

Weighted Interval Scheduling: Memoization

Memoization. Store results of each sub-problem in a cache; lookup as needed.

Input: $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$

Sort jobs by finish times so that $f_1 \leq f_2 \leq \dots \leq f_n$.

Compute $p(1), p(2), \dots, p(n)$

for $j = 1$ to n

$M[j] = \text{empty}$ \leftarrow global array

$M[j] = 0$

M-Compute-Opt(j) {

if ($M[j]$ is empty)

$M[j] = \max(w_j + \text{M-Compute-Opt}(p(j)), \text{M-Compute-Opt}(j-1))$

return $M[j]$

}

Weighted Interval Scheduling: Running Time

Claim. Memoized version of algorithm takes $O(n \log n)$ time.

- Sort by finish time: $O(n \log n)$.
- Computing $p(\cdot)$: $O(n)$ after sorting by start time.
- $M\text{-Compute-Opt}(j)$: each invocation takes $O(1)$ time and either
 - (i) returns an existing value $M[j]$
 - (ii) fills in one new entry $M[j]$ and makes two recursive calls
- Progress measure $\Phi = \#$ nonempty entries of $M[\]$.
 - initially $\Phi = 0$, throughout $\Phi \leq n$.
 - (ii) increases Φ by 1 \Rightarrow at most $2n$ recursive calls.
- Overall running time of $M\text{-Compute-Opt}(n)$ is $O(n)$. ▪

Remark. $O(n)$ if jobs are pre-sorted by start and finish times.

Weighted Interval Scheduling: Finding a Solution

Q. Dynamic programming algorithms computes optimal value. What if we want the solution itself?

A. Do some post-processing.

```
Run M-Compute-Opt(n)
Run Find-Solution(n)

Find-Solution(j) {
    if (j = 0)
        output nothing
    else if ( $v_j + M[p(j)] > M[j-1]$ )
        print j
        Find-Solution(p(j))
    else
        Find-Solution(j-1)
}
```

- # of recursive calls $\leq n \Rightarrow O(n)$.

Weighted Interval Scheduling: Bottom-Up

Bottom-up dynamic programming. Unwind recursion.

Input: $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$

Sort jobs by finish times so that $f_1 \leq f_2 \leq \dots \leq f_n$.

Compute $p(1), p(2), \dots, p(n)$

```
Iterative-Compute-Opt {  
    M[0] = 0  
    for j = 1 to n  
        M[j] = max(vj + M[p(j)], M[j-1])  
}
```

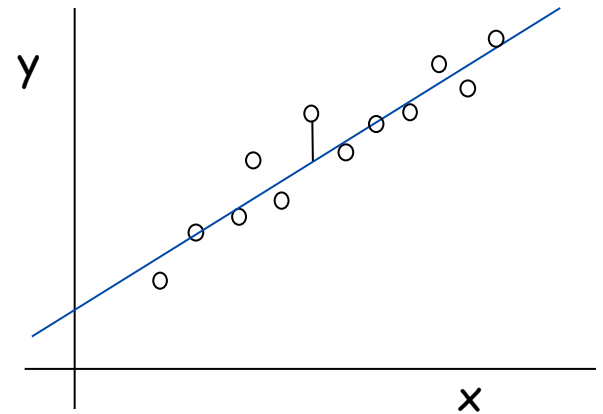
6.3 Segmented Least Squares

Segmented Least Squares

Least squares.

- Foundational problem in statistic and numerical analysis.
- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.
- Find a line $y = ax + b$ that minimizes the sum of the squared error:

$$SSE = \sum_{i=1}^n (y_i - ax_i - b)^2$$



Solution. Calculus \Rightarrow min error is achieved when

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2}, \quad b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

Segmented Least Squares

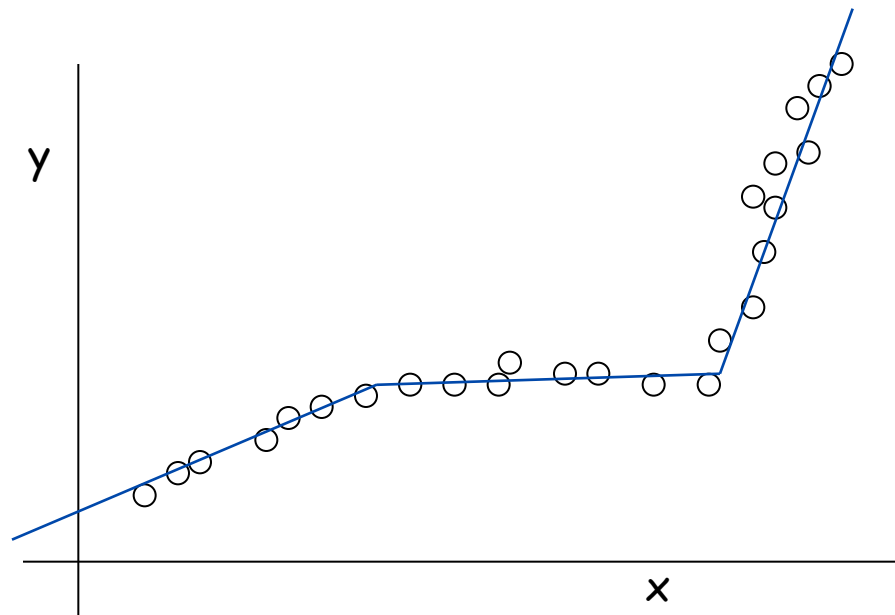
Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ with
- $x_1 < x_2 < \dots < x_n$, find a sequence of lines that minimizes $f(x)$.

Q. What's a reasonable choice for $f(x)$ to balance accuracy and parsimony?

↑
number of lines

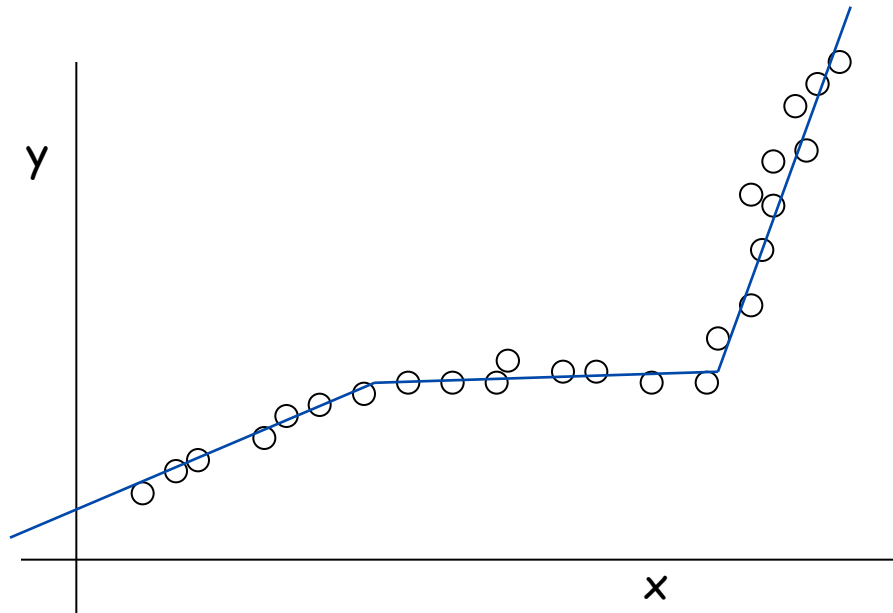
↑
goodness of fit



Segmented Least Squares

Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ with
- $x_1 < x_2 < \dots < x_n$, find a sequence of lines that minimizes:
 - the sum of the sums of the squared errors E in each segment
 - the number of lines L
- Tradeoff function: $E + c L$, for some constant $c > 0$.



Dynamic Programming: Multiway Choice

Notation.

- $OPT(j)$ = minimum cost for points p_1, p_2, \dots, p_j .
- $e(i, j)$ = minimum sum of squares for points p_i, p_{i+1}, \dots, p_j .

To compute $OPT(j)$:


- Last segment uses points p_i, p_{i+1}, \dots, p_j for some i .
- $Cost = e(i, j) + c + OPT(i-1)$.

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \min_{1 \leq i \leq j} \{ e(i, j) + c + OPT(i-1) \} & \text{otherwise} \end{cases}$$

Segmented Least Squares: Algorithm

INPUT: n, p_1, \dots, p_N, c

```
Segmented-Least-Squares() {  
    M[0] = 0  
    for j = 1 to n  
        for i = 1 to j  
            compute the least square error  $e_{ij}$  for  
            the segment  $p_i, \dots, p_j$   
  
    for j = 1 to n  
        M[j] =  $\min_{1 \leq i \leq j} (e_{ij} + c + M[i-1])$   
  
    return M[n]  
}
```

Running time. $O(n^3)$.  can be improved to $O(n^2)$ by pre-computing various statistics

- Bottleneck = computing $e(i, j)$ for $O(n^2)$ pairs, $O(n)$ per pair using previous formula.

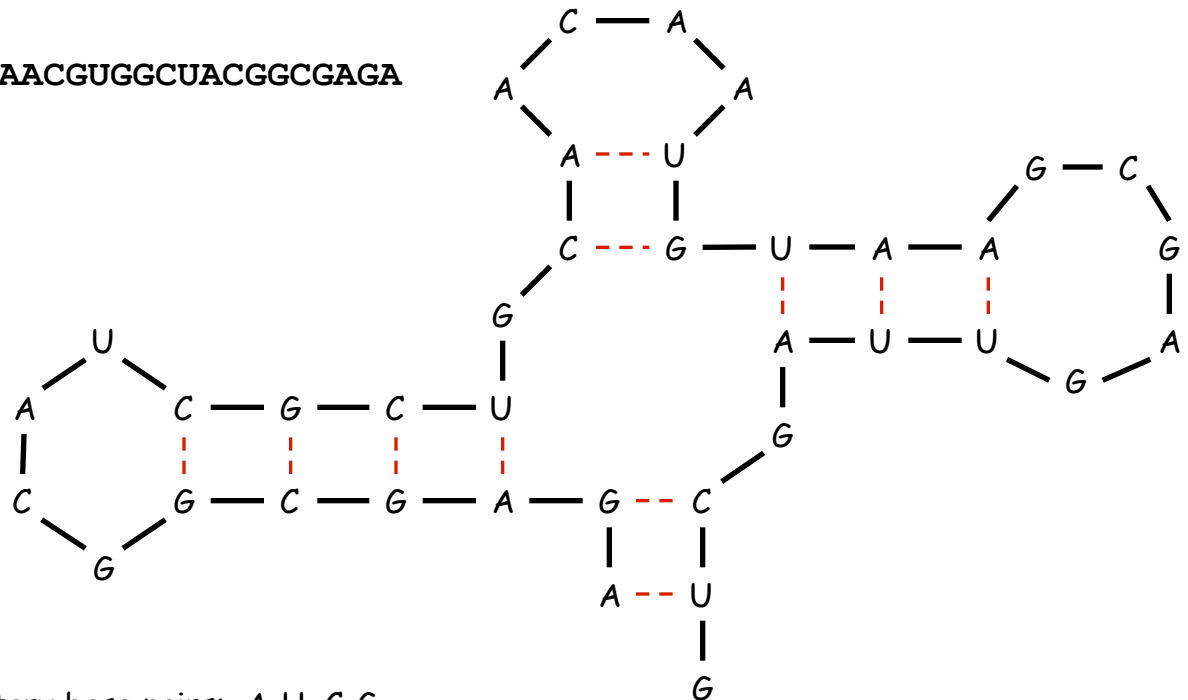
6.5 RNA Secondary Structure

RNA Secondary Structure

RNA. String $B = b_1b_2\dots b_n$ over alphabet $\{A, C, G, U\}$.

Secondary structure. RNA is single-stranded so it tends to loop back and form base pairs with itself. This structure is essential for understanding behavior of molecule.

Ex: GUCGAUUGAGCGAUGUAACAACGUGGCUACGGCGAGA



complementary base pairs: A-U, C-G

RNA Secondary Structure

Secondary structure. A set of pairs $S = \{ (b_i, b_j) \}$ that satisfy:

- [Watson-Crick.] S is a matching and each pair in S is a Watson-Crick complement: A-U, U-A, C-G, or G-C.
- [No sharp turns.] The ends of each pair are separated by at least 4 intervening bases. If $(b_i, b_j) \in S$, then $i < j - 4$.
- [Non-crossing.] If (b_i, b_j) and (b_k, b_l) are two pairs in S , then we cannot have $i < k < j < l$.

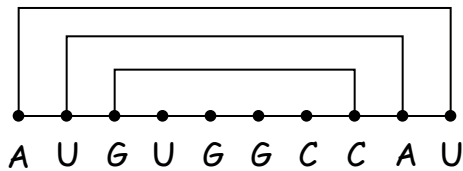
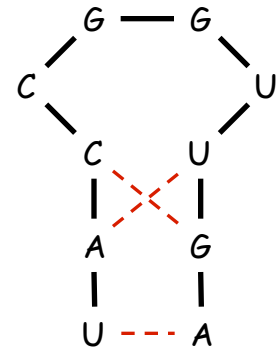
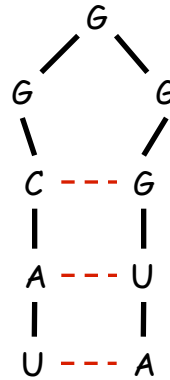
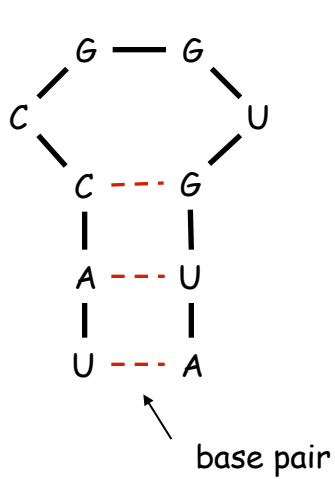
Free energy. Usual hypothesis is that an RNA molecule will form the secondary structure with the optimum total free energy.

↑
approximate by number of base pairs

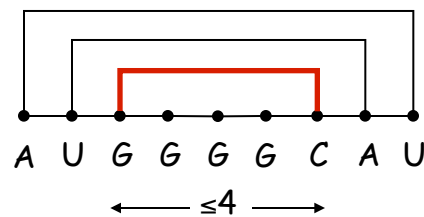
Goal. Given an RNA molecule $B = b_1b_2\dots b_n$, find a secondary structure S that maximizes the number of base pairs.

RNA Secondary Structure: Examples

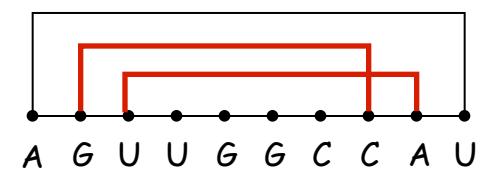
Examples.



ok



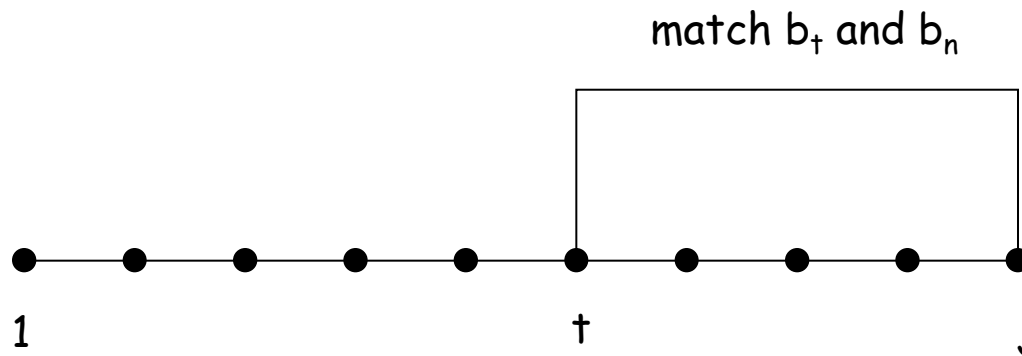
sharp turn



crossing

RNA Secondary Structure: Subproblems

First attempt. $\text{OPT}(j)$ = maximum number of base pairs in a secondary structure of the substring $b_1b_2\dots b_j$.



Difficulty. Results in two sub-problems.

- Finding secondary structure in: $b_1b_2\dots b_{t-1}$. $\leftarrow \text{OPT}(t-1)$
- Finding secondary structure in: $b_{t+1}b_{t+2}\dots b_{j-1}$. \leftarrow need more sub-problems

Dynamic Programming Over Intervals

Notation. $\text{OPT}(i, j)$ = maximum number of base pairs in a secondary structure of the substring $b_i b_{i+1} \dots b_j$.

- Case 1. If $j-i \leq 4$.
 - $\text{OPT}(i, j) = 0$ by no-sharp turns condition.
- Case 2. Base b_j is not involved in a pair.
 - $\text{OPT}(i, j) = \text{OPT}(i, j-1)$
- Case 3. Base b_j pairs with b_t for some $i \leq t < j - 4$.
 - non-crossing constraint decouples resulting sub-problems
 - $\text{OPT}(i, j) = 1 + \max_t \{ \text{OPT}(i, t-1) + \text{OPT}(t+1, j-1) \}$

↑
take max over t such that $i \leq t < j-4$ and
 b_t and b_j are Watson-Crick complements

Remark. Same core idea in CKY algorithm to parse context-free grammars.

Bottom Up Dynamic Programming Over Intervals

Q. What order to solve the sub-problems?

A. Do shortest intervals first.

```
RNA( $b_1, \dots, b_n$ ) {  
  for  $k = 5, 6, \dots, n-1$   
    for  $i = 1, 2, \dots, n-k$   
       $j = i + k$   
      Compute  $M[i, j]$   
  
  return  $M[1, n]$   
}
```

using recurrence

i

4	0	0	0	↗
3	0	0	↗	↗
2	0	↗	↗	↗
1	↗	↗	↗	↗
	6	7	8	9

j

Running time. $O(n^3)$.

Dynamic Programming Summary

Recipe.

- Characterize structure of problem.
- Recursively define value of optimal solution.
- Compute value of optimal solution.
- Construct optimal solution from computed information.

Dynamic programming techniques.

- Binary choice: weighted interval scheduling.
- Multi-way choice: segmented least squares. ←
- Adding a new variable: knapsack.
- Dynamic programming over intervals: RNA secondary structure.

Viterbi algorithm for HMM also uses DP to optimize a maximum likelihood tradeoff between parsimony and accuracy

↖ CKY parsing algorithm for context-free grammar has similar structure

Top-down vs. bottom-up: different people have different intuitions.