A Linear Map Between Strong and Weak Orderings and Modular Periodicity

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Abstract

The horse numbers, Fubini numbers, or Ordered Bell numbers count the total weak orderings $(<, >, =)$ on a set of elements. *Constrained horse numbers* count orderings of elements such that a subset are in a specific strong ordering relative to each other. Constrained horse numbers are expressed as a sum of horse numbers with weightings given by *the signed Stirling numbers of the first kind*. Considering the case of fully ordered constraint, a recurrence for the horse numbers is determined. Additionally, cardinality for unions and intersections of strong ordering or equivalence constraints on subsets is given. The new recurrence is used to connect the sequence of counts of strong and weak orderings by a matrix transformation. Finally the Horse numbers are proven to be eventually periodic modulus any integer by invoking the eventual modular periodicity of the *Stirling numbers of the second kind* in their first argument.

1 Introduction

The cardinality of totally ordered sets is considered. No incomparable element is allowed or calculated for.

1.1 Total Weak Ordering

If the finite set is $\{d, e, a, b, c \dots\}$, then a weak ordering may be applied with symbols $<, >, =$, such as $a < d < e \cdots$, or $(c = d) < a \cdots$. The number of such orderings are known as the horse numbers, Fubini numbers, or ordered Bell numbers $H(n)$. A horse race is a combinatorial setting where ties may occur, hence horse numbers.

1.2 Total Strong Ordering

A strong ordering is a permutation of a set of elements. Permutation in the sense of only allowing the relations $<$ and $>$. Strong orderings are counted by the factorial.

1.3 Stirling Numbers of the First Kind

Given in a table of identities in *Concrete Mathematics* [1], the signed Stirling numbers of the first kind $s(l,m)$ give the coefficient on x^m in the falling factorial $(x)_l$ where $(x)_l = (x)(x-1)\cdots(x-l+1)$. The formula is $(x)_l = \sum_{m=0}^{l} s(l,m)x^m$. Usually these numbers appear when counting permutations with a set number of cycles. Here they appear for rigged horse races, and a recurrence for the ordered Bell numbers.

1.4 Stirling Numbers of the Second Kind, Inverse Matrix Property

The Stirling numbers of the second kind are here introduced given two properties that are useful. Usually these numbers may count the number of partitions of a set. Stated in *Advanced combinatorics; the art of finite and infinite expansions* [2] if $s(n,k)$ the signed Stirling numbers of the first kind and $S(n,k)$ are treated as matrices (even infinite matrices, for $n,k \geq 0$) with rows indexed by n and columns by k, then they are inverses of each other. Denote such matrices M_s for the signed Stirling numbers of the first kind, and M_S for the Stirling numbers of the second kind.

The second important property of $S(n,k)$ are that they are eventually periodic in n modulus any finite natural number for fixed k. This may be shown using the following formula lifted from the paper *Stirling matrix via Pascal matrix* [3].

$$S(n,k) = \frac{1}{(k-1)!}\sum_{t=1}^{k}(-1)^{k-t}\binom{k-1}{t-1}t^{n-1} \tag{1}$$

The only dependence on n for fixed k is a finite summation of power functions of n. The individual power functions must repeat because integers under multiplication and modulus a fixed integer form a finite group. The finite sum of these eventually periodic sequences weighted with constants is then also eventually periodic in the same modular equivalence. Therefore modulus any finite natural number the sum will be eventually periodic in n.

1.5 Constrained or Rigged Weak Ordering

Constrained horse numbers $B_k(n)$ count weak orderings such that k elements of the finite set are chosen, and constrained to follow a specific strong ordering. If the elements under total weak ordering are $x_1, x_2, \ldots, x_n$, such a strong ordering could be $x_1 < x_2 < \cdots < x_k$. Another type of constraint may be imposed by equality, for example $x_1 = x_2 = \cdots = x_k$. Both of these types will be counted, with equality being the easier case.

1.6 Shift Operators

Shift operators are used to formally show the number of orderings where $x_1, x_2, \ldots, x_k$ have a strong but not specific ordering can be expressed using $s(l,m)$, the Stirling numbers of the first kind.

Consider expressing the counting in terms of the left, right shift operators T_+, T_- on the sequence $H(0), H(1), \ldots, H(n+k)$. On a sequence $F(n)$ shift operators are defined such that $T_+F(n) = F(n+1)$ and $T_-F(n) = F(n-1)$. Importantly $T_+T_- = T_-T_+ = I$ (I being the identity operation), so any product of shift operators may be abbreviated T_a where a is an integer and $T_aF(n) = F(n+a)$ so long as $F(n+a)$ is in the domain. In the case $F(n+a)$ is not part of the finite sequence $T_aF(n) = 0$. The shift operator is linear and commutes with integers acting as scalars.

2 General Rigged Orderings

2.1 Strong Ordering of $k \leq n$ Elements

Theorem 1.

$$B_k(n) = \frac{1}{k!} \sum_{j=0}^{k} s(k, k-j) H(n-j) \tag{2}$$

Proof. First remove or ignore the counting of k elements to be counted in strong ordering $x_1, x_2, x_3, \ldots, x_k$: $T_{-k}H(n)$.

- Reintroduce the element x_1 by increasing the argument to $H(n-k+1)$, then subtract any case where $x_1 \in \emptyset$. That is $(T_+)T_{-k}H(n)$
- Reintroduce the element x_2 by increasing the argument, then subtract any case where $x_2 \in \{x_1\}$. That is $(T_+ - 1I)T_{-k}T_+H(n)$
- Reintroduce the element x_3 by increasing the argument, then subtract any case where $x_3 \in \{x_1, x_2\}$. That is $(T_+ - 2I)(T_+ - 1I)T_+T_{-k}H(n)$
- $\cdots$
- Reintroduce the element x_k by increasing the argument, then subtract any case where $x_k \in \{x_1, x_2, \ldots, x_{k-1}\}$. The total is $(T_+ - (k-1)I) \cdots (T_+ - 2I)(T_+ - 1I)T_+T_{-k}H(n)$.

Now all elements are counted, but those from $x_1, x_2, \ldots, x_k$ have had mutual equalities removed from the count, such that any counted ordering has $x_1, x_2, \ldots, x_k$ strongly ordered. The falling factorial appears with argument T_+ and k terms.

$$(T_+ - (k-1)I) \cdots (T_+ - 2I)(T_+ - 1I)T_+T_{-k}H(n) \tag{3}$$

By commuting shift operators and integer scalars, the count is:

$$T_{-k}(T_+)_kH(n) \tag{4}$$

Where $(x)_n$ is the falling factorial of x with n terms. As discussed in the introduction 1.3, Stirling number $s(n,a)$ expresses the integer coefficient of x^a in $(x)_n$. The count is now expressed as follows.

$$T_{-k}[\sum_{j=0}^{k} s(k,j)T_j]H(n) \tag{5}$$

The formula applies because T_a commute and repetition of T_a may be treated as would exponentiation of a polynomial variable. The effect of the shift operators is now trivial upon $H(n)$.

$$\sum_{j=0}^{k} s(k,j)H(n-k+j) \tag{6}$$

By reindexing the sum.

$$\sum_{j=0}^{k} s(k,k-j)H(n-j) \tag{7}$$

The number of arrangements of $x_1, x_2, \ldots, x_n$ where $x_1, x_2, \ldots, x_k$ are strongly ordered have been counted. Given the strongly ordered subset count it is straightforward to determine $B_k(n)$ by dividing by $k!$, since there are $k!$ total strong orderings of $x_1, x_2, \ldots, x_k$, and only one is desired per the definition of $B_k(n)$.

$$B_k(n) = \frac{1}{k!}\sum_{j=0}^{k} s(k,k-j)H(n-j)$$

$\square$

Note constrained horse numbers may be efficiently computed by applying a discrete Fourier domain convolution. This is the case because $B_k(n)$ has an expression as a weighted sum of $H(m)$, $m \leq n$, with constant coefficients for fixed k.

Another interesting notation is given using the relation between falling factorials and binomial coefficients on the equation for strong orderings (4) after it has been divided by $k!$ to pick a specific ordering. The notation is inspired by the following manipulations.

$$B_k(n) = \frac{T_+(T_+-1)\cdots(T_+-k+1)}{k!}H(n-k) \tag{8}$$

$$B_k(n) = \frac{T_+!}{k!(T_+-k)!}H(n-k) \tag{9}$$

$$B_k(n) = \binom{T_+}{k}H(n-k) \tag{10}$$

If a useful interpretation can be given to this expression, it should be undertaken, to qualify the meaning of division here. There is ambiguity between inversion of a linear operation or division in the sense of counting. For now this is only notational.

2.2 Union and Intersection of Disjoint Strong Constraints

Corollary 2. *The cardinality of weakly ordered arrangements of the elements* $x_1, x_2, \dots, x_n$ *under condition* $[x_{a_1} < x_{a_1+1} < \cdots < x_{a_1+A_1-1}] \wedge [x_{a_2} < x_{a_2+1} < \cdots < x_{a_2+A_2-1}] \wedge \cdots \wedge [x_{a_N} < x_{a_N+1} < \cdots < x_{a_N+A_N-1}]$ *of* N *specifically strongly ordered subsets of sizes* $A_1, A_2, \dots, A_N$ *where* $\{x_{a_j}, x_{a_j+1}, \dots, x_{a_j+A_j-1}\} \cap \{x_{a_i}, x_{a_i+1}, \dots, x_{a_i+A_i-1}\} = \emptyset \ \forall i, j$*:*

$$\frac{1}{A_1! A_2! \cdots A_N!} [(T_+)_{A_1} (T_+)_{A_2} \cdots (T_+)_{A_N}] T_{-(\sum_{j=1}^{N} A_j)} H(n) \tag{11}$$

This is equivalent to the following by the notation used earlier (10).

$$[\prod_{j=1}^{N} \binom{T_+}{A_j}] H(n - \sum_{j=1}^{N} A_j) \tag{12}$$

Proof. The procedure of the proof of the first theorem 1 may be repeated for any amount of disjoint subsets which are strongly ordered, because counting only involves the number of elements from the respective subset already reintroduced, and the total count. The subsets being disjoint allows this to remain trivial. An additional provision of allowing $H(0), H(1), \dots, H(n + (\sum_{j=1}^{N} A_j))$ under shift operation is needed. Factorial division for the size of each strongly ordered subset again accomplishes specifying a strong ordering, rather than over counting all strong orderings of a subset. The resulting formula is given above (12). □

The cardinality for union of conditions immediately follows by the inclusion-exclusion principle.

2.3 Union and Intersection of Equality Constraints

Corollary 3. *The first theorem exemplified that dealing with conditions* $x_a = x_b$ *is a simple reduction in the effective number of elements being ordered, for* $x_a = x_b$*,* $a \neq b$*, this is* $H(n-1)$*. For an intersection of such equalities, the number of elements that are removed from counting is determined by counting the number of equivalence classes introduced by the intersection of equality constraints,* k*, the cardinality is then* $H(n+k-m)$*, where* m *is the number of elements included non-trivially (*$x_a = x_a$ *is trivial) in the intersection of equality constraints. The cardinality for union of equality constraints is easily expressed via the inclusion-exclusion principle.*

3 Horse Numbers

3.1 A Complete Alternating Recurrence

Corollary 4.

$$H(n) = n! - \sum_{j=1}^{n} s(n, n-j) H(n-j) \tag{13}$$

Proof. $B_n(n) = 1$, if the set is $x_1, x_2, \ldots, x_n$, and $x_1 < x_2 < \cdots < x_n$ the number of arrangements is 1. It follows from the first theorem (1) :

$$1 = \frac{1}{n!} \sum_{j=0}^{n} s(n, n-j) H(n-j) \tag{14}$$

Which may be rearranged after the substitution $s(n, n) = 1$ to the result.

$$H(n) = n! - \sum_{j=1}^{n} s(n, n-j) H(n-j)$$

$\square$

4 Linear Transformation Between Strong and Weak Orderings, Modular Periodicity

4.1 Linear Transformation

Theorem 5. *The infinite vectors f_n and H_n with entries $n!$ and $H(n)$ respectively obey the following relation with other symbols as defined in the introduction 1.4:*

$$M_S f_n = H_n \tag{15}$$

$$M_s H_n = f_n \tag{16}$$

Proof. The above corollary 4 may be written as a matrix equation by moving the sum to the the left hand side (note $s(n, n)$ is 1, and $s(n, k)$ is zero for $k > n$). An infinite lower triangular matrix such as $s(n, k)$ multiplied on a vector may be abbreviated by such a sum.

$$M_s H_n = f_n$$

The first relation then immediately follows given the inverse of M_s is M_S.

$$M_S M_s H_n = M_S f_n \tag{17}$$

$$H_n = M_S f_n$$

$\square$

In light of this theorem, the Stirling numbers of the first kind could be appropriately labeled as the weak Stirling numbers (those that act naturally on H_n). The Stirling numbers of the second kind are then the strong Stirling numbers.

4.2 Modular Periodicity

Theorem 6. *The Horse numbers $H(n)$ are eventually periodic under any finite natural number modular equivalence K.*

Proof. Consider the existing relation.

$$H_n = M_S f_n$$

The entries of f_n are simply $n!$. Consider this relation modulus some natural number K. The vector entries f_n are certainly zero for $n \geq K$ in the modular equivalence K.

Therefore modulus K the relation stands in a simplified form:

$$H(n) = \sum_{k=0}^{K-1} S(n,k)f(k) \ (mod \ K) \tag{18}$$

For all $n \geq 0$, $H(n)$ $(mod\ K)$ is written as a finite sum of $S(n,k)$ with coefficients independent of n. Each $S(n,k)$ $(mod\ K)$ for $0 \leq k \leq K$ is eventually periodic in n for any k as was shown in the introduction 1.4. The finite sum of such eventually periodic functions scaled by constants is also eventually periodic, this eventually periodic sum gives $H(n)$ $(mod\ K)$.

□

5 Remarks

5.1 Primes

The function $F(n) = \sum_{j=0}^{n} H(j)s(n,j)$ $(mod\ K)$ defined on natural numbers has its first zero for $n = K$ if K is prime or $K = 4$. This follows from the fact that $F(n) = n!$ immediately by the complete alternating recurrence for $H(n)$ 4.

5.2 Infinite Matrices

Are there generalizations to be made that may be made of linear transformations between infinite sequences which preserve eventual periodicity?

The matrix entries M_s form Sierpiński triangles sometimes with defects when plotted modulus some natural number.

The diagonals of the infinite matrix M_s appear to be eventually periodic modulus any natural number. Can this be proven? Does it relate the result on $H(n)$?

5.3 Shift Operators

It is interesting to consider where else in combinatorics the shift operator may yield simplified understanding or proofs. Interpreting the binomial form (10) is expected to be useful to this end.

The shift operator also allows for certain recurrences such as the discrete coupled oscillators problem of physics to be solved as a linear differential equation after separation of variables is done. This occurs by letting the shift operator act like a differential operator on indices n as $T_{\pm} = e^{\pm \frac{d}{dn}}$, revealing the dispersion relation of the system. Are there interesting combinatorial problems that can be solved with tools of differential equations by such an abuse of notation?

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