Algebraic Geometry I

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Compiled on November 13, 2024

Work in progress! Unfinished document!

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About These Notes

1 Sheaves

Sheaf theory is supposed to keep track of local vs. global information on topological spaces.

Definition 1.1. Let X be a topological space. Define a poset Ouv_X with objects the open sets of X ordered by inclusion.

Let $\mathcal{C}$ be a category. A $\mathcal{C}$-valued *presheaf* on X is a functor $\mathcal{F}\colon \mathrm{Ouv}_X^{\mathrm{op}} \to \mathcal{C}$.

We will mostly be interested in presheaves of abelian groups, rings, or other algebraic structures. Sometimes one requires that $\mathcal{F}(\emptyset)$ is a terminal object of $\mathcal{C}$, but we generally will not assume this.

Given such a presheaf $\mathcal{F}$ and some open set $U \subseteq X$, we will call the elements of $\mathcal{F}(U)$ *local sections of* $\mathcal{F}$ over U. We write $\Gamma(U, \mathcal{F}) := \mathcal{F}(U)$ for the *space of sections* over U. If $U = X$, then an element of $\mathcal{F}(X)$ will be known as a *global section* of $\mathcal{F}$ and $\Gamma(X, \mathcal{F})$ as the *space of global sections*.

Given open sets $V \subseteq U \subseteq X$, we will write the induced map of the inclusion as $\rho_{UV}\colon \mathcal{F}(U) \to \mathcal{F}(V)$ and call it a *restriction map*. If $s \in \mathcal{F}(U)$ is a section, then we will often denote its *restriction* $\rho_{UV}(s) \in \mathcal{F}(V)$ to V by $s|_V$.

We will take the term "presheaf on X" sans further qualifiers to mean Ab-valued presheaf on X.

Example 1.2. Let X and Y be spaces.

1. Define a presheaf $\mathcal{F}$ of sets on X by putting

$$\mathcal{F}(U) := \{f\colon X \to Y \mid f \text{ continuous}\}$$

for any open $U \subseteq X$ with restriction maps given by restriction of domain.

2. Letting $Y = \mathbb{R}$ in the last definition, we obtain the *presheaf C_X of continuous functions* on X. Note that in this case pointwise addition and multiplication make C_X into a presheaf of rings on X, although we will often consider it as simply as a presheaf of abelian groups.

3. Let G be an abelian group. Define the *constant presheaf* $\mathbb{G}$ with values in G as $\mathbb{G}(U) := G$ for all $U \subseteq X$ open with all restriction maps the identity of G.

2 Schemes

Definition 2.1. A *ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a space and $\mathcal{O}_X \in \mathrm{Sh}_{\mathrm{CRing}}(X)$ a sheaf of rings on X. A *morphism of ringed spaces* $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ where $f\colon X \to Y$ is a continuous function and $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ a map of sheaves of rings.

Remark 2.2. Given morphisms of ringed space $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(g, g^\sharp)\colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$, their composite is the morphism $(g \circ f, g^\sharp \circ f^\sharp)\colon (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$ where $g^\sharp \circ f^\sharp\colon \mathcal{O}_Z \to (g \circ f)_*\mathcal{O}_X$ is given by

$$\mathcal{O}_Z \xrightarrow{g^\sharp} g_*\mathcal{O}_Y \xrightarrow{g_*(f^\sharp)} g_*(f_*\mathcal{O}_X) = (g \circ f)_*\mathcal{O}_X$$

using functoriality of pushforwards with respect to morphisms of sheaves.

Note that an isomorphism of ringed spaces is a map $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces such that f is a homeomorphism and $f^\sharp$ an isomorphism of sheaves.

In many cases (though not always) $f^\sharp$ will be naturally "induced" by f.

Example 2.3.

1. If X is a space and $\mathcal{O}_X = C_X$ its sheaf of continuous functions, then $(X, \mathcal{O}_X)$ is a ringed space. Given a continuous map $f\colon X \to Y$, we obtain a morphism $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ by

$$\begin{aligned} f^\sharp|_U\colon \mathcal{O}_Y(U) &\to (f_*\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U)) \\ (\phi\colon U \to \mathbb{R}) &\mapsto (f^{-1}(U) \xrightarrow{f} U \xrightarrow{\phi} \mathbb{R}) \end{aligned}$$

for all $U \subseteq X$ open.

2. If X is a smooth manifold and $\mathcal{O}_X = C^\infty_X$ its sheaf of smooth functions, then $(X, \mathcal{O}_X)$ is a ringed space. Given a smooth map $f\colon X \to Y$, we define a map of sheaves $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ by composition with f as above and therefore obtain a morphism $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces.

3. If X is a complex manifold and $\mathcal{O}_X$ its sheaf of holomorphic functions, then $(X, \mathcal{O}_X)$ is a ringed space and any holomorphic map $f\colon X \to Y$ induces a map of ringed spaces as above.

4. Let k be an algebraically closed field. A subset $X \subseteq k^n$ is an *affine algebraic set* if $X = V(\mathfrak{a}) = \{(t_1, \dots, t_n) \in k^n \mid f(t_1, \dots, t_n)\} = 0 \text{ for all } f \in \mathfrak{a}\}$ where $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$ is an ideal. The set X then becomes a space by equipping it with the subspace topology of the Zariski topology on $k^n \cong \operatorname{MaxSpec} k[x_1, \dots, x_n] \subset \operatorname{Spec} k[x_1, \dots, x_n]$.

   We call a function $h\colon U \to k$ defined on an open subset $U \subseteq X$ *regular* if for each $x \in U$ there exists an open neighborhood $V_x \subseteq U$ of x and polynomials $g_1, g_2 \in k[x_1, \dots, x_n]$ such that for all $y \in V_x$, we can express h as $h(y) = g_1(y)/g_2(y)$ (in particular g_2 does not vanish on V_x).

   We then obtain a ringed space $(X, \mathcal{O}_X)$ by letting $\mathcal{O}_X$ be the *sheaf of regular functions* on X, i.e.

   $$\mathcal{O}_X(U) := \{h\colon U \to k \mid h \text{ regular}\}$$

   together with the obvious restriction maps. We call this ringed space the *ringed space associated with the affine algebraic set X*.

Note that in examples 2 and 3, we cannot expect a general continuous map to induce a morphism of ringed spaces in the same way, since composing a smooth (respectively, holomorphic) map with a continuous function may not yield a smooth (respectively, holomorphic) map again.

Remark 2.4. A regular function $h\colon U \to k$ is continuous with respect to the Zariski topologies on its domain and codomain; this follows from the fact that polynomials are continuous.

We should thus ask whether any continuous map $f\colon X \to Y$ between affine algebraic sets induces a $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ via composition as in example 1. The answer is no in general, but if it does, we call it a *regular function*.

Example 2.5. Consider the ringed spaces $(\mathbb{R}^n, \mathcal{C}_{\mathbb{R}^n})$ and $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$ and define a morphism $(f, f^\sharp)\colon (\mathbb{R}^n, \mathcal{C}_{\mathbb{R}^n}) \to (\mathbb{R}^n, \mathcal{C}_{\mathbb{R}^n})$ by $f = \mathrm{id}_{\mathbb{R}^n}$ and taking $f^\sharp\colon \mathcal{C}^\infty_{\mathbb{R}^n} \to (\mathrm{id}_{\mathbb{R}^n})_* \mathcal{C}_{\mathbb{R}^n} = \mathcal{C}_{\mathbb{R}^n}$ to be the inclusion. Note in particular that f is a homeomorphism but $(f, f^\sharp)$ is not an isomorphism.

Similarly, we obtain a map $(\mathbb{C}^n, \mathcal{O}^{\mathrm{hol}}_{\mathbb{C}^n}) \to (\mathbb{C}^n_{\mathrm{Zar}}, \mathcal{O}^{\mathrm{reg}}_{\mathbb{C}^n})$ from the ringed space of homolomorphic functions on $\mathbb{C}^n$ to the ringed space of regular functions on $\mathbb{C}^n$ equipped with the Zariski topology.

Definition 2.6. A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that the stalks $\mathcal{O}_{X,x}$ are local rings for all $x \in X$.

Example 2.7. Let $(X, \mathcal{O}_X)$ be as in example 1 above. Then $(X, \mathcal{O}_X)$ is a locally ringed space. To see this, note that the stalk of $\mathcal{O}_X$ at any point $x \in X$ is given by

$$\mathcal{O}_{X,x} = \{(h\colon U \to \mathbb{R}) \mid x \in U \subseteq X \text{ open}, h \in \mathcal{O}_X(U)\}/\sim$$

where $(h\colon U \to \mathbb{R}) \sim (h'\colon V \to \mathbb{R})$ if $h|_W = h'|_W$ for some open $x \in W \subseteq U \cap V$. Let $\mathfrak{m}_x := \{[h\colon U \to \mathbb{R}] \in \mathcal{O}_{X,x} \mid h(x) = 0\}$ be the set of germs vanishing at x. Obviously $\mathfrak{m}_x$ is a proper ideal, and it is in fact the unique maximal ideal of $\mathcal{O}_{X,x}$: To see this, it suffices to show that every element $g \in \mathcal{O}_{X,x} \setminus \mathfrak{m}_x$ is invertible. But a continuous function that does not vanish at x does not vanish on a full neighborhood of x and is therefore invertible on such a neighborhood.

Analogous reasoning shows that the ringed spaces from examples 2 through 4 above are also locally ringed.

Definition 2.8. A *morphism of locally ringed spaces* is a morphism of ringed spaces $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ if the induced map on stalks $f^\sharp_x\colon \mathcal{O}_{Y,f(x)} \to (f_*\mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$ is a *morphism of local rings*, i.e. $(f^\sharp_x)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$.

Composition of morphisms of locally ringed spaces is given by composition of morphisms of ringed spaces.

Remark 2.9. Note that being a morphism of local rings is a condition over being a morphism of rings which are local. If $\phi\colon A \to B$ is a ring map where A and B are local, then $\phi^{-1}(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$ will always hold, but the reverse inclusion might not: Take for example $A = \mathbb{Z}_{(p)}$ and $B = Q(A) = \mathbb{Q}$ together with the canonical map.

Remark 2.10. If $\phi\colon A \to B$ is a ring homomorphism and $\mathfrak{q} \subset B$ a prime ideal, then $\mathfrak{p} := \phi^{-1}(\mathfrak{q})$ is a prime ideal of A. Moreover, ϕ induces a ring homomorphism $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$.

Example 2.11. Let A be a ring and $\mathcal{O}_{\operatorname{Spec} A}$ its structure sheaf. Then the pair $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is a ringed space, and in fact a locally ringed space: We have shown that $\mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}} \cong A_{\mathfrak{p}}$. By the previous remark, any ring homomorphism $\phi\colon A \to B$ then induces a morphism of locally ringed spaces $(f, f^{\sharp})\colon (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \to (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$.

Definition 2.12. An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A.

Example 2.13. The following are important examples of affine schemes:

1. $(\operatorname{Spec} \mathbb{Z}, \mathcal{O}_{\operatorname{Spec} \mathbb{Z}})$. For $D(a)$ a basic open set, we have $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}(D(a)) \cong \mathbb{Z}_a$.

2. $(\operatorname{Spec} k, \mathcal{O}_{\operatorname{Spec} k})$ for k a field. In this case $\operatorname{Spec} k$ consists of a single point and $\mathcal{O}_{\operatorname{Spec} k}(\operatorname{Spec}(k)) = k$.

3. $\mathbb{A}^n_A := (\operatorname{Spec} A[x_1, \dots, x_n], \mathcal{O}_{\operatorname{Spec} A[x_1,\dots,x_n]})$ for A any ring.

4. $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for A a discrete valuation ring. In this case $\operatorname{Spec} A = \{(0), \mathfrak{m}\}$ where $\mathfrak{m}$ is the unique maximal ideal with the open sets being the empty set, $\operatorname{Spec} A$ itself, and $\{(0)\}$. We then have $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A$ and $\mathcal{O}_{\operatorname{Spec} A}(\{(0)\}) = \mathcal{O}_{\operatorname{Spec} A,(0)} = Q(A)$.

5. $(\operatorname{Spec} k[x]/(x^2), \mathcal{O}_{\operatorname{Spec} k[x]/(x^2)})$ where k is a field ($k[x]/(x^2)$ is known as the *ring of dual numbers* over k). In this case $\operatorname{Spec} k[x]/(x^2)$ again consists of a single point, namely (x).

Note that $(\operatorname{Spec} k, \mathcal{O}_{\operatorname{Spec} k})$ and $(\operatorname{Spec} k[x]/(x^2), \mathcal{O}_{\operatorname{Spec} k[x]/(x^2)})$ consist both of one point, yet are different: $\mathcal{O}_{\operatorname{Spec} k}(\operatorname{Spec} k) = k$ while $\mathcal{O}_{\operatorname{Spec} k[x]/(x^2)}(\operatorname{Spec} k[x]/(x^2)) = k[x]/(x^2)$.

Example 2.14. Consider the locally ringed spaces $(\mathbb{C}^n, \mathcal{O}^{\mathrm{hol}}_{\mathbb{C}^n})$ and $(\mathbb{A}^n_{\mathbb{C}}, \mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}})$ where $\mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}}$ is the structure sheaf. We define a map $(f, f^\sharp)\colon (\mathbb{C}^n, \mathcal{O}^{\mathrm{hol}}_{\mathbb{C}^n}) \to (\mathbb{A}^n_{\mathbb{C}}, \mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}})$ as follows: f is the map

$$\begin{aligned} f\colon \mathbb{C}^n &\cong \operatorname{MaxSpec}(\mathbb{C}[x_1, \dots, x_n]) \hookrightarrow \mathbb{A}^n_{\mathbb{C}} \\ (t_1, \dots, t_n) &\mapsto (x_1 - t_1, \dots, x_n - t_n) \end{aligned}$$

which is continuous because polynomials are continuous in the standard topology on $\mathbb{C}^n$. Letting $A := \mathbb{C}[x_1, \dots, x_n]$, we define $f^\sharp\colon \mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}} \to f_* \mathcal{O}^{\mathrm{hol}}_{\mathbb{C}^n}$ as

$$f^\sharp|_U(s) = \left(U \cap \mathbb{C}^n \xrightarrow{s} \coprod_{\mathfrak{m} \in U \cap \mathbb{C}^n} A_{\mathfrak{m}} \to \mathbb{C} \right)$$

for all $(s\colon U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}) \in \mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}}(U)$ sections over the open set $U \subseteq \mathbb{A}^n_{\mathbb{C}}$ where the map $\coprod_{\mathfrak{m} \in U \cap \mathbb{C}^n} A_{\mathfrak{m}} \to \mathbb{C}$ is given component-wise by the maps $A_{\mathfrak{m}} \twoheadrightarrow A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}} \cong \mathbb{C}$. Holomorphicity of $f^\sharp|_U(s)$ comes down to the fact that s is locally representable as a quotient of polynomials which are of course holomorphic.

Lemma 2.15. *Let $(f, f^\sharp)\colon (\operatorname{Spec} A, \mathcal{O}_A) \to (\operatorname{Spec} B, \mathcal{O}_B)$ be a morphism of locally ringed spaces. Then f is of the form $f = \operatorname{Spec} \phi$ where $\phi := f^\sharp(\operatorname{Spec} A)\colon B \cong \mathcal{O}_B(\operatorname{Spec} B) \to (f_* \mathcal{O}_A)(\operatorname{Spec} B) = \mathcal{O}_A(\operatorname{Spec} A) \cong A$ is the map on global sections.*

Proof. Since passing to stalks is commutative with the map on global sections, we have a commutative diagram

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d] & A \arrow[d] \\ B_{f(\mathfrak{q})} \arrow[r, "f^\sharp_{\mathfrak{q}}"'] & A_{\mathfrak{q}} \end{tikzcd}$$

for any $\mathfrak{q} \in \operatorname{Spec} A$. But $(f, f^\sharp)$ is local, so $(f^\sharp_{\mathfrak{q}})^{-1}(\mathfrak{q}B_{\mathfrak{q}}) = f(\mathfrak{q})B_{f(\mathfrak{q})}$ and we conclude that $\phi^{-1}(\mathfrak{q}) = f(\mathfrak{q})$, whence $f = \operatorname{Spec} \phi$. ■

Corollary 2.16. *Let $(f, f^\sharp), (g, g^\sharp)\colon (\operatorname{Spec} A, \mathcal{O}_A) \to (\operatorname{Spec} B, \mathcal{O}_B)$ be two mor-*

phisms of locally ringed spaces. If $f^\sharp(\operatorname{Spec} B) = g^\sharp(\operatorname{Spec} B)$, then $(f, f^\sharp) = (g, g^\sharp)$.

Solution. By the previous lemma we have $f = g$. To show that $f^\sharp = g^\sharp$, it suffices to show that they agree on all distinguished open sets $D(b) \subseteq \operatorname{Spec} B$, and this follows from the fact that $D(b) = \operatorname{Spec} B_b$ by restriction. ■

Proposition 2.17. *Let A, B be two rings. Then there exists a bijection*

$$\left\{\begin{matrix}\text{ring homomorphisms} \\ A \to B\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{morphisms of locally ringed spaces} \\ (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})\end{matrix}\right\}$$

Proof. Given a morphism $(f, f^\sharp)\colon (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, we obtain a ring homomorphism $\phi\colon A \to B$ as

$$\begin{aligned}\phi = f^\sharp\colon A \cong \mathcal{O}_A(\operatorname{Spec} A) \to (f_*\mathcal{O}_B)(\operatorname{Spec} A) &= \mathcal{O}_B(f^{-1}(\operatorname{Spec} A)) \\ &= \mathcal{O}_B(\operatorname{Spec} B) \\ &\cong B\end{aligned}$$

In the other direction, if we start with a ring homomorphism $\psi\colon A \to B$, we obtain a morphism $(g, g^\sharp)\colon (\operatorname{Spec} B, \mathcal{O}_B) \to (\operatorname{Spec} A, \mathcal{O}_A)$ via $g = \operatorname{Spec} \psi$ and defining $g^\sharp\colon \mathcal{O}_A \to g_*\mathcal{O}_B$ as follows: For any distinguished open set $D(a) \subseteq \operatorname{Spec} A$, we have $\mathcal{O}_A(D(a)) \cong A_a$ as well as

$$(g_*\mathcal{O}_B)(D(a)) = \mathcal{O}_B(g^{-1}(D(a))) = \mathcal{O}_B(D(\psi(a))) = B_{\psi(a)}$$

so we can take $g^\sharp\colon \mathcal{O}_A(D(a)) \to (g_*\mathcal{O}_B)(D(a))$ to be the natural map $A_a \to B_{\psi(a)}$ induced by ψ. It is not hard to check that this glues together to a sheaf homomorphism (in particular since the $D(a)$ form a basis of the Zariski topology on $\operatorname{Spec} A$), so $(g, g^\sharp)$ is a morphism of ringed spaces. To see that it is in fact a morphism of locally ringed spaces, let $\mathfrak{q} \in \operatorname{Spec} B$ be a prime ideal and define $\mathfrak{p} := \psi^{-1}(\mathfrak{q})$. Since the map $g^\sharp_\mathfrak{p}\colon \mathcal{O}_{A,\mathfrak{p}} \to \mathcal{O}_{B,\mathfrak{q}}$ is compatible with $\psi = g^\sharp(\operatorname{Spec} A)$, we find that $(g^\sharp_\mathfrak{p})^{-1}(\mathfrak{q}B_\mathfrak{q}) = (\psi^{-1}(\mathfrak{q}))A_{\psi^{-1}(\mathfrak{q})} = \mathfrak{p}A_\mathfrak{p}$ so $g^\sharp$ is local.

Finally, we will show that these two constructions are mutually inverse. One direction is easy: If we start with a ring homomorphism $\phi\colon A \to B$, then construct a morphism of locally ringed spaces $(f, f^\sharp)$, we recover ϕ as $\phi = f^\sharp(\operatorname{Spec} A)$. On the other hand, if we start with $(f, f^\sharp)$ and apply our construction to $f^\sharp(\operatorname{Spec} A)$ to get another morphism $(g, g^\sharp)$, the previous corollary implies that $(f, f^\sharp) = (g, g^\sharp)$ since $f^\sharp(\operatorname{Spec} A) = g^\sharp(\operatorname{Spec} B)$ by construction. ■

Definition 2.18. A *scheme* is a ringed space $(X, \mathcal{O}_X)$ that is locally isomorphic to an affine scheme, i.e. for all points $x \in X$ there exists an open neighborhood $U \ni x$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A.

Definition 2.19. We define AffSch, Sch, and LocRingSpc to be the categories with objects the affine schemes, schemes, and locally ringed spaces, respectively, and morphisms all morphisms of locally ringed spaces.

We also define a category RingSpc which has as objects all ringed spaces and as morphisms all morphisms of ringed spaces.

We thus have a chain of subcategory inclusions

$$\mathrm{AffSch} \hookrightarrow \mathrm{Sch} \hookrightarrow \mathrm{LocRingSpc} \hookrightarrow \mathrm{RingSpc}$$

of which the first two are full.

Remark 2.20. Proposition 2.17 implies that we have an equivalence of categories

$$\begin{aligned} \mathrm{CRing}^{\mathrm{op}} &\xrightarrow{\simeq} \mathrm{AffSch} \\ A &\mapsto (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \end{aligned}$$

Remark 2.21. Recall the example from Remark 2.9 and note that the induced map $(\operatorname{Spec} Q(A), \mathcal{O}_{\operatorname{Spec} Q(A)}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ *is* a morphism of local rings.

2.1 From Classical Algebraic Geometry to Scheme Theory

Let k be an algebraically closed field and $X \subseteq k^n$ an affine algebraic set, say $X = V(\mathfrak{a})$ for some ideal $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$. Then X comes with a sheaf, its sheaf of regular functions $\mathcal{O}_X$ (see example 2.3), which gives a ringed space $(X, \mathcal{O}_X)$. In general $(X, \mathcal{O}_X)$ is not a scheme, but we can associate to it a scheme as follows: Consider the ideal $I(X) \subseteq k[x_1, \dots, x_n]$ given by

$$I(X) := \{f \in k[x_1, \dots, x_n] \mid f(t_1, \dots, t_n) = 0 \text{ for all } (t_1, \dots, t_n) \in X\}$$

By Hilbert's Nullstellensatz, we then have $I(X) = \sqrt{\mathfrak{a}}$. We then obtain the *affine coordinate ring* $A(X) := k[x_1, \dots, x_n]/I(X)$ of X, and from this an affine scheme $(\operatorname{Spec} A(X), \mathcal{O}_{A(X)})$.

Proposition 2.22. *There exists a morphism of locally ringed spaces* $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (\operatorname{Spec} A(X), \mathcal{O}_{A(X)})$.

Proof. Noting that

$$\operatorname{MaxSpec} A(X) = \{\mathfrak{m} \in \operatorname{MaxSpec} k[x_1, \dots, x_n] \mid I(X) \subseteq \mathfrak{m}\}$$

we see that under the bijection $k^n \leftrightarrow \operatorname{MaxSpec} k[x_1, \dots, x_n]$, $(t_1, \dots, t_n) \mapsto (x_1 - t_1, \dots, x_n - t_n)$ we in fact have $X \cong \operatorname{MaxSpec} A(X)$, and so define f to be the inclusion $X \cong \operatorname{MaxSpec} A(X) \hookrightarrow \operatorname{Spec} A(X)$.

If $U \subseteq \operatorname{Spec} A(X)$ is open and $s\colon U \to \coprod_{\mathfrak{p} \in U} A(X)_{\mathfrak{p}}$ is any section in $\mathcal{O}_{A(X)}(U)$, we obtain a section $t \in (f_* \mathcal{O}_X)(U) = \mathcal{O}_X(U \cap \operatorname{MaxSpec} A(X))$ via

$$t\colon U \cap \operatorname{MaxSpec} A(X) \xrightarrow{s|_{U \cap \operatorname{MaxSpec} A(X)}} \coprod_{\mathfrak{m} \in \operatorname{MaxSpec} A(X)} A_{\mathfrak{m}} \to k$$

where the last map is obtained by observing that $A(X)_{\mathfrak{m}} / \mathfrak{m} A(X)_{\mathfrak{m}} \cong k$ for all $\mathfrak{m} \in \operatorname{MaxSpec} A(X)$ and applying this isomorphism to each component. Since s is locally of the form $s = \bar{g}_1 / \bar{g}_2$ for $\bar{g}_1, \bar{g}_2 \in A(X)$, t is locally of the form $t = g_1 / g_2$ for g_1, g_2 lifts of $\bar{g}_1, \bar{g}_2$, respectively, and therefore $f^\sharp(s) := t$ is well-defined.

One then checks that $(f, f^\sharp)$ does in fact define a morphism of locally ringed spaces. In fact, if $\mathfrak{m} \in \operatorname{MaxSpec} k[x_1, \dots, x_n]$ is the maximal ideal corresponding to $(t_1, \dots, t_n) \in k^n$, then $\mathcal{O}_{X,\mathfrak{m}} \cong \mathcal{O}_{Y,\mathfrak{m}} \cong A(X)_{\mathfrak{m}}$. ∎

Remark 2.23. To define $(\operatorname{Spec} A(X), \mathcal{O}_{A(X)})$, we only that X is an affine algebraic set, not that $X = V(\mathfrak{a})$ for a given ideal $\mathfrak{a}$. If we remember this information, we can consider $\operatorname{Spec} k[x_1, \dots, x_n]/\mathfrak{a}$. We then have morphisms

$$(X, \mathcal{O}_X) \xrightarrow{(f, f^\sharp)} (\operatorname{Spec} A(X), \mathcal{O}_{A(X)}) \xrightarrow{(g, g^\sharp)} (\operatorname{Spec} k[x_1, \dots, x_n]/\mathfrak{a}, \mathcal{O}_{k[x_1, \dots, x_n]/\mathfrak{a}})$$

where g is a continuous bijection.

Example 2.24. Let k be a field and $A := k[x]/(x^2)$ its ring of dual numbers. Then $\operatorname{Spec} k$ and $\operatorname{Spec} A$ both consist of a single point. Define two maps $(f, f^\sharp)\colon (\operatorname{Spec} k, \mathcal{O}_k) \to (\operatorname{Spec} A, \mathcal{O}_A)$ and $(g, g^\sharp)\colon (\operatorname{Spec} A, \mathcal{O}_A) \to (\operatorname{Spec} k, \mathcal{O}_k)$ as follows: f and g must be the unique maps. The map $f^\sharp$ is given by the quotient map $f^\sharp\colon \mathcal{O}_A(\operatorname{Spec} A) \cong A \twoheadrightarrow A/(x) \cong k \cong (f_* \mathcal{O}_k)(\operatorname{Spec} A)$, and its counterpart $g^\sharp\colon \mathcal{O}_k(\operatorname{Spec} k) \cong k \hookrightarrow A \cong (g_* \mathcal{O}_A)(\operatorname{Spec} k)$ is the canonical inclusion. Then $(g,g^\#) \circ (f,f^\#) = \mathrm{id}_{(\operatorname{Spec} k, \mathcal{O}_k)}$, but $(f,f^\#) \circ (g,g^\#) \neq \mathrm{id}_{(\operatorname{Spec} A, \mathcal{O}_A)}$: $f^\# \circ g^\#$ is the composite $A \twoheadrightarrow k \hookrightarrow A \neq \mathrm{id}_A$.

Exercise 2.25. TODO

Here are two special cases of this:

Example 2.26.

1. Let $(X, \mathcal{O}_X)$ be any scheme. Since there is a unique ring map $\mathbb{Z} \to \Gamma(X, \mathcal{O}_X)$, there is a unique morphism of schemes $X \to \operatorname{Spec} \mathbb{Z}$, i.e. $\operatorname{Spec} \mathbb{Z}$ is a terminal object of Sch.

2. If k is a field and A a k-algebra, then the inclusion $k \hookrightarrow A$ corresponds to a morphism $(\operatorname{Spec} A, \mathcal{O}_A) \to (\operatorname{Spec} k, \mathcal{O}_k)$.

Next, we want to briefly discuss how to create new schemes out of old via gluing.

Proposition 2.27 ([Vak25, Exercise 4.4.A]). *Suppose we are given schemes X_i, open subschemes $X_{ij} \subseteq X_i$ with $X_{ii} = X_i$, and isomorphisms $f_{ij}\colon X_{ij} \to X_{ji}$ with f_{ii} the identity such that the* cocycle condition $f_{ik}|_{X_{ij} \cap X_{ik}} = f_{jk}|_{X_{ji} \cap X_{jk}} \circ f_{ij}|_{X_{ij} \cap X_{ik}}$ *is satisfied for all i, j, k. Then there is a unique scheme X along with open subschemes isomorphic to the X_i respecting the gluing data in the obvious sense.*

Proof. TODO (maybe?) ■

Example 2.28. Let k be a field and let $X_1 = X_2 = \mathbb{A}^1_k$ and $U_1 = U_2 = \mathbb{A}^1_k \setminus \{0\} = \operatorname{Spec}(k[x]_x)$. There are two interesting choices of morphism $(\phi, \phi^\#)\colon (U_1, \mathcal{O}_{U_1}) \to (U_2, \mathcal{O}_{U_2})$:

1. $(\phi, \phi^\#) = \mathrm{id}_{(U_1, \mathcal{O}_{U_1})}$. In this case we obtain the *affine line with two origins* .

2. $(\phi, \phi^\#)$ is given by the ring isomorphism $k[x]_x \to k[x]_x$, $x \mapsto 1/x$. In this case we obtain the *projective line* $\mathbb{P}^1_k$ over the field k.

To make sense of this second example, assume that k is algebraically closed and only consider maximal ideals. Then $\phi^{-1}((x-t)) = (x-1/t)$, so $\mathbb{P}^1_k$ identifies with $k^2 \setminus /k^\times$, points of which we write as $[t_1 : t_2]$ (these are the familiar homogeneous coordinates).

References

[Vak25] Ravi Vakil. *The Rising Sea: Foundations of Algebraic Geometry*. Google-Books-ID: N2Xx0AEACAAJ. Princeton University Press, Oct. 21, 2025. 632 pp. ISBN: 978-0-691-26867-5 (cit. on p. 11).

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