SINGULARITIES IN THE GENERALIZED CONSTANTIN–LAX–MAJDA EQUATION

ON A SEARCH FOR FINITE-TIME SINGULARITIES IN THE GENERALIZED CONSTANTIN–LAX–MAJDA EQUATION

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A Project Submitted to Math 745

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Project Proposal

For my term project I will compute and analyse solutions of the Generalized Constantine-Lax-Majda equation. The Constantine-Lax-Majda (CLM) equation was proposed as a one-dimensional model for the three-dimensional vorticity equation and has the form

$$\omega_t - u_x\omega = 0, \qquad u_x = H\omega$$

where ω is the vorticity of the fluid and u is the fluid velocity. For this analysis the equations will be examined on the periodic domain $x \in [-\pi, \pi]$. The Hilbert transform of the vorticity is defined as

$$H\omega(x,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \omega(y,t)\cot\left(\frac{x-y}{t}\right)\, dy,$$

which has a singularity at $x = y$ and so must be evaluated using the Cauchy Principle Value. The CLM equation was later expanded on by De Gregorio to include the convection term $u\omega_x$. The De Gregorio equation has the form

$$\omega_t + u\omega_x - u_x\omega = 0, \qquad u_x = H\omega.$$

The De Gregorio equation was later generalized by Okamoto et al. to include the real parameter a. The result is the Generalized Constantin-Lax-Majda equation (GCLM)

$$\omega_t + au\omega_x - u_x\omega = 0, \qquad u_x = H\omega.$$

The parameter a determines the relative strength of the convection term. When $a = 0$ the GCLM is equal to the CLM, and when $a = 1$ it is equal to the De Gregorio equation. The CLM equation was proposed in 1985 and has been shown to exhibit finite time blow-up. While the De Gregorio equation was proposed shortly after in 1989, questions still remain as to whether or not the equation permits finite time blow-up. The goal of this analysis will be to examine the affect the parameter a has on the finite time blow-up of the GCLM equation. I will explore solutions of the GCLM for values of $a \in [-1, 1]$.

Abstract

The Constantine-Lax-Majda (CLM) equation was proposed as a one-dimensional model for the three-dimensional vorticity equation. The CLM equation was later expanded on by De Gregorio to include the convection term $u\omega_x$. The De Gregorio equation was generalized by Okamoto et al. to include the real parameter a which determines the relative strength of the convection term. The result is the Generalized Constantin-Lax-Majda equation (GCLM). In this analysis we examine the affect the parameter $a \in [-1, 1]$ has on the finite time blow-up of the GCLM equation on the periodic domain $x \in [-\pi, \pi]$.

The analysis found RESULTS RESULTS RESULTS. CONCLUSION CONCLUSION CONCLUSION.

Nearly all values of a resulted in blow-up or blow-up like behaviour. The results for most a values are inconclusive as the resolution was insufficient to properly capture the singular behaviour. However, the results indicate that all values of $a < 0.8$ would result in a finite-time singularity. The primary effect a had was on how quickly the singularity formed.

Contents

1 Introduction

The three-dimensional Euler equations are a simplified form of the Navier-Stokes equations which describe the flow of inviscid, incompressible fluids. The Euler equations are

$$u_t + u \cdot \nabla u + \nabla p = 0, \tag{1.1a}$$

$$\nabla \cdot u = 0, \tag{1.1b}$$

where u is the velocity field and p is the scalar pressure. By substituting in the vorticity ($\omega = \nabla \times u$) these equations can be re-expressed in the three-dimensional vorticity equation

$$\omega_t + (u\nabla)\omega = (\omega\nabla)u, \tag{1.2}$$

where the velocity can be computed from the vorticity field with the equation DOUBLE CHECK THIS

$$u = -\nabla \times (\nabla^{-1}\omega). \tag{1.3}$$

A central question in fluid dynamics is whether or not finite-time singularities can form in fluid flows with smooth initial conditions [2]. As first demonstrated by Beale, Kato, and Majda, singularities can form if and only if the maximum vorticity of the flow becomes infinite [1]. The vorticity equation allows for the direct modelling of the vorticity and so is a very useful tool for studying singularity formation in the Euler equations. WHY USE 1D MODELS.

1.1 The CLM Equation

The Constantin-Lax-Majda (CLM) equation was the first one-dimensional model of the three-dimensional vorticity equation for an incompressible fluid [2]. The CLM equation is

$$\omega_t = H(\omega)\omega \tag{1.4}$$

where H is the Hilbert transform DOUBLE CHECK THIS

$$H\omega(x,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \omega(y,t)\cot\left(\frac{x-y}{2}\right)\,dy. \tag{1.5}$$

The CLM equation was defined on the unbounded domain R^1 DOUBLE CHECK THIS. A significant advantage of this model was that its simplicity. This allowed Constantin et al. to prove the following Theorem [2].

Theorem 1.1. *Suppose $\omega_0(x)$ is a smooth function decaying sufficiently rapidly as $|x| \to \infty$ ($\omega_0 \in H^1(\mathbb{R})$ suffices). Then the solution to the model vorticity equation in (1.4) is given by*

$$\omega(x,t) = \frac{4\omega_0(x)}{(2 - tH\omega_0(x))^2 + t^2\omega_0^2(x)}. \tag{1.6}$$

The CLM model is a very simple model. The biggest advantage it has is that it has an explicit solution which allows for ...[2].

However, De Gregorio noted three shortcomings of the model [3].

1. asdf

1.2 The De Gregorio Equation

The De Gregorio equation was proposed as a one-dimensional model that improves on the CLM equation by ... [3]

1.3 The Generalized CLM Equation

The De Gregorio equation was later generalized to the GCLM [5].

$$H\omega(x,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \omega(y,t)\cot\left(\frac{x-y}{2}\right)\, dy. \tag{1.7}$$

1.4 Blow-up Criteria

Okamoto et al. (2008) prove Theorem 3.1 and Theorem 3.2.

Theorem 1.2. *Let $a \in \mathbb{R}$ be given. For all $\omega_0 \in H^1(S^1)/\mathbb{R}$, there exists a $T > 0$ depending only on a and $\|\omega_{0,x}\|$ such that there exists a unique solution $\omega \in C^0([0,T]; H^1(S^1)/\mathbb{R}) \cap C^1([0,T]; L^2(S^1)/\mathbb{R})$ of (GCLM ref) with $\omega(0,x) = w_0$.*

1.5 Overview

In this paper we will conduct a search for singularity formation in the GCLM by

2 Numerical Methods

We use the same numerical methods as [5] which are described below. We use the notation $\hat{\omega}_k$ for the Fourier coefficients of the vorticity. There derivatives The numerical integration techniques below describe the procedure to calculate the value of ω at step $n+1$ from the value at step number n. To distinguish the notation from the Fourier coefficient we use $\omega^{(n)}$, $u^{(n)}$, and $t^{(n)}$ to denote the vorticity, velocity, and time, respectively, at step n.

2.1 Domain and Grid

The vorticity $\omega($ is represented in physical space on a grid of $N = 2^14$ equidistant points on the domain $x \in [\pi, pi]$. Because the domain is periodic we must avoid double counting the grid point on the boundary. Therefore we set $x_0 = -\pi$ and $x_N = \pi - h$, where h is the uniform distance between points x_i and x_{i+1}. The vorticity is represented in Fourier space with the truncated Fourier series

$$\omega(t, x) = \sum_{k=-N/2}^{N/2-1} \hat{\omega}_k e^{ikx}. \tag{2.1}$$

Transformation from physical to Fourier space, and vice versa, are performed using the standard Matlab fft and ifft functions.

We compute the value of $u^{(n)}$ in Fourier Space using the formula

$$u^{(n)}(t, x) = 4\ln(2)\hat{\omega}_0^{(n)} + \sum_{k=-N/2, k\neq 0}^{N/2-1} \frac{\hat{\omega}_k^{(n)} \exp(ikx)}{k}\, dv. \tag{2.2}$$

Compute u_x in Fourier space. Compute ω_x in Fourier space.

The full derivation of each formula is shown in the Appendix.

2.2 Time Stepping

When modelling time dependent PDEs it is best practice to use explicit time stepping methods for non-linear terms in order to reduce computation cost, and use implicit methods for linear term to improve stability [6]. As described in prior sections, both the convection term and stretching term of the GCLM are non-linear. Therefore, as is used in [5] and [4], we perform time stepping using the explicit fourth-order Runge-Kutta (RK4) method. The RK4 time stepping is performed with the formula

$$\omega^{(n+1)} = \omega^{(n)} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{2.3}$$

where the k_i terms are calculated using

$$k_1 = f(\omega^{(n)}, t^{(n)}) \tag{2.4a}$$

$$k_2 = f(\omega^{(n)} + dt \cdot k_1/2, t^{(n)} + dt/2) \tag{2.4b}$$

$$k_3 = f(\omega^{(n)} + dt \cdot k_2/2, t^{(n)} + dt/2) \tag{2.4c}$$

$$k_4 = f(\omega^{(n)} + dt \cdot k_3, t^{(n)} + dt) \tag{2.4d}$$

where $dt = 1E - 4$ is the time step size. The function f in the above equations is given by

$$f(\omega^{(n)}, t^{(n)}) = u_x\omega - au\omega_x \tag{2.5}$$

The derivatives u_x and ω_x are computed in Fourier space. The

2.3 Complete Numerical Method

Combining the methods described above, we use the following procedure

1. Convert $\omega^(n)$ from Physical space to Fourier space using FFT.

2. Calculate u_x

2.4 Initial Conditions

Okamoto et al. (2008) showed that any solution that satisfies (GCLM ref) must also satisfy

$$\frac{d}{dt}\int_{-\pi}^{\pi} \omega(t,x)\,dx = \int_{-\pi}^{\pi} (-au\omega_x + u_x\omega)\,dx = (a+1)\int_{-\pi}^{\pi} u_x\omega\,dx = (a+1)(H\omega,\omega), \tag{2.6}$$

where $(\cdot, \cdot)$ denotes the L^2 inner product (Okamoto et al., 2008). Therefore, we may assume without loss of generality that the initial condition satisfies

$$\int_{-\pi}^{\pi} \omega(0, x)\, dx = 0.$$

3 Results

these are the results

4 Conclusion

this is the conclusion

5 Appendix

5.1 Hilbert Transform

The Hilbert transform is defined as

$$H\omega(x, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \omega(y, t) \cot\left(\frac{x - y}{2}\right)\, dy.$$

The solution $\omega(t, y)$ is represented as the truncated Fourier series

$$\omega(t, y) = \sum_{k=-N/2}^{N/2-1} \omega_k(t) \exp(iky).$$

Substituting this into the Hilbert transform

$$H\omega(x,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-N/2}^{N/2-1}\omega_k(t)\exp(iky)\cot\left(\frac{x-y}{2}\right)\,dy.$$

We can swap the integral and sum to obtain

$$H\omega(x,t) = \frac{1}{2\pi}\sum_{k=-N/2}^{N/2-1}\omega_k(t)\int_{-\pi}^{\pi}\exp(iky)\cot\left(\frac{x-y}{2}\right)\,dy.$$

We now apply a substitution $u = y - x$. Therefore,

$$H\omega(x,t) = -\frac{1}{2\pi}\sum_{k=-N/2}^{N/2-1}\omega_k(t)\int_{-\pi-x}^{\pi-x}\exp(ik(u+x))\cot\left(\frac{u}{2}\right)\,du.$$

$$H\omega(x,t) = -\frac{1}{2\pi}\sum_{k=-N/2}^{N/2-1}\omega_k(t)\exp(ikx)\int_{-\pi-x}^{\pi-x}\exp(iku)\cot\left(\frac{u}{2}\right)\,du.$$

Because the integrand is periodic we can shift the domain by x

$$H\omega(x,t) = -\frac{1}{2\pi}\sum_{k=-N/2}^{N/2-1}\omega_k(t)\exp(ikx)\int_{-\pi}^{\pi}\exp(iku)\cot\left(\frac{u}{2}\right)\,du.$$

Converting the exponential to its trigonometric form

$$H\omega(x,t) = -\frac{1}{2\pi}\sum_{k=-N/2}^{N/2-1}\omega_k(t)\exp(ikx)\int_{-\pi}^{\pi}[\cos(ku)+i\sin(ku)]\cot\left(\frac{u}{2}\right)\,du,$$

$$H\omega(x,t) = -\frac{1}{2\pi}\sum_{k=-N/2}^{N/2-1}\omega_k(t)\exp(ikx)\left[\int_{-\pi}^{\pi}\cos(ku)\cot(\frac{u}{2})du + i\int_{-\pi}^{\pi}\sin(ku)\cot\left(\frac{u}{2}\right)\,du\right].$$

The first integral is odd and so is equal to zero. The second integral is even and so can be simplified to a half integral. Therefore,

$$H\omega(x,t) = -\frac{2i}{2\pi}\sum_{k=-N/2}^{N/2-1}\omega_k(t)\exp(ikx)\left[\int_0^{\pi}\sin(ku)\cot\left(\frac{u}{2}\right)\,du\right].$$

The remaining integral is a known identity [CITATION]

$$\int_0^{\pi}\sin(ku)\cot(\frac{u}{2})\,du = \pi\,\mathrm{sgn}(k),$$

where $\mathrm{sgn}(k)$ is the signum function. Substituting in to the Hilbert transform

$$H\omega(x,t) = -i\sum_{k=-N/2}^{N/2-1}\omega_k(t)\exp(ikx)\,\mathrm{sgn}(k).$$

5.2 Hilbert Transform Implementation

The Hilbert Transform is implemented in Matlab using the code shown below. The Fast Fourier Transform in Matlab formats the Fourier coefficients in order 0,...,N/2 -1 then -N/2,...,-1.

```
% Hilbert Transform Physical
function h = ht(w_t, N)
    % convert w_t to Fourier space
    w_c = fft(w_t,N);
    % wave numbers in fft format
    k = [0:N/2-1, -N/2:-1];
    % signum of wavenumbers
    sgn_k = sign(k);
    % multiply Fourier coefficients by -i sgn_k
    w_c = (-1i*sgn_k).*w_c;
```

```
    % convert w_c back to physical space
    h = ifft(w_c,N);
end
```

5.3 Velocity Field

The velocity field is defined as

$$u(t,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \omega(t,y) \log\left|\sin\left(\frac{x-y}{2}\right)\right| \, dy.$$

The solution $\omega(t,y)$ is represented as the truncated Fourier series

$$\omega(t,y) = \sum_{k=-N/2}^{N/2-1} \omega_k(t) \exp(iky).$$

Substituting this into the velocity field

$$u(t,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sum_{k=-N/2}^{N/2-1} \omega_k(t) \exp(iky) \log\left|\sin\left(\frac{x-y}{2}\right)\right| \, dy.$$

We can swap the integral and sum to obtain

$$u(t,x) = \frac{1}{\pi} \sum_{k=-N/2}^{N/2-1} \omega_k(t) \int_{-\pi}^{\pi} \exp(iky) \log\left|\sin\left(\frac{x-y}{2}\right)\right| \, dy.$$

We now apply a substitution $v = y - x$. Therefore,

$$u(t,x) = -\frac{1}{\pi} \sum_{k=-N/2}^{N/2-1} \omega_k(t) \int_{-\pi-x}^{\pi-x} \exp(ik(v+x)) \log\left|\sin\left(\frac{v}{2}\right)\right| \, dv.$$

Because the integrand is periodic we can shift the domain by x

$$u(t,x) = -\frac{1}{\pi}\sum_{k=-N/2}^{N/2-1} \omega_k(t)\exp(ikx)\int_{-\pi}^{\pi} \exp(ikv)\log\left|\sin\left(\frac{v}{2}\right)\right|\,dv.$$

Converting the exponential to its trigonometric form

$$u(t,x) = -\frac{1}{\pi}\sum_{k=-N/2}^{N/2-1} \omega_k(t)\exp(ikx)\int_{-\pi}^{\pi} (\cos(kv) + i\sin(kv))\log\left|\sin\left(\frac{v}{2}\right)\right|\,dv.$$

$$u(t,x) = -\frac{1}{\pi}\sum_{k=-N/2}^{N/2-1} \omega_k(t)\exp(ikx)\left[\int_{-\pi}^{\pi} \cos(kv)\log\left|\sin\left(\frac{v}{2}\right)\right|\,dv + i\int_{-\pi}^{\pi} \sin(kv)\log\left|\sin\left(\frac{v}{2}\right)\right|\,dv\right].$$

The second integral is odd and so is equal to zero. The first integral is even and so can be simplified to a half integral. Therefore,

$$u(t,x) = -\frac{2}{\pi}\sum_{k=-N/2}^{N/2-1} \omega_k(t)\exp(ikx)\int_{0}^{\pi} \cos(kv)\log\left|\sin\left(\frac{v}{2}\right)\right|\,dv.$$

The remaining integral is a known identity [CITATION]

$$\int_{0}^{\pi} \cos(kv)\log\left|\sin\left(\frac{v}{2}\right)\right|\,dv = -2\pi\ln(2), \qquad k = 0,$$

$$\int_{0}^{\pi} \cos(kv)\log\left|\sin\left(\frac{v}{2}\right)\right|\,dv = -\frac{\pi}{2|k|}, \qquad k \neq 0.$$

Substituting in to the velocity field

$$u(t,x) = 4\ln(2)w_0(t) + \sum_{k=-N/2, k\neq 0}^{N/2-1} \frac{\omega_k(t)\exp(ikx)}{k}\,dv.$$

5.4 Velocity Implementation

```
% Calculate velocity from vorticity
function u = calc_u(w_t,N)
    % convert w_t to Fourier space
    w_c = fft(w_t,N);
    % wave numbers in fft format
    k = [0:N/2-1, -N/2:-1];
    % exclude k=0
    points = (k ~= 0);
    % calc k=0 val
    w_c(~points) = 4*log(2)*w_c(~points);
    % divide Fourier coefficients by k~=0
    w_c(points) = -w_c(points)./abs(k(points));
    % convert w_c back to physical space
    u = real(ifft(w_c, N));
end
```

References

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