# Horse Numbers and Rigged Variants

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**Abstract**

*The Horse numbers, Fubini numbers, or Ordered Bell numbers* count the total weak orderings $(<, >, =)$ on a set of $n$ elements. *Constrained Horse numbers or Bill numbers* count orderings of $n$ elements such that $k$ are in a specific strong ordering relative to each other. Bill numbers are expressed as a sum of Horse numbers with weightings given by *the signed Stirling numbers of the first kind*. Considering the case of fully ordered constraint, a recurrence for the Horse numbers is determined. Additionally, cardinality for unions and intersections of strong ordering or equivalence constraints on subsets is given.

## 1 Introduction

The cardinality of totally ordered sets is considered. No incomparable element is allowed or calculated for.

## 1.1 Total Weak Ordering

If the finite set is $\{d, e, a, b, c \dots\}$, then a weak ordering may be applied with symbols $<, >, =$, such as $a < d < e \dots$, or $(c = d) < a \dots$. The number of such orderings are known as the Horse numbers, Fubini numbers, or ordered Bell numbers $H(n)$. A horse race is a combinatorial setting where ties may occur, hence Horse numbers. Fubini's name is associated in relation to his theorem for integrals, where integrals over two variables may be taken as one (analogues to equality ordering), or performed separately in some strong order. Counting the ways to perform these integrals is an equivalent problem.

## 1.2 Total Strong Ordering

A strong ordering is a permutation of a set of elements. Permutation in the sense of only allowing the relations $<$ and $>$. Strong orderings are counted by the factorial.

## 1.3 Stirling Numbers of the First Kind

Given in a table of identities in *Concrete Mathematics* [1], the signed Stirling numbers of the first kind $s(l, m)$ give the coefficient on $x^m$ in the falling factorial $(x)_l$ where $(x)_l = (x)(x-1)...(x-l+1)$. The formula is $(x)_l = \sum_{m=0}^{l} s(l, m)x^m$. Usually, as is the case in *Concrete Mathematics* [1] these numbers appear when counting permutations with a set number of cycles.

## 1.4 Constrained Weak Ordering

Constrained Horse numbers or Bill numbers $B_k(n)$ count weak orderings such that $k$ elements of the finite set are chosen, and constrained to follow a specific strong ordering. If the elements under total weak ordering are $x_1, x_2, \dots, x_n$, such a strong ordering could be $x_1 < x_2 < \cdots < x_k$. Another type of constraint may be imposed by equality, for example $x_1 = x_2 = \cdots = x_k$. Both of these types will be counted, with equality being the easier case.

## 1.5 Example by Hand: Single Strong Ordering

The case $B_2(n)$ is given a short expression in terms of the Horse numbers. $B_2(n)$ counts when $x_1 < x_2$, or any other two elements are placed in a specific strong ordering, and provides inspiration for the generalized case.

**Theorem 1.**

$$B_2(n) = \frac{H(n) - H(n-1)}{2} \tag{1}$$

*Proof.* The expression $[x_1 < x_2] \vee [x_1 > x_2] \vee [x_1 = x_2]$ is a tautology on over any set $x_1, x_2 \dots x_n$ that is weakly ordered, so long as $n \geq 2$. The three conditions are mutually exclusive, so their union has cardinality that may be expressed as a sum of each of the three relation's cardinalities. Consider each case:

- Count for $x_1 < x_2$, this is $B_2(n)$.
- Count for $x_1 = x_2$, this is $H(n-1)$.
- Count for $x_1 > x_2$, this is $B_2(n)$ again by symmetry.

The sum of these counts is $H(n)$ because the expression as a whole is a tautology.

$$H(n) = 2B_2(n) + H(n-1) \tag{2}$$

Finally as was to be shown

$$B_2(n) = \frac{H(n) - H(n-1)}{2}$$

□

### 1.6 Shift Operators

Shift operators are used to formally show the number of orderings where $x_1, x_2, \ldots, x_k$ have a strong but not specific ordering can be expressed using $s(l, m)$, the Stirling numbers of the first kind.

Consider expressing the counting in terms of the left, right shift operators $T_+, T_-$ on the sequence $H(0), H(1), \ldots, H(n+k-1)$. Shift operators are defined such that $T_+F(n) = F(n+1)$ and $T_-F(n) = F(n-1)$. Importantly $T_+T_- = T_-T_+ = I$ ($I$ being the identity operation), so any product of shift operators may be abbreviated $T_a$ where $a$ is an integer and $T_aF(n) = F(n+a)$ so long as $F(n+a)$ is in the domain. In the case $F(n+a)$ is not part of the finite sequence $T_aF(n) = 0$. The shift operator is linear and commutes with integers acting as scalars.

## 2 General Rigged Orderings

### 2.1 Strong Ordering of $k \leq n$ Elements

**Theorem 2.**

$$B_k(n) = \frac{1}{k!} \sum_{j=0}^{k} s(k, k-j) H(n-j) \tag{3}$$

*Proof.* First remove or ignore the counting of $k$ elements to be counted in strong ordering $x_1, x_2, x_3, \ldots, x_k$: $T_{-k}H(n)$.

- Reintroduce the element $x_1$ by increasing the argument to $H(n-k+1)$, then subtract any case where $x_1 \in \emptyset$. That is $(T_1)T_{-k}H(n)$
- Reintroduce the element $x_2$ by increasing the argument, then subtract any case where $x_2 \in \{x_1\}$. That is $(T_1 - 1I)T_{-k}T_1H(n)$
- Reintroduce the element $x_3$ by increasing the argument, then subtract any case where $x_3 \in \{x_1, x_2\}$. That is $(T_1 - 2I)(T_1 - 1I)T_1T_{-k}H(n)$
- ...
- Reintroduce the element $x_k$ by increasing the argument, then subtract any case where $x_k \in \{x_1, x_2, \dots, x_{k-1}\}$. The total is $(T_1 - (k-1)I)\dots(T_1 - 2I)(T_1 - 1I)T_1T_{-k}H(n)$.

Now all elements are counted, but those from $x_1, x_2, \dots, x_k$ have had mutual equalities removed from the count, such that any counted ordering has $x_1, x_2, \dots, x_k$ strongly ordered. The falling factorial appears with argument $T_1$ and $k$ terms.

$$(T_1 - (k-1)I)\dots(T_1 - 2I)(T_1 - 1I)T_1T_{-k}H(n) \tag{4}$$

By commuting shift operators and integer scalars, the count is $T_{-k}(T_1)_kH(n)$ where $(x)_n$ is the falling factorial of x with $n$ terms. As discussed in the introduction 1.3, Stirling number $s(n, a)$ expresses the integer coefficient of $x^a$ in $(x)_n$. The count is now expressed as follows.

$$T_{-k}[\sum_{j=0}^{k} s(k, j)T_j]H(n) \tag{5}$$

The formula applies because $T_a$ commute and repetition of $T_a$ may be treated as would exponentiation of a polynomial variable. The effect of the shift operators is now trivial upon $H(n)$.

$$\sum_{j=0}^{k} s(k, j)H(n - k + j) \tag{6}$$

By reindexing the sum.

$$\sum_{j=0}^{k} s(k, k - j)H(n - j) \tag{7}$$

The number of arrangements of $x_1, x_2, \dots, x_n$ where $x_1, x_2, \dots, x_k$ are strongly ordered have been counted. Given the strongly ordered subset count it is straightforward to determine $B_k(n)$ by dividing by $k!$, since there are $k!$ total strong orderings of $x_1, x_2, \dots, x_n$.

$$B_k(n) = \frac{1}{k!} \sum_{j=0}^{k} s(k, k-j) H(n-j)$$

□

*Note constrained Horse numbers may be efficiently computed by applying a discrete Fourier domain convolution. This is the case because $B_k(n)$ has an expression as a weighted sum of $H(m)$, $m \leq n$, with constant coefficients for fixed $k$.*

## 2.2 Union and Intersection of Disjoint Strong Constraints

**Corollary 3.** *The cardinality of weakly ordered arrangements of the elements $x_1, x_2, \ldots, x_n$ under condition $[x_{a_1} < x_{a_1+1} < \cdots < x_{a_1+A_1-1}] \wedge [x_{a_2} < x_{a_2+1} < \cdots < x_{a_2+A_2-1}] \wedge \cdots \wedge [x_{a_N} < x_{a_N+1} < \cdots < x_{a_N+A_N-1}]$ of $N$ specifically strongly ordered subsets of sizes $A_1, A_2 \cdots A_N$ where $\{x_{a_j}, x_{a_j+1}, \ldots, x_{a_j+A_j-1}\} \cap \{x_{a_i}, x_{a_i+1}, \ldots, x_{a_i+A_i-1}\} = \emptyset \ \forall i, j$:*

$$\frac{1}{A_1! A_2! \ldots A_N!} [(T_1)_{A_1} (T_1)_{A_2} \ldots (T_1)_{A_N}] T_{-(\sum_{j=1}^{N} A_j)} H(n) \tag{8}$$

*Proof.* The procedure of the proof of the second theorem (2) may be repeated for any amount of disjoint subsets which are strongly ordered, because counting only involves the number of elements from the respective subset already reintroduced, and the total count. The subsets being disjoint allows this to remain trivial. An additional provision of allowing $H(0), H(1) \ldots H(n + (\sum_{j=1}^{N} A_j))$ under shift operation is needed. Factorial division for the size of each strongly ordered subset again accomplishes specifying a strong ordering, rather than over counting all strong orderings of a subset. The resulting formula is given above (8). □

*The cardinality for union of conditions immediately follows by the inclusion-exclusion principle.*

## 2.3 Union and Intersection of Equality Constraints

**Corollary 4.** *The first theorem exemplified that dealing with conditions $x_a = x_b$ is a simple reduction in the effective number of elements being ordered, for $x_a = x_b$, $a \neq b$, this is $H(n-1)$. For an intersection of such equalities, the number of elements that are removed from counting is determined by counting the number of equivalence classes introduced by the intersection of equality constraints, $k$, the cardinality is then $H(n+k-m)$, where $m$ is the number of elements included non-trivially ($x_a = x_a$ is trivial) in the intersection of equality constraints. The cardinality for union of equality constraints is easily expressed via the inclusion-exclusion principle.*

# 3 Horse Numbers

## 3.1 A Complete Alternating Recurrence

**Corollary 5.**

$$H(n) = n! - \sum_{j=1}^{n} s(n, n-j)H(n-j) \tag{9}$$

*Proof.* $B_n(n) = 1$, if the set is $x_1, x_2 \ldots x_n$, and $x_1 < x_2 < \cdots < x_n$ the number of arrangements is 1. It follows from the second theorem (2) :

$$1 = \frac{1}{n!}\sum_{j=0}^{n} s(n, n-j)H(n-j) \tag{10}$$

Which may be rearranged after the substitution $s(n, n) = 1$ to the result.

$$H(n) = n! - \sum_{j=1}^{n} s(n, n-j)H(n-j)$$

□

# 4 Remarks

It is interesting to consider where else in combinatorics the shift operator may yield simplified understanding or proofs. Here the treatment revealed an over arching structure allowed the existing counting properties of $H(n)$ to be utilized, without considering a recurrence or explicit definition for $H(n)$. More concretely, $B_k(n)$ has some recurrence that could be shown to yield the second theorem (2), by defining $B_k(n)$ then making a connection to a recurrence for $H(n)$ or $s(l, m)$. The possibility to make adjustments to counting allowed by the shift operator means the second theorem (2) is revealed without such low level considerations.

The shift operator also allows for certain recurrences such as the discrete coupled oscillators problem of physics to be solved as a linear differential equation after separation of variables is done. This occurs by letting the shift operator act like a differential operator on indices $n$ as $T_\pm = e^{\pm \frac{d}{dn}}$, revealing the dispersion relation of the system. Are there interesting combinatorial problems that can be solved with tools of differential equations by such an abuse of notation?

# 5 Acknowledgments

Nathan Constantinides is acknowledged for checking the single strong ordering case (1) numerically for small $n$.

# References

[1] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, Concrete Mathematics, Second Edition, *Pearson Education* (1994), 259-264.

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