Nonlinear wave dispersion and analysis of the cubic Boussinesq equation

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1 Introduction

In applied mathematics, there exist several kinds of partial differential equations that govern physical behavior. One of these systems are known as wave equations. Most commonly, a wave equation is expressed as a closed form equation $u(x,t)$, where x is the spatial variable, t the time variable. Although they exist in more than one dimension, it is typically applied to a just the real number line. The most well known of these is the **one dimensional wave equation**. Defined for $t \geq 0$ and $x \in [0, L]$, the PDE is:

$$u_{tt} = c^2 u_{xx} \tag{1}$$

With boundary and initial conditions

$$\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \\ u_(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}$$

The partial differential equation defined above is second order in space and time, as well as linear in all components. The solution can be found both analytically and numerically. While the solution comes naturally, studying **nonlinear** wave equations can become more of a challenge. There are additional terms, as well as nonlinearity that is introduced, complicating the overall problem. In this paper, we introduce a certain nonlinear wave equation. We will derive it as well as nondimentionalize it. Following, we will provide both an analytical and numerical method - something that is often rare with these sorts of equations.

2 Cubic Boussinesq equation

2.1 Derivation

To define the equation that will be solved, we introduce the physical phenomena of water waves on an **incompressible** and **irrotational** flow in the (x, z) plane, where x is horizontal position, and z is vertical displacement. The resulting boundary conditions for $z = \eta(x,t)$ are:

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} - w = 0 \tag{2a}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + w^2) + g\eta \tag{2b}$$

Where u is the horizontal flow velocity, or $\frac{\partial \phi}{\partial x}$ and w is the vertical flow velocity, or $\frac{\partial \phi}{\partial z}$. We now introduce the parameter $-h$, or the mean water depth.

We denote the horizontal velocity potential $\frac{\partial \phi_b}{\partial x}$ at $-h$ as u_b. We can now write a system of PDE's given the boundary conditions as:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[(h+\eta)u_b = \frac{1}{6}h^3\frac{\partial^3 u_b}{\partial x^3} \tag{3a}$$

$$\frac{\partial u_b}{\partial t} + u_b\frac{\partial u_b}{\partial x} + g\frac{\partial \eta}{\partial x} = \frac{1}{2}h^2\frac{\partial^3 u_b}{\partial t \partial x^2} \tag{3b}$$

This set of equations, when the right hand side is set to zero, can be reduced to a single partial differential equation:

$$\frac{\partial^2 \eta}{\partial t^2} - gh\frac{\partial^2 \eta}{\partial x^2} - gh\frac{\partial^2}{\partial x^2}\left(\frac{3}{2}\frac{\eta^2}{h} + \frac{1}{3}h^2\frac{\partial^2 \eta}{\partial x^2}\right) = 0 \tag{4}$$

We are now equipped to nondimentionalize this equation in a form that will be further analyzed.

2.2 Nondimentionalization

Consider the dimensions of the following parameters:

$$[h] = length \tag{5}$$

$$[\eta] = length \tag{6}$$

$$[g] = length/time^2 \tag{7}$$

$$[x] = length \tag{8}$$

$$\tag{9}$$

We introduce the following *dimensionless* quantities such that:

$$u = \frac{1}{2}\frac{\eta}{h} \tag{10a}$$

$$\tau = \sqrt{3}t\sqrt{\frac{g}{h}} \tag{10b}$$

$$\xi = \sqrt{3}\frac{x}{h} \tag{10c}$$

Upon introducing these nondimentional quantities and substituting them into (4), we arrive at the **quadratic** Boussinesq equation:

$$\frac{\partial^2 u}{\partial \tau^2} - \frac{\partial^2 u}{\partial d\xi^2} - \frac{\partial^2}{\partial \xi^2}\left(3u^2 + \frac{\partial^2 u}{\partial \xi^2}\right) = 0 \tag{11}$$

Where $u(x,t)$ represents vertical displacement, τ is our now dimensionless time, and ξ is dimensionless horizontal displacement. For the sake of our analysis, we will substitute these parameters for the typical (t,x) and consider them dimensionless from this point. Additionally, we will analyze a u^3 term, making the equation nonlinear to 3rd degree. Now that the equation has been defined and nondimensionalized, we will apply an analytical method to find a closed form solution.

3 Analytical methods

As mentioned prior, searching for analytical methods that solve nonlinear dispersive wave equations can be rather challenging. In *The tanh method: a tool for solving certain classes of nonlinear evolution and wave equations*, the authors outline an analytical method known as the **tanh method**. It allows a closed form solution to the **cubic Boussinesq equation**. We will outline the general steps of the method, then proceed to apply it to the cubic BE.

3.1 Tanh method

Any general nth order PDE can be thought of as a polynomial in the partial derivatives (Here, P is some polynomial operator):

$$P(u, u_t, u_x, u_{tt}, u_{xx}, ...) = 0 \tag{12}$$

P subsequently acts on the function and its derivatives. The general strategy here is to reduce our PDE into an ODE and solve it. Due the physical nature of traveling waves, we introduce a transformational variable such that $\xi = kx - \lambda t$. All traveling wave equations contain such a relation. Now let $u(x,t) = U(\xi)$, which will reduce it to an ordinary differential equation. As we change variables (considering the transformation variable $\xi = kx - \lambda t$), we must compute the following subsequent differential operators:

$$\frac{\partial}{\partial t} = -\lambda \frac{d}{d\xi} \qquad \frac{\partial}{\partial x} = k\frac{d}{d\xi} \qquad \frac{\partial^2}{\partial x^2} = k^2 \frac{d}{dx^2} \tag{13}$$

And so on. This reduces our PDE into an ODE (Q is again, some polynomial operator), such that:

$$Q(U, U', U'', ...) = 0 \tag{14}$$

We now introduce the namesake ***tanh*** function. Let:

$$Y(x,t) = tanh(\xi) \tag{15}$$

From here one can see that tanh will operate on the traveling wave parameter $\xi = kx - \lambda t$. Due to the change of variables, we must recalculate the following differential operators such that:

$$Y = tanh(\xi) \implies \xi = \tanh^{-1}(Y) \implies \frac{d\xi}{dY} = \frac{1}{1-Y^2} \implies \frac{d}{d\xi} = (1-Y^2)\frac{d}{dY} \tag{16}$$

Applying this computation to each order of ξ and some elementary calculus, we get:

$$\frac{d^2}{d\xi^2} = -2Y(1-Y^2)\frac{d}{dY} + (1-Y^2)^2\frac{d^2}{dY^2} \tag{17a}$$

$$\frac{d^3}{d\xi^3} = 2(1-Y^2)(3Y^2-1)\frac{d}{dY} - 6Y(1-Y^2)^2\frac{d^2}{dY^2} + (1-Y^2)^3\frac{d^3}{dY^3} \tag{17b}$$

The next step is to express the solution as a linear combination of coefficients a_i and powers of Y^i such that:

$$u(x,t) = U(\xi) = \sum_{i=1}^{m} a_i Y^i \tag{18}$$

Where each a_i is found by grouping powers Y^i. In essence, this is an analog to assuming a power series solution to an ordinary differential equation then grouping powers to find the coefficients. Now that the method is defined, we can apply it to our specific equation

3.2 Applying to cubic BE

As aforementioned, we will be analytically solving the **cubic Boussinesq equation**. Similar to its quadratic counterpart, it contains a u^3, instead of a u^2 term. The cubic Boussinesq equation is defined as:

$$u_{tt} - u_{xx} - u_{xxxx} + 2(u^3)_{xx} = 0 \tag{19}$$

At first glance, the question as to the absence of initial data arises. This is quelled later, when the subsequent constants that are generated are set to 0, to align with the wave behavior prescribed by the PDE. As mentioned in the prior section, we transform $u(x,t)$ such that:

$$u(x,t) = U(\xi), \quad where \quad \xi = kx - \lambda t \tag{20}$$

Using the differential operators that were previously computed, we can transform the cubic BE to an ODE, such that:

$$u_{tt} - u_{xx} - u_{xxxx} + 2(u^3)_{xx} = 0 \implies (\lambda^2 - k^2)U'' - k^4 U'''' + 6k^2(U^2U')' = 0 \tag{21}$$

Since each term contains a derivative with respect to ξ, we can integrate across the ODE twice, yielding:

$$(\lambda^2 - k^2)U - k^4 U'' + 2k^2 U^3 = c_0 + c_1\xi = 0 \tag{22}$$

By setting the constants equal to zero, we are left with a second order nonlinear homogenous ordinary differential equation. We can now apply our finite sum solution to determine each a_i. Before this, we must determine the extent of 'finite' in the sum. Again, assume that $U(\xi)$ can be written as:

$$U(\xi) = \sum_{i=1}^{m} a_i \gamma^i, \quad \gamma = tanh(\xi) \tag{23}$$

Therefore we can rewrite (23), with the computed differentials with respect to γ as:

$$(\lambda^2 - k^2)U - k^4\left(- 2\gamma(1-\gamma^2)\frac{dU}{d\gamma} + (1-\gamma^2)\frac{d^2U}{d\gamma^2}\right) + 2k^3U^3 = 0 \tag{24}$$

Now, knowing that $U(\gamma) = \sum_{i=1}^{m} a_i\gamma^i, \quad \gamma = tanh(\xi)$, we must be able to determine the magnitude of m. We balance the highest order linear term with the highest order nonlinear term. Note that by this reasoning, it follows that:

$$3m = 4 + m - 2 \implies m = 2 \tag{25}$$

Applying $m = 2$ to the finite sum, the result becomes:

$$\sum_{i=1}^{m} a_i\gamma^i \implies \sum_{i=1}^{2} a_i\gamma^i \implies U(\gamma) = a_0 + a_1\gamma \tag{26}$$

Now that we have computed the finite sum, we can take its respective nth order derivatives such that:

$$\frac{dU}{d\gamma} = a_1 \tag{27a}$$

$$\frac{d^2U}{d\gamma^2} = 0 \tag{27b}$$

Substituting Y and its subsequent derivatives into (24), the result is:

$$(\lambda^2 - k^2)(a_0 + a_1\gamma) - k^4(-2\gamma(1-\gamma^2)(a_1)) + 2k^3(a_0 + a_1\gamma) = 0 \tag{28}$$

Now, as is done with ODEs, we couple powers of γ^n and solve the resulting system of nonlinear equations for a_i, giving:

$$(\lambda^2 - k^2)a_0 + 2k_0^{a3} = 0 \tag{29a}$$

$$(\lambda^2 - k^2)a_1 + 2k^4a_1 + 6k^2a_0^2a_1 = 0 \tag{29b}$$

$$6k^2a_1^2a_0 = 0 \tag{29c}$$

$$-2k^4a_1 + 2o^2a_1^3 = 0 \tag{29d}$$

Solving the subsequent system for powers of a_i, we get:

$$a_0 = 0 \qquad\qquad a_1 = \mp k \qquad\qquad \lambda = \mp k\sqrt{1-2k^2} \tag{30}$$

We will now assemble our solution from the ground up.

Recall now that:

$$\xi = kx - \lambda t \tag{31a}$$

$$Y = \tanh(\xi) = U(Y) = a_0 + a_1 Y \tag{31b}$$

$$U(\xi) = a_0 + a_1 \tanh(kx - \lambda t) \tag{31c}$$

$$u(x,t) = \pm k(\tanh\Big(k(x \pm \sqrt{1-2k^2}t)\Big) \tag{31d}$$

Our resulting analytical solution is (31d), where the given restriction $|k| \leq \frac{1}{\sqrt{2}}$. This implies:

$$u(x,t) = \pm\frac{1}{2}\tanh\left(\frac{1}{2}(x \pm \frac{t}{\sqrt{2}})\right) \tag{32}$$

Note that from here, we can plot this analytical solution to be able to view the plot in three dimensions. Before this, we seek to prove that $u(x,t)$ satisfies the relation prescribed in the partial differential equation. We will take the '+' solution. Note that:

$$u_{tt} = -\frac{1}{8}\operatorname{sech}^2\left(\frac{\frac{t}{\sqrt{2}}+x}{2}\right)\tanh\left(\frac{\frac{t}{\sqrt{2}}+x}{2}\right) \tag{33}$$

$$u_{xx} = -\frac{1}{4}\operatorname{sech}^2\left(\frac{\frac{t}{\sqrt{2}}+x}{2}\right)\tanh\left(\frac{\frac{t}{\sqrt{2}}+x}{2}\right) \tag{34}$$

$$u_{xxxx} = -\frac{1}{2}\operatorname{sech}^4\left(\frac{\frac{t}{\sqrt{2}}+x}{2}\right)\tanh\left(\frac{\frac{t}{\sqrt{2}}+x}{2}\right) - \frac{1}{4}\operatorname{sech}^4\left(\frac{\frac{t}{\sqrt{2}}+x}{2}\right)\tanh^3\left(\frac{\frac{t}{\sqrt{2}}+x}{2}\right) \tag{35}$$

$$2(u^3)_{xx} = u_{tt} = -\frac{3}{16}\operatorname{sech}^2\left(\frac{\frac{t}{\sqrt{2}}+x}{2}\right)\tanh\left(\frac{\frac{t}{\sqrt{2}}+x}{2}\right)\left(\tanh^2\left(\frac{\frac{t}{\sqrt{2}}+x}{2}\right)\operatorname{sech}^2\left(\frac{\frac{t}{\sqrt{2}}+x}{2}\right)\right) \tag{36}$$

From here, we assign these relations to the partial differential equation and take note that indeed:

$$u_{tt} - u_{xx} - u_{xxxx} + 2(u^3)_{xx} = 0 \tag{37}$$

Now that we have proved the validity of the tanh method, we will proceed with a brief discussion of the surface plot and a way to apply a numerical method for solution.

3.3 Surface plot

As is the case with single dimensional PDEs, a discussion of the plot in $\mathbb{R}^3$ is often relevant. We plot the analytical solution to the Boussinesq equation on the interval $x \in [-1, 1]$ and $t \in [0, 4]$. It is evident from the plot that traveling wave behavior is being exhibited. Note from the figure below:

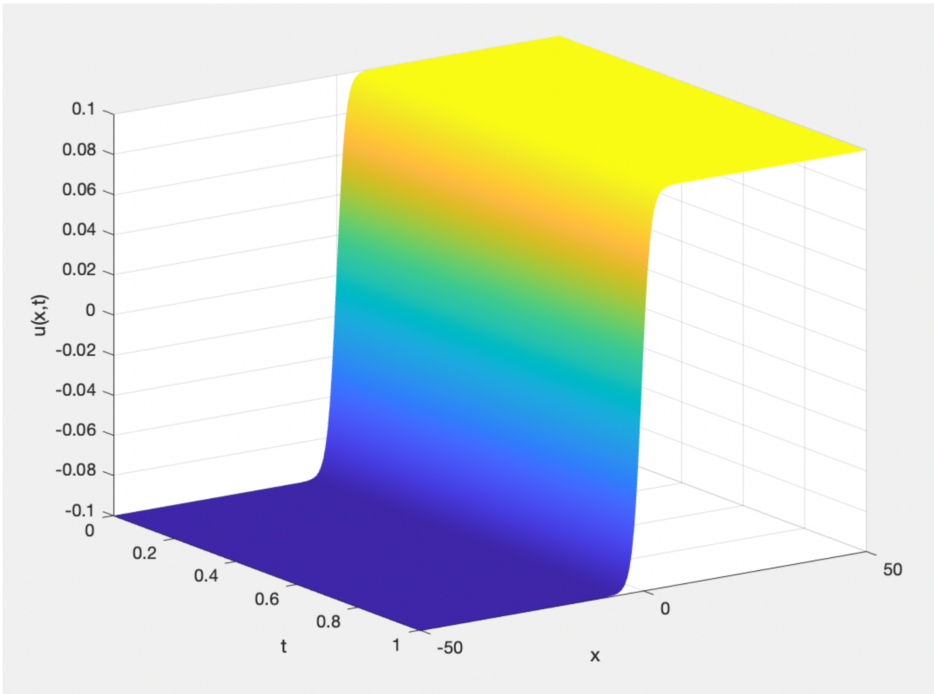

Figure 1: Surface plot for $u(x,t)$, the analytical solution to the Boussinesq equation

Note that the equation attains its maximum on the boundaries, and then diffuses quickly to zero. From here we will offer a brief discussion about a class of numerical method to approximate the solution to the Boussinesq equation.

4 Numerical method

For many linear and nonlinear partial differential equations it is often necessary to employ a numerical method to approximate the particular solution. Many of the PDEs studied do not exhibit analytical solutions so the use of numerical methods is necessary. In the case of the Boussinesq equation, both are applicable. Reconsider the cubic Boussinesq equation such that:

$$u_{tt} - u_{xx} - u_{xxxx} + 2(u^3)_{xx} = 0 \tag{38}$$

In *Traveling Wave Analysis of Partial Differential Equations* by Grifits and Schiesser, they discuss in detail a method for numerically aproximating the solution to the Boussinesq equation. Unfortunately, in an attempt to replicate the code they generalized towards any Boussinesq equation to our equation, we were unable to get the code to work. The method is so challenging due to the 4th ordered spatial term. The core of the method is handling this term by successfully differentiating $\frac{\partial^2 u}{\partial x^2}$ twice. In future discussions we will focus on getting said code to work.

5 Conclusion

Nonlinear wave equations are an intriguing class of partial differential equations. Because of the existence of nonlinear terms it can often be rather challenging to seek an analytical solution. The tanh method is unique in that is one of few analytical methods that provides a closed form solution. While all nonlinear wave equations exhibit traveling wave solutions, the solutions to each specific PDE can vary greatly, adding additional complexity to the study. In future discussions, we will attempt to expand upon the study of numerical methods for Boussinesq equations. The numerical methods here are even more helpful as they can be compared exactly to the analytical solution.

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References

[1] Maarten W Dingemans (2015) *Water Wave Propagation Over Uneven Botttoms*, Self, Advanced Series on Ocean Engineering, Vol 13.

[2] Graham W. Griffiths and William E. Schiesser (2012) *Traveling Wave Analysis of Partial Differential Equations*, Elsevier, 1st ed.

[3] W Malfliet (2004) *The tanh method: a tool for solving certain classes of nonlinear evolution and wave equations*, Journal of Computational and Applied Mathematics, New York, Vol 165.

[4] A.J.M., Jawada, M.D.Petkovic, P.Laketa, and A.Biswas (2018) *Dynamics of shallow water waves with Boussinesq equation*, Scientia Iranica, US, Vol 20.