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SOME PACKING AND COVERING THEOREMS

BY

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1. Introduction. A family F of convex bodies is said to be *packed* in a larger convex body T if no members of F have any points in common and if their union is contained in T. The family F is said to *cover* T if T is contained in their union, where now members of F may intersect one another. To avoid complications with respect to boundary points we shall regard the members of F as being open if they are being packed in T, and closed if they are being used to cover T.

The "potato sack" theorem of Auerbach, Banach, Mazur, and Ulam states that any family of k-dimensional convex bodies with diameters at most D and with total volume S may be packed in some k-dimensional cube whose size depends only on D and S. A key lemma in the proof Kosiński [3] gives of this states that any family of k-dimensional parallelepipeds (rectangular, here and in general) with edges at most D in length and with total volume V can be packed in a k-dimensional parallelepiped with edges of length $3D, 3D, \ldots, 3D, (V+D^k)/D^{k-1}$. In § 2 it is shown that this parallelepiped may be replaced by one with edges of length $2D, 2D, \ldots, 2D, 2(V+D^k)/D^{k-1}$; this yields a more efficient packing when $k \geqslant 3$. In § 3 a corresponding covering problem is treated. Finally, in § 4 and § 5 various special results for the case $k = 2$ are obtained, which in some instances are best possible.

2. Refinement of Kosiński's lemma. We first outline a proof of our refinement for the case $k = 2$; when $k = 1$, the result is trivially true.

Let there be given a set of rectangles with edges at most D in length and with total area V. Increase the size of these rectangles by no more than is necessary to change them into rectangles of size $D/2^j$ by $D/2^k$, where j and k are non-negative integers. The total area of these enlarged rectangles is certainly less than $4V$.

Now consider the set of all rectangles of base $D/2^j$ and height $D/2^k$, where $j, k = 0, 1, 2, \ldots, 0 \leqslant k \leqslant j$ but not $k = j = 0$. This set may be packed, with room to spare, in a D by $2D$ rectangle by the scheme illustrated in Fig. 1.

Disregard temporarily any of the enlarged original rectangles of size D by D and place each of the remaining ones in the rectangular position of the same size situated in a larger rectangle of size D by $2D$ according to this scheme. It may happen that several enlarged rectangles are placed in the same position in which case we proceed as follows:

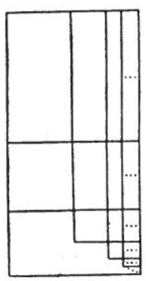

Fig. 1

If there are two or more rectangles placed in the first position, of size $D/2$ by D, these are combined two at a time to form D by D rectangles, which are disregarded temporarily, until at most one $D/2$ by D rectangle remains. (At most a finite number of rectangles can be placed in any one position originally.) Next, if there are two or more rectangles placed in the second position, of size $D/2$ by $D/2$, these are combined two at a time to form $D/2$ by D rectangles which are then placed in the first position until at most one rectangle remains in the second position. It may now be necessary to repeat this procedure for the first position again before proceeding to the third position, of size $D/4$ by D.

All positions in a given column are treated, working from the top down, before proceeding to the next column. At any given position, if two or more rectangles have been placed there they are combined two at a time and placed in the preceding position until at most one rectangle remains in that position. It may then be necessary to go through this procedure again for some of the earlier positions before continuing the next position. Each enlarged rectangle which was originally placed in the D by $2D$ rectangle is shifted at most a finite number of times before being placed in a position where it remains unmoved for the remainder of the process. Ultimately, therefore, each of the enlarged rectangles, with the exception of those used to form D by D squares, can be assigned a position in the D by $2D$ rectangle such that different ones do not overlap.

From the hypothesis on the total area of the rectangles it follows that there remain fewer than $4V/D^2$ squares of side D to be packed. These can certainly be packed in a $2D$ by $(2V/D+D)$ rectangle. This shows that the original set of rectangles may be packed in a $2D$ by $2(V+D^2)/D$ rectangle.

The argument in the k-dimensional case is completely analogous. After the "shifting" process has been carried out so that there is no overlapping between the enlarged parallelepipeds placed in a larger one with sides of length $2D, \ldots, 2D, D$, there remain fewer than $2^kV/D^k$ k-dimensional cubes of side D. These can be packed in a parallelepiped with sides of length $2D, \ldots, 2D$, $(2V/D^{k-1}+D)$. This suffices to complete the proof of the following result.

THEOREM 1. *Let there be given a set of k-dimensional parallelepipeds with edges at most D in length and with total volume V. Such a set may be packed in a k-dimensional parallelepiped with edges of length $2D, \ldots$ $\ldots, 2D$, $2(V+D^k)/D^{k-1}$.*

This, in conjunction with Kosiński's lemma 1 (also proved earlier by Macbeath [4]), which states that any k-dimensional convex body of volume S is contained in some parallelepiped of volume not more than $k!\,S$, provides a proof of the original theorem stated in § 1.

By a somewhat different argument it can be shown that a set of k-dimensional cubes of total volume V can be packed in a cube of volume 2^kV, but we will omit the proof of this. Notice that this result would not be true if 2^kV were replaced by anything less than $2^{k-1}V$ as is shown by the example of two cubes each of volume $V/2$.

3. A covering theorem. The technique employed in the proof of Theorem 1 can be modified so as to yield a result on the covering of a cube by parallelepipeds.

THEOREM 2. *A k-dimensional cube of side D can be covered by a set of parallelepipeds satisfying the hypothesis of Theorem 1 if*

$$V \geqslant c_k(2D)^k, \quad \textit{where} \quad c_k = \frac{2\cdot 4 \ldots 2^k}{1\cdot 3 \ldots (2^k-1)} - 1 < 2.463.$$

We outline the proof for the case $k=2$; when $k=1$ the result is trivially true.

Let the sides of the original rectangles be decreased by no more than is necessary to change them into rectangles of size $D/2^j$ by $D/2^k$, where j and k are non-negative integers. The total area of these reduced rectangles is certainly greater than $V/4$.

The identical procedure as described in the proof of Theorem 1 is now used to place many of these reduced rectangles into positions of size $D/2^j$ by $D/2^k$ in a larger rectangle of size D by $2D$ in such a way that no two reduced rectangles overlap and those rectangles which have not been so placed have been combined to form squares of side D.

From elementary results on partitions of numbers (see, e.g., Riordan [5], p. 111-113) it follows that the total area of the rectangular positions of base $D/2^j$ and height $D/2^k$, where $j, k = 0, 1, 2, \ldots, 0 \leqslant k \leqslant j$ but not $k=j=0$, is

$$D^2[1-1/2)^{-1}(1-1/2^2)^{-1}-1] = c_2D^2.$$

Hence, when the above packing and shifting process has been carried out, at least one D by D square will have been formed from the reduced rectangles if $V/4 \geqslant c_2D^2$, or if $V \geqslant c_2(2D)^2$.

The rectangles may now be restored to their original sizes. This suffices to complete the proof for this case.

A similar argument may be applied in the k-dimensional case. After the packing and shifting process has been carried out so that there is no overlapping between the reduced parallelepipeds placed in part of a larger one with sides of length $2D, \ldots, 2D, D$, at least one k-dimensional cube of side D will have been formed if $V/2^k \geqslant c_k D^k$, or if $V \geqslant c_k(2D)^k$.

If we restrict ourselves to coverings in which the sides of the covering parallelepipeds are parallel to the sides of the cube being covered, then a best possible result would be to show that the conclusion of Theorem 2 holds if only $V \geqslant (2^k-1)D^k$. This follows from the fact that with this restriction it is impossible to cover a cube of side D with 2^k-1 smaller cubes.

Macbeath [4] has shown that any k-dimensional convex body of volume V can be used to cover some k-dimensional parallelepiped of volume at least V/k^k. (It seems likely that the factor $1/k^k$ could be replaced by $k!/k^k$ but this appears to have been proved only for certain special cases. See, e.g., Bambah and Roth [1] and Bielecki and Radziszewski [2].) The following result, which can be considered the covering analogue of the "potato sack" theorem, is proved by combining Macbeath's result with Theorem 2.

THEOREM 3. *A k-dimensional cube of side D can be covered by any set of k-dimensional convex bodies with diameters at most D and total volume S, if $S \geqslant c_k(2kD)^k$.*

4. Some sharper results on packing when $k = 2$. It follows from the result stated at the end of § 2 that any set of squares of total area A can be packed in a square of area $4A$. We now show, among other things, that the factor 4 may be replaced by 2.

THEOREM 4. *Let there be given a set of squares of total area A, the largest of which has side D. Such a set may be packed in any rectangle of area $2A$ and shorter side B, if $D \leqslant B$.*

For convenience we split the proof into two parts. We treat first the case in which $D \leqslant B/2$.

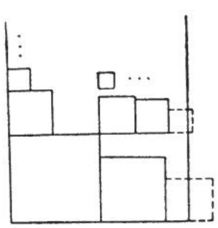

Fig. 2

Place the squares in a rectangle of base B according to the scheme illustrated in Fig. 2, starting in the lower left corner and continuing according to decreasing height. Whenever a square would go outside the rectangle, as indicated by the dotted lines, it is used to start a new row. The second square in each row is placed at distance D from the left edge of the rectangle. This process is continued until all the squares have been packed in the rectangle with base B. We now obtain an estimate of how high such a rectangle must be.

Let w_i denote the length of the side of the first square in the $(i+1)$-st row and let A_i denote the total area of the squares in the i-th row plus $(w_{i-1}^2 - w_i^2)$. That is, the area of the first square in each row is counted with the preceding row. When there are only a finite number of rows w_i and A_i will equal zero from some point on.

From the definition of the packing procedure it follows that

$$A_i \geqslant (B-D)w_i \quad \text{for} \quad i = 1, 2, \ldots$$

When summed over i this becomes

$$A - D^2 \geqslant (B-D)\sum_{i=1}^{\infty} w_i,$$

since the area of every square except the largest has been included. This implies that the set of rectangles can be packed in a rectangle of base B and height

$$D + \sum_{i=1}^{\infty} w_i \leqslant D + (A-D^2)/(B-D).$$

This last quantity is less than or equal to $2A/B$ if $B(BD-A) \leqslant 2D(BD-A)$. But this inequality holds since $D \leqslant B/2$ and $BD \leqslant B^2/2 \leqslant 2A/2 = A$, using the fact that B is the shorter side of a rectangle of area $2A$. This suffices to complete the proof for this case.

The remaining possibility to be treated is that for some positive integer n the inequalities $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n \geqslant B/2 > d_{n+1} \geqslant \ldots$ hold, where d_i denotes the length of the side of the i-th largest square.

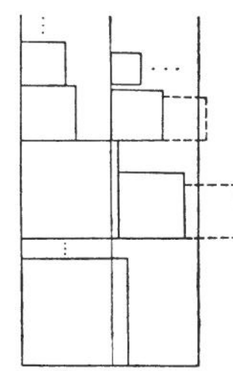

Fig. 3

The n largest squares can be packed in a rectangle of base B in such a way that the total height required is $\delta = \sum_{i=1}^{n} d_i$ (see Fig. 3). The packing process continues according to the scheme illustrated in Fig. 3, starting with the square of side d_{n+1} which is placed next to the one side d_n and proceeding according to decreasing height. The only difference between the procedure for packing the remaining squares and that described in the proof for the first case is that now the second square in each new row is placed at distance $B/2$ from the left edge of the rectangle.

Let w_i denote the length of the side of the first square in the $(n+i)$-th row and let A_i denote the area of those parts of the squares in the $(n-1+i)$-th row whose distance from the left edge of the rectangle is greater than $B/2$, plus w_i^2, for $i = 1, 2, \ldots$ As before, when there are only a finite number of rows these quantities equal zero from some point on.

From the definition of the packing procedure it follows that

$$A_i \geqslant \frac{1}{2} B w_i \quad \text{for} \quad i = 1, 2, \ldots$$

When summed over i, this implies that

$$A - \delta B/2 \geqslant \sum_{i=1}^{\infty} A_i \geqslant \frac{1}{2} B \sum_{i=1}^{\infty} w_i,$$

since an area at least equal to $\delta B/2$ formed by the n largest squares isn't included in the sum. Hence, the set of squares may be packed in a rectangle of base B and height

$$\delta + \sum_{i=1}^{\infty} w_i \leqslant \delta + \frac{A - \delta B/2}{B/2} = 2A/B,$$

which completes the proof of the theorem.

By an argument similar to that used in the first part of the proof of Theorem 4 it can be shown that any set of squares of total area A, the largest of which has side D, can be packed in a square of side $D + (A - D^2)^{1/2}$. This is clearly best possible, in a sense. Unfortunately, we are unable to obtain as strong results on the packing of a set of rectangles in a larger rectangle. We state without proof the closest analogue to Theorem 4 we have found. The packing procedure is similar to that described above.

THEOREM 5. *Let there be given a set of rectangles with sides at most D in length and with total area A. Such a set may be packed in any rectangle of area $2A$ and shorter side B, provided that $D \leqslant (\sqrt{2} - 1) B$.*

From this it follows that if $D \leqslant (2 - \sqrt{2}) A^{1/2}$, then the set may be packed in some larger rectangle of area $2A$.

The best general result in this direction that we have been able to prove is the following

THEOREM 6. *Let there be given a set of rectangles with sides at most D in length and with total area A. Such a set may be packed in any rectangle of area S and shorter side B, if $D \leqslant B$ and $S \geqslant 2A + B^2/2$.*

Let each rectangle in the original set have its longer side horizontal. Those rectangles whose base is at least as great as $B/2$ are placed one above another in a rectangle of base B. The height required for this is certainly no more than $2\delta/B$, where δ denotes the total area of these rectangles. The remaining rectangles are packed above these according to decreasing height by a procedure which differs from that described

in the first part of the proof of Theorem 4 only in the following respects. No space is left between the first and second rectangles in a row and as soon as any rectangle crosses a vertical line bisecting the large rectangle with base B a new row is started with the next rectangle. The details involved in showing that this construction implies the required result do not differ greatly from those in the proof of Theorem 4 and are omitted for that reason.

In concluding this section we mention the following unsolved problems.

1. What is the smallest number S such that any set of squares of total area one may be packed in a rectangle of base one and height S (**P 583**)?

That $S \geqslant \sqrt{3}$ follows from considering a set of three squares each with area $1/3$. Perhaps $S = \sqrt{3}$, but the most we can show is that $S \leqslant 2$.

2. What is the smallest number T such that any set of squares of total area one may be packed in some rectangle of area T (**P 584**)?

That $T > 1.2$ follows from considering two squares of area x^2 and y^2 where $x \geqslant y$, $x^2 + y^2 = 1$, and the value of $x(x+y)$ is maximal. Theorem 4 implies that $T \leqslant 2$.

3. What is the area R of the smallest rectangle in which can be packed the set of rectangles of total area one and sides of length $1/n$ and $1/(n+1)$, for $n = 1, 2, \ldots$? It can be shown that $R \leqslant 113/96$. Is $R > 1$ (**P 585**)?

5. A sharper result on coverings when $k = 2$. The technique used in § 4 can be modified to yield a result which is considerably stronger than Theorem 2 when $k = 2$.

THEOREM 7. *Let there be given a set of rectangles with sides at most D in length and with total area A. Such a set may be used to cover any rectangle of area S and shorter side B, where $D \leqslant B$, if*

$$A \geqslant S(1 + D/B) + BD,$$

and hence certainly if $A \geqslant 3S$.

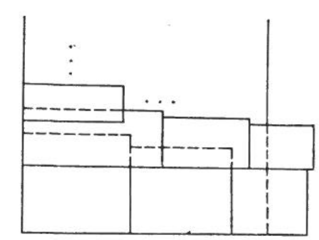

Fig. 4

Let each rectangle in the original set have its longer side horizontal. Place these rectangles in a larger one of base B according to the scheme illustrated in Fig. 4, starting in the lower left corner and continuing according to decreasing height. Whenever a covering rectangle goes outside the larger rectangle a new row is started with the next rectangle. The new row is directly above the last rectangle in the preceding row, as indicated in the diagram. This process is continued until the set of rectangles is exhausted.

Let b_i and h_i denote the base and height of the rectangle in the i-th row which goes outside the large rectangle being covered. If there is no such rectangle, e.g., if the covering process terminates after a finite number of rows perhaps in the middle of a row, then let b_i and h_i equal zero. Also, let A_i denote the total area of those rectangles placed in the i-th row.

From the definition of the covering procedure it follows that

$$Bh_i \geqslant A_{i+1} - b_{i+1}h_{i+1} \quad \text{for} \quad i = 1, 2, \ldots$$

Therefore,

$$B\sum_{i=1}^{\infty} h_i \geqslant A - (A_1 - b_1 h_1) - \sum_{i=1}^{\infty} b_i h_i \geqslant A - DB - D\sum_{i=1}^{\infty} h_i,$$

upon summing over i and using the hypothesis on the size of the rectangles. Thus, a rectangle of area S and shorter side B may certainly be covered by the original set of rectangles if $\sum_{i=1}^{\infty} h_i \geqslant S/B$, and hence if $(A - DB)/(B+D) \geqslant S/B$, or if $A \geqslant S(1+D/B)+BD$. This completes the proof of the theorem.

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SOME NEW CONSTRUCTIONS OF 4-TUPLE SYSTEMS

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In 1852 Steiner [12] posed the following problem: For which integer N is it possible to form a system V_3 of triples of numbers $1, \ldots, N$ in such a way that every pair of numbers appears in exactly one triple? Assuming this is solved we can ask for the possibility of forming quadruples such that any triple not in V_3 should appear in exactly one quadruple, and that no quadruple should contain a triple from V_3. Does this impose a new condition on the number N? Steiner carries on stating analogous problems for quintuples, sixtuples, etc.

The generalized Steiner's problem is as follows: Given four positive integers $v > l > k, \lambda$ we consider the proposition $P(v, l, k, \lambda)$ meaning that for every set S having v elements there exists a system V of subsets of S having l elements each and such that every subset of S having k elements is contained in exactly λ sets of the system. We shall call V a *realization* of $P(v, l, k, \lambda)$. Sometimes it is also called *tactical configuration* (or, briefly, *configuration*). Configurations with $k = 2$ are known as *balanced incomplete block designs* (BIBD). The number of l-element subsets belonging to a realization of $P(v, l, k, \lambda)$ and containing some fixed h elements is equal to $\lambda\binom{v-h}{k-h}\Big/\binom{l-h}{k-h}$ whence $P(v, l, k, \lambda)$ implies that

$$(1) \qquad \lambda\binom{v-h}{k-h}\Big/\binom{l-h}{k-h} \text{ is an integer for } h = 0, 1, \ldots, k-1.$$

It is also known that in general proposition (1) does not imply proposition $P(v, l, k, \lambda)$. For instance if $l = 6$, $k = 2$ and $\lambda = 1$ (see [13]) or if $l = 5$, $k = 2$ and $\lambda = 2$ and also for some other BIBD (see [2], [8] and [10]) it does not. Nevertheless, for some l, k, and λ formula (1) does imply $P(v, l, k, \lambda)$, e.g. for $l = 3$, $k = 2$ and every λ ([6], [9] and [11]), for $l = 4$, $k = 2$ and every λ (see [4]), for $l = 5$, $k = 2$, $\lambda = 1, 4, 20$ (except possibly $v = 141$; see [4]), $l = 4$, $k = 3$ for every λ ([3] and [5]), etc. (see [1] and [14]).