Useful Identities

0.1 Converting Phase to Initial Position and Velocity

Simple harmonic oscillators follow sinusoidal paths, written as $x(t) = sin(\omega t + \phi)$. Using angle sum identities for trig functions we can also write this motion $x(t)$ in terms of an initial velocity parameter and an initial displacement parameter.

$$x(t) = Asin(\omega t + \phi) \tag{1}$$
$$= A(cos(\omega t)sin(\phi) + sin(\omega t)cos(\phi)) \tag{2}$$

We now name the parameters $\alpha = Asin(\phi), \beta = Acos(\phi)$, and rewrite our equation of motion. Why does β encode initial $(t = 0)$ velocity? Why does α encode initial displacement?

$$x(t) = \alpha cos(\omega t) + \beta sin(\omega t) \tag{3}$$

0.2 Euler's Formula

Sine and cosine are solutions to the SHM differential equation. Another equally good solution is the exponential of the imaginary unit times the time variable. We can show equivalence between these two states by expanding our coefficents to be complex numbers and by using Euler's Formula. Below I show rough reasoning for Euler's Formula using Taylor series.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ... \tag{4}$$

Now instead of plugging in z to the exponent we plug in $i\theta$.

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + ... \tag{5}$$

We now use the fact that exponentiation distributes over multiplication, and the identity $i^n =$
$1, \quad n\ is\ a\ multiple\ of\ 4;$
$i, \quad n + 1\ is\ a\ multiple\ of\ 4;$
$-1, n + 2\ is\ a\ multiple\ of\ 4;$
$-i, n + 3\ is\ a\ multiple\ of\ 4;$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + ... = (1 - \frac{\theta^2}{2!} + ...) + i(\theta - \frac{\theta^3}{3!} + ...) = cos(\theta) + isin(\theta) \tag{6}$$

Oberserve that grouping every other term, in other words all the real terms together and all the imaginary terms together, gives us the Taylor series for both sine and cosine