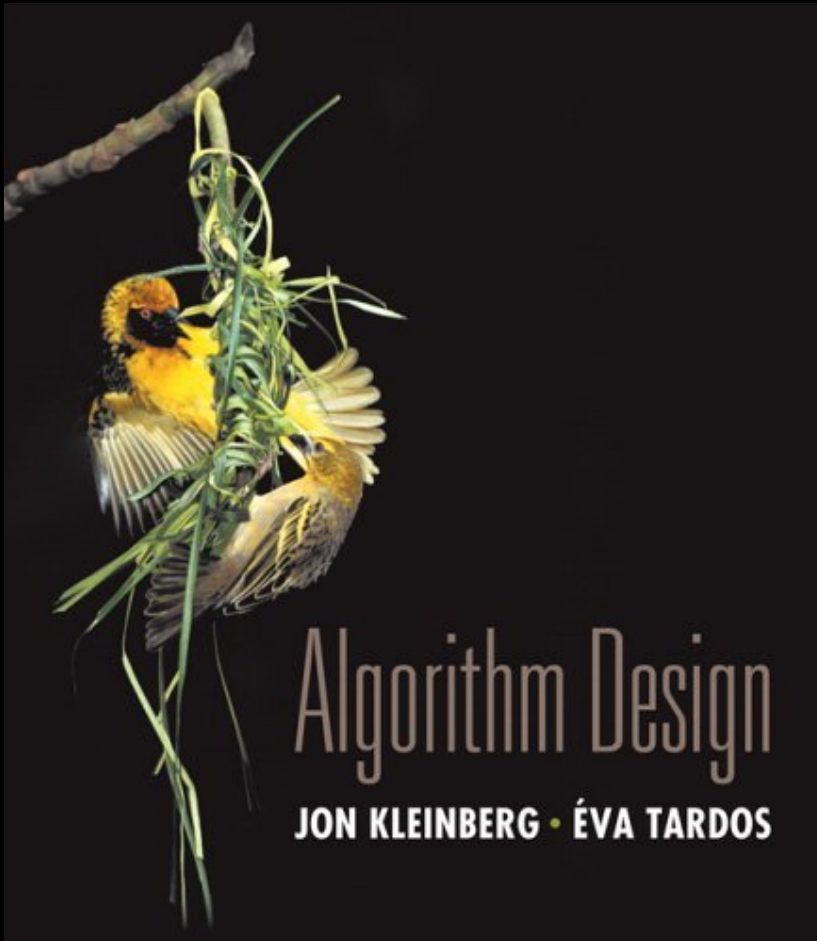


Chapter 4

Greedy Algorithms



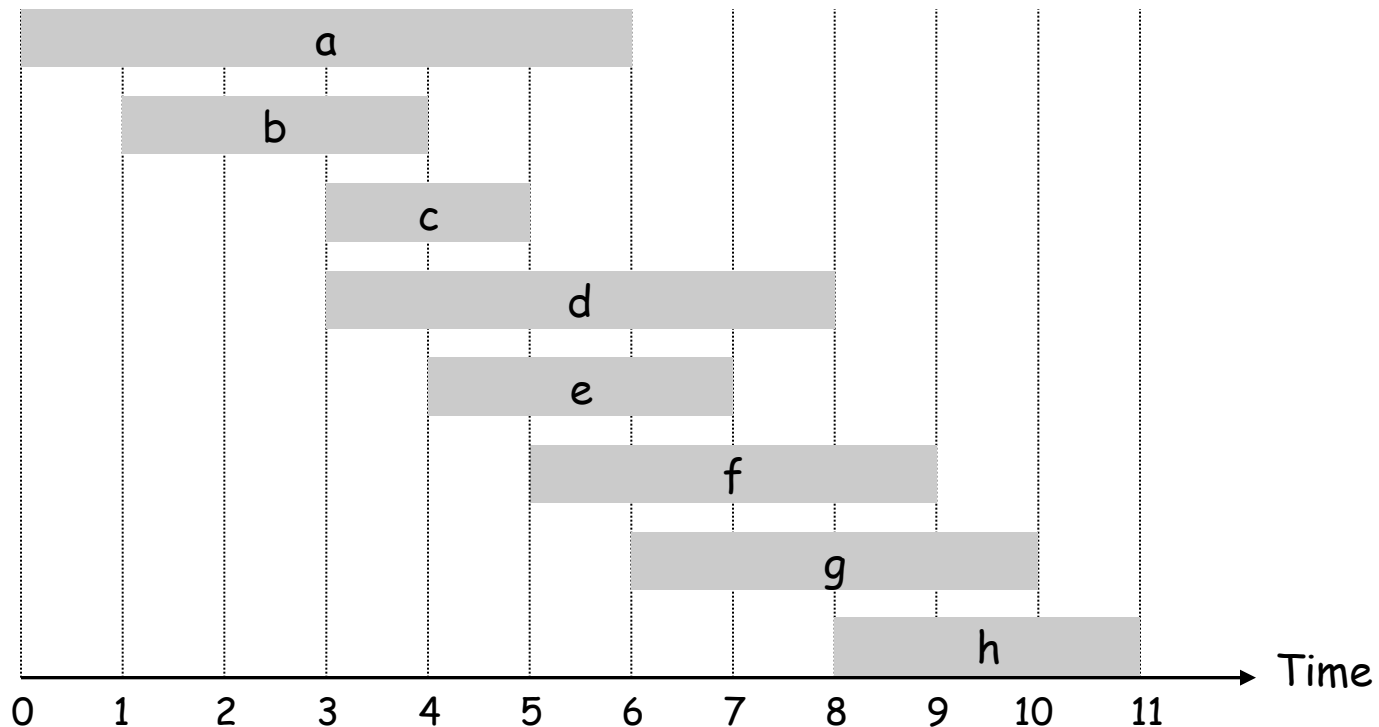
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4.1 Interval Scheduling

Interval Scheduling

Interval scheduling.

- Job j starts at s_j and finishes at f_j .
- Two jobs **compatible** if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

- [Earliest start time] Consider jobs in ascending order of start time s_j .
- [Earliest finish time] Consider jobs in ascending order of finish time f_j .
- [Shortest interval] Consider jobs in ascending order of interval length $f_j - s_j$.
- [Fewest conflicts] For each job, count the number of conflicting jobs c_j . Schedule in ascending order of conflicts c_j .

Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.



breaks earliest start time



breaks shortest interval



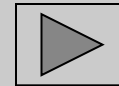
breaks fewest conflicts

Interval Scheduling: Greedy Algorithm

Greedy algorithm. Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

```
Sort jobs by finish times so that  $f_1 \leq f_2 \leq \dots \leq f_n$ .
```

```
    ↙ jobs selected  
A ←  $\phi$   
for j = 1 to n {  
    if (job j compatible with A)  
        A ← A  $\cup$  {j}  
}  
return A
```



Implementation. $O(n \log n)$.

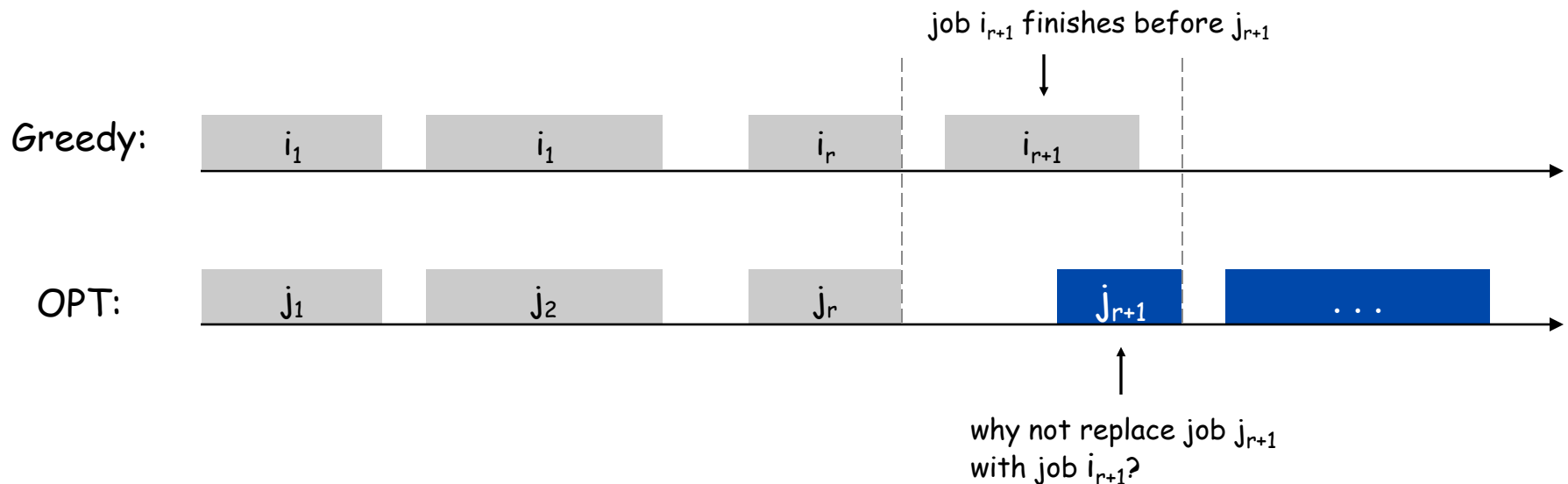
- Remember job j^* that was added last to A.
- Job j is compatible with A if $s_j \geq f_{j^*}$.

Interval Scheduling: Analysis

Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

- Assume greedy is not optimal, and let's see what happens.
- Let i_1, i_2, \dots, i_k denote set of jobs selected by greedy.
- Let j_1, j_2, \dots, j_m denote set of jobs in the optimal solution with $i_1 = j_1, i_2 = j_2, \dots, i_r = j_r$ for the largest possible value of r .

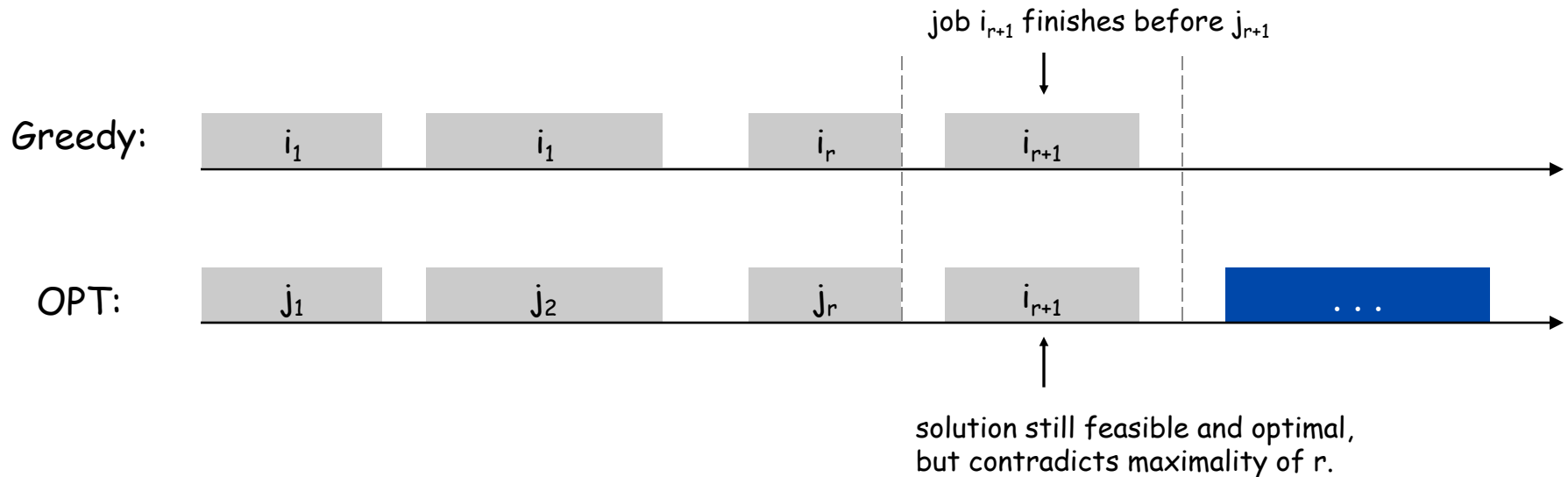


Interval Scheduling: Analysis

Theorem. Greedy algorithm is optimal.

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- Assume greedy is not optimal, and let's see what happens.
- Let i_1, i_2, \dots, i_k denote set of jobs selected by greedy.
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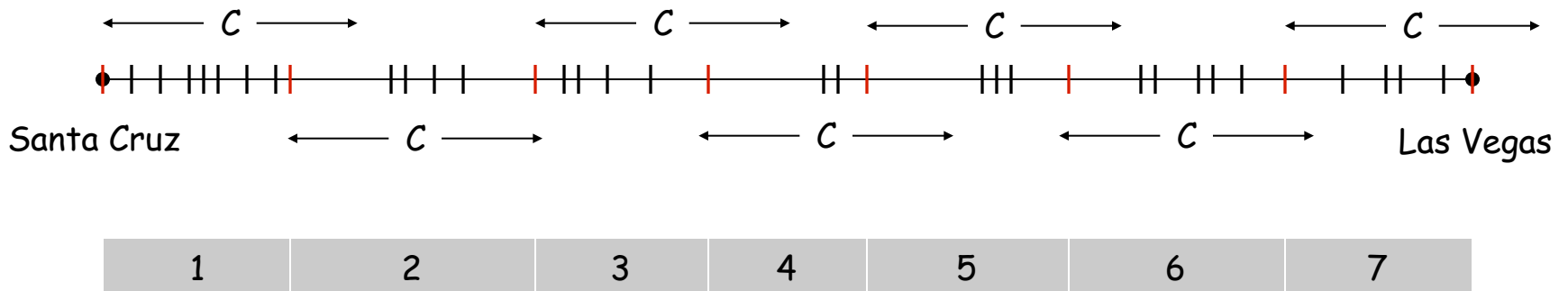
Selecting Breakpoints

Selecting Breakpoints

Selecting breakpoints.

- Road trip from Santa Cruz to Las Vegas along fixed route.
- Refueling stations at certain points along the way.
- Fuel capacity = C .
- Goal: makes as few refueling stops as possible.

Greedy algorithm. Go as far as you can before refueling.



Selecting Breakpoints: Greedy Algorithm

Truck driver's algorithm.

```
Sort breakpoints so that:  $0 = b_0 < b_1 < b_2 < \dots < b_n = L$ 
```

```
 $S \leftarrow \{0\}$   $\leftarrow$  breakpoints selected
```

```
 $x \leftarrow 0$   $\leftarrow$  current location
```

```
while ( $x \neq b_n$ )
```

```
    let  $p$  be largest integer such that  $b_p \leq x + C$ 
```

```
    if ( $b_p = x$ )
```

```
        return "no solution"
```

```
     $x \leftarrow b_p$ 
```

```
     $S \leftarrow S \cup \{p\}$ 
```

```
return  $S$ 
```

Implementation. $O(n \log n)$

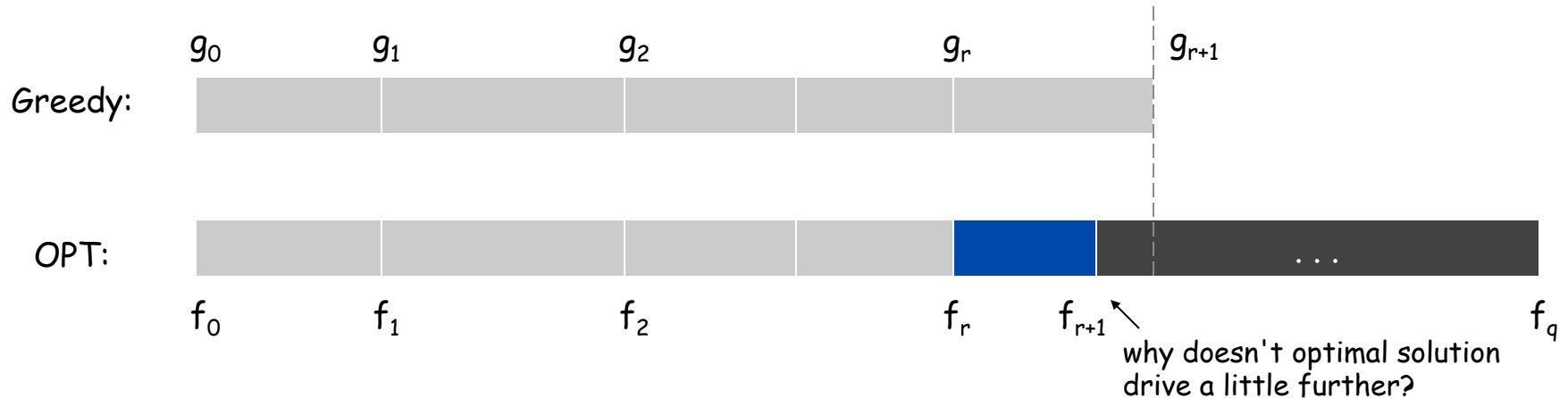
- Use binary search to select each breakpoint p .

Selecting Breakpoints: Correctness

Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

- Assume greedy is not optimal, and let's see what happens.
- Let $0 = g_0 < g_1 < \dots < g_p = L$ denote set of breakpoints chosen by greedy.
- Let $0 = f_0 < f_1 < \dots < f_q = L$ denote set of breakpoints in an optimal solution with $f_0 = g_0, f_1 = g_1, \dots, f_r = g_r$ for largest possible value of r .
- Note: $g_{r+1} > f_{r+1}$ by greedy choice of algorithm.

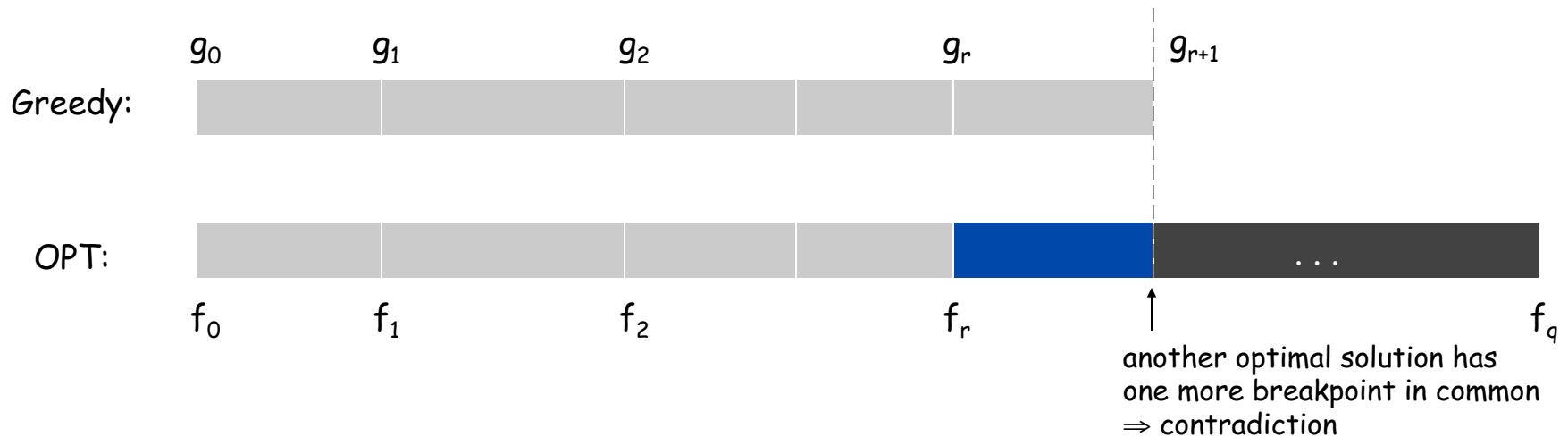


Selecting Breakpoints: Correctness

Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

- Assume greedy is not optimal, and let's see what happens.
- Let $0 = g_0 < g_1 < \dots < g_p = L$ denote set of breakpoints chosen by greedy.
- Let $0 = f_0 < f_1 < \dots < f_q = L$ denote set of breakpoints in an optimal solution with $f_0 = g_0, f_1 = g_1, \dots, f_r = g_r$ for largest possible value of r .
- Note: $g_{r+1} > f_{r+1}$ by greedy choice of algorithm.



4.2 Scheduling to Minimize Lateness

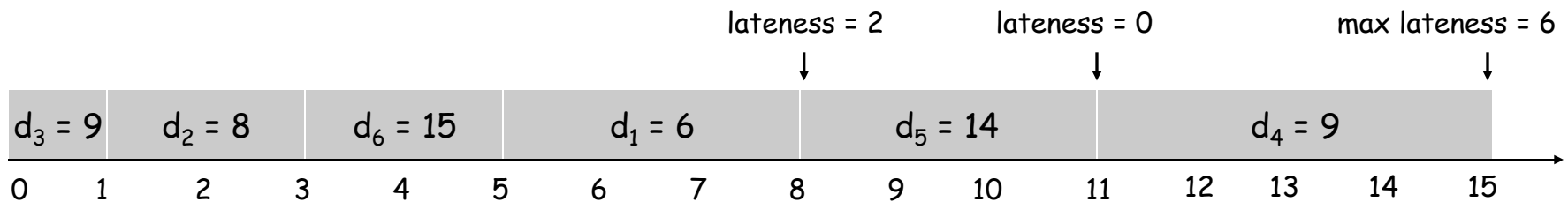
Scheduling to Minimizing Lateness

Minimizing lateness problem.

- Single resource processes one job at a time.
- Job j requires t_j units of processing time and is due at time d_j .
- If j starts at time s_j , it finishes at time $f_j = s_j + t_j$.
- Lateness: $\ell_j = \max \{ 0, f_j - d_j \}$.
- Goal: schedule all jobs to minimize **maximum** lateness $L = \max \ell_j$.

Ex:

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [Shortest processing time first] Consider jobs in ascending order of processing time t_j .
- [Earliest deadline first] Consider jobs in ascending order of deadline d_j .
- [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [Shortest processing time first] Consider jobs in ascending order of processing time t_j .

	1	2
t_j	1	10
d_j	100	10

counterexample

- [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

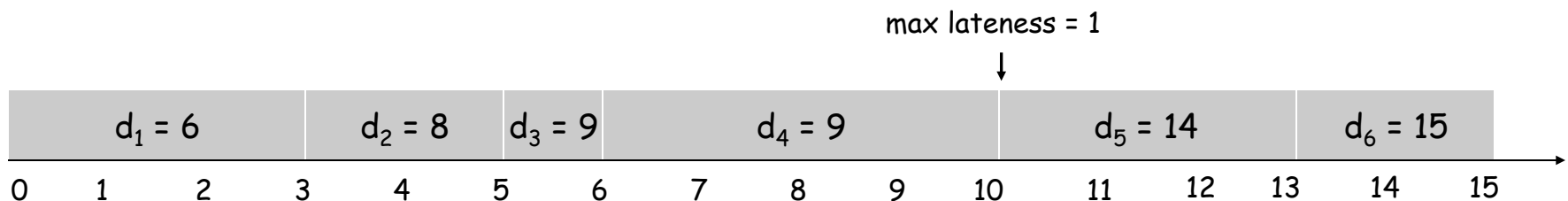
	1	2
t_j	1	10
d_j	2	10

counterexample

Minimizing Lateness: Greedy Algorithm

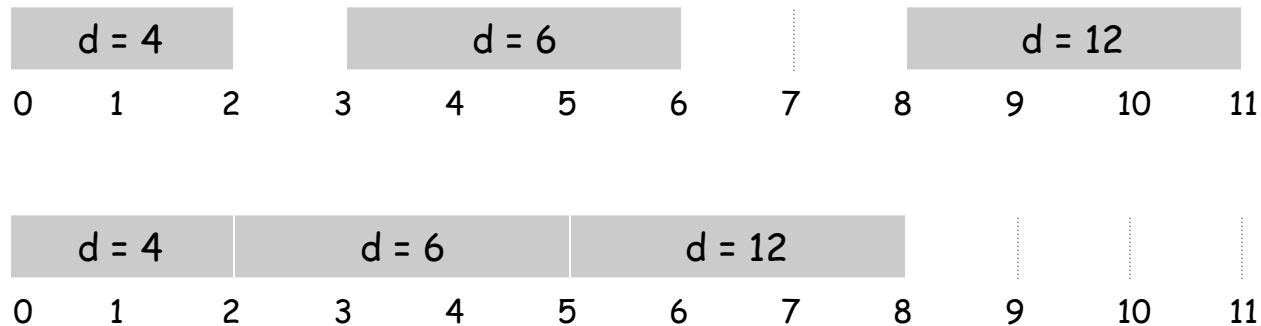
Greedy algorithm. Earliest deadline first.

```
Sort n jobs by deadline so that  $d_1 \leq d_2 \leq \dots \leq d_n$   
  
 $t \leftarrow 0$   
for  $j = 1$  to  $n$   
    Assign job  $j$  to interval  $[t, t + t_j]$   
     $s_j \leftarrow t, f_j \leftarrow t + t_j$   
     $t \leftarrow t + t_j$   
output intervals  $[s_j, f_j]$ 
```



Minimizing Lateness: No Idle Time

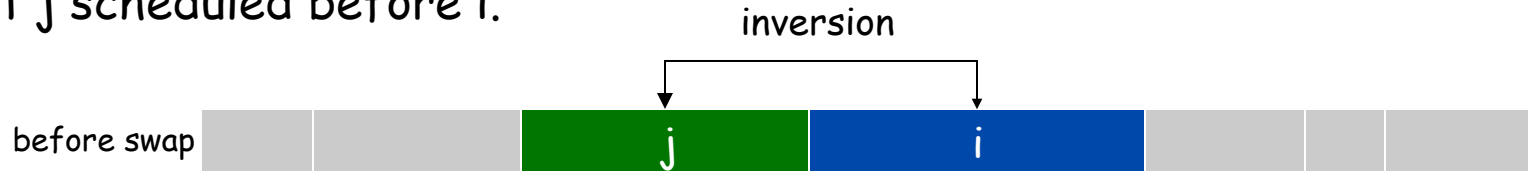
Observation. There exists an optimal schedule with no **idle time**.



Observation. The greedy schedule has no idle time.

Minimizing Lateness: Inversions

Def. An **inversion** in schedule S is a pair of jobs i and j such that: $i < j$ but j scheduled before i .

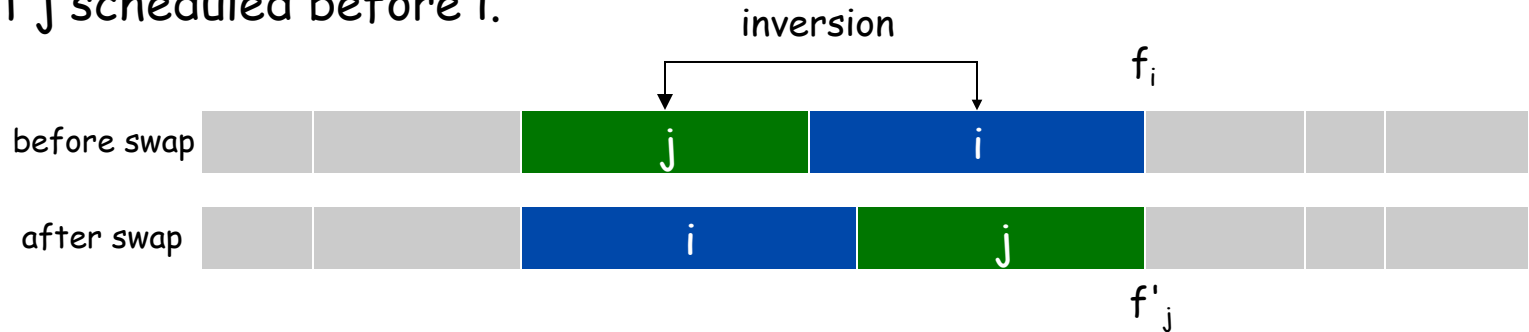


Observation. Greedy schedule has no inversions.

Observation. If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.

Minimizing Lateness: Inversions

Def. An **inversion** in schedule S is a pair of jobs i and j such that: $i < j$ but j scheduled before i .



Claim. Swapping two adjacent, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

Pf. Let ℓ be the lateness before the swap, and let ℓ' be it afterwards.

- $\ell'_k = \ell_k$ for all $k \neq i, j$
- $\ell'_i \leq \ell_i$
- If job j is late:

$$\begin{aligned}
 \ell'_j &= f'_j - d_j && \text{(definition)} \\
 &= f_i - d_j && (j \text{ finishes at time } f_i) \\
 &\leq f_i - d_i && (i < j) \\
 &\leq \ell_i && \text{(definition)}
 \end{aligned}$$

Minimizing Lateness: Analysis of Greedy Algorithm

Theorem. Greedy schedule S is optimal.

Pf. Define S^* to be an optimal schedule that has the fewest number of inversions, and let's see what happens.

- Can assume S^* has no idle time.
- If S^* has no inversions, then $S = S^*$.
- If S^* has an inversion, let i - j be an adjacent inversion.
 - swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions
 - this contradicts definition of S^* ▪

Greedy Analysis Strategies

Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.

Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

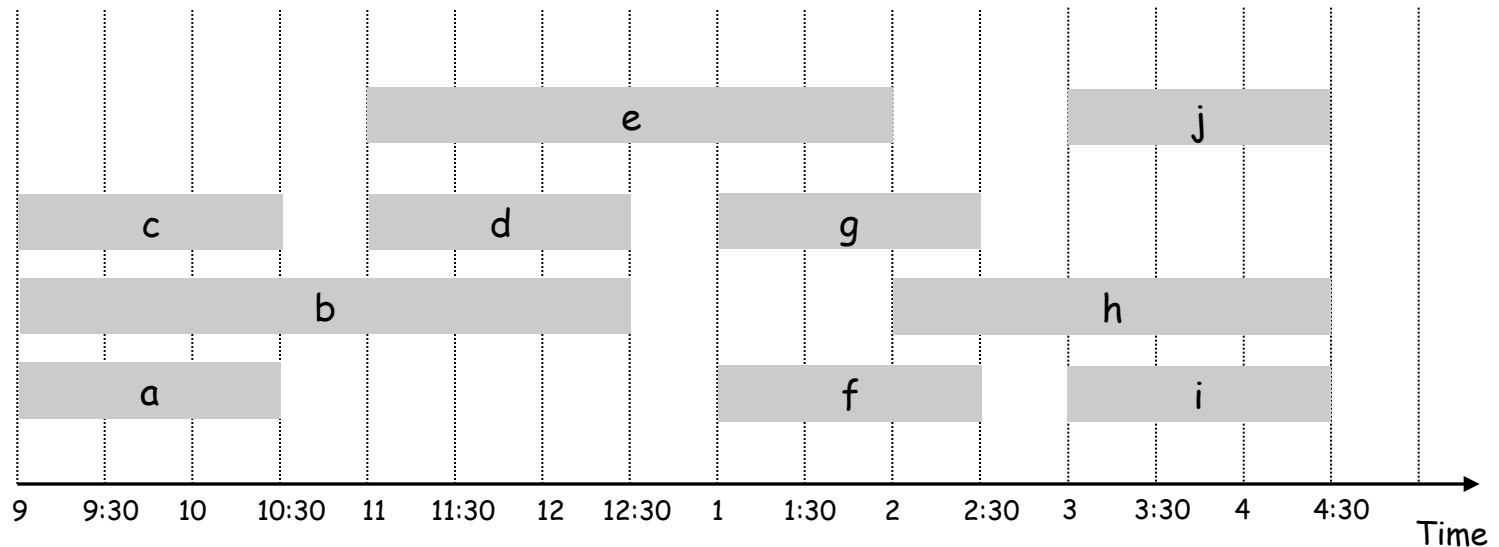
4.1 Interval Partitioning

Interval Partitioning

Interval partitioning.

- Lecture j starts at s_j and finishes at f_j .
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses 4 classrooms to schedule 10 lectures.

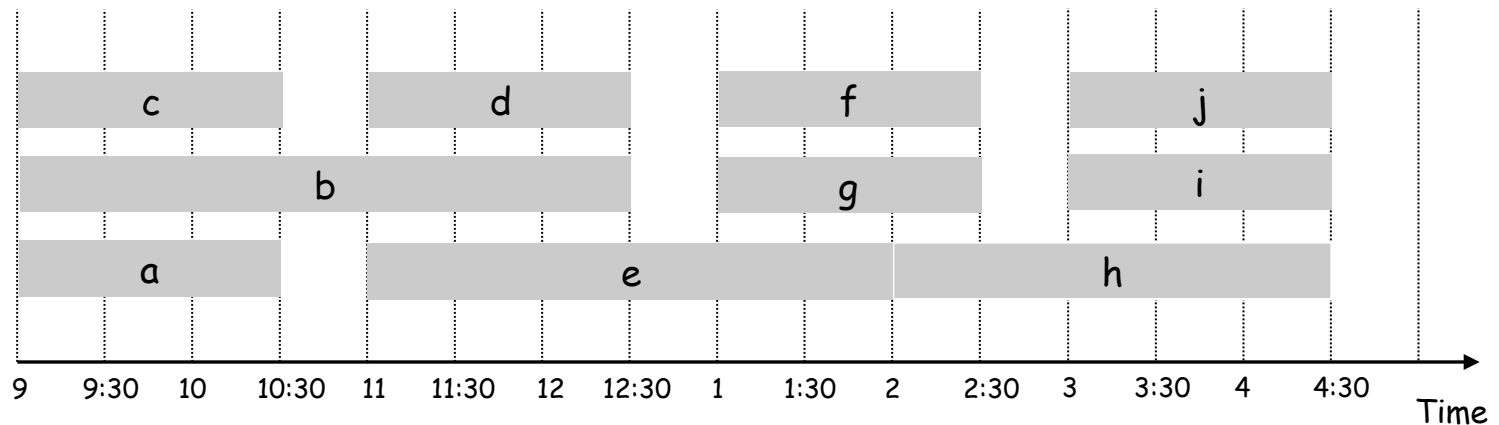


Interval Partitioning

Interval partitioning.

- Lecture j starts at s_j and finishes at f_j .
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses only 3.



Interval Partitioning: Lower Bound on Optimal Solution

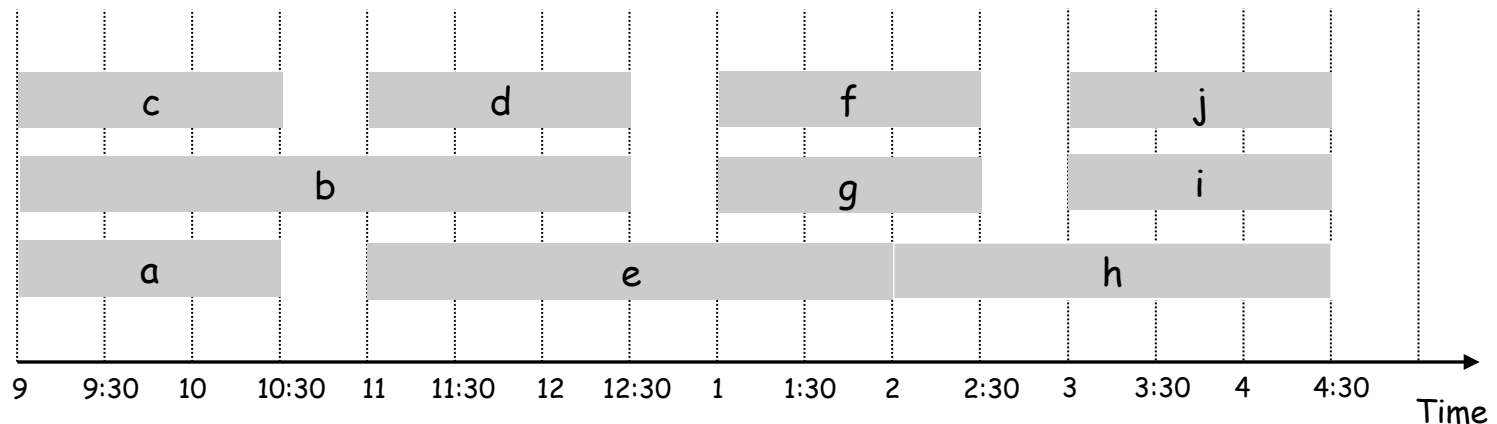
Def. The **depth** of a set of open intervals is the maximum number that contain any given time.

Key observation. Number of classrooms needed \geq depth.

Ex: Depth of schedule below = 3 \Rightarrow schedule below is optimal.

↑
a, b, c all contain 9:30

Q. Does there always exist a schedule equal to depth of intervals?



Interval Partitioning: Greedy Algorithm

Greedy algorithm. Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that  $s_1 \leq s_2 \leq \dots \leq s_n$ .  
 $d \leftarrow 0$   $\leftarrow$  number of allocated classrooms  
  
for  $j = 1$  to  $n$  {  
    if (lecture  $j$  is compatible with some classroom  $k$ )  
        schedule lecture  $j$  in classroom  $k$   
    else  
        allocate a new classroom  $d + 1$   
        schedule lecture  $j$  in classroom  $d + 1$   
         $d \leftarrow d + 1$   
}
```

Implementation. $O(n \log n)$.

- For each classroom k , maintain the finish time of the last job added.
- Keep the classrooms in a priority queue.

Interval Partitioning: Greedy Analysis

Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Greedy algorithm is optimal.

Pf.

- Let d = number of classrooms that the greedy algorithm allocates.
- Classroom d is opened because we needed to schedule a job, say j , that is incompatible with all $d-1$ other classrooms.
- Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s_j .
- Thus, we have d lectures overlapping at time $s_j + \varepsilon$.
- Key observation \Rightarrow all schedules use $\geq d$ classrooms. ■