

## Electrostatic Potential Energy

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### 0.1 Potential Energy of a Charge Density

We start by finding the total energy of any charge density as an integral over space.

Let  $\rho_f$  be a function of  $x, y, z$  with units  $\frac{C}{m^3}$  that fully determines the configuration of charge in the system. Consider assembling this charge density from infinity in a manner that allows a single real value  $a \in [0, 1]$  to define the progress of arranging the charges, so the intermediate charge density during assembly of the system will be  $\rho_i$  where

$$\rho_i = a\rho_f \quad (1)$$

In addition assembling charge to a varying voltage will require energy

$$U = \int_0^Q V dq \quad (2)$$

Coulomb's law in voltage charge density form is given by

$$V = \int_{Space} \frac{\rho}{4\pi\epsilon_0 r} d\tau \quad (3)$$

where  $r$  is the distance to the point which the voltage is defined and  $\rho$  is the charge density.

Starting with equations 2 and 3.

$$U = \int_0^Q \int_{Space} \frac{\rho_i}{4\pi\epsilon_0 r} d\tau dq \quad (4)$$

Then doing a change of variable on the outer integral

$$q = \int_{Space} \rho_i dm = a \int_{Space} \rho_f dm \quad (5)$$

$$dq = \left( \int_{Space} \rho_f dm \right) \cdot da. \quad (6)$$

Noting that  $\rho_f dm$  and equation 3 must be nested together because each  $\rho_f dm$  of charge is allocated at a different location and thus potential.

$$= \int_0^1 \int_{Space} \rho_f \int_{Space} \frac{\rho_i}{4\pi\epsilon_0 r} d\tau dm da \quad (7)$$

## 0.2 Potential Energy of an Electric Field

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Then substituting equation 1 inside the integral and un-nesting  $a$  since it is not a function of position

$$= \int_0^1 a \int_{Space} \rho_f \int_{Space} \frac{\rho_f}{4\pi\epsilon_0 r} d\tau dm da \quad (8)$$

Notice that the integral  $a da$  is just  $\frac{1}{2}$  and the inner most integral is just the final voltage of the complete configuration  $V_f$  (equation 3) which results in the potential energy of a charge configuration expressed in terms of density:

$$U = \frac{1}{2} \int_{Space} \rho_f V_f dm \quad (9)$$

## 0.2 Potential Energy of an Electric Field

Start with the potential energy of some electric charge density

$$U = \frac{1}{2} \int_{Space} \rho V d\tau \quad (10)$$

Substituting Maxwell's equation for the divergence of the E-field

$$= \frac{\epsilon_0}{2} \int_{Space} (\nabla \cdot E) V d\tau \quad (11)$$

Then using the definition of the potential  $V$

$$= \frac{\epsilon_0}{2} \int_{Space} (\nabla^2 V) V d\tau \quad (12)$$

Using the first auxiliary identity<sup>1</sup>

$$= -\frac{\epsilon_0}{2} \int_{Space} \frac{\nabla^2 (V^2)}{2} - \nabla V \cdot \nabla V d\tau \quad (13)$$

Rearranging and using Maxwell's equation for the divergence of the E-Field with the definition of norm

$$= \frac{\epsilon_0}{2} \int_{Space} |E|^2 d\tau - \frac{\epsilon_0}{2} \int_{Space} \frac{\nabla^2 (V^2)}{2} d\tau \quad (14)$$

Rewriting the Laplacian

$$= \frac{\epsilon_0}{2} \int_{Space} |E|^2 d\tau - \frac{\epsilon_0}{2} \int_{Space} \frac{\nabla \cdot \nabla (V^2)}{2} d\tau \quad (15)$$

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<sup>1</sup>See appendix

### 0.3 Laplacian Identity and Gradient Product Rule

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Using the product rule<sup>2</sup> on  $\nabla(V^2)$

$$= \frac{\epsilon_0}{2} \int_{Space} |E|^2 d\tau - \frac{\epsilon_0}{2} \int_{Space} \nabla \cdot (V \cdot \nabla V) d\tau \quad (16)$$

By The Divergence Theorem

$$= \frac{\epsilon_0}{2} \int_{Space} |E|^2 d\tau - \frac{\epsilon_0}{2} \int_{Surface} (V \cdot \nabla V) \cdot da \quad (17)$$

Inspecting the surface integral term in this equation, it is evident that it will approach 0 as we expand the enclosed volume in an otherwise neutral environment.  $V$  falls off like  $\frac{1}{r}$  and  $\nabla V$  falls off as  $\frac{1}{r^2}$  ( $\nabla V$  is proportional to the E-field). For the limit of an infinitely far surface the surface elements  $da$  will grow as  $r^2$  which means the integrand acts like  $\frac{1}{r}$ . The radius can be expanded to infinity without enclosing any net charge so the integral will be 0. Finally the energy of an E-field is:

$$U = \frac{\epsilon_0}{2} \int_{Space} |E|^2 d\tau \quad (18)$$

## Appendix

### 0.3 Laplacian Identity and Gradient Product Rule

Identity for the Laplacian used in 0.2

$$\nabla^2(f) \cdot f = \frac{\nabla^2(f^2)}{2} - \nabla f \cdot \nabla f \quad (19)$$

Expanding the right side of the equation

$$\begin{aligned} \nabla^2(f) \cdot f &= \frac{\nabla^2(f^2)}{2} - \left(\frac{df}{dx}\right)^2 - \left(\frac{df}{dy}\right)^2 - \left(\frac{df}{dz}\right)^2 \\ &= \frac{1}{2} \left( \frac{d^2(f^2)}{dx^2} + \frac{d^2(f^2)}{dy^2} + \frac{d^2(f^2)}{dz^2} \right) - \left(\frac{df}{dx}\right)^2 - \left(\frac{df}{dy}\right)^2 - \left(\frac{df}{dz}\right)^2 \\ &= \frac{1}{2} \left( \frac{d}{dx} \left( 2f \frac{df}{dx} \right) + \frac{d}{dy} \left( 2f \frac{df}{dy} \right) + \frac{d}{dz} \left( 2f \frac{df}{dz} \right) \right) - \left(\frac{df}{dx}\right)^2 - \left(\frac{df}{dy}\right)^2 - \left(\frac{df}{dz}\right)^2 \\ &= f \frac{df^2}{dx^2} + \left(\frac{df}{dx}\right)^2 + f \frac{df^2}{dy^2} + \left(\frac{df}{dy}\right)^2 + f \frac{df^2}{dz^2} + \left(\frac{df}{dz}\right)^2 - \left(\frac{df}{dx}\right)^2 - \left(\frac{df}{dy}\right)^2 - \left(\frac{df}{dz}\right)^2 \\ &= f \frac{df^2}{dx^2} + f \frac{df^2}{dy^2} + f \frac{df^2}{dz^2} \\ &= \nabla^2(f) \cdot f \end{aligned}$$

The Product Rule for The Gradient

$$\nabla(f \cdot g) = \nabla g \cdot f + \nabla f \cdot g \quad (20)$$

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<sup>2</sup>Also in the appendix

Expanding the left side to show equality

$$\begin{aligned}\nabla(f \cdot g) &= \frac{d}{dx}(f \cdot g)\hat{i} + \frac{d}{dy}(f \cdot g)\hat{j} + \frac{d}{dz}(f \cdot g)\hat{k} \\ &= (g\frac{df}{dx} + f\frac{dg}{dx})\hat{i} + (g\frac{df}{dy} + f\frac{dg}{dy})\hat{j} + (g\frac{df}{dz} + f\frac{dg}{dz})\hat{k} \\ &= \nabla g \cdot f + \nabla f \cdot g\end{aligned}$$