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A New Algorithm for the Expansion of Egyptian Fractions*

M. N. BLEICHER

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

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A rational number p/q is said to be written in Egyptian form if it is presented as a sum of reciprocals of distinct positive integers, $n_1, n_2, \ldots, n_k$. The new algorithm here presented is based on the continued fraction expansion of the original fraction. It has the advantage of relatively short length, while keeping the n_i below the very reasonable bound of q^2. This method also ties in the best lower approximations to p/q with the sub-sums of the Egyptian expansion. Because it is based on the continued fraction the method is extendable to irrational numbers.

I. INTRODUCTION

The problem of Egyptian Fractions is raised by the oldest extant mathematical manuscript, the Rhind Papyrus written by Ahmose [5, 23, 40]. With the manuscript is a table[1] of how to express ratios of integers as sums of distinct fraction with numerator one, unit fractions. (The Egyptians had one exception, the fraction 2/3). Fibonacci [24] in 1202 published an algorithm for expressing any rational number between zero and one in the Egyptian form.[2] Sylvester [38] among others rediscovered this algorithm and/or extended the work toward the representants of irrational numbers [9, 10, 28, 29, 34].

Others, [9, 39, 41] have studied the problem of representing integers by such sums.

The question of length of the representation is a natural one and has been raised in many contexts. Reasonably simple necessary and sufficient conditions are known for a fraction to have a representation of length 2

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[1] Found on an accompanying leather scroll; see [2, pp. 414–415] and The mathematical leather scroll, *J. Egyptian Archeol.* **13** (1927), 232.

[2] For an explanation at the Fibonacci–Sylvester methods and some additional historical comments, see [2]. For further history see [7, 29 and 8], which also give some bibliography not included in this work.

or 3 [13, 18, 19, or 35]. Rav [25] found a necessary and sufficient condition for a fraction to be representable if k steps, namely:

Given m/n, a reduced fraction, the equation

$$\frac{m}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k}$$

is solvable with $x_1, x_2, \ldots, x_k$ distinct positive integers if and only if there is a fraction $M/N = m/n$ and k distinct divisors $N_1, \ldots, N_k$ of N such that $(N_1, \ldots, N_k) = 1$ and $N_1 + N_2 + \cdots + N_k = 0 \bmod M$. Sierpinski [32] considered the question when the condition of positivity of the x_i's is discarded. Stewart and Webb [37] considered the case where the x_i were regarded to be distinct and positive, but the numerators were allowed to take on a number of values positive and negative.

Stewart [36], Graham [12, 13] and others [1, 30], have considered the problem of what fractions can be represented if the x_i's are restricted to be in certain sets, e.g., the positive odd integers. Erdös and Strauss (cf. [8]) have conjectured that fraction of the form $4/n$ can always be represented in 3 or fewer steps and although a considerable amount of work has been done [1, 3, 16–18, 20, 25, 30, 35] this remains unanswered. Sierpiński [33] has conjectured that fraction of the form $5/n$ can also be represented in 3 steps and a number of authors [9, 10, 21, 25] have worked on this conjecture.

Several algorithms are now known for the expansion of an arbitrary fraction into an Egyptian expansion. There is the first algorithm due to Fibonacci [24] and also Sylvester [38] which will be referred to as the Fibonacci–Sylvester (F–S) algorithm. There is an algorithm given by Erdös [8], an algorithm given by Golomb [11] and an algorithm given by Bleicher [2], hereafter called the Farey Series algorithm.

Ideally one would like an algorithm which would yield a short expansion in which the denominators do not get too large. Let p/q be a reduced fraction with $0 < p < q$. Denote by $N(p, q)$ the length of the shortest expansion of p/q. Erdös has shown by using his algorithm that independent of p, $N(p, q) \leqslant 8 \ln q/\ln \ln q$ for q large enough, where $\ln x$ is the natural logarithm of x. For the F–S algorithm, the best result known to the author is that the length is at most $p \leqslant q - 1$, although for large q it is much less. Golomb [8] has shown that for his algorithm the fraction $(q - 1)/q$ is expanded into exactly $q - 1$ terms and the same is true for the Farey Series algorithm. For all but the Erdös algorithm, p is also an upper bound for the number of terms.

With regard to the size of the denominators, they grow exponentially for the F–S algorithm and become large relative to the size of q, for the Erdös algorithm they are bounded by $4q^2 \ln q/\ln \ln q$ for large q, and in the Golomb algorithm and Farey series algorithm they are bounded by $q(q-1)$.

In this paper we give a new algorithm, Section III, for representing a fraction in the Egyptian style, hereafter called the Continued Fraction (C.F.) Algorithm.

This algorithm is based on the Farey Series algorithm and the continued fraction expansion of p/q and is quite distinct in theory and in practice from those of Fibonacci–Sylvester, Erdös and Golomb.

In Section IV we show that the expansion according to the C.F. algorithm of any fraction with denominator at most q, $(q > 2)$, can be done in at most $2(\ln q)^2/\ln \ln q$ steps, while keeping all denominators less than q^2. If we take the numerator into consideration we get that p/q can be expanded in at most $1 + a_2 + a_4 + \cdots + a_{2k} \leqslant p$, where a_i are the partial quotients in the continued fraction expansion of p/q.

Thus the C.F. algorithm gives a short expansion while keeping the denominators small. Although for large q with p large compared with q the Erdös algorithm bound is lower, but the denominators grow larger, while for all q if p is small compared to q the C.F. Algorithm has a lower bound.

In Section V we compare the algorithms by means of several examples.

Since the theory of Farey Series and Continued Fractions have a vast literature and many generalizations, the new algorithms would seem to give the best hope for bringing the Egyptian Fraction problem into a better context and allows the generalization of the Egyptian fraction problems to be solved in a more unified style.

Since the C.F. algorithm is based on the continued fraction of p/q and is defined inductively on the length of the continued fraction, it yields an extension to real numbers, Section VI, which allows every real number r to be expanded as an infinite sub-sum of the harmonic series

$$r = \sum_{i=1}^{\infty} \frac{1}{n_i}, \qquad n_i < n_{i+1}.$$

Furthermore, all lower best approximations[3] to r occur as partial sums of the expansion. Salzer [29, 30] (see also Sierpiński [34]) has dealt with the best approximation aspect of the F–S algorithm.

II. The Farey Series Algorithm

We recall that the Farey Series of order n, F_n consists of all the reduced fractions a/b with $0 \leqslant a \leqslant b \leqslant n$, arranged in increasing order. Further-

[3] More precisely best approximations of the second kind as defined by Khintchine [15, p. 30].

more, if a/b and c/d are adjacent fractions in F_n and $a/b < c/d$ then $(c/d) - (a/b) = 1/bd$ and $b \neq d$. Also as the order of the Farey Series is increased the first fraction to appear between a/b and c/d is $(a + c)/(b + d)$ (for details and proofs, see [2, pp. 416–420]).

The Farey Series algorithm can now be stated as follows: Given the reduced fraction p/q, construct the Farey series of order q. Let r/s be the adjacent fraction to p/q which is less than p/q in F_q. From the properties of Farey Series we see that $(p/q) = (1/qs) + (r/s)$ and $s < q$, $r < p$. We now repeat the process on r/s, etc. and continue until the denominator is 1, which must happen in at most $q - 1$ steps. When the denominator is 1, the numerator is zero, so that the last term can be ignored and we have an Egyptian fraction representation for p/q with at most $q - 1$ steps and with all the denominators less than or equal to $q(q - 1)$.

This form of the algorithm can be made more efficient for actual computation by only constructing the relevant part of the Farey series; for details, see [2, pp. 424–434].

The Fibonacci–Sylvester Algorithm yields very large denominators in relation to the size of q, in fact they grow at least exponentially. For amplification of these remarks and some numerical examples see Sylvester [38] or [2, pp. 424–434].

If the continued fraction expansion of p/q is known then the above algorithm becomes much simpler since from the continued fraction one can easily find all the relevant terms of the Farey Series. Using the notation of Khintchine [15] let

$$p/q = [0; a_1, a_2, \ldots, a_n]. \tag{1}$$

Since $p/q < 1$ if we require that $a_n \geqslant 2$ then this expression is unique (this and all other information about continued fractions which we use can be found in Khintchine [15, pp. 1–33]). Let p_i/q_i denote the i-th convergent to p/q. Then for i even and $i < n$ we have $p_i/q_i < p/q$, while for i odd and $i < n$ we have $p/q < p_i/q_i$ and $p/q = p_n/q_n$. We can now explain the Farey Series algorithm from a different viewpoint. Let the fraction p/q be given and have continued fraction expansion given by (1). If $n = 1$ then $p/q = 1/a_1$ and we are done, if $n > 1$ we suppose the algorithm has been explained for all fractions with shorter continued fraction expansion. If n is odd then we have $p_{n-1}/q_{n-1} < p/q$, where $p_{n-1}/q_{n-1} = [0; a_1, a_2, \ldots, a_{n-1}]$ and p_{n-1}/q_{n-1} is the Farey fraction adjacent to p/q in the Farey series order q. Thus $p/q = 1/qq_{n-1} + p_{n-1}/q_{n-1}$. Since p_{n-1}/q_{n-1} has a continued fraction of length $n - 1$, or $n - 2$ if $a_{n-1} = 1$, we are done by induction. If n is even we have

$$p_{n-2}/q_{n-2} < p/q < p_{n-1}/q_{n-1}.$$

In this case the intermediate values of the convergents [15, pp. 19–20] enter in.

The intermediate values are the fractions

$$\frac{p_{n-2}}{q_{n-2}} < \frac{p_{n-2} + p_{n-1}}{q_{n-2} + q_{n-1}} < \cdots < \frac{p_{n-2} + ip_{n-1}}{q_{n-2} + iq_{n-1}}$$
$$< \cdots < \frac{p_{n-2} + a_n p_{n-1}}{q_{n-2} + a_n q_{n-1}} = \frac{p}{q}. \tag{2}$$

Either by noting that each intermediate value is obtained from the previous one and the fraction p_{n-1}/q_{n-1} by adding numerators and denominators and also that successive convergents of a continued fraction are adjacent in the Farey series (or by direct computation using $p_{n-1}q_{n-2} - p_{n-2}q_{n-1} = 1$), we see that the positive difference of two successive intermediate values is 1 divided by the product of their denominators. Thus we obtain

$$\frac{p}{q} = \frac{p_{n-2}}{q_{n-2}} + \sum_{i=0}^{a_n - 1} \frac{1}{(q_{n-2} + iq_{n-1})(q_{n-2} + (i+1)\, q_{n-1})}. \tag{3}$$

Since we can assume inductively that p_{n-2}/q_{n-2} has an expansion as an Egyptian fraction with denominators not exceeding $q_{n-2}\,(q_{n-2} - 1)$, we obtain an Egyptian fraction expansion of p/q in which no denominator exceeds $q(q-1)$. In fact this yields the identical expansion as the Farey series, for the relevant terms of the Farey series are precisely the intermediate values.

Since except for the last step when n is odd, the algorithm only uses even steps in the induction and since for $n = 0$ the fraction $p_0/q_0 = 0/1$ is not needed we have a total of $1 + a_2 + a_4 + \cdots + a_n$ steps when n is even.

We sum up the discussion in the above paragraphs in the following two theorems.

THEOREM 1. *The Farey Series algorithm yields Egyptian fraction expansions for every fraction p/q between* 0 *and* 1 *in which the number of terms is at most $p - 1$ and the size of the denominators is at most $q(q-1)$.*

THEOREM 2. *The Farey Series algorithm yields an Egyptian fraction Expansion for every fraction between* 0 *and* 1. *If $p/q = [0; a_1, a_2, \ldots, a_n]$ then the number of terms in the Egyptian fraction expansion obtained from the algorithm is at most $1 + a_2 + a_4 + \cdots + a_{n^*}$ where $n^* = 2[n/2]$.*

If one takes differences of nonsuccessive terms in the series of formula (2), one does not get a numerator of 1. However, by picking the right size step in the right place the numerator can be cancelled into the denominator. There are two difficulties in this process. The first is to pick the jumps large enough and often enough to make a substantial saving in the order of magnitude of the number of steps required. The second, more difficult problem is to show that all the denominators remain distinct after the cancellations and that none of them become equal to any of the denominators chosen at a previous step in the inductive process. We shall give an algorithm, the Continued Fraction Algorithm, for doing this.

III. The Continued Fraction Algorithm for Egyptian Fractions

In this section we define the Continued Fraction Algorithm and prove that it yields an Egyptian Fraction expansion with small denominators. Namely, we prove

THEOREM 3. *For every reduced fraction p/q, $0 < p < q$ the Continued Fraction Algorithm yields an Egyptian Fraction expansion*

$$\frac{p}{q} = \sum_{i=1}^{k} \frac{1}{n_i}$$

with $n_1 < n_2 < n_3 < \cdots < n_k \leqslant q(q-1)$ and $k \leqslant p$.

In the next section we estimate the size of k more carefully.

Proof. The inequality $k \leqslant p$ will follow from the fact that the Continued Fraction Algorithm yields no more steps than the Farey Series Algorithm, and Theorem 2.

The algorithm is explained inductively on n, the length of the continued fraction expansion of p/q given by (1).

Case 1. $n = 1$. In this case $p/q = 1/q$, yielding the desired representation.

Case 2. $n > 1$ and n is odd. In this case

$$p_{n-1}/q_{n-1} < p/q = p_n/q_n < p_{n-2}/q_{n-2}\,,$$

so that

$$p/q = 1/qq_{n-1} + p_{n-1}/q_{n-1}\,.$$

By the induction we already have an expansion of p_{n-1}/q_{n-1}. Since

$q_{n-1} < q$ and by induction we see that the denominators are less than $q(q-1)$.

Case 3. $n > 1$ and n is even.
Let P be the least prime such that

(1) $(P, q_{n-1}) = 1$.
(2) $2q_{n-1} + 1$ is not a power of P.

LEMMA

1. $P - 1 \equiv 0 \bmod q_{n-1}$ *only if*
   (a) $P = 2,\ q_{n-1} = 1$;
   (b) $P = 3,\ q_{n-1} = 2$;
   (c) $P = 5,\ q_{n-1} = 4$.
2. $P^2 - 1 \equiv 0 \bmod q_{n-1}$ *only if*
   (a) $P = 2,\ q_{n-1} = 1{,}3$;
   (b) $P = 3,\ q_{n-1} = 2$ or 8;
   (c) $P = 5,\ q_{n-1} = 4, 6, 24$;
   (d) $P = 7,\ q_{n-1} = 12$.
3. $P^3 - 1 \equiv 0 \bmod q_{n-1}$ *only if*
   (a) $P = 2,\ q_{n-1} = 1, 7$;
   (b) $P = 3,\ q_{n-1} = 2$;
   (c) $P = 5,\ q_{n-1} = 4$.

Proof. Let P be the k-th prime.

Since P is either the smallest or second smallest prime not dividing q_{n-1}, q_{n-1} is the least as big as $\prod'_{p<r} p$ where r is the $(k-1)$-st prime. For $P = 11$,

$$q_{n-1} \geqslant 2 \cdot 3 \cdot 5 = 30 > P$$

for each increase of P to the next prime the lower bound for q_{n-1} is multiplied at least by 7 while P is at most doubled. Thus for q_{n-1} to divide $P - 1$ we need only consider $P < 7$. Some arithmetic now settles the part of the Lemma dealing with $P - 1 \equiv 0 \bmod q_{n-1}$.

Similarly for $P^2 - 1 \equiv 0 \bmod q_{n-1}$, since $2 \cdot 3 \cdot 5 \cdot 7 > 13^2 > 13^2 - 1$ we need only consider $P < 11$. For $P^3 - 1 \equiv 0 \bmod q_{n-1}$, since $2 \cdot 3 \cdot 4 \cdot 5 \cdot 11 \cdot 13 = 30\,030 > 19^3 = 6859$ we need only consider $P < 17$. We now check if any q_{n-1} dividing $P^2 - 1$ $(P^3 - 1)$ requires P, for $P < 11$ $(P < 17)$. The indicated arithmetic establishes the lemma.

Let $t = [\log_P a_n]$, so that

$$P^t \leqslant a_n < P^{t+1} \tag{4}$$

Case 3.0. $t = 0$. In this case, a_n must be small compared to q_{n-1} and therefore also compared to q and we proceed as in the Farey Series Algorithm to get a representation of p/q as a sum of the a_n difference of the intermediate fraction plus the inductive representation for p_{n-2}/q_{n-2}.

Case 3.1. $t = 1$. In this case, choose d to satisfy

$$0 \leqslant d < P, \qquad q - dq_{n-1} \equiv 0 \mod P. \tag{5}$$

This can be done since $(q_n, P) = 1$. Thus

$$q - dq_{n-1} = (a_n - d)\, q_{n-1} + q_{n-2} \equiv 0 \mod P, \tag{6}$$

since $q = a_n q_{n-1} + q_{n-2}$.

We now expand d and $a_n - d$ base P to get

$$\begin{gathered} a_n - d = b_1 P + b_0 = (b_1 b_0)_P \\ 0 \leqslant b_1 < P \qquad \text{and} \qquad 0 \leqslant b_0 < P. \end{gathered} \tag{7}$$

Looking at (2) we replace (3) of the Farey Series Algorithm by

$$\begin{aligned} \frac{p}{q} = {} & \sum_{k=0}^{d-1} \frac{p_{n-2} + (a_n - k)\, p_{n-1}}{q_{n-2} + (a_n - k)\, q_{n-1}} - \frac{p_{n-2} + (a_n - k - 1)\, q_{n-1}}{q_{n-2} + (a_n - k - 1)\, q_{n-1}} \\ & + \sum_{m=0}^{b_1-1} \frac{p_{n-2} + ((m+1)\, P + b_0)\, p_{n-1}}{q_{n-2} + ((m+1)\, P + b_0)\, q_{n-1}} - \frac{p_{n-2} + (mP + b_0)\, p_{n-1}}{q_{n-2} + (mP + b_0)\, q_{n-1}} \\ & + \sum_{l=0}^{b_0-1} \frac{p_{n-2} + (l+1)\, p_{n-1}}{q_{n-2} + (l+1)\, q_{n-1}} - \frac{p_{n-2} + lp_{n-1}}{q_{n-2} + lq_{n-1}} + \frac{p_{n-2}}{q_{n-2}}. \end{aligned} \tag{8}$$

The equality holds since for $k = 0$ we get p/q on the R.H.S. and every other term occurs once with a plus sign and once with a minus sign, where it should be noticed that the negative term of $k = d - 1$ cancels the positive term of $m = b_l - 1$ since $a_n - d = (b_1 b_0)_P$.

We next note that

$$\begin{aligned} & \frac{p_{n-2} + (i+j)\, p_{n-1}}{q_{n-2} + (i+j)\, q_{n-1}} - \frac{p_{n-2} + ip_{n-1}}{q_{n-2} + iq_{n-1}} \\ & \qquad = \frac{j}{(q_{n-2} + (i+j)\, q_{n-1})(q_{n-2} + iq_{n-1})}, \end{aligned} \tag{9}$$

where the verification uses the fact that $q_{n-2} p_{n-1} - p_{n-2} q_{n-1} = 1$.

Thus (8) simplifies to

$$\frac{p}{q} = \sum_{k=0}^{d-1} \frac{1}{\{q_{n-2} + (a_n - k)\, q_{n-1}\}\{q_{n-2} + (a_n - k - 1)\, q_{n-1}\}}$$

$$+ \sum_{m=0}^{b_1-1} \frac{P}{\{q_{n-2} + ((m+1)\, P + b_0)\, q_{n-1}\}\{q_{n-2} + (mP + b_0)\, q_{n-1}\}}$$

$$+ \sum_{m=0}^{b_0-1} \frac{1}{\{q_{n-2} + (l+1)\, q_{n-1}\}\{q_{n-2} + lq_{n-1}\}} + \frac{p_{n-2}}{q_{n-2}}. \tag{10}$$

To see that the middle sum really involves unit fractions we note that

$$\begin{aligned} &q_{n-2} + ((m+1)\, P + b_0)\, q_{n-1} \\ &\qquad = \{q_{n-2} + (a_n - d)\, q_{n-1}\} - \{(b_1 - m - 1)\, Pq_{n-1}\}. \end{aligned} \tag{11}$$

By (6) the first term of the R.H.S. is divisible by P while the second term clearly is divisible by P. Thus the L.H.S. is also divisible by P. It follows that the fractions of (10) all reduce to unit fractions. Also since

$$[q_{n-2} + ((m+1)\, P + b_0)\, q_{n-1}]/P > q_{n-1} \geqslant q_{n-2},$$

we see that none of the reduced denominators of (10) are used in the inductive expansion of p_{n-2}/q_{n-2} since all those denominators are less than $q_{n-2}(q_{n-2} - 1)$. Also, the denominators used here are at most

$$q_n(q_n - q_{n-1})^2 < q_n(q_n - 1);$$

hence, if we can show that all the terms in all three sums are distinct from one another, we are through with this case.

No two terms from the same sum can be equal since the numerators are constant and the terms in the denominator change monotonically.

It is also clear that no terms of the first and third sums can be equal since the denominators of the first sum are much larger.

We must verify that no term of the second sum can equal any term of either the first or third sum. But the denominators of the first sum are much larger than those of the middle sum and hence since $P > 1$ there can be no equalities between terms of the first and second sum.

If a term of the middle sum of (10) equals a term of the third sum, then counting powers of P we see that

$$\{q_{n-2} + (l+1)\, q_{n-1}\}\{q_{n-2} + lq_{n-1}\} \equiv 0 \mod P$$

and thus, by (6) and (7), $l + 1 = b_0$.

Thus if two terms are equal, we get

$$P/\{q_{n-2} + ((m+1)P + b_0)\, q_{n+1}\}\{q_{n-2} + (mP + b_0)\, q_{n-1}\}$$
$$= 1/\{q_{n-2} + b_0 q_{n-1}\}\{q_{n-2} + (b_0 - 1)\, q_{n-1}\}. \tag{12}$$

Cross multiplying and considering the equality modulo q_{n-1} we get

$$Pq_{n-2}^2 \equiv q_{n-2}^2 \mod q_{n-1}. \tag{13}$$

Since q_{n-1}, q_{n-2} are successive convergents of a continued fraction they are relatively prime and (13) reduces to

$$P - 1 \equiv 0 \mod q_{n-1}. \tag{14}$$

By Lemma 1, we see that the only possibilities are:

$$\begin{array}{lllll} P = 2, & q_{n-1} = 1, & b_0 = 1, & m = 0, & q_{n-1} = 1, \\ P = 3, & q_{n-1} = 2, & 1 \leqslant b_0 \leqslant 2, & 0 \leqslant m \leqslant 1, & q_{n-2} = 1, \\ P = 5, & q_{n-1} = 4, & 1 \leqslant b_0 \leqslant 4, & 0 \leqslant m \leqslant 3, & q_{n-2} = 1, 3, \end{array}$$

where the possibilities for q_{n-2} are restricted by $(q_{n-1}, q_{n-2}) = 1$ and $q_{n-2} \leqslant q_{n-1}$. Using the fact that $P \mid \{q_{n-2} + b_0 q_{n-1}\}$ to calculate b_0, some arithmetic shows that equality cannot occur.

Case 3.2. $t \geqslant 2$. In this case, pick d to satisfy

$$0 \leqslant d < P^{t+1} \tag{15}$$

and

$$(a_n - d)\, q_{n-1} + q_{n-2} \equiv 0 \mod P^{t-1}. \tag{16}$$

This can be done since $(P, q_{n-1}) = 1$.

We next expand d and $a_n - d$ to the base P to obtain

$$d = (c_s c_{s-1} \cdots c_1 c_0)_P, \qquad 0 \leqslant c_i < P, \quad s \leqslant t - 2, \tag{17}$$

where either $d = 0$ or $c_s \neq 0$ and

$$a_n - d = (b_r b_{r-1} \cdots b_1 b_0) \qquad 0 \leqslant b^c < P \quad t - 1 \leqslant r \leqslant t \quad b_r \neq 0. \tag{18}$$

The inequality on r follows from the fact that $P^{t+1} > a_n \geqslant P^t$ and $P^{t-1} > d$.

Now we prove two Lemmas on the divisibility by P of the factors of the denominators of the intermediate fractions.

LEMMA 2. *Let* $0 \leqslant k \leqslant b_j$; $0 \leqslant j + i \leqslant t - 1$; *then*

$$q_{n-2} + (kb_{j-1} \cdots b_0)_P q_{n-1} \equiv 0 \mod P^{i+j} \tag{19}$$

if and only if either $i = 0$ *or both* $k = b_j$ *and* $b_{j+1} = b_{j+2} = \cdots = b_{j+i-1} = 0$.

Proof. We know by choice of d that

$$q_{n-2} + (b_r \cdots b_0)_P \, q_{n-1} \equiv 0 \mod P^{t-1}. \tag{20}$$

Subtracting the L.H.S. of (19) from the L.H.S. of (20) we get

$$(b_r \cdots b_{j+1}(b_j - k) \, 0 \cdots 0)_P \, q_{n-1} \,. \tag{21}$$

It is clear that the L.H.S. of (19) is divisible by P^{j+i} if and only if (21) is, which holds if and only if the conditions of the lemma are satisfied, since $(p, q_{n-1}) = 1$. This lemma is proved.

Similarly for $j \leqslant s$ and $0 \leqslant k \leqslant c_j$, we get

$$q_n - (kc_{j-1} \cdots c_0)_P \, q_{n-1} = q_n - dq_{n-1} + ((c_s \cdots c_j 0 \cdots 0)_P - kP^j) \, q_{n-1}$$

which leads to

LEMMA 3. *Let* $0 \leqslant k < P$, $0 \leqslant j \leqslant s$, *and* $0 \leqslant i \leqslant s - j$, *then a necessary and sufficient condition that*

$$q_{n-2} + ((b_r \cdots b_0)_P + (c_s \cdots (c_j - k) \, 0 \cdots 0)_P \, q_{n-1} \equiv 0 \mod^{j+i} P \tag{22}$$

is either $i = 0$ *or both*

$$k = c_j \qquad \textit{and} \qquad c_{j+1} = c_{j+2} = \cdots = c_{j+i-1} = 0.$$

Lemmas 2 and 3 tell us that for certain values of i we can take $j = P^i$, $0 \leqslant i \leqslant t$, and the right-hand side of (9) will reduce to a unit fraction. Thus we are led to consider the following expression for p/q:

$$\begin{aligned} \frac{p}{q} = \sum_{j=0}^{s} \sum_{k=0}^{c_j - 1} \Bigg\{ & \frac{p_{n-2} + (a_n - (kc_{j-1} \cdots c_1 c_0)_P) \, p_{n-1}}{q_{n-2} + (a_n - (kc_{j-1} \cdots c_1 c_0)_P) \, q_{n-1}} \\ & - \frac{p_{n-2} + (a_n - ((k+1) \, c_{j-1} \cdots c_1 c_0)_P) \, p_{n-1}}{q_{n-2} + (a_n - ((k+1) \, c_{j-1} \cdots c_1 c_0)_P) \, q_{n-1}} \Bigg\} \\ & + \sum_{l=0}^{r} \sum_{m=0}^{b_l - 1} \Bigg\{ \frac{p_{n-2} + ((m+1) \, b_{l-1} \cdots b_0)_P \, p_{n-1}}{q_{n-2} + ((m+1) \, b_{l-1} \cdots b_0)_P \, q_{n-1}} \\ & - \frac{p_{n-2} + (mb_{l-1} \cdots b_0)_P \, p_{n-1}}{q_{n-2} + (mb_{l-1} \cdots b_0)_P \, q_{n-1}} \Bigg\} + \frac{p_{n-2}}{q_{n-2}} \,. \end{aligned} \tag{23}$$

To see that the equality is correct we observe that for $j = 0$, $k = 0$ we have p/q as the positive term, for $l = 0$, $m = 0$ we have p_{n-2}/q_{n-2} as the negative term. Every other term in the double sums occurs once with a "+" and once with a "−", where formulas (16)–(18) are used to see that the smallest negative term of the first double sum is the same as the largest positive term in the second double sum.

Using formula (9) and the recursion formula for $q = q_n$ formula (23) reduces to

$$\frac{p}{q} = \sum_{j=0}^{s} \sum_{k=0}^{c_j-1} \frac{P^j}{\{q - (kc_{j-1} \cdots c_0)_P\, q_{n-1}\}\{q - ((k+1)\, c_{j-1} \cdots c_0)_P\, q_{n-1}\}}$$
$$+ \sum_{l=0}^{r} \sum_{m=0}^{b_l-1} \frac{P^l}{\left\{ \begin{array}{r}\{q_{n-2} + ((m+1)\, b_{l-1} \cdots b_0)_P\, q_{n-1}\} \\ \times \{q_{n-2} + (mb_{l-1} \cdots b_0)_P\, q_{n-1}\}\end{array} \right\}} + \frac{p_{n-2}}{q_{n-2}}. \tag{24}$$

We see from Lemmas 2 and 3 that all the fractions on the right hand side of (24), with the sole exception of p_{n-2}/q_{n-2} reduce to fractions with numerator 1. If $l = r = t$ then both terms in the denominator are dvisible P^{t-1} and since $t - 1 \geqslant 1$ the denominator is divisible by P^t.

The Distinctness of the Terms

The terms from the first double sum are all distinct since as j and k increase, the denominators get smaller and the numerators get bigger.

The second double sum is more troublesome since both the numerator and the denominator are increasing; however, it is clear that for a fixed value of l, distinct values of m yield distinct fractions. We suppose now that for two different values, l and x, with $l > x$ we have equal fractions, and we derive a contradiction. Thus, suppose

$$P^l/\{q_{n-2} + ((m+1)\, b_{l-1} \cdots b_0)_P\, q_{n-1}\}\{q_{n-2} + (mb_{l-1} \cdots b_0)_P\, q_{n-1}\}$$
$$= P^x/\{q_{n-2} + ((y+1)\, b_{x-1} \cdots b_0)_P\, q_{n-1}\}\{q_{n-2} + (yb_{x-1} \cdots b_0)_P\, q_{n-1}\} \tag{25}$$

which is equivalent to

$$P^l\{q_{n-2} + ((y+1)\, b_{x-1} \cdots b_0)_P\, q_{n-1}\}\{q_{n-2} + (yb_{x-1} \cdots b_0)_P\, q_{n-1}\}$$
$$= P^x\{q_{n-2} + ((m+1)\, b_{l-1} \cdots b_0)_P\, q_{n-1}\}\{q_{n-2} + (mb_{l-1} \cdots b_0)_P\, q_{n-1}\}. \tag{26}$$

If $l \leqslant t - 1$, we see from Lemma 2 that the right-hand side is divisible by P^{x+2l} and also that $q_{n-2} + (yb_{x-1} \cdots b_0)_P\, q_{n-1}$ is divisible by p^x but no higher power of P. It follows that P^l divides

$$q_{n-2} + ((y+1)\, b_{x-1} \cdots b_0)_P\, q_{n-1}\,.$$

Again by Lemma 2 we conclude that

$$y + 1 = b_x \qquad \text{and} \qquad b_x = b_{x+1} = \cdots = b_{l-1} = 0. \tag{27}$$

Since $b_l \geqslant m + 1$, $b_l \neq 0$; hence by Lemma 2, P^l is the highest power of P dividing $q_{n-2} + ((y+1)\, b_{x-1} \cdots b_0)_P\, q_{n-1}$. Thus in order for equality in (26) to hold, P^l must be the highest power of P dividing $q_{n-2} + ((m+1)\, b_{l-1} \cdots b_0)_P\, q_{n-1}$.

For $l = t$ the right hand side of (26) is divisible by at least $P^{x+2(t-1)}$, while the left hand side is divisible by P^{t+x+x^*} where x^* is the power of P dividing $\{q_{n-2} + ((y+1)\, b_{x-1} \cdots b_0)_P\, q_{n-1}\}$. Thus $x^* \geqslant t - 2$.

For $x \leqslant t - 3$

$$y + 1 = b_x\,, \qquad b_{x+1} = b_{x+2} = \cdots = b_{t-3} = 0. \tag{28}$$

When $l = t$ we shall consider three cases:

Case 1. $b_{t-1} \neq 0$;
Case 2. $b_{t-1} = 0,\ b_{t-2} \neq 0$;
Case 3. $b_{t-1} = 0,\ b_{t-2} = 0$;

In case 3 we see that $x \leqslant t - 3$ and (27) holds. So Case 3 for $l = t$ will be treated with the case $l \leqslant t - 1$, when it becomes necessary to divide the proof into cases.

Looking at (26) modulo q_{n-1}, we get $P^l q_{n-2}^2 \equiv P^x q_{n-2}^2 \bmod q_{n-1}$. Since q_{n-1} and q_{n-2} are successive convergents of a continued fraction they are relatively prime and the factor q_{n-2}^2 can be canceled to yield $P^l \equiv P^x$. By choice of P, $(P, q_{n-1}) = 1$, thus P^x may be canceled, yielding

$$P^{l-x} - 1 \equiv 0 \mod q_{n-1}\,. \tag{29}$$

We now show that $m = 0$. Suppose $m \neq 0$. Let

$$B = \{q_{n-2} + (b_x \cdots b_0)_P\, q_{n-1}\}. \tag{30}$$

Since $q_{n-2} \leqslant q_{n-1}$ we see that $B \leqslant P^{x+1} q_{n-1}$ with equality possible only if $q_{n-2} = q_{n-1} = 1$ and $P = 2$.

We rewrite (26) as

$$P^{l-x} = \frac{\{(m+1)\, P^l q_{n-1} + \cdots + B\}\{mP^l q_{n-1} + \cdots + B\}}{\{B - wP^x q_{n-1}\}\{B - (w+1)\, P^x q_{n-1}\}}\,, \tag{31}$$

where $w = b_x - (y + 1)$. Since all the factors are positive on the right hand side of (31) it is decreasing in B; thus

$$P^{l-x} \geqslant \frac{\{(m+1)\,P^l + \cdots + P^{x+1}\}\{mP^l + \cdots + P^{x+1}\}}{\{P^x(P-w)\}\{P^x(P-w-1)\}} \tag{32}$$

with equality possible only for $P = 2$. Thus

$$1 \geqslant \frac{(m+1)(m)\,P^{l-x}}{(P-w)(P-w-1)}.$$

This is impossible, unless $l - x = 1$. In this case since $P^{x+1} = P^l$, (32) yields

$$1 \geqslant \frac{(m+2)(m+1)\,P}{(P-w)(P-w-1)} \geqslant \frac{(m+2)(m+1)}{P-1} \tag{33}$$

again with equality possible only for $P = 2$. Since $m \geqslant 1$ this can only happen for $P > 7$. But it follows from (29) with $l - x = 1$ and Lemma 1 that $P \leqslant 5t$, which is a contradiction. Thus $m > 0$ is impossible.

We now handle Case 1 of $l = t$.

Suppose $x = t - 1$. If $w = 0$, then (27) holds. This will be handled with $l \leqslant t - 1$. Thus we may suppose $w \geqslant 1$. From (29) and Lemma 1, we deduce that $P \leqslant 5$. The first inequality of (33) is possible only for $P = 5$, $w = 1, m = 0$. Formula (31) becomes

$$P = (4P^t + B)B/(B - 4P^{t-1})(B - 8P^{t-1}),$$

where Lemma 1 tells us that $q_{n-1} = 4$.

In solving this quadratic equation for the integer B, the discriminant Δ^2 must be a perfect square, but

$$\Delta^2 = 64P^{2(t-1)}(8),$$

which is not a perfect square. Thus the equality (25) is impossible in Case 1 when $x = t - 1$.

We suppose $x \leqslant t - 2$. In this case, (32) yields

$$1 \geqslant \left(1 + \frac{b_{t-1}}{P} + \cdots + \frac{P}{P^{t-x}}\right)\left(\frac{b_{t-1}P^{t-1} + \cdots + P^{x+1}}{(P-w)(P-w-1)\,P^x}\right). \tag{34}$$

This is impossible for $t - 1 \geqslant x + 2$, i.e., for $x \leqslant t - 3$. Thus we need only consider $x = t - 2$. In this case (34) becomes

$$1 \geqslant \left(1 + \frac{b_{t-1}+1}{P}\right)(b_{t-1}+1)\,\frac{P}{(P-w)(P-w-1)},$$

which is the same as

$$1 \geqslant \frac{(P + b_{t-1} + 1)(b_{t-1} + 1)}{(P - w)(P - w - 1)}. \tag{35}$$

Sum $b_{t-1} \geqslant 1$ and $w \geqslant 0$; (35) yields $1 \geqslant 2(P + 2)/P(P - 1)$ which implies $P \geqslant 5$. For $P = 5$, (35) can hold only if $w = 0$, $b_{t-1} = 1, 2$ or $w = 1$, $b_{t-1} = 1$. It follows from (29) with $l - x = 2$ and Lemma 1 that $P = 5$. Thus we need to eliminate the possibility that (31) holds for $P = 5$, $q_{n-1} = 4$, $q_{n-2} = 1, 3$, $l = t$, $x = t - 2$, $m = 0$ and either $w = b_x - (y + 1) = 0$, $b_{t-1} = 1, 2$ or $w = b_x - (y + 1) = 1$, $b_{t-1} = 1$. In this case, (31) simplifies to

$$25 = \frac{(5^t q_{n-1} + b_{t-1} 5^{t-1} q_{n-1} + B)(b_{t-1} 5^{t-1} q_{n-1} + B)}{(B - w 5^{t-2} q_{n-1})(B - (w + 1)\, 5^{t-2} q_{n-1})}.$$

This is a quadratic equation for q_{n-1}. We find the discriminant is

$$\Delta^2 = B^2 \cdot 4 \cdot 5^{2(t-1)}\{(wP - b_{t-1})^2 - 24(w(w + 1) - b_{t-1}(P + b_{t-1}))\}$$

Thus

$$\delta^2 = \frac{\Delta^2}{B^2 \cdot 4 \cdot 5^{2(t-1)}} \equiv w(w + 1) \qquad \text{mod } 5.$$

Since $w = 1$ does not yield a quadratic residue we must have $w = 0$. For $w = 0$, δ^2 becomes $25 b_{t-1}^2 + 24 \cdot 5 \cdot b_{t-1}$ which is not a square for $b_{t-1} = 1$ or 2. Thus Case 1 of $l = t$ cannot occur.

We next turn to Case 2 of $l = t$ which yields to similar methods. First we note that since $b_{t-1} = 0$, $x \leqslant t - 2$.

When $x \leqslant t - 4$ (32) yields the contradictory inequality

$$P^{t-x} > \frac{P^{t-x} \cdot (b_{t-2})}{(P - w)(P - w - 1)} \cdot P^{t-x-2} > b_{t-2} P^{t-x} \geqslant P^{t-x}.$$

For $x = t - 3$, (32) yields

$$1 \geqslant \frac{\left(1 + \dfrac{b_{t-2} + 1}{P}\right)}{P - w} \cdot \frac{(b_{t-2} + 1)}{P - w - 1} \cdot P. \tag{36}$$

But since $0 \leqslant w \leqslant P - 1$, $1 \leqslant b_{t-2} \leqslant P - 1$, (36) can not hold for $P = 2, 3$ and for $P = 5$ only for the following values of w and b_{t-2}.

$$\begin{aligned} P = 5, \qquad & w = 0, \qquad 1 \leqslant b_{t-2} \leqslant 2, \\ & w = 1, \qquad b_{t-2} = 1. \end{aligned} \tag{37}$$

Since (29) holds, we know from Lemma 1 that the only possibility is with $P = 5$, $q_{n-1} = 4$. We can now use the quadratic formula to solve (31) for B. But for none of the permissible values of w and b_{t-2} do we get an integer. Hence $x \neq t - 3$.

We now consider $x = t - 2$.

In this case, (37) yields

$$1 \geqslant \frac{\left(1 + \frac{1}{P}\right)}{P - w} \cdot \frac{P}{(P - w - 1)} = \frac{P + 1}{(P - w)(P - w - 1)} \tag{38}$$

with equality possible only for $P = 2$.

Since (26) holds with $l - x = 2$ Lemma 1 says we need only consider $P \leqslant 7$. Also for $w = 0$ we have $y + 1 = b_{t-2} = b_x$ and since $b_{t-1} = 0$, Formula (27) holds and this will be handled later with $l \leqslant t - 1$.

Thus the only possibilities for (38) are

$$\begin{aligned} P &= 5, \qquad w = 1, \\ P &= 7, \qquad w = 1, 2, 3. \end{aligned}$$

Equation (31) becomes

$$P^2 = (P^t q_{n-1} + B)B/(B - wP^{t-2}q_{n-1})(B - (w + 1)\, P^{t-2}q_{n-1}).$$

We can solve this for B by the quadratic formula, the discriminant is

$$\Delta^2 = 4q_{n-1}^2 P^{2(t-1)}(P^2(w + 1) - w(w + 1)).$$

This last expression is not a square for any of the permissible values of P and w, as an examination of the quantity in parentheses will show.

We have thus shown that if $l = t$ for no value of x can (25) hold in Cases 1 and 2.

We now handle Case 3 of $l = t$ together with the case $l < t$, by assuming only the conditions of formula (27).

Since $m = 0$, from (27) we see that

$$q_{n-2} + ((y + 1)\, b_{x-1} \cdots b_0)_P\, q_{n-1} = q_{n-2} + (mb_{l-1} \cdots b_0)_P\, q_{n-1}\,.$$

and

$$((m + 1)\, b_{l-1} \cdots b_0)_P = (yb_{x-1} \cdots b_0)_P + P^l + P^x.$$

In this case after cancellation, (26) reduces to

$$q_{n-2}(P^{l-x} - 1) = q_{n-1}(P^l + P^x + (1 - P^{l-x})(yb_{x-1} \cdots b_0)_P) \tag{39}$$

Considering (39) modulo $P^{l-x} - 1$ we get

$$q_{n-1}(P^l + P^x) \equiv 0 \mod (P^{l-x} - 1).$$

Since $(P^{l-x} - 1, P^l + P^x) = 1$ or 2, respectively, as P is 2 or odd respectively we deduce that

$$2q_{n-1} \equiv 0 \mod P^{l-x} - 1. \tag{40}$$

Combining (29) and (40) we conclude that

$$q_{n-1} = P^{l-x} - 1 \qquad \text{or} \qquad 2q_{n-1} = P^{l-x} - 1.$$

The condition $2q_{n-1} = P^{l-x} - 1$ is forbidden by our choice of P. We now show that $q_{n-1} = P^{l-x} - 1$ is not possible under these circumstances. Using $y + 1 = b_x$ and $q_{n-1} = P^{l-x} - 1$ (39) can be simplified to

$$q_{n-2} = 2P^l + (b_x \cdots b_0)_P(1 - P^{l-x}). \tag{41}$$

For P odd, this implies q_{n-2} is even, but so is $q_{n-1} = P^{l-x} - 1$, this contradicts $(q_{n-1}, q_{n-2}) = 1$.

For $P = 2$ (41) becomes

$$q_{n-2} + (b_x \cdots b_0)_P(2^{l-x} - 1) = 2^{l+1}. \tag{42}$$

Since $q_{n-2} \leqslant q_{n-1} = 2^{l-x} - 1$ and $(b_x \cdots b_0)_P \leqslant P^{x-1} - 1$, we obtain from (42) the inequality

$$2^{l-x} - 1 + (2^{x+1} - 1)(2^{l-x} - 1) \geqslant 2^{l+1}. \tag{43}$$

But (43) simplifies to the contradictory

$$2^{l+1} - 2^{x+1} \geqslant 2^{l+1}.$$

This completes the proof that no two terms of the second double sum of (24) can be equal.

We are now faced with the easier task of showing no term of the first double sum of (24) can equal any term of the second.

If $d = 0$ in formula (17), then there is only one double sum in (24) and we are finished. If $d \neq 0$, we suppose some term in the first double sum of (24) is equal to some term in the second double sum. Setting these terms equal, cross multiplying, and using the relations $q = a_n q_{n-1} + q_{n-2}$ and $a_n = (b_r \cdots b_0)_P + (c_s \cdots c_0)_P$, we obtain the following

$$\begin{aligned} P^j\{q_{n-2} + (mb_{l-1} \cdots b_0)\, q_{n-1})\}\{q_{n-2} + ((m+1)\, b_{l-1} \cdots b_0)_P\, q_{n-1}\} \\ = P^l\{q_{n-2} + ((b_r \cdots b_0)_P + (c_s \cdots c_{j+1}(c_j - k)\, 0 \cdots 0)_P)\, q_{n-1}\} \\ \cdot \{q_{n-2} + ((b_r \cdots b_0)_P + (c_s \cdots c_{j+1}(c_j - k - 1)\, 0 \cdots 0)_P)\, q_{n-1}\}. \end{aligned} \tag{44}$$

We know that $j \leqslant s < r$ from formulas (17) and (18).

Dividing both sides of (44) by P^{l+j} yields

$$AB = CD, \tag{45}$$

where

$$A = P^{-j}\{q_{n-2} + (b_r \cdots b_0)_P\, q_{n-1} + (c_s \cdots c_{j+1}(c_j - k)\, 0 \cdots 0)_P\, q_{n-1}\}, \tag{46}$$

$$B = q_{n-2} + (b_r \cdots b_0)_P\, q_{n-1} + (c_s \cdots c_{j+1}(c_j - k - 1)\, 0 \cdots 0)_P\, q_{n-1}, \tag{47}$$

$$C = P^{-l}\{q_{n-2} + (m-1)\, P^l q_{n-1} + (b_{l-1} \cdots b_0)_P\, q_{n-1}\}, \tag{48}$$

$$D = q_{n-2} + ((m+1)\, b_{l-1} \cdots b_0)_P\, q_{n-1}\,. \tag{49}$$

Since $r \geqslant 1$ and since $m + 1 \leqslant b_l$, it is clear that $B \geqslant D$. On the other hand, using the fact that $j < r$ we get

$$A > b_r P^{r-j} q_{n-1} \geqslant P q_{n-1} > b_l q_{n-1} > C,$$

where for the last inequality we used $q_{n-1} \leqslant q_{n-2}$ and $(m+1) \leqslant b_l$. Thus $A > C$ and $B \geqslant D$ which contradicts (45). It follows that the terms of (24) are all distinct.

None of the terms in any double sum of (24) can equal any term in the inductively assumed expansion of p_{n-2}/q_{n-2}, since even after reducing the fractions the denominators are greater than $(q_{n-2})^2$ while those in the inductively supposed expansion of p_{n-2}/q_{n-2} are less than q_{n-2}^2. Also the denominators in the double sums are all less than $q(q - q_{n-1}) \leqslant q(q-1)$. Thus Theorem 3 is established.

IV. The Length of the Continued Fraction Algorithm

We begin this section by estimating the size of P in terms of each of q_{n-1} and q.

Let $\theta(x)$ be defined, as usual, by

$$\theta(x) = \sum_{P^* \leqslant X} \ln P^* = \ln \prod_{P^* \leqslant X} P^* \tag{50}$$

where P^* takes on the values of all primes not exceeding x.

Since P is either the smallest or second smallest prime not dividing q_{n-1}, if we choose X such that

$$\theta(X) \geqslant \ln q_{n-1} + \ln X, \tag{51}$$

then we know that $P \leqslant X$.

Rosser and Schoenfeld [27] have shown that

$$x(1 - (1/\ln x)) \leqslant \theta(x), \qquad \text{for} \quad x \geqslant 41. \tag{52}$$

LEMMA 4. *For* $\ln q_{n-1} \geqslant 36$ *we have* $P \leqslant \frac{3}{2} \ln q_{n-1}$.

Proof. For $\ln q_{n-1} \geqslant 36$ we have $\frac{3}{2} \ln q_{n-1} \geqslant 41$.
Using formula (52) we see that

$$\theta(\tfrac{3}{2} \ln q_{n-1}) \geqslant \tfrac{3}{2} \ln q_{n-1} \left(1 - \frac{1}{\ln \ln q_{n-1} + \ln \frac{3}{2}}\right).$$

By the remarks preceding (51), we see that it is sufficient to show that

$$\tfrac{3}{2} \ln q_{n-1} \left(1 - \frac{1}{\ln \ln q_{n-1} + \ln \frac{3}{2}}\right) \geqslant \ln q_{n-1} + \ln \ln q_{n-1} + \ln \tfrac{3}{2},$$

which is equivalent to

$$\frac{\ln q_{n-1}}{\ln \ln q_{n-1}} \geqslant \frac{2\left(1 + \dfrac{\ln \frac{3}{2}}{\ln \ln q_{n-1}}\right)}{\left(1 - \dfrac{3}{\ln \ln q_{n-1} + \ln \frac{3}{2}}\right)}. \tag{53}$$

But the L.H.S. is increasing in q_{n-1} as is the denominator of the R.H.S.; while the numerator is decreasing. Thus if (53) holds for a value of q_{n-1} it holds for all larger values. A computation shows that it is valid for $\ln q_{n-1} \geqslant 36$. Lemma 4 is established.

The first value of q_{n-1} to require a given prime P is the product, say Q, of all the primes less than P, unless for some prime $P_1 < P$ and some integer e such that $(P_1^e - 1)/2 = Q$ all the other primes less than P divide Q. The divisibility of $(P_1^e < 1)/2$ by all primes less than P can easily be tested by congruences for small P and e.

In this way we can compute Table I below which shows all primes which occur for q_{n-1} with $\ln q_{n-1} \leqslant 49$ as well as the first values of q_{n-1} for which that prime is needed.

In Table I the first column lists the prime P involved, the second column is the product of all primes less than P, called Q, the third column lists any values of q_{n-1}, with $3/2 \ln q_{n-1} \leqslant P - 1$, which require that prime and which are not multiples of Q, the fourth column lists the logarithm of the least q_{n-1} requiring that prime, and the fifth column lists nonmultiples of Q not listed in column 3 which satisfy $4/3 \ln q_{n-1} < P$.

The "—" in the third column indicates that P is needed only for multiplies of Q in the range $\ln q_{n-1} \leqslant 40.97$.

TABLE I

P	πp^* $p^* < P = Q$	$\frac{3}{2} \ln q_{n-1} \leqslant P - 1$	$\ln q_{n-1}$	$\frac{4}{3} \ln q_{n-1} < P$
2	1		0	
3	2		.69 ···	
5	6	4	1.38 ···	
7	30	12, 40	2.48 ···	
11	210		5.34 ···	1200
13	2310		7.74 ···	
17	30030		10.30 ···	
19	510510	—	13.14 ···	
23	9699690	—	16.07 ···	
29	223092870	—	19.53 ···	
31	6459693230	—	22.58 ···	
37	200560490130	—	26.02 ···	
41	304256253527210	—	33.34 ···	
47		—	37.06 ···	
53			40.91 ···	
57			44.88 ···	

Lemma 17, together with Table I, yields the following:

LEMMA 5. *Except for the following*:

$P = 2, \quad q_{n-1} = 1,$
$P = 3, \quad q_{n-1} = 2,$
$P = 5, \quad q_{n-1} = 4, 6,$
$P = 7, \quad q_{n-1} = 12, 30, 40,$
$P = 11, \quad q_{n-1} = 210, 420, 630,$
$P = 13, \quad q_{n-1} = 2310,$

it always happens that $\frac{3}{2} \ln q_{n-1} \geqslant \ln q_{n-1}$.

Proof. Lemma 4 and Table I.

We now estimate P in terms of $\ln q$.

COROLLARY. *For all possible* P,

$$P - 1 \leqslant 2 \ln q.$$

Proof. Since $\ln q \geqslant \ln q_{n-1}$ we need only consider the exceptions in the preceding lemma. But, recalling that $a_n \geqslant 2$,

$$q = a_n q_{n-1} + q_{n-2} \geqslant 2q_{n-1} + 1.$$

Thus $\ln q \geqslant \ln q_{n-1} + .69 \cdots$. A glance at Table I shows that

$$2 \ln q > P - 1$$

for the exceptional cases.

We next prove, by means of several Lemmas, the following main theorem.

THEOREM 4. *If p/q is a reduced fraction with $0 \leqslant p < q$ and $q > 2$, and*

$$\frac{p}{q} = \sum_{i=1}^{k} \frac{1}{n_i}$$

is the Egyptian Fraction Expansion of p/q obtained by the Continued Fraction Algorithm, then

$$n_i \leqslant q(q-1)$$

and

$$k \leqslant \min \left\{ \frac{2(\ln q)^2}{\ln \ln q}, p \right\}. \tag{54}$$

Proof. We have already shown that $k \leqslant p$ in Theorem 1, thus it is sufficient to show that

$$k \leqslant \frac{2(\ln q)^2}{\ln \ln q}.$$

LEMMA 6. *Let $f(x) = (x^2/\ln x)$, then for $x > 1$,*

$$Df(x) = \frac{x}{\ln x}\left(2 - \frac{1}{\ln x}\right) \geqslant 0 \qquad \textit{for} \quad x \geqslant e^{1/2} = 1.64 \cdots$$

$$D^2f(x) = \frac{2}{\ln x} - \frac{3}{(\ln x)^2} + \frac{2}{(\ln x)^3} > 0 \qquad \textit{for} \quad x > 1,$$

$$D^3f(x) = \frac{-2}{x(\ln x)^2} + \frac{6}{x(\ln x)^3} - \frac{6}{x(\ln x)^4} < 0 \quad \textit{for} \quad x > 1,$$

and the only minimum of f occurs at $x = e^{1/2}$ with value $f(e^{1/2}) = 2e$.

Proof. Except for $D^2f > 0$ and $D^3f < 0$ the computation is straightforward. To see that $D^2f > 0$, notice that

$$D^2f = \frac{2}{\ln x}\left(1 - \frac{1}{\ln x}\right)^2 + \frac{1}{(\ln x)^2}.$$

For $D^3 f < 0$, note that

$$D^3 f = \frac{-2}{x(\ln x)^2}\left(1 - \frac{\sqrt{3}}{\ln x}\right)^2 - \frac{2(2\sqrt{3} - 3)}{x(\ln x)^3}.$$

Lemma 6 is proved.

LEMMA 7. *For $q \leqslant 16$, formula* (54) *holds.*

Proof. For $3 \leqslant q \leqslant 11$,

$$k \leqslant p \leqslant q - 1 \leqslant 10 < 2(2e) \leqslant 2f(\ln q) = \frac{2(\ln q)^2}{\ln \ln q}.$$

For $12 \leqslant q \leqslant 14$, a computation shows

$$k \leqslant q - 1 \leqslant 13 < 2f(\ln 12) \leqslant 2f(\ln q).$$

For $q = 15, 16$ a short computation shows

$$k \leqslant q - 1 \leqslant \frac{2(\ln q)^2}{\ln \ln q},$$

which proves the lemma.

Having disposed of the lowest values of q, we proceed inductively as in the definition of the algorithm. As before we let n be the length of the continued fraction expansion of p/q as given by formula (1).

LEMMA 8. *If $n = 1$ then formula* (54) *holds.*

Proof. For $n = 1$, $k = 1$ and

$$1 \leqslant 2(2e) \leqslant 2f(\ln q).$$

Since the minimum of $f(\ln q)$ is $2e$ by Lemma 6.

LEMMA 9. *If $n > 1$ and n is odd, then formula* (54) *holds.*

Proof. Let k' be the length of the expansion of p_{n-1}/q_{n-1}. In this case $k = 1 + k'$.

For $q_{n-1} \leqslant 10$, $k' \leqslant 9$ so that

$$k = 1 + k' \leqslant 10 \leqslant 2(2e) \leqslant 2f(\ln q).$$

For $q_{n-1} \geqslant 11$, $\ln q_{n-1} > e^{1/2} > 1$ so that $f(\ln q)$ is increasing in q_{n-1} by Lemma 6. Also since $q = q_n = a_n q_{n-1} + q_{n-2}$, we see that $q \geqslant 23$ and

$$q_{n-1} = (q - q_{n-2})/a_n < q/a_n \leqslant q/2. \tag{55}$$

From the induction hypothesis and the above comments we see that

$$k = 1 + k' \leqslant 1 + 2f(\ln q_{n-1}) < 1 + 2f(\ln q/2). \tag{56}$$

Thus it suffices to show that for $q \geqslant 23$

$$1 + 2f(\ln q/2) \leqslant 2f(\ln q). \tag{57}$$

Formula (57) is equivalent to

$$\frac{\ln \ln q}{(\ln q)^2} + 2\left\{\frac{\ln q - \ln 2}{\ln q} \cdot \frac{\ln \ln q}{\ln q} \cdot \frac{\ln q - \ln 2}{\ln(\ln q - \ln 2)} - 1\right\} \leqslant 0. \tag{58}$$

In order to prove the validity of (58) we need some lemmas.

LEMMA 10. *Let* $g(z) = z/\ln z$, *then*

$$Dg(z) = \frac{1}{\ln z} - \frac{1}{(\ln z)^2}$$

$$Dg < 0 \qquad \textit{for} \quad z < e$$

$$Dg > 0 \qquad \textit{for} \quad z > e$$

$$D^2g(z) = \frac{-1}{z(\ln z)^2} + \frac{2}{z(\ln z)^3} \leqslant 0 \qquad \textit{for} \quad z \geqslant e^2$$

and the unique minimum of g *is at* $z = e$ *with value* $g(e) = e$.

Proof. Straightforward.

LEMMA 11. *Let* $g(z) = z/\ln z$; *then*

$$g(x - \ln 2)/g(x) \leqslant 1 \qquad \textit{for} \quad x \geqslant x_0 = 3.0 \cdots, \tag{59}$$

where x_0 *is the unique solution for equality to hold.*

Proof. Since, by the previous lemma, $g(z)$ has a unique minimum at $z = e$ with negative derivative before that point and positive derivative after that point, the only possibility for

$$g(x - \ln 2) = g(x)$$

is when $x - \ln 2 < e < x$ and there is only one such value. A computation shows $x_0 = 3.0...$. Since x_0 is the only value for equality in (59), continuity guarantees the proper inequality holds for all $x \geqslant x_0$ since it holds for $x = e + \ln 2 > x_0$. Lemma 11 is proved.

We return now to formula (58). Since we need only consider $q \geqslant 23$ and since $\ln 23 = 3.1... > x_0$, we see from Lemma 11, that it is sufficient to prove

$$\frac{\ln \ln q}{(\ln q)^2} + 2\left\{\frac{\ln q - \ln 2}{\ln q} - 1\right\} \leqslant 0$$

which is equivalent to

$$\frac{1}{\ln q}\left\{\frac{\ln \ln q}{\ln q} - 2 \ln 2\right\} \leqslant 0. \tag{60}$$

By Lemma 10, $\ln \ln q/\ln q \leqslant 1/e < 2 \ln 2$; thus (60) is valid, and therefore so is (57) and Lemma 9 is established.

We proceed now to the more complicated case when n is even. Similar to the definition of the algorithm, we shall need to consider separately the three possibilities $a_n < 2 \ln q$, $2 \ln q \leqslant a_n < P^2$ and $P^2 \leqslant a_n$.

LEMMA 12. *If n is even and $a_n \leqslant 2 \ln q$ then formula* (54) *holds.*

Proof. We write a for a_n. Let k'' be the number of terms in the expansion of p_{n-2}/q_{n-2}. Then

$$k \leqslant a + k'' \tag{61}$$

We first consider $q \leqslant 54$.

By Lemma 7 we need only consider $q \geqslant 17$. For $q_{n-2} \leqslant 9$ we know $k'' \leqslant 8$ also from $a \leqslant 2 \ln q \leqslant 2 \ln 54$; we see that $a \leqslant 7$. It follows that for $q_{n-2} \leqslant 9$, $k \leqslant 7 + 8 < 2f(\ln 17) \leqslant 2f(\ln q)$.

Thus we may assume that $q_{n-2} \geqslant 10$ and consequently $q_{n-1} \geqslant 11$. By formula (55) we see that $q \geqslant 2q_{n-1} + q_{n-2} \geqslant 32$.

For $q \geqslant 32$, $2f(\ln q) \geqslant 2f(\ln 32) \geqslant 19$. Thus for $q_{n-2} \leqslant 13$.

$$k \leqslant a + k'' \leqslant 7 + 12 \leqslant 2f(\ln q).$$

Thus we are left to consider $q_{n-2} \geqslant 14$, so that $q_{n-1} \geqslant 15$. From formula (55) we deduce that $a \leqslant (54 - 14)/15$ and so $a = 2$. But for $a = 2$ and $q_{n-2} \leqslant 18$,

$$k \leqslant a + k'' \leqslant 2 + 17 < 2f(\ln q).$$

If $q_{n-2} \geqslant 19$, then by (55) $q \geqslant 59$.

We shall now consider $q \geqslant 55$.

We next prove a Lemma which estimates k'' in terms of q_{n-1}. This Lemma is valid for all the cases on a_n.

LEMMA 13. *For* $q_{n-1} \geqslant 3$,

$$k'' \leqslant 2f(\ln q_{n-1}). \tag{62}$$

Proof. If $q_{n-2} \leqslant 5$ then $k'' \leqslant 4 \leqslant 2(2e) \leqslant 2f(\ln q_{n-1})$ for all values of $q_{n-1} \geqslant 3$.

If $q_{n-2} \geqslant 6$ then $\ln \ln q_{n-2} \geqslant \frac{1}{2}$ so that by Lemma 6 and the induction hypothesis we again obtain

$$k'' \leqslant 2f(\ln q_{n-2}) < 2f(\ln q_{n-1}).$$

Lemma 13 is proved.

When $\ln \ln q_{n-1} \geqslant \frac{1}{2}$, using (55) and Lemma 6 we obtain from (62)

$$k'' \leqslant 2f(\ln q - \ln a).$$

But for $\ln \ln q_{n-1} \leqslant \frac{1}{2}$, $q_{n-1} \leqslant 5$, and since $q \geqslant 55$,

$$\ln q - \ln a \geqslant \ln q - \ln 2 \ln q \geqslant 1,$$

so by Lemma 6 again $k'' \leqslant q_{n-1} \leqslant 5 \leqslant 2(2e) \leqslant 2f(\ln q - \ln a)$.

Thus it is sufficient to show that

$$a + 2f(\ln q - \ln a) \leqslant 2f(\ln q) \tag{63}$$

for $q \geqslant 55$.

LEMMA 14. *The function* $a + 2f(\ln q - \ln a)$ *is a convex function of* a *for* $\ln q - \ln a > e^{1/2}$. *In particular, it is convex in the range* $2 \leqslant a \leqslant 2 \ln q$ *for* $q \geqslant 55$.

Proof. In the desired range

$$\ln q - \ln a \geqslant \ln q - \ln \ln q - \ln 2 \geqslant \ln 55 - \ln 2 \ln 55 > e^{1/2}.$$

By Lemma 6, for $\ln q - \ln q \geqslant e^{1/2}$, both $f'(\ln q - \ln a) \geqslant 0$ and $f''(\ln q - \ln a) > 0$. But

$$D_a^2 f(\ln q - \ln a) = f''(\ln q - \ln a) \cdot \frac{1}{a^2} + f'(\ln q - \ln a) \cdot \frac{1}{a^2}.$$

Thus in the desired range

$$D_a^2(a + 2f(\ln q - \ln a) = 2D_a^2 f(\ln q - \ln a) > 0.$$

The convexity is established, and Lemma 14 is proved.

From the convexity of the left hand side of (63) it is sufficient to establish the inequality for the extreme values of a, namely, $a = 2$ and $a = 2 \ln q$.

We consider first $a = 2$. In this case, (63) is equivalent to

$$\frac{\ln \ln q}{(\ln q)^2} + \frac{(\ln q - \ln 2)}{\ln q} \cdot \frac{\ln \ln q}{\ln q} \frac{\ln q - \ln 2}{\ln(\ln q - \ln a)} - 1 \leqslant 0.$$

But by Lemma 11 and the fact that $\ln q \geqslant \ln 55 > 4 > x_0$, it is sufficient to show

$$\frac{1}{\ln q} \left\{ \frac{\ln \ln q}{\ln q} - \ln 2 \right\} \leqslant 0.$$

By Lemma 10, $(\ln \ln q / \ln q) \leqslant 1/e$. Since $1/e < \ln 2 = .69...$, the desired inequality follows.

We now examine the other extreme value when $a = 2 \ln q$. Setting $x = \ln q$, formula (63) is equivalent to

$$\frac{\ln x}{x} + \frac{x - \ln 2x}{x} \cdot \frac{\ln x}{x} \cdot \frac{x - \ln 2x}{\ln(x - \ln 2x)} - 1 \leqslant 0. \tag{64}$$

In a manner analogous to the proof of Lemma 11, we see that for $x \geqslant 4$, the inequality

$$(\ln x/x) \cdot \{(x - \ln 2x)/\ln(x - \ln 2x)\} \leqslant 1$$

is valid.

Thus, since $x = \ln q \geqslant \ln 55 > 4$, it suffices to show that

$$(\ln x/x) + \{(x - \ln 2x)/x\} - 1 \leqslant 0, \tag{65}$$

which is equivalent to the obviously true $-\ln 2/x \leqslant 0$.

The validity of (63) is established for all $q \geqslant 55$ and Lemma 12 follows.

We next consider the case $2 \ln q \leqslant a \leqslant P^2$. We prove

LEMMA 15. *If n is even and $2 \ln q \leqslant a < P^2$, then formula (54) holds.*

Proof. According to the Corollary of Lemma 5, $P - 1 \leqslant 2 \ln q \leqslant a$, but since $P - 1$, and a are integers while $2 \ln q$ is irrational, this implies $P \leqslant a$. Thus in the process of the algorithm we are in the case $t = 1$.

LEMMA 16. *For $2 \ln q \leqslant a$ and $q \geqslant 17$, with the exception of*

$$\begin{array}{lll} P = 5, & q_{n-1} = 4, & 29 \leqslant q \leqslant 42, \\ P = 7, & q_{n-1} = 12, & 121 \leqslant q \leqslant 190, \\ P = 11, & q_{n-1} = 210, & 3571 \leqslant q \leqslant 3827, \end{array}$$

we have $P \leqslant \frac{4}{3} \ln q$.

Proof. Again using the result of Rosser and Schoenfeld [27], formula (52), we first show that for $\ln q \geqslant 41$, $x = \frac{4}{3} \ln q$ we have

$$\theta(x) \geqslant \ln q_{n-1} + \ln x. \tag{66}$$

Since $a \geqslant 2 \ln q$, $q_{n-1} \leqslant q/2 \ln q$ and thus (66) is implied by

$$\theta(x) \geqslant \ln q - \ln 2 \ln q + \ln x. \tag{67}$$

Substituting $x = \frac{4}{3} \ln q$ and using formula (52), we see that (66) is implied by

$$\tfrac{4}{3} \ln q(1 - (1/\ln(\tfrac{4}{3} \ln q)) > \ln q + \ln(\tfrac{3}{2}).$$

The above is implied by

$$\frac{\ln q}{3} - \frac{\frac{4}{3} \ln q}{\ln \frac{4}{3} \ln q} \geqslant \ln(\tfrac{3}{2}),$$

which holds if

$$\ln q \geqslant 46$$

It follows, as in the proof of Lemma 4, that for $\ln q \geqslant 46$, $P \leqslant x = \frac{4}{3} \ln q$. For $\ln q \leqslant 46$, we use Table I. Since $a \geqslant P$, the minimal value of q for a given value P is greater than the value of Q in Table I for the next value of P, for $P \geqslant 11$. Thus it is a simple arithmetic verification that for $P > 11$, the Lemma holds.

For $P = 2, 3$ we see that

$$P \leqslant \tfrac{4}{3} \ln 17 \leqslant \tfrac{4}{3} \ln q.$$

For $P = 5$, $a \geqslant 5$.
Lemma 16 holds for $q \geqslant e^{15/4} = 42.521$. The least possible value of q_{n-1} is $q_{n-1} = 4$. Thus $q \geqslant 4.5 + 1 = 21$, but since $a \geqslant 2 \ln q$, $a \geqslant 2 \ln 21 = 6.08\ldots$. Thus $a \geqslant 7$, and $q \geqslant 4.7 + 1 = 29$. Thus the exceptions for $q_{n-1} = 4$ are correct. If $q_{n-1} = 6$, then since $a \geqslant P$, $q \geqslant 31$, and since $a \geqslant 2 \ln q$, $a \geqslant 7$. Thus $q \geqslant 6 \cdot 7 + 1 = 43 > e^{15/4}$. So $q_{n-1} = 6$ yields no exceptions of $P = 5$. If $q_{n-1} > 6$, $q_{n-1} \geqslant 18$; so $q > e^{15/4}$.

For $P = 7$, $a \geqslant 7$ and Lemma 16 holds for $q > e^{21/4} = 190.5\ldots$. The least value of q_{n-1} is $q_{n-1} = 12$.

Thus $q \geqslant 85$ and $a \geqslant 2 \ln q > 8$, so that $a \geqslant 9$. But for $a \geqslant 9$, $q \geqslant 109$, and $a \geqslant 2 \ln q > q$. Hence $a \geqslant 10$ and $q \geqslant 121$. Thus for $q_{n-1} = 12$, the exceptions for $P = 7$ are correct. The next possibility for q_{n-1} is $q_{n-1} = 30$, but since $a \geqslant 7$, $q \geqslant 210 > e^{21/4}$ and $7 \leqslant \frac{4}{3} \ln q$.

For $P = 11$, $q_{n-1} \geqslant 210$; so $q \geqslant 2311$. But then

$$a \geqslant 2 \ln q \geqslant 2 \ln 2311 > 15,$$

so $q \geqslant 3381$. It follows that $a \geqslant 17$ and $q \geqslant 3571$. On the other hand, if $q > e^{33/4} = 3827.6...$, then $\frac{4}{3} \ln q > 11$, so that the exceptions are correct for $P = 11$.

Lemma 16 is established.

We now turn to the proof of Lemma 15. Again let k'' be the number of terms in the expansion of p_{n-2}/q_{n-2}. Then

$$k \leqslant b_0 + b_1 + c_0 + k'', \tag{68}$$

where $a = c_0 + (b_1 b_0)_P < P^2$. Each of c_0, b_0, $b_1 \leqslant P - 1$, but not all of them can be $P - 1$, since the sum is less. than P^2. Thus

$$c_0 + b_0 + b_1 \leqslant 3P - 4. \tag{69}$$

If $q_{n-1} \leqslant 6$, then $P = 2, 3$ or 5, so that $k'' \leqslant q_{n-2} - 1 \leqslant 4$ and $c_0 + b_0 + b_1 \leqslant 11$. Thus $k \leqslant 15$ and formula (54) holds since

$$k \leqslant 15 \leqslant 2f(\ln 17) \leqslant 2f(\ln q).$$

For $7 \leqslant q_{n-1} \leqslant 11$, we see from Table I that $P = 2, 3$. Consequently,

$$k \leqslant (3P - 4) + k'' \leqslant 5 + 9 = 14.$$

But $14 \leqslant 2f(\ln 17) \leqslant 2f(\ln q)$.

For $q_{n-1} = 12.$, $P = 7$, we see that $a \geqslant 7$, $q \geqslant 12a + 1$ and $a \geqslant 2 \ln q$ imply, as in the proof of Lemma 16, that $a \geqslant 10$, $q \geqslant 121$.

But then

$$k \leqslant (3P - 4) + k'' \leqslant 17 + 10 = 27.$$

While

$$27 < 28 < 2f(\ln 121) \leqslant 2(f(\ln q).$$

Thus for $q_{n-1} = 12$, Lemma 15 holds.

Also for $P = 2$ we have

$$2 \ln q \leqslant a < P^2 = 4$$

Thus $2 \ln q \leqslant 3$, whence $q \leqslant 4$.

So that $P = 2$ is completely taken care of for Lemma 15.

Also for $P = 3$, since $q_{n-1} > 12$, $q_{n-1} \geqslant 14$. Thus $q \geqslant 3.14 + 1 = 43$, so that $a \geqslant 2 \ln q = 7.5\ldots$. Thus $a \geqslant 8$ and $q \geqslant 8.14 + 1 = 113$. But then $9 \leqslant 2 \ln q \leqslant a < P^2 = 9$ is impossible so that $P = 3$ is taken care of for Lemma 15.

For $P = 5$, $q_{n-1} \geqslant 18$, $a \geqslant 5$ so that $q \geqslant 91$.

Since $a \geqslant 2 \ln q \geqslant 2 \ln 91$, we see that $a \geqslant 10$ and $q \geqslant 181$. On the other hand, if $q_{n-1} = 18$, then $k'' \leqslant 16$ so that

$$k \leqslant 11 + 16 = 27 \leqslant 2f(\ln 181).$$

So that Lemma 15 holds. If $P = 5$ and $q_{n-1} > 18$, then $q_{n-1} \geqslant 24$.

If $P = 5$ and $q_{n-1} \geqslant 24$, then since $a \geqslant 10$ we deduce that $q \geqslant 241$. But if $q_{n-2} \leqslant 23$, $k'' \leqslant 22$ so that $k \leqslant 11 + 22 < 2f(\ln 241)$. For $q_{n-1} > 24$ we see that for $P = 5$, $q_{n-1} \geqslant 36$ and $q \geqslant 361$.

For $P = 7$, $q_{n-1} \geqslant 30$ and since $a \geqslant P$, $q \geqslant 211$ thus $a \geqslant 2 \ln 211 > 10$ so that $a \geqslant 11$ and $q \geqslant 331$.

For $P = 11$, $q_{n-1} \geqslant 210$, and we have $q \geqslant 331$ and $P < \frac{4}{3} \ln q$ except as noted in Lemma 16. But for the exceptions $a = 17$ or 18, and $q_{n-2} < q_{n-1} = 210$. Thus

$$k \leqslant 18 + k'' \leqslant 18 + 2f(\ln 209) < 2f(\ln 3571).$$

For $P > 11$, $q_{n-1} > 210$, so that $q \geqslant 331$.

From the last few paragraphs we see that it is sufficient to prove Lemma 15 under the assumptions that $q_{n-1} \geqslant 30$,

$$P \leqslant \tfrac{4}{3} \ln q \qquad \text{and} \qquad q \geqslant 331.$$

We wish to show that

$$k \leqslant 2f(\ln q).$$

By (68) and Lemma 13 it suffices to show that

$$b_0 + c_0 + \frac{a - b_0 - c_0}{P} + 2f(\ln q - \ln a) \leqslant 2f(\ln q) \tag{70}$$

since $a < P^2 \leqslant \frac{4}{3}(\ln q)^2$ and $q \geqslant 331$ we deduce that

$$\ln q - \ln a \geqslant \ln 331 - 2 \ln(\tfrac{4}{3} \ln 331) > e > e^{1/2}.$$

We see from Lemma 14 that the left hand side is a convex function of a.

Thus it suffices to prove (70) for the extreme values of a; in particular, for $a = 2 \ln q$ and $a = P^2$.

For $a = 2 \ln q$, formula (70) becomes

$$b_0 + c_0 + \frac{2 \ln q - b_0 - c_0}{P} + 2f(\ln q - \ln 2 \ln q) \leqslant 2f(\ln q) \quad (71)$$

But since $b_0 + c_0 \leqslant 2(P-1) < 2P \leqslant \frac{8}{3} \ln q$, we see that it is sufficient to show

$$\tfrac{8}{3} \ln q + \frac{2 \ln q - \frac{8}{3} \ln q}{P} + 2f(\ln q - \ln 2 \ln q) \leqslant 2f(\ln q). \quad (72)$$

Dropping the negative term from the L.H.S. of (72) it suffices to show

$$\tfrac{8}{3} \ln q + 2\{f(\ln q - \ln 2 \ln q) - f(\ln q\} \leqslant 0. \quad (73)$$

The inequality (73) is equivalent to, after the substitution $x = \ln q$,

$$\frac{4}{3} \frac{\ln x}{x} + \left(1 - \frac{\ln 2x}{x}\right)\left(\frac{\ln x}{x}\right)\left(\frac{x - \ln 2x}{\ln(x - \ln 2x)}\right) - 1 \leqslant 0. \quad (74)$$

For $x \leqslant 8$, since $x \geqslant \ln 331$ we see from Lemma 10, since

$$x - \ln 2x \geqslant \ln 331 - \ln 2 \ln 331 > e$$

that

$$\frac{\ln x}{x} \cdot \frac{x - \ln 2x}{\ln(x - \ln 2x)} < 1.$$

Thus for $x \leqslant 8$ then inequality (74) is implied by

$$\frac{4}{3} \frac{\ln x}{x} + 1 - \frac{\ln 2x}{x} - 1 \leqslant 0$$

which is equivalent to $x \leqslant 8$. Thus (74) holds when $x \leqslant 8$.

For $x - \ln 2x > e^2$ we use the estimate from the mean value theorem, namely,

$$\frac{x - \ln 2x}{\ln(x - \ln 2x)} \leqslant \frac{x}{\ln x} - \ln 2x \left(\frac{1}{\ln x} - \frac{1}{(\ln x)^2}\right), \quad (75)$$

where Lemma 10 tells us thet $g'(z)$ is decreasing in the range

$$e^2 \leqslant x - \ln 2x \leqslant z \leqslant x$$

and thus we may use $g'(x)$ in (75). Using (75) in (74) we find that for (74) is implied by

$$\frac{4}{3}\frac{\ln x}{x} + \left(1 - \frac{\ln 2x}{x}\right)\left(1 - \frac{\ln 2x}{x} + \frac{\ln 2x}{x \ln x}\right) - 1 \leqslant 0. \tag{76}$$

Formula (76) is equivalent to

$$-\frac{2}{3} + \frac{\ln 2x - 1}{x} + \frac{1}{\ln x} - \frac{4}{3}\frac{\ln 2}{\ln 2x} - \frac{\ln 2}{x \ln x} \leqslant 0. \tag{77}$$

Discarding the last two terms and using the fact that $x - \ln 2x \geqslant e^2$ implies $x \geqslant 10$, we see that formula (77) is implied by the fact that

$$-\frac{2}{3} + \frac{\ln 2x - 1}{x} + \frac{1}{\ln x} \leqslant -\frac{2}{3} + \frac{\ln 20 - 1}{10} + \frac{1}{\ln 10} < 0.$$

Thus we may restrict our attention to showing that formula (74) holds for $8 \leqslant x$ and $x^2 \geqslant \ln 2x \leqslant e^2$. But in this case formula (74) is implied by

$$\frac{4}{3}\frac{\ln x}{x} + \left(1 - \frac{\ln 2 + \ln x}{x}\right)\frac{\ln 8}{8} \cdot \frac{e^2}{2} - 1 \leqslant 0,$$

which is equivalent to

$$\frac{\ln x}{x}\left(\frac{4}{3} - \frac{e^2 \ln 8}{16}\right) - \frac{(\ln 2)\, e^2 \ln 8}{16x} - 1 + \frac{e^2 \ln 8}{16} \leqslant 0 \tag{78}$$

Since $x^2 - \ln 2x \leqslant e^2 < 7.4 < 11 - \ln 22$, we see that $x < 11$. Thus (78) follows from the fact that

$$\frac{\ln 8}{8}\left(\frac{4}{3} - \frac{e^2 \ln 8}{16}\right) - \frac{(\ln 2)\, e^2 \ln 8}{16 \cdot 11} - 1 + \frac{e^2 \ln 8}{16} < 0.$$

Thus (74) holds for all $x \geqslant \ln 331$, so that formula (70) holds for the extreme value $a = 2 \ln q$.

We now handle the other extreme when $a = P^2$. For $a = P^2$, formula (70) becomes

$$b_0 + c_0 + \frac{P^2 - b_0 - c_0}{P} + 2f(\ln q - 2 \ln P) \leqslant 2f(\ln q). \tag{79}$$

Since the L.H.S. of (79) is increasing in b_0 and c_0 and $b_0, c_0 \leqslant P - 1$, it suffices to show

$$3P - 4 + \frac{2}{P} + 2f(\ln q - 2 \ln P) \leqslant 2f(\ln q). \tag{80}$$

Since $P \geqslant 2$, the $-4 + 2/P$ may be discarded. Putting $x = \ln q$, it is sufficient to prove

$$3P + 2\left\{\frac{(x - 2\ln P)^2}{\ln(x - 2\ln P)} - \frac{x^2}{\ln x}\right\} \leqslant 0. \tag{81}$$

Formula (81) is equivalent to

$$\frac{3}{2}\frac{P\ln x}{x} + \frac{x - 2\ln P}{x}\left(\frac{\ln x}{x}\cdot\frac{x - 2\ln P}{\ln(x - 2\ln P)}\right) - 1 \leqslant 0. \tag{82}$$

Since $q \geqslant 331$, $x > 5.8$ and $x - 2\ln P > x - 2\ln \frac{4}{3}x \geqslant 1.7$. Also for $x = 331$,

$$\frac{\ln x}{x}\cdot\frac{x - 2\ln\frac{4}{3}x}{\ln(x - 2\ln\frac{4}{3}x)} < 1.$$

We deduce from Lemma 10 that

$$\frac{\ln x}{x}\cdot\frac{x - 2\ln P}{\ln(x - 2\ln P)} < 1$$

for $q \geqslant 331$.

Thus (82) is implied by

$$\tfrac{3}{2}(P\ln x/x) - (2\ln P/x) \leqslant 0, \tag{83}$$

which is equivalent to

$$\tfrac{3}{4}(P/\ln P) \leqslant x/\ln x \tag{84}$$

Since $P \geqslant 5 > e$, $P/\ln P$ is increasing in P. Since $P \leqslant \frac{4}{3}\ln q = \frac{4}{3}x$, formula (84) is implied by the obviously true

$$\tfrac{3}{4}(\tfrac{4}{3}x/\ln\tfrac{4}{3}x) < x/\ln x. \tag{85}$$

Thus (70) holds for the extreme value $a = P^2$, and since we have previously shown that it holds for the other extreme value $a = 2\ln q$, we see that Lemma 15 is established.

We now turn our attention to the final case.

LEMMA 17. *If n is even and $P^2 \leqslant a_n$ then formula* (54) *holds.*

We begin by establishing the inequality for $q_{n-1} \leqslant 7$. The number of steps in the expansion of p/q is

$$k \leqslant 2t(P - 1) + k''. \tag{86}$$

but for $q_{n-1} \leqslant 7$, $P = 2$, 3, or 5.

Also,

$$t \leqslant \frac{\ln a}{\ln P} < \frac{\ln q}{\ln P}$$

and (86) yields

$$k \leqslant (2 \ln q/\ln P)(P-1) + k''.$$

We want to show

$$k'' + 2 \ln q \left(\frac{P-1}{\ln P}\right) \leqslant \frac{2(\ln q)^2}{\ln \ln q}. \tag{87}$$

But (87) is equivalent to

$$(k''/2 \ln q) + (P-1)/\ln P \leqslant \ln q/\ln \ln q. \tag{88}$$

For $P = 2$, $q_{n-1} = 1, 3, 5$ or 7 we see that $k'' \leqslant 5$. Since $q \geqslant 17$ and $\ln q/\ln \ln q \geqslant e$ it is a simple calculation to see that (88) holds for $P = 2$.

For $P = 3$, $q_{n-1} = 2$, $k'' = 0$ and again (89) holds.

For $P = 5$, $q_{n-1} = 4$ or 6, $k'' \leqslant 4$. But $a \geqslant P^2 = 25$ so that $q \geqslant 101$ and (88) follows from the fact that

$$(4/2 \ln 101) + (4/\ln 5) \leqslant 3 \leqslant \ln 101/\ln \ln 101.$$

Thus we may suppose that $q_{n-1} \geqslant 8$ and $\ln \ln q_{n-1} > 0$. Also by Lemma 12 we may suppose $a \geqslant 2 \ln q \geqslant 2 \ln 17 > 5$, so $q \geqslant 49$.

Using Lemma 13, we deduce

$$k \leqslant 2t(P-1) + 2f(\ln q_{n-1}). \tag{89}$$

But

$$t \leqslant \frac{\ln a}{\ln P} \leqslant \frac{\ln q - \ln q_{n-1}}{\ln P}$$

and formula (89) yields

$$k \leqslant 2(\ln q - \ln q_{n-1}) \frac{P-1}{\ln P} + \frac{2(\ln q_{n-1})^2}{\ln \ln q_{n-1}}. \tag{90}$$

When $P - 1 \leqslant \ln q$, since $P - 1/\ln P$ is increasing for $P \geqslant 2$, we get

$$\begin{aligned} k &\leqslant \frac{2(\ln q - \ln q_{n-1}) \ln q}{\ln \ln q} + \frac{2(\ln q_{n-1})^2}{\ln \ln q_{n-1}} \\ &= \frac{2(\ln q)^2}{\ln \ln q} + 2 \ln q_{n-1} \left\{\frac{\ln q_{n-1}}{\ln \ln q_{n-1}} - \frac{\ln q}{\ln \ln q}\right\}. \end{aligned} \tag{91}$$

This last term is negative since for $\ln q_{n-1} \geqslant e$ the function $z/\ln z$ increases in the interval $e \leqslant z \leqslant \ln q$ (Lemma 10), while for

$$\ln 8 \leqslant \ln q_{n-1} \leqslant e$$

we get

$$\frac{\ln q_{n-1}}{\ln \ln q_{n-1}} \leqslant \frac{\ln 8}{\ln \ln 8} < \frac{\ln 49}{\ln \ln 49} \leqslant \frac{\ln q}{\ln \ln q}.$$

Since the last term is negative, formula (91) yields formula (54) when $P - 1 < \ln q$.

We must now show that formula (90) yields the desired result for $q \geqslant 49$, $q_{n-1} \geqslant 8$, $P - 1 \geqslant \ln q$. By Lemma 16 with the exceptions of $P = 7$, $q_{n-1} = 12$, $121 \leqslant q \leqslant 190$ and $P = 11$, $q_{n-1} = 210$, $3571 \leqslant q \leqslant 3827$ we may suppose $\frac{4}{3} \ln q > P$. But if $P = 7$, $a \geqslant P^2 = 49$ and $q_{n-1} = 12$, then $q \geqslant (12)(49) + 1 > 190$; while if $P = 11$, $a \geqslant 121$ and $q_{n-1} \geqslant 210$, then $q \geqslant (121)(210) \geqslant 3827$ so that the exceptions are eliminated and we suppose $P \leqslant \frac{4}{3} \ln q$.

In this case (90) yields

$$k \leqslant \frac{2(\ln q - \ln q_{n-1}) \frac{4}{3} \ln q_{n-1}}{\ln(\frac{4}{3} \ln q_{n-1})} + \frac{2(\ln q_{n-1})^2}{\ln \ln q_{n-1}}. \tag{92}$$

Replacing $\ln(\frac{4}{3} \ln q_{n-1})$ by the smaller $\ln \ln q_{n-1}$, formula (92) yields

$$k \leqslant \frac{2}{3} \left\{ \frac{4(\ln q)(\ln q_{n-1}) - (\ln q_{n-1})^2}{\ln \ln q_{n-1}} \right\}. \tag{93}$$

Setting $y = \ln q_{n-1}$, the R.H.S. of (93) is

$$F(y) = \frac{2}{3} \left\{ \frac{4(\ln q)\, y - y^2}{\ln y} \right\},$$

so that

$$F'(y) = \frac{2}{3} \left\{ \frac{4 \ln q}{\ln y} \left(1 - \frac{1}{\ln y}\right) - \frac{y}{\ln y} \left(2 - \frac{1}{\ln y}\right) \right\}.$$

Thus $F'(y)$ is positive iff

$$4 \ln q \geqslant y \left(\frac{2 \ln y - 1}{\ln y - 1} \right).$$

Since $\ln q > y$ this is true for $\ln y \geqslant \frac{3}{2}$. Thus if $\ln \ln q_{n-1} \geqslant \frac{3}{2}$ we increase the R.H.S. of (93) if $\ln q_{n-1}$ is replaced by $\ln q$. But making this substitute yields precisely (54). We are left only with the possibility when

$$\ln q \leqslant P - 1 \leqslant \tfrac{4}{3} \ln q_{n-1}, \qquad q \geqslant 49,$$

and $8 \leqslant q_{n-1} \leqslant e^{e^{3/2}} < 89$.

Since $q_{n-1} < 89$, $P = 2, 3, 5, 7$, (see Table I). Since $P - 1 > \ln q \geqslant \ln 49 > 3$, $P = 5, 7$. But $a \geqslant P^2 \geqslant 25$, thus $q \geqslant (8)(25) = 200$. Since $P - 1 \geqslant \ln 200$, $P = 7$. For $P = 7$, $a \geqslant 49$, $q_{n-1} \geqslant 12$ so that $q \geqslant 589$ and hence $6 = P - 1 < \ln q \leqslant 6.3$. Thus all exceptions are eliminated and Lemma 17 hold.

Theorem 4 is now an immediate consequence of Lemmas 8, 9, 12, 15 and 17 and of Theorem 3.

COROLLARY. *If p/q is a reduced fraction with $0 < p < q$ and $q > 2$, and if*

$$\frac{p}{q} = \sum_{i=1}^{k} \frac{1}{n_i}$$

is the Egyptian Fraction expansion of p/q obtained by the Continued Fraction Algorithm, then

$$n_1 < n_2 < \cdots < n_k < q(q-1)$$

and

$$k = \min \left\{ \frac{2(\ln q)^2}{\ln \ln q}, 1 + a_2 + a_4 + \cdots + a_{n^*} \right\},$$

where $p/q = [0; a_1, a_2, \ldots, a_n]$ is the continued fraction of p/q and $n^* = 2[n/2]$.

Proof. Theorems 3 and 4.

V. EXAMPLES AND COMPARISON

In this section we expand some fraction by the different algorithms to give a better idea of the comparative strengths and weakness of the different algorithms.

As might be expected from Theorem 4, for p of order $\ln q$ or smaller the continued fraction algorithm, and the Fibonacci–Sylvester algorithm are usually shortest, with the C.F. algorithm giving much smaller denominators.

Examples I

1. Let $p/q = 4/23 = [0; 5, 1, 3]$. All three algorithms yields the same expansion, namely,

$$\frac{4}{23} = \frac{1}{6} + \frac{1}{6(23)}\,.$$

2. Let $p/q = 5/121 = [0; 24, 1, 4]$. The C.F. algorithm yields

$$\frac{5}{121} = \frac{1}{25} + \frac{1}{1225} + \frac{1}{3477} + \frac{1}{7081} + \frac{1}{11737}\,.$$

The Erdös algorithms yields

$$\frac{5}{121} = \frac{1}{48} + \frac{1}{72} + \frac{1}{180} + \frac{1}{1452} + \frac{1}{4354} + \frac{1}{8712} + \frac{1}{87120}\,.$$

The F–S algorithm yields

$$\frac{5}{121} = -\frac{1}{25} + \frac{1}{757} + \frac{1}{763308} + \frac{1}{873960180913} + \frac{1}{15,276,184,876,404,402,665,313.}\,.$$

3. $p/q = p/(p! + 1) = [0; (p-1)!\ p]$.
Since $q_{n-1} = (p-1)!$, $P \geqslant p$. Thus the C.F. expansion is the same as the Farey series expansion, namely,

$$\frac{p}{p!+1} = \sum_{i=0}^{p-1} \frac{1}{\{i(p-1)!+1\}\{(i+1)(p-1)!+1\}}\,.$$

There are p terms, the largest denominator is

$$q(q - q_{n-1}) = (p! + 1)(p! - (p-1)! + 1).$$

The F–S expansion also has p terms; the largest denominator is bigger than $\{(p-1)! + 1\}^2$. This follows from the more general result that $p/(k! + 1)$ has p terms in it F–S expansion[4] and the fact that the denominators, n_i, of the F–S expansion satisfy $n_{i+1} \geqslant n_i(n_i + 1) + 1$.

The Erdös expansion depends heavily on the arithmetic properties of p, but has at most $p + 3$ terms. The largest denominator is $(p! + 1)(p!)$ if $p + 1$ is prime and is $(p! + 1)(p + 1)!$ if $p + 1$ is not prime.

[4] This was pointed out to me by Prof. Peter Fillmore in a letter dated March 28, 1970.

For the numerator of moderate size compared with the denominator, the best algorithm depends on the particular choices of p and q.

Examples II

1. $p/q = 17/101 = [0; 5, 1, 16]$. All the algorithms yield the same

$$\frac{17}{101} = \frac{1}{6} + \frac{1}{606}.$$

2. $p/q = 19/123 = [0; 6, 2, 9]$.
The C.F. expansion is

$$\frac{19}{123} = \frac{1}{7} + \frac{1}{91} + \frac{1}{1599}.$$

The F–S expansion is

$$\frac{19}{123} = \frac{1}{7} + \frac{1}{87} + \frac{1}{8,323}.$$

The Erdös expansion is

$$\frac{19}{123} = \frac{1}{10} + \frac{1}{30} + \frac{1}{60} + \frac{1}{369} + \frac{1}{2952} + \frac{1}{14,760}.$$

3. $p/q = 59/121 = [0; 2, 19, 1, 2]$
By the C.F. algorithm,

$$\frac{59}{121} = \frac{1}{3} + \frac{1}{7} + \frac{1}{91} + \frac{1}{3120} + \frac{1}{9680}.$$

By the F–S algorithm,

$$\frac{59}{121} = \frac{1}{3} + \frac{1}{7} + \frac{1}{88} + \frac{1}{20,328}.$$

By the Erdös algorithm,

$$\frac{59}{121} = \frac{1}{3} + \frac{1}{8} + \frac{1}{40} + \frac{1}{240} + \frac{1}{10890} + \frac{1}{87190}.$$

To get long expansions by the Continued Fraction Algorithm we take an example with q_{n-1} divisible by the first few primes, this necessarily make q large.

4. $p/q = 20/46291 = [0; 2310, 20]$.

The C–F expansion is

$$\frac{20}{46201} = \sum_{i=1}^{20} \frac{1}{\{2310i + 1\}\{2310(i - 1) + 1\}}.$$

The largest denominator is (46201)(43891).

The Erdös expansion has only 10 terms. The largest denominator is (46201) $9!/35$ = (46201)(10468).

For p close to q we consider first the fraction 21/23 since it is the first fraction which can not be expanded into four terms.

Examples III

1. $(21)/23) = [0; 1, 10, 2]$.

By the C.F. algorithm,

$$\frac{21}{23} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{40} + \frac{1}{110} + \frac{1}{253}.$$

By the F–S algorithm,

$$\frac{21}{23} = \frac{1}{2} + \frac{1}{3} + \frac{1}{11} + \frac{1}{304} + \frac{1}{230,734}.$$

By the Erdös algorithm,

$$\frac{21}{23} = \frac{1}{2} + \frac{1}{3} + \frac{1}{24} + \frac{1}{46} + \frac{1}{66} + \frac{1}{552}.$$

2. $p/q = (q - 1)/q = [0; 1, q - 1]$.

For the C.F. algorithm,

$$q_{n-1} = q_{n-1} = 1, \qquad P = 2 \qquad \text{and} \qquad t = \log_2(q - 1).$$

If $t \geqslant 2$, i.e., $q \geqslant 5$, then $d = 2^{t-1} - 1 = (1, 1 \cdots 1)_2$ with $t - 1$ ones. So that the number of terms is at least $t = \log_2(q - 1)$ and at most $2 \log_2(q - 1)$. The largest denominator is $q(q - 1)$.

For the Erdös method we cannot evaluate this so easily for arbitrary q, but if $q = n! - 1$, $n > 3$, then the number of terms is $2(n - 1)$, which is less than $2 \log_2(n! - 1)$. The largest denominator is $q(q + 1)/(n + 3)$.

It seems to be very difficult to get a general formula for the Fibonacci–Sylvester method here.

It is frequently the case that trial and error, or a bit of insight in applying the algorithms yields a much better expansion (shorter or smaller denominators) than any of the algorithms.

VI. Real Numbers

Let r be a real irrational number $0 < r < 1$. Let the continued fraction for r be given by

$$\Gamma = [0; a_1, a_2, \ldots].$$

Let p_n/q_n, n even, be a convergent to r. The expansion of p_n/q_n by the C.F. algorithm has the form

$$\frac{p_n}{q_n} = \sum_{i=1}^{k_i} \frac{1}{n_i} + \frac{p_{n-2}}{q_{n-2}}.$$

Since $(p_n/q_n) \to r$, and since the C.F. expansion of p_n/q_n contains the expansion of p_{n-2}/q_{n-2}, we see that we obtain an infinite expansion for Γ, say,

$$r = \sum_{i=1}^{\infty} \frac{1}{n_i},$$

in which $p_n/q_n = \sum_{n_i < q_n^2} 1/n_i$. This last remark is because, in the proof of the algorithm, we showed that the new terms introduced in going from p_{n-2}/q_{n-2} to p_n/q_n satisfied

$$q_{n-2}^2 < n_i < q_n^2.$$

We summarize this discussion in the following theorem:

Theorem 5. *Let r be a real number, $0 < r < 1$. Then the Continued Fraction Algorithm yields an Egyptian Fraction expansion for r*

$$r = \sum_{i=1}^{\infty} \frac{1}{n_i} \qquad n_1 < n_2 < \cdots.$$

This sum is finite or infinite according as r is rational or irrational. Further, all lower best approximation (in the sense of continued fractions[5]) occur as partial sums of the expansion. In fact, if p/q is a lower best approximation, then

$$\frac{p}{q} = \sum_{n_i < q^2} \frac{1}{n_i}.$$

The proof follows from the above remarks and the fact that the continued fraction has the best approximations as is its convergents.

[5] Again of the second kind according to Khintchine [15, p. 30].

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