# Analytical Methods for Weakly Nonlinear Oscillators and the Two-Timing Approach: The Duffing Equation


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## Abstract

- The simple harmonic oscillator, a mass attached to a spring, is governed by the second order linear differential equation, $\ddot{x} + b\dot{x} + kx = 0$ where $x$ is vertical position, $m$ is mass, $b$ the damping coeffecient, and $k$ the spring constant
- Its general solution without damping is found easily, such that $x(t) = A\cos t + B\sin t$
- The **Duffing equation** is governed by the *nonlinear* differential equation, $\ddot{x} + x + \epsilon x^3 = 0$. A loose approximation of its solution can be found using perturbation theory. While it works, the solution contains a **secular** term, and grows exponentially in time
- To soothe this, the method of **two-timing** is introduced, defining 'slow' and 'fast' time, and the secular term is removed
- From here, the efficacy of $\epsilon$ can be determined


## The Duffing Equation

Nonlinear oscillators are subset of oscillatory differential equations. The Duffing equation is defined by $\ddot{x} + x + x^3 = 0$. The equation is termed *weakly nonlinear* as $\epsilon <<< 1$. The solution to the Duffing equation must be started with perturbation theory, assuming $x(t)$ is analytic. This equation also has a limit cycle about the origin, a circular orbit in which trajectories get 'caught'. Below is the phase plane for the Duffing equation. [1]

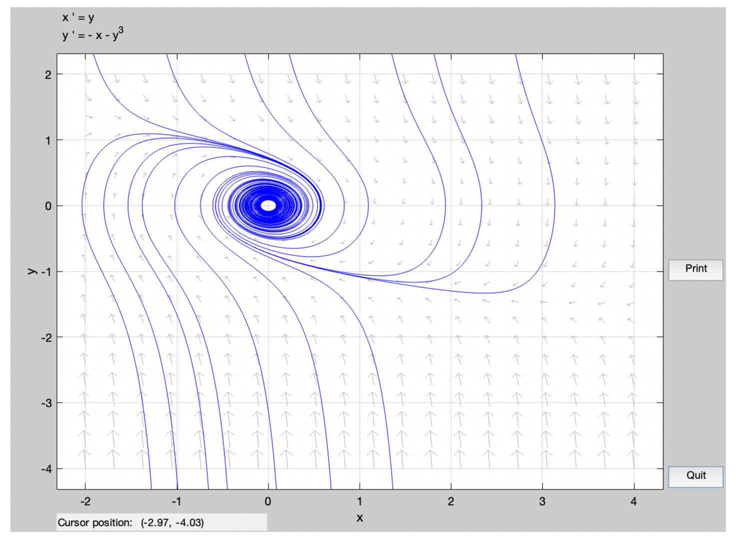

Figure 1: Phase plane for the Duffing equation

## Breakdown of Motion

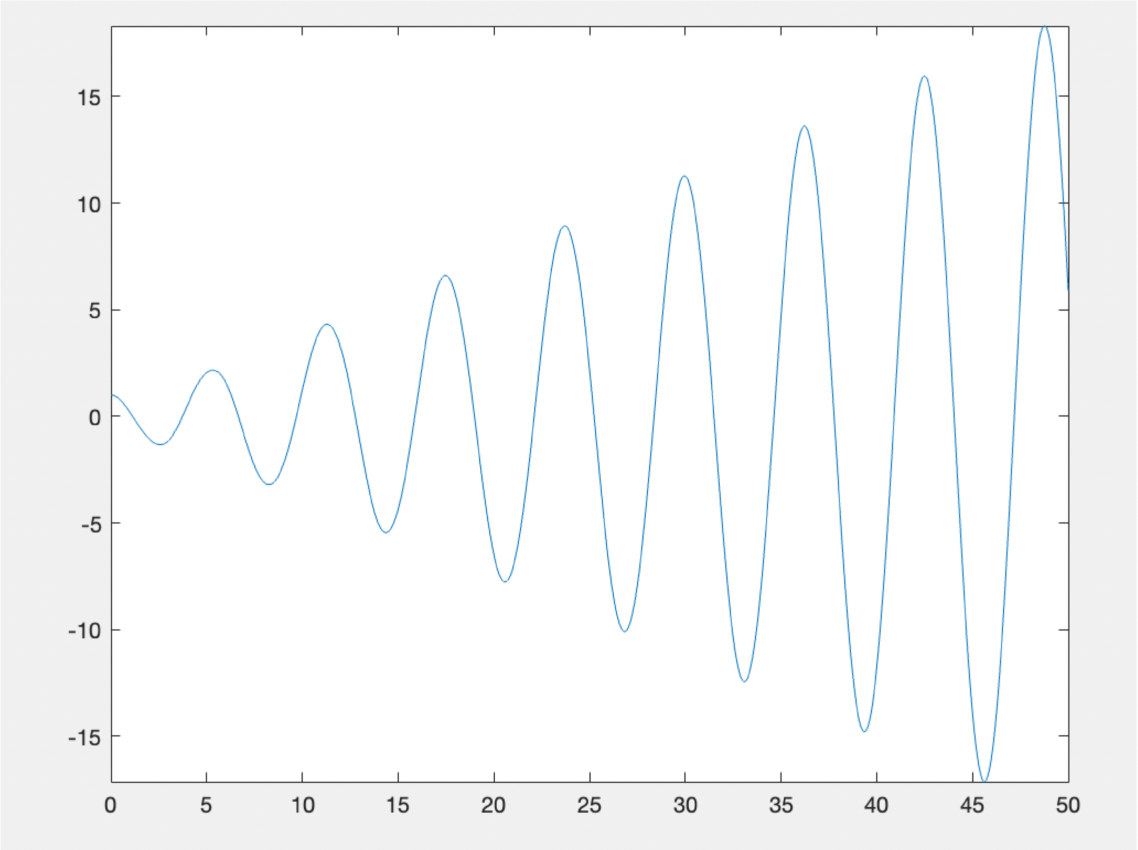

Figure 2: The failure of perturbation theory for the Duffing equation

## Applying a Typical Perturbation Method

We attempt to find a solution using perturbation theory. Let x have a power series expansion in powers of $\epsilon$ and functions of t, such that:

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_1(t) + O(\epsilon^3) \tag{1}$$

Substitution into the Duffing equation up to powers of linear $\epsilon$ gives:

$$\ddot{x}_0 + x_0 = 0 \implies x_0(t) = \cos(t) \tag{2}$$

$$x_1(t) = \frac{1}{32}(\cos(3t) - \cos(t)) - \frac{3}{8}t\sin(t) \tag{3}$$

Finding $x_0(t)$ and $x_1(t)$ allows us to substitute them back into the perturbation approximation [2]. This gives a first order approximation of the solution to the Duffing equation:

$$x(t) = \cos(t) + \epsilon(x_1(t) = \tfrac{1}{32}(\cos(3t) - \cos(t)) - \tfrac{3}{8}t\sin(t)) \tag{4}$$

## Important Result

The first order approximation produces a **secular term**, note the 't' in the analytical solution. When plotting the solution, we can immediately see the issue. The graph of $x(t)$ grows exponentially in time, going against the motion prescribed by the differential equation. To soothe this, we must employ other methods

## Two-Timing

To remove the secular term, we begin by introducing a second time scale, scaled by a parameter $\alpha$, and reapply perturbation theory

$$\tau = \alpha t \tag{5}$$

$$\alpha = \alpha_0 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + O(\alpha^3) \tag{6}$$

$$\implies \alpha^2\ddot{x}(\tau) + x(\tau) + \epsilon x^3(\tau) \tag{7}$$

$$x_0(\tau) = \cos(\tau) \tag{8}$$

$$x_1(\tau) = \frac{1}{32}(\cos(3\tau) - \cos(\tau)) + (\alpha_1 - \frac{3}{8})\tau\sin(\tau) \tag{9}$$

$$\implies x(t) \approx \cos((1+\tfrac{3}{8}\epsilon)t) + \tfrac{1}{32}\epsilon(\cos((3+\tfrac{9}{8}\epsilon)t) - \cos((1+\tfrac{3}{8})t)) + O(\epsilon^2) \tag{10}$$

## Results

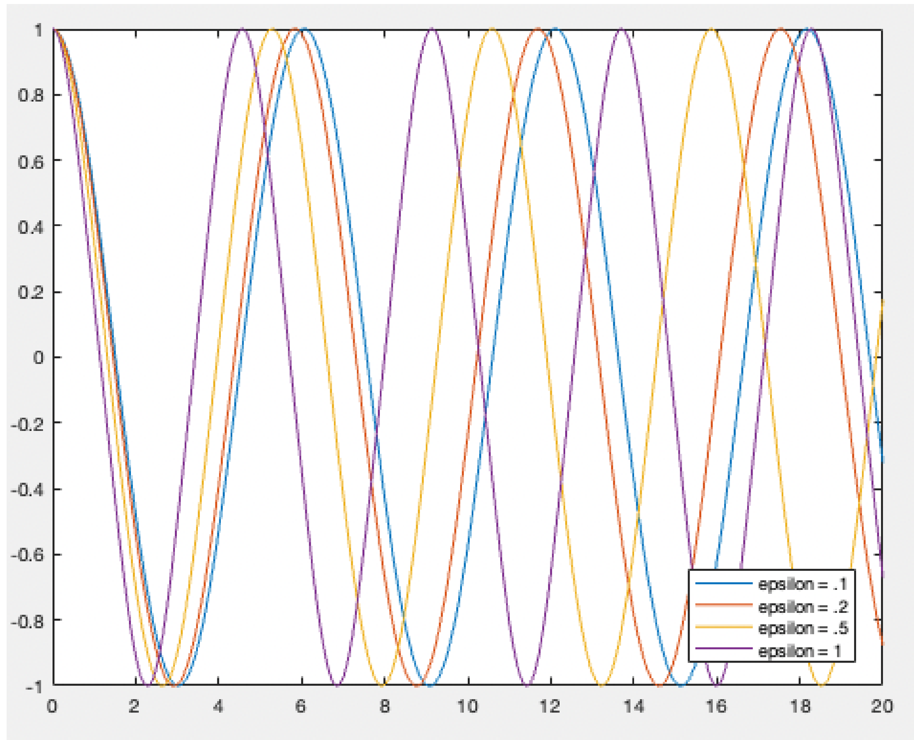


Figure 3: Various analytical solutions to the Duffing equation

## Conclusion

The first order two-timed solution is represented by (10). By scaling time in two different dimensions, we are able to derive a solution that is more synonymous with the motion of the oscillator. Although, we must still be conscious of $\epsilon$, as the motion breaks down if the parameter is tended too high.

## Additional Information

From here, my research will continue by looking at examples of other oscillators and the underpinnings of perturbation theory. The next step is to time oscillators three-fold, adding an additional parameter to ensure the solution is even more exact.

## References


[1] S. Strogatz. *Nonlinear Dynamics and Chaos.* Westview Press, 2nd edition, 2015.

[2] H. A Salas. An elementary solution to a duffing equation. *Hindawi*, 2022, May 2022.



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