2.6 The fast Fourier transform

We have so far seen how divide-and-conquer gives fast algorithms for multiplying integers and matrices; our next target is *polynomials*. The product of two degree-d polynomials is a polynomial of degree $2d$, for example:

$$(1 + 2x + 3x^2) \cdot (2 + x + 4x^2) = 2 + 5x + 12x^2 + 11x^3 + 12x^4.$$

More generally, if $A(x) = a_0 + a_1 x + \cdots + a_d x^d$ and $B(x) = b_0 + b_1 x + \cdots + b_d x^d$, their product $C(x) = A(x) \cdot B(x) = c_0 + c_1 x + \cdots + c_{2d} x^{2d}$ has coefficients

$$c_k \;=\; a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0 \;=\; \sum_{i=0}^{k} a_i b_{k-i}$$

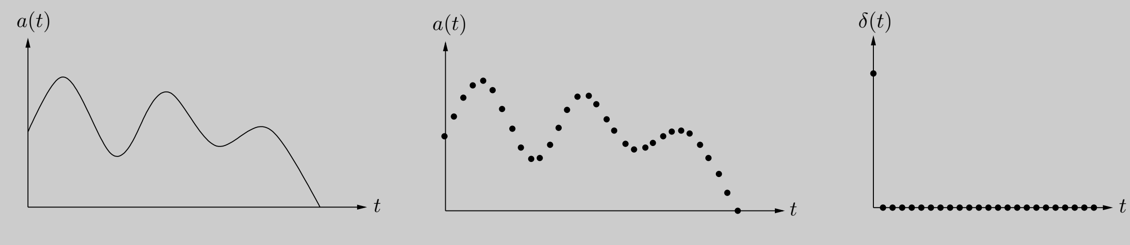
(for $i > d$, take a_i and b_i to be zero). Computing c_k from this formula takes $O(k)$ steps, and finding all $2d + 1$ coefficients would therefore seem to require $\Theta(d^2)$ time. *Can we possibly multiply polynomials faster than this?*

The solution we will develop, the fast Fourier transform, has revolutionized—indeed, defined—the field of signal processing (see the following box). Because of its huge importance, and its wealth of insights from different fields of study, we will approach it a little more leisurely than usual. The reader who wants just the core algorithm can skip directly to Section 2.6.4.

Why multiply polynomials?

For one thing, it turns out that the fastest algorithms we have for multiplying integers rely heavily on polynomial multiplication; after all, polynomials and binary integers are quite similar—just replace the variable x by the base 2, and watch out for carries. But perhaps more importantly, multiplying polynomials is crucial for *signal processing.*

A *signal* is any quantity that is a function of time (as in Figure (a)) or of position. It might, for instance, capture a human voice by measuring fluctuations in air pressure close to the speaker's mouth, or alternatively, the pattern of stars in the night sky, by measuring brightness as a function of angle.



(a) (b) (c)

In order to extract information from a signal, we need to first *digitize* it by sampling (Figure (b))—and, then, to put it through a *system* that will transform it in some way. The output is called the *response* of the system:

$$\text{signal} \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow \text{response}$$

An important class of systems are those that are *linear*—the response to the sum of two signals is just the sum of their individual responses—and *time invariant*—shifting the input signal by time t produces the same output, also shifted by t. Any system with these properties is completely characterized by its response to the simplest possible input signal: the *unit impulse* $\delta(t)$, consisting solely of a "jerk" at $t = 0$ (Figure (c)). To see this, first consider the close relative $\delta(t - i)$, a shifted impulse in which the jerk occurs at time i. Any signal $a(t)$ can be expressed as a linear combination of these, letting $\delta(t - i)$ pick out its behavior at time i,

$$a(t) = \sum_{i=0}^{T-1} a(i)\delta(t - i)$$

(if the signal consists of T samples). By linearity, the system response to input $a(t)$ is determined by the responses to the various $\delta(t-i)$. And by time invariance, these are in turn just shifted copies of the *impulse response* $b(t)$, the response to $\delta(t)$.

In other words, the output of the system at time k is

$$c(k) \;=\; \sum_{i=0}^{k} a(i)b(k - i),$$

exactly the formula for polynomial multiplication!