binatorics of Rigged Horse Racing via Symmetric Groups and a New Recurrence for The Ordered Bell I

The Horse Numbers, Fubini Numbers, or Ordered Bell numbers $H(n)$ count the weak orderings $(<, =)$ on a set of elements. If the finite set is $\{d, e, a, b, c...\}$, then some constraint may be applied when counting weak orderings of this set, such as $a < d$, or $c = d$. Let $B_i(n)$ be the number of ways to order $x_1, \ldots, x_n$ such that $x_1 < x_2 < \cdots < x_i$. The case $B_2(n)$ is given a short expression in terms of the Horse Numbers $B_2(n) = \frac{H(n)-H(n-1)}{2}$. Then $B_k(n)$ is expressed as a linear sum of $H(k), k \leq N$, by formalizing the permutation symmetry, and applying Vieta's formulas. $B_k(n) = \frac{1}{k!}\sum_{j=0}^{n} s(k, k-j)H(n-j)$ where $s(n, k)$ are the signed Stirling numbers of the first kind. Thus M constrained Horse Numbers may be computed in $MlogM$ operations given already computed Horse Numbers, by applying a Fourier domain convolution. Considering the case of full ordered constraint $B_n(n)$, $H(n)$ is given a recurrent definition relative to the number of strong orderings $n!$, $H(n) = n! + \sum_{j=0}^{n-1} s(n, j)H(j)$.

Theorem 0.1

$$B_2(n) = \frac{H(n) - H(n-1)}{2}.$$

Proof:

The expression $[x_1 < x_2] \vee [x_1 > x_2] \vee [x_1 = x_2]$ over $n \geq 2$ weakly ordered variables is a tautology, the cardinality of a set satisfying the condition must be $H(n)$. These three conditions are mutually exclusive, so their union has cardinality as a direct sum of each condition's cardinality. The cardinality of each restriction is:

1. Count orderings where $x_1 < x_2$, $B_2(n)$.
2. Count orderings where $x_1 = x_2$, $H(n-1)$.
3. Count orderings where $x_1 > x_2$, $B_2(n)$ by symmetry.

The sum is $H(n) = 2B_1(n) + H(n-1)$, as a result:

$$B_2(n) = \frac{H(n) - H(n-1)}{2}.$$

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Theorem 0.2

$$B_k(n) = \frac{1}{k!}\sum_{j=0}^{n} s(k,k-j)H(n-j).$$

Where $s(n,k)$ are the signed Stirling numbers of the first kind.

Proof: Consider the following condition on n weakly ordered variables $x_1, x_2, ..., x_n$: $(x_1, x_2, ..., x_k$ *are strongly ordered*$) \vee (\vee_{i=1}^{k} \vee_{j=i+1}^{k} [x_i = x_j])$. This condition is a tautology, as some subset of k variables, must either be strongly ordered, or have equality between at least two of the variables.

Note the second term in the condition $(\vee_{i=1}^{k}\vee_{j=i+1}^{k}[x_i = x_j])$ is exhaustive, as it is an enumeration of all the pairings of variables $x_1, x_2, ..., x_k$.

Since the condition is a tautology on n variables with weak orderings, its cardinality must be $H(n)$. The two terms are mutually exclusive, so the cardinality of their union is the sum of individual cardinalities:
$H(n) = k!B_k(n) + | \vee_{i=1}^{k} \vee_{j=i+1}^{k}[x_i = x_j]|$
$B_k(n)$ enters by symmetry, the number of orderings where $x_1, x_2, ..., x_k$ are strongly ordered is the $k!$ ways to rearrange the strong ordering $x_1 < x_2 < ... < x_k$.

Given some true assignment of $\vee_{i=1}^{k} \vee_{j=i+1}^{k} [x_i = x_j]$, the numbers of weak orderings given the equality constraints is $H(n-a)$ where a is the number of variables that were made redundant by equality with each other, as given equality of two or more variables, when ordered weakly, the two or more variables as ordered as if they are one.

Now consider the expression for the cardinality of a set union:

$$|\vee_{i=1}^{k}\vee_{j=i+1}^{k}[x_i = x_j]| = \sum_{1\leq i\leq \frac{k^2-k}{2}} S_i - \sum_{1\leq j<i\leq \frac{k^2-k}{2}} S_i \wedge S_j + \sum_{1\leq k<j<i\leq \frac{k^2-k}{2}} S_i \wedge S_j \wedge S_k ... +$$

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$$(-1)^{\frac{k^2-k}{2}-1} | \wedge_{i=1}^{\frac{k^2-k}{2}} S_i|$$

S_i are $x_a = x_b$ for any $a \neq b$.

This formula will yield the sign of respective terms, positive or negative, when counting the cardinality of $\vee_{i=1}^{k} \vee_{j=i+1}^{k} [x_i = x_j]$.

Now the combination of $x_a = x_b$ for any $a \neq b$ via $\wedge$ must be considered through the lens of symmetric groups. The symmetric group is over the set $x_1, x_2, ..., x_k$. The bijection are equalities between two group elements, and the group operation is the composition of equalities by $\wedge$, which is a composition of functions. Cycle notation represents the situation well, $(x_1, x_2, ..., x_{k-2})(x_{k-1}, x_{k-2})$ for example is selection of equalities such that $x_1, x_2, ..., x_{k-2}$ are all equal, and x_{k-1}, x_{k-2} are equal. The Stirling Numbers of the first kind count the number of such arrangements, given restriction to a certain number of cycles (equality groups). The number of cycles determines a in . In the language of symmetric groups $x_a = x_b$ is a two cycle, or transposition. These transpositions are composed via $\wedge$. An important result of the theory of symmetric groups is that any permutation (collection of cycles, assigment to) will not have unique decomposition into transpositions, any decomposition of a given permutation will always be even, or always be odd (J Gallian, Contemporary Abstract Algebra, pg 103 Th 5.5). This means that the counting of is simplified, as all terms are positive sum, with sign determined by . The number of terms for a given $H(n-k+a)$ will be $|s(k,a)|$ since the Stirling numbers of the first kind count the number of permutations of k with a cycles. Since the form $S_a \wedge S_b \wedge ... \wedge S_m$ of M terms will always for some choice of $a, b, ..., m$ yield $H(n - M)$, the sign of any permutation that yields M redundant variables is set to the sign of $S_a \wedge S_b \wedge ... \wedge S_m$, since a permutation will always be decomposed into only even or only odd numbers of transpositions, and even and odd numbers of transpositions are given a corresponding sign by . As a result, the number is known, and the sign is known. Since the sign alternates with the parity of the cycles, the signed Stirling numbers of the first kind are applicable, the final result is

$$B_k(n)k! - \sum_{j=0}^{n} s(k, k-j)H(n-j) = H(n).$$

This leads immediately to the desired formula. ∎

Curious connection between products and Vitaes Formulas for Stirling numbers and the form of the Union cardinality