Proof for Prof. Gerlach

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1 The Problem

Suppose we are given a polygon in the plane or a polyhedron in space. If we label the unit normal to the k different edges (faces) of this polygon (polyhedron) $\hat{n}_1, \hat{n}_2, ..., \hat{n}_k$ and define $\vec{n}_i = \alpha_i \hat{n}_i$, where α_i is the length (area) of the corresponding edge (face), then show that $\sum_{i=1}^{k} \vec{n}_i = 0$.

2 The Solution in Two Dimensions

Call the area enclosed by the polygon R, and the polygon itself (oriented counterclockwise) ∂R. We will relabel the α_i as l_i to indicate that these are edge lengths in the 2-D case. Suppose we define a constant (non-zero) vector field $\vec{F}$. Now one formulation of Green's Theorem in the plane–utilizing the 2-D divergence rather than the so-called scalar curl–is stated as:

$$\iint_R div(\vec{F})\, dA = \oint_{\partial R} \vec{F} \cdot d\hat{n}.,$$

where $\hat{n}$ is the outward-pointing normal. Now since $\vec{F}$ is constant, its divergence is zero; thus, the left-hand side of the above theorem is zero, and we will reason from the right-hand side to obtain the desired identity. Calling the ith oriented edge of our polygon ∂R_i, we see that ∂R decomposes into the union $\partial R = \bigcup_{i=1}^{k} \partial R_i$ and hence

$$\oint_{\partial R} \vec{F} \cdot d\hat{n} = \sum_{i=1}^{k} \int_{\partial R_i} \vec{F} \cdot d\hat{n} = \sum_{i=1}^{k} \int_{\partial R_i} \vec{F} \cdot \hat{n}_i \, ds.$$

Since both $\vec{F}$ and the unit outward normal $\hat{n}_i$ are constant along each edge ∂R_i, we can bring this portion out of the integral to obtain

$$\oint_{\partial R_i} \vec{F} \cdot d\hat{n} = \vec{F} \cdot \hat{n}_i \int_{\partial R_i} ds,$$

where $\int_{\partial R_i} ds = l_i$, the length of the edge ∂R_i, and $\vec{F} \cdot \hat{n}_i = \|\vec{F}\| \cos\theta_i$, where θ_i is the angle that the vector $\vec{F}$ makes with the vector $\hat{n}_i$ in the plane. Thus, our original integral becomes

$$\oint_{\partial R} \vec{F} \cdot d\hat{n} = \sum_{i=1}^{k} \|\vec{F}\| l_i \cos\theta_i = \|\vec{F}\| \sum_{i=1}^{k} l_i \cos\theta_i,$$

where we have taken $\|\vec{F}\|$ out of the sum because $\vec{F}$ was assumed to be constant everywhere. Now we know from before that this expression must equal zero, and we assumed $\vec{F}$ nonzero, so we conclude that the sum must be zero:

$$\sum_{i=1}^{k} l_i \cos\theta_i = 0,$$

regardless of our choice of $\vec{F}$. Let us choose, then, two different possibilities for the vector field $\vec{F}$ and see what happens. First, take $\vec{F} = \hat{\imath}$, the unit vector the x-direction. Then the angle θ_i is precisely the angle that $\hat{n}_i$ makes with the x-axis, so that $l_i \cos\theta_i$ is the component of the length-scaled vector $\vec{n}_i$ in the x-direction. Thus our sum becomes

$$\sum_{i=1}^{k} (\vec{n}_i)_x = 0.$$

Similarly, if we take $\vec{F} = \hat{\jmath}$, the unit vector in the y-direction, we obtain

$$\sum_{i=1}^{k} l_i \cos\phi_i = 0,$$

where ϕ_i is the angle that the vector $\hat{n}_i$ makes with the y-axis, or if we express it in terms of θ_i from before, we have $\cos\phi_i = \cos(\theta_i - \frac{\pi}{2}) = \cos(-(\frac{\pi}{2} - \theta_i)) = \cos(\frac{\pi}{2} - \theta_i) = \sin\theta_i$, yielding the analogous identity

$$\sum_{i=1}^{k} l_i \sin\theta_i = \sum_{i=1}^{k} (\vec{n}_i)_y = 0.$$

Now the vector $\vec{n}_i$ is simply the sum of its components: $\vec{n}_i = (\vec{n}_i)_x \hat{\imath} + (\vec{n}_i)_y \hat{\jmath}$, so we finally obtain the identity

$$\sum_{i=1}^{k} \vec{n}_i = \sum_{i=1}^{k} (\vec{n}_i)_x \hat{\imath} + \sum_{i=1}^{k} (\vec{n}_i)_y \hat{\jmath} = 0\hat{\imath} + 0\hat{\jmath} = 0$$

as was to be shown. ■

3 The Solution in m Dimensions

Suppose we now have a polytope in $\mathbb{R}^m$ whose interior region we label D and whose boundary we label ∂D, consisting of $(m-1)$-facets, or just facets, as we shall refer to them. Suppose also that for the ith facet, we call $\vec{n}_i = h_i \hat{n}_i$ the outward-facing normal vector to this facet and h_i is the (hyper)volume of the facet. Then, defining the divergence of a vector field $\vec{F}$ to be

$$div(\vec{F}) = \sum_{i=1}^{m} \frac{\partial F_i}{\partial x_i},$$

we have the analog of the divergence theorem in $\mathbb{R}^m$ to be

$$\int_D div(\vec{F})\, dH = \oint_{\partial D} \vec{F} \cdot d\hat{N},$$

where it is understood that dH is an infinitesimal m-volume element and $d\hat{N}$ is the infinitesimal outward pointing normal to ∂D. As before, if we define $\vec{F}$ to be constant everywhere, then the expression on the left-hand side is equal to zero. Labelling the k different facets of our polytope ∂D_i, we have:

$$\oint_{\partial D} \vec{F} \cdot d\hat{N} = \sum_{i=1}^{k} \int_{\partial D_i} \vec{F} \cdot d\hat{N}$$

$$= \sum_{i=1}^{k} \int_{\partial D_i} (\vec{F} \cdot \hat{n}_i)\, dH,$$

where dH is now an infinitesimal $(m-1)$-volume element, and since $\vec{F}$ and $\hat{n}_i$ are constant along a facet, this is

$$\sum_{i=1}^{k} (\vec{F} \cdot \hat{n}_i) \int_{\partial D_i} dH$$

$$= \sum_{i=1}^{k} \|\vec{F}\| h_i \cos\theta_i,$$

because $\vec{F} \cdot \hat{n}_i = \|\vec{F}\| \cos\theta_i$, where θ_i is the angle between the vectors $\vec{F}$ and $\hat{n}_i$, and $\int_{\partial D_i} dH$ is simply h_i as we defined it, the volume of the ith facet. Since $\vec{F}$ is constant everywhere, we can bring it out of the sum and divide by the magnitude ($\vec{F}$ assumed nonzero) to obtain that

$$\sum_{i=1}^{k} h_i \cos\theta_i = 0.$$

If we now define, one at time, $\vec{F}_j = \hat{e}_j$, the jth standard basis vector for $\mathbb{R}^m$, and if we call θ_{ij} the angle that $\hat{n}_i$ makes with $\hat{e}_j$, then the jth component of the vector $\vec{n}_i$ is $proj_{\hat{e}_j}(\vec{n}_i) = h_i \cos\theta_{ij}$, and applying the above summation identity, we obtain the following:

$$\sum_{i=1}^{k} \vec{n}_i = \sum_{i=1}^{k} \sum_{j=1}^{m} h_i \cos\theta_{ij} \hat{e}_j$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{k} h_i \cos\theta_{ij} \right) \hat{e}_j$$

$$= \sum_{j=1}^{m} 0 \hat{e}_j = 0,$$

which was to be shown. ■