Manuscript: Shift Invariant Methods for Discrete Oscillators

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Abstract

Discretely indexed linearly coupled oscillator systems are reformulated using the derivative form of the translation operator. The result is a shift invariant PDE that may be solved by determining a dispersion relation. For physics education such a solution yields two distinct levels of mathematical depth, ansatz or full PDE. Both levels allow better explanation of the dispersion relation of many coupled oscillators common in undergraduate curricula.

1 Many coupled oscillators

Consider a 1d system that is coupled linearly and homogeneously. Here the specific example of equal masses m with positions $x_n(t)$ connected by identical springs of constant k is taken. The governing equations of motion (one for each oscillator) follow.

$$m\frac{dx_n^2}{dt^2} = k(x_{n-1} - 2x_n + x_{n+1}) \tag{1}$$

1.1 Discrete coupling as a PDE

The index n is taken to be continuous, to allow the powerful tools of differential equations to be involved. Continuous in n, $x(n,t)$ will pass through the value of $x_n(t)$ for integral values of n. Additionally introduce the translation operator T acting on the variable n.

The equations of motion are now of the form:

$$m\frac{d^2x(n,t)}{dt^2} = k(T_1x(n,t) - 2x(n,t) + T_{-1}x(n,t)) \tag{2}$$

Now employ the derivative as the generator of translation to replace T.

$$m\frac{d^2x(n,t)}{dt^2} = k(e^{\frac{d}{dn}} - 2 + e^{\frac{-d}{dn}})x(n,t) \tag{3}$$

Apply separation of variables $x(n,t) = C(t)A(n)$ with λ the separation constant.

$$\frac{m}{k}\frac{\frac{d^2C(t)}{dt^2}}{C(t)} = \frac{[(e^{\frac{d}{dn}} - 2 + e^{\frac{-d}{dn}})A(n)]}{A(n)} = \lambda \tag{4}$$

This yields two ODEs. The first is simple.

$$\frac{d^2C(t)}{dt^2} = \frac{k}{m}\lambda C(t) \tag{5}$$

Note that this is the equation for the simple harmonic oscillator, but with the factor lambda, the solutions are as follows:

$$C(t) = Ae^{\sqrt{\lambda}\sqrt{\frac{k}{m}}t} \tag{6}$$

If λ is negative one recognizes this is an oscillating form. Define parameter $\sqrt{\frac{k}{m}} = \omega_0$ and use the notation for a harmonic oscillator which is defined by its angular frequency ω. Take $\lambda = -(\frac{\omega}{\omega_0})^2$, yielding:

$$e^{\sqrt{-(\frac{\omega}{\omega_0})^2}\omega_0 t} = e^{\pm i\omega t} \tag{7}$$

1.2 Spatial ODE

Now move to find $A(n)$ in a similar fashion.

$$(e^{\frac{d}{dn}} - 2 + e^{\frac{-d}{dn}})A(n) = \lambda A(n) \tag{8}$$

$$(2cosh(\frac{d}{dn}) - 2 - \lambda)A(n) = 0 \tag{9}$$

As for any other linear ODE, we replace derivatives with powers of r.

$$2cosh(r) - 2 - \lambda = 0 \tag{10}$$

Solutions of the separated ODEs are now known and need physical interpretation.

$$r = acosh(1 + \frac{\lambda}{2}) \tag{11}$$

$$x(n,t) = Ae^{\sqrt{\lambda}\sqrt{\frac{k}{m}}t} \cdot e^{\pm rn} \tag{12}$$

1.3 Assembling the base solution

Let $i\theta = r$ and re-express λ in terms of ω (note $cosh(i\theta) = cos(\theta)$). This leads finally to a basis x_ω of solutions determined by ω.

$$\theta = acos\left(1 - \frac{1}{2}(\frac{\omega}{\omega_0})^2\right) \tag{13}$$

$$x_\omega(n,t) = Ae^{\pm i\omega t} \cdot e^{\pm i\theta n} \tag{14}$$

This is the expected family of non-evanescent solutions, shown in *Waves (Draft), Ch. 2 Eqn. 56* by D. Morin.

2 Remarks

2.1 Further utility of the approach

One can expand the exponentiated derivatives of the translation operators to any order and truncate. The first interesting approximation simply yields the non-dispersive wave equation, since $cosh(\frac{d}{dn}) \approxeq 1+\frac{1}{2}\frac{d^2}{dn^2}$. In this approximation the n dependence is governed by $\frac{d^2}{dn^2}A(n) = \lambda A(n)$. Such approximations are physically viewable as being qualified by long wavelength disturbances on the coupled oscillators. Constant coefficient linear couplings featuring more than one left or right coupling can be solved with the translation operator approach. Such problems result in a closed form dispersion relation involving phasor exponentiations of the wave number.

2.2 For a physics course

In the context of a physics course, such a method allows instructors to prepare students for the mathematics of translation generation by the wave momentum operator, and Bloch's Theorem, seen later in the context of quantum mechanics. This method simultaneously solves the salient problem of many coupled oscillators in an introductory wave physics course, usually not discussed in detail due to complexity. Simplifying assumptions may be made, such as starting from phasor ansatz, rather than showing separation of variables. Phasor ansatz for n will yield the dispersion relation without involving exponentiation of derivatives.

3 Conflict of interest statement

The author has no conflicts of interest to report.