Rigged Horse Numbers and Modular Periodicity

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Abstract

The *r-Fubini numbers* for fixed r are expressed as a finite sum of *Fubini numbers* weighted with the *signed Stirling numbers of the first kind* by way of the translation operator. Then eventual modular periodicity of r-Fubini numbers is shown and their maximum period is determined to be the Carmichael function of the modulus.

1 Introduction

The cardinality of totally ordered sets is considered. No incomparable elements are accounted for.

1.1 Orderings weak and strong

If all elements of a finite set are given an ordering under the trichotomy $<, >, =$, this is a weak total ordering. The number of such orderings are named the *horse numbers*, *Fubini numbers*, or *ordered Bell numbers* denoted $H(n)$. A strong ordering is any weak ordering that contains no equality relation. In these pages all orderings are assumed to be total.

1.2 Stirling numbers of the first and second kind

The *Stirling numbers* are here introduced with the addendum of three useful properties. The *signed Stirling numbers of the first kind* count partitions with cyclic ordering (the sign gives the parity of permutation), and *Stirling numbers of the second kind* count ways to partition a set into unordered groups. Stated in *Advanced combinatorics* [1] if $s(n, k)$ the signed Stirling numbers of the first kind and $S(n, k)$ the Stirling numbers of the second kind are treated as matrices then they are inverses of each other. This applies even to the infinite matrices, where $n, k \geq 0$. Rows are associated to n and columns to k for finite and infinite cases. Denote such matrices M_s for the signed Stirling numbers of the first kind, and M_S for the Stirling numbers of the second kind. In the form of an equation:

$$M_s M_S = M_S M_s = I \tag{1}$$

The second important property is given by *Concrete Mathematics* [2]. The Stirling numbers of the first kind give the coefficient for fixed powers of the argument of the falling factorial.

The third important property, specific to $S(n, k)$, are that they are eventually periodic in n modulo $K \in \mathbb{N}$ for fixed k. Eventual modular periodicity for a sequence $f(n)$ means for sufficiently large n the following holds with fixed $r \in \mathbb{Z}$.

$$f(n) \ = f(n + r) \ (mod \ K) \tag{2}$$

This will be shown using the following formula lifted from the paper *Stirling matrix via Pascal matrix* [3] and re-expressed under multiplication by unity, $\frac{k}{k}$.

$$S(n, k) = \frac{1}{k!} \sum_{t=0}^{k} (-1)^{k-t} \binom{k}{t} t^n \tag{3}$$

1.3 r-Fubini numbers or rigged weak orderings

r-Fubini numbers $H_r(n)$ count weak orderings such that r elements of the finite set of cardinality n are distinguished, and constrained to follow a specific strong ordering. If the elements under total weak ordering are $x_1, x_2, \ldots, x_n$, such a strong ordering inducing r-Fubini counting could be $x_1 < x_2 < \cdots < x_r$. Here these permutations are indexed by denoting the number of total elements to be ordered n, and size of the distinguished subset that follow a fixed strong ordering r.

Others choose to denote the number of distinguished strongly ordered elements, and the number of undistinguished elements. These numbers have been studied by Rácz who derived an expression in terms of the r-Lah numbers and factorials [4] for them. Asgari and Jahangiri [5] proved the eventual periodicity of the r-Fubini numbers modulo any natural number, which will also be shown here more briefly. Asgari and Jahangiri also gave calculations for the period.

1.4 Summation, shifting, and scaling of eventually periodic sequences

Simple proofs are given for important operations that preserve eventual modular periodicity in the appendix A, these operations are scaling by an integer, addition of eventually periodic sequences, and index shifting.

1.5 Shift operators

Shift operators are used to formally show r-Fubini numbers may be expressed using $s(l, m)$, the signed Stirling numbers of the first kind.

Computation of the r-Fubini numbers uses the left and right shift operators T_+, T_- on the sequence $H(0), H(1), \ldots, H(n+r)$. Shift operators applied on a sequence $F(n)$ are linear operators, defined such that $T_+F(n) = F(n+1)$ and $T_-F(n) = F(n-1)$. Importantly $T_+T_- = T_-T_+ = I$ (I being the identity operation), so any product of shift operators may be abbreviated T_a, $a \in \mathbb{Z}$, and $T_aF(n) = F(n+a)$ so long as $n + a \in \mathbb{N} \cup \{0\}$. In their use here, the shift operators will not leave the sequence with an index outside of $\mathbb{N} \cup \{0\}$.

2 General rigged orderings

2.1 Strong ordering of $r \leq n$ Elements

Theorem 1.

$$H_r(n) = \frac{1}{r!} \sum_{j=0}^{r} s(r, r-j) H(n-j) \tag{4}$$

Proof. The proof first counts the case where the subset $x_1, x_2, x_3, \ldots, x_r$ are strongly ordered, then gives them a single ordering by dividing by $r!$. First remove or ignore the counting of r elements to be counted in strong ordering which is $T_{-r}H(n)$. Then proceed to reintroduce the elements subtracting out permutations where there is any mutual equality in the distinguished subset of size $r \in \mathbb{N} \cup \{0\}$.

- Reintroduce the element x_1 by increasing the argument to $H(n-r+1)$, then subtract any case where $x_1 \in \emptyset$. That is $T_+T_{-r}H(n)$.
- Reintroduce the element x_2 by increasing the argument, then subtract any case where $x_2 \in \{x_1\}$. That is $(T_+ - 1I)T_+T_{-r}H(n)$.
- Reintroduce the element x_3 by increasing the argument, then subtract any case where $x_3 \in \{x_1, x_2\}$. That is $(T_+ - 2I)(T_+ - 1I)T_+T_{-r}H(n)$.
- $\cdots$

- Reintroduce the element x_r by increasing the argument, then subtract any case where $x_k \in \{x_1, x_2, \ldots, x_{r-1}\}$. The total is $(T_+ - (r-1)I) \cdots (T_+ - 2I)(T_+ - 1I)T_+T_{-r}H(n)$.

Now all elements are included with their respective ordering, with those from $x_1, x_2, \ldots, x_r$ without mutual equalities, such that any counted ordering has $x_1, x_2, \ldots, x_r$ strongly ordered. The falling factorial appears with argument T_+ and r terms. The notation $(x)_n$ is the falling factorial of x with n terms.

$$T_{-r}(T_+)_r H(n) \tag{5}$$

Given in a table of identities in *Concrete Mathematics* [2], the signed Stirling numbers of the first kind $s(l, m)$ give the coefficient on x^m in the falling factorial $(x)_l$. The count is now expressed as follows.

$$T_{-r}[\sum_{j=0}^{r} s(r, j)T_j]H(n) \tag{6}$$

The formula applies because repetition of T_+ may be treated as would multiplication of a polynomial variable. The effect of the shift operators is now trivial upon $H(n)$.

$$\sum_{j=0}^{r} s(r, j)H(n - r + j) \tag{7}$$

By re-indexing the sum.

$$\sum_{j=0}^{r} s(r, r - j)H(n - j) \tag{8}$$

The number of arrangements of $x_1, x_2, \ldots, x_n$ where $x_1, x_2, \ldots, x_r$ are strongly ordered have been counted. Given the strongly ordered subset count it is straightforward to determine $H_r(n)$ by dividing by $r!$.

$$H_r(n) = \frac{1}{r!} \sum_{j=0}^{r} s(r, r - j)H(n - j)$$

□

An interesting notation is apparent.

$$H_r(n) = \frac{T_+(T_+ - 1) \cdots (T_+ - r + 1)}{r!} H(n - r) \tag{9}$$

$$H_r(n) = \binom{T_+}{r} H(n - r) \tag{10}$$

It appears that the fully removed operator $\binom{T_+}{r}T_{-r}$ could be used in repeated application to create counting for multiple disjoint strongly ordered subsets of different sizes. Additionally different subsets could be counted relative to T_+ via subtraction such as using a binomial coefficient or other count instead of the successive integers.

3 Horse numbers

3.1 A complete alternating recurrence

Corollary 2.

$$H(n) = n! - \sum_{j=1}^{n} s(n, n-j)H(n-j) \tag{11}$$

Proof. $H_n(n) = 1$, if the set is $x_1, x_2, \ldots, x_n$, and $x_1 < x_2 < \cdots < x_n$ the number of arrangements is 1. It follows from the first theorem (1) :

$$1 = \frac{1}{n!} \sum_{j=0}^{n} s(n, n-j)H(n-j) \tag{12}$$

Which may be rearranged after the substitution $s(n, n) = 1$ to the result.

$$H(n) = n! - \sum_{j=1}^{n} s(n, n-j)H(n-j)$$

□

4 Linear transformation between strong and weak orderings

Corollary 3. *The infinite vectors f_n and H_n with entries $n!$ and $H(n)$ respectively obey the following relation with matrices M_s and M_S as defined in the introduction 1.2:*

$$M_S f_n = H_n \tag{13}$$
$$M_s H_n = f_n \tag{14}$$

Proof. The above equation (12) may be written as a matrix equation by multiplication by $n!$ and reindexing.

$$\sum_{j=0}^{n} s(n, j)H(j) = n! \tag{15}$$

The infinite lower triangular matrix $s(n, k)$ multiplied on a vector may be expressed as exactly the sum derived.

$$M_s H_n = f_n$$

The first relation then immediately follows given the previously stated inverse of M_s being M_S (1).

$$M_S M_s H_n = M_S f_n \tag{16}$$
$$H_n = M_S f_n$$

□

In light of this theorem, the signed Stirling numbers of the first kind could be appropriately labeled as the weak Stirling numbers (those that act naturally on H_n). The Stirling numbers of the second kind are then the strong Stirling numbers.

5 Modular periodicity

Proposition 4. *The Horse numbers $H(n)$ are eventually periodic modulo $K \in \mathbb{N}$, with maximum possible period $\lambda(K)$, the Carmichael function of K.*

Proof. Consider the existing relation (13).

$$H_n = M_S f_n$$

The entries of f_n are simply $n!$. Modulo K the vector entries f_n are certainly zero for $n \geq K$.

The relation stands in a simplified form:

$$H(n) = \sum_{k=0}^{K-1} S(n,k)k! \ (mod\ K) \tag{17}$$

$H(n)$ $(mod\ K)$ is written as a finite sum of $S(n,k)$ with coefficients independent of n. Each $S(n,k)$ $(mod\ K)$ for $0 \leq k \leq K-1$ contributes sums of exponential dependence in n for fixed k by the explicit form for the Stirling numbers of the second kind (3). The finite sum of such eventually periodic exponentially generated sequences scaled by constants is also eventually periodic 1.4. Due to Carmichael all such exponentials of any $j \in \mathbb{Z}$, $0 \leq j \leq K$ follow:

$$j^{R+\lambda(n)} = j^R \ (mod\ K) \tag{18}$$

The onset of eventual periodicity must occur after $H(R)$, $R = \max(R_1, R_2, \ldots)$ increments of the argument, where $K = p_1^{R_1} p_2^{R_2} \ldots$, where p_i are prime. All the exponentials each with fixed coefficients will have entered periodicity after R increments again due to Carmichael. The longest possible period given $H(n)$ $(mod\ K)$ is a weighted sum of j^n is clearly $\lambda(K)$.

□

Corollary 5. *$H_r(n)$ the r-Fubini numbers are eventually periodic in n modulo K for fixed r with maximum period $\lambda(K)$ and periodicity after at most $r-1+R$ for $r>0$, where R is the maximum exponent in the prime power decomposition of K.*

Proof. $H(n)$ is eventually periodic in accordance with the above proposition 4. It follows that the operation $\binom{T_+}{r}T_{-r}$ on $H(n)$ to generate $H_r(n) = \binom{T_+}{r}T_{-r}H(n)$ given in theorem 1 preserve the structure of $H(n)$ as a weighted sum of exponentials of n, which again have maximum period $\lambda(K)$ modulo K. The periodic onset may be delayed by shift operations, hence the addition of $r-1$. $\square$

6 Remarks

6.1 Exact periodicity for $H_r(n)$

It appears that if one can prove the coefficient on t^n in $H(n)$ $(mod\ K)$ is nonzero and coprime to K for sufficiently many $t \in \mathbb{Z}/K\mathbb{Z}$ for all K an exact period follows. Given the coefficients of t^n are nonzero where t are generators for all cycles of the decomposition of $\mathbb{Z}/K\mathbb{Z}$ into prime power cyclic groups it would follow by the recurrence for $\lambda(K)$ in terms of distinct prime powers, and the period of sums of sequences 1.4 that $\lambda(K)$ is the period.

6.2 Infinite matrices

The matrix entries M_s form Sierpiński triangles sometimes with defects when plotted modulo some natural number.

The diagonals of the infinite matrix M_s appear to be eventually periodic modulo any natural number.

6.3 Shift operators

It is interesting to consider where else in combinatorics the shift operator may yield simplified understanding or proofs. Interpreting the binomial form (10) is expected to be useful to this end.

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A Properties of Eventual Modular Periodicity

For each proof it is unstated that n is sufficiently large but finite.

Proposition 6. *Consider $g(n)$ (mod K), $K \in \mathbb{N}$, formed by scaling an eventually modularly periodic sequence $f(n)$ with period r by factor $m \in \mathbb{Z}$. The sequence $g(n) = mf(n)$ will also be eventually periodic modulo K.*

Proof.

$$g(n+r) = mf(n+r)\ (mod\ K) \tag{19}$$
$$g(n+r) = mf(n)\ (mod\ K) \tag{20}$$
$$g(n+r) = g(n)\ (mod\ K) \tag{21}$$

□

Proposition 7. *Consider $f(n)$ formed by summing two eventually period sequences modulo K: $h(n), g(n)$ with period x, y respectively, both in $\mathbb{Z}$. $f(n) = h(n) + g(n)$ (mod K) will also be eventually periodic.*

Proof.

$$f(n + lcm(x, y)) = h(n + lcm(x, y)) + g(n + lcm(x, y))\ (mod\ K) \tag{22}$$
$$f(n + lcm(x, y)) = h(n) + g(n)\ (mod\ K) \tag{23}$$
$$f(n + lcm(x, y)) = f(n)\ (mod\ K) \tag{24}$$

□

Proposition 8. *If $f(n) \pmod{K}$, is eventually periodic, then so is $f(n+w) \pmod{K}$ where $w \in \mathbb{Z}$.*

Proof. For clarity let $n + w = m$.

$$f(n+w) \ = f(m) \ (mod\ K) \tag{25}$$

$$f(m+r) = f(m) \ (mod\ K) \tag{26}$$

$$f(n+w+r) = f(n+w) \ (mod\ K) \tag{27}$$

$\square$

As is given below, in shift operator notation: If $f(n) \pmod{K}$ is eventually periodic, then so is $T_w f(n) \pmod{K}$ where $w \in \mathbb{Z}$.

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