

Matching Residents to Hospitals

Goal. Given a set of preferences among hospitals and medical school students, design a **self-reinforcing** admissions process.

Unstable pair: applicant x and hospital y are **unstable** if:

- x prefers y to its assigned hospital.
- y prefers x to one of its admitted students.

Stable assignment. Assignment with no unstable pairs.

- Natural and desirable condition.
- Individual self-interest will prevent any applicant/hospital deal from being made.

Stable Matching Problem

Goal. Given n men and n women, find a "suitable" matching.

- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

	favorite ↓ 1 st	2 nd	least favorite ↓ 3 rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	favorite ↓ 1 st	2 nd	least favorite ↓ 3 rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Perfect matching: everyone is matched monogamously.

- Each man gets exactly one woman.
- Each woman gets exactly one man.

Stability: no incentive for some pair of participants to undermine assignment by joint action.

- In matching M , an unmatched pair m - w is **unstable** if man m and woman w prefer each other to current partners.
- Unstable pair m - w could each improve by eloping.

Stable matching: perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n men and n women, find a stable matching if one exists.

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

	favorite ↓		least favorite ↓
	1 st	2 nd	3 rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	favorite ↓		least favorite ↓
	1 st	2 nd	3 rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

A. No. Bertha and Xavier will hook up.

	favorite ↓		least favorite ↓
	1 st	2 nd	3 rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	favorite ↓		least favorite ↓
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Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-A, Y-B, Z-C stable?

A. Yes.

	favorite ↓		least favorite ↓
	1 st	2 nd	3 rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	favorite ↓		least favorite ↓
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Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Roommate Problem

Q. Do stable matchings always exist?

A. Not obvious a priori.

is core of market nonempty?

Stable roommate problem.

- $2n$ people; each person ranks others from 1 to $2n-1$.
- Assign roommate pairs so that no unstable pairs.

	1 st	2 nd	3 rd
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

$A-B, C-D \Rightarrow B-C$ unstable
 $A-C, B-D \Rightarrow A-B$ unstable
 $A-D, B-C \Rightarrow A-C$ unstable

Observation. Stable matchings do not always exist for stable roommate problem.

Propose-And-Reject Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.



```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```


Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference.

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

Claim. Algorithm terminates after at most n^2 iterations of while loop.

Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals. ■

	1 st	2 nd	3 rd	4 th	5 th
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

	1 st	2 nd	3 rd	4 th	5 th
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

$n(n-1) + 1$ proposals required

Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf. (by contradiction)

- Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
- Then some woman, say Amy, is not matched upon termination.
- By Observation 2, Amy was never proposed to.
- But, Zeus proposes to everyone, since he ends up unmatched. ▪

Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. (by contradiction)

- Suppose A - Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching S^* .

- Case 1: Z never proposed to A .
 - $\Rightarrow Z$ prefers his GS partner to A .
 - $\Rightarrow A$ - Z is stable.

men propose in decreasing
order of preference

S^*

Amy-Yancey

Bertha-Zeus

...

- Case 2: Z proposed to A .
 - $\Rightarrow A$ rejected Z (right away or later)
 - $\Rightarrow A$ prefers her GS partner to Z .
 - $\Rightarrow A$ - Z is stable.

← women only trade up

- In either case A - Z is stable, a contradiction. ▪

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of *Gale-Shapley* yield the same stable matching? If so, which one?

An instance with two stable matchings.

- A-X, B-Y, C-Z.
- A-Y, B-X, C-Z.

	1 st	2 nd	3 rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1 st	2 nd	3 rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of *Gale-Shapley* yield the same stable matching? If so, which one?

Def. Man m is a **valid partner** of woman w if there exists some stable matching in which they are matched.

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of *Gale-Shapley* yield the same stable matching? If so, which one?

Def. Man m is a **valid partner** of woman w if there exists some stable matching in which they are matched.

Man-optimal assignment. Each man receives best valid partner.

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

Def. Man m is a **valid partner** of woman w if there exists some stable matching in which they are matched.

Man-optimal assignment. Each man receives best valid partner.

Claim. All executions of GS yield **man-optimal** assignment, which is a stable matching!

- No reason a priori to believe that man-optimal assignment is perfect, let alone stable.
- Simultaneously best for each and every man.

Man Optimality

Claim. GS matching S^* is man-optimal.

Pf. (by contradiction)

- Suppose some man is paired with someone other than best partner. Men propose in decreasing order of preference \Rightarrow some man is rejected by valid partner.
- Let Y be **first** such man, and let A be **first** valid woman that rejects him.
- Let S be a stable matching where A and Y are matched.
- When Y is rejected, A forms (or reaffirms) engagement with a man, say Z , whom she prefers to Y .
- Let B be Z 's partner in S .
- Z not rejected by any valid partner at the point when Y is rejected by A . Thus, Z prefers A to B .
- But A prefers Z to Y .
- Thus A - Z is unstable in S . ▪

S	
Yancey	Amy
Zeus	Bertha
...	

↑
since this is first rejection
by a valid partner

Stable Matching Summary

Stable matching problem. Given preference profiles of n men and n women, find a **stable** matching.

no man and woman prefer to be with each other than assigned partner

Gale-Shapley algorithm. Finds a stable matching in $O(n^2)$ time.

Man-optimality. In version of *GS* where men propose, each man receives best valid partner.

w is a valid partner of m if there exist some stable matching where m and w are paired

Q. Does man-optimality come at the expense of the women?

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds **woman-pessimal** stable matching S^* .

Pf.

- Suppose A - Z matched in S^* , but Z is not worst valid partner for A .
- There exists stable matching S in which A is paired with a man, say Y , whom she likes less than Z .
- Let B be Z 's partner in S .
- Z prefers A to B . \leftarrow man-optimality
- Thus, A - Z is an unstable in S . ▪

S	
Amy	Yancey
Bertha	Zeus
...	