First Curvature-Based Protrusion Tracking Framework

First Author[a,*], Second Author[a,b], Third Author[b]

[a]*First Address*
[b]*Second Address*

Abstract

A submitted program is expected to satisfy the following criteria: it must be of benefit to other physicists, or be an exemplar of good programming practice, or illustrate new or novel programming techniques which are of importance to computational physics community; it should be implemented in a language and executable on hardware that is widely available and well documented; it should meet accepted standards for scientific programming; it should be adequately documented and, where appropriate, supplied with a separate User Manual, which together with the manuscript should make clear the structure, functionality, installation, and operation of the program.

Your manuscript and figure sources should be submitted through Editorial Manager (EM) by using the online submission tool at https://www.editorialmanager.com/comphy/.

In addition to the manuscript you must supply: the program source code; a README file giving the names and a brief description of the files/directory structure that make up the package and clear instructions on the installation and execution of the program; sample input and output data for at least one comprehensive test run; and, where appropriate, a user manual.

A compressed archive program file or files, containing these items, should be uploaded at the "Attach Files" stage of the EM submission.

For files larger than 1Gb, if difficulties are encountered during upload the author should contact the Technical Editor at cpc.mendeley@gmail.com.

Keywords: keyword1; keyword2; keyword3; etc.

PROGRAM SUMMARY/NEW VERSION PROGRAM SUM-

*Corresponding author.
E-mail address: firstAuthor@somewhere.edu

MARY
Program Title:
CPC Library link to program files: (to be added by Technical Editor)
Developer's repository link: (if available)
Code Ocean capsule: (to be added by Technical Editor)
Licensing provisions(please choose one): CC0 1.0/CC By 4.0/MIT/Apache-2.0/BSD 3-clause/BSD 2-clause/GPLv3/GPLv2/LGPL/CC BY NC 3.0/MPL-2.0
Programming language:
Supplementary material:
*Journal reference of previous version:**
*Does the new version supersede the previous version?:**
*Reasons for the new version:**
*Summary of revisions:**
Nature of problem(approx. 50-250 words):
Solution method(approx. 50-250 words):
Additional comments including restrictions and unusual features (approx. 50-250 words):

References

[1] Reference 1

[2] Reference 2

[3] Reference 3

* Items marked with an asterisk are only required for new versions of programs previously published in the CPC Program Library.

1. Mean Curvature Distribution Derivation (Exponential Decay Case)

Suppose our protrusion can be modeled by the following structure:

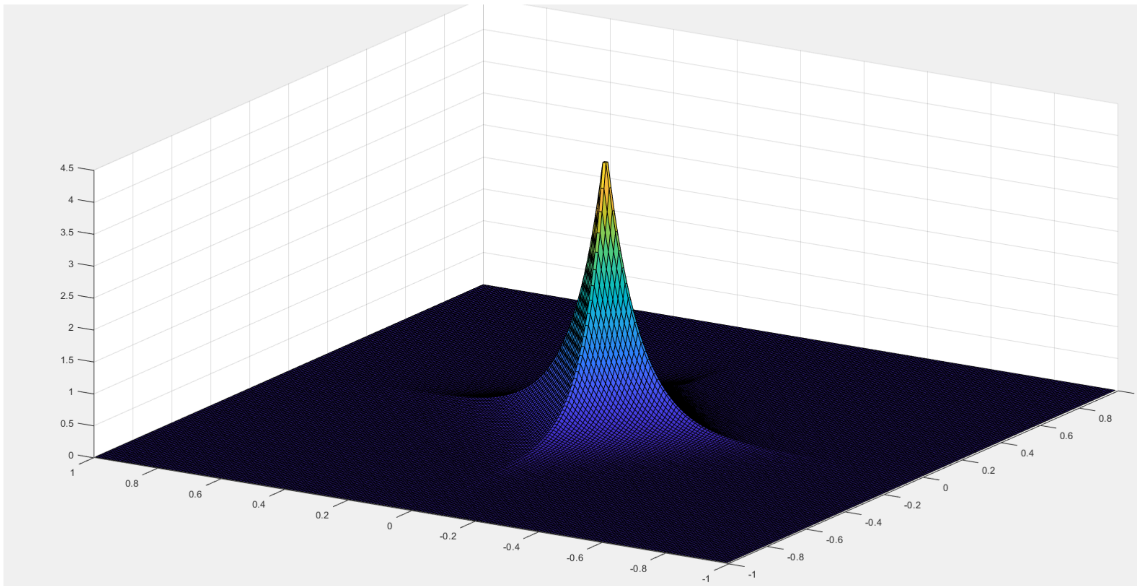

Figure 1: Protrusion modelled with Exponential Decay

$$U, V iid \sim U(0,1) \tag{1}$$

$$Z = e^{-\lambda(U+V)} \tag{2}$$

$$\mathbf{r} = U\mathbf{U}_1 + V\mathbf{U}_2 + Z\mathbf{U}_3 \tag{3}$$

$$\mathbf{r}_U = \mathbf{U}_1 - \lambda e^{-\lambda(U+V)}\mathbf{U}_3 \tag{4}$$

$$\mathbf{r}_V = \mathbf{U}_2 - \lambda e^{-\lambda(U+V)}\mathbf{U}_3 \tag{5}$$

A visualization is provided in Figure 1. We utilize the following properties from Barret O'Neill (todo, cite here):

$$E = \mathbf{r}_U \cdot \mathbf{r}_U \tag{6}$$
$$F = \mathbf{r}_U \cdot \mathbf{r}_V \tag{7}$$
$$G = \mathbf{r}_V \cdot \mathbf{r}_V \tag{8}$$
$$EG - F^2 = ||\mathbf{r}_U \times \mathbf{r}_V||^2 \tag{9}$$
$$L = S(\mathbf{r}_U) \cdot \mathbf{r}_U \tag{10}$$
$$M = S(\mathbf{r}_U) \cdot \mathbf{r}_V = S(\mathbf{r}_V) \cdot \mathbf{r}_U \tag{11}$$
$$N = S(\mathbf{r}_V) \cdot \mathbf{r}_V \tag{12}$$
$$H = \frac{GL + EN - 2FM}{2(EG - F^2)} \tag{13}$$

where H is the mean curvature, E, F, and G are the coefficients from the first fundamental form, and L, M, and N are the coefficients from the second fundamental form. The shape operator is defined as:

$$S(\mathbf{V}) = -\nabla_{\mathbf{V}} \mathbf{U} \tag{14}$$

where ∇_V is the covariant derivative with respect to a tangent vector $\mathbf{V}$, and $\mathbf{U}$ is the unit normal vector, which is given by:

$$\mathbf{U} = \frac{\mathbf{r}_U \times \mathbf{r}_V}{||\mathbf{r}_U \times \mathbf{r}_V||} \tag{15}$$

For our specific mapping $\mathbf{r}$, we obtain:

$$E = 1 + \lambda^2 e^{-2\lambda(U+V)} \tag{16}$$
$$F = \lambda^2 e^{-2\lambda(U+V)} \tag{17}$$
$$G = 1 + \lambda^2 e^{-2\lambda(U+V)} \tag{18}$$
$$EG - F^2 = 1 + 2\lambda^2 e^{-2\lambda(U+V)} \tag{19}$$
$$\mathbf{U} = \frac{\lambda e^{-\lambda(U+V)}(\mathbf{U}_1 + \mathbf{U}_2) + \mathbf{U}_3}{\sqrt{1 + 2\lambda^2 e^{-2\lambda(U+V)}}} \tag{20}$$

To calculate the coefficients of the second fundamental form, we begin by calculating the shape operator:

$$S(\mathbf{r}_U) = -\nabla_{\mathbf{U}_1 - \lambda e^{\lambda(U+V)}\mathbf{U}_3} \frac{\lambda e^{-\lambda(U+V)}(\mathbf{U}_1 + \mathbf{U}_2) + \mathbf{U}_3}{\sqrt{1 + 2\lambda^2 e^{-2\lambda(U+V)}}} \tag{21}$$

$$= -\nabla_{\mathbf{U}_1} \frac{\lambda e^{-\lambda(U+V)}(\mathbf{U}_1 + \mathbf{U}_2) + \mathbf{U}_3}{\sqrt{1 + 2\lambda^2 e^{-2\lambda(U+V)}}} \tag{22}$$

The above reduction happens because there is no Z variable in the unit normal vector. Furthermore, a covariant derivative with respect to $\mathbf{U}_1$ is just a partial derivative with respect to U.

$$S(\mathbf{r}_U) = -\frac{\partial}{\partial U} \frac{\lambda e^{-\lambda(U+V)}(\mathbf{U}_1 + \mathbf{U}_2) + \mathbf{U}_3}{\sqrt{1 + 2\lambda^2 e^{-2\lambda(U+V)}}} \tag{23}$$

$$= -\frac{\big(- \lambda^2(1 + 2\lambda^2 e^{-2\lambda(U+V)})e^{-\lambda(U+V)} + 2\lambda^4 e^{-3\lambda(U+V)}\big)(\mathbf{U}_1 + \mathbf{U}_2) + 2\lambda^3 e^{-2\lambda(U+V)}\mathbf{U}_3}{(1 + 2\lambda^2 e^{-2\lambda(U+V)})^{\frac{3}{2}}} \tag{24}$$

Note that $S(\mathbf{r}_U) = S(\mathbf{r}_V)$ due to the symmetry of the above derivation. Furthermore, we observe $L = M = N$ once again due to symmetry. Thus:

$$L = M = N = \frac{\lambda^2}{\sqrt{1 + 2\lambda^2 e^{-2\lambda(U+V)}}} \tag{25}$$

Finally, we derive an expression for the mean curvature.

$$H = \frac{\lambda^2}{(1 + 2\lambda^2 e^{-2\lambda(U+V)})^{\frac{3}{2}}} \tag{26}$$

When $\lambda = 1$ the histogram of H as a random variable (estimated with Monte-Carlo), resembles Figure 2.

2. Mean Curvature Distribution Derivation (Gaussian Decay Case)

We now wish to extend the previous derivation to protrusions with thinner tails and mass held closer to their centers.

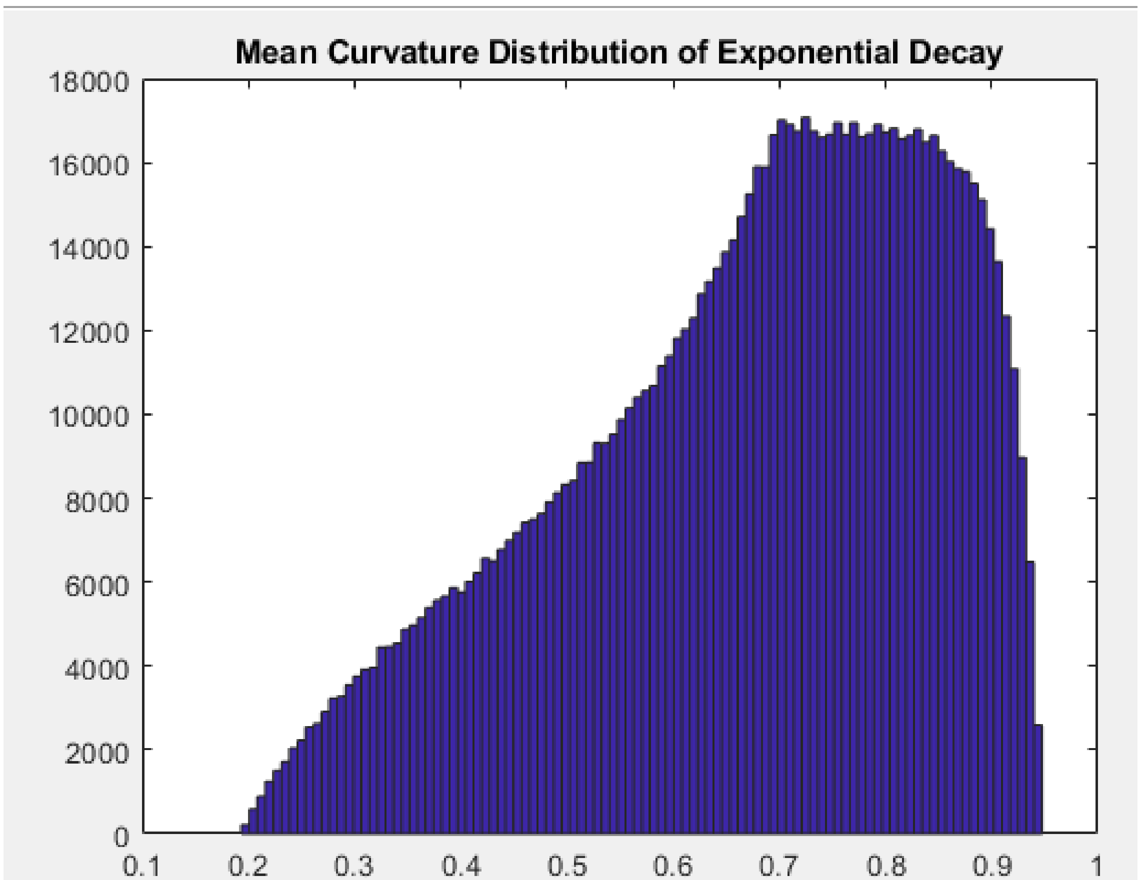

Figure 2: Histogram of Mean Curvature with Exponential Decay

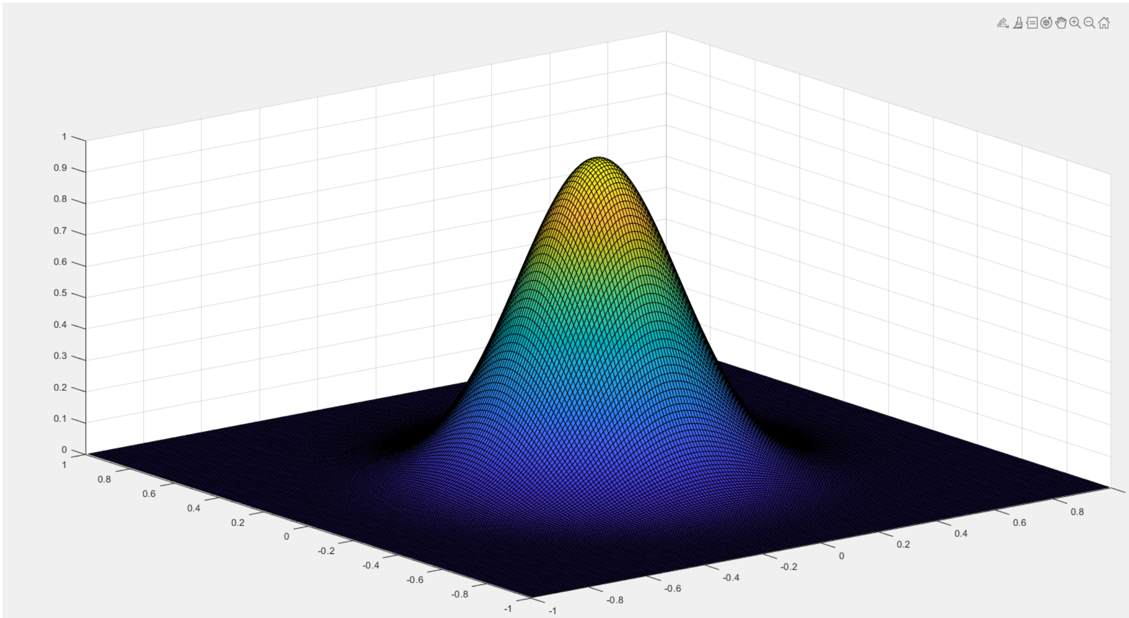

Figure 3: Protrusion modelled with Gaussian Decay

$$U, V iid \sim U(0,1) \tag{27}$$
$$Z = e^{-\lambda(U^2+V^2)} \tag{28}$$
$$\mathbf{r} = U\mathbf{U}_1 + V\mathbf{U}_2 + Z\mathbf{U}_3 \tag{29}$$
$$\mathbf{r}_U = \mathbf{U}_1 - 2\lambda U e^{-\lambda(U^2+V^2)}\mathbf{U}_3 \tag{30}$$
$$\mathbf{r}_V = \mathbf{U}_2 - 2\lambda V e^{-\lambda(U^2+V^2)}\mathbf{U}_3 \tag{31}$$

A visualization is provided in Figure 3. For our new mapping $\mathbf{r}$, we obtain:

$$E = 1 + 4\lambda^2 U^2 e^{-2\lambda(U^2+V^2)} \tag{32}$$
$$F = 4\lambda^2 UV e^{-2\lambda(U^2+V^2)} \tag{33}$$
$$G = 1 + 4\lambda^2 V^2 e^{-2\lambda(U^2+V^2)} \tag{34}$$
$$EG - F^2 = 1 + 4\lambda^2 e^{-2\lambda(U^2+V^2)}(U^2+V^2) \tag{35}$$
$$\mathbf{U} = \frac{2\lambda e^{-\lambda(U^2+V^2)}(U\mathbf{U}_1 + V\mathbf{U}_2) + \mathbf{U}_3}{\sqrt{1 + 4\lambda^2 e^{-2\lambda(U^2+V^2)}(U^2+V^2)}} \tag{36}$$

To calculate the coefficients of the second fundamental form, we begin by calculating the shape operator:

$$S(\mathbf{r}_U) = -\nabla_{\mathbf{U}_1 - 2\lambda U e^{-\lambda(U^2+V^2)}\mathbf{U}_3} \frac{2\lambda e^{-\lambda(U^2+V^2)}(U\mathbf{U}_1 + V\mathbf{U}_2) + \mathbf{U}_3}{\sqrt{1 + 4\lambda^2 e^{-2\lambda(U^2+V^2)}(U^2+V^2)}} \tag{37}$$

$$\tag{38}$$

Once again, this covariant derivative reduces to a partial derivative with respect to U.

$$S(\mathbf{r}_U) = -\frac{\partial}{\partial U} \frac{2\lambda e^{-\lambda(U^2+V^2)}(U\mathbf{U}_1 + V\mathbf{U}_2) + \mathbf{U}_3}{\sqrt{1 + 4\lambda^2 e^{-2\lambda(U^2+V^2)}(U^2+V^2)}} \tag{39}$$

Due to the complexity of the answer, $S(\mathbf{r}_U)$ is presented component by component:

$$-\frac{2\lambda(1-2\lambda U^2)e^{-\lambda(U^2+V^2)}\left(1 + 4\lambda^2 e^{-2\lambda(U^2+V^2)}(U^2+V^2)\right) + 8\lambda^3 U^2 e^{-3\lambda(U^2+V^2)}(1 - 2\lambda(U^2+V^2))}{(1 + 4\lambda^2 e^{-2\lambda(U^2+V^2)}(U^2+V^2))^{\frac{3}{2}}} \tag{40}$$

$$-\frac{-4\lambda^2 UV e^{-\lambda(U^2+V^2)}\left(1 + 4\lambda^2 e^{-2\lambda(U^2+V^2)}(U^2+V^2)\right) + 8\lambda^3 UV e^{-3\lambda(U^2+V^2)}(1 - 2\lambda(U^2+V^2))}{(1 + 4\lambda^2 e^{-2\lambda(U^2+V^2)}(U^2+V^2))^{\frac{3}{2}}} \tag{41}$$

$$\frac{4\lambda^2 U e^{-2\lambda(U^2+V^2)}(1 - 2\lambda(U^2+V^2))}{(1 + 4\lambda^2 e^{-2\lambda(U^2+V^2)}(U^2+V^2))^{\frac{3}{2}}} \tag{42}$$

To reduce the complexity of this problem, for large enough λ, we assume that $e^{-3\lambda(U^2+V^2)} \to 0$, as this takes a relatively small number to the third power. Thus:

$$S(\mathbf{r}_U) \approx \frac{-2\lambda(1-2\lambda U^2)e^{-\lambda(U^2+V^2)}}{\sqrt{1 + 4\lambda^2 e^{-2\lambda(U^2+V^2)}(U^2+V^2)}}\mathbf{U}_1 + \tag{43}$$

$$\frac{4\lambda^2 UV e^{-\lambda(U^2+V^2)}}{\sqrt{1 + 4\lambda^2 e^{-2\lambda(U^2+V^2)}(U^2+V^2)}}\mathbf{U}_2 + \tag{44}$$

$$\frac{4\lambda^2 U e^{-2\lambda(U^2+V^2)}(1 - 2\lambda(U^2+V^2))}{(1 + 4\lambda^2 e^{-2\lambda(U^2+V^2)}(U^2+V^2))^{\frac{3}{2}}}\mathbf{U}_3 \tag{45}$$

Once again utilizing $e^{-3\lambda(U^2+V^2)} \to 0$, and the symmetry of $S(\mathbf{r}_U)$ and $S(\mathbf{r}_V)$:

$$L = S(\mathbf{r}_U) \cdot \mathbf{r}_U \approx \frac{-2\lambda(1-2\lambda U^2)e^{-\lambda(U^2+V^2)}}{\sqrt{1+4\lambda^2 e^{-2\lambda(U^2+V^2)}(U^2+V^2)}} \tag{46}$$

$$M = S(\mathbf{r}_U) \cdot \mathbf{r}_V \approx \frac{4\lambda^2 UV e^{-\lambda(U^2+V^2)}}{\sqrt{1+4\lambda^2 e^{-2\lambda(U^2+V^2)}(U^2+V^2)}} \tag{47}$$

$$N = S(\mathbf{r}_V) \cdot \mathbf{r}_V \approx \frac{-2\lambda(1-2\lambda V^2)e^{-\lambda(U^2+V^2)}}{\sqrt{1+4\lambda^2 e^{-2\lambda(U^2+V^2)}(U^2+V^2)}} \tag{48}$$

$$\tag{49}$$

Finally, (applying $e^{-3\lambda(U^2+V^2)} \to 0$ once more), we obtain:

$$H = \frac{2\big(\lambda(U^2+V^2)-1\big)e^{-\lambda(U^2+V^2)}}{(1+4\lambda^2 e^{-2\lambda(U^2+V^2)}(U^2+V^2))^{\frac{3}{2}}} \tag{50}$$

When $\lambda = 1$ the histogram of H as a random variable (estimated with Monte-Carlo), resembles Figure 4.

References

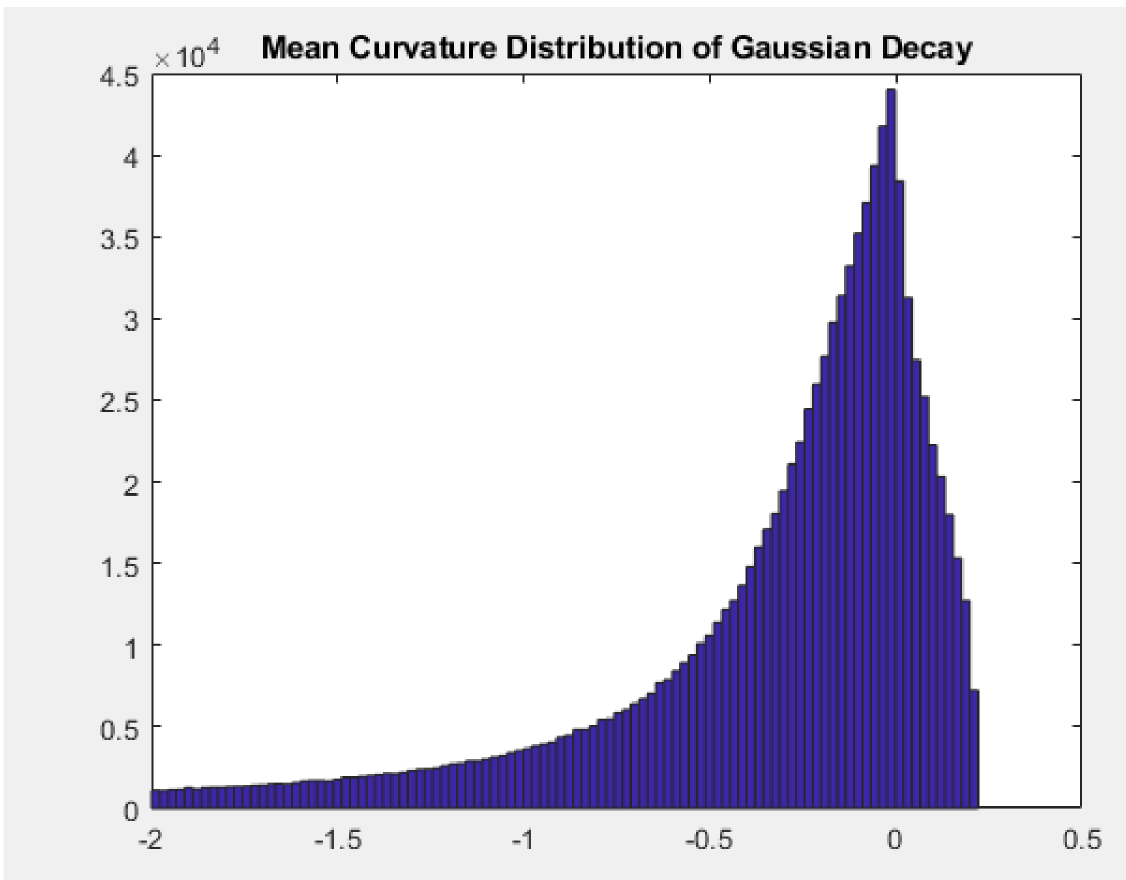

Figure 4: Histogram of Mean Curvature with Exponential Decay