Large Exponents in Shooting Method for Thin Flexures

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Abstract. In designing flexures for torsion balances, optical suspensions, or pendulums for scientific use fiber structures become thin and semi-analytic calculations of their bending become infeasible with float64. Semi-analytic calculations can be more efficent than finite element methods allowing faster design optimization. We provide simple analytical results which show that failure of float64 semi-analytic bending simulation is due to small angle exponential growth of the bending angle. The analytic solutions are used to provide timesaving guesses for applying the shooting method to bending with an arbitrary precision implementation of RK45 which resolves cases where float64 implementations fail.

Keywords: key words

1. Introduction

1.1. Shear-free beams

Consider a thin flexure suspending a weight. The weight experiences gravitational force $F_{w,0} = mg$ and a side force G_0. The fiber angle θ and moment M a distance s along the neutral axis depend on boundary conditions and material parameter E. The geometry of deformation can be determined by the relations

$$\frac{dM}{ds} = F_{w,0}\sin(\theta(s)) + G_0\cos(\theta(s)) \quad (1)$$

$$\frac{d\theta}{ds} = \frac{M(s)}{EI(s)} \quad (2)$$

on the moment and angle [1]. The geometry of the unbent material is encoded in $I(s)$ the second moment of area of the cross section.

The geometry can be solved for using traditional ordinary differential equation (ODE) solvers and one sided boundary conditions. Then to impose a boundary condition on the other end of the flexure, $s = L$, the shooting method can be used to acheive final bending angle θ_0. This approach is effective but fails in very thin cases where the sine term dominates.

For design of precision measurement suspensions in Kibble balances, torsion balances, and optical mounts when flexures become thin finite element analysis can be used. For increased computational speed semi-analytic numerics as described above is effective and has been implemented [1] for non-thin cases.

1.2. Floating point representation

A floating point number is roughly represented on a computer by two integers as $N = m \cdot 2^l$. Importantly in float64 computing the maximum magnitude for the exponent limits calculations to thos which have order of magnitude between 10^{-308} and 10^{308}. For the calculation of bending the limitations of float64's exponent prevent calculations that appear in practical cases. Format float128 extends this range enough for many applications we discuss. To do our calculations we use a library mpmath for python which implements floating point calculations with an exponent range that exceeds any physical application of the bending model.

2. Analytical solutions

2.1. Large exponents in bending

The problem that arises with exponent limitations in bending calculation is confusing because it involves scales like 10^{-308}. If something is so small it goes unmeasured. Numerically this scale is of practical use because a symmetry breaking numerical trick of this scale is needed when $G_0 = 0$. We must represent such small values in our solver which is not possible with a float64. A small initial condition on M is needed for semi-analytic bending. For example $M(0) \sim 10^{-440}$ with parameters from an optical suspension made of fused silica [2]. Without a small nonzero initial M the problem is symmetric and would not favor a leftward or rightward bend geometry. If $G_0 \neq 0$ then this numerical trick is not needed as the side force breaks the left-right symmetry of the solver.

When one tries to solve for the bending of a material in our application, the boundary conditions $M(0)$ and $\theta(0)$ are set. Then the shooting method is used on the final angle to reach a solution with $\theta(L) = \theta_0$.

According to the equation for the variation of θ (2) when the geometry gets extremely thin the rate of change of θ will become extremely large. The result is that to get reasonable resulting $M(L)$ and $\theta(L)$ in a shooting approach the initial conditions must be extremely small. Then the final value for θ does not grow beyond $\frac{\pi}{2}$ which is where oscillations in θ and M begin.

2.2. Small angle solutions

Sine term only To illustrate beyond a qualitative understanding that there is an exponential increase present in bending, consider the region where θ is small and $F_{w,0}\sin(\theta)$ dominates. The small angle holds true for part of all bending geometries with $\theta(0) = 0$. Additionally consider a constant width geometry $I(s) = C$. Then the equations for bending simplify to

$$\frac{d^2\theta}{ds^2} = \frac{F_{w,0}}{EC}\theta. \quad (3)$$

The solution is trivially exponential. The scaling of θ from the initial $\theta(0)$ to $\theta(l)$ is easily quantified.

$$\theta(l) = \exp\left(\sqrt{\frac{F_{w,0}}{EC}}l\right)\theta(0) \quad (4)$$

Now to estimate the total scaling over a varying geometry, for example Figure **??**, approximate the geometry as piecewise-constant. The end angle will be

$$\theta(L) = \theta(0)\exp\left(\sqrt{\frac{F_{w,0}}{E}}\int_0^L \frac{1}{\sqrt{I(s)}}ds\right). \quad (5)$$

Calculating this scale factor for parameters of an optical suspension made of fused silica [2] the scale factor is order 10^{440}.

Cosine term only The same assumptions are made as in the sine case, only differing by considering a dominant $G_0 \cos(\theta)$ term instead, and removing the piecewise-constant approximation. The approximate solution for small θ and $\theta(0) = 0$ is

$$\theta(L) = \int_0^L \frac{G_0 s}{EI(s)} ds. \tag{6}$$

General case We do not provide analysis for the general case as it is empirically determined that applying the cosine approximation, and then scaling by the scale factor of the sine approximation provides a sufficient initial guess at G_0 for shooting the bending ODE when both terms contribute.

3. Shooting with guessing algorithm

We assume the bending angle is less than $\frac{\pi}{2}$ to avoid an oscillating regime on θ and M which could make shooting for solutions more difficult.

3.1. Guessing

The shooting method alone is theoretically sufficient to find bending solutions, but it is much faster to provide an initial guess based off of approximate solutions. If such an initial guess is not provided naive shooting would search an interval that appears to be exponentially large in the integral of $\frac{1}{\sqrt{I(s)}}$.

Sine term only guess The scale factor

$$\exp\left(\sqrt{\frac{F_{w,0}}{E}} \int_0^L \frac{1}{\sqrt{I(s)}} ds\right)$$

is calculated for the bending geometry. Then because there is error due to the piecewise approximation, the arbitrary precision RK45 (APRK45) solver is used to determine an ancillary bending angle $\theta^\star$ for an initial

$$M^\star(0) = \exp\left(-\sqrt{\frac{F_{w,0}}{E}} \int_0^L \frac{1}{\sqrt{I(s)}} ds\right).$$

Then the guess for the shooting stage is determined as $M(0) = \frac{\theta_0}{\theta^\star} M^\star(0)$. This is simply leveraging the linear nature of the approximate ODE to rescale the purely analytic estimate to better acheive θ_0 under the small angle approximation.

Cosine term only guess When a side force is present it determines the final bending angle and no numerical artifact needs to be introduced to break symmetry to cause a bend. Instead of shooting the initial condition $M(0)$, we have $M(0) = 0$ and are shooting G_0. In this case we can use the cosine only approximate solution to figure that a good guess for G_0 to produce a bending through angle θ_0 is $G_0 = (\int_0^L \frac{s}{EI(s)})^{-1} \theta_0$.

Both terms significant In this case, G_0 breaks symmetry but the cosine only solution does not provide a good guess. The math is hard so we don't do it and instead take a guess which works well empirically. Instead find the scale factor for the sine only solution, and multiply it on the initial guess for G_0 produced for a cosine only estimate. That is $G_0 = (\int_0^L \frac{s}{EI(s)})^{-1} \exp(-\sqrt{\frac{F_{w,0}}{E}} \int_0^L \frac{1}{\sqrt{I(s)}} ds)\theta_0$. Then run an iteration of APRK45 on this guess, and refine it by approximate linearity as in the sine term only estimate.

3.2. Shooting

Then a shooting step is done starting with the guess for either G_0 or $M(0)$ with the APRK45 solver for side force significant and side force insignificant cases respectively. We have found that Anderson-Björck root finding method [3] was best applied to the shooting step, converging quickly for a wide swath of cases.

3.3. Speed and error

The fixed step-size arbitrary precision solver leverages the RK45 coefficients to provide calculable and sufficiently small error. The geometry $I(s)$ is sampled at a fixed interval and cubic spline interpolated to feed the solver. Errors for our parameters were much smaller than our tolerance with RK45. Time to make a single bending calculation for 100 sample points along the fiber was about 5 seconds on a personal computer with python3 implementation. Implementing the arbitrary precision solver in a compiled language would greatly boost this performance to order ten calculations a second. More easily the algorithm could be implemented in float128 with existing libraries, possibly still limiting the application to only some cases of thin flexures.

4. References are examples. We need to find the appropriate references and put it in RefClean.bib

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