Manuscript: Shift Invariant Methods for Discrete Oscillators

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Abstract

Discretely indexed linearly coupled oscillator systems are reformulated using the derivative form of the translation operator. The result is a shift invariant PDE that may be solved by determining a dispersion relation. For physics education such a solution yields two distinct levels of mathematical depth, ansatz or full PDE. Both levels allow better explanation of the dispersion relation of many coupled oscillators common in undergraduate curricula.

1 Many coupled oscillators

Consider a 1d system that is coupled linearly and homogeneously. Here the specific example of equal masses m with positions $x_n(t)$ connected by identical springs of constant k is taken. The governing equations of motion (one for each oscillator) follow.

$$m\frac{dx_n^2}{dt^2} = k(x_{n-1} - 2x_n + x_{n+1}) \tag{1}$$

1.1 Phasor ansatz

Use the common trial solution $x_n(t) = e^{i\omega t + i\theta n}$ in the equations of motion (1). Introduce constant $\omega_0^2 = \frac{k}{m}$.

$$-\frac{\omega^2}{\omega_0^2} e^{i\omega t + i\theta n} = (e^{i\theta} + e^{-i\theta} - 2) e^{i\omega t + i\theta n} \tag{2}$$

$$2\cos\theta = 2 - \frac{\omega^2}{\omega_0^2} \tag{3}$$

The system's dispersion relation has been determined immediately, since θ is the wavenumber for the discrete positions. Other ansatz $x_n(t) = e^{\pm i\omega t \pm i\theta n}$ work equally well due to even symmetry of cosine and the second derivative of phasors. From this collection of solutions real solutions may be constructed.

1.2 Discrete coupling as a PDE

The index n is taken to be continuous, to allow the powerful tools of differential equations to be involved. Continuous in n, $x(n,t)$ will pass through the value of $x_n(t)$ for integral values of n. Additionally introduce the translation operator T acting on the variable n. The equations of motion are now of the form:

$$m\frac{d^2 x(n,t)}{dt^2} = k(T_1 x(n,t) - 2x(n,t) + T_{-1} x(n,t)) \tag{4}$$

Now employ the derivative as the generator of translation to replace T.

$$m\frac{d^2 x(n,t)}{dt^2} = k(e^{\frac{d}{dn}} - 2 + e^{\frac{-d}{dn}}) x(n,t) \tag{5}$$

Apply separation of variables $x(n,t) = C(t)A(n)$ with λ the separation constant.

$$\frac{m}{k}\frac{\frac{d^2 C(t)}{dt^2}}{C(t)} = \frac{[(e^{\frac{d}{dn}} - 2 + e^{\frac{-d}{dn}})A(n)]}{A(n)} = \lambda \tag{6}$$

This yields two ODEs. The first is simple.

$$\frac{d^2 C(t)}{dt^2} = \frac{k}{m}\lambda C(t) \tag{7}$$

Note that this is the equation for the simple harmonic oscillator, but with the factor λ, the solutions are as follows.

$$C(t) = e^{\sqrt{\lambda}\sqrt{\frac{k}{m}}t} \tag{8}$$

If λ is negative one recognizes this is an oscillating form. Substitute in ω_0 and use the notation for a harmonic oscillator which is defined by its angular frequency ω. Take $\lambda = -\frac{\omega^2}{\omega_0^2}$, yielding:

$$e^{\sqrt{-\frac{\omega^2}{\omega_0^2}}\omega_0 t} = e^{\pm i\omega t} \tag{9}$$

1.3 Spatial ODE

Now move to find $A(n)$ in a similar fashion.

$$(e^{\frac{d}{dn}} - 2 + e^{\frac{-d}{dn}})A(n) = \lambda A(n) \tag{10}$$

$$(2\cosh(\frac{d}{dn}) - 2 - \lambda)A(n) = 0 \tag{11}$$

As for any other linear ODE, we replace derivatives with powers of r.

$$2\cosh(r) - 2 - \lambda = 0 \tag{12}$$

Solutions of the separated ODEs are now known and need physical interpretation.

$$r = \text{acosh}(1 + \frac{\lambda}{2}) \tag{13}$$

$$x(n,t) = C_0 e^{\sqrt{\lambda}\sqrt{\frac{k}{m}}t} \cdot e^{\pm rn} \tag{14}$$

1.4 Assembling the base solution

Let $i\theta = r$ and re-express λ in terms of ω (note $\cosh(i\theta) = \cos(\theta)$). This leads finally to a basis x_ω of solutions determined by ω.

$$\theta(\omega) = \text{acos}\left(1 - \frac{1}{2}\frac{\omega^2}{\omega_0^2}\right) \tag{15}$$

$$x_\omega(n,t) = e^{\pm i\omega t} \cdot e^{\pm i\theta n} \tag{16}$$

This is the expected family of non-evanescent solutions, exemplified by *Waves (Draft), Ch. 2 Eqn. 56* by D. Morin.

2 Remarks

2.1 For a physics course

In the context of a wave physics course, such a method allows instructors to justify the nontrivial dispersion relation $\theta(\omega)$, since phasor ansatz will yield $\theta(\omega)$ through algebra. Alternatively to prepare students for the mathematics of translation generation by the wave momentum operator, and Bloch's Theorem, seen later in the context of quantum mechanics, one may involve the translation operator.

2.2 Further utility of the approach

One can expand the exponentiated derivatives of the translation operators to any order and truncate. The first interesting approximation simply yields the non-dispersive wave equation, since $\cosh(\frac{d}{dn}) \approxeq 1 + \frac{1}{2}\frac{d^2}{dn^2}$. In this approximation the n dependence is governed by $\frac{d^2}{dn^2}A(n) = \lambda A(n)$. Such approximations are physically viewable as being qualified by long wavelength disturbances on the coupled oscillators. Constant coefficient linear couplings featuring more than one left or right coupling can be solved with the translation operator approach. These couplings result in a closed form dispersion relation involving phasor exponentiations of the wave number.

3 Conflict of interest statement

The author has no conflicts of interest to report.