Hénon-like Maps and Renormalisation

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We generalise the period doubling operator on the space of Hénon-like maps introduced by de Carvalho, Lyubich and Martens to those of arbitrary stationary combinatorics.

Hénon-like Maps

Let $\mathcal{H}_\Omega(\bar\varepsilon)$ denote the space Hénon-like maps $F \in C^\omega(B,B)$ satisfying the following properties:

- F is an orientation preserving diffeomorphism onto its image;
- F is expressible as

$$F(x,y) = (f(x) - \varepsilon(x,y), x) \tag{1}$$

- $f\colon J \to J$ is a real-analytic unimodal map
- $\varepsilon\colon B \to \mathbb{R}$ is real-analytic and non-zero
- f and ϵ have holomorphic extensions to Ω_x and Ω respectively;
- $\|\varepsilon\|_\Omega \le \bar\varepsilon$. We call ε an *$\bar\varepsilon$-thickening*.

Let $\mathcal{U}$ denote the space of unimodal maps on J. Let υ be a unimodal permutation on $W = \{0, 1, \ldots, p-1\}$.

Then there exists a subspace $\mathcal{U}_\upsilon \subset \mathcal{U}$ on which the *renormalisation operator* of type υ, $\mathcal{R}_\mathcal{U}\colon \mathcal{U}_\upsilon \to \mathcal{U}$, is defined.

Theorem

For each υ there exist $C, \bar\varepsilon_0 > 0$ and a polydisk $\Omega \subset \mathbb{C}$, such that: For any $\bar\varepsilon \in (0, \bar\varepsilon_0)$ there is a subspace $\mathcal{H}_{\Omega,\upsilon}(\bar\varepsilon)$ of $\mathcal{H}_\Omega(\bar\varepsilon)$ which contains $\mathcal{U}_\upsilon$ and an operator

$$\mathcal{R}\colon \mathcal{H}_{\Omega,\upsilon}(\bar\varepsilon) \to \mathcal{H}_\Omega(C\bar\varepsilon^p) \subset \mathcal{H}_\Omega(\bar\varepsilon)$$

which is a continuous extension of $\mathcal{R}_\mathcal{U}$.

The Renormalisation Operator

More precisely, given $F(x,y) = (\phi(x,y), x) \in \mathcal{H}_\Omega(\bar\varepsilon)$ there is a map $H(x,y)$, called the *horizontal diffeomorphism*, defined on a subdomain of Ω. Then F is *pre-renormalisable* if $G = HF^pH^{-1}$ has an invariant square symmetric about the diagonal $\{x = y\}$. Define the *renormalisation of F* by

$$\mathcal{R}F = IGI^{-1} = IHF^pH^{-1}I^{-1} = \Psi^{-1}F^p\Psi \tag{2}$$

where I is a suitable affine map and $\Psi = H^{-1} \circ I^{-1}$ is a non-affine map called the *Scope Map*.

It is know that the unimodal renormalisation operator $\mathcal{R}_\mathcal{U}$ has a unique hyperbolic fixed point f_* with codim.-one stable manifold.

The Renormalisation Operator ctd.

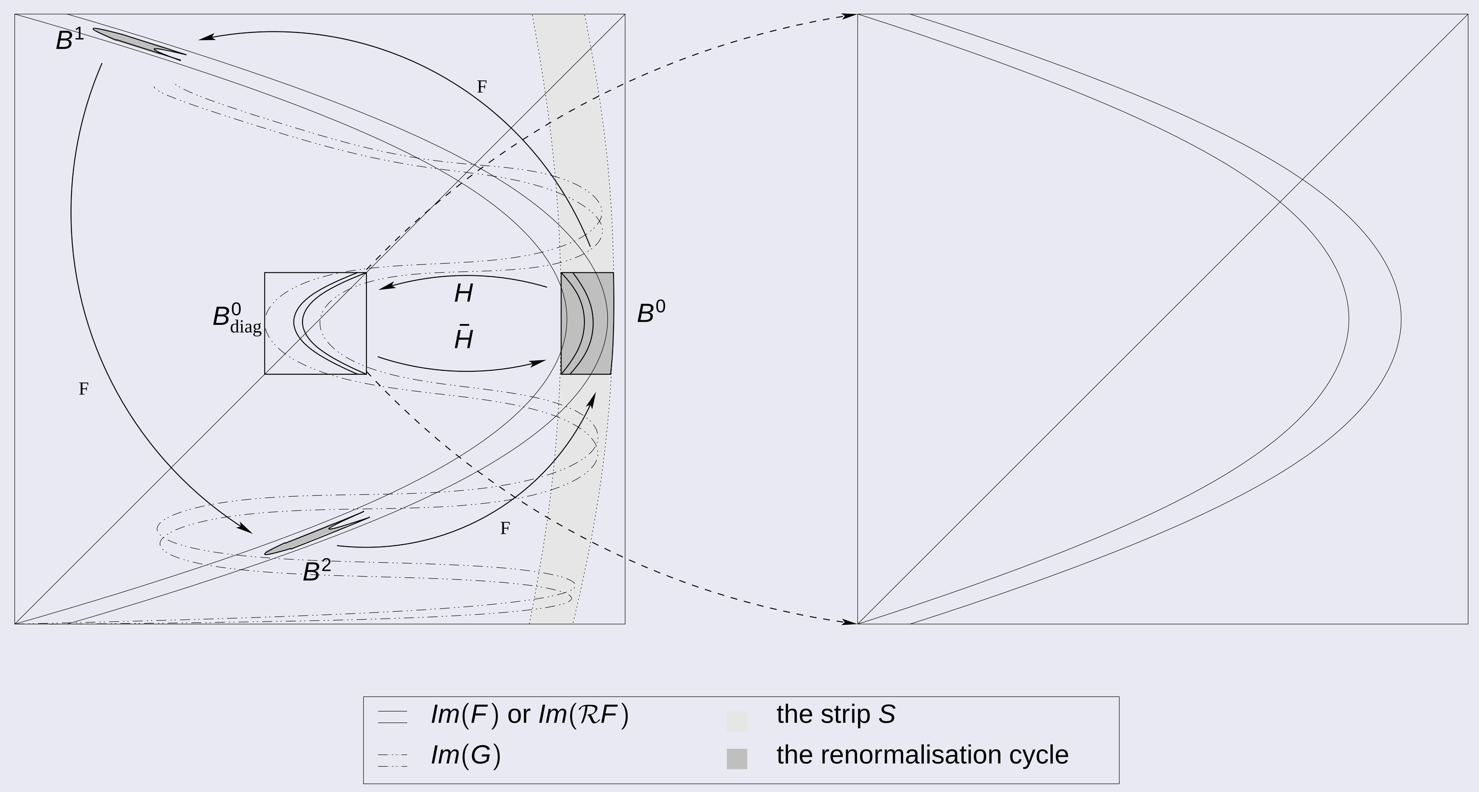

Figure: A renormalisable Hénon-like map whose combinatorial type is period tripling. Here the dashed lines represent the image of the square B under the pre-renormalisation G.

Theorem

For each υ there exists a $\bar\varepsilon_0 > 0$ such that for all $0 < \bar\varepsilon < \bar\varepsilon_0$

- *the operator $\mathcal{R}\colon \mathcal{H}_{\Omega,\upsilon}(\bar\varepsilon) \to \mathcal{H}_\Omega(\bar\varepsilon)$ has a unique fixed point $F_* = (f_*, \pi_x)$ where f_* is the fixed point of $\mathcal{R}_\mathcal{U}$;*
- *F_* is hyperbolic and has a codimension-one stable manifold.*

Infinitely Renormalisable Maps

Let $\mathcal{I}_{\Omega,\upsilon}(\bar\varepsilon) \subset \mathcal{H}_\Omega(\bar\varepsilon)$ denote the space of infinitely renormalisable maps. Let

- W^n denote the set of words $\mathbf{w}$ of length n,
- W^* denote the set of words $\mathbf{w}$ of arbitrary finite length,
- $\bar W$ denote the set of words $\mathbf{w}$ of infinite length.

We endow both W^* and $\bar W$ with the structure of a p-adic adding machine, denoted by $\mathbf{w} \mapsto 1 + \mathbf{w}$.

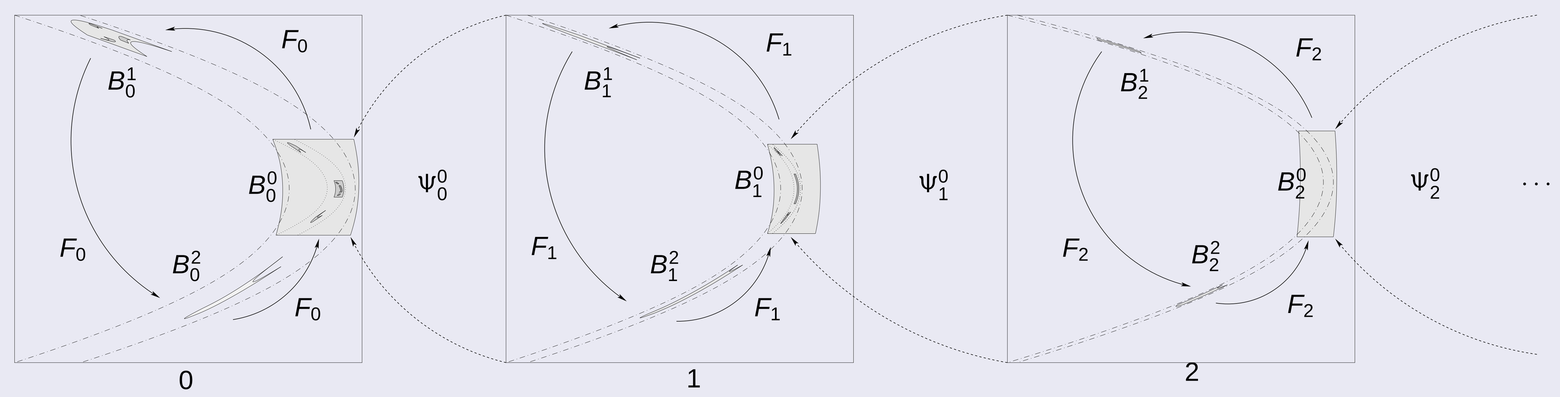

Figure: Scope maps for a period-3 infinitely renormalisable Hénon-like map.

For $F \in \mathcal{I}_\Omega(\bar\varepsilon)$ let $F_n = \mathcal{R}^nF$. Let $\Psi_n = \Psi(F_n)$ be called the *Scope Map of height n*. For $w \in W$, $\mathbf{w} = w_0 \ldots w_{m-1} \in W^m$ let

- $\Psi_n^w = F_n^{\circ w} \circ \Psi_n$;
- $\Psi_n^\mathbf{w} = \Psi_n^{w_0} \circ \cdots \circ \Psi_{n+m}^{w_{m-1}}$;
- $B_n^\mathbf{w} = \Psi_n^\mathbf{w}(B)$.

The maps $\Psi_n^\mathbf{w}$ are called the *$\mathbf{w}$-Scope Maps at height n*. The $B_n^\mathbf{w}$ are called the *canonical boxes of height n* and the collection $\underline{B}$ of all canonical boxes is called the *canonical boxing*.

Universality, Non-Rigidity and Unbounded Geometry

Theorem (Invariant Cantor Set)

Let $F \in \mathcal{I}_\Omega(\bar\varepsilon)$. Then there exists an F-invariant Cantor set $\mathcal{O} \subset B$ upon which F acts as the p-adic adding machine. Moreover, there exists a unique F-invariant measure μ whose support is the Cantor set $\mathcal{O}$.

For an $F \in \mathcal{I}_{\Omega,\upsilon}(\varepsilon)$ we now define the *Average Jacobian* to be

$$b(F) = \exp \int \log \mathrm{Jac}_z F d\mu(z) \tag{3}$$

For the Hénon maps $F(x,y) = (a - x^2 - by, x)$ this is just b.

Theorem (Universality)

Let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar\varepsilon)$. Then

$$F_n(x,y) = (f_n(x) - a(x)yb^{p^n}(1 + O(\rho^n)), x) \tag{4}$$

where b is the average Jacobian of F, $f_n \in \mathcal{U}$ and $f_n \to f_*$ exponentially and $a \in C^\omega(I, \mathbb{R})$ and $0 < \rho < 1$ are universal.

This universal limiting behaviour also occurs in the unimodal theory. However, in contrast, we get the following.

Theorem (Non-Rigidity)

Let $F_0, F_1 \in \mathcal{I}_\Omega(\bar\varepsilon)$ be two infinitely renormalisable Hénon-like maps. Let us denote

- *their respective average Jacobians by b_0, b_1;*
- *their respective Cantor sets by $\mathcal{O}_0, \mathcal{O}_1$.*

Then any conjugation $\Gamma\colon \mathcal{O}_0 \to \mathcal{O}_1$ sending τ_0 to τ_1 is at most C^α where

$$\alpha \le \frac{1}{2}\left(1 + \frac{\log b_0}{\log b_1}\right) \tag{5}$$

The boxing $\underline{B} = \{B^\mathbf{w}\}_{\mathbf{w} \in W_p^*}$ has *bounded geometry* if there exist constants $0 < \kappa < 1 < C$, such that for all $\mathbf{w} \in W_p^*$ and $w, \tilde w \in W_p$,

$$C^{-1}\operatorname{dist}(B^{\mathbf{w}w}, B^{\mathbf{w}\tilde w}) < \operatorname{diam}(B^{\mathbf{w}w}) < C\operatorname{dist}(B^{\mathbf{w}w}, B^{\mathbf{w}\tilde w}),$$

$$\kappa \operatorname{diam}(B^\mathbf{w}) < \operatorname{diam}(B^{\mathbf{w}w}) < (1-\kappa)\operatorname{diam}(B^\mathbf{w}).$$

We will say that $\mathcal{O}$ has *bounded geometry* if there exists a boxing $B^\mathbf{w}$ of $\mathcal{O}$ with bounded geometry. Otherwise $\mathcal{O}$ has *unbounded geometry*.

Theorem (Generic Unbounded Geometry)

Given a one-parameter family $F_b \in \mathcal{I}_\Omega(\bar\varepsilon)$ such that $b(F_b) = b$ there exists $b_0 > 0$ and $S \subset (0, b_0]$, a dense G_δ set with full (relative) Lebesgue measure such that F_b has unbounded geometry whenever $b \in S$.