

Exactly self-similar blow-up of the generalized De Gregorio equation

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Abstract

We study exactly self-similar blow-up profiles for the generalized De Gregorio model for the three-dimensional Euler equation:

$$w_t + auw_x = u_x w, \quad u_x = Hw$$

We show that for any $\alpha \in (0, 1)$ such that $|a\alpha|$ is sufficiently small, there is an exactly self-similar C^α solution that blows up in finite time. This simultaneously improves on the result in [9] by removing the restriction $1/\alpha \in \mathbb{Z}$ and [8, 3], which only deals with asymptotically self-similar blow-ups.

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1 introduction

The famous millennium problem of the global regularity of the motion of incompressible fluids in three-dimensional space concerns the Navier–Stokes equation:

$$\begin{aligned} u_t + u \cdot \nabla u &= -p + \nu \Delta u, \\ \nabla \cdot u &= 0, \end{aligned}$$

where $u(x, t)$ is the velocity field and $p(x, t)$ is the scalar pressure. The coefficient $\nu \geq 0$ reflects the viscosity of the fluid. For inviscid fluid ($\nu = 0$) the equation reduces to the Euler equation. The vanishing of $\nabla \cdot u$ captures the incompressibility condition. The global wellposedness of the Euler and Navier–Stokes equations in three dimensions for smooth and decaying initial data is wide open, attracting a great deal of research efforts. The interested reader is referred to the surveys [1, 5, 10, 11, 12].

Let $\omega = \nabla \times u$ be the vorticity, which then satisfy

$$\omega_t + (u \cdot \nabla) \omega = \omega \cdot \nabla u + \nu \Delta \omega.$$

The second term on the left-hand side, known as the advection term, has the effect of transporting the vorticity. Since u is divergence free, the advection term will not affect the L^p norms of the vorticity. The first term on the right-hand side, known as the vortex stretching term, is only present in the three-dimensional case. Schematically $\nabla u \approx \omega$, so the vortex stretching term may be as bad as ω^2 in the worst-case scenario, and may cause blow-up of the equation. This is thought to be the crux of the millennium problem.

1.1 The generalized De Gregorio Model

In [6, 7] De Gregorio proposed a one-dimensional equation to model the competition between advection and vortex stretching in the Euler equation. The equation belongs to the family

$$w_t + auw_x = u_x w, \quad u_x = Hw$$

where H denotes the Hilbert transform, and a is a real parameter quantifying the relative strength of advection, modeled by uw_x , and vortex stretching, modeled by $u_x w$. Note that the relation $u_x = Hw$ also mimics the Biot–Savart law relating the velocity and the vorticity. It turns out that this

equation also models a variety of other equations, including the surface quasi-geostrophic equation, see [9]. Other similar 1D models of the Euler equation can be found in [13].

What De Gregorio studied in [6] is the case $a = 1$, which mirrors the Euler equation. The special case $a = 0$ had appeared in Constantin–Lax–Majda [4] and had been known to develop finite-time singularity for all non-trivial initial data. The generalization to arbitrary a was done by Okamoto–Sakajo–Wunsch [15]. When $a < 0$, advection and vortex stretching cooperate to cause a blow-up, as shown in [2]. When $a > 0$, they fight against each other and the picture is more interesting. For small a , smooth blow-up solutions were found by Elgindi–Jeong [9], Elgindi–Ghoul–Masmoudi [8] and Chen–Hou–Huang [3]. In general, C^α blow-ups were found for $|a\alpha|$ small enough. A numerical investigation of the behavior of the solution with different values of a can be found in Lushnikov–Silantyev–Siegel [14].

1.2 The result

This note improves on the known results on the C^α blow-ups. The ones constructed in [9] is exactly self-similar, i.e., of the form

$$w(x, t) = \frac{1}{1-t} W\left(\frac{x}{(1-t)^{(1+\lambda)/\alpha}}\right)$$

where $W \in C^\alpha$, but with the restriction that $1/\alpha \in \mathbb{Z}$. The construction in [8, 3] works for all $\alpha \in (0, 1)$, as long as $|a\alpha| \ll 1$, but the solution is not exactly self-similar, but only asymptotically so. In this note we fill the gap by constructing exactly self-similar blow-up solutions for all $\alpha \in (0, 1)$, provided that $|a\alpha| \ll 1$. Specifically we show that

Theorem 1. *There is $c > 0$ such that if $|a\alpha| < c$, then there are $W(\cdot; a) \in C^\alpha$ and $\lambda(a) \in \mathbb{R}$ such that*

$$w(x, t) = \frac{1}{1-t} W\left(\frac{x}{(1-t)^{1+\lambda(a)}}; a\right)$$

is a self-similar solution. Moreover, $W(\cdot; a)$ and $\lambda(a)$ are analytic in a .

1.3 The method

We mostly follow [9]. Plugging the ansatz in the equation

$$w_t + auw_x = u_x w, \quad u_x = Hw = -|\nabla|^{-1}w_x$$

we get the steady-state equation for W :

$$F := \left(\frac{1+\lambda}{\alpha} x + aU \right) W_x + (1 - U_x)W = 0, \quad U_x = HW.$$

The explicit solution when $a = 0$ (see (4.2)–(4.3) of [9])

$$\begin{aligned} (\bar{W}, \bar{\lambda}) &= \left(-\frac{2 \sin(\alpha\pi/2)|x|^\alpha \operatorname{sgn} x}{|x|^{2\alpha} + 2 \cos(\alpha\pi/2)|x|^\alpha + 1}, 0 \right), \\ \bar{U}_x &= H\bar{W} = \frac{2(\cos(\alpha\pi/2)|x|^\alpha + 1)}{|x|^{2\alpha} + 2 \cos(\alpha\pi/2)|x|^\alpha + 1} \end{aligned}$$

is odd and differentiable with respect to $|x|^\alpha$. It suggests the change of variable $\tilde{f}(x) = f(x^{1/\alpha})$, and the need to study the Hilbert transform

$$\begin{aligned} \widetilde{Hf}(x) &= Hf(x^{1/\alpha}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)dy}{x^{1/\alpha} - y} = \frac{1}{\pi} \int_0^{\infty} \frac{2yf(y)dy}{x^{2/\alpha} - y^2} \\ &= \frac{1}{\alpha\pi} \int_0^{\infty} \frac{2z^{2/\alpha-1}\tilde{f}(z)dz}{x^{2/\alpha} - z^{2/\alpha}} \quad (\text{with } f \text{ odd and } z = y^\alpha) \end{aligned}$$

in the new variable. In terms of the new variable we need to solve

$$F = (1 + \lambda)xW_x + a\widetilde{UW}_x + (1 - \widetilde{HW})\tilde{W} = 0, \quad U_x = HW.$$

In Section 3 we will generalize the estimates in [9] from the discrete range $1/\alpha \in \mathbb{Z}$ to the full range $\alpha \in (0, 1)$, using some delicate analysis of the integral kernel. Armed with these estimates, in Section 4 we will use the implicit function theorem to show that a perturbation of the explicit solution above exists as long as $|a\alpha|$ is small enough. The argument mostly follows [9], but is easier if we let F map $X := \{W \in H^2 : xW_x \in H^2\}$ to H^2 . This way all terms in F are automatically in H^2 : for the most difficult term $UW_x = (U/x)(xW_x)$ we will need the Hardy inequalities collected in Section 2. Then we only need the $H^2 \rightarrow X$ bound of $(dF)^{-1}$ for the implicit function theorem to work and give us the desired solution for small $|a\alpha|$.

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2 Hardy inequalities

Here we record some useful Hardy-type inequalities in Sobolev spaces.

Lemma 1 (Kufner–Persson Theorem 4.3). *If $p \in (1, \infty)$, $\gamma < (p - 1)/p$, $k \geq 1$ and $f(0) = \dots = f^{(k-1)}(0) = 0$, then*

$$\|x^{\gamma-k}f(x)\|_{L^p} \lesssim_{k,p,\gamma} \|x^\gamma f^{(k)}\|_{L^p}.$$

Definition 1. *For any integer $k \geq 0$, any number $p \in (1, \infty)$ and any weight $w(x) \geq 0$ define*

$$\|f\|_{W^{k,p}(w)} = \sum_{j=0}^{\infty} \|wf^{(j)}\|_{L^p}, \quad \|f\|_{\bar{W}^{k,p}(w)} = \|f\|_{W^{k,p}(w)} + \|xf'(x)\|_{W^{k,p}(w)}.$$

Let $W_-^{k,p}(w)$ and $\bar{W}_-^{k,p}(w)$ denote the subspace of odd functions in $W^{k,p}(w)$.

Definition 2.

$$If(x) = \int_0^x \frac{f(y) - f(0) - yf'(0)}{y^2} dy.$$

Lemma 2. *If $k \geq 0$ and $p \in (1, \infty)$ then*

$$\|(If)'\|_{W^{k,p}} \lesssim_{k,p,\gamma} \|f''\|_{W^{k,p}} \leq \|f\|_{W^{k+2,p}}.$$

Proof. Without loss of generality assume $f(0) = f'(0) = 0$. Then the result follows from Lemma 1 because $(x^{-2}f(x))^{(k)}$ is a linear combination of $x^{-2-j}f^{(k-j)}(x)$, $0 \leq j \leq k$. \square

Lemma 3. *If $k \geq 0$ and $p \in (1, \infty)$ then*

$$\begin{aligned} \left\| g(x) \int_0^x f(y) dy \right\|_{W^{k,p}} &\lesssim_k \sum_{l=0}^k \left\| g^{(l)}(x) \int_0^x f(y) dy \right\|_{L^p} \\ &+ \sum_{\substack{n \geq 1 \\ m+n \leq k}} \|g^{(m)} f^{(n-1)}\|_{L^p} \\ &\lesssim_{k,p,\gamma} \sum_{l=0}^k \sup |xg^{(l)}(x)| \|f\|_{L^p} \\ &+ 1_{k \geq 1} \|g\|_{C^{k-1}} \|f\|_{W^{k-1,p}}. \end{aligned} \quad (\text{by Lemma 1})$$

Remark 1. If we discount the term where all the derivatives fall on the integral, we have $1 \leq n \leq k - 1$ in the summation, so we only need the $W^{k-2,p}$ norm of f .

Lemma 4. If $k \geq 0$ and $p \in (1, \infty)$ then

$$\left\| \frac{xIf(x)}{1+x^2} \right\|_{W^{k,p}} \lesssim_{k,p,\gamma} \|f'\|_{W^{\max(k-1,1),p}} \leq \|f\|_{W^{\max(k,2),p}}.$$

Proof. By Lemma 3 (note that we have subtracted the term with all k derivatives hitting If) and Lemma 2,

$$\begin{aligned} \left\| \frac{d^k}{dx^k} \frac{xIf(x)}{1+x^2} - 1_{k \geq 1} \frac{x(If)^{(k)}(x)}{1+x^2} \right\|_{L^p} &\lesssim_{k,p,\gamma} \|(If)'\|_{W^{\max(k-2,0),p}} \\ &\lesssim_{k,p,\gamma} \|f'\|_{W^{\max(k-1,1),p}}. \end{aligned}$$

If $k = 0$ then there is nothing more to prove. If $k \geq 1$ then similarly

$$\left\| \frac{d^{k-1}}{dx^{k-1}} \frac{x(If)'(x)}{1+x^2} - 1_{k \geq 1} \frac{x(If)^{(k)}(x)}{1+x^2} \right\|_{L^p} \lesssim_{k,p,\gamma} \|f'\|_{W^{\max(k-1,1),p}}.$$

Also,

$$\begin{aligned} \left\| \frac{x(If)'(x)}{1+x^2} \right\|_{W^{k-1,p}} &= \left\| \frac{f(x) - f(0)}{(1+x^2)x} - \frac{f'(0)}{1+x^2} \right\|_{W^{k-1,p}} \\ &\lesssim_k \|1/(1+x^2)\|_{C^{k-1}} \|(f(x) - f(0))/x\|_{W^{k-1,p}} \\ &\quad + \|1/(1+x^2)\|_{W^{k-1,p}} |f'(0)| \\ &\lesssim_{k,p,\gamma} \|f'\|_{W^{\max(k-1,1),p}}. \quad (\text{by Lemma 1 and } W^{1,p} \subset C^0) \end{aligned}$$

□

3 Hilbert transform on Hölder functions

In this section we generalize the bounds for the Hilbert transform in [9].

Definition 3. For a function f let $\tilde{f}(x) = f(x^{1/\alpha})$ ($x \geq 0$).

For example,

$$\tilde{\tilde{W}}(x) = -\frac{2 \sin(\alpha\pi/2)x}{x^2 + 2 \cos(\alpha\pi/2)x + 1}, \quad \widetilde{H}\tilde{W}(x) = \frac{2(\cos(\alpha\pi/2)x + 1)}{x^2 + 2 \cos(\alpha\pi/2)x + 1}.$$

Remark 2. In this section we only consider functions defined on $\{x : x \geq 0\}$, unless stated otherwise.

Definition 4. For $r \geq 1$ (not necessarily an integer) define

$$H^{(r)}(f)(x) = \frac{1}{\pi} \int_0^\infty \frac{2ry^{2r-1}}{x^{2r} - y^{2r}} f(y) dy$$

and

$$\tilde{H}f = H^{(1/\alpha)}f$$

so that

$$\tilde{H}f(x) = (HEf(\cdot^\alpha))(x^{1/\alpha})$$

where $Ef(x) = f(|x|) \operatorname{sgn} x$ is the odd extension of f . Then for odd V ,

$$\tilde{H}\tilde{V}(x) = HEV(x^{1/\alpha}) = HV(x^{1/\alpha}) = \widetilde{HV}(x).$$

Before bounding this operator, we need some elementary inequalities.

Lemma 5. If $r \geq 1$ and $t \geq 0$ then

$$\frac{2rt^{2r-1}}{1-t^{2r}} \leq \frac{2t}{1-t^2} \leq \frac{2rt}{1-t^{2r}} \leq \frac{2rt^{2r-1}}{1-t^{2r}} + 2r.$$

Proof. First we show the first two inequalities. Clearing the denominator and canceling the factor $2t$ give $rt^{2r-2} - rt^{2r} \leq 1 - t^{2r} \leq r - rt^2$. The first inequality is equivalent to $(r-1)t^{2r} + 1 \geq rt^{2r-2}$, and the second equivalent to $t^{2r} + r - 1 \geq rt^2$. Both follow from Young's inequality. The last inequality is nothing but $|t - t^{2r-1}| \leq |1 - t^{2r}|$, which holds because t and t^{2r-1} are always between 1 and t^{2r} . \square

Lemma 6. For $r \geq 1$ we have $\|H^{(r)}\|_{L^2 \rightarrow L^2} \leq Cr$ for some constant C .

Proof. We have

$$H^{(1)}f(x) = \frac{1}{\pi} \int_0^\infty \frac{2yf(y)dy}{x^2 - y^2} = \frac{1}{\pi} \int_0^\infty \left(\frac{1}{x-y} - \frac{1}{x+y} \right) f(y) dy = HEf(x)$$

where $Ef(x) = f(|x|) \operatorname{sgn} x$, so $H^{(2)}$ is an isometry on $L^2(\mathbb{R}^+)$.

For general r we have

$$\pi H^{(r)} f(x) = \int_0^\infty \frac{2ry^{2r-1}}{x^{2r} - y^{2r}} f(y) dy = \int_0^\infty \frac{2rt^{2r-1}}{1 - t^{2r}} f(tx) dt \quad (y = tx)$$

so

$$\pi(H^{(r)} - H^{(1)})f(x) = \int_0^\infty \left(\frac{2rt^{2r-1}}{1 - t^{2r}} - \frac{2t}{1 - t^2} \right) f(tx) dt.$$

Since $\|f(tx)\|_{L_x^2} = \|f\|_{L^2}/\sqrt{t}$, we have

$$\|H^{(r)}\|_{L^2 \rightarrow L^2} \leq 1 + \frac{1}{\pi} \int_0^\infty \left(\frac{2t}{1 - t^2} - \frac{2rt^{2r-1}}{1 - t^{2r}} \right) \frac{dt}{\sqrt{t}}.$$

Note that the integrand is nonnegative by Lemma 5. By the same lemma,

$$\int_0^2 \left(\frac{2t}{1 - t^2} - \frac{2rt^{2r-1}}{1 - t^{2r}} \right) \frac{dt}{\sqrt{t}} \leq 2r \int_0^2 \frac{dt}{\sqrt{t}} = 4\sqrt{2}r.$$

For $t \geq 2$,

$$\frac{2t}{1 - t^2} - \frac{2rt^{2r-1}}{1 - t^{2r}} \leq \frac{2rt^{2r-1}}{t^{2r} - 1} \leq \frac{8r}{3t}$$

so

$$\int_2^\infty \left(\frac{2t}{1 - t^2} - \frac{2rt^{2r-1}}{1 - t^{2r}} \right) \frac{dt}{\sqrt{t}} \leq \int_2^\infty \frac{8rdt}{3t\sqrt{t}} = \frac{8\sqrt{2}}{3}r$$

and then

$$\|H^{(r)}\|_{L^2 \rightarrow L^2} \leq 1 + \frac{20\sqrt{2}}{3\pi}r \leq \left(1 + \frac{20\sqrt{2}}{3\pi} \right) r.$$

□

Lemma 7. For $r \geq 1$, if $f(0) = 0$ then for $k = 1, 2$ and $x \neq 0$,

$$(H^{(r)} f)^{(k)}(x) = \frac{1}{\pi} \int_0^\infty \frac{2rx^{2r-k}y^{k-1}}{x^{2r} - y^{2r}} f^{(k)}(y) dy.$$

Proof. Using $H^{(1)}f = HEf$, where $Ef(x) = f(|x|) \operatorname{sgn} x$, we see that the identity holds for $r = 1$.

For $r > 1$ we have (note the singularity of the kernel has been subtracted)

$$\pi((H^{(r)} - H^{(1)})f)'(x) = \int_0^\infty \partial_x \left(\frac{2ry^{2r-1}}{x^{2r} - y^{2r}} - \frac{2y}{x^2 - y^2} \right) f(y) dy.$$

By Euler's theorem on homogeneous functions ($xF_x + yF_y = 0$ for F homogeneous of degree 0) applied to the starred equality,

$$\begin{aligned}\partial_x \frac{y^{2r-1}}{x^{2r} - y^{2r}} &= \frac{1}{y} \partial_x \frac{y^{2r}}{x^{2r} - y^{2r}} \stackrel{*}{=} -\frac{1}{x} \partial_y \frac{y^{2r}}{x^{2r} - y^{2r}} = -\frac{1}{x} \partial_y \frac{x^{2r}}{x^{2r} - y^{2r}} \\ &= -\partial_y \frac{x^{2r-1}}{x^{2r} - y^{2r}}\end{aligned}$$

so

$$\pi((H^{(r)} - H^{(1)})f)'(x) = - \int_0^\infty \partial_y \left(\frac{2rx^{2r-1}}{x^{2r} - y^{2r}} - \frac{2x}{x^2 - y^2} \right) f(y) dy.$$

Since $f(0) = 0$ we can integrate by parts to get the identity for $k = 1$.

Similarly,

$$\partial_x \frac{x^{2r-1}}{x^{2r} - y^{2r}} = \frac{1}{y} \partial_x \frac{x^{2r-1}y}{x^{2r} - y^{2r}} = -\frac{1}{x} \partial_y \frac{x^{2r-1}y}{x^{2r} - y^{2r}} = -\partial_y \frac{x^{2r-2}y}{x^{2r} - y^{2r}}$$

so

$$\pi((H^{(r)} - H^{(1)})f)''(x) = - \int_0^\infty \partial_y \left(\frac{2rx^{2r-2}y}{x^{2r} - y^{2r}} - \frac{2y}{x^2 - y^2} \right) f'(y) dy.$$

Integrating by parts we get the identity for $k = 2$. This time we don't need $f'(0) = 0$ because the parenthesis vanishes at $y = 0$. \square

Lemma 8. *For $r \geq 1$, if $f(0) = 0$ then $\|(H^{(r)}f)''\|_{L^2} \leq Cr\|f''\|_{L^2}$ for some constant C .*

Proof. Since $f(0) = 0$, taking the second derivative commutes with E and H . Then $(H^{(1)}f)'' = (HEf)'' = HEf'' = H^{(1)}(f'')$, so $\|(H^{(1)}f)''\|_{L^2} = \|H^{(1)}(f'')\|_{L^2} = \|f''\|_{L^2}$.

For $r > 1$, we change variable as before to get

$$\|(H^{(r)}f)''\|_{L^2} \leq \left(1 + \frac{1}{\pi} \int_0^\infty \left(\frac{2rt}{1-t^{2r}} - \frac{2t}{1-t^2} \right) \frac{dt}{\sqrt{t}} \right) \|f\|_{H^2}.$$

Note that the integrand is nonnegative by Lemma 5. By the same lemma,

$$\int_0^2 \left(\frac{2rt}{1-t^{2r}} - \frac{2t}{1-t^2} \right) \frac{dt}{\sqrt{t}} \leq 2r \int_0^2 \frac{dt}{\sqrt{t}} = 4\sqrt{2}r.$$

For $t \geq 2$,

$$\frac{2rt}{1-t^{2r}} - \frac{2t}{1-t^2} \leq \frac{2t}{t^2-1} \leq \frac{8}{3t}$$

so

$$\int_2^\infty \left(\frac{2rt}{1-t^{2r}} - \frac{2t}{1-t^2} \right) \frac{dt}{\sqrt{t}} \leq \int_2^\infty \frac{8dt}{3t\sqrt{t}} = \frac{8\sqrt{2}}{3}$$

and then

$$\|(H^{(r)}f)''\|_{L^2} \leq \left(1 + \frac{1}{\pi} \left(4\sqrt{2}r + \frac{8\sqrt{2}}{3} \right) \right) \|f''\|_{L^2} \leq \left(1 + \frac{20\sqrt{2}}{3\pi} \right) r \|f\|_{H^2}.$$

□

By Lemma 6 and Lemma 8 we get

Lemma 9. *For $r \geq 1$ we have $\|H^{(r)}\|_{H_0^2 \rightarrow H^2} \leq Cr$ for some constant C , where H_0^2 denotes the space of H^2 functions vanishing at 0.*

Remark 3. *To bound higher derivatives of $H^{(r)}f$ in L^2 , more derivatives of f at 0 need to vanish.*

4 Hölder steady states for nonzero a

We first define the spaces which we are going to work with.

Definition 5. *Let $\bar{H}^2 = \{f \in H^2 : xf_x \in H^2\}$ with $\|f\|_{\bar{H}^2} = \|f\|_{H^2} + \|xf_x\|_{H^2}$. We subscript a space by 0 to indicate the subspace of functions that vanish at 0. For example, $\bar{H}_0^2 = \{f \in \bar{H}^2 : f(0) = 0\}$.*

By Lemm 2.2 of [9], for $0 < \alpha \leq 1$ and $f \in H_0^2$ we have $(\tilde{H}f)_x(0) = f_x(0) \cot(\alpha\pi/2)$.

Definition 6. *Let $X = \{V \in \bar{H}_0^2 : V_x(0) = 0\}$ and $Y = \{V \in H_0^2 : V_x(0) + 2 \sin(\alpha\pi/2) \tilde{H}V(0) = 0\}$.*

We will solve the steady-state equation $F = 0$ (see Section 1.3) using the implicit function theorem, so we need to find its differential.

$$dF_{(\bar{W},0,0)}(V, \mu, 0) = LV + \mu x \tilde{\bar{W}}_x$$

where

$$LV = V + xV_x - \tilde{\bar{W}} \tilde{H}V - V \tilde{H} \tilde{\bar{W}}.$$

Lemma 10. If $0 < \alpha \leq 1$, then L is an isomorphism from X to Y and

$$\begin{aligned} L^{-1}f(x) &= \frac{x(1-x^2)\sin(\alpha\pi/2)}{(1+2x\cos(\alpha\pi/2)+x^2)^2} \\ &\quad \times \int_0^x \frac{1-y^2}{y} \sin \frac{\alpha\pi}{2} g(y) dy + \left(\frac{1+y^2}{y} \cos \frac{\alpha\pi}{2} + 2 \right) h(y) dy \\ &\quad - \frac{x((1+x^2)\cos(\alpha\pi/2)+2x)}{(1+2x\cos(\alpha\pi/2)+x^2)^2} \\ &\quad \times \int_0^x - \left(\frac{1+y^2}{y} \cos \frac{\alpha\pi}{2} + 2 \right) g(y) + \frac{1-y^2}{y} \sin \frac{\alpha\pi}{2} h(y) dy \end{aligned}$$

where

$$g(x) = \frac{f(x)}{x} - \frac{f'(0)}{1+2x\cos(\alpha\pi/2)+x^2},$$

and

$$\begin{aligned} h(x) &= \frac{\tilde{H}f(x)}{x} - \frac{(1-x^2)\tilde{H}f(0)}{x(1+2x\cos(\alpha\pi/2)+x^2)} \\ &= \frac{\tilde{H}f(x)-\tilde{H}f(0)}{x} + \frac{2(\cos(\alpha\pi/2)+x)}{1+2x\cos(\alpha\pi/2)+x^2}\tilde{H}f(0) \\ &= \frac{\tilde{H}f(x)-\tilde{H}f(0)-x(\tilde{H}f)'(0)}{x} - \frac{2x(\cos\alpha\pi+x\cos(\alpha\pi/2))}{1+2x\cos(\alpha\pi/2)+x^2}\tilde{H}f(0). \end{aligned}$$

The norm of L^{-1} is bounded, uniformly in α .

This will be proved towards the end of the section.

Lemma 11. For $0 < \alpha \leq 1$ and $n = 0, \pm 1$,

$$\begin{aligned} T_{n,\alpha}f &= \frac{x}{1+x^2} \int_0^x y^n \left(\frac{f(y)}{y} - \frac{f'(0)}{1+2y\cos(\alpha\pi/2)+y^2} \right) dy, \\ S_{n,\alpha}f &= \frac{x^{1-\max(n,0)}}{1+x^2} \int_0^x y^n \left(\frac{\tilde{H}f(y)-\tilde{H}f(0)}{y} - \frac{2(\cos(\alpha\pi/2)+y)\tilde{H}f(0)}{1+2y\cos(\alpha\pi/2)+y^2} \right) dy \end{aligned}$$

are bounded from Y to H^2 , with $\|T_{n,\alpha}\|_{Y \rightarrow H^2} \leq C$ and $\|S_{n,\alpha}\|_{Y \rightarrow H^2} \leq C/\alpha$.

Proof. Clearly $T_{n,\alpha}f(0) = S_{n,\alpha}f(0) = 0$, so it remains to bound the H^2 norm.

For $T_{n,\alpha}$ we have

$$\begin{aligned}(T_{n,1} - T_{n,\alpha})f(x) &= \frac{xf'(0)}{1+x^2} \int_0^x \frac{2(1-\cos(\alpha\pi/2))y^{n+1}}{(1+y^2)(1+2y\cos(\alpha\pi/2)+y^2)} dy \\ &= \frac{xf'(0)}{1+x^2} g_{n,\alpha}(x)\end{aligned}$$

where all higher derivatives of $g_{n,\alpha}$ are bounded, uniformly in $\alpha \in [0, 1]$. $T_{n,1}$ has the desired bound, as shown in (4.15) of [9], so does $T_{n,\alpha}$ because $x/(1+x^2) \in H^\infty$ and $|f'(0)| \lesssim \|f\|_{H^2}$.

For $S_{-1,\alpha}$, we rewrite the integrand as

$$I\tilde{H}f(y) - \frac{2(\cos \alpha\pi + y \cos(\alpha\pi/2))}{1+2y\cos(\alpha\pi/2)+y^2} \tilde{H}f(0).$$

For the first term, by Lemma 4 and Lemma 9,

$$\left\| \frac{xI\tilde{H}f(x)}{1+x^2} \right\|_{H^2} \lesssim \|\tilde{H}f\|_{H^2} = \|H^{(1/\alpha)}f\|_{H^2} \lesssim \|f\|_{H^2}/\alpha.$$

For the second term we have

$$\begin{aligned}&\frac{x}{1+x^2} \int_0^x \frac{2(\cos \alpha\pi + y \cos(\alpha\pi/2))}{1+2y\cos(\alpha\pi/2)+y^2} dy \\ &= \frac{x}{1+x^2} \left(\cos \frac{\alpha\pi}{2} \ln(1+2x \cos \frac{\alpha\pi}{2} + x^2) - 2 \sin \frac{\alpha\pi}{2} \arctan \frac{x \sin \frac{\alpha\pi}{2}}{1+x \cos \frac{\alpha\pi}{2}} \right)\end{aligned}$$

whose \bar{H}^2 norm is bounded, uniformly in α , so the second term has the desired bound because $|\tilde{H}f(0)| \lesssim \|\tilde{H}f\|_{H^1} \lesssim \|f\|_{H^1}/\alpha$.

For $S_{n,\alpha}$ ($n = 0, 1$) we use the old integrand. By Lemma 3, Hardy's inequality and Lemma 9, the contribution of the first term in $S_{0,\alpha}$ is

$$\lesssim \left\| \frac{\tilde{H}f(x) - \tilde{H}f(0)}{x} \right\|_{H^1} + \|(\tilde{H}f)'\|_{H^1} \lesssim \|\tilde{H}f\|_{H^2} \lesssim \|f\|_{H^2}/\alpha$$

and the contribution of the first term in $S_{1,\alpha}$ is

$$\lesssim \|x/(1+x^2)\|_{C^2} \left\| \frac{1}{x} \int_0^x \tilde{H}f(y) dy \right\|_{H^2} \lesssim \|\tilde{H}f\|_{H^2} \lesssim \|f\|_{H^2}/\alpha.$$

For the second term in $S_{0,\alpha}$ we have

$$\frac{x}{1+x^2} \int_0^x \frac{2(\cos(\alpha\pi/2) + y)}{1+2y\cos(\alpha\pi/2)+y^2} dy = \frac{x \ln(1+2x\cos(\alpha\pi/2)+x^2)}{1+x^2}$$

and

$$\begin{aligned} & \frac{1}{1+x^2} \int_0^x \frac{2y(\cos(\alpha\pi/2) + y)}{1+2y\cos(\alpha\pi/2)+y^2} dy \\ &= \frac{1}{1+x^2} \left(2x - \int_0^x \frac{2y\cos(\alpha\pi/2)}{1+2y\cos(\alpha\pi/2)+y^2} dy \right) \\ &= \frac{1}{1+x^2} \left(2x - \cos \frac{\alpha\pi}{2} \ln \left(1+2x\cos \frac{\alpha\pi}{2} + x^2 \right) \right) \\ &+ \frac{2}{1+x^2} \frac{\cos^2 \frac{\alpha\pi}{2}}{\sin \frac{\alpha\pi}{2}} \arctan \frac{x \sin \frac{\alpha\pi}{2}}{1+x \cos \frac{\alpha\pi}{2}} \end{aligned}$$

whose H^2 norms are bounded, uniformly in α , so they also have the desired bound as before. \square

Then we upgrade the H^2 norm to the \bar{H}^2 norm.

Lemma 12. *For $0 < \alpha \leq 1$, $n = 0, \pm 1$, $T_{n,\alpha}$ and $S_{n,\alpha}$ are bounded from Y to \bar{H}_0^2 , with $\|T_{n,\alpha}\|_{Y \rightarrow \bar{H}^2} \leq C$ and $\|S_{n,\alpha}\|_{Y \rightarrow \bar{H}^2} \leq C/\alpha$.*

Proof. Since

$$(T_{n,1} - T_{n,\alpha})f = \frac{xf'(0)}{1+x^2} g_{n,\alpha}(x)$$

where all higher derivatives of $xg'_{n,\alpha}$ are bounded, $T_{n,\alpha}$ can be controlled as before.

For $S_{n,\alpha}$, as before we can fall the derivative on the integral and it remains to bound

$$\frac{x^{\min(n,0)+1}}{1+x^2} \left(\tilde{H}f(x) - \tilde{H}f(0) - \frac{2x(\cos(\alpha\pi/2) + x)\tilde{H}f(0)}{1+2x\cos(\alpha\pi/2)+x^2} \right)$$

whose H^2 norm is $\lesssim \|\tilde{H}f\|_{H^2} \lesssim \|f\|_{H^2}/\alpha$ by Lemma 9. \square

Proof of Lemma 10. The formal expression has been derived in Section 4.2 of [9]. The \bar{H}^2 bound of $L^{-1}f$ comes from those of $T_{n,\alpha}f$ and $S_{n,\alpha}f$. Note that the only terms in L^{-1} not covered by them are

$$\frac{-x^3 \sin(\alpha\pi/2)}{(1+2x\cos(\alpha\pi/2)+x^2)^2} \int_0^\infty y \cos(\alpha\pi/2) h(y) dy$$

and

$$-\frac{x^3 \cos(\alpha\pi/2)}{(1+2x\cos(\alpha\pi/2)+x^2)^2} \int_0^\infty -y \sin(\alpha\pi/2) h(y) dy$$

which cancel each other. To show the bound is uniform in α , it suffices to note that each appearance of $S_{n,\alpha}$ (via h) in L^{-1} is accompanied by a factor of $\sin(\alpha\pi/2) < 2\alpha$, which offsets the worse bounds of $S_{n,\alpha}$. Finally, $L^{-1}f(0) = (L^{-1}f)'(0) = 0$ because the integrals vanish at 0, and they are multiplied by a factor of x . \square

Recall that in terms of the new variable we need to solve

$$F := (1+\lambda)xW_x + a\widetilde{UW}_x + (1-\tilde{H}\tilde{W})\tilde{W} = 0, \quad U_x = HW.$$

Lemma 13. $F(W, \lambda, a) : \bar{H}_0^2 \times \mathbb{R} \times \mathbb{R} \rightarrow H_0^2$ is analytic.

Proof. Multiplicative closedness of H^2 takes care of every term but \widetilde{UW}_x , which we deal with now. Since $UW_x = (U/x)(xW_x)$, in terms of the variable $y = x^\alpha$ we have $\widetilde{UW}_x(y) = (\tilde{U}(y)/y^{1/\alpha})(\alpha y \tilde{W}_y)$. For the second factor we have $\|y\tilde{W}_y\|_{H^2} \leq \|\tilde{W}\|_{\bar{H}^2}$. To bound the first factor we start from the identity $xU_x = xHW$. In terms of y it is $\alpha y \tilde{U}_y = y^{1/\alpha} \widetilde{HW}$, so

$$\frac{\tilde{U}(y)}{y^{1/\alpha}} = \frac{1}{y^{1/\alpha}} \int_0^y \frac{\widetilde{HW}(z)}{\alpha z^{1-1/\alpha}} dz.$$

Taking the derivative and integrating by parts we get

$$\frac{d}{dy} \frac{\tilde{U}(y)}{y^{1/\alpha}} = \frac{\widetilde{HW}(y)}{\alpha y} - \frac{1}{\alpha y^{1+1/\alpha}} \int_0^y \frac{\widetilde{HW}(z)}{\alpha z^{1-1/\alpha}} dz = \frac{1}{\alpha y^{1+1/\alpha}} \int_0^y \widetilde{HW}'(z) z^{1/\alpha} dz$$

and similarly,

$$\frac{d^2}{dy^2} \frac{\tilde{U}(y)}{y^{1/\alpha}} = \frac{1}{\alpha y^{2+1/\alpha}} \int_0^y \widetilde{HW}''(z) z^{1+1/\alpha} dz.$$

By Lemma 1,

$$\|\tilde{U}(y)/y^{1/\alpha}\|_{L^2} \lesssim C_\alpha \|\widetilde{HW}\|_{L^2}/\alpha, \quad \|(\tilde{U}(y)/y^{1/\alpha})''\|_{L^2} \lesssim C'_\alpha \|\widetilde{HW}''\|_{L^2}/\alpha.$$

The order of C_α and C'_α can be found in Chapter 0 of Kufner–Persson: Since the weight are $y^{\pm 2-2/\alpha}$, it corresponds to $\epsilon = \pm 2 - 2/\alpha$ (and $p = 2$) in (0.4), so $C_\alpha \lesssim 1/(1-\epsilon) \leq 1/(2/\alpha - 1) \leq \alpha$ for $\alpha \in (0, 1)$. Hence

$$\|\tilde{U}(y)/y^{1/\alpha}\|_{H^2} \lesssim \|\widetilde{HW}\|_{H^2} \lesssim \|\tilde{W}\|_{H^2}/\alpha$$

by Lemma 9, so

$$\|\widetilde{UW}_x\|_{H^2} \lesssim \|\tilde{W}\|_{\bar{H}^2} \cdot \alpha \|\tilde{W}\|_{H^2} / \alpha \leq \|\tilde{W}\|_{\bar{H}^2}^2.$$

□

In particular,

$$dF_{(\tilde{W}, 0, 0)}(V, \mu, 0) = \mu x \tilde{\bar{W}}_x + x V_x - \tilde{\bar{W}} \tilde{H} V - V \tilde{H} \tilde{\bar{W}} + V = LV + \mu x \tilde{\bar{W}}_x.$$

Lemma 14. $dF_{(\tilde{W}, 0, 0)}(\cdot, \cdot, 0)$ is an isomorphism from $X \times \mathbb{R}$ to H_0^2 .

Proof. It suffices to show that $(x \tilde{\bar{W}}_x)_x + 2\tilde{H}(x \tilde{\bar{W}}_x)$ does not vanish at 0. This is because $(x \tilde{\bar{W}}_x)_x(0) = \tilde{\bar{W}}_x(0) = -2 \sin(\alpha\pi/2)$ and $\tilde{H}(x \tilde{\bar{W}}_x)(0) = H(x \tilde{\bar{W}}_x)/\alpha = 0$, □

Proof of Theorem 1. Let $W = \alpha\Omega$, $U = \alpha\Upsilon$ and $F = \alpha\Phi$. Then

$$\Phi = (1 + \lambda)x \tilde{\bar{\Omega}}_x + a\alpha \widetilde{\Upsilon\Omega}_x + (1 - \alpha \tilde{H} \tilde{\bar{\Omega}})\Omega.$$

Note that every term in Φ is analytic from \bar{H}_0^2 to H_0^2 , with all derivatives uniform in α (the bound for $\alpha \tilde{H} \tilde{\bar{\Omega}}$ comes from Lemma 9). Also

$$d\Phi_{(\tilde{\bar{\Omega}}, 0, 0)}(V, \mu, 0) = LV + \mu x \tilde{\bar{W}}_x / \alpha$$

where

$$\frac{(x \tilde{\bar{W}}_x)_x + 2\tilde{H}(x \tilde{\bar{W}}_x)}{\alpha} = -2 \frac{\sin(\alpha\pi/2)}{\alpha}$$

is bounded and bounded away from 0, uniformly in α . Hence $d\Phi_{(\tilde{\bar{\Omega}}, 0, 0)}(\cdot, \cdot, 0)$ is invertible and its inverse is analytic, with all derivatives bounded, uniformly in α . Now by the implicit function theorem, there is $c > 0$ such that if $|a\alpha| < c$, then there are $W(\cdot; a) \in \bar{H}_0^2$ and $\lambda(a) \in \mathbb{R}$ such that $\Phi(W(\cdot; a), \lambda(a), a) = 0$. Then

$$w(x, t) = \frac{\operatorname{sgn} x}{1-t} W\left(\left|\frac{x}{(1-t)^{1+\lambda(a)}}\right|^{\alpha}; a\right)$$

is a self-similar solution. Since $W(\cdot; a) \in \bar{H}_0^2 \subset C^1$, $w(\cdot, t) \in C^\alpha$. Finally, the analyticity of $W(\cdot; a)$ and $\lambda(a)$ in a follows from Lemma 13. □

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