\lambda(n)$ is used for the final result.

□

5 Remarks

5.1 Infinite matrices

Remark 14. The matrix entries M_s form Sierpiński triangles sometimes with defects when plotted modulo some natural number.

The diagonals of the infinite matrix M_s appear to be eventually periodic modulo any natural number.

5.2 Shift operators

Remark 15. It is interesting to consider where else in combinatorics the shift operator may yield simplified understanding or proofs. Interpreting the binomial form (13) is expected to be useful to this end.

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A Properties of eventual modular periodicity

For each proof it is unstated that n is sufficiently large but finite.

Proposition 16. *Consider $g(n) \pmod{K}$, $K \in \mathbb{N}$, formed by scaling an eventually modularly periodic sequence $f(n)$ with period r by factor $m \in \mathbb{Z}$. The sequence $g(n) = mf(n)$ will also be eventually periodic modulo K.*

Proof.

$$g(n+r) = mf(n+r) \pmod{K} \tag{24}$$

$$g(n+r) = mf(n) \pmod{K} \tag{25}$$

$$g(n+r) = g(n) \pmod{K} \tag{26}$$

□

Proposition 17. *Consider $f(n)$ formed by summing two eventually period sequences modulo K: $h(n), g(n)$ with period x, y respectively, both in $\mathbb{N}$. $f(n) = h(n) + g(n) \pmod{K}$ will also be eventually periodic.*

Proof.

$$\begin{aligned} f(n + \mathrm{lcm}(x,y)) &= h(n + \mathrm{lcm}(x,y)) + g(n + \mathrm{lcm}(x,y)) \pmod{K} && (27) \\ f(n + \mathrm{lcm}(x,y)) &= h(n) + g(n) \pmod{K} && (28) \\ f(n + \mathrm{lcm}(x,y)) &= f(n) \pmod{K} && (29) \end{aligned}$$

□

Proposition 18. *If $f(n) \pmod{K}$, is eventually periodic, then so is $f(n+w) \pmod{K}$ where $w \in \mathbb{Z}$.*

Proof. For clarity let $n + w = m$.

$$\begin{aligned} f(n+w) &= f(m) \pmod{K} && (30) \\ f(m+r) &= f(m) \pmod{K} && (31) \\ f(n+w+r) &= f(n+w) \pmod{K} && (32) \end{aligned}$$

□

The above property may be expressed in shift operator notation: If $f(n) \pmod{K}$ is eventually periodic, then so is $T_w f(n) \pmod{K}$ where $w \in \mathbb{Z}$.

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