Algebraic Geometry I

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Contents

About These Notes

1 Sheaves

2 Schemes

Definition 2.1. A *ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a space and $\mathcal{O}_X \in \mathrm{Sh}_{\mathrm{CRing}}(X)$ a sheaf of rings on X. A *morphism of ringed spaces* $(f, f^\#): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ where $f: X \to Y$ is a continuous function and $f^\#: \mathcal{O}_Y \to f_* \mathcal{O}_X$ a map of sheaves of rings.

Remark 2.2. Given morphisms of ringed space $(f, f^\#): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(g, g^\#): (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$, their composite is the morphism $(g \circ f, g^\# \circ f^\#): (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$ where $g^\# \circ f^\#: \mathcal{O}_Z \to (g \circ f)_* \mathcal{O}_X$ is given by

$$\mathcal{O}_Z \xrightarrow{g^\#} g_* \mathcal{O}_Y \xrightarrow{g_*(f^\#)} g_*(f_* \mathcal{O}_X) = (g \circ f)_* \mathcal{O}_X$$

using functoriality of pushforwards with respect to morphisms of sheaves.

Note that an isomorphism of ringed spaces is a map $(f, f^\#): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces such that f is a homeomorphism and $f^\#$ an isomorphism of sheaves.

In many cases (though not always) $f^\#$ will be naturally "induced" by f.

Example 2.3.

1. If X is a space and $\mathcal{O}_X = \mathcal{C}_X$ its sheaf of continuous functions, then $(X, \mathcal{O}_X)$ is a ringed space. Given a continuous map $f: X \to Y$, we obtain

a morphism $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ by

$$f^\sharp|_U \colon \mathcal{O}_Y(U) \to (f_*\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U))$$
$$(\phi\colon U \to \mathbb{R}) \mapsto (f^{-1}(U) \xrightarrow{f} U \xrightarrow{\phi} \mathbb{R})$$

for all $U \subseteq X$ open.

2. If X is a smooth manifold and $\mathcal{O}_X = C_X^\infty$ its sheaf of smooth functions, then $(X, \mathcal{O}_X)$ is a ringed space. Given a smooth map $f\colon X \to Y$, we define a map of sheaves $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ by composition with f as above and therefore obtain a morphism $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces.

3. If X is a complex manifold and $\mathcal{O}_X$ its sheaf of holomorphic functions, then $(X, \mathcal{O}_X)$ is a ringed space and any holomorphic map $f\colon X \to Y$ induces a map of ringed spaces as above.

4. Let k be an algebraically closed field. A subset $X \subseteq k^n$ is an *affine algebraic set* if $X = V(\mathfrak{a}) = \{(t_1, \dots, t_n) \in k^n \mid f(t_1, \dots, t_n)\} = 0$ for all $f \in \mathfrak{a}\}$ where $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$ is an ideal. The set X then becomes a space by equipping it with the subspace topology of the Zariski topology on $k^n \cong \operatorname{MaxSpec} k[x_1, \dots, x_n] \subset \operatorname{Spec} k[x_1, \dots, x_n]$.

   We call a function $h\colon U \to k$ defined on an open subset $U \subseteq X$ *regular* if for each $x \in U$ there exists an open neighborhood $V_x \subseteq U$ of x and polynomials $g_1, g_2 \in k[x_1, \dots, x_n]$ such that for all $y \in V_x$, we can express h as $h(y) = g_1(y)/g_2(y)$ (in particular g_2 does not vanish on V_x).

   We then obtain a ringed space $(X, \mathcal{O}_X)$ by letting $\mathcal{O}_X$ be the *sheaf of regular functions* on X, i.e.

$$\mathcal{O}_X(U) := \{h\colon U \to k \mid h \text{ regular}\}$$

   together with the obvious restriction maps. We call this ringed space the *ringed space associated with the affine algebraic set X* .

Note that in examples 2 and 3, we cannot expect a general continuous map to induce a morphism of ringed spaces in the same way, since composing a smooth/holomorphic map with a continuous function may not yield a smooth/holomorphic map again, respectively.

Remark 2.4. A regular function $h\colon U \to k$ is continuous with respect to the Zariski topologies on its domain and codomain; this follows from the fact that polynomials are continuous.

We should thus ask whether any continuous map $f\colon X \to Y$ between affine algebraic sets induces a $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ via composition as in example 1. The answer is no in general, but if it does, we call it a *regular function*.

Example 2.5. Consider the ringed spaces $(\mathbb{R}^n, C_{\mathbb{R}^n})$ and $(\mathbb{R}^n, C^\infty_{\mathbb{R}^n})$ and define a morphism $(f, f^\sharp)\colon (\mathbb{R}^n, C_{\mathbb{R}^n}) \to (\mathbb{R}^n, C_{\mathbb{R}^n})$ by $f = \mathrm{id}_{\mathbb{R}^n}$ and taking $f^\sharp\colon C^\infty_{\mathbb{R}^n} \to (\mathrm{id}_{\mathbb{R}^n})_* C_{\mathbb{R}^n} = C_{\mathbb{R}^n}$ to be the inclusion. Note in particular that f is a homeomorphism but $(f, f^\sharp)$ is not an isomorphism.

Similarly, we obtain a map $(\mathbb{C}^n, \mathcal{O}^{\mathrm{hol}}_{\mathbb{C}^n}) \to (\mathbb{C}^n_{\mathrm{Zar}}, \mathcal{O}^{\mathrm{reg}}_{\mathbb{C}^n})$ from the ringed space of homolomorphic functions on $\mathbb{C}^n$ to the ringed space of regular functions on $\mathbb{C}^n$ equipped with the Zariski topology.

Definition 2.6. A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that the stalks $\mathcal{O}_{X,x}$ are local rings for all $x \in X$.

Example 2.7. Let $(X, \mathcal{O}_X)$ be as in example 1 above. Then $(X, \mathcal{O}_X)$ is a locally ringed space. To see this, note that the stalk of $\mathcal{O}_X$ at any point $x \in X$ is given by

$$\mathcal{O}_{X,x} = \{(h\colon U \to \mathbb{R}) \mid x \in U \subseteq X \text{ open}, h \in \mathcal{O}_X(U)\}/\sim$$

where $(h\colon U \to \mathbb{R}) \sim (h'\colon V \to \mathbb{R})$ if $h|_W = h'|_W$ for some open $x \in W \subseteq U \cap V$. Let $\mathfrak{m}_x := \{[h\colon U \to \mathbb{R}] \in \mathcal{O}_{X,x} \mid h(x) = 0\}$ be the set of germs vanishing at x. Obviously $\mathfrak{m}_x$ is a proper ideal, and it is in fact the unique maximal ideal of $\mathcal{O}_{X,x}$: To see this, it suffices to show that every element $g \in \mathcal{O}_{X,x} \setminus \mathfrak{m}_x$ is invertible. But a continuous function that does not vanish at x does not vanish on a full neighborhood of x and is therefore invertible on such a neighborhood.

Analogous reasoning shows that the ringed spaces from examples 2 through 4 above are also locally ringed.

Definition 2.8. A *morphism of locally ringed spaces* is a morphism of ringed spaces $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ if the induced map on stalks $f^\sharp_x\colon \mathcal{O}_{Y,f(x)} \to (f_*\mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$ is a *morphism of local rings*, i.e. $(f^\sharp_x)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$.

Composition of morphisms of locally ringed spaces is given by composition of morphisms of ringed spaces.

Remark 2.9. Note that being a morphism of local rings is a condition over being a morphism of rings which are local. If $\phi\colon A \to B$ is a ring map where A and B are local, then $\phi^{-1}(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$ will always hold, but the reverse inclusion might not: Take for example $A = \mathbb{Z}_{(p)}$ and $B = Q(A) = \mathbb{Q}$ together with the canonical map.

Remark 2.10. If $\phi\colon A \to B$ is a ring homomorphism and $\mathfrak{q} \subset B$ a prime ideal, then $\mathfrak{p} := \phi^{-1}(\mathfrak{q})$ is a prime ideal of A. Moreover, ϕ induces a ring homomorphism $A_\mathfrak{p} \to B_\mathfrak{q}$.

Example 2.11. Let A be a ring and $\mathcal{O}_{\operatorname{Spec} A}$ its structure sheaf. Then the pair $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is a ringed space, and in fact a locally ringed space: We have shown that $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} \cong A_\mathfrak{p}$. By the previous remark, any ring homomorphism $\phi\colon A \to B$ then induces a morphism of locally ringed spaces $(f, f^\sharp)\colon (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \to (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$.

Definition 2.12. An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A.

Example 2.13. The following are important examples of affine schemes:

1. $(\operatorname{Spec} \mathbb{Z}, \mathcal{O}_{\operatorname{Spec} \mathbb{Z}})$. For $D(a)$ a basic open set, we have $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}(D(a)) \cong \mathbb{Z}_a$.
2. $(\operatorname{Spec} k, \mathcal{O}_{\operatorname{Spec} k})$ for k a field. In this case $\operatorname{Spec} k$ consists of a single point and $\mathcal{O}_{\operatorname{Spec} k}(\operatorname{Spec}(k)) = k$.
3. $\mathbb{A}^n_A := (\operatorname{Spec} A[x_1, \ldots, x_n], \mathcal{O}_{\operatorname{Spec} A[x_1, \ldots, x_n]})$ for A any ring.
4. $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for A a discrete valuation ring. In this case $\operatorname{Spec} A = \{(0), \mathfrak{m}\}$ where $\mathfrak{m}$ is the unique maximal ideal with the open sets being the empty set, $\operatorname{Spec} A$ itself, and $\{(0)\}$. We then have $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A$ and $\mathcal{O}_{\operatorname{Spec} A}(\{(0)\}) = \mathcal{O}_{\operatorname{Spec} A, (0)} = Q(A)$.
5. $(\operatorname{Spec} k[x]/(x^2), \mathcal{O}_{\operatorname{Spec} k[x]/(x^2)})$ where k is a field ($k[x]/(x^2)$ is known as the *ring of dual numbers* over k). In this case $\operatorname{Spec} k[x]/(x^2)$ again consists

of a single point, namely (x).

Note that $(\operatorname{Spec} k, \mathcal{O}_{\operatorname{Spec} k})$ and $(\operatorname{Spec} k[x]/(x^2), \mathcal{O}_{\operatorname{Spec} k[x]/(x^2)})$ consist both of one point, yet are different: $\mathcal{O}_{\operatorname{Spec} k}(\operatorname{Spec} k) = k$ while $\mathcal{O}_{\operatorname{Spec} k[x]/(x^2)}(\operatorname{Spec} k[x]/(x^2)) = k[x]/(x^2)$.

Example 2.14. Consider the locally ringed spaces $(\mathbb{C}^n, \mathcal{O}^{\mathrm{hol}}_{\mathbb{C}^n})$ and $(\mathbb{A}^n_{\mathbb{C}}, \mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}})$ where $\mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}}$ is the structure sheaf. We define a map $(f, f^\sharp)\colon (\mathbb{C}^n, \mathcal{O}^{\mathrm{hol}}_{\mathbb{C}^n}) \to (\mathbb{A}^n_{\mathbb{C}}, \mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}})$ as follows: f is the map

$$\begin{aligned} f\colon \mathbb{C}^n \cong \operatorname{MaxSpec}(\mathbb{C}[x_1, \dots, x_n]) &\hookrightarrow \mathbb{A}^n_{\mathbb{C}} \\ (t_1, \dots, t_n) &\mapsto (x_1 - t_1, \dots, x_n - t_n) \end{aligned}$$

which is continuous because polynomials are continuous in the standard topology on $\mathbb{C}^n$. Letting $A := \mathbb{C}[x_1, \dots, x_n]$, we define $f^\sharp\colon \mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}} \to f_* \mathcal{O}^{\mathrm{hol}}_{\mathbb{C}^n}$ as

$$f^\sharp|_U(s) = \left(U \cap \mathbb{C}^n \xrightarrow{s} \coprod_{\mathfrak{m} \in U \cap \mathbb{C}^n} A_{\mathfrak{m}} \to \mathbb{C} \right)$$

for all $(s\colon U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}) \in \mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}}(U)$ sections over the open set $U \subseteq \mathbb{A}^n_{\mathbb{C}}$ where the map $\coprod_{\mathfrak{m} \in U \cap \mathbb{C}^n} A_{\mathfrak{m}} \to \mathbb{C}$ is given component-wise by the maps $A_{\mathfrak{m}} \twoheadrightarrow A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}} \cong \mathbb{C}$. Holomorphicity of $f^\sharp|_U(s)$ comes down to the fact that s is locally representable as a quotient of polynomials which are of course holomorphic.

Proposition 2.15. *Let A, B be two rings. Then there exists a bijection*

$$\left\{ \begin{matrix} \text{ring homomorphisms} \\ A \to B \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{morphisms of locally ringed spaces} \\ (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \end{matrix} \right\}$$

Proof. TODO. ■

Definition 2.16. A *scheme* is a ringed space $(X, \mathcal{O}_X)$ that is locally isomorphic to an affine scheme, i.e. for all points $x \in X$ there exists an open neighborhood $U \ni x$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A.

Definition 2.17. We define AffSch, Sch, and LocRingSpc to be the categories with objects the affine schemes, schemes, and locally ringed spaces, respectively, and morphisms all morphisms of locally ringed spaces.

We also define a category RingSpc which has as objects all ringed spaces and as morphisms all morphisms of ringed spaces.

We thus have a chain of subcategory inclusions

$$\mathrm{AffSch} \hookrightarrow \mathrm{Sch} \hookrightarrow \mathrm{LocRingSpc} \hookrightarrow \mathrm{RingSpc}$$

of which the first two are full.

Remark 2.18. Proposition 2.15 implies that we have an equivalence of categories

$$\begin{aligned} \mathrm{CRing}^{\mathrm{op}} &\xrightarrow{\simeq} \mathrm{AffSch} \\ A &\mapsto (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \end{aligned}$$

Remark 2.19. Recall the example from Remark 2.9 and note that the induced map $(\operatorname{Spec} Q(A), \mathcal{O}_{\operatorname{Spec} Q(A)}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ *is* a morphism of local rings.

Index