

P1.1

We begin by massaging this complex exponential into a different form using Euler's formula for ease of use in the questions that follow

$$\frac{1}{2}e^{j\frac{\pi}{4}} = \frac{1}{2}(\cos(\frac{\pi}{4}) + j\sin(\frac{\pi}{4})) = \frac{1}{2}(\frac{\sqrt{2}}{2}) + j\frac{1}{2}(\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{4} + j\frac{\sqrt{2}}{4}$$

(a)

$$\text{Re}\{z\} = \frac{\sqrt{2}}{4}$$

(b)

$$\text{Im}\{z\} = \frac{\sqrt{2}}{4}$$

(c)

$$|z| = \sqrt{(\frac{\sqrt{2}}{4})^2 + (\frac{\sqrt{2}}{4})^2} = \sqrt{\frac{2}{16} + \frac{2}{16}} = \sqrt{\frac{4}{16}} = \frac{2}{4} = \frac{1}{2}$$

$$|z| = \frac{1}{2}$$

As a side note on this problem, any complex exponential without a coefficient (e^{jx} for some x) has a magnitude of 1. Thus, the magnitude simply becomes the leading coefficient, which is $\frac{1}{2}$ in this case.

(d)

$$\angle z = \frac{\pi}{4}$$

(e)

$$z^* = \frac{\sqrt{2}}{4} - j\frac{\sqrt{2}}{4}$$

(f)

$$z + z^* = (\frac{\sqrt{2}}{4} + j\frac{\sqrt{2}}{4}) + (\frac{\sqrt{2}}{4} - j\frac{\sqrt{2}}{4}) = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$$

$$z + z^* = \frac{\sqrt{2}}{2}$$

P1.2

(a)

$$\operatorname{Re}\{z\} = \frac{z + z^*}{2} = \frac{(\operatorname{Re}\{z\} + j\operatorname{Im}\{z\}) + (\operatorname{Re}\{z\} - j\operatorname{Im}\{z\})}{2} = \frac{2\operatorname{Re}\{z\}}{2} = \operatorname{Re}\{z\}$$

(b)

$$j\operatorname{Im}\{z\} = \frac{z - z^*}{2} = \frac{(\operatorname{Re}\{z\} + j\operatorname{Im}\{z\}) - (\operatorname{Re}\{z\} - j\operatorname{Im}\{z\})}{2} = \frac{2j\operatorname{Im}\{z\}}{2} = j\operatorname{Im}\{z\}$$

P1.3(a) According to Euler's formula, $\operatorname{Re}\{e^{j\theta}\} = \cos(\theta)$. Therefore, by P1.2(a),

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

For clarity you can also prove this directly like so

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \frac{(\cos(\theta) + j\sin(\theta)) + (\cos(\theta) - j\sin(\theta))}{2} = \frac{2\cos(\theta)}{2} = \cos(\theta)$$

(b) According to Euler's formula, $\operatorname{Im}\{e^{j\theta}\} = \sin(\theta)$. Therefore, by P1.2(b),

$$j\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{j2}$$

For clarity you can also prove this directly like so

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \frac{(\cos(\theta) + j\sin(\theta)) - (\cos(\theta) - j\sin(\theta))}{2j} = \frac{2j\sin(\theta)}{2j} = \sin(\theta)$$

P1.4

(a)(i)

$$z^* = re^{-j\theta}$$

(a)(ii)

$$z^2 = (re^{j\theta})^2 = r^2 e^{j2\theta}$$

$$z^2 = r^2 e^{j2\theta}$$

(a)(iii)

$$jz = jre^{j\theta} = e^{\frac{\pi}{2}} re^{j\theta} = re^{j\theta + \frac{\pi}{2}}$$

$$jz = re^{j\theta + \frac{\pi}{2}}$$

(a)(iv)

$$zz^* = re^{j\theta} re^{-j\theta} = r^2 e^{j\theta - j\theta} = r^2 e^0 = r^2$$

$$zz^* = r^2$$

(a)(v)

$$\frac{z}{z^*} = z \frac{1}{z^*} = re^{j\theta} \frac{1}{re^{-j\theta}} = re^{j\theta} re^{j\theta} = r^2 e^{j\theta + j\theta} = r^2 e^{j2\theta} =$$

$$zz^* = r^2 e^{j2\theta}$$

(a)(vi)

$$\frac{1}{z} = \frac{1}{re^{j\theta}} = re^{-j\theta}$$

$$\frac{1}{z} = re^{-j\theta}$$

P1.5

$$\begin{aligned} 2 \sin\left(\frac{\alpha}{2}\right) e^{j\frac{\alpha-\pi}{2}} &= 2 \frac{e^{j\frac{\alpha}{2}} - e^{-j\frac{\alpha}{2}}}{2j} e^{j(\frac{\alpha}{2}-\frac{\pi}{2})} = \frac{e^{j(\frac{\alpha}{2}+\frac{\alpha}{2}-\frac{\pi}{2})} - e^{j(\frac{\alpha}{2}-\frac{\alpha}{2}-\frac{\pi}{2})}}{j} = \frac{e^{j(\alpha-\frac{\pi}{2})} - e^{-j\frac{\pi}{2}}}{j} \\ &= \frac{e^{j\alpha} e^{-j\frac{\pi}{2}} - e^{-j\frac{\pi}{2}}}{j} = \frac{e^{-j\frac{\pi}{2}}(e^{j\alpha} - 1)}{j} = \frac{(\cos(\frac{\pi}{2}) - j \sin(\frac{\pi}{2}))(e^{j\alpha} - 1)}{j} = \frac{-j(e^{j\alpha} - 1)}{j} = (1 - e^{j\alpha}) \end{aligned}$$

P1.6