Dear Benson,

Here is a template for a Latex file to keep as a log of our project. On the second page I have added a comments section, where we can put comments, rough work, calculations, questions, etc. These should be added as a list, with the authors name (BG or PH resp.) and the date of the comment.

–PeH, February 16, 2015.–

Comments.

PH – 2015/02/01 The sections "Cross Sections" and onwards are rough work, and will eventually (probably) be scrapped.

BG – 2015/02/07 We started the investigation of the viability of a greedy algorithm. The algorithm works as follows:

1. Assume the longer side of a rectangle R has a length $\frac{a}{b}$, and shorter side with a length $\frac{a\prime}{b\prime}$
2. Fill R with a piece with a side length $\frac{1}{n_0} \times \frac{1}{n_0+1}$ where $\frac{1}{n_0}$ satisfies $\frac{1}{n_0} \leq \frac{a\prime}{b\prime} < \frac{1}{n_0-1}$
3. Cut the remaining area into 2 rectangles:
   $R_0 : \frac{1}{n+1} \times (\frac{a\prime}{b\prime} - \frac{1}{n_0})$
   $R_1 : (\frac{a}{b} - \frac{1}{n+1}) \times \frac{a\prime}{b\prime}$
4. Apply Step 1 to rectangle R_0.
5. Stop when $\frac{a\prime}{b\prime} - \sum \frac{1}{n_i} = 0$
6. Apply Step 1 to next left-most rectangle.

Studying $R_0 : \frac{1}{n_0+1} \times (\frac{a\prime}{b\prime} - \frac{1}{n_0})$, we ask ourselves, is $\frac{a\prime}{b\prime} - \frac{1}{n_0} > \frac{1}{n_0+1}$?
By hypothesis, $\frac{1}{n_0} \leq \frac{a\prime}{b\prime} < \frac{1}{n_0-1} \Rightarrow 0 \leq \frac{a\prime}{b\prime} - \frac{1}{n_0} < \frac{1}{n_0-1} - \frac{1}{n_0}$

$\frac{1}{n_0-1} - \frac{1}{n_0} = \frac{n_0-(n_0-1)}{n_0(n_0-1)} = \frac{1}{n_0(n_0-1)}$ Is $\frac{1}{n_0(n_0-1)} < \frac{1}{n_0+1}$?

$n_0 + 1 < n_0^2 - n_0 \Rightarrow n_0^2 - 2n_0 - 1 > 0$

Hence, $\frac{a\prime}{b\prime} - \frac{1}{n_i} < \frac{1}{n_i(n_i-1)} < \frac{1}{n_i+1}$ for $n_i \geq 2$. This implicates that the shorter side will be on the vertical edges, which will approach 0 as $\frac{1}{n_i}$ decreases. When it reaches 0, the left side will be completely covered with smaller rectangles.

BG – 2015/02/16 I coded a rough simulation of the greedy algorithm to answer the question of whether or not some rectangle will be required to be used twice. Very quickly I found a case in which the same rectangle is called upon consecutively. Take the case when you have a rectangle with dimensions $\frac{1}{n} \times \frac{1}{n+x}$. The algorithm will tile it with $\frac{1}{n+x} \times \frac{1}{n+x+1}$ and the remaining area will be, $\frac{1}{n} \times (\frac{1}{n} - \frac{1}{n+x+1})$. In this case $\frac{1}{n+x} \times \frac{1}{n+x+1}$ will be used again, provided $\frac{1}{n+x+1}$ is still the shorter side. Specifically, one case I came across was $\frac{1}{11} \times \frac{1}{140}$ which would be tiled with $\frac{1}{140} \times \frac{1}{141}$ leaving a rectangle of $\frac{130}{1551} \times \frac{1}{140}$. I will play around with the algorithm to see how we can go around this.

On Tiling the Unit Square

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Abstract

We consider the problem, originally due to L. Moser, of tiling the unit square with rectangles of side lengths $\frac{1}{n} \times \frac{1}{n+1}$ for $n = 1, 2, \ldots$. This is the result of a University of Toronto Mentoring Program 2015.

1 Introduction

We consider the following:

> *Problem:* Can the unit square $[0,1]^2$ be tiled with rectangles of sides $\frac{1}{n} \times \frac{1}{n+1}$ for $n = 1, 2, \ldots$

A more general formulation is the following:

> *Problem:* Find the smallest square $[0, 1+\epsilon]^2$, $\epsilon \geq 0$, into which the rectangles of sides $\frac{1}{n} \times \frac{1}{n+1}$, $n = 1, 2, \ldots$, can be packed.

According to [3, Section D11] these problems are originally due to L. Moser. The best result for the second problem, to the authors knowledge, is due to Balint [1, 2]. See also Paulhus [5].

2 Preliminaries

Let $T_0 = [0,1]^2$ denote the unit square. For each $n \in \mathbb{N}$, let T_n denote the tile of side lengths $\frac{1}{n} \times \frac{1}{n+1}$. In our considerations we shall make the following hypotheses about any tiling of T_0:

> *Hypothesis 1:* The tiles are added parallel to the sides of T_0.
>
> *Hypothesis 2:* For each $n \in \mathbb{N}$, the free space $T_0 \setminus \bigcup_{n=1}^{n} T_i$ is connected and simply connected.

Assume a tiling is given. Let

$$V_0 = T_0, \qquad V_n = V_0 \setminus \bigcup_{i=1}^{n} T_i, \quad i = 1, 2, \ldots \tag{2.1}$$

Then, by the above hypotheses, for each positive integer n the region V_n is a polygon with sides aligned with the horizontal and vertical directions. Let V_n have vertices $v_n^1, \ldots, v_n^{r_n}$. Consider the set of all horizontal and vertical lines through the v_n^j. Take their complement in V_n. This complement consists of finitely many rectangles $F_n^1, \ldots, F_n^{k_n}$ with sides parallel to the sides of T_0. Furthermore, $V_n = \bigcup_{i=1}^{k_n} F_n^i$.

We make the following assumptions:

> *Assumption 1:* For each positive integer n, the tile T_n is positioned in V_{n-1} so that at least one vertex of T_n lies at one of the vertices of V_{n-1}.
>
> *Assumption 2:* For each positive integer n, if the tile T_n fits into one of the free spaces F_{n-1}^j, then it must be positioned in one of them.

Now we ask the following questions:

1. Does a tiling exist satisfying Assumptions 1 and 2?
2. Does such a tiling have the property that for each positive integer n, each F_n^i except one, is eventually tiled by finitely many tiles?

Since each F_n^i has side-lengths of the form $\sum \frac{1}{n_i} - \sum \frac{1}{m_j}$ we therefore ask, firstly, if the sides can be 'tiled' by a finite number of intervals of lengths $\frac{1}{l_1}, \frac{1}{l_2}, \ldots, \frac{1}{l_k}$, where $l_1, l_2, \ldots, l_k$ are distinct positive integers and $\min_r l_r > \max(n_i, m_j)$. In the case when we do not have the second contraint we know the following, originally due to Fibonacci and rediscovered by Sylvester.

Theorem 2.1. *Any positive rational number $\frac{a}{b}$ can be expressed as a finite sum of unit fractions, i.e. there exist distinct positive integers $n_1, n_2, \ldots, n_k$ such that*

$$\frac{a}{b} = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k} \tag{2.2}$$

2.1 Cross Sections

Assume a tiling exists. Take a line parallel to the horizontal or vertical sides of T_0 and intersect it with T_0. Assume that the tiles $T_{i_1}, T_{i_2}, \ldots$ intersect this line horizontally and the tiles T_{j_1}, T_{j_2} intersect it vertically. Then

$$\sum \frac{1}{i_k} + \sum \frac{1}{j_k - 1} = 1 \tag{2.3}$$

The problem of finding positive integers i_k, j_k satisfying this equation is classical. (See Klee and Wagon [4, Chapter 2.15].)

Assume there does exist a tiling of the unit square. Note that if we take a horizontal section, i.e. intersect with $\{y = y_0\}$ we find the length decomposes as

$$1 = |l(S(y_0))| = \sum |l(S_{n_i(y_0)}(y_0))| = \sum \frac{1}{n_i(y_0)} \tag{2.4}$$

3 Perimeter Representation

Observe that if a tiling exists then the perimeter of the free space must tend to zero, i.e. $\lim_{n\to\infty} |P(V_n)| = 0$.

3.1 Winding Numbers: How to tell adding a tile is admissible.

References

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