The Combinatorics of Rigged Horse Racing and a Complete Alternating Recurrence for The Ordered Bell Numbers

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ABSTRACT

The Horse Numbers, Fubini Numbers, or Ordered Bell numbers $H(n)$ count the total weak orderings $(<,=)$ on a set of elements. If the finite set is $\{d, e, a, b, c \dots\}$, then some constraint may be applied when counting weak orderings of this set, such as $a < d < e \dots$, or $(c = d) < a \dots$. Let $B_k(n)$ be the number of ways to order $x_1, x_2, \dots, x_n$ such that $x_1 < x_2 < \dots < x_k$. The case $B_2(n)$ is given a short expression in terms of the Horse Numbers $B_2(n) = \frac{H(n)-H(n-1)}{2}$. Then $B_k(n)$ is expressed as a linear sum of $H(k), k \leq N$ by counting that yields the form of The Stirling Numbers of the First Kind. $B_k(n) = \frac{1}{k!}\sum_{j=0}^{k} s(k, k-j)H(n-j)$ where $s(l, m)$ are The Signed Stirling numbers of the first kind. Thus M constrained Horse Numbers may be computed in $MlogM$ operations given already computed Horse Numbers, by applying a Fourier domain convolution. Considering the case of full ordered constraint $B_n(n)$, $H(n)$ is given a recurrent definition relative to the number of strong orderings $n!$, $H(n) = n! - \sum_{j=1}^{n} s(k, k-j)H(n-j)$. Additionally, cardinality for unions and intersections of conditions with $=, <$ on subsets of the elements $x_1, x_2, \dots, x_n$ are given.

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0.1 Theorem

$$B_2(n) = \frac{H(n) - H(n-1)}{2} \tag{1}$$

Proof The expression $[x_1 < x_2] \vee [x_1 > x_2] \vee [x_1 = x_2]$ is a tautology on over any set $x_1, x_2 \ldots x_n$ that is weakly ordered, so long as $n \geq 2$. The three conditions are mutually exclusive, so their union has cardinality that may be expressed as a sum of each of the three relation's cardinalities. Consider each case:

- Count for $x_1 < x_2$, this is $B_2(n)$.
- Count for $x_1 = x_2$, this is $H(n-1)$.
- Count for $x_1 > x_2$, this is $B_2(n)$ again by symmetry.

The sum of these counts is $H(n)$ because the expression as a whole is a tautology.

$$H(n) = 2B_2(n) + H(n-1) \tag{2}$$

Finally as was to be shown:

$$B_2(n) = \frac{H(n) - H(n-1)}{2} \tag{3}$$

0.2 Theorem

$$B_k(n) = \frac{1}{k!} \sum_{j=0}^{k} s(k, k-j) H(n-j) \tag{4}$$

Proof The proof is presented in a combinatoric setting, by first counting the number of weak orderings such that the set $x_1, x_2 \ldots x_k$ is a strongly ordered subset, denoted. Next translation operators are used to formally show the result can be expressed using $s(l, m)$, The Stirling Numbers of the First Kind. Finally the count is divided by $k!$, since $x_1 < x_2 < \cdots < x_k$ is only one of $k!$ strong orderings of the subset $x_1, x_2, \ldots, x_k$.

- Remove or ignore the counting of $k-1$ elements $x_2, x_3, \ldots, x_k$, this is $H(n-k+1)$
- Reintroduce the element x_2 by increasing the argument to $H(n-k+2)$, then subtract any case where $x_2 = x, x \in \{x_1\}$. The total is $H(n-k+2) - H(n-k+1)$
- Reintroduce the element x_3 by increasing the argument to $H(n-k+3) - H(n-k+2)$, then subtract any case where $x_3 = x, x \in \{x_1, x_2\}$. The total is $H(n-k+3) - H(n-k+2) - 2[H(n-k+2) - H(n-k+1)]$.
- ...
- Reintroduce the element x_k by increasing the argument $P(j+1)$, where $P(j)$ is the expression for the previous step's count, then subtract any case where $x_k = x, x \in \{x_1, x_2, \ldots, x_{k-1}\}$. The total is $P(j+1) - (k-1)[P(j)]$.

The number of weak orderings of $x_1, x_2, \ldots, x_n$ where the subset $x_1, x_2, \ldots, x_k$ is strongly ordered has been counted. Consider expressing the counting in terms of the translation operators $e^{\pm \frac{d}{dn}}$, where $e^{\pm \frac{d}{dn}} f(n) = f(n \pm 1)$.

- Remove or ignore the counting of $k-1$ elements $x_2, x_3, \ldots, x_k$, $(e^{-\frac{d}{dn}})^{k-1} H(n)$.
- Reintroduce the element x_2 by increasing the argument to $H(n-k+2)$, then subtract any case where $x_2 = x, x \in \{x_1\}$. That is $(e^{\frac{d}{dn}} - 1) e^{(1-k)\frac{d}{dn}} H(n)$

- Reintroduce the element x_3 by increasing the argument, then subtract any case where $x_1 = x, x \in \{x_1\}$. That is $(e^{\frac{d}{dn}} - 2)(e^{\frac{d}{dn}} - 1)e^{(1-k)\frac{d}{dn}} H(n)$
- ...
- Reintroduce the element x_k by increasing the argument, then subtract any case where $x_k = x, x \in \{x_1, x_2, \ldots, x_{k-1}\}$. The total is $(e^{\frac{d}{dn}} - k + 1) \ldots (e^{\frac{d}{dn}} - 2)(e^{\frac{d}{dn}} - 1)e^{(1-k)\frac{d}{dn}} H(n)$.

The falling factorial appears with argument $e^{\frac{d}{dn}}$ and k terms, this is made obvious by separating $e^{(1-k)\frac{d}{dn}}$.

$$[(e^{\frac{d}{dn}} - k + 1) \ldots (e^{\frac{d}{dn}} - 2)(e^{\frac{d}{dn}} - 1)e^{\frac{d}{dn}}]e^{(-k)\frac{d}{dn}} H(n) \tag{5}$$

So the count is $e^{-k\frac{d}{dn}}(e^{\frac{d}{dn}})_k H(n)$ where $(x)_n$ is the falling factorial of x with n terms. The Signed Stirling Numbers of the First Kind $s(l, m)$ give the coefficient on x^m for $(x)_l$, or $(x)_l = \sum_{m=0}^{l} s(l, m)x^m$. The count is now expressed as follows:

$$e^{-k\frac{d}{dn}}[\sum_{j=0}^{k} s(k, j)e^{j\frac{d}{dn}}]H(n) \tag{6}$$

The effect of the translation operators is now trivial upon $H(n)$.

$$\sum_{j=0}^{k} s(k, j)H(n - k + j) \tag{7}$$

By reindexing the sum.

$$\sum_{j=0}^{k} s(k, k - j)H(n - j) \tag{8}$$

The number of arrangements of $x_1, x_2, \ldots, x_n$ where $x_1, x_2, \ldots, x_k$ are strongly ordered have been counted. It is now straightforward to count the arrangements where $x_1 < x_2 < \cdots < x_k$ or B_k by dividing by $k!$.

$$B_k(n) = \frac{1}{k!} \sum_{j=0}^{k} s(k, k - j)H(n - j) \tag{9}$$

The number of weak orderings of $x_1, x_2, \ldots, x_n$ where the subset $x_1, x_2, \ldots, x_k$ is strongly ordered has been counted. Consider expressing the counting in terms of the left, right shift operators T_+, T_- on the one sided sequence $H(0), H(1), ..., H(n + k - 1)$, where $T_+T_- = T_-T_+ = I$, so any product of shift operators may be abbreviated T_a where a is an integer. The shift operator is linear and commutes with integers acting as scalars. As an analogy, in continuous arguments, the discrete shift operator resembles the exponential of the derivative $e^{\pm\frac{d}{dn}}$, where $e^{\pm\frac{d}{dn}} f(n) = f(n \pm 1)$.

- Remove or ignore the counting of $k - 1$ elements $x_2, x_3, \ldots, x_k$, $T_{1-k}H(n)$.
- Reintroduce the element x_2 by increasing the argument to $H(n - k + 2)$, then subtract any case where $x_2 = x, x \in \{x_1\}$. That is $(T_1 - 1)T_{1-k}H(n)$
- Reintroduce the element x_3 by increasing the argument, then subtract any case where $x_3 = x, x \in \{x_1, x_2\}$. That is $(T_1 - 2I)(T_1 - 1I)T_{1-k}H(n)$

- ...
- Reintroduce the element x_k by increasing the argument, then subtract any case where $x_k = x, x \in \{x_1, x_2, \ldots, x_{k-1}\}$. The total is $(T_1 - (k-1)I) \ldots (T_1 - 2I)(T_1 - 1I)T_{1-k}H(n)H(n)$.

The falling factorial appears with argument T_1 and k terms, this is made obvious by separating T_{1-k}.

$$(T_1 - (k+1)I) \ldots (T_1 - 2I)(T_1 - 1I)T_1 T_{-k} H(n) \tag{10}$$

So the count is $T_{-k}(T)_k H(n)$ where $(x)_n$ is the falling factorial of x with n terms. The Signed Stirling Numbers of the First Kind $s(l, m)$ give the coefficient on x^m for $(x)_l$, or $(x)_l = \sum_{m=0}^{l} s(l, m)x^m$. The count is now expressed as follows:

$$T_{-k}[\sum_{j=0}^{k} s(k, j)T_j]H(n) \tag{11}$$

The formula applies because T_a commute and are linear. The effect of the translation operators is now trivial upon $H(n)$.

$$\sum_{j=0}^{k} s(k, j)H(n - k + j) \tag{12}$$

By reindexing the sum.

$$\sum_{j=0}^{k} s(k, k - j)H(n - j) \tag{13}$$

The number of arrangements of $x_1, x_2, \ldots, x_n$ where $x_1, x_2, \ldots, x_k$ are strongly ordered have been counted. It is now straightforward to count the arrangements where $x_1 < x_2 < \cdots < x_k$ or B_k by dividing by $k!$.

$$B_k(n) = \frac{1}{k!} \sum_{j=0}^{k} s(k, k - j)H(n - j) \tag{14}$$

0.3 Theorem

$$H(n) = n! - \sum_{j=1}^{n} s(n, n - j)H(n - j) \tag{15}$$

Proof $B_n(n) = 1$, if the set is $x_1, x_2 \ldots x_n$, and $x_1 < x_2 < \cdots < x_n$ the number of arrangements is 1. It follows from the second theorem:

$$1 = \frac{1}{n!} \sum_{j=0}^{n} s(n, n - j)H(n - j) \tag{16}$$

Which may be rearranged to the result.

$$H(n) = n! - \sum_{j=1}^{n} s(n, n - j)H(n - j) \tag{17}$$

0.4 Theorem

The cardinality of weakly ordered arrangements of the elements $x_1, x_2, \dots, x_n$ where $[x_{a_1} < x_{a_1+1} < \dots < x_{a_1+A_1-1}] \wedge [x_{a_2} < x_{a_2+1} < \dots < x_{a_2+A_2-1}] \wedge \dots \wedge [x_{a_N} < x_{a_N+1} < \dots < x_{a_N+A_N-1}]$ where $x_{a_j}, x_{a_j+1}, \dots, x_{a_j+A_j-1}$ is disjoint from $x_{a_i}, x_{a_i+1}, \dots, x_{a_i+A_i-1}$ for all i, j:

$$\frac{1}{A_1!A_2!\dots A_N!}[(e^{\frac{d}{dn}} - A_1 + 1)\dots(e^{\frac{d}{dn}} - 2)(e^{\frac{d}{dn}} - 1)e^{\frac{d}{dn}}]\dots[(e^{\frac{d}{dn}} - A_N + 1)\dots (e^{\frac{d}{dn}} - 2)(e^{\frac{d}{dn}} - 1)e^{\frac{d}{dn}}]e^{-(\sum_{j=1}^{N} A_j)\frac{d}{dn}}H(n) \tag{18}$$

Proof The procedure of the second theorem may be repeated for any amount of disjoint subsets which are strongly ordered, because counting only required knowing the number of elements from the subset already reintroduced, and the total count. The subsets being disjoint allows this to remain trivial.

Notes The cardinality for union of conditions immediately follows by the inclusion-exclusion principle.

The first theorem exemplified that dealing with conditions $x_a = x_b$ is a simple reduction in the effective number of elements being ordered. The caveat must be included that there are cases where a condition $x_a = x_b$ has no effect on the count. For example $[x_1 = x_2 \wedge x_2 = x_3 \wedge x_3 = x_1] \equiv [x_1 = x_2 \wedge x_2 = x_3]$, because of transitivity of equality.

It is interesting to consider where else in combinatorics the translation operator yields simplified understanding or proofs. Here the treatment revealed an over arching structure, allowed the existing counting properties of H(n) to be utilized, without considering the a recurrence or explicit definition for H(n). More concretely, B_k has some recurrence that could be shown to yield the second theorem, by defining B_k very , then making a connection to a recurence for H or $s(l, m)$. The possibility to make adjustments to counting meant a formula relative to H was much more immediate. In the field of physics there is momentum as the generator of translation. Additionally, the homogenous discrete linear coupled oscillators problem may be solved by conversion to a linear differential equation via translation operator description.