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Benjamin Schreyer

0.1 Finding the Solutions for N Linearly Coupled Oscillators

We start with the equations of motion for the N'th oscillator, assuming oscillating masses are of mass m, with coupling springs with linear constant k.

$$m\frac{dx_n^2}{dt^2} = k(x_{n-1} - 2x_n + x_{n+1}) \tag{1}$$

Assume solutions are of the form $x_n = A(n)e^{i\omega t}$. This is essentially equivalent to applying separation of variables. Then simplify the EOM by plugging in. Also let $w_0 = \sqrt{\frac{k}{m}}$.

We make x a continuous function of n as it will later let us expand it as a Taylor series. This is reasonable because for any set of N points that do not overlap in their domain coordinate, one can find a smooth curve that passes through them.

$$-m\omega^2 A(n)e^{iwt} = ke^{iwt}(A(n-1) - 2A(n) + A(n+1)) \tag{2}$$

$$-\frac{\omega^2}{\omega_0^2}A(n) = A(n-1) + A(n+1) - 2A(n) \tag{3}$$

$$-\frac{\omega^2}{\omega_0^2}A(n) = \Sigma_{k=0}^{\infty}\frac{d^kA}{dn^k}|_n(-1)^k\frac{1}{k!} + \Sigma_{k=0}^{\infty}\frac{d^kA}{dn^k}|_n\frac{1}{k!} - 2A(n) \tag{4}$$

$$-\frac{\omega^2}{\omega_0^2}A(n) = (e^{\frac{d}{dn}} + e^{-\frac{d}{dn}})A(n) - 2A(n) \tag{5}$$

$$-\frac{\omega^2}{\omega_0^2}A(n) = 2cosh(\frac{d}{dn})A(n) - 2A(n) \tag{6}$$

$$0 = 2cosh(\frac{d}{dn})A(n) - 2A(n) + \frac{\omega^2}{\omega_0^2}A(n) \tag{7}$$

Note this last line is a linear ODE that can be solved with a characteristic function/polynomial which is given as follows (Assuming solutions of the form $e^{\alpha n}$):

$$2cosh(\alpha) - 2 + \frac{\omega^2}{\omega_0^2} = 0 \tag{8}$$

Now make the convenient substitution $i\theta = \alpha$ since we are expecting to see oscilatting forms in the n coordinate (Note $cos(a) = cosh(ia)$).

$$2cos(\theta) - 2 + \frac{\omega^2}{\omega_0^2} = 0 \tag{9}$$

$$\theta = acos(1 - \frac{\omega^2}{2\omega_0^2}) \tag{10}$$

We now have a solutions of the form

$$\theta = acos(1 - \frac{\omega^2}{2\omega_0^2}) \tag{11}$$

$$x = Ae^{i\theta n}e^{i\omega t} \tag{12}$$