

Rigged Horse Numbers and their Modular Periodicity

Benjamin Schreyer

Departments of Computer Science and Physics

University of Maryland

College Park, Maryland 20742

USA

Plasma Physics Division

U.S. Naval Research Laboratory

Washington, D.C. 20375

USA

benontheplanet@gmail.com

Abstract

The permutations of horse racing, where ties are possible, are counted by the *Fubini numbers*, also called the *horse numbers*. The *r-horse numbers* are a counting of such horse race finishes where some subset of r horses agree to finish the race in a specific strong ordering. The *r-horse numbers* for fixed r are expressed as a sum of r index shifted sequences of Fubini numbers weighted with the *signed Stirling numbers of the first kind*. Then eventual modular periodicity of *r-Fubini numbers* is shown and their maximum period is determined to be the *Carmichael function* of the modulus. The maximum period is attained in the case of an odd modulus for Fubini numbers.

1 Introduction

1.1 Contributions

A new combinatorial counting and proof is given for the linear composition of Fubini numbers to yield *r-Fubini numbers*, which has analogues structure to a form for the *r-Bell numbers* studied by Nyul and Kereskényi-Balogh [1]. For fixed r , *r-Fubini numbers* are a constant coefficient sum of the Fubini numbers under index shifts. Proofs for the eventual modular

periodicity of Fubini and r -Fubini numbers are formulated, which give an upper bound for their modular eventual period. The upper bound is the Carmichael function $\lambda(K)$ where K is the modulus. When K is odd it is shown that $\lambda(K)$ is the exact period. The results apply to the combinatorial problem of r -horse numbers under a division by $r!$ to specify a permutation of the r elements.

1.2 Orderings weak and strong

Definition 1. *Fubini numbers* are denoted $F(n)$, which count weak orderings of n elements.

In the case of a horse race, the ordering is weak, equality determines ties, and $<$ and $>$ determine clear succession. Other common names for the counting of weak orderings are the horse numbers, or ordered Bell numbers. Without the possibility of a tie, such orderings are regular permutations (strong).

1.3 Rigged weak orderings

Definition 2. *Relative strong ordering* among elements of a set demands that for elements in the relative strong ordering, their relations may be described only by $<$ or $>$. An element in a relative strong ordering may be set equal to another element, but that element cannot come from the elements participating in the relative strong ordering.

Definition 3. The r -Fubini numbers $F_r(n)$ count weak orderings such that r elements of the finite set of cardinality n are distinguished, and constrained to have relative strong ordering between the r elements.

Definition 4. The r -horse numbers $H_r(n)$ count weak orderings such that r elements of the finite set of cardinality n are distinguished, and constrained to follow a specific relative strong ordering.

A simple relation exists between the r -Fubini numbers and r -horse numbers. Every permutation counted by $H_r(n)$ has $r!$ rearrangements of the rigging that yield all cases where the r elements feature no tied elements.

$$r!H_r(n) = F_r(n) \tag{1}$$

The r -horse numbers count a simple nontrivial restriction on weak permutations. A restriction where two or more horses agree to tie is simply counted by reducing the number of elements in the permutation, since the horses act as a single unit in any permutation. The nontrivial counting is further exemplified. Consider arbitrary elements under total weak ordering x_1, x_2, \dots, x_n . A strong ordering of a subset of elements inducing r -horse counting could be $x_1 < x_2 < \dots < x_r$. In this text r -horse counted permutations are indexed by denoting the number of total elements to be ordered n , and the size of the subset that follow a fixed strong ordering r .

For the r -Fubini numbers, the same type of indexing is used, defining n as the total number of permuted elements, and r as the number which are constrained to have no ties. Others choose to instead express the number of elements constrained to be relatively strongly ordered and the number of unconstrained elements. Related orderings and r -Fubini numbers have been studied by R      [2] who references an expression in terms of the r -Stirling numbers of the second kind and factorials for $F_r(n)$. Asgari and Jahangiri [3] proved the eventual periodicity of the r -Fubini numbers modulo an arbitrary natural number. Asgari and Jahangiri also gave calculations for the period.

1.4 Stirling numbers of the first and second kind

Definition 5. *Signed Stirling numbers of the first kind*, labeled $s(n, k)$, count partitions of n elements into k cycles (the sign gives the parity of permutation). Two index sequences may naturally be arranged into a matrix with indexing of rows n and columns k , for the case of $s(n, k)$. Let \hat{s} denote the infinite matrix of $s(n, k)$.

Definition 6. *Stirling numbers of the second kind*, labeled $S(n, k)$, count ways to partition a set into unordered groups. The matrix of $S(n, k)$, labeled \hat{S} , is indexed as is \hat{s} above.

The Stirling numbers are here introduced with the addendum of three useful properties, which are expressed with notation I as the identity operation. The first proposition is stated in *Advanced Combinatorics* [4, Eqn. 6f, p. 144].

Proposition 1. *The matrices \hat{s} and \hat{S} are inverses of each other. This applies even to the infinite matrices, where $n, k \geq 0$:*

$$\hat{s}\hat{S} = \hat{S}\hat{s} = I \quad (2)$$

Both \hat{s} and \hat{S} are additionally lower triangular.

The second important property is listed in *Concrete Mathematics* [5, Eqn. 6.13, p. 263]. The notation $(x)_n$ is the falling factorial of x with n multiplicative terms $(x)_n = x(x-1)\cdots(x-n+1)$.

Proposition 2. *The Stirling numbers of the first kind give the coefficient for fixed powers of the argument of the falling factorial.*

$$(x)_n = \sum_{k=0}^n s(n, k)x^k \quad (3)$$

In order that the last property may be introduced a further definition is needed.

Definition 7. *Eventual modular periodicity* for a sequence $f(n)$ is defined as follows. A sequence $f(n)$ is *eventually modular periodic* if there exists a $q \in \mathbb{N} \cup \{0\}$ such that, for large enough n :

$$f(n) \equiv f(n+q) \pmod{K}, K \in \mathbb{N} \quad (4)$$

The third important property, specific to $S(n, k)$ with fixed k , is their eventual modular periodicity. This may be shown using the following formula from the paper *Stirling matrix via Pascal matrix* [6, p. 55] which is re-expressed under multiplication by unity, specifically $\frac{k}{k}$.

Lemma 3.

$$S(n, k) = \frac{1}{k!} \sum_{t=0}^k (-1)^{k-t} \binom{k}{t} t^n \quad (5)$$

The finite period of modular exponentiation determines eventual modular periodicity for $S(n, k)$ for fixed k .

1.5 The Carmichael function

Definition 8. $(\mathbb{Z}/K\mathbb{Z})^\times$ is the group of integers coprime to $K \in \mathbb{N}$ under multiplication modulo K .

Definition 9. $\lambda(K)$ is the *Carmichael function*, which gives the exponent (least common multiple of all periods of exponentiations) of integers under multiplication modulo a natural number, often in the context of the group $(\mathbb{Z}/K\mathbb{Z})^\times$.

The Carmichael function has the following recurrence [7] using $\varphi(n)$ *Euler's totient function*.

$$\lambda(n) = \begin{cases} \varphi(n) & \text{if } n \in \{1, 2, 4\} \text{ or } n \text{ is an odd prime power;} \\ \frac{1}{2}\varphi(n) & \text{if } n = 2^r, r \geq 3; \\ \text{lcm}(\lambda(n_1), \lambda(n_2), \dots) & \text{if } n = n_1 n_2 \cdots \text{ where } n_1, n_2, \dots \text{ are powers of unique primes.} \end{cases} \quad (6)$$

The previously stated equation for $S(n, k)$ has dependence on n exclusively as a sum of exponentiations of integers (5) by n . Importantly under a modulus K , the function $\lambda(K)$ is the maximum eventual period of exponentiation of integers. Two properties of the Carmichael function stated further are leveraged.

Proposition 4. $\forall a \in \{0, 1, \dots, K-1\}$

$$a^R \equiv a^{\lambda(K)+R} \pmod{K} \quad (7)$$

Where R is the greatest exponent in the factorization of K into unique prime powers.

For coprime elements to K a stronger statement may be made.

Proposition 5. $\forall b \in (\mathbb{Z}/K\mathbb{Z})^\times$

$$b^{\lambda(K)} \equiv 1 \pmod{K} \quad (8)$$

1.6 Operations preserving eventual periodicity

The common operations scaling by an integer, addition of eventually periodic sequences, and index shifting preserve eventual periodicity. Upper bounds for the eventual period are preserved for scaling and shifting. To include addition of sequences, the least common multiple of the sequences' eventual periods must be considered.

1.7 Shift operators

Shift operators are used to formally show r -Fubini numbers may be expressed using the signed Stirling numbers of the first kind.

Definition 10. Operators $E^{\pm 1}$ are the right and left shift operators respectively. Often multiple single shift operations are abbreviated E^m , $m \in \mathbb{Z}$.

Computation of the r -Fubini numbers uses the left and right shift operators E, E^{-1} on the sequence $C(0), C(1), \dots, C(n+r)$. Shift operators may be viewed through the lens of countably infinitely indexed vectors as sequences, which immediately yield matrix representations of shift operators. Zero is placed in the first index of the vector sequence when E^{-1} is applied. Shift operators applied on a sequence $C(n)$ are linear operators, defined such that $EC(n) = C(n+1)$ and $E^{-1}C(n) = C(n-1)$. In vector notation, indexing by subscript:

$$(E\vec{C})_n = \vec{C}_{n+1} \tag{9}$$

$$(E^{-1}\vec{C})_n = \begin{cases} 0 & \text{if } n = 0; \\ \vec{C}_{n-1} & \text{if } n \neq 0 \end{cases} \tag{10}$$

Vectors will always be indicated by the upper arrow. Sequences also distribute over addition of shift operators so $(AE^a + BE^b)C(n) = AC(n+a) + BC(n+b)$. Importantly $EE^{-1} = E^{-1}E = I$, operator I being the identity operation or zero shift. Under combinatorially reasonable indexing of sequences ($F_r(r-c)$, $c > 0$ is clearly not meaningful), the shift operators will not leave a sequence indexed outside of $\mathbb{N} \cup \{0\}$.

2 General rigged orderings of $r \leq n$ elements

The following lemma is needed in the proof of the novel expression for the r -Fubini numbers $F_r(n)$, which immediately yields an expression for $H_r(n)$. Let $G(n)$ be the number of orderings of some set which contains n elements. The orderings counted may observe the restriction of no ties (relative strong ordering) between some subset of m of those n elements. For subsets cardinality m of a finite set of n elements, $n \geq m$.

Lemma 6. *Permutations formed by including an additional element in an ordered set $\{x_1, x_2, \dots, x_n\}$ such that the new element x' satisfies $x' \notin \{x_1, x_2, \dots, x_m\}$ are counted by $G(n+1) - mG(n) = (E - mI)G(n)$.*

Proof. Consider adding the new element, with no restriction. The new number of orderings is $G(n+1)$, since the new element has no special constraint upon it, the number of elements is simply increased.

To restrict counting to permutations with x' such that $x' \notin \{x_1, x_2, \dots, x_m\}$, note that for each ordering counted by $G(n)$, when x' is introduced, it may be set equal to one of x_1, x_2, \dots, x_m to form a unique disallowed permutation. The multiplication principle determines that there are $mG(n)$ new disallowed permutations. The permutations are unique as each includes the new element x' , and any two elements in the strongly ordered subset constraining the counting of $G(n)$ will not be equal. Equality within the subset would cause degeneracy and under counting. Such degeneracy (if it was present) would cause under counting rather than over counting because the permutations are to be excluded. Using the multiplication principle to exclude cases where $x' \in \{x_1, x_2, \dots, x_m\}$, the lemma follows. \square

The counting above may be repeated for an increasingly large subset that follows a strong ordering. The application of the counting above yields a new counting that is compatible with the lemma. Adding the element x' creates again a count of weakly ordered permutations such that a subset of elements follows a strong ordering, exactly what the lemma demands of $G(n)$.

2.1 Counting with shift operators

Theorem 7.

$$H_r(n) = \frac{1}{r!} \sum_{j=0}^r s(r, r-j) F(n-j) \quad (11)$$

Proof. The proof first counts the case where the subset $\{x_1, x_2, x_3, \dots, x_r\}$ is strongly ordered, then gives them a single ordering by dividing by $r!$.

To begin counting, first remove or ignore the counting of the r relatively strongly ordered elements, leaving weak permutations of $n-r$ elements which is $E^{-r}F(n)$. Elements of the distinguished subset are added back to the counted set with no ties within the set $\{x_1, x_2, x_3, \dots, x_r\}$. The result is subtraction of ascending integer multiples of the identity operation from E in accordance with Lemma 6. The first subtraction is redundant (hence \emptyset), for the first step, since no elements in the strongly ordered subset exist in the remaining elements at first.

- Add x_1 , then do not count cases $x_1 \in \emptyset$, the count is $EE^{-r}F(n)$
- Add x_2 , do not count $x_2 \in \{x_1\}$, the count is $(E-1I)EE^{-r}F(n)$
- Add x_3 , do not count $x_3 \in \{x_1, x_2\}$, the count is $(E-2I)(E-1I)EE^{-r}F(n)$
- ...

- Add x_r , do not count $x_r \in \{x_1, x_2, \dots, x_{r-1}\}$, $(E-(r-1)I) \cdots (E-2I)(E-1I)EE^{-r}F(n)$

Now all elements are included with their respective ordering, with those from x_1, x_2, \dots, x_r without mutual equalities, such that all counted orderings have x_1, x_2, \dots, x_r relatively strongly ordered. The falling factorial appears with argument E and r terms.

$$F_r(n) = (E)_r E^{-r} F(n) \quad (12)$$

The count is now expressed as follows via the expansion previously introduced for falling factorials (3).

$$F_r(n) = \left[\sum_{j=0}^r s(r, j) E^j \right] E^{-r} F(n) \quad (13)$$

The formula applies because repetition of E may be treated as would multiplication of a polynomial variable. The effect of the shift operators is now trivial upon $F(n)$.

$$F_r(n) = \sum_{j=0}^r s(r, j) F(n - r + j) \quad (14)$$

Then re-index the sequences.

$$F_r(n) = \sum_{j=0}^r s(r, r - j) F(n - j) \quad (15)$$

The number of arrangements of x_1, x_2, \dots, x_n where x_1, x_2, \dots, x_r are relatively strongly ordered have been counted. Given $F_r(n)$, dividing by $r!$ allows a straightforward determination of $H_r(n)$.

$$H_r(n) = \frac{1}{r!} \sum_{j=0}^r s(r, r - j) F(n - j)$$

□

An interesting notation is apparent.

$$H_r(n) = \frac{E(E-1) \cdots (E-r+1)}{r!} F(n-r) \quad (16)$$

$$H_r(n) = \binom{E}{r} F(n-r) \quad (17)$$

2.2 A useful alternating recurrence

Corollary 8.

$$F(n) = n! - \sum_{j=1}^n s(n, n-j)F(n-j) \quad (18)$$

Proof. $H_n(n) = 1$, if the set is $\{x_1, x_2, \dots, x_n\}$, and $x_1 < x_2 < \dots < x_n$ the number of arrangements is 1. The next equation then follows from Theorem 7:

$$1 = \frac{1}{n!} \sum_{j=0}^n s(n, n-j)F(n-j) \quad (19)$$

This expression may be rearranged after the substitution $s(n, n) = 1$ to the result.

$$F(n) = n! - \sum_{j=1}^n s(n, n-j)F(n-j)$$

□

3 Linear transformation between strong and weak orderings

Corollary 9. *The infinite vectors \vec{f} and \vec{F} with entries $n!$ and $F(n)$ respectively obey the following relation with matrices \hat{s} and \hat{S} as defined in the introduction 1.4:*

$$\hat{S}\vec{f} = \vec{F} \quad (20)$$

$$\hat{s}\vec{F} = \vec{f} \quad (21)$$

Proof. The above equation (19) may be written as a matrix equation by multiplication by $n!$ and re-indexing the last indices of both sequences.

$$\sum_{j=0}^n s(n, j)F(j) = n! \quad (22)$$

The infinite lower triangular matrix of $s(n, k)$ multiplied on a vector may be expressed as exactly the sum derived.

$$\hat{s}\vec{F} = \vec{f}$$

The first relation then immediately follows, given the previously stated inverse of \hat{s} being \hat{S} (2).

$$\begin{aligned}\hat{S}\hat{s}\vec{F} &= \hat{S}\vec{f} \\ \vec{F} &= \hat{S}\vec{f}\end{aligned}\tag{23}$$

□

The signed Stirling numbers of the first kind act naturally on \vec{F} as a matrix. The Stirling numbers of the second kind act correspondingly on \vec{f} . It has been given that the infinite vectors \vec{f} and \vec{F} , indexed with $\mathbb{N} \cup \{0\}$, are related by lower triangular matrix multiplication. Recall that \vec{f} and \vec{F} contain the strongly and weakly (respectively) ordered permutation counts for successive (by cardinality) sets of distinguished elements. Since \hat{S} and \hat{s} are lower triangular, the relations may be truncated to hold for finite vectors containing the count of weak and strong permutations for $0, 1, \dots, N$ elements, where N is finite.

4 Modular periodicity

Asgari and Jahangiri [3] show eventual modular periodicity for $F_r(n)$ and an explicit calculation for the eventual period. Below we determine eventual modular periodicity for $F(n)$, $F_r(n)$ via the perspective of exponentially generated sequences and transformations of such sequences that preserve their structure. In addition, an upper bound for the period is proven to be the Carmichael function $\lambda(K)$ of the modulus. Two conditions for $\lambda(K)$ to be the exact modular period follow, with the case of odd K proven directly by the results of Asgari and Jahangiri's periodicity calculation [3].

4.1 Fubini numbers modulo K

Theorem 10. *The Fubini numbers $F(n)$ are eventually periodic modulo $K \in \mathbb{N}$, with maximum possible modular period $\lambda(K)$.*

Proof. Consider the first relation of Corollary 9 (20).

$$\vec{F} = \hat{S}\vec{f}$$

The entries of \vec{f} , \vec{f}_n , are simply $n!$ and modulo K the vector entries \vec{f}_n are certainly zero for $n \geq K$.

The relation stands in a simplified form:

$$F(n) \equiv \sum_{k=0}^{K-1} S(n, k)k! \pmod{K}\tag{24}$$

$F(n) \pmod{K}$ is written as a finite sum of $S(n, k)$ with coefficients independent of n . Each $S(n, k) \pmod{K}$ for $0 \leq k \leq K - 1$ contributes sums of exponential dependence in n for fixed k by the explicit form for the Stirling numbers of the second kind (24). The finite sum of such eventually modular periodic exponentially generated sequences scaled by constants is also eventually modular periodic, as summarized in the introduction 1.6. Due to Carmichael $\forall j \in \mathbb{Z}, 0 \leq j \leq K$:

$$j^{R+\lambda(K)} \equiv j^R \pmod{K} \quad (25)$$

The onset of eventual modular periodicity must occur after $F(R)$, $R = \max(R_1, R_2, \dots)$ increments of the argument, where $K = p_1^{R_1} p_2^{R_2} \dots$, with p_i unique and prime. All the exponentials each with fixed coefficients will have entered periodicity after R increments. The longest possible eventual period of $F(n) \pmod{K}$ is clearly $\lambda(K)$, since $F(n) \pmod{K}$ may be expressed as a weighted sum of integer powers j^n . □

4.2 Exact Carmichael periodicity for odd K

Theorem 11. *For $F(n) \pmod{K}$ if K is odd then the eventual period of the sequence is $\lambda(K)$.*

Proof. This can be seen immediately in the last theorem in Asgari and Jahangiri's paper [3], noting that Euler's totient $\varphi(n)$ is equal to the Carmichael function for odd prime power arguments.

According to Asgari and Jahangiri [3], the eventual period for this case is $\text{lcm}(\varphi(p_1^{k_1}), \varphi(p_2^{k_2}), \dots)$ where $p_i^{k_i}$ are the unique prime powers composing K . When the argument of $\lambda(n)$ is an odd prime power, it is equal to $\varphi(n)$. This allows the following expression for the eventual period q :

$$q = \text{lcm}(\lambda(p_1^{k_1}), \lambda(p_2^{k_2}), \dots) \quad (26)$$

$$q = \lambda(K) \quad (27)$$

Where the recurrence for $\lambda(n)$ (6) is used for the final result. □

4.3 Extension to r -Fubini numbers

Corollary 12. *$F_r(n)$ the r -Fubini numbers are eventually periodic in n modulo K for fixed r with maximum period $\lambda(K)$ and periodicity onset after at latest $r - 1 + R$ for $r > 0$. Here R is the maximum exponent in the unique prime power decomposition of K .*

Proof. Fubini numbers $F(n)$ are eventually modular periodic in accordance with Theorem 10. Next observe the operation $r! \binom{E}{r} E^{-r}$ on $F(n)$ to generate $F_r(n) = r! \binom{E}{r} E^{-r} F(n)$ per Theorem 7 preserves the structure of $F(n)$ as a weighted sum of exponentials of n . Such a sum again has maximum period modulo K of $\lambda(K)$. The periodic onset may be delayed by shift operations, hence the addition of $r - 1$ to the onset index. \square

5 Remarks

5.1 Infinite matrices

Remark 13. The matrix entries \hat{s} form Sierpiński triangles sometimes with defects when plotted modulo some natural number. The diagonals of the infinite matrix \hat{s} appear to be eventually periodic modulo an arbitrary natural number.

5.2 Operators

Remark 14. It appears that the fully removed operator $r! \binom{E}{r} E^{-r}$ could be used in repeated application to create counting for multiple disjoint strongly ordered subsets of different sizes among weakly ordered elements. Additionally, different subsets could be counted relative to E via subtraction such as using a binomial coefficient or other sequence instead of successive integers.

Remark 15. It is interesting to consider where else in combinatorics the shift operator may yield simplified understanding or proofs. Interpreting the binomial form used to find $H_r(n)$ (17) is expected to be useful to this end. It may be the case that a multiplication principle for multi-index sequences should include operator multiplication.

5.3 Analogy between ordered and unordered Bell numbers

Remark 16. The formula for r -Bell numbers (the unordered counterpart to r -Fubini numbers) given by Nyul and Kereskényi-Balogh [1], $B_r(n) = \sum_{j=0}^r s(r, r-j) B(n-j)$ and the formula for r -Fubini numbers given here (15) are remarkably similar. Note the indexing for r, n between this work and that of Nyul and Kereskényi-Balogh differ but are equivalent. Explaining this similarity should yield insight into other problems susceptible to operator counting.

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