Horse Numbers and Rigged Variants

Benjamin G. Schreyer
UMD?
UMD ADDR
MORE
Un. St.
benontheplanet@gmail.com

AAAA
Department of and Lorum
Sit Amet Technology University of Management and
53300 Lorem Ipsum
Dolor
email@email.com

Abstract

The Horse Numbers, Fubini Numbers, or Ordered Bell numbers $H(n)$ count the total weak orderings $(<, >, =)$ on a set of elements. Let $B_k(n)$ n elements such that k are strongly ordered relative to each other with weightings given by *The Signed Stirling numbers of the first kind* which count expressing permutations as a set number of cycles, with sign determined by permutation parity. Considering the case of fully ordered constraint $B_n(n)$, a recurrence for The Horse Numbers is determined. Additionally, cardinality for unions and intersections of strong ordering or equivalence constraints on subsets is given.

1 Introduction

The cardinality of totally ordered sets is considered. No incomparable element is allowed or calculated for.

1.1 Total Weak Ordering

If the finite set is $\{d, e, a, b, c \dots\}$, then a weak ordering may be applied with symbols $<, >, =$, such as $a < d < e \dots$, or $(c = d) < a \dots$. The number of such orderings are known as The Horse numbers, Fubini numbers, or ordered Bell numbers $H(n)$. A horse race is a combinatorial setting where ties may occur, hence Horse numbers. Fubini's name is associated in relation to his theorem for integrals, where integrals over two variables may be taken as one (analogues to equality ordering), or performed separately in some strong order. Counting the ways to perform these integrals is an equivalent problem.

1.2 Total Strong Ordering

A strong ordering is a permutation of a set of elements. Permutation in the sense of only allowing the relations $<$ and $>$, counted by the factorial.

1.3 Constrained Weak Ordering

The case $B_k(n)$ is such that k elements of the finite set are chosen, and constrained to follow a specific strong ordering. If the elements under total weak ordering are $x_1, x_2, \dots, x_n$, such a strong ordering could be $x_1 < x_2 < \dots < x_k$. Another type of constraint may be imposed by equality, for example $x_1 = x_2 = \dots = x_k$. Both of these types will be counted, with equality being the easier case.

1.4 Example by Hand: Single Strong Ordering

The case $B_2(n)$ is given a short expression in terms of the Horse Numbers. This is when $x_1 < x_2$, or any other two elements are placed in a strong ordering, and provides inspiration for the generalized case.

Theorem 1.

$$B_2(n) = \frac{H(n) - H(n-1)}{2} \tag{1}$$

Proof. The expression $[x_1 < x_2] \vee [x_1 > x_2] \vee [x_1 = x_2]$ is a tautology on over any set $x_1, x_2 \dots x_n$ that is weakly ordered, so long as $n \geq 2$. The three conditions are mutually exclusive, so their union has cardinality that may be expressed as a sum of each of the three relation's cardinalities. Consider each case:

- Count for $x_1 < x_2$, this is $B_2(n)$.
- Count for $x_1 = x_2$, this is $H(n-1)$.
- Count for $x_1 > x_2$, this is $B_2(n)$ again by symmetry.

The sum of these counts is $H(n)$ because the expression as a whole is a tautology.

$$H(n) = 2B_2(n) + H(n-1) \tag{2}$$

Finally as was to be shown:

$$B_2(n) = \frac{H(n) - H(n-1)}{2} \tag{3}$$

□

1.5 Shift Operators

Shift operators are used to formally show the number of orderings where $x_1, x_2, \ldots, x_k$ have a strong but not specific ordering can be expressed using $s(l,m)$, The Stirling Numbers of the First Kind. Then the count is divided by $k!$, since $x_1 < x_2 < \cdots < x_k$ is only one of $k!$ strong orderings of the subset $x_1, x_2, \ldots, x_k$.

Consider expressing the counting in terms of the left, right shift operators T_+, T_- on the one sided sequence $H(0), H(1), \ldots, H(n+k-1)$, where $T_+T_- = T_-T_+ = I$ (I being the identity operation), so any product of shift operators may be abbreviated T_a where a is an integer. The shift operator is linear and commutes with integers acting as scalars.

2 General Rigged Orderings

2.1 Strong Ordering of $k \leq n$ Elements

Theorem 2.

$$B_k(n) = \frac{1}{k!} \sum_{j=0}^{k} s(k, k-j) H(n-j) \tag{4}$$

Proof. First remove or ignore the counting of k elements to be counted in strong ordering $x_1, x_2, x_3, \ldots, x_k$: $T_{-k}H(n)$.

- Reintroduce the element x_1 by increasing the argument to $H(n-k+1)$, then subtract any case where $x_1 \notin \emptyset$. That is $(T_1)T_{-k}H(n)$
- Reintroduce the element x_2 by increasing the argument, then subtract any case where $x_2 \notin \{x_1\}$. That is $(T_1 - 1)T_{1-k}H(n)$
- Reintroduce the element x_3 by increasing the argument, then subtract any case where $x_3 \notin \{x_1, x_2\}$. That is $(T_1 - 2I)(T_1 - 1I)T_{1-k}H(n)$
- ...

- Reintroduce the element x_k by increasing the argument, then subtract any case where $x_k \notin \{x_1, x_2, \ldots, x_{k-1}\}$. The total is $(T_1 - (k-1)I) \ldots (T_1 - 2I)(T_1 - 1I)T_{1-k}H(n)$.

Now all elements are counted, but those from $x_1, x_2, \ldots, x_k$ have had mutual equalities removed from the count, such that any counted ordering has $x_1, x_2, \ldots, x_k$ strongly ordered. The falling factorial appears with argument T_1 and k terms.

$$(T_1 - (k+1)I) \ldots (T_1 - 2I)(T_1 - 1I)T_1 T_{-k} H(n) \tag{5}$$

By commuting shift operators and integer scalars, the count is $T_{-k}(T_1)_k H(n)$ where $(x)_n$ is the falling factorial of x with n terms. The Signed Stirling Numbers of the First Kind $s(l,m)$ give the coefficient on x^m for $(x)_l$, or $(x)_l = \sum_{m=0}^{l} s(l,m)x^m$. The count is now expressed as follows:

$$T_{-k}[\sum_{j=0}^{k} s(k,j)T_j]H(n) \tag{6}$$

The formula applies because T_a commute and repetition of T_a may be treated as exponentiation of a polynomial variable. The effect of the shift operators is now trivial upon $H(n)$.

$$\sum_{j=0}^{k} s(k,j)H(n-k+j) \tag{7}$$

By reindexing the sum.

$$\sum_{j=0}^{k} s(k,k-j)H(n-j) \tag{8}$$

The number of arrangements of $x_1, x_2, \ldots, x_n$ where $x_1, x_2, \ldots, x_k$ are strongly ordered have been counted. It is now straightforward to count the arrangements where $x_1 < x_2 < \cdots < x_k$ or B_k by dividing by $k!$.

$$B_k(n) = \frac{1}{k!}\sum_{j=0}^{k} s(k,k-j)H(n-j) \tag{9}$$

□

Note M constrained Horse Numbers may be computed in $MlogM$ operations given already computed Horse Numbers, by applying a Fourier domain convolution to compute $B_k(n)$ which has an expression as a weighted sum of $H(m)$, $m \leq n$, with constant coefficients for fixed k.

2.2 Union and Intersection of Strong Constraints

Corollary 3. *The cardinality of weakly ordered arrangements of the elements* $x_1, x_2, \ldots, x_n$ *under condition* $[x_{a_1} < x_{a_1+1} < \cdots < x_{a_1+A_1-1}] \wedge [x_{a_2} < x_{a_2+1} < \cdots < x_{a_2+A_2-1}] \wedge \cdots \wedge [x_{a_N} < x_{a_N+1} < \cdots < x_{a_N+A_N-1}]$ *of* N *specifically strongly ordered subsets of sizes* $A_1, A_2 \cdots A_N$ *where* $\{x_{a_j}, x_{a_j+1}, \ldots, x_{a_j+A_j-1}\} \cap \{x_{a_i}, x_{a_i+1}, \ldots, x_{a_i+A_i-1}\} = \emptyset \ \forall i, j$*:*

$$\frac{1}{A_1!A_2!\ldots A_N!}[(T_1)_{A_1}(T_1)_{A_2}\ldots(T_1)_{A_N}]T_{-(\sum_{j=1}^{N} A_j)}H(n) \tag{10}$$

Proof. The procedure of the second theorem may be repeated for any amount of disjoint subsets which are strongly ordered, because counting only involves the number of elements from the respective subset already reintroduced, and the total count. The subsets being disjoint allows this to remain trivial. An additional provision of allowing $H(0), H(1) \ldots H(n+(\sum_{j=1}^{N} A_j))$ under shift operation is needed. Factorial division for the size of each strongly ordered subset again accomplishes specifying a strong ordering, rather than over counting all strong orderings of a subset. □

The cardinality for union of conditions immediately follows by the inclusion-exclusion principle.

2.3 Union and Intersection of Equality Constraints

Corollary 4. *The first theorem exemplified that dealing with conditions* $x_a = x_b$ *is a simple reduction in the effective number of elements being ordered, for* $x_a = x_b$*,* $a \neq b$*, this is* $H(n-1)$*. For an intersection of such equalities, the number of elements that are removed from counting is determined by counting the number of equivalence classes introduced by the intersection of equality constraints,* k*, the cardinality is then* $H(n+k-m)$*, where* m *is the number of elements included non-trivially in the intersection of equality constraints. The cardinality for union of equality constraints is easily expressed via the inclusion-exclusion principle.*

3 Horse Numbers

3.1 A Complete Alternating Recurrence

Corollary 5.

$$H(n) = n! - \sum_{j=1}^{n} s(n, n-j)H(n-j) \tag{11}$$

Proof. $B_n(n) = 1$, if the set is $x_1, x_2 \ldots x_n$, and $x_1 < x_2 < \cdots < x_n$ the number of arrangements is 1. It follows from the second theorem:

$$1 = \frac{1}{n!} \sum_{j=0}^{n} s(n, n-j) H(n-j) \tag{12}$$

Which may be rearranged to the result.

$$H(n) = n! - \sum_{j=1}^{n} s(n, n-j) H(n-j) \tag{13}$$

□

4 Remarks

It is interesting to consider where else in combinatorics the shift operator may yield simplified understanding or proofs. Here the treatment revealed an over arching structure allowed the existing counting properties of $H(n)$ to be utilized, without considering the a recurrence or explicit definition for $H(n)$. More concretely, $B_k(n)$ has some recurrence that could be shown to yield the second theorem, by defining $B_k(n)$ then making a connection to a recurence for $H(n)$ or $s(l, m)$. The possibility to make adjustments to counting allowed by the shift operator meant the second theorem is revealed without such low level considerations.

The shift operator also allows for certain recurrences such as the discrete coupled oscillators problem of physics as a linear differential equation after separation of variables is done. This occurs by letting the shift operator act on continuous indices n as $T_+ = e^{\frac{d}{dn}}$.

5 Acknowledgments

Fill in!

References

[1] P. S. Chee and G. L. Chia, Construction of magic squares by compositions, *Menemui Mat.* **5** (1983), 51–62.

[2] G. L. Chia, Self-complementary magic squares of doubly even orders, *Discrete Math.* **341** (2018), 1359–1362.

[3] G. L. Chia and Angeline P. L. Lee, Self-complementary magic squares, *Ars Combinat.* **114** (2014), 449–460.

[4] J. M. Kudrle and S. B. Menard, Magic square, in C. J. Colbourn and J. H. Dinitz, eds., *The CRC Handbook of Combinatorial Designs*, 2nd ed., CRC Press, 2007, pp. 524–528.

[5] W. W. Rouse Ball and H. S. M. Coxeter, *Mathematical Recreations and Essays*, 12th ed., University of Toronto Press, 1974.

2020 *Mathematics Subject Classification*: 06A05.
Keywords: Horse Numbers, Ordered Bell Numbers, Fubini Numbers, Stirling Numbers of the First Kind, Weak Ordering.

(Concerned with sequence A000670.)

Received —- – 20–; revised versions received —y 1- 20–; —- -8 20–; —- – 20–. Published in *Journal of Integer Sequences*, —–y – 20–.

Return to Journal of Integer Sequences home page.