Matching Residents to Hospitals

Goal. Given a set of preferences among hospitals and medical school students, design a self-reinforcing admissions process.

Unstable pair: applicant x and hospital y are unstable if:

- x prefers y to its assigned hospital.
- y prefers x to one of its admitted students.

Stable assignment. Assignment with no unstable pairs.

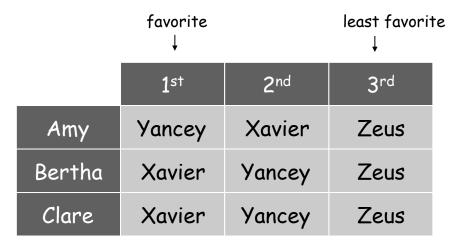
- Natural and desirable condition.
- Individual self-interest will prevent any applicant/hospital deal from being made.

Goal. Given n men and n women, find a "suitable" matching.

- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

	favorite ↓		least favorite ↓
	1 ^{s†}	2 nd	3 rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile



Women's Preference Profile

Perfect matching: everyone is matched monogamously.

- Each man gets exactly one woman.
- Each woman gets exactly one man.

Stability: no incentive for some pair of participants to undermine assignment by joint action.

- In matching M, an unmatched pair m-w is unstable if man m and woman w prefer each other to current partners.
- Unstable pair m-w could each improve by eloping.

Stable matching: perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n men and n women, find a stable matching if one exists.

Q. Is assignment X-C, Y-B, Z-A stable?

	favorite ↓		least favorite
	1 ^{s†}	2 nd	3 rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

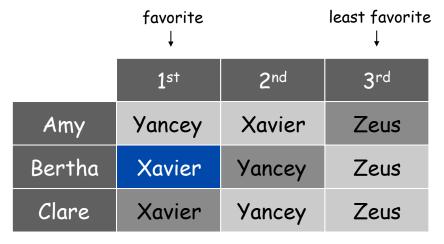
	favorite ↓		least favorite ↓
	1 ^{s†}	2 nd	3 rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

- Q. Is assignment X-C, Y-B, Z-A stable?
- A. No. Bertha and Xavier will hook up.

	favorite ↓		least favorite
	1 ^{s†}	2 nd	3 rd
Xavier	Amy	Bertha	Clare
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Men's Preference Profile

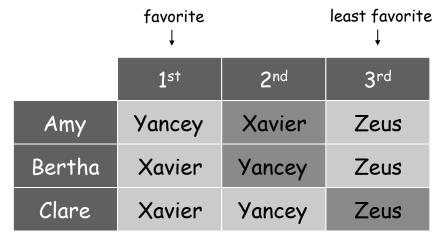


Women's Preference Profile

- Q. Is assignment X-A, Y-B, Z-C stable?
- A. Yes.

	favorite ↓		least favorite
	1 ^{s†}	2 nd	3 rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile



Women's Preference Profile

Stable Roommate Problem

- Q. Do stable matchings always exist?
- A. Not obvious a priori.

Stable roommate problem.

- 2n people; each person ranks others from 1 to 2n-1.
- Assign roommate pairs so that no unstable pairs.

	1 st	2 nd	3 rd
Adam	В	С	D
Bob	С	Α	D
Chris	Α	В	D
Doofus	Α	В	С

A-B, C-D \Rightarrow B-C unstable A-C, B-D \Rightarrow A-B unstable A-D, B-C \Rightarrow A-C unstable

Observation. Stable matchings do not always exist for stable roommate problem.

Propose-And-Reject Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
   Choose such a man m
   w = 1<sup>st</sup> woman on m's list to whom m has not yet proposed
   if (w is free)
        assign m and w to be engaged
   else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
   else
        w rejects m
}
```

Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference.

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

Claim. Algorithm terminates after at most n^2 iterations of while loop. Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals.

	1 st	2 nd	3 rd	4 th	5 th
Victor	Α	В	С	D	Е
Wyatt	В	С	D	Α	Е
Xavier	С	D	Α	В	Е
Yancey	D	Α	В	С	Е
Zeus	Α	В	С	D	Е

	1 ^{s†}	2 nd	3 rd	4 th	5 th
Amy	W	X	У	Z	V
Bertha	X	У	Z	V	W
Clare	У	Z	V	W	X
Diane	Z	V	W	X	У
Erika	V	W	X	У	Z

n(n-1) + 1 proposals required

Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf. (by contradiction)

- Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
- Then some woman, say Amy, is not matched upon termination.
- By Observation 2, Amy was never proposed to.
- But, Zeus proposes to everyone, since he ends up unmatched. ■

Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. (by contradiction)

- Suppose A-Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching 5*.
- Case 1: Z never proposed to A.
- men propose in decreasing order of preference

Amy-Yancey

5*

 \Rightarrow Z prefers his GS partner to A.

Bertha-Zeus

 \Rightarrow A-Z is stable.

- Case 2: Z proposed to A.
 - \Rightarrow A rejected Z (right away or later)
 - ⇒ A prefers her GS partner to Z. ← women only trade up
 - \Rightarrow A-Z is stable.
- In either case A-Z is stable, a contradiction.

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

An instance with two stable matchings.

- A-X, B-Y, C-Z.
- A-Y, B-X, C-Z.

	1 ^{s†}	2 nd	3 rd
Xavier	Α	В	С
Yancey	В	Α	С
Zeus	Α	В	С

	1 ^{s†}	2 nd	3 rd
Amy	У	X	Z
Bertha	X	У	Z
Clare	X	У	Z

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

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Man-optimal assignment. Each man receives best valid partner.

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Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

Man-optimal assignment. Each man receives best valid partner.

Claim. All executions of GS yield man-optimal assignment, which is a stable matching!

- No reason a priori to believe that man-optimal assignment is perfect, let alone stable.
- Simultaneously best for each and every man.

Man Optimality

Claim. GS matching S* is man-optimal.

Pf. (by contradiction)

- Suppose some man is paired with someone other than best partner. Men propose in decreasing order of preference \Rightarrow some man is rejected by valid partner.
- Let Y be first such man, and let A be first valid woman that rejects him.
- Let 5 be a stable matching where A and Y are matched.
- When Y is rejected, A forms (or reaffirms)
 engagement with a man, say Z, whom she prefers to Y.
- Let B be Z's partner in S.
- Z not rejected by any valid partner at the point when Y is rejected by A. Thus, Z prefers A to B.
- But A prefers Z to Y.
- Thus A-Z is unstable in S.

Yancey-Amy Zeus-Bertha

since this is first rejection

by a valid partner

Stable Matching Summary

Stable matching problem. Given preference profiles of n men and n women, find a stable matching.

no man and woman prefer to be with each other than assigned partner

Gale-Shapley algorithm. Finds a stable matching in $O(n^2)$ time.

Man-optimality. In version of GS where men propose, each man receives best valid partner.

w is a valid partner of m if there exist some stable matching where m and w are paired

Q. Does man-optimality come at the expense of the women?

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S*.

Pf.

- Suppose A-Z matched in S*, but Z is not worst valid partner for A.
- There exists stable matching S in which A is paired with a man, say Y, whom she likes less than Z.
- Let B be Z's partner in S.
- Z prefers A to B. ← man-optimality
- Thus, A-Z is an unstable in S. ■

Amy-Yancey
Bertha-Zeus