Generating Functions for Horses

Benjamin Schreyer

University of Maryland

Motivations

Motivation 1

Before the photo-finish, ties were common in races. With ties we call these arrangements weak orderings. If we have n horses in a race we define that there are $H(n)$ ways for the horses to be ranked. Further we want to more easily count outcomes of horse races where some horses have "rigged" the race.

Since there are now ties not just permuting $x_1, x_2, \ldots, x_n$:

$$n! \leq H(n) = \sum_{i=1}^{n} \binom{n}{i} H(n-i).$$

With $[ab]$ denoting a tie example orderings could be:

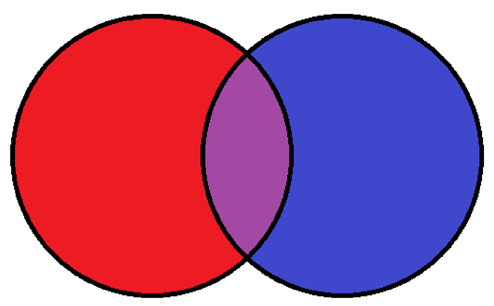
$$x_1x_2x_3 \text{ or } x_3[x_1x_2] \text{ or } [x_2x_1x_3].$$

Rigged: constrain x_2 beats x_1 or $x_1 < x_2$, denoted $H_2(n)$. Generalize to $x_1 < x_2 < \cdots < x_r$ counted by $H_r(n)$.

Motivation 2

We seek simple countings for other combinatorial problems.

Throughout we denote the set $\{1, 2, 3, \ldots, k\}$ as $[k]$. Inclusion-exclusion principle hints r simple constraints (C_i, $i \in [r]$) $\rightarrow$ sum over only r terms.



$$|A| + |B| - |A \cap B|$$

Methods

Inclusion-exclusion principle

Inclusion-exclusion acts as a manual way to find few-term counting formulae by hand which help find generating function generalizations.

$$|\bigcup_{i=1}^{r} C_i| = \sum_{i=1}^{r} |C_i| - \sum_{1 \leq i < j \leq r} |C_i \cap C_j| + \sum_{1 \leq i < j < k \leq r} |C_i \cap C_j \cap C_k| \cdots$$

Generating functions

We can represent a sequence as the polynomial coefficients of a function.

$$s_n : s_0, s_1, s_2, \ldots \sim s(x) = s_0 + s_1 x + s_2 x^2 \cdots$$

Manipulating a sequence by shifts is trivial to represent with generating functions.

$$s_{n \mp 1} \sim x^{\pm 1} s(x).$$

Rigged horse numbers

Counting $H_2(n)$

With small r $H_1(n), H_2(n)$, etc can be calculated by a cases approach:

$$\#x_2 < x_1 + \#x_1 < x_2 + \#x_1 \text{ tied } x_2 = H(n)$$

$$H_2(n) + H_2(n) + H(n-1) = H(n)$$

$$H_2(n) = \frac{1}{2}(H(n) - H(n-1))$$

This gives us insight into the generalization that follows.

Weak orderings with r named elements in ascending order

We consider first weak orderings of $[n]$ where $[r]$ are not part of the same group (are not tied). These are called the r-Fubini numbers $F_r(n)$ [1]. To solve the rigged horse numbers problem divide $r!$ overcounting.

$H_r(n) = \frac{1}{r!} F_r(n).$

$H_r(n)$ for $n = 5, r = 3$:

$$12354, 12[354], \cancel{[12]543}, 14253, \cancel{24513}, 1[24][35] \ldots$$

Counting factor by factor

Consider weak orderings of $[n]$ with generating function $H(x)$. To constrain $[r]$ to not be tied, remove the r elements and reintroduce them subtracting out cases where they are tied with each other. As we add elements there are more cases to tie elements of $[r]$ which need to be excluded.

$$F_2(x) = (\frac{1}{x} - 1)(\frac{1}{x})x^2 H(x), F_3(x) = (\frac{1}{x} - 2)(\frac{1}{x} - 1)(\frac{1}{x})x^3 H(x) \ldots$$

Generalize by multiplying out r **factors** [2]:

$$F_r(n) = \sum_{i=0}^{r} (-1)^i s(r, r-i) H(n-i).$$

$s(i, j)$ are the unsigned Stirling numbers of the first kind which are the magnitude of the coefficients of the falling factorial.

References

[1] Gabriella Rácz. "The r-Fubini-Lah numbers and polynomials". In: *Australas. J Comb.* 78 (2020), pp. 145–153.

[2] Benjamin Schreyer. *Rigged Horse Numbers and their Modular Periodicity.* 2024. eprint: `arXiv:2409.03799`.

[3] Kereskényiné Balogh Zsófia and Nyul Gábor. "Stirling numbers of the second kind and Bell numbers for graphs". In: vol. 58. 2014, pp. 264–274.

Other orderings

Permutations with r named elements "deranged"

A commonly studied problem is to count the number of permutations where no element is mapped to itself. We say that all elements are deranged. We can pose a generalization where we consider permutations of $[n]$ where $[r]$ are not mapped to themselves $P_r(n)$.

For $n = 3, r = 2$:

$$\cancel{123}, 213, 312, \cancel{132}, \cancel{321}, 231$$

Generalized we have:

$$P_r(n) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (n-i)!$$

Partitions with r named elements in separate blocks

If we are interested in partitions of $[n]$ where $[r]$ are never part of the same partition we have the r-Bell numbers $B_r(n)$. The formula we give has been derived by connecting partitions to graphs and then considering chromatic polynomials [3].

For $n = 5, r = 3$:

$$\cancel{\{\{1,2,3,4,5\}\}}, \{\{1, 4\}, \{3, 5\}, \{2\}\}, \cancel{\{\{1,3\},\{2,4,5\}\}} \ldots$$

Generalized we have:

$$B_r(n) = \sum_{i=0}^{r} (-1)^i s(r, r-i) B(n-i).$$

Conclusions

Counting problems with distinguished item constraints are not easy to think about. We have shown and applied a counting strategy with the following advantages.

- explainable steps
- polynomial arithmetic only

We provide a formula for counting the rigged horse numbers which has $r \leq n$ terms.

Future Research

- Can we apply strategy for multiple disjoint sets of named elements?
- Are there other interesting problems we can easily count this way?
- Are there examples where it is useful to manipulate a multivariate generating function?
- We would like to find inclusion-exclusion only arguments for the countings of $B_r(n)$ and $F_r(n)$ we exhibited ($25 each).