Tiling the Unit Square

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Introduction

Observation: Since $\frac{1}{k}\cdot\frac{1}{k+1} = \frac{1}{k} - \frac{1}{k+1}$ holds for all integers k, it follows that $\sum_{k=1}^{\infty}\frac{1}{k}\cdot\frac{1}{k+1} = \lim_{n\to\infty}(\frac{1}{1}-\frac{1}{2})+\ldots+(\frac{1}{n}-\frac{1}{n+1}) = 1$

This led L Moser to state the following:

Problem: Can the unit square $[0,1]^2$ be tiled with rectangles of sides $\frac{1}{n}\cdot\frac{1}{n+1}$ for $n = 1, 2, \ldots$?

More generally, we can ask:

Problem: Is there a rectangle of unit area which can be tiled with rectangles of sides $\frac{1}{n}\cdot\frac{1}{n+1}$ for $n = 1, 2, \ldots$?

Hypothesis & Approach

Definition:

- *Tile* T_k - The kth rectangle to be added, with dimensions $\frac{1}{k}$ by $\frac{1}{k+1}$
- *Free Space* - The region $T_0 \setminus \bigcup_{i=1}^{n} T_i$. We further subdivide this into rectangles $F_n(1), F_n(2), \ldots, F_n(m_n)$ and denote those with boundary touching the tiled region by $F_n^1, F_n^2, \ldots, F_n^{d_n}$

We looked for an algorithm that adds the tiles $T_1, T_2, \ldots$ in order. The following hypotheses are made and notation is used in our approach:

- The tiles are added parallel to the sides of T_0
- For each $n \in \mathbb{N}$, the free space is connected and simply connected
- The intersection of the tiled space with horizontal or vertical lines is connected

Observe that a tile placed inside a free tile will only decrease the perimeter if the tile touches the boundary of the unit tile.

2 by 1/2 Rectangle

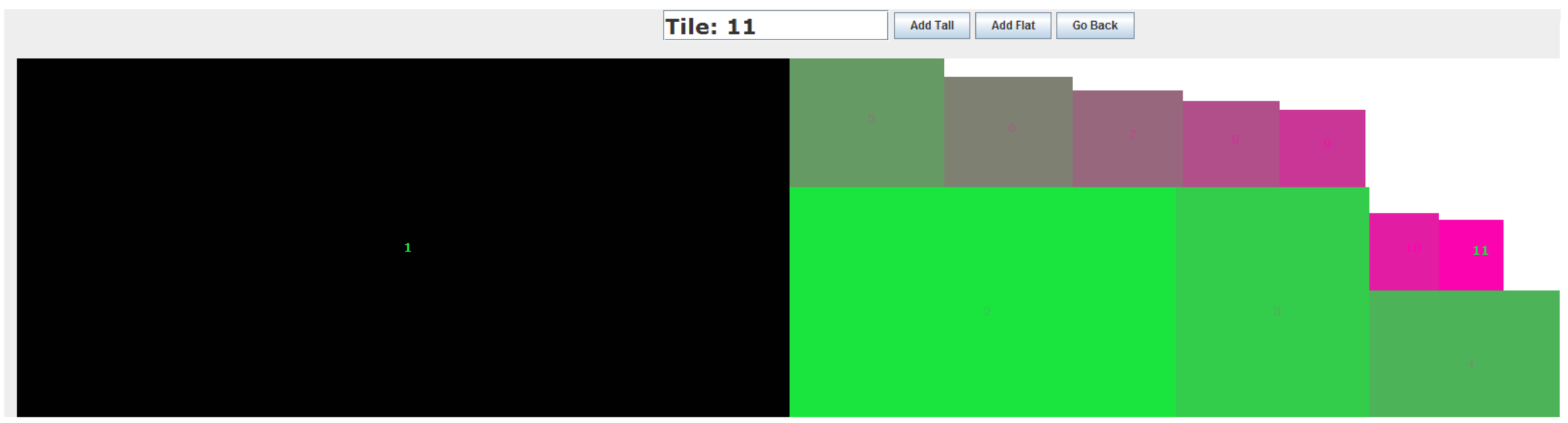

Figure 2: 2 by $\frac{1}{2}$ - 11 Tiles

Algorithm

Rotational Algorithm

Assume tiles $T_1, T_2, \ldots, T_n$ have been added satisfying hypotheses into $F_{n-1}^{r_{n-1}}$.

1. If T_{n+1} fits in F_n^1 or $F_n^{d_n}$ and decreases perimeter, add the tile inside it
2. Otherwise add T_{n+1} to first F_n^i that is fits inside
3. If T_{n+1} doesn't fit inside any boundary free tile F_n^i go to previous level, moving T_n from $F_{n-1}^{r_{n-1}}$ to the next boundary free tile in which it fits

Perimeter Decreasing - As the perimeter approaches 0, so does the area

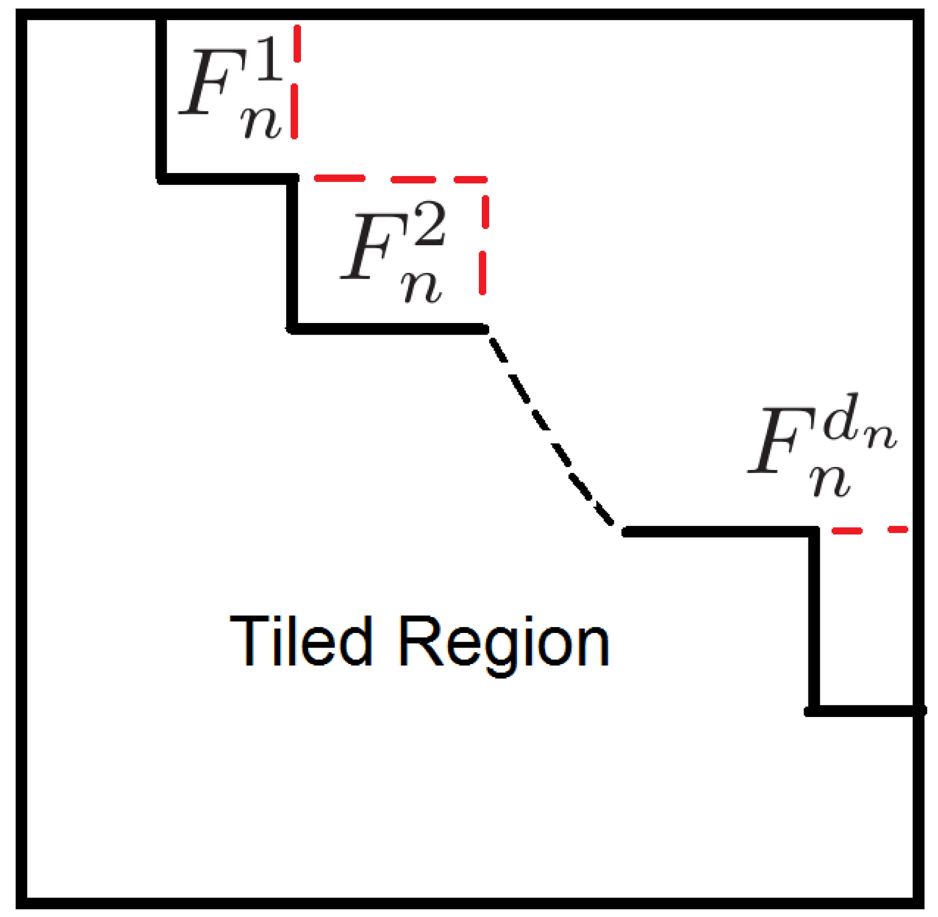

Figure 1: Example Configuration

Conclusion & Results

- No configuration satisfying our hypothesis was found to exist
- Simulated through nearly 40,000 configurations witih depth first search through all positions and rotations for T_k
- Surprisingly, **no** tiling with more than 12 tiles was found to be achievable
- Other rectangles of area 1 ($\frac{a}{b}\cdot\frac{b}{a}$) were explored with similar results
- A $\frac{4}{3}$ by $\frac{3}{4}$ was tiled with up to 14 tiles
- A 2 by $\frac{1}{2}$ was tiled with up to 11 tiles
- An algorithm to tile the unit square should less conservative and use regions outside of the defined freespace

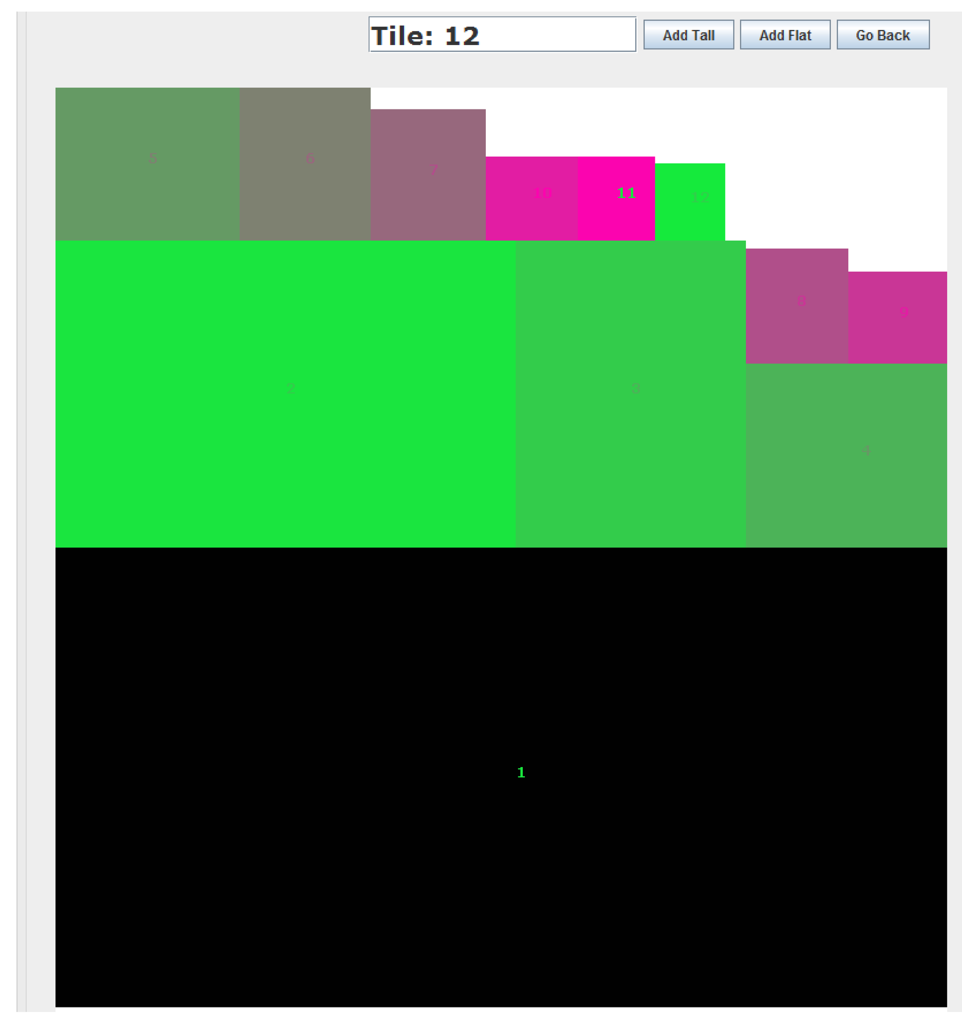

Figure 3: Unit Square - 12 Tiles

References

[1] A. Meir and L. Moser. On packing of squares and cubes. *J. Combinatorial Theory*, 5:126–134, 1968.

[2] J. W. Moon and L. Moser. Some packing and covering theorems. *Colloq. Math.*, 17:103–110, 1967.