Analytical Methods for Weakly Nonlinear Oscillators and the Two-Timing Approach

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1.1 Non-damped homogeneous oscillators

Differential equations have infinite use within the scope of applied mathematics. One class of solutions follow equations that exhibit oscillatory behavior. It is first important to start by defining a general closed form solution of a simple periodic equation. Consider the second order homogeneous constant coefficient differential equation:

$$\ddot{x} + \lambda x = 0\,, \lambda > 0 \tag{1}$$

Initial and boundary conditions have been omitted for convenience. It is rather easy from here to see the closed form solution:

$$x(t) = A\cos\sqrt{\lambda}t + B\sin\sqrt{\lambda}t \tag{2}$$

Note the periodic behavior of the general solution, with period $\frac{2\pi}{\sqrt{\lambda}}$. While the analytical solution ends here, we gain a bit of insight by writing the second order equation as a first order system. The dynamical system becomes:

$$\begin{cases} \dot{x_1} = y \\ \dot{x_2} = -\lambda x \end{cases} \tag{3}$$

From solving this autonomous system we wind up with the same result but, by denoting the roots of the characteristic with r, get the equation

$$r = \frac{\sqrt{-4\lambda}}{2} \tag{4}$$

This displays a phase in the x(t), y(t) plane with circles centered around the origin - giving more evidence to the periodic behavior of the general solution of the harmonic oscillator. In the following sections, I will illustrate other examples of harmonic oscillators before attempting to solve those that are weakly non-linear.

1.2 Damped harmonic oscillators

One of the most prevalent physical representations of oscillators equations is the mass spring system. Consider a mass attached to a spring with force constant k, and friction. It is governed by Newton's Second Law:

$$F = ma = m\frac{d^2x}{dt^2} = m\ddot{x} \tag{5}$$

The force contributed by the spring, dictated by Hooke's law, is a proportionality constant multiplied by distance, x:

$$F_{spring} = -kx \tag{6}$$

The force due to friction is a constant b multiplied by the velocity

$$F_{fric} = -b\dot{x} \tag{7}$$

Finally the second order constant coefficient homogeneous differential equation becomes

$$m\ddot{x} + b\dot{x} + kx = 0 \implies \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0 \tag{8}$$

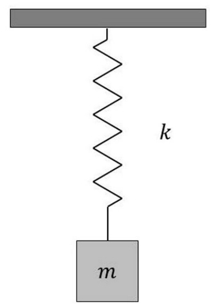

Figure 1: Mass spring system

It is important at this step to take note of the behavior of the damped oscillator. We will denote the roots of the general characteristic equation as λ, such that:

$$\lambda = \frac{\frac{-b}{m} \pm \sqrt{\frac{b^2}{m^2} - 4\frac{k}{m}}}{2} \tag{9}$$

Using some intuition of the motion described of the mass, we will assume that b, m, and $k > 0$. It is clear from here that $4\frac{k}{m} > \frac{b^2}{m^2}$, giving way to a complex eigenvalue. Knowing that $\frac{-b}{m} < 0$, the general closed form solution of the damped harmonic oscillator is:

$$x(t) = e^{\lambda_{re}}(A\cos\lambda_{im} + B\sin\lambda_{im}) \tag{10}$$

We know the real part of the eigenvalue is less than zero, giving evidence to exponential damping, eventually approaching its initial equilibrium. If we think about applying this system in the real world, this makes sense. The mass bobs up and down and gets closer to its initial equilibrium point.

The multiplication of periodic terms gives way to oscillation about the resting position of the mass. Let m = 1, b = 4, x(0) = 1, x(1) = 0. If we vary the spring constant k, with values ranging from 5 to 7, we can take note of the change in behavior. A snapshot of various closed form solutions in the (t,x(t)) plane is displayed below:

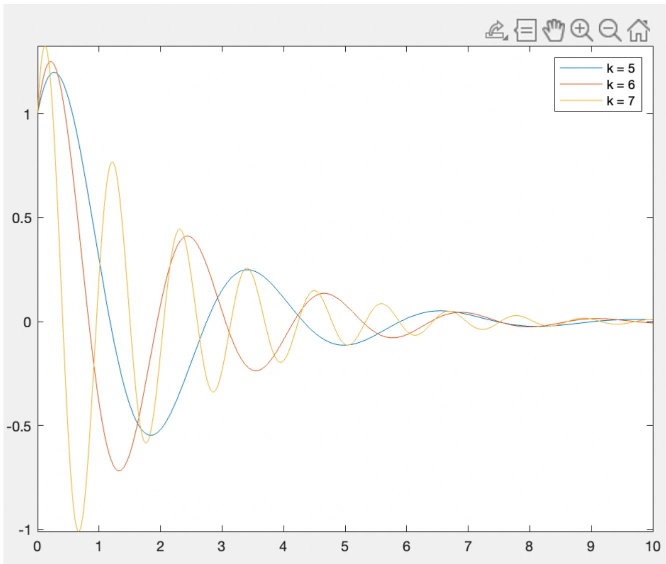

Figure 2: Particular solutions to the damped harmonic oscillator, with varied values of k

The various plots for x(t) confirm the behavior of the physical hypothesis. When k is increased, the period of the oscillating motion increases. To proceed, we will now examine cases in which second order oscillators have non-linear terms.

2 Non-linear oscillators and the Duffing equation

In the previous section, we discussed the notions of linear harmonic oscillators, both undamped and damped cases. Physically, these systems have closed form solutions that closely match the visual behavior of the movement. Both the dynamics and method of solution complicate when nonlinear terms are introduced into the second order equation. Consider the **Duffing equation**, a second order nonlinear homogeneous ordinary differential equation that models the displacement of a mass on a spring with nonlinear potential energy term:

$$\ddot{x} + +\epsilon x^3 \;\; \epsilon > 0 \tag{11}$$

The phase plane for the Duffing equation can provide us with some insight. We will notice that the dynamical system possesses a limit cycle about the origin, a characteristic attribute of nonlinear harmonic oscillators. The phase plane is displayed below:

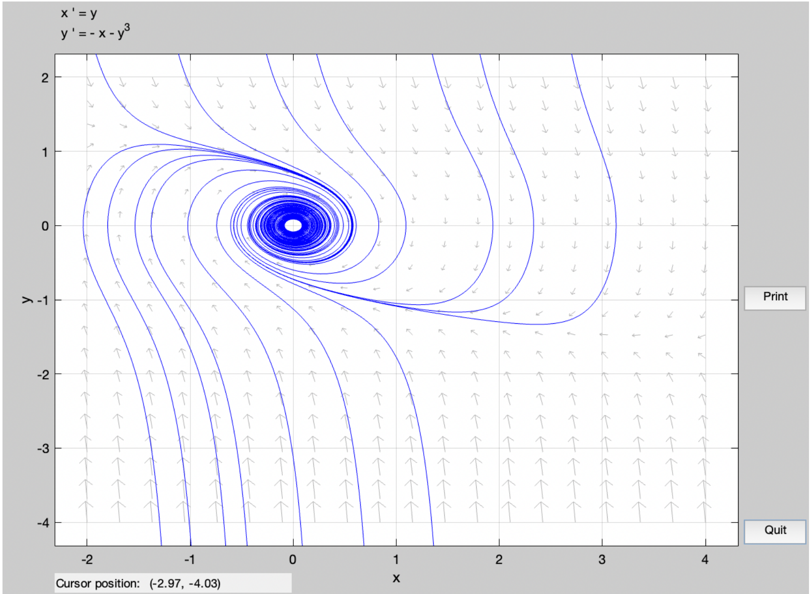

Figure 3: Phase plane for the Duffing equation, $\epsilon = .1$

Unlike simple harmonic oscillators, nonlinear oscillators are often difficult to analytically solve, requiring non-traditional techniques. Despite this, we can implement methods of perturbation theory to approximate the particular solution. In the following sections we will illustrate the peculiar behavior of this and other nonlinear oscillators.

2.1 Physical representation

Reconsider the simple harmonic oscillator:

$$\ddot{x} + 2\beta\dot{x} + \omega^2 x \tag{12}$$

In this oscillator β represents the opposing force due to friction, and ω^2 is the spring constant k. Up until this point, we assumed that the potential energy U(x) could be represented by:

$$U(x) = \frac{1}{2}x^2 \tag{13}$$

While this is a good approximation, it truncates higher order terms, which produces error. Adding in higher order terms in the Taylor series gives:

$$U(x) = \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + O(x^5) \tag{14}$$

If we now substitute the new expression for potential energy into the second order equation it yields:

$$\ddot{x} + 2\beta\dot{x} + \omega^2 x\frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + O(x^5) = 0 \implies \ddot{x} + \omega^2 x + \epsilon x^3 \tag{15}$$

Now that we have derived the Duffing equation, we will attempt to approximate its solution.

2.2 Using the Poincaré-Lindstedt method

We will begin solution by using typical perturbation techniques. Recall that the Duffing equation is:

$$\ddot{x} + x + \epsilon x^3, \;\; \epsilon << 0, x(0) = 1, \dot{x}(0) = 0 \tag{16}$$

The parameter **perturbation** ϵ represents a small deviation from the classical simple harmonic oscillator, which is why we denote the oscillator as **weakly** nonlinear, since the nonlinear term is quite small. Let x have a power series expansion in powers of ϵ and functions of t, such that:

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_1(t) + O(\epsilon^3) \tag{17}$$

Substitution into the Duffing equation up to powers of linear ϵ gives:

$$(\ddot{x}_0(t) + \epsilon \ddot{x}_1(t) + ...) + (x_0(t) + \epsilon x_1(t) + ...) + \epsilon(x_0(t) + \epsilon x_1(t) + ...)^3 = 0 \tag{18}$$

$$\ddot{x}_0 + x_0 = 0 \implies x_0(t) = \cos(t) \tag{19a}$$

$$\epsilon \ddot{x}_1 + \epsilon x_1 + \epsilon x_0^3 = 0 \implies \ddot{x}_1 + x_1 = -\cos^3 t \implies x_1(t) = \tfrac{1}{32}(\cos(3t) - \cos(t)) - \tfrac{3}{8} t \sin(t)$$

$$x_0(t) = \cos(t) \tag{20a}$$

$$x_1(t) = \frac{1}{32}(\cos(3t) - \cos(t)) - \frac{3}{8} t \sin(t) \tag{20b}$$

Therefore, the equation expanded about the first order of ϵ is:

$$x(t) = \cos(t) + \epsilon(x_1(t) = \frac{1}{32}(\cos(3t) - \cos(t)) - \frac{3}{8} t \sin(t)) \tag{21}$$

When plotting this with a small value of epsilon, there is an issue:

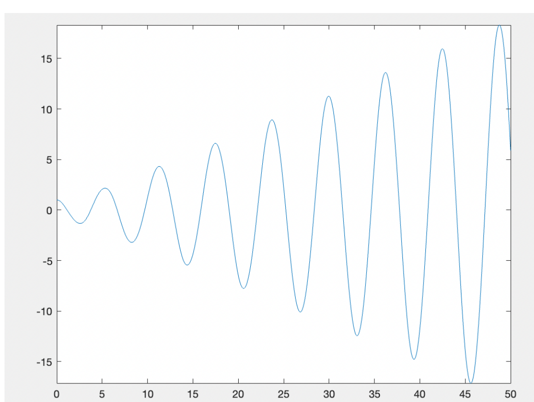

Figure 4: Behavior of the first order perturbation expansion of the Duffing equation, no two-timing

The particular solution possesses a **secular term**, $t\sin(t)$, growing exponentially as time increases. This goes against the physical mechanics perscribed by the harmonic oscillator. We will fix this by implementing a method known as **two-timing**, creating a new time scale by scaling the original time t by a constant α. The efficacy of this method lies in scaling time to be "slow", so that the approximation works to a higher order of accuracy:
Let the new time scale be:

$$\tau = \alpha t \tag{22a}$$

$$\alpha = \alpha_0 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + O(\alpha^3) \tag{22b}$$

We choose $\alpha_0 = 1$, as that is the original angular frequency of the first term of the 'broken' solution. Non-dimentionalizing the original Duffing equation yields:

$$\frac{dx}{d\tau}\frac{d\tau}{dt} = \alpha\frac{dx}{dt} \implies \frac{d}{dt}(\frac{dx}{dt}) = \alpha^2\frac{d^2x}{dt^2} \tag{23}$$

Now the new Duffing equation with time scale τ is:

$$\alpha^2\ddot{x}(\tau) + x(\tau) + x^3(\tau) \tag{24}$$

Using the same techniques to find function of τ up to order ϵ gives:

$$x_0(\tau) = \cos(\tau) \tag{25a}$$

$$x_1(\tau) = \frac{1}{32}(\cos(3\tau) - \cos(\tau)) + (\alpha_1 - \frac{3}{8})\tau\sin(\tau) \tag{25b}$$

The beauty in two-timing is apparent now, as we can choose $\alpha_1 = \frac{3}{8}$, and be rid of the secular term. To finish, t is re substituted into the particular solution. We now have a first order approximation of the solution of the Duffing equation:

$$x(t) \approx \cos((1+\frac{3}{8}\epsilon)t) + \frac{1}{32}\epsilon(\cos((3+\frac{9}{8}\epsilon)t) - \cos((1+\frac{3}{8})t)) + O(\epsilon^2) \tag{26}$$

We can clearly conclude that the oscillator is better approximated for smaller perturbations in the original motion, a key notion in perturbation theory.

2.3 Van der Pol oscillator

The discussion of harmonic oscillators thus far has revolved around two cases - linear, and nonlinear in the oscillators potential energy term. This leads to an thought-provoking question - what about those that possess a non-linear term in its first derivative, such that the damping force is nonlinear? This question is extensively answered with the example of the **Van der Pol oscillator**, governed by the equation:

$$\ddot{x} + \epsilon(1-x^2)\dot{x} + x = 0, x(0) = 1, \dot{x}(0) = 1, 0 < \epsilon << 1 \tag{27}$$

Again in this case we retain the perturbation ϵ, representing a small change in the nonlinear term of the second order differential equation. If ϵ were to equal 0, then the Van der Pol oscillator would be reduced to the classical simple harmonic oscillator:

$$\ddot{x} + x = 0 \implies x(t) = A\cos t + B\sin t \tag{28}$$

Before attempting to use perturbation theory to approximate a closed form solution to this equation, it will be important to perform a phase plane analysis of this oscillator. Rewriting the second order equation as a system of first order equations gives:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + \epsilon(1 - x^2)y \end{cases} \tag{29}$$

The phase plane of this nonlinear system of differential equations is displayed below for $\epsilon = 2$:

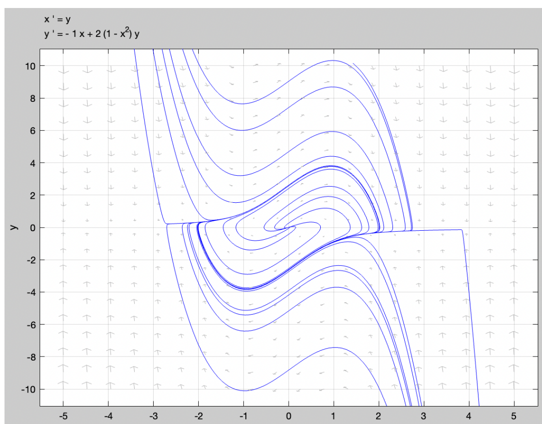

Figure 5: Phase plane for the Van der Pol oscillator, $\epsilon = 2$

The phase plane is rather interesting. For small enough starting conditions, the oscillator gets caught in a limit cycle that revolves around the origin. This behavior of closed orbits is further evidenced when attempting a first order perturbation approximation of the Van der Pol oscillator. Assume that x(t) has its usual expansion in powers of epsilon and that it has initial conditions x(0) = 1 (0) = 1

$$x(t) = x_0(t) + \epsilon x_1(t) + O(\epsilon^2) \tag{30}$$

Substituting (30) into (27) up to powers of linear ϵ gives:

$$\ddot{x}_0 + \epsilon\ddot{x}_1 - \epsilon\dot{x}_0 + \epsilon x_0^2\dot{x}_0 + x_0 + \epsilon x_1 = 0 \tag{31}$$

Collecting powers of epsilon gives:

$$\ddot{x}_0 + x_0 = 0 \tag{32a}$$

$$\ddot{x}_1 + \dot{x}_0 + x_0^2\dot{x}_0 + x_1 = 0 \tag{32b}$$

It is rather clear that the solution to x_0, with $x(0) = x_0(0)$ and $\dot{x}(0) = \dot{x}_0(0)$, is:

$$x_0(t) = \cos t + \sin t \tag{33}$$

With this in mind, we can assemble the ODE for x_1, such that:

$$\ddot{x}_1 - (\cos t - \sin t) + (\cos t + \sin t)^2((\cos t - \sin t)) + x_1 = 0 \tag{34}$$

$$\ddot{x}_1 + x_1 = (\cos t - \sin t) - (\cos t + \sin t)^2((\cos t - \sin t)) \tag{35}$$

The equation above is second order and non-homogeneous - and rather difficult to analytically solve. In Nasser M. Abbasi's 2020 paper *Solving the Van Der Pol nonlinear differential equation using first order approximation perturbation method*, he analytically solves it by implementing the method of undetermined coefficients. I concede all credit to Nasser for the following solution of x_1, with the only substitution being the initial conditions I prescribed at the beginning of the problem:

$$x_1(t) = \frac{-1}{8}\cos t - \frac{3}{4}\sin t + \frac{3}{8}\cos 3t + \frac{1}{4}\sin 3t \tag{36}$$

Which gives way to the first order approximation of the Van der Pol oscillator as:

$$x(t) \approx \cos t + \sin t + \epsilon(\frac{-1}{8}\cos t - \frac{3}{4}\sin t + \frac{3}{8}\cos 3t + \frac{1}{4}\sin 3t) \tag{37}$$

Note that there is no secular term here, removing the need for a two-time approach. Plotting x(t) for various values of ϵ, we can see how the approximation begins to break down for values too high, again showing that perturbation theory is far more effective for smaller values of :

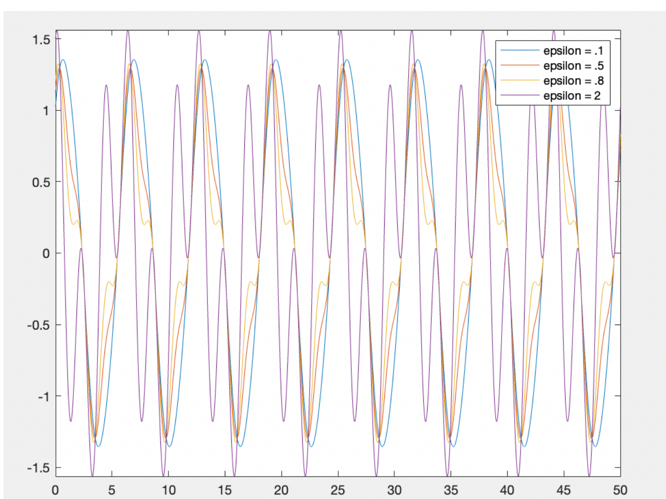

Figure 6: Particular solutions for the first order approximation of the Van der Pol oscillator, with varied values of ϵ

3 Limitations

The limitations of perturbation theory are clear from these approximations, for which there are several. We only implemented a first order perturbation expansion up to a linear power of ϵ. A more accurate solution could be derived by finding more terms in the power series expansion of x(t), but it would become increasingly difficult to solve the resulting ordinary differential euqations analytically. Additionally, it is important to remember that these are **weakly** nonlinear oscillators - we are to assume that ϵ is relatively small, representing a small perturbation to the classical simple harmonic oscillator. For large values of ϵ, or greater changes to the simple equation will result in worse approximations. One way to sidestep this would be to implement a two-timing approach in which α is taken up to order $O(\alpha^2)$. The resulting takeaway is that for higher order expansions about ϵ, there is higher order in accuracy of solution compared to the true solution.

4 Acknowledgements

I would like to thank Dr. Fauci for all of her help throughout the semester, and in assisting my journey through applied mathematics.

References

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