Small examples/interesting for r-Fubini numbers

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From the paper, note the r-Fubini numbers are just $r!$ scalings of the cases where counting is done under constraint $x_1 < x_2 < \cdots < x_r$.

0.1 Counting with shift operators

Theorem 1.

$$H_r(n) = \frac{1}{r!}\sum_{j=0}^{r} s(r, r-j)H(n-j) \tag{1}$$

0.2 A useful alternating recurrence

Corollary 2.

$$H(n) = n! - \sum_{j=1}^{n} s(n, n-j)H(n-j) \tag{2}$$

1 Linear transformation between strong and weak orderings

Corollary 3. *The infinite vectors $\vec{f}$ and $\vec{H}$ with entries $n!$ and $H(n)$ respectively obey the following relation with matrices $\hat{s}$ and $\hat{S}$.*

$$\hat{S}\vec{f} = \vec{H} \tag{3}$$

$$\hat{s}\vec{H} = \vec{f} \tag{4}$$

2 DFA

Asgari and Jahangiri show eventual modular periodicity for $H_r(n)$ and an explicit calculation for the eventual period. They have a DFA!

3 Small r cases using set operations

Vertical bars denote cardinality of a set of n elements, under the restriction within the bars. A subscript on the bars indicates removing one of the n elements, equivalent to always setting it equal to some other element.

3.1 $r = 2$

$$H_2(n) = |x_1 < x_2| \tag{5}$$

Use the reordering of x_1, x_2 to express in terms of the number of strong orderings of x_1, x_2.

$$H_2(n) = \frac{1}{2!}|\neg(x_1 = x_2)| \tag{6}$$

Re-express not, then use the fact that equality constraints are equivalent to removing elements.

$$H_2(n) = \frac{1}{2!}[|none| - |x_1 = x_2|] \tag{7}$$

$$H_2(n) = \frac{1}{2!}[H(n) - H(n-1)] \tag{8}$$

3.2 $r = 3$

Start from where (6) started.

$$H_3(n) = \frac{1}{3!}|\neg[x_1 = x_2 \vee x_1 = x_3 \vee x_2 = x_3]| \tag{9}$$

Invert the not.

$$H_3(n) = \frac{1}{3!}\left(H(n) - |[x_1 = x_2 \vee x_1 = x_3 \vee x_2 = x_3]|\right) \tag{10}$$

Expand on the first $\vee$.

$$H_3(n) = \frac{1}{3!}\left(H(n) - |[x_1 = x_2| - |x_1 = x_3 \vee x_2 = x_3]| + |[x_1 = x_2] \wedge [x_1 = x_3 \vee x_2 = x_3]|\right) \tag{11}$$

Resolve the first and second set cardinalitys.

$$H_3(n) = \frac{1}{3!}\left(H(n) - H(n-1) - (2H(n-1) - |x_1 = x_2 \wedge x_2 = x_3|) + |[x_1 = x_2] \wedge [x_1 = x_3 \vee x_2 = x_3]|\right) \tag{12}$$

The first $\wedge$ cardinality just removes two elements.

$$H_3(n) = \frac{1}{3!}\left(H(n) - 3H(n-1) + H(n-2) + |[x_1 = x_2] \wedge [x_1 = x_3 \vee x_2 = x_3]|\right) \tag{13}$$

Now assume $x_1 = x_2$ in the right hand operand of the remaining $\wedge$. The result is, remove the element x_1 WLOG (could have done x_2), and simplify the $\vee$. This gives a subscript -1.

$$H_3(n) = \frac{1}{3!}\left(H(n) - 3H(n-1) + H(n-2) + |[x_2 = x_3 \vee x_2 = x_3]|_{-1}\right) \tag{14}$$

Finally the last constrained cardinality is just removing two elements.

$$H_3(n) = \frac{1}{3!}\left(H(n) - 3H(n-1) + 2H(n-2)\right) \tag{15}$$
