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An Algorithm for Packing Squares

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An algorithm is presented that can be used to pack sets of squares (or rectangles) into rectangles. The algorithm is applied to three open problems and will show how the best known results can be improved by a factor of at least 6×10^6 in the first two problems and 2×10^6 in the third. © 1998 Academic Press, Inc.

PROBLEM NUMBER 1

Moser originally noted that since $\sum_{i=2}^{\infty} 1/i^2 = \pi^2/6 - 1$ it is reasonable to ask whether the set of squares with sides of length 1/2, 1/3, 1/4, ..., which we will call the reciprocal squares, can be packed in a rectangle of area $\pi^2/6 - 1$. Failing that, find the smallest ε such that the reciprocal squares can be packed in a rectangle of area $(\pi^2/6 - 1) + \varepsilon$. The problem also appears as part of D5 in *Unsolved Problems in Geometry* [1]. The history of record holders is as follows:

- A. Meir and L. Moser (1968), a square of side 5/6 which shows that $\varepsilon < 1/20$ (they also show that this is the smallest possible square) [7].
- Derek Jennings (1994), a rectangle of dimensions $47/60 \times 5/6$ which shows that $\varepsilon < 1/127$ [5].
- Karen Ball (1996), a rectangle of dimensions $629/1000 \times 31/30$ which shows that $\varepsilon < 1/198$ [4].

Richard Guy showed me some of his efforts and convinced me that the results in the literature could be much improved. While I was trying to automate some of Guy's packing techniques I discovered a simple algorithm which allows me to pack the squares in a very efficient manner.

Throughout the paper the width of a rectangle will always refer to the shorter side and the length will always refer to the longer side. Moreover, when it is said that a rectangle has dimensions $a \times b$ the width will always be written first.

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Pack the reciprocal squares in decreasing order of size into a rectangle of dimensions $1/2 \times 2(\pi^2/6 - 1)$. Start by placing the 1/2 square in an obvious manner, leaving the task of packing the squares 1/3, 1/4, 1/5... into a rectangle of dimensions $1/2 \times (\pi^2/3 - 2 - 1/2)$.

First place the square 1/3 into a corner and cut the remaining area into two rectangular pieces. Note that there are two choices of how to make the cut: we will cut as in Fig. 1, leaving two rectangles, $R_1 = (\pi^2/3 - 17/6) \times 1/2$ and $R_2 = 1/6 \times 1/3$.

Now place the square 1/4 in a corner of the smallest width rectangle into which it will fit. Since 1/4 will not fit in R_2 we must place it in R_1. This exemplifies Rule 1:

Rule 1. Always place the next square in a corner of the smallest width rectangle into which it will fit.

Cut R_1 into two smaller rectangular pieces and once again we have two choices. Rule 2 will tell us which way to make the cut:

Rule 2. After placing a square into the corner of a rectangle, always cut the remaining area into two rectangular pieces by cutting from the free corner of the square to the longer side of the original rectangle.

Hence, after placing square 1/4 all the remaining area will be found in the rectangles with dimensions $1/6 \times 1/3$, $1/4 \times 1/2$, and $(\pi^2/3 - 37/12) \times 1/2$ (see Fig. 2).

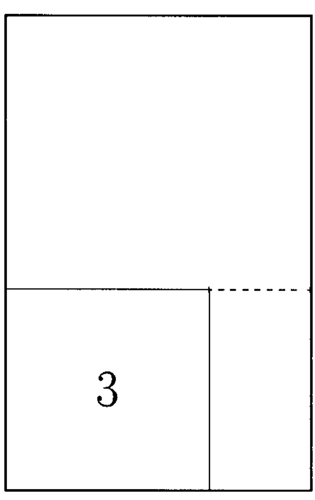

Figure 1

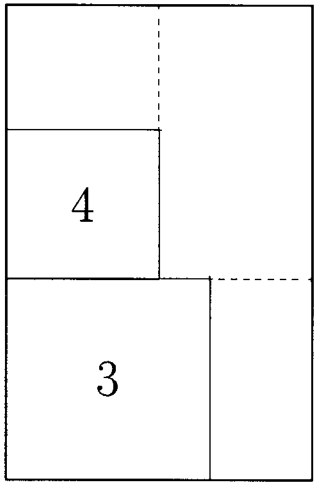

Figure 2

Next place square 1/5 which will not fit in the first rectangle but will fit in the other two. Applying Rule 1 we must place square 1/5 in the rectangle $(\pi^2/3 - 37/12) \times 1/2$ (since $\pi^2/3 - 37/12 < 1/4$). After applying Rule 2 we are left with Fig. 3.

Square 1/6 now fits perfectly into the rectangle $R = 1/6 \times 1/3$ which is where it goes by Rule 1. Apply Rule 2 to replace rectangle R by two new

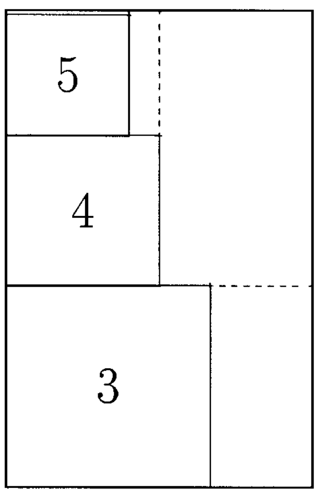

Figure 3

rectangles, one with the dimensions $S = 1/6 \times 1/6$ and one with the dimensions $0 \times 1/6$ which we may forget since it has no area.

Apply Rule 1 and find that square 1/7 goes on top of square 1/6 in Fig. 4 and into rectangle S (which is actually a square!). Since rectangle S has no longer side, cut to an arbitrary side since the set of resulting rectangles will be the same with either choice.

There are a few things to note:

- When we place a square in the corner of a rectangle it does not matter which corner we use. In the figures the square is always drawn in the lower left-hand corner of the chosen rectangle.
- Where the rectangles lie in relation to one another is not important to the algorithm. Once a cut is made we may consider the two resulting rectangles as completely disjoint objects. If we follow the algorithm we will never split a square between two or more rectangles.
- On occasion a square will fit equally well into two or more rectangles. In this case choose to place the square in the rectangle with the smallest length. Note that using the other option, placing the square in the rectangle with the largest length when the widths are the same, gives rise to a second algorithm which appears to be just as good as the first.

If we continue using the algorithm, after we have placed square 1/1000 we will have the packing shown in Fig. 5. The squares 1/3, through 1/99 are labelled with the numbers 3 through 99. The smaller squares contain only a dot and the smallest squares are barely visible.

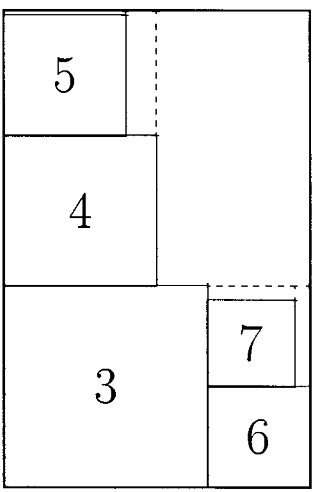

Figure 4

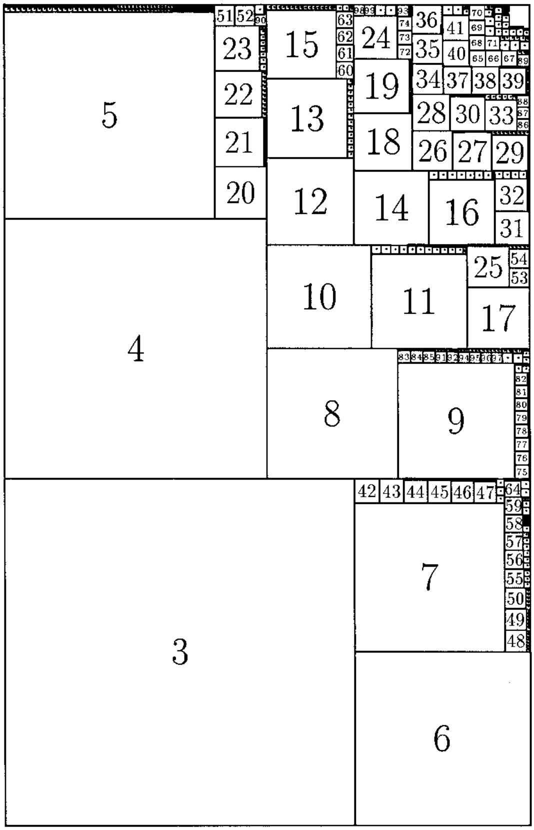

Fig. 5. After placing square 1/1000.

I apologize to the observant reader who notices imperfections in the figures presented. In particular, some of the smaller squares appear to overlap with other squares. This is due to a limitation of the graphical language used to create the pictures.

Of course, we can use the algorithm to pack even more squares, as my computer did, but first a lemma.

LEMMA 1. *All the reciprocal squares from n to infinity can be packed in a rectangle of length l and width w provided*

$$n \geqslant \frac{1+l}{wl}.$$

Proof. We will pack the squares from n to infinity in rows of length no greater than l.

Define $n_0 = n$. In the first row will go the squares n_0 to $n_1 - 1$ and in the second row will go squares n_1 to $n_2 - 1$ and in the ith row will go the squares n_{i-1} to $n_i - 1$. Take $n_{i+1} = n_i \lfloor (1+l) \rfloor$ for $i \geqslant 0$ and hence $n_i \leqslant n_0 (1+l)^i$ for $i \geqslant 0$. Then

$$\sum_{k=n_i}^{n_{i+1}-1} \frac{1}{k} \leqslant \frac{n_{i+1} - n_i}{n_i} \leqslant l$$

and we can be sure that n_{i-1} to $n_i - 1$ will fit in the ith row. Hence all the squares from n to infinity will fit in our rectangle provided.

$$\begin{aligned} w &\geqslant \sum_{i=0}^{\infty} \frac{1}{n_i} \\ &\geqslant \frac{1}{n_0} \sum_{i=0}^{\infty} \left(\frac{1}{1+l} \right)^i \\ &= \frac{1+l}{n_0 l}, \end{aligned}$$

which gives the result.

More sophisiticated packings are possible. ∎

After my computer packed the 10^9th square the largest rectangle R had a length and width each greater than 0.00001903. By the lemma, squares 2761408696 and on will fit in R. We are left with the task of finding a place to pack the squares $10^9 + 1$ to 2761408695. These can be packed in a rectangle of length 1/2 and width $1.606553066 \times 10^{-9}$ as shown in Fig. 6. Adjoining this strip to the original rectangle of dimensions $1/2 \times 2(\pi^2/6 - 1)$ shows that $\varepsilon < 1/1244918662$.

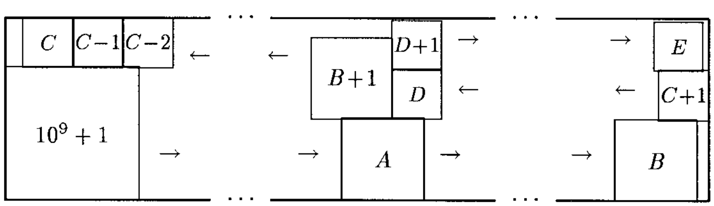

Fig. 6. Pack the squares $1/(10^9 + 1)$ to $1/E = 1/2761408695$ in the manner shown where $A = 1622971324$, $B = 1648721271$, $C = 2675827341$, and $D = 2718281828$. The highest horizontal edge belongs to the square $C - 2$ so that a rectangle of width $1/(C-2) + 1/(10^9 + 1) <$ $1.606553066 \times 10^{-9}$ is large enough to acommodate this finite set of squares. The reader will find Remark 1 useful in verifying these numbers.

Remark 1. A useful tool for ad hoc packing of reciprocal squares is

$$\frac{1}{n+1}+\cdots+\frac{1}{n+m}=\ln\frac{n+m}{n}-\frac{m}{2n(n+m)}+\frac{m(2n+m)}{12n^2(n+m)^2}-\cdots,$$

where the coefficients $-1/2$, $1/12$, $-1/120$, $1/1344$, $-1/1920$, ... in terms $k=1, 2, 4, 6, 8, \ldots$ are $2^{1-k}B_k$ with B_k the kth Bernoulli number.

PROBLEM NUMBER 2

A similar question is finding the smallest ε such that the set of squares of sides $1/(2n+1)$ for $n=1, 2, 3, \ldots$ can be packed in a rectangle of area $\pi^2/8-1+\varepsilon$. The record holders are:

- Jennings (1995), a rectangle of dimensions $4/9\times 8/15$ which shows that $\varepsilon<1/299$ [6].
- Bálint, a rectangle of size $15182/43407\times 71/105$ which shows that $\varepsilon<1/365$ [3].

We will try to fit all of the odd reciprocal squares into a rectangle of dimensions $1/3\times 3(\pi^2/8-1)$. Apply the algorithm directly and you will construct the packing shown in Fig. 7 and will discover that the square 1/33 cannot be placed! This is an example of a phenomenon that any square packer will note. The hardest squares to fit are the largest squares.

In this case it is not hard to fix the problem, we simply change the order in which we place the squares. In particular, place square 1/11 before placing square 1/9. We get the packing shown in Fig. 8 after placing square 1/999. The squares 1/3 through 1/99 are labelled with the odd numbers 3 through 99. As before smaller squares contain only a dot.

The corresponding lemma in this case is:

LEMMA 2. *All the odd reciprocal squares from n to infinity can be packed in a rectangle of length l and width w, provided*

$$n\geqslant\frac{1+2l}{2wl}.$$

Proof. We will pack the squares from n to infinity in rows of length no greater than l.

Define $n_0=n$. In the first row will go the squares n_0 to n_1-2 and in the second row will go squares n_1 to n_2-2 and in the ith row will go the

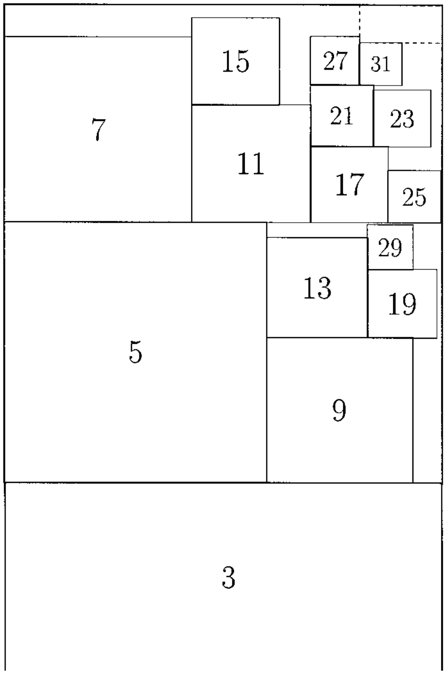

Fig. 7. After placing square 1/31. The largest width rectangle created by the algorithm is shown with the dashed lines and has width less than 1/33. We cannot continue.

squares n_{i-1} to $n_i - 2$. Assume that n_i is odd for all i. Take $n_{i+1} = n_i \lfloor (1+2l) \rfloor$ for $i \geqslant 0$ and hence $n_i \leqslant n_0 (1+2l)^i$ for $i \geqslant 0$. Then

$$\sum_{k=(n_i-1)/2}^{(n_{i+1}-1)/2-1} \frac{1}{2k+1} \leqslant \frac{n_{i+1}-n_i}{2n_i} \leqslant l$$

and we can be sure that n_{i-1} to $n_i - 2$ will fit in the ith row. Hence all the squares from n to infinity will fit in our rectangle, provided

$$w \geqslant \sum_{i=0}^{\infty} \frac{1}{n_i} \geqslant \frac{1}{n_0} \sum_{i=0}^{\infty} \left(\frac{1}{1+2l}\right)^i = \frac{1+2l}{2n_0 l}$$

which gives the result. ∎

After the computer placed square $10^9 - 1$ we find that the largest unfillled rectangle has length and width each greater than 0.000013293. Apply the lemma to find that squares 2829668375 and on will fit in R and we are left with the task of placing squares $10^9 + 1$ to 2829668373. Pack these in a strip of dimensions $(1.344586785 \times 10^{-9}) \times 1/3$ (similar to the

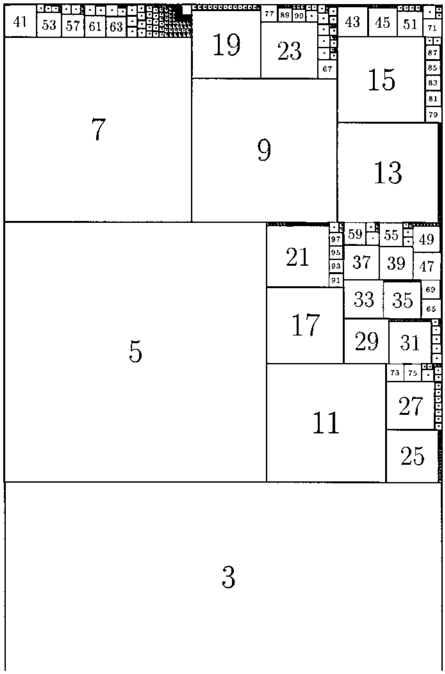

Fig. 8. After placing the odd squares 1/3 to 1/999.

packing in Fig. 6 except only two rows are required). Adjoin this strip to the original rectangle which shows that $\varepsilon < 1/2231168737$.

PROBLEM NUMBER 3

Possibly the most aesthetically pleasing of the three problems is to ask for the smallest ε such that the set of rectangles of dimensions $1/(n+1) \times 1/n$ will fit in a square of side $1+\varepsilon$ (since $\sum_{n=1}^{\infty} 1/n(n+1) = 1$). Some records have been set in the past:

- Meir and Moser (1968), a square of side $1 + 1/30$ [7],
- Jennings (1995), a square of side $1 + 1/203$ [6],
- Bálint, a square of side $1 + 1/500$ [3],

although Bálint's result is already mentioned in [2]. If we allow ourselves to pack in rectangles instead of squares, Bálint has shown a packing in an area of less that 1.0024 [3].

We will try to pack these rectangles, in decreasing order of size, into the unit square using a slightly different algorithm. To avoid confusion we will call the rectangle of dimensions $1/(n+1) \times 1/n$ the P_n-rectangle and the unused area inside the unit square will be contained in *boxes*. Starting with $n=1$ and one box which is the unit square follow these two rules:

Rule 1. Place the rectangle P_n in a corner of the smallest width box into which it will fit under either orientation. If P_n fits equally well in two or more boxes choose to place it in the box with the shortest length.

Rule 2. After placing a rectangle in the corner of a box, always cut the remaining area into two rectangular pieces by cutting from the corner of the rectangle to the longer side of the original box.

After placing rectangle P_{1000} the packing is as shown in Fig. 9. Rectangles P_1 and P_2 are placed in the obvious way so that the rest of the unit square shown in the figure has dimensions $1/2 \times 2/3$. Rectangles P_3 through P_{99} are labelled 3 through 99 and as usual the smaller P-rectangles contain a dot.

Using this algorithm the computer placed rectangles P_1 to P_{10^9} at which time the largest box, B, had length and width greater than 0.000018831. We can overestimate the area of a P_n-rectangle by assuming it is the square of side $1/n$ to which we can apply Lemma 1 directly. Doing so we find that rectangles $P_{2820079889}$ and on will fit into B. If we make our original box slightly larger, say to have side $1+1/(10^9+1)$, one can easily find a home

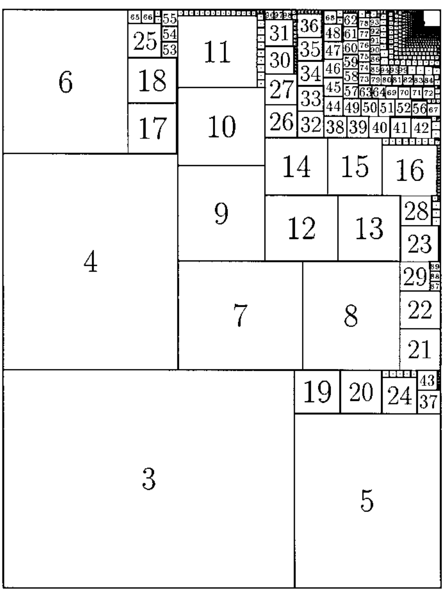

Fig. 9. After placing P_{1000}.

for P_{10^9+1} through $P_{2820079888}$ in the resulting gnomon. Hence we have exhibited a packing in a square of side $1+1/(10^9+1)$.

If we allow ourselves to pack in a rectangle we can fit P_{10^9+1} through $P_{2820079888}$ in a strip of dimensions $1/(10^9+1)\times 1$ which gives a packing in a rectangle of area $1+1(10^9+1)$.

DISCUSSION

The results presented in this paper can easily be improved. One can improve the lemmas, which I left simple purposely so as not to distract from the power of the algorithm. One can also ask the computer to pack even further! The runs presented, up to 10^9 in each case, took only a few hours to complete.

The point is that there is no reason for us to stop where we did. This would suggest that in all three cases perfect packings exist, that is, $\varepsilon=0$. It also appears that this algorithm will construct perfect packings, although that may be a difficult thing to prove.

ACKNOWLEDGMENT

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