Rigged Horse Numbers and their Modular Periodicity

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Abstract

The permutations of horse racing, where ties are possible, are counted by the *Fubini numbers*, also called the *horse numbers*. The *r-Fubini numbers* are a counting of such horse race finishes where some subset of r horses agree to finish the race in a specific strong ordering. The r-Fubini numbers for fixed r are expressed as a sum of r index shifted sequences of Fubini numbers weighted with the signed Stirling numbers of the first kind. Then eventual modular periodicity of r-Fubini numbers is shown and their maximum period is determined to be the Carmichael function of the modulus. The maximum is attained in the case of an odd modulus.

1 Introduction

1.1 Contributions

A new expression is found for the r-Fubini numbers. For fixed r, r-Fubini numbers are a constant coefficient sum of the Fubini numbers under index shifts. Novel proofs for the eventual modular periodicity of Fubini and r-Fubini numbers are formulated, which give an upper bound for their modular eventual period. The upper bound is the Carmichael function $\lambda(K)$ where K is the modulus. When K is odd it is shown that $\lambda(K)$ is the exact period.

1.2 Orderings weak and strong

Definition 1. *Fubini numbers* are denoted $H(n)$, which count weak orderings of n elements.

In the case of a horse race, the ordering is weak, equality determines ties, and $<$ and $>$ determine clear succession. Other common names for the counting of weak orderings are the horse numbers, or ordered Bell numbers. Without possibility of a tie such orderings are regular permutations (strong).

1.3 The r-Fubini numbers or rigged weak orderings

Definition 2. *Relative strong ordering* among elements of a set demands that for elements in the relative strong ordering, their relations may be described only $<$ or $>$. An element in a relative strong ordering may be set equal to another element, but that element cannot come from the elements participating in the relative strong ordering.

Definition 3. The *r-Fubini numbers* $H_r(n)$ count weak orderings such that r elements of the finite set of cardinality n are distinguished, and constrained to follow a specific relative strong ordering.

The r-Fubini numbers count the simplest nontrivial restrictions on counting weak permutations. A restriction where two or more horses agree to tie is simply counted by reducing the number of elements in the permutation, since the horses act as a single unit in any permutation. The nontrivial counting is further exemplified. Consider arbitrary elements under total weak ordering $x_1, x_2, \ldots, x_n$. A strong ordering of a subset of distinguished elements inducing r-Fubini counting could be $x_1 < x_2 < \cdots < x_r$. In this text r-Fubini counted permutations are indexed by denoting the number of total elements to be ordered n, and size of the distinguished subset that follow a fixed strong ordering r.

Others choose to instead express the number of distinguished relatively strongly ordered elements and the number of undistinguished elements. Orderings counted by r-Fubini numbers have been studied by Rácz [1] who derived an expression in terms of the r-Lah numbers and factorials for $H_r(n)$. Asgari and Jahangiri [2] proved the eventual periodicity of the r-Fubini numbers modulo an arbitrary natural number. Asgari and Jahangiri also gave calculations for the period.

1.4 Stirling numbers of the first and second kind

Definition 4. *Signed Stirling numbers of the first kind* $s(n,k)$ count partitions of n elements into k cycles (the sign gives the parity of permutation). Two index sequences may naturally be arranged into a matrix with indexing of rows n and columns k, for the case of $s(n,k)$. Let $\hat{s}$ denote the matrix of $s(n,k)$.

Definition 5. *Stirling numbers of the second kind*, $S(n,k)$ count ways to partition a set into unordered groups. The matrix of $S(n,k)$, labeled $\hat{S}$, is indexed as is $\hat{s}$ above.

The Stirling numbers are here introduced with the addendum of three useful properties, which are expressed with notation I as the identity operation. Stated in *Advanced combinatorics* [3].

Proposition 1. *$\hat{s}$ and $\hat{S}$ are inverses of each other. This applies even to the infinite matrices, where $n, k \geq 0$:*

$$\hat{s}\hat{S} = \hat{S}\hat{s} = I \tag{1}$$

$\hat{s}$ and $\hat{S}$ are additionally both lower triangular.

The second important property is listed in *Concrete Mathematics* [4]. The notation $(x)_{\underline{n}}$ is the falling factorial of x with n multiplicative terms $(x)_{\underline{n}} = x(x-1)\cdots(x-n+1)$.

Proposition 2. *The Stirling numbers of the first kind give the coefficient for fixed powers of the argument of the falling factorial.*

$$(x)_{\underline{n}} = \sum_{k=0}^{n} s(n,k)x^k \tag{2}$$

Before the last property is introduced a further definition is needed.

Definition 6. *Eventual modular periodicity* for a sequence $f(n)$ is defined as follows. A sequence $f(n)$ is *eventually modular periodic* if there exists T such that, for large enough n, $f(n) = f(n+T)$.

$$f(n) \ = f(n+T) \pmod{K} \tag{3}$$

Where $K \in \mathbb{N}$.

The third important property, specific to $S(n,k)$ with fixed k, is their eventual periodicity in n modulo $K \in \mathbb{N}$. This may be shown using the following formula from the paper *Stirling matrix via Pascal matrix* [5] which is re-expressed under multiplication by unity, $\frac{k}{k}$.

Lemma 3.

$$S(n,k) = \frac{1}{k!}\sum_{t=0}^{k}(-1)^{k-t}\binom{k}{t}t^n \tag{4}$$

The finite period of modular exponentiation determines eventual modular periodicity for $S(n,k)$ for fixed k.

1.5 The Carmichael function

Definition 7. $(\mathbb{Z}/K\mathbb{Z})^\times$ is the group of integers coprime to $K \in \mathbb{N}$ under multiplication modulo K.

Definition 8. $\lambda(K)$ is the *Carmichael function*, which gives the exponent of integers under multiplication, often in the context of the group $(\mathbb{Z}/K\mathbb{Z})^\times$.

The Carmichael function has the following recurrence [6] using $\varphi(n)$ *Euler's totient function.* For p prime and $r \geq 1$, $\varphi(p^r) = p^{r-1}(p-1)$.

$$\lambda(n) = \begin{cases} \varphi(n) & \text{if } n \in \{1,2,4\} \text{ or n is an odd prime power} \\ \frac{1}{2}\varphi(n) & \text{if } n = 2^r, r \geq 3 \\ \operatorname{lcm}(\lambda(n_1), \lambda(n_2), \ldots) & \text{if } n = n_1 n_2 \cdots \text{ where } n_1, n_2, \ldots \text{ are powers of unique primes} \end{cases} \tag{5}$$

The previously stated equation for $S(n,k)$ has dependence on n exclusively as a sum of exponentiations of integers (4) by n. Importantly under a modulus K, $\lambda(K)$ is the maximum eventual period of exponentiation of integers. Two properties of the Carmichael function stated further are leveraged.

Proposition 4. $\forall a \in \{0, 1, \ldots, K-1\}$

$$a^R = a^{\lambda(K)+R} \pmod{K} \tag{6}$$

Where R is the greatest exponent in the factorization of K into unique prime powers.

For coprime elements to K a stronger statement may be made.

Proposition 5. $\forall b \in (\mathbb{Z}/K\mathbb{Z})^\times$

$$b^{\lambda(K)} = 1 \pmod{K} \tag{7}$$

1.6 Summation, shifting, and scaling of eventually modular periodic sequences

Simple proofs are given for important operations that preserve eventual modular periodicity in the appendix A. Relevant operations are scaling by an integer, addition of eventually periodic sequences, and index shifting. Upper bounds for the eventual period are preserved for scaling and shifting. To include addition of sequences the least common multiple of the sequences' eventual periods must be considered.

1.7 Shift operators

Shift operators are used to formally show r-Fubini numbers may be expressed using the signed Stirling numbers of the first kind.

Definition 9. $T_\pm$ are the right and left shift operators respectively. Often multiple single shift operations are abbreviated T_m, $m \in \mathbb{Z}$.

Computation of the r-Fubini numbers uses the left and right shift operators T_+, T_- on the sequence $H(0), H(1), \ldots, H(n+r)$. Shift operators may be viewed through the lens of countably infinitely indexed vectors as sequences, which immediately yield matrix representation of shift operators. Zero is placed in the first index of the vector sequence when T_+ is applied. Shift operators applied on a sequence $F(n)$ are linear operators, defined such that $T_+F(n) = F(n+1)$ and $T_-F(n) = F(n-1)$. In vector notation, indexing by subscript:

$$(T_-\vec{F})_n = \vec{F}_{n+1} \tag{8}$$

$$(T_+\vec{F})_n = \begin{cases} 0 & \text{if } n = 0 \\ \vec{F}_{n-1} & \text{if } n \neq 0 \end{cases} \tag{9}$$

Vectors will always be indicated by the upper arrow. Sequences also distribute over addition of shift operators so $(AT_a + BT_b)F(n) = AF(n+a) + BF(n+b)$. Importantly $T_+T_- = T_-T_+ = I$ (I being the identity operation or zero shift), so all products of shift operators may be abbreviated T_a, $a \in \mathbb{Z}$. For abbreviated products $T_aF(n) = F(n+a)$ so long as $n + a \in \mathbb{N} \cup \{0\}$. In their use here the shift operators will not leave the sequence with an index outside of $\mathbb{N} \cup \{0\}$.

2 General rigged orderings of $r \leq n$ elements

The following lemma is needed in the proof of the novel expression for $H_r(n)$ the r-Fubini numbers. Let $F(n)$ be the number of orderings of some set which contains n elements. The orderings counted may observe the restriction of no ties (relative strong ordering) between some subset of those n elements.

Lemma 6. *Permutations formed by including an additional element in an ordered set $\{x_1, x_2, \ldots x_n\}$ such that the new element x' satisfies $x' \notin \{x_1, x_2, \ldots, x_m\}$ are counted by $F(n+1) - mF(n) = (T_+ - m)F(n)$.*

Proof. Consider adding the new element, with no restriction. The new number of orderings is $F(n+1)$, since no new element was distinguished, the number of elements is simply increased.

To restrict counting to permutations with x' such that $x' \notin \{x_1, x_2, \ldots, x_m\}$, note that for each ordering counted by $F(n)$, when x' is introduced, it may be set equal to one of $x_1, x_2, \ldots, x_m$ to form a unique disallowed permutation. The permutations are unique as each includes the new element x' and any two elements in the strongly ordered subset counted by $F(n)$ will not be equal which would cause degeneracy and under counting. Such degeneracy (if it was present) would cause under counting rather than over counting because the permutations are to be excluded. Using the multiplication rule to exclude cases where $x' \in \{x_1, x_2, \ldots, x_m\}$ the lemma follows. $\square$

The counting above may me repeated for an increasingly large subset that follows a strong ordering. The application of the counting above yields a new counting that is compatible with the lemma. Adding the element x' creates again a count of weakly ordered permutations such that a subset of elements follows a strong ordering, exactly what the lemma demands of $F(n)$.

2.1 Counting with shift operators

Theorem 7.

$$H_r(n) = \frac{1}{r!}\sum_{j=0}^{r} s(r, r-j)H(n-j) \tag{10}$$

Proof. The proof first counts the case where the subset $\{x_1, x_2, x_3, \ldots, x_r\}$ are relatively strongly ordered, then gives them a single ordering by dividing by $r!$.

To begin counting first remove or ignore the counting of the r distinguished elements, leaving weak permutations of $n-r$ elements which is $T_{-r}H(n)$. Elements of the distinguished subset are added back to the counted set with no ties within the set $\{x_1, x_2, x_3, \ldots, x_r\}$. The result is subtraction of ascending integer multiple of the identity operation from T_+ in accordance with Lemma 6.

- Reintroduce the element x_1, then do not count cases $x_1 \in \emptyset$. That is $T_+T_{-r}H(n)$. The subtraction is redundant (hence $\emptyset$), for the first step, since no elements in the strongly ordered subset exist in the remaining elements.
- Reintroduce the element x_2, do not count cases $x_2 \in \{x_1\}$. That is $(T_+ - 1I)T_+T_{-r}H(n)$.
- Reintroduce the element x_3, do not count cases $x_3 \in \{x_1, x_2\}$. That is $(T_+ - 2I)(T_+ - 1I)T_+T_{-r}H(n)$.
- $\cdots$
- Reintroduce the element x_r, do not count cases $x_r \in \{x_1, x_2, \ldots, x_{r-1}\}$. The total is $(T_+ - (r-1)I)\cdots(T_+ - 2I)(T_+ - 1I)T_+T_{-r}H(n)$.

Now all elements are included with their respective ordering, with those from $x_1, x_2, \ldots, x_r$ without mutual equalities, such that all counted orderings have $x_1, x_2, \ldots, x_r$ relatively strongly ordered. The falling factorial appears with argument T_+ and r terms.

$$T_{-r}(T_+)_{\underline{r}}H(n) \tag{11}$$

The count is now expressed as follows via the expansion previously introduced for falling factorials (2).

$$T_{-r}[\sum_{j=0}^{r} s(r,j)T_j]H(n) \tag{12}$$

The formula applies because repetition of T_+ may be treated as would multiplication of a polynomial variable. The effect of the shift operators is now trivial upon $H(n)$.

$$\sum_{j=0}^{r} s(r,j)H(n-r+j) \tag{13}$$

By re-indexing the sum.

$$\sum_{j=0}^{r} s(r,r-j)H(n-j) \tag{14}$$

The number of arrangements of $x_1, x_2, \ldots, x_n$ where $x_1, x_2, \ldots, x_r$ are relatively strongly ordered have been counted. Given the strongly ordered subset count it is straightforward to determine $H_r(n)$ by dividing by $r!$.

$$H_r(n) = \frac{1}{r!}\sum_{j=0}^{r} s(r,r-j)H(n-j)$$

□

An interesting notation is apparent.

$$H_r(n) = \frac{T_+(T_+-1)\cdots(T_+-r+1)}{r!}H(n-r) \tag{15}$$

$$H_r(n) = \binom{T_+}{r}H(n-r) \tag{16}$$

It appears that the fully removed operator $\binom{T_+}{r}T_{-r}$ could be used in repeated application to create counting for multiple disjoint strongly ordered subsets of different sizes. Additionally different subsets could be counted relative to T_+ via subtraction such as using a binomial coefficient or other sequence instead of successive integers.

2.2 A useful alternating recurrence

Corollary 8.

$$H(n) = n! - \sum_{j=1}^{n} s(n, n-j)H(n-j) \tag{17}$$

Proof. $H_n(n) = 1$, if the set is $\{x_1, x_2, \ldots, x_n\}$, and $x_1 < x_2 < \cdots < x_n$ the number of arrangements is 1. It follows from the Theorem 7:

$$1 = \frac{1}{n!}\sum_{j=0}^{n} s(n, n-j)H(n-j) \tag{18}$$

Which may be rearranged after the substitution $s(n, n) = 1$ to the result.

$$H(n) = n! - \sum_{j=1}^{n} s(n, n-j)H(n-j)$$

□

3 Linear transformation between strong and weak orderings

Corollary 9. *The infinite vectors $\vec{f}$ and $\vec{H}$ with entries $n!$ and $H(n)$ respectively obey the following relation with matrices $\hat{s}$ and $\hat{S}$ as defined in the introduction 1.4:*

$$\hat{S}\vec{f} = \vec{H} \tag{19}$$

$$\hat{s}\vec{H} = \vec{f} \tag{20}$$

Proof. The above equation (18) may be written as a matrix equation by multiplication by $n!$ and re-indexing.

$$\sum_{j=0}^{n} s(n, j)H(j) = n! \tag{21}$$

The infinite lower triangular matrix of $s(n, k)$ multiplied on a vector may be expressed as exactly the sum derived.

$$\hat{s}\vec{H} = \vec{f}$$

The first relation then immediately follows given the previously stated inverse of $\hat{s}$ being $\hat{S}$ (1).

$$\begin{aligned} \hat{S}\hat{s}\vec{H} &= \hat{S}\vec{f} \\ \vec{H} &= \hat{S}\vec{f} \end{aligned} \tag{22}$$

□

The signed Stirling numbers of the first kind act naturally on $\vec{H}_n$ as a matrix. The Stirling numbers of the second kind correspondingly on $\vec{f}_n$. It has been given that the infinite vectors, indexed with $\mathbb{N} \cup \{0\}$, $\vec{f}$ and $\vec{H}$ are related by lower triangular matrix multiplication. Recall that $\vec{f}$ and $\vec{H}$ contain the strongly and weakly (respectively) ordered permutation counts for successive (by cardinality) sets of distinguished elements. Since $\hat{S}$ and $\hat{s}$ are lower triangular, the relations may be truncated to hold for finite vectors containing the count of weak and strong permutations for $0...N$ elemnts, where N is finite.

4 Modular periodicity

Asgari and Jahangiri [2] show eventual modular periodicity for $H_r(n)$ and an explicit calculation for the eventual period. Below we determine eventual modular periodicity for $H(n)$, $H_r(n)$ via the perspective of exponentially generated sequences and transformations of such sequences that preserve their structure. In addition an upper bound for the period is proven to be the Carmichael function $\lambda(K)$ of the modulus. Two conditions for $\lambda(K)$ to be the exact modular period follow, with the case of odd K proven directly by the results of Asgari and Jahangiri's periodicity calculation [2].

4.1 Fubini numbers modulo K

Theorem 10. *The Fubini numbers $H(n)$ are eventually periodic modulo $K \in \mathbb{N}$, with maximum possible modular period $\lambda(K)$.*

Proof. Consider the existing relation (19).

$$\vec{H} = \hat{S}\vec{f}$$

The entries of $\vec{f}$, $\vec{f}_n$, are simply $n!$. Modulo K the vector entries $\vec{f}_n$ are certainly zero for $n \geq K$.

The relation stands in a simplified form:

$$H(n) = \sum_{k=0}^{K-1} S(n,k)k! \pmod{K} \tag{23}$$

$H(n) \pmod K$ is written as a finite sum of $S(n,k)$ with coefficients independent of n. Each $S(n,k) \pmod K$ for $0 \le k \le K-1$ contributes sums of exponential dependence in n for fixed k by the explicit form for the Stirling numbers of the second kind (23). The finite sum of such eventually modular periodic exponentially generated sequences scaled by constants is also eventually modular periodic as summarized in the introduction 1.6. Due to Carmichael $\forall j \in \mathbb{Z}$, $0 \le j \le K$:

$$j^{R+\lambda(K)} = j^R \pmod K \tag{24}$$

The onset of eventual modular periodicity must occur after $H(R)$, $R = \max(R_1, R_2, \ldots)$ increments of the argument, where $K = p_1^{R_1} p_2^{R_2} \ldots$, with p_i unique and prime. All the exponentials each with fixed coefficients will have entered periodicity after R increments. The longest possible eventual period given $H(n) \pmod K$ is a weighted sum of integer powers j^n is clearly $\lambda(K)$.

□

4.2 A Condition for exact Carmichael periodicity

Let C_i be the unique cyclic groups in the decomposition by direct product of the group $(\mathbb{Z}/K\mathbb{Z})^\times$ into cycles. An element of $(\mathbb{Z}/K\mathbb{Z})^\times$ may be written as a tuple $(b_1, b_2, b_3, \ldots)$ where b_i are residues in each cycle. Let a_i denote a generator which exists for each C_i such that $\langle a_i \rangle = C_i$.

Theorem 11. *The Carmichael function $\lambda(K)$ is exactly the period of $H(n) \pmod K$ if $\forall C_i$, a_i is part of the cyclic decomposition of a base exponentiated to the n'th power in the sum for $H(n) \pmod K$.*

Proof. The definition of the exponent as the least common multiple of finite orders of group elements is extended to the sum of exponentiations, since the modular period of a sum of sequences is their least common multiple. So if all a_i are exponentiated in the sum, the period is $\mathrm{lcm}(|C_1|, |C_2|, \ldots) = \lambda(K)$ per Equation 5 (Note the order and exponent of a cyclic group are equal). □

4.3 Extension to r-Fubini numbers

Corollary 12. *$H_r(n)$ the r-Fubini numbers are eventually periodic in n modulo K for fixed r with maximum period $\lambda(K)$ and periodicity onset after at latest $r - 1 + R$ for $r > 0$, where R is the maximum exponent in the unique prime power decomposition of K.*

Proof. $H(n)$ is eventually modular periodic in accordance with Theorem 10. Next note the operation $\binom{T_+}{r} T_{-r}$ on $H(n)$ to generate $H_r(n) = \binom{T_+}{r} T_{-r} H(n)$ per Theorem 7 preserves the structure of $H(n)$ as a weighted sum of exponentials of n, which again have maximum period modulo K of $\lambda(K)$. The periodic onset may be delayed by shift operations, hence the addition of $r - 1$ to the onset index. □

4.4 Exact Carmichael periodicity for odd K

Theorem 13. *For $H_r(n) \pmod K$ if K is odd then the eventual period of the sequence is $\lambda(K)$.*

Proof. This can be seen immediately in the last theorem in Asgari and Jahangiri's paper [2], noting that Euler's totient $\varphi(n)$ is equal to the Carmichael function for odd prime power arguments.

The eventual period for this case is $\mathrm{lcm}(\varphi(p_1^{k_1}), \varphi(p_2^{k_2}), \ldots)$ where $p_i^{k_i}$ are the unique prime powers composing K. When the argument of $\lambda(n)$ is an odd prime power, it is equal to $\varphi(n)$. This allows the following expression for the eventual period T:

$$T = \mathrm{lcm}(\lambda(p_1^{k_1}), \lambda(p_2^{k_2}), \ldots) \tag{25}$$

$$T = \lambda(K) \tag{26}$$

Where the recurrence for $\lambda(n)$ (5) is used for the final result.

□

5 Remarks

5.1 Infinite matrices

Remark 14. The matrix entries M_s form Sierpiński triangles sometimes with defects when plotted modulo some natural number. The diagonals of the infinite matrix M_s appear to be eventually periodic modulo an arbitrary natural number.

5.2 Shift operators

Remark 15. It is interesting to consider where else in combinatorics the shift operator may yield simplified understanding or proofs. Interpreting the binomial form used to find $H_r(n)$ (16) is expected to be useful to this end.

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A Properties of eventual modular periodicity

For each proof it is unstated that n is sufficiently large but finite.

Consider $g(n) \pmod K$, $K \in \mathbb{N}$, formed by scaling an eventually modular periodic sequence $f(n)$ with period r by factor $m \in \mathbb{Z}$.

Proposition 16. *The sequence $g(n) = mf(n)$ is eventually periodic modulo K.*

Proof.

$$g(n+r) = mf(n+r) \pmod K \tag{27}$$
$$g(n+r) = mf(n) \pmod K \tag{28}$$
$$g(n+r) = g(n) \pmod K \tag{29}$$

□

Consider $f(n)$ formed by summing two eventually period sequences modulo K: $h(n), g(n)$ with period x, y respectively, both in $\mathbb{N}$.

Proposition 17. *$f(n) = h(n) + g(n) \pmod K$ is eventually periodic.*

Proof.

$$f(n + \mathrm{lcm}(x,y)) = h(n + \mathrm{lcm}(x,y)) + g(n + \mathrm{lcm}(x,y)) \pmod K \tag{30}$$
$$f(n + \mathrm{lcm}(x,y)) = h(n) + g(n) \pmod K \tag{31}$$
$$f(n + \mathrm{lcm}(x,y)) = f(n) \pmod K \tag{32}$$

□

Proposition 18. *If $f(n) \pmod K$, is eventually periodic, then so is $f(n+w) \pmod K$ where $w \in \mathbb{Z}$.*

Proof. For clarity let $n + w = m$.

$$f(n+w) \ = f(m) \pmod K \tag{33}$$
$$f(m+r) = f(m) \pmod K \tag{34}$$
$$f(n+w+r) = f(n+w) \pmod K \tag{35}$$

□

The above property may be expressed in shift operator notation: If $f(n) \pmod K$ is eventually periodic, then so is $T_w f(n) \pmod K$ where $w \in \mathbb{Z}$.

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