PLACEHOLDER LOGO

Tiling the Unit Square

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PLACEHOLDER LOGO

Introduction

Through fractional decomposition, $\frac{1}{k} \cdot \frac{1}{k+1}$ can be rewritten as $\frac{1}{k} - \frac{1}{k+1}$

As a consequence, $\sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{1}{n+1}$ can be evaluated as a telescoping series.

$$\sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{1}{n+1} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{\infty} = 1$$

This unintuitive result begs the answer to the following:

Problem: Can the unit square $[0,1]^2$ be tiled with rectangles of sides $\frac{1}{n} \times \frac{1}{n+1}$ for $n = 1, 2, \ldots$

Relaxing the constraint gives a more general problem:

Problem: Find the smallest square $[0, 1+\epsilon]^2$, $\epsilon \geq 0$, into which the rectangles of sides $\frac{1}{n} \times \frac{1}{n+1}$, $n = 1, 2, \ldots$, can be packed.

This question was originally proposed by L. Moser. The first problem remains unsolved to this day, and for the latter, the best solution so far is by Balint.

Hypothesis & Approach

Since the summation shows that an infinte number of tiles add up to an area of 1, we worked on an inductive hypothesis, which would show that the free space always fits the next tile to be positioned. The following hypotheses are made and definition are used in our approach:

- The tiles are added parallel to the sides of T_0.
- For each $n \in \mathbb{N}$, the free space is connected and simply connected.
- *Tile* T_k - The kth rectangle to be added, with dimensions $\frac{1}{k}$ by $\frac{1}{k+1}$
- *Free Space* - The rectangular spacing that remains to be tiled, $T_0 \setminus \bigcup_{n=1}^{n} T_i$. The free space can be further separated into individual rectangles, denoted by $F_n^1, F_n^2, \ldots, F_n^{k_n}$.

We attack the problem by assuming that there exists a tiling where the free spaces are in the proportion of roughly $\frac{2}{n+1}$ by $\frac{2}{n+1}$, where the next tile to be placed is T_{k+1}

$F_N^1 \sim \frac{2}{N+1} \times \frac{2}{N}$

$F_N^1 \sim \frac{2}{N+2} \times \frac{2}{N+1}$

$\ldots$

$F_N^N \sim \frac{2}{N+N} \times \frac{2}{N+N-1}$

By always tiling a rectangle T_k in a free space with approximately double the dimensions of T_k, the proportionaity of the free space can be shown to be preserved.

What remains is to discern whether or not the initial conditions of the hypotheses are possible to reach.

A computer program was written to run through all the possible configurations of the first few tiles.

Algorithm

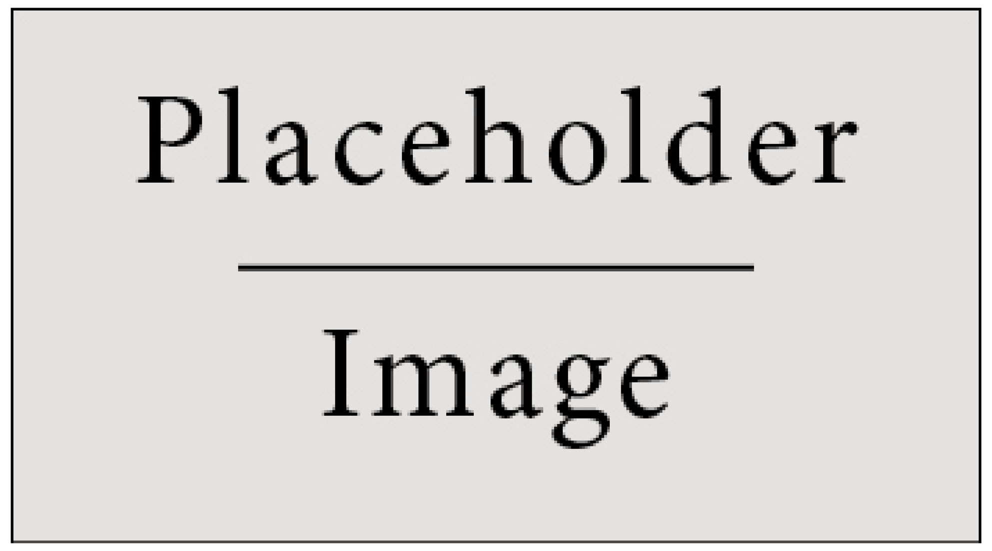


Figure 1: Figure caption

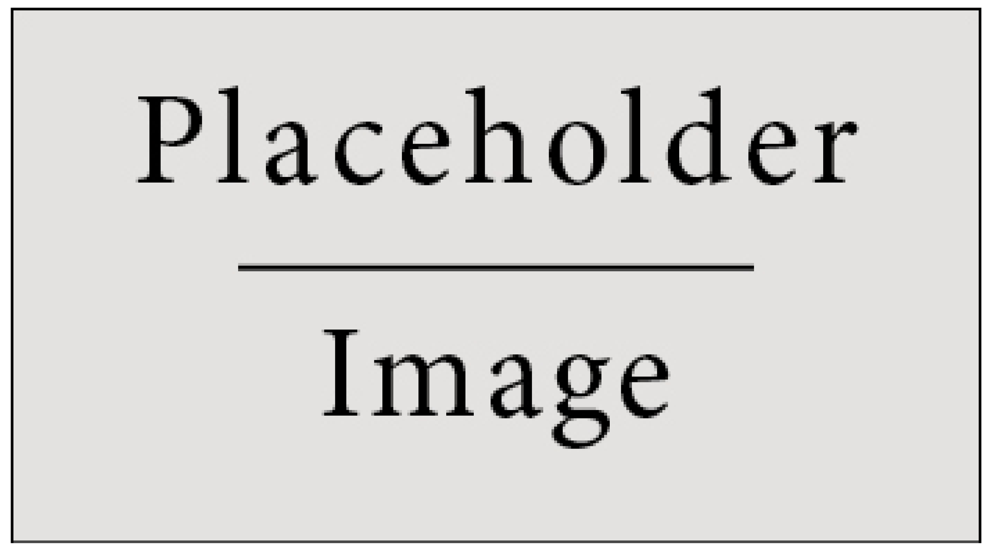
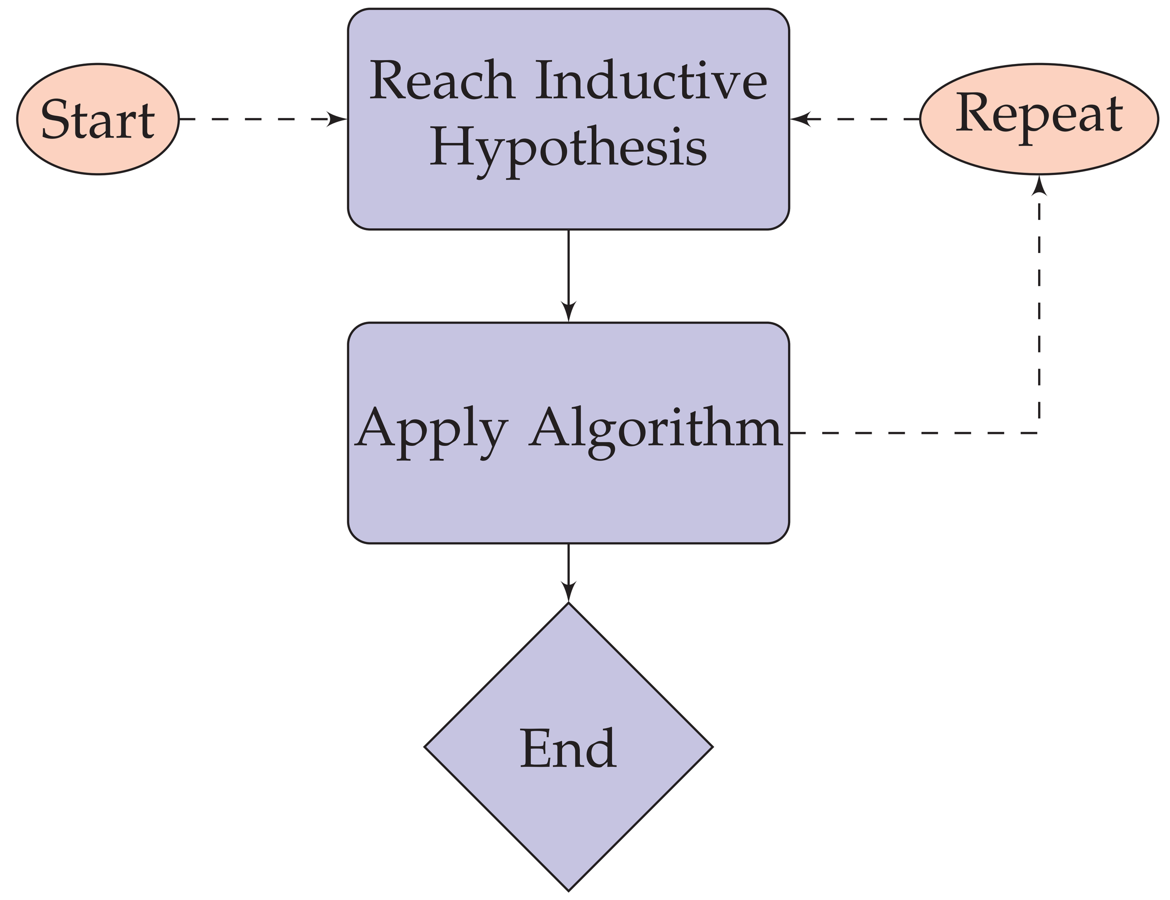


Figure 2: Figure caption

Conclusion & Results



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Future Investigation

Techniques and concepts used in the attempt to tile the square can be applied towards rectangles with varying side lengths. Any rectangle with side lengths $\frac{a}{b}$ by $\frac{b}{a}$ has an area of 1. It is worth looking into whether a rectangle of $\frac{3}{2}$ by $\frac{1}{2}$ or $\frac{4}{3}$ by $\frac{3}{4}$ would be more optimal for tiling. Other tiling challenges also remain to be tackled, such tiling an area of $\frac{\pi^2}{6}$ with tiles of $\frac{1}{k^2}$

References

[1] Marc M. Paulhus. An algorithm for packing squares. *J. Combin. Theory Ser. A* **82**, 13(2):147–157, 1998.