Universitat Politècnica de Catalunya

# Fluid Mechanics

*Derivations*

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# 1 Introduction

The difference between a fluid and a solid is that the fluid can not support a shear force by static deformation, it experiences a gradual deformation, reaching a steady-state but never equilibrium. A fluid is considered a continuum, constituted by an infinite amount of infinitely small particles.

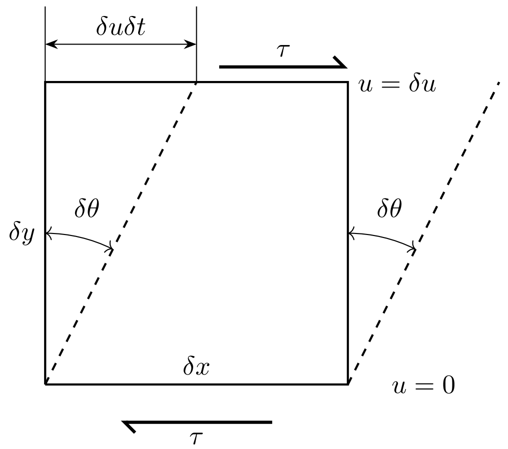
However, average values are considered instead of fluctuations at such a small scale. The fluid particle to consider should have less size than the mean free path $\lambda$. It should be small enough for the problem but big enough that it allows collision to reach equilibrium.

$$\lambda = \frac{K_B T}{\pi \sqrt{2} d^2 P} \qquad \frac{1}{n} \;, \quad \lambda^3 << \Delta\theta << L^3$$

## 1.1 Concepts, variables and fluid statics

## 1.2 Shear stress and deformation

Shear stress $\tau$ is the viscous force per unit area, it has the same direction as the force that is being applied. The expression for $\tau$ can be derived using the following diagram.




As is observed, the shear force $\tau$ is proportional to the change in the angle $\theta$ in time:

$$\tau \propto \frac{\delta\theta}{\delta t}$$

This change can be expressed as a derivative of the angle $\theta$ in time. With some trigonometry one arrives to the expression of the shear force that depends on velocity and position.

$$\tau = \mu \frac{d\theta}{dt} = \mu \frac{dudt}{dydt} = \boxed{\tau = \mu \frac{du}{dy}}$$

Where $\mu$ is a constant called the dynamic viscosity that depends on each fluid. Another constant is also defined: $\nu = \frac{\mu}{\rho}$, the kinematic viscosity. These give an idea to how easy it is to make a fluid flow and how "sticky" it is.

## 1.3 Boundary layer

The boundary layer is the region of a fluid where the effect of a nearby solid surface is noticeable. It creates a gradient of velocities of this fluid.

## 1.4 Inviscid vs viscous flow

In gases the origin of viscosity is the exchange of particles between two fluid particles. In liquids there are still interactions between molecules, the same goes for temperature, that is thermal conduction.

An ideal fluid in a fluid with no viscosity and no thermal conduction. There are a few parameters that are used to indicate how close a certain fluid is to being ideal or behaving in an ideal manner depending on other variables.

- **Reynolds number**. High numbers yield low viscosity and low numbers give high viscosity.

$$Re = \frac{\rho L V}{\mu}$$

- **Mach Number**. Those below 0.3 can be considered incompressible.

$$M = \frac{V}{V_s}$$

# 2 Fluid Statics

## 2.1 The Fundamental law of Hydrostatics

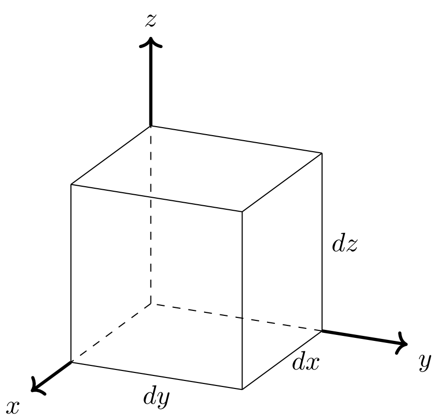
Fluid statics studies the balance between pressure force and other forces such as gravitational force. In fluid statics the velocity of the fluid is 0 at each point of space and time.

$$\vec{v} = \vec{0} \qquad\qquad \vec{v}(x, y, z, t) = 0$$

Consider a small fluid particle of sides $dx$, $dy$, $dz$ and volume $d\theta$. This would be a at a scale where curvature is not relevant, therefore it can be drawn as a cube. For this it will be necessary to know the Taylor series of a multivariable function:

$$f(\vec{r} + d\vec{r}) \cong f(\vec{r}) + \partial_x f(\vec{r})dx + \partial_y f(\vec{r})dy + \partial_z f(\vec{r})dz + \ldots$$

Considering that the fluid particle is small enough, this approximation is very accurate. Every coefficient following these first ones is much much smaller than the ones preceding it. Consider $dx$ the limit when $x$ goes to 0, but not in the mathematically rigorous fashion.




From the diagram, the force acting upon the x axis is the difference of pressure on the wall at $x_0$ and the wall at $x_0 + dx$:

$$dF_{px} = P(x_0, y_0, z_0)dydz - P(x_0 + dx, y_0, z_0)$$

Using a first order Taylor series approximation:

$$dF_{px} = (P(x_0, y_0, z_0) - P(x_0, y_0, z_0) - \partial_x P(x_0, y_0, z_0)dx)dydz$$

$$dF_{px} = -\partial_x P(x_0, y_0, z_0)dxdydz = -\partial_x P(x_0, y_0, z_0)d\theta$$

By analogy, the differential of pressure in each axis can be derived to obtain:

$$dF_{py} - \partial_y P(x_0, y_0, z_0)d\theta$$

$$dF_{pz} - \partial_z P(x_0, y_0, z_0)d\theta$$

Therefore, expressing the three components of the differential of force in one vector, one obtains:

$$d\vec{F}_p = \left( -\frac{\partial}{\partial x} P(x_0, y_0, z_0)\hat{\imath} - \frac{\partial}{\partial y} P(x_0, y_0, z_0)\hat{\jmath} - \frac{\partial}{\partial z} P(x_0, y_0, z_0)\hat{k} \right) d\theta$$

Where one should recognize the sum of partial derivatives ad the gradient and can express the latter in the following way:

$$d\vec{F}_p = \vec{\nabla} P d\theta \quad \Longrightarrow \quad \boxed{\frac{d\vec{F}_p}{d\theta} = -\vec{\nabla} P}$$

The result is the fundamental law of hydrostatics. This equation can be specified to the case where there is a gravitational field and the fluid has a density $\rho$.

$$d\vec{F}_p = d\vec{F}_g \quad \Longrightarrow \quad \boxed{\rho\vec{g} = -\vec{\nabla} P}$$

### 2.1.1 Fundamental law of Hydrostatics for liquids

In liquids $\rho$ is considered to be constant (liquids are practically incompressible).

$$\frac{\partial P(z)}{\partial z} = -\rho g \quad \Longrightarrow \quad \Delta P = -\rho g \Delta z$$

### 2.1.2 Fundamental law of Hydrostatics for gases

In gases one cannot consider the density $\rho$ constant. En example could be done considering that the density will decrease linearly with altitude as it does in the standard atmosphere model.The equation for the pressure distribution in this atmosphere could be the following considering the law of ideal gases:

$$\rho\vec{g} = \vec{\nabla} P, \quad \vec{g} = g\hat{k} \quad \Longrightarrow \quad -\frac{\partial P(z)}{\partial z} - \frac{\partial P(z)}{\partial y} - \frac{\partial P(z)}{\partial x} = \rho g \hat{k} \quad \Longrightarrow \quad -\frac{\partial P(z)}{\partial z} = \rho g \quad \Longrightarrow \quad \frac{dP(z)}{dz} = -\rho g$$

$$P(z) = R_{air}\rho(z)T(z) \quad \Longrightarrow \quad \frac{dP(z)}{dz} = -\rho g = -g\frac{P(z)}{R_{air}T(z)}$$

Supposing now that the temperature decreases linearly with altitude (as it does in the standard atmosphere model), one can arrive at the equation for the pressure distribution in this atmosphere, on submerged under any other fluid if the correct constants are known ($R_{fluid}$):

$$T(z) = T_0 - \beta z$$

$$\frac{dP(z)}{P(z)} = -g\frac{dz}{R_{air}(T_0 - \beta z)} \quad \Longrightarrow \quad \int_{P_0}^{P(z)} \frac{dP(z)}{P(z)} = \int_{z_0}^{z} -g\frac{dz}{R_{air}(T_0 - \beta z)}$$

$$\ln\left(\frac{P(z)}{P_0}\right) = -\frac{g}{R_{air}}\int_{z_0}^{z}\frac{dz}{T_0 - \beta z} = \frac{g}{R_{air}T_0}\frac{T_0}{\beta}\ln\left(1 - \frac{\beta}{T_0}z\right) = \frac{g}{R_{air}\beta}\ln\left(1 - \frac{\beta}{T_0}z\right)$$

$$\boxed{P(z) = P_0\left(1 - \frac{\beta}{T_0}z\right)^{\frac{g}{R_{air}}}}$$

## 2.2 Forces on a planar surface

Free Surface

From the diagram one can observe the following relation:

$$P = P_0 + \rho g(h_s - z)$$

One obtains the force acting upon the surface $S$ by adding up all differential areas $dS$, therefore integration over that surface:

$$\vec{F} = -\int_S P\hat{r}dS = -\hat{r}\int_S PdS = -\hat{r}\int_S [P_0 + \rho g(h_s - z)] = -\hat{r}\left[P_0 + \rho gh\right]\int_S dS + \hat{r}\rho g\int_S zdS$$

$$\vec{F} = S(P_0 + \rho g h_{cg} = S(P_0 + \gamma g h_{cg}) = \boxed{\vec{F} - \hat{r}SP_{cg}}$$

## 2.3 Forces on a curved surface

As the topic in concern is hydrostatics, the sum of all forces must be equal to 0:

$$\sum F = 0$$

One must consider the weight of the fluid above the surface that is being acted on

$$\vec{F} = \vec{F_h} + \vec{F_v}$$

## 2.4 Forces on a completely submerged object

Any object that is submerged in a fluid can be divided into multiple parts and it can be analyzed their vertical and horizontal forces. Considering the gradient in pressure for one horizontal side of the object is exactly the same as for the other, the net horizontal force experienced will be 0. This is seen any time an object is submerged in a fluid, it will not acquire a horizontal acceleration in hydrostatic conditions. It's also a consequence of conservation of energy: energy cannot be created from nowhere, therefore the object cannot start moving in static conditions. A person for example does not feel a force from the air pushing them horizontally unless there is wind, regardless of their geometry. Objects falls down because of their potential energy, so this justifies their vertical movement (if in a gravitational field and certain conditions are met). Any object that is completely submerged experiences a buoyancy force equal to the weight of the fluid displaced. This is also known as Archimedes's principle.

$$F = \rho g V_{sub}$$

For objects submerged in two different fluids (air and water is the case of any boat that does not sink), one must take into account the buoyancy force from both liquids.

$$F = (\rho_1 V_1) + \rho_2 V_2$$

# 3 Fluid Dynamics

## 3.1 Eulerian vs Lagrangian description of flow

- **Euler**: The dynamics is described in a fixed coordinate system:

$$\vec{v}(\vec{x}, t), \quad p(\vec{x}, t), \quad \rho(\vec{x}, t), \quad \ldots$$

- **Lagrange**: The dynamics is described in a moving coordinate system following an individual particle:

$$\vec{v_p}(t), \quad p_p(t), \quad \rho(\vec{x}, t), \quad \ldots$$

## 3.2 Stokes Derivative

The stokes derivative is a way of approximating the time derivative of a function in a discrete fashion using multidimensional Taylor series expansion of the first order. The formal mathematical description of a derivative is the following:

$$\lim_{h \to 0} \frac{F(x_0 + h) - F(x_0)}{h}$$

In the multivariable Stokes Derivative case, the step to be taken is not in one specific direction as it would be in the case of a partial derivative, but a step in all 4 dimensions of space and time:

$$\lim_{\Delta t \to 0} \frac{F(x_0 + \Delta t, y_0 + \Delta t, z_0 + \Delta t, t + \Delta t)}{\Delta t}$$

Therefore the Stokes Derivative will be defined and written (in capital letters) in the following manner:

$$\frac{DF}{DT} = \lim_{\Delta t \to o} \frac{F\left(x(t + \Delta t), y(t + \Delta t), x(t + \Delta t), t + \Delta t\right)}{\Delta t}$$

Taylor expansion can express what the functions evaluated at forward step $\Delta t$:

$$x(t+\Delta t) \cong x(t) + \Delta t \frac{\partial x(t)}{\partial t} + \ldots$$

$$x(t+\Delta t) \approx x(t) + \Delta t v_x(t)$$

Using the latter, the definition of the Stokes Derivative can be expanded:

$$\frac{DF}{DT} = \lim_{\Delta t \to 0} \frac{F\left(x(t)+\Delta t v_x(t), y(t)+\Delta t v_y(t), z(t)+\Delta t v_z(t), t+\Delta t\right)}{\Delta t}$$

$$\frac{DF}{DT} = \lim_{\Delta t \to 0} \frac{\frac{\partial F}{\partial x}\Delta t v_x(t) + \frac{\partial F}{\partial y}\Delta t v_y(t) + \frac{\partial F}{\partial z}\Delta t v_z(t) + \frac{\partial F}{\partial T}\Delta t}{\Delta t} = \boxed{\frac{DF}{DT} = \frac{\partial F}{\partial t} + v_x \frac{\partial F}{\partial x} + v_y \frac{\partial F}{\partial y} + v_z \frac{\partial F}{\partial z}}$$

This derivative is used to calculate velocity and acceleration in a fluid, knowing that these are both derivatives of the particles position. If the velocity or acceleration field is knows one could also obtain the position function, but would have to probably numerically approximate the values in a discrete fashion, considering the analytic solution to the partial differential equation would be extremely hard to obtain or directly unobtainable.

$$\vec{a}(x,y,z,t) = \frac{D\vec{v}(x,y,z,t)}{DT} = \frac{D}{DT}(v_x\hat{i} + v_y\hat{j} + v_z\hat{k}) = \frac{Dv_x}{DT}\hat{i} + \frac{Dv_y}{DT}\hat{j} + \frac{Dv_z}{DT}\hat{j}$$

If one expands on the partial derivatives they would obtain:

$$\begin{cases} a_x = \partial_t v_x + v_x \partial_x v_x + v_y \partial_y v_x + v_z \partial_z v_x \\ a_y = \partial_t v_y + v_x \partial_x v_y + v_y \partial_y v_y + v_z \partial_z v_y \\ a_z = \partial_t v_z + v_x \partial_x v_z + v_y \partial_y v_z + v_z \partial_z v_z \end{cases}$$

Or what is equivalent:

$$\boxed{\vec{a} = \frac{\partial v}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v}}$$

Where $(\vec{v} \cdot \vec{\nabla} \neq \vec{\nabla} \cdot \vec{v})$, or in other words, is not the divergence of the velocity field.

## 3.3 Particle flow visualization

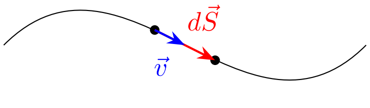
### 3.3.1 Pathlines

A pathline is the actual position a particle follows in time.

$$\frac{dx}{dt} = v_x\left(x(t), y(t), z(t), t\right) \qquad \frac{dx}{dt} = v_y\left(x(t), y(t), z(t), t\right) \qquad \frac{dz}{dt} = v_z\left(x(t), y(t), z(t), t\right)$$

### 3.3.2 Streamlines

Streamlines are lines tangent to all the velocity vectors of each fluid particle at a certain moment in time. Pathlines take into account the time that passes for a particle to change positions, while streamlines are a frozen instant in time. It is like considering a snapshot of the fluid and drawing a line tangent to all the velocity vectors of each fluid particle.




As observed in the diagram, for streamlines vectors $d\vec{S}$ and $\vec{v}$ are always parallel. Another way of expressing that is via the cross product, this will allow expansion and derivation of a relation that yields streamlines:

$$d\vec{S} \times \vec{v} = 0$$

$$(dx, dy, dz)(v_x, v_y, v_z) = 0 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx & dy & dz \\ v_x & v_y & v_z \end{vmatrix} = 0$$

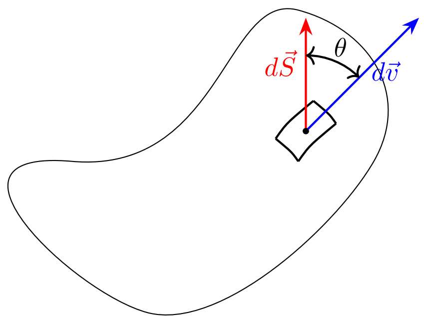
That yields a system of equation once the determinant has been expanded and separated in axes.

$$\begin{cases} v_z dy - v_y z = 0 \\ v_x dz - v_z dz = 0 \\ vy dx - v_x dy = 0 \end{cases} \implies \boxed{\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z}}$$

If the flow is steady both pathlines and streamlines will be the same.

## 3.4 Volume rate of flow and mass flux

The volume rate of flow $Q$ and mass flux $\dot{m}$ is always defined relative to a certain arbitrary surface $S$. It does not make sense to describe these how much fluid flows through without defining what surface it is penetrating.




From the diagram one can deduce an expression for the differential of volume flowing through the surface $S$:

$$dV = v dt dS \cos\theta = (\vec{v} \cdot \hat{n}) dS dt$$

To now obtain the volume rate of flow one must integrate over the surface $S$:

$$Q = \int_S (\vec{v} \cdot \hat{n}) dS = \boxed{Q = \int_S V_n dS}$$

The mass flux $\dot{m}$ can be obtained via the latter result, considering $\rho = \frac{m}{v}$:

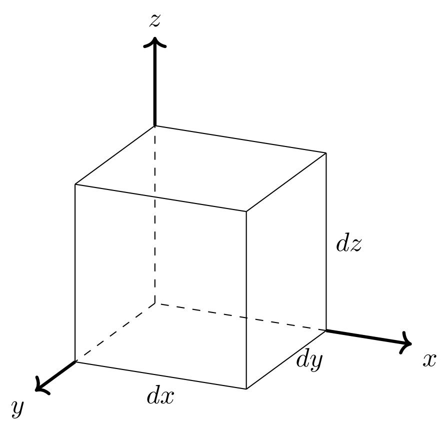
$$\dot{m}_s \rho = Q, \implies \boxed{\dot{m} = \int_S \rho V_n dS}$$

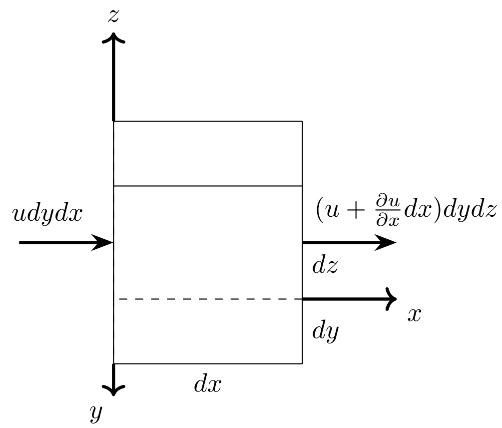
## 3.5 Velocity Divergence

The definition of the velocity divergence could be the ratio between the amount of kinetic energy that enters a given volume and the amount that leaves. One could argue that this should always be the same, but they would not be taking into account compressibility effects (the density can augment in the volume of control). The formal definition of the divergence of a vector field is the following:

$$\vec{\nabla} \cdot F(x, y, z) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}$$

To derive this, one can consider the control volume cube of a differential of $d\theta$, with sides $dx$, $dy$, $dz$. The volume rate of flow $dQ$ is the difference between the flow through the first wall and the second.



The latter can be generalized for the three dimensions and one would obtain the following result.

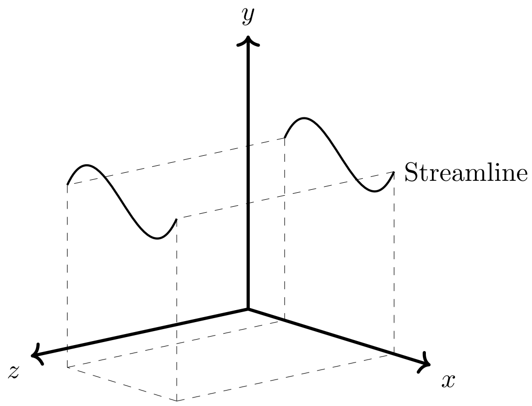
$$dQ = -v_x dydz + (v_x + dx\partial_x v_x)dydz - v_y dxdz + (v_y + dy\partial_y v_y)dxdz - v_z dxdy + (v_z + dz\partial_z v_z)dxdy$$

$$dQ = \partial_x v_x dxdydz + \partial_y v_y dxdydz + \partial_z v_z dxdydz = d\theta\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

$$\boxed{\frac{dQ}{d\theta} = \vec{\nabla}\cdot\vec{V}}$$

### 3.6 Streamfunction

The streamfunction $\psi$ is a way to encode the velocity of a 2D incompressible flow at each point in one single number.The flow being 2D does not mean that it cannot be a three dimensional problem, however the streamlines are identical in the third dimension. If the flow is incompressible, therefore $\vec{\nabla}\cdot\vec{V} = 0$.




$$\vec{\nabla}\cdot\vec{V} = 0 \implies \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(v_x, v_y, v_z) = 0 = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right)\cdot(v_x, v_y, 0) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Therefore, one can impose the following for the divergence of $\vec{V}$ to be 0:

$$\begin{cases} v_x(x,y) = \partial_y\psi(x,y) \\ v_y(x,y) = -\partial_x\psi(x,y) \end{cases} \implies \vec{\nabla}\cdot\vec{V} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\psi(x,y) - \frac{\partial}{\partial y}\frac{\partial}{\partial x}\psi(x,y) = 0$$

Therefore, another way of expressing the streamfunction is:

$$\boxed{\vec{V} = \vec{\nabla}\times\psi(x,y)\hat{z}}$$

To compute the streamfunction for a given velocity field one can use the fundamental theorem of calculus:

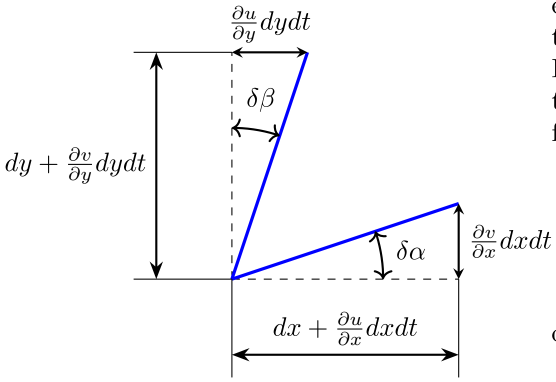
$$\psi_2 - \psi_1 = \int_{\vec{r}_1}^{\vec{r}_2} d\vec{r}\cdot\vec{\nabla}\psi = \int_{\vec{r}_1}^{\vec{r}_2}\left(\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy\right) = \int_{\vec{r}_1}^{\vec{r}_2}(-v_y dx + v_x dy)$$

For the integration limits any path that leads from $\vec{r}_1$ to $\vec{r}_2$ can be used. The simplest approach in most cases however is to use Cartesian coordinates and move from $(x_1, y_1)$ to $(x_2, y_2)$ moving a first step in $x$ maintaining $y$ constant and then in $y$ maintaining $x$ constant (or vice versa)

$$\int_{\vec{r}_1}^{\vec{r}_2}(-v_y dx + v_x dy) = \int_{(x_1,y_1)}^{(x_2,y_1)}[-v_y(x, y_1)]\,dx + \int_{(x_2,y_1)}^{(x_2,y_2)}[-v_x(x_2, y)]\,dy$$

## 3.7 Rotation and Deformation

To understand the motion of a fluid particle one must first understand deformation and rotation, seeing as the motion is analyzed by decomposing the motion into these two categories.




The diagram corresponds to a fluid particle experiencing a rotation and a possible deformation. From the diagram one can obtain the angular velocity $\vec{\omega}$. Firstly observe that if one considers positive rotation to be counterclockwise (regular convention) the following expression is obtained:

$$\omega_z = \frac{1}{2}\left(\frac{d\alpha}{dt} - \frac{d\beta}{dt}\right)$$

Assuming small values of d$\alpha$, using taylor expansion one can obtain:

$$d\alpha = \tan d\alpha = \frac{\partial v}{\partial x}dt$$

Using some trigonometry one can obtaun the following:

$$\frac{d\alpha}{dt} = \frac{\partial v}{\partial x} \quad \frac{d\beta}{dt} = \frac{\partial u}{\partial y} \quad \Longrightarrow \quad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

The latter can be generalized for the other dimensions in the following manner:

$$\frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} - \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}\right] = \frac{1}{2}\vec{\nabla}\times\vec{v}$$

$$\boxed{\vec{\omega} = \frac{1}{2}\vec{\nabla}\times\vec{v}}$$

The tensor for the motion of a particle $\bar{\bar{J}}$ can be written in the following manner:

$$\bar{\bar{J}} = \begin{pmatrix} \partial_x u & \partial_y u & \partial_z u \\ \partial_x v & \partial_y v & \partial_z v \\ \partial_x w & \partial_y w & \partial_z w \end{pmatrix} = \frac{1}{2}\left(\bar{\bar{J}} - \bar{\bar{J}}^T\right) + \frac{1}{2}\left(\bar{\bar{J}} + \bar{\bar{J}}^T\right)$$

It is decomposed in these two other matrices, symmetric and asymmetric. If one expands on these two different matrices:

$$\frac{1}{2}\left(\bar{\bar{J}} - \bar{\bar{J}}^T\right) = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ -\frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ -\frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) & -\frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

$$\frac{1}{2}\left(\bar{\bar{J}} + \bar{\bar{J}}^T\right) = \begin{pmatrix} \partial_x u & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \partial_y v & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & \partial_z w \end{pmatrix}$$

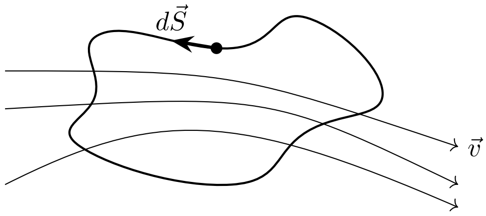
The first matrix corresponds to a purely rotational matrix, the second one corresponds to a deformation. Therefore one can express the motion of a particle as a combination of both these transformations. The shear deformation itself can also be expressed as a combination of two other matrices: a uniform expansion proportional to $\vec{\nabla}\cdot\vec{v}$ and a shear without volume change.

## 3.8 Circulation

The circulation in a quantity measuring the tendency of V to be aligned against a closed path C.

The circulation $\Gamma$ is defined as:

$$\Gamma = -\oint_C \vec{V} \cdot dS$$




## 3.9 Stokes Theorem

Stokes Theorem allows change from line integrals along a closed path to surface integrals:

$$\oint_C d\vec{r} \cdot \vec{A} = \iint_S \left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n} dS \quad \Longrightarrow \quad \Gamma_c = -\oint_C \vec{v} \cdot d\vec{S} = -\iint_s \left(\vec{\nabla} \times \vec{v}\right) \cdot \hat{n} dS$$

Where $S$ is an arbitrary surface that has $C$ as it's boundary and $\hat{n}$ is the normal vector to the surface $S$.

As is observed, Stokes theorem connects vorticity (curl) with circulation:

$$\hat{n}\left(\vec{\nabla} \times \vec{v}\right) = \frac{\oint \vec{v} d\vec{r}}{dS} \qquad d\Gamma = -\left(\vec{\nabla} \times \vec{v}\right) dS$$

## 3.10 Velocity Potential

If the flow is irrotational ($\vec{\nabla} \times \vec{v} = 0$), there exists a function $\phi$ whose gradient is equal to the velocity field.

$$\vec{\nabla} \times \left(\vec{\nabla}\phi\right) = 0 \quad \Longrightarrow \quad \vec{v} = \vec{\nabla}\phi$$

As in the streamfunction, one can obtain the velocity potential in the following fashion, by moving using Cartesian coordinates, first in $x$ and then in $y$ or vice versa.

$$\phi_2 - \phi_1 = \int_{\vec{r}_1}^{\vec{r}_2} d\vec{r} \cdot \vec{\nabla}\phi = \int_{\vec{r}_1}^{\vec{r}_2} d\vec{r} \cdot \vec{v}$$

Lines of constant potential are called equipotential lines, and those tangent to the velocity are always perpendicular to the equipotential ones.

## 3.11 Cylindrical Coordinates

For all problems in cylindrical coordinates, one can consider the following relation between these and the Cartesian coordinates:

$$\begin{cases} x = \rho \cos\phi \\ y = \rho \sin\phi \\ z = z \end{cases}$$

The Nabla or Del operator $\vec{\nabla}$ can be written considering the change in angle $\phi$ depends on the radius $\rho$. The case is that this quantity $\phi$ must be expressed as $\rho\phi$:

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{\partial}{\partial\rho\phi} + \hat{z}\frac{\partial}{\partial z} = \boxed{\frac{\partial}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{z} = \vec{\nabla}}$$

Therefore, expressions for the gradient, divergence and curl can be obtained in cylindrical coordinates using the latter operator.

### 3.11.1 Gradient

The gradient of a vector field $f$ is described as $\vec{\nabla} f$. Using the definition for $\vec{\nabla}$ obtained in the latter section, one reaches:

$$\vec{\nabla} f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$$

### 3.11.2 Divergence

The divergence of a vector field $f$ is described as $\vec{\nabla} \cdot f$. If once again one uses the definition of $\vec{\nabla}$ in cylindrical coordinates they obtain:

$$\begin{aligned}
\vec{\nabla} \cdot \vec{f} = \left(\frac{\partial}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{z}\right) \cdot \left(f_\rho \hat{\rho} + f_\phi \hat{\phi} + f_z \hat{z}\right) = \hat{\rho}\left[\frac{\partial}{\partial \rho}\left(f_\rho \hat{\rho} + f_\phi \hat{\phi} + f_z \hat{z}\right)\right] + \\
\frac{1}{\rho}\hat{\phi}\left[\frac{\partial}{\partial \phi}\left(f_\rho \hat{\rho} + f_\phi \hat{\phi} + f_z \hat{z}\right)\right] + \hat{z}\left[\frac{\partial}{\partial z}\left(f_\rho \hat{\rho} + f_\phi \hat{\phi} + f_z \hat{z}\right)\right] = \\
\hat{\rho}\left[\left(\hat{\rho}\frac{\partial f_\rho}{\partial \rho} + f_\rho \frac{\partial \hat{\rho}}{\partial \rho}\right) + \left(\hat{\phi}\frac{\partial f_\phi}{\partial \rho} + f_\phi \frac{\partial \hat{\phi}}{\partial \rho}\right) + \left(\hat{z}\frac{\partial f_z}{\partial \rho} + f_z \frac{\partial \hat{\rho}}{\partial \rho}\right)\right] + \\
\frac{1}{\rho}\hat{\phi}\left[\left(\hat{\phi}\frac{\partial f_\rho}{\partial \phi} + f_\rho \frac{\partial \hat{\rho}}{\partial \phi}\right) + \left(\hat{\phi}\frac{\partial f_\phi}{\partial \phi} + f_\phi \frac{\partial \hat{\phi}}{\partial \phi}\right) + \left(\hat{z}\frac{\partial f_z}{\partial \phi} + f_z \frac{\partial \hat{\rho}}{\partial \phi}\right)\right] + \\
\hat{z}\left[\left(\hat{\phi}\frac{\partial f_\rho}{\partial z} + f_\rho \frac{\partial \hat{\rho}}{\partial z}\right) + \left(\hat{\phi}\frac{\partial f_\phi}{\partial z} + f_\phi \frac{\partial \hat{\phi}}{\partial z}\right) + \left(\hat{z}\frac{\partial f_z}{\partial z} + f_z \frac{\partial \hat{\rho}}{\partial z}\right)\right]
\end{aligned}$$

Considering the following relations:

$$\begin{array}{ccc}
\frac{\partial}{\partial \rho}(\hat{\rho}) = 0 & \frac{\partial}{\partial \rho}\left(\hat{\phi}\right) = 0 & \frac{\partial}{\partial \rho}(\hat{z}) = 0 \\
\frac{\partial}{\partial \phi}(\hat{\rho}) = \hat{\phi} & \frac{\partial}{\partial \phi}\left(\hat{\phi}\right) = -\hat{\rho} & \frac{\partial}{\partial \phi}(\hat{z}) = 0 \\
\frac{\partial}{\partial z}(\hat{\rho}) = 0 & \frac{\partial}{\partial z}\left(\hat{\phi}\right) = 0 & \frac{\partial}{\partial z}(\hat{z}) = 0
\end{array}$$

$$\hat{\rho} \cdot \hat{\rho} = \hat{\phi} \cdot \hat{\phi} = \hat{z} \cdot \hat{z} = 1$$

$$\hat{\rho} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{z} = \hat{\rho} \cdot \hat{z} = 0$$

The divergence of the vector field $f$ is:

$$\vec{\nabla} \cdot \vec{f} = \frac{\partial f_\rho}{\partial \rho} + \frac{f_\rho}{\rho} + \frac{1}{\rho}\frac{\partial f_\phi}{\partial \phi} + \frac{\partial f_z}{\partial z} = \boxed{\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho f_\rho) + \frac{1}{\rho}\frac{\partial f_\phi}{\partial \phi} + \frac{\partial f_z}{\partial z} = \vec{\nabla} \cdot \vec{f}}$$

### 3.11.3 Curl

The curl is derived in the same way as the divergence, taking into account the Nabla operator and constructing the vector product, expanding on the determinant:

### 3.11.4 Stokes Derivative

# 4 Equations in Integral Form

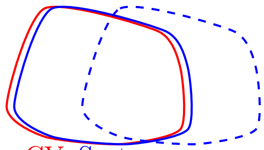
## 4.1 Reynolds Transport Theorem

The Reynolds Transport Theorem deals with extensive properties, those whose sum of the partial parts gives the total. Examples of properties that are extensive can be energy, mass or volume. Some properties that are not extensive are density, temperature or pressure. The theorem has been used to help derive the laws of conservation such as the conservation of momentum, mass, energy, etc.

Any property $B$ that is extensive can be written in the following manner:

$$B = \int_{V_{sys}} dB = \int_{V_{sys}} \rho\beta d\theta$$

Where $\beta$ is an intensive property that corresponds to the amount of $B$ per unit mass.




The derivatives of these values $B$ can be expressed in the rigorous mathematical way:

$$\dot{B}_{sys}(t_0) = \lim_{\Delta t\to 0} \frac{B_{sys}(t_0+\Delta t) - B_{sys}(t_0)}{\Delta t} \qquad \dot{B}_{cv}(t_0) = \lim_{\Delta t\to 0} \frac{B_{cv}(t_0+\Delta t) - B_{cv}(t_0)}{\Delta t}$$

From the diagram, one can obtain the relation between the values of the system at a second state and the control volume:

$$B_2(t_0) = B_{cv}(t_0) \qquad B_{cv}(t_0+\Delta t) = B_2(t_0+\Delta t) - dB_{out} + dB_{in}$$

The differential in volume will be given by the section that a given fluid flows thought, and the velocity of that section times $\Delta t$, to give a "width":

$$d\theta_{out} = A_{out}v_{out}\Delta t \qquad d\theta_{in} = A_{in}v_{in}\Delta t$$

The differential amount of $B$ that leaves the system can be expressed as the density times the amount of $B$ per unit area multiplied times the differential in volume

$$dB_{out} = A_{out}v_{out}dt\rho_{out}\beta_{out} \qquad dB_{in} = A_{in}v_{in}dt\rho_{in}\beta_{in}$$

From all these previous equations, one can obtain the relation between the change per unit time for $B_{sys}$ and $B_{cv}$:

$$\frac{d}{dt}\left(B_{sys}(t_0)\right) = \frac{d}{dt}\left(B_{cv}(t_0)\right) + (\beta\rho vA)_{out} - (\beta\rho vA)_{in}$$

To generalize the case, consider adding each differential of volume coming into the control volume and going out.

$$\theta_{out} = \int_S (\hat{n}\cdot\vec{v})\Delta t dA_{out} \quad \theta_{in} = \int_S (\hat{n}\cdot\vec{v})\Delta t dA_{in}$$

These values of $\theta$ can be compressed into one single one like following:

$$\oint_{CS} \beta\rho(\vec{v}\cdot\hat{n}dA)$$

The latter surface integral contains how much $B$ comes into the system. Therefore the change in the extensive property $B$ in time can be expressed as:

$$\frac{d}{dt}\left(B_{sys}(t_0)\right) = \frac{d}{dt}\left(B_{cv}(t_0)\right) + \oint_{CS} \beta\rho(\vec{v}\cdot\hat{n}dA)$$

The generalization for a moving control volume consists in taking the relative velocity $\vec{v} - \vec{V}_{cs}$, and expressing the time derivative of $B$ as:

$$\frac{d}{dt}B_{cv}(t_0) = \frac{d}{dt}\int\limits_{CV(t)} \rho\beta d\theta$$

Therefore the end expression for the change in the extensive property $B$ in a system will be the following:

$$\boxed{\frac{d}{dt}\left(B_{sys}\right) = \frac{d}{dt}\int\limits_{CV(t)} \beta\rho d\theta + \oint\limits_{CS} \beta\rho\left[\left(\vec{v} - \vec{v}_{cv}\right)\cdot\hat{n}\right] dA}$$

If the control volume is not moving (fixed in space), the time derivative outside the integral can be written inside and the velocity of the control volume $\vec{v}_{cv}$ is 0:

$$\boxed{\frac{d}{dt}\left(B_{sys}\right) = \int\limits_{CV(t)} \frac{\partial}{\partial t}\left(\beta\rho\right) d\theta + \oint\limits_{CS} \beta\rho\left(\vec{v}\cdot\hat{n}\right) dA}$$

## 4.2 Mass Conservation

The Reynolds Transportation Theorem is purely mathematical, but it will be used in this subsection to derive the laws of conservation of mass. The extensive property $B$ will be expressed as the mass, and the intensive property $\beta$ will correspond to the amount of mass per unit mass, therefore $\beta = 1$. The change in mass per unit time has to be 0, considering no mass is destroyed or created.

$$\left(\frac{dm}{dt}\right)_{sys} = \frac{d}{dt}\int\limits_{CV(t)} \rho d\theta + \oint\limits_{CS} \rho\left[\left(\vec{v} - \vec{v}_{cv}\right)\cdot\hat{n}\right] dA$$

If one assumes that the control volume is fixed in space, the latter equation simplifies to:

$$\left(\frac{dm}{dt}\right)_{sys} = \int\limits_{CV(t)} \frac{\partial}{\partial t}\left(\rho\right) d\theta + \oint\limits_{CS} \rho\left(\vec{v}\cdot\hat{n}\right) dA$$

If one now assumes that the flow is quasi 1D, meaning all quantities like $\rho$ only depend on the $x$ value and on $t$.

$$\rho(x,t) \quad P(x,t) \quad \vec{v}(x,t) = \hat{i}v_x(x,t), \quad \ldots$$

$$\oint\limits_{CS} \rho\left(\vec{v}\cdot\hat{n}\right) dA = \sum_{i\in out} \left(\rho v A\right)_i - \sum_{i\in in} \left(\rho v A\right)_i = 0$$

$$\left(\frac{dm}{dt}\right)_{sys} = \int\limits_{CV(t)} \frac{\partial}{\partial t}\left(\rho\right) d\theta + \sum_{i\in out} \left(\rho v A\right)_i - \sum_{i\in in} \left(\rho v A\right)_i = 0$$

If one now continues to assume conditions, in this case that the flow is static (not depending on time), they reach the following:

$$\boxed{\sum_{i\in out} \left(\rho v A\right)_i - \sum_{i\in in} \left(\rho v A\right)_i = 0}$$

## 4.3 Momentum conservation

The Reynolds Transportation Theorem provides equations for the time derivative of an extensive property. To obtain an equation that relates forces, one can make use of the following property:

$$\frac{d\vec{P}}{dt} = \sum \vec{F}$$

Therefore, if the extensive property $B$ is given as $\vec{P}$, one can obtain the sum of forces acting upon control volume taken. $\beta$ will therefore be $\vec{v}$ (the amount of momentum per unit mass).

$$\boxed{\frac{d\vec{P}_{sys}}{dt} = \frac{d}{dt}\int\limits_{CV(t)} \rho\vec{v}d\theta + \oint\limits_{CS} \rho\vec{v}\left(\vec{v}\cdot\hat{n}\right)dA = \sum\vec{F}}$$

As in the mass conservation, assuming steady flow the time derivative of the volume integral will be 0, leaving only the surface contributions. If one assumes the case is quasi 1D, they can simplify the equation in the following manner:

$$\boxed{\sum_{i\in out}\left(\dot{m}\vec{v}\right)_i - \sum_{i\in in}\left(\dot{m}\vec{v}\right)_i = \sum\vec{F}}$$

The forces acting upon the control volume consist in surface forces and others:

- Pressure force
- Shear Force
- Weight
- Ficticious forces

Pressure forces are given assuming a control surface of any shape, understanding pressure is always perpendicular and of opposite direction to the surface if the normal is taken outwards, therefore in the case of the closed control volume:

$$d\vec{F_P} = -P\hat{n}dA \quad \Longrightarrow \quad \vec{F_P} = -\oint\limits_{CS} P\hat{n}dA$$

In this integral, the only relevant fact is the difference of contributions on each individual side, not the absolute value. Therefore one can substract a constant from the pressure and obtain an equaly valid expression:

$$\vec{F_P} = -\oint\limits_{CS}(P-P_0)\hat{n}dA$$

Gravitational force is defined in a constant gravitational field as $m\vec{g}$, therefore generalizing the case for any shape, knowing the density distribution $\rho$:

$$\vec{F_g} = m\vec{g} \quad \Longrightarrow \quad \vec{F_g} = \vec{g}\int\limits_{CV}\rho d\theta = -g\hat{z}\int\limits_{CV}\rho d\theta$$

Fictitious forces are given in acceleration form:

$$a_{rel} = \frac{d^2R}{dt^2} + \frac{d\Omega}{dt}\times r + 2\Omega\times v + \Omega\times(\Omega\times r) \quad \Longrightarrow \quad \vec{F}_{rel} = \int\limits_{CV} a_{rel}\rho d\theta$$

Viscous forces are left untouched in this unit, further details are given when deriving the equations in differential form.

$$Shear\ Force = \vec{F_\mu}$$

By substituting all of the latter values in the equation obtain before, one obtains the complete integral form of the momentum conservation:

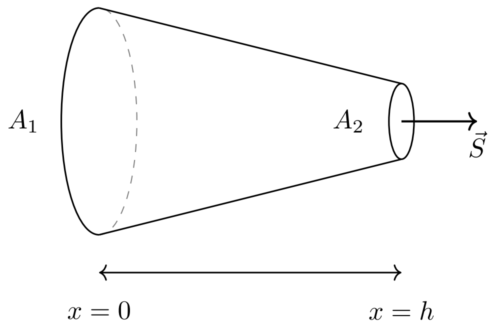
$$\frac{d}{dt}\int\limits_{CV(t)}\rho\vec{v}d\theta + \oint\limits_{CS}\rho\vec{v}\left(\vec{v}\cdot\hat{n}\right)dA = g\hat{z}\int\limits_{CV}\rho d\theta - \oint\limits_{CS}(P-P_0)\hat{n}dA - \int\limits_{CV}a_{rel}\rho d\theta + \vec{F_\mu}$$

$$\frac{d}{dt}\int\limits_{CV(t)}\rho\vec{v}d\theta + \oint\limits_{CS}\rho\vec{v}\left(\vec{v}\cdot\hat{n}\right)dA + \oint\limits_{CS}(P-P_0)\hat{n}dA = -\int\limits_{CV}\left(g\hat{z}-a_{rel}\right)\rho d\theta + \vec{F_\mu}$$

$$\boxed{\frac{d}{dt}\int\limits_{CV(t)}\rho\vec{v}d\theta + \oint\limits_{CS}\left[\rho\vec{v}\left(\vec{v}\cdot\hat{n}\right) + \left(P-P_0\right)\hat{n}\right]dA = -\int\limits_{CV}\left(g\hat{z}-a_{rel}\right)\rho d\theta + \vec{F_\mu}}$$

## 4.4 Bernoulli's equation

In order to derive Bernoulli's equation, consider a CV fixed in spaced around a streamline (like a small streamtube). The streamline is directed into an arbitrary direction $\hat{S}$ and is inclined an angle $\theta$, the control volume has a variable $A(s)$ and a length $ds$.




One can assume this is a quasi 1D situation with an inlet and an outlet. The volume of the CV is given by:

$$V_{CV} = \frac{h}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right)$$

$$d\theta_{CV} = \frac{ds}{3}\left(A + (A + dA) + \sqrt{A(A + dA)}\right)$$

Knowing $dA$ is a second order differential, it can be disregarded:

$$d\theta_{CV} = AdS$$

Applying the conservation of mass equations:

$$\frac{\partial}{\partial t}\rho d\theta_{CV} + \rho_2 v_2 A_2 - \rho_1 v_1 A_1 = 0$$

Where the inlet and outlet mass flux can be written as a differential:

$$d(\rho v A) = -AdS\frac{\partial \rho}{\partial t}$$

Using the momentum equation with the same assumptions as the ones used to evaluate the conservation of mass.

$$\int\limits_{CV} \frac{\partial}{\partial t}\left(\rho \vec{v}\right) d\theta_{CV} + \left(\vec{v}\rho v A\right)_2 - \left(\vec{v}\rho v A\right)_1 = \sum \vec{F}$$

The integral can be discarded, as considering the total volume is still a differential, one could express the equation in the following form:

$$\frac{\partial}{\partial t}\left(\rho \vec{v}\right) d\theta_{CV} + d\left(\vec{v}\rho v A\right) = \sum \vec{F}$$

By projecting the latter equation in the direction of $\hat{S}$:

$$\frac{\partial}{\partial t}(\rho v_s) d\theta_{CV} + (v_s \rho A) dv_s + v_s d(v_s \rho A) \quad \Longrightarrow \quad dSA\frac{\partial}{\partial t}(\rho v_s) + (v_s \rho A) dv_s + v_s d(v_s \rho A) = \sum \vec{F}$$

The following step is to evaluate $\sum \vec{F}$. The gravitational force projected in $\hat{S}$ is given by:

$$d\vec{F}_g = -\hat{z} g \rho d\theta_{CV} \sin\theta = -\hat{z} g \rho AdS \sin\theta = -g\rho AdS\hat{z}$$

Pressure force is given by:

$$dF_{P,S} = -dP(A + dA) = -AdP$$

Therefore, putting everything together:

$$dSA\frac{\partial}{\partial t}(\rho v_s) + (v_s \rho A) dv_s + v_s d(v_s \rho A) = -\rho g A dz - AdP$$

$$\rho\frac{\partial}{\partial t}(v_s) AdS + v_s\frac{\partial}{\partial t}(\rho) AdS + (v_s \rho A) dv_s + v_s d(v_s \rho A) = -\rho g A dz - AdP$$

$$\rho\frac{\partial}{\partial t}(v_s) AdS + v_s\left(\frac{\partial}{\partial t}(\rho) AdS + d(v_s \rho A)\right) + (v_s \rho A) dv_s = -\rho g A dz - AdP$$

$$\frac{\partial}{\partial t}(v_s)\rho AdS + dv_s (v_s \rho A) = -\rho g A dz - AdP$$

Rearranging and dividing by $\rho$, one obtains the differential form of Bernoulli's equation:

$$\boxed{\frac{\partial}{\partial t}(v_s)dS + vdv = -gdz - \frac{dP}{\rho}}$$

The integral form of the equation will be given by:

$$\boxed{\int_1^2 \frac{\partial}{\partial t}(v)dS + \frac{1}{2}\left(v_2^2 - v_1^2\right) = g(z_2 - z_1) - \int_1^2 \frac{dP}{\rho}}$$

# 5 Equations in differential form

## 5.1 Conservation of Mass in Differential Form

Starting from a fixed control volume:

$$\int\limits_{CV} \frac{\partial}{\partial t}(\rho)d\theta + \oint\limits_{CS} \rho\vec{v}\cdot\hat{n}dA$$

Using the divergence theorem, stating the following:

$$\int\limits_{\partial R} \vec{F}\cdot\hat{n}dS = \int\limits_{R} \vec{\nabla}\cdot\vec{F}dV$$

One can express the control surface integral in the following manner:

$$\oint\limits_{CS} \rho\vec{v}\cdot\hat{n}dA = \int\limits_{CV} \vec{\nabla}\cdot(\rho\vec{v})d\theta \quad \xRightarrow{\text{Infinitesimal CV}} \quad \int\limits_{CV} \left(\frac{\partial}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v})\right) d\theta = 0$$

There are many functions that evaluate to 0, but when bounds are moved there is only 1 function that always computes to 0 regardless of the integration limits, the 0 function. Therefore:

$$\boxed{\frac{\partial}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0}$$

If the flow is steady one obtains:

$$\vec{\nabla}\cdot(\rho\vec{v}) = 0$$

If it is steady and incompressible:

$$\vec{\nabla}\cdot\vec{v} = 0$$

## 5.2 Conservation of Momentum in Differential Form

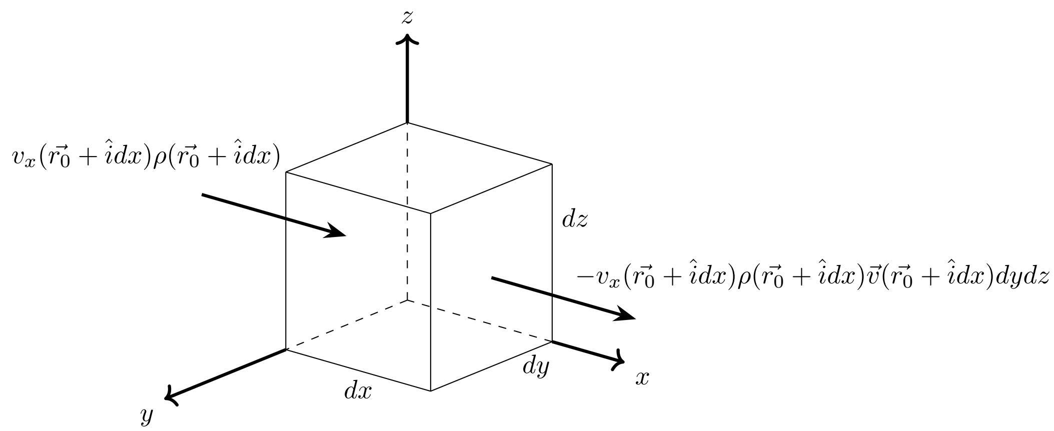
Starting from the momentum conservation equation in integral form, one applies the same steps using the divergence theorem as applied in the latter section:

$$\frac{d}{dt}\int\limits_{CV} \rho\vec{v}d\theta + \oint\limits_{CS} \rho\vec{v}(\vec{v}\cdot\hat{n})dA = \sum\vec{F}$$

Evaluating the control volume integral, considering it is fixed in space and taking the differential form:

$$\frac{d}{dt}\int\limits_{CV} \rho\vec{v}d\theta = \int\limits_{CV} \frac{\partial}{\partial t}(\rho\vec{v})\,d\theta \quad \xRightarrow{\text{Infinitesimal CV}} \quad d\theta\left(\vec{v}\frac{\partial}{\partial t}(\rho) + \rho\frac{\partial}{\partial t}(\vec{v})\right)$$

Consider now a small volume $d\theta$ that can be considered a cube taking into account that at such small size curvature is not noticeable. Using a first order Taylor vector expansion one can compute the fluxes entering and exiting the cube:




Using first order Taylor expansion:

$$v_x\left(\vec{r_0}+\hat{i}dx\right)\rho\left(\vec{r_0}+\hat{i}dx\right)\vec{v}\left(\vec{r_0}+\hat{i}dx\right)=v_x\left(\vec{r_0}\right)\rho\left(\vec{r_0}+\hat{i}dx\right)\vec{v}\left(\vec{r_0}\right)+\frac{\partial}{\partial x}\left(v_x\rho\vec{v}\right)dxdydz$$

Therefore the flux contribution from faces A and B will be:

$$\frac{\partial}{\partial x}\left(v_x\rho\vec{v}\right)dxdydz$$

The same will happen for all the other faces, therefore the total flux will be the following:

$$\left(\frac{\partial}{\partial x}\left(v_x\rho\vec{v}\right)dxdydz+\frac{\partial}{\partial y}\left(v_y\rho\vec{v}\right)dxdydz+\frac{\partial}{\partial z}\left(v_z\rho\vec{v}\right)dxdydz\right)$$

$$d\theta\left[v_x\rho\frac{\partial}{\partial x}\left(\vec{v}\right)+\vec{v}\frac{\partial}{\partial x}\left(\rho v_x\right)+v_y\rho\frac{\partial}{\partial y}\left(\vec{v}\right)+\vec{v}\frac{\partial}{\partial y}\left(\rho v_y\right)+v_z\rho\frac{\partial}{\partial z}\left(\vec{v}\right)+\vec{v}\frac{\partial}{\partial z}\left(\rho v_z\right)\right]$$

$$d\theta\left[\vec{v}\left(\vec{\nabla}\cdot\left(\rho\vec{v}\right)\right)+\rho\left(v_x\frac{\partial}{\partial x}\vec{v}+v_y\frac{\partial}{\partial y}\vec{v}+v_z\frac{\partial}{\partial z}\vec{v}\right)\right]$$

By adding the part obtained from the volume integral and the surface flux one obtains (after rearranging):

$$\frac{d}{dt}\int\limits_{CV}\rho\vec{v}d\theta+\oint\limits_{CS}\rho\vec{v}(\vec{v}\cdot\hat{n})dA\xrightarrow{\text{InfinitesimalCV}}d\theta\left[\vec{v}\left(\underbrace{\frac{\partial}{\partial t}\rho+\vec{\nabla}\cdot\left(\rho\vec{v}\right)}_{\text{Continuity Equation}}\right)+\rho\left(\frac{\partial}{\partial t}\vec{v}+v_x\frac{\partial}{\partial x}\vec{v}+v_y\frac{\partial}{\partial y}\vec{v}+v_z\frac{\partial}{\partial z}\vec{v}\right)\right]$$

Therefore, given the continuity equation has to be satisfied for mass conservation considering incompressible flows:

$$\frac{d}{dt}\int\limits_{CV}\rho\vec{v}d\theta+\oint\limits_{CS}\rho\vec{v}(\vec{v}\cdot\hat{n})dA\quad\xrightarrow{\text{Infinitesimal CV}}\quad\rho d\theta\frac{D\vec{v}}{Dt}=\sum\vec{F}$$

This result is intuitive, at least when it comes to units. The Stokes derivative gives the acceleration of the fluid, whereas the density $\rho$ and the volume (infinitesimal) $d\theta$ give a mass. What is happening is simply Newton's second law, expressed in a different way. When evaluating the forces that act on a fluid, it can be considered the pressure force $\vec{F}_p$, the gravitational force $\vec{F}_g$ and the viscous forces $\vec{F}_\mu$. Neglecting the latter, one arrives to the Euler Equation:

### 5.2.1 Euler Equation

Let it be considered the two forces just discussed:

$$\vec{F}_p=-\int\limits_{CS}P\hat{n}dA\quad\xrightarrow{\text{Infinitesimal CV}}\quad-\nabla\vec{P}d\theta$$

$$\vec{F}_g = \int\limits_{CV} \rho d\theta \quad \xrightarrow[\quad]{\text{Infinitesimal CV}} \quad \rho\vec{g}d\theta$$

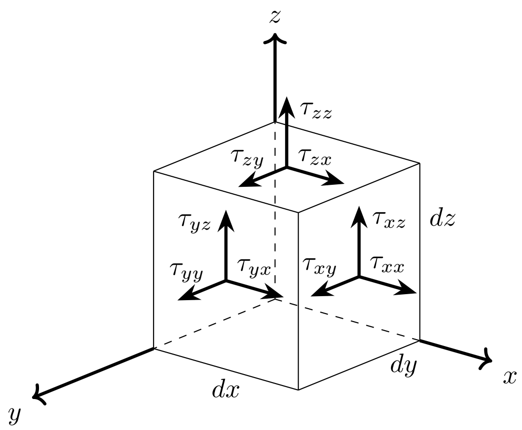
Adding the latter results one obtains the Euler Equation, valid for ideal (inviscous and incompressible) flows:

$$d\theta\rho\frac{D\vec{v}}{Dt} = \vec{g}\rho d\theta - \nabla\vec{P}d\theta \quad \Longrightarrow \quad \boxed{\rho\frac{D\vec{v}}{Dt} = \vec{g}\rho - \nabla\vec{P}}$$

### 5.2.2 Navier-Stokes Equation

As stated before-hand, the Navier-Stokes equation is the momentum conservation for viscous flow. In the case of this document, only the incompressible version of the equation will be derived and shown.

Consider there are, for every surface, 3 different directions in which stress can act for each surface in the infinitesimal cube.




Using as usual a first order Taylor expansion, all the 0 order terms cancel out (as usual).

$$\vec{F}_{\mu,x} = d\theta\left(\frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{xy}) + \frac{\partial}{\partial z}(\tau_{xz})\right)$$

To account for all the information about each surface direction, the shear stress is organised in a tensor:

$$\bar{\bar{\tau}} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

Therefore:

$$\vec{F}_\mu = d\theta\Bigg[\hat{i}\left(\frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx})\right) + \hat{j}\left(\frac{\partial}{\partial x}(\tau_{yx}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{yz})\right) + \hat{k}\left(\frac{\partial}{\partial x}(\tau_{zx}) + \frac{\partial}{\partial y}(\tau_{zy}) + \frac{\partial}{\partial z}(\tau_{zz})\right)\Bigg]$$

The latter can be written in the following manner:

$$\vec{F}_\mu = d\theta\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) \cdot \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = d\theta\left(\vec{\nabla}\cdot\bar{\bar{\tau}}\right)$$

The general conservation of momentum is therefore:

$$\boxed{\rho\frac{D}{Dt}\vec{v} = \vec{g}\rho - \vec{\nabla}P + \vec{\nabla}\cdot\bar{\bar{\tau}}}$$

However, this just adds new variables that are not known to the equation, therefore the shear stress tensor $\bar{\bar{\tau}}$ is usually expressed as a function of the velocity. There are many ways to express it, but the most usual and the one that gives the standard Navier-Stokes equations is the follwing:

$$\bar{\bar{\tau}} = \mu \nabla \vec{v}$$

The standard Navier-Stokes equation for incompressible fluids:

$$\rho \frac{D}{Dt}\vec{v} = \vec{g}\rho - \vec{\nabla} P + \vec{\nabla} \cdot \mu \nabla \vec{v}$$

$$\boxed{\rho \frac{D}{Dt}\vec{v} = \vec{g}\rho - \vec{\nabla} P + \mu \nabla^2 \vec{v}}$$

If the equations are fully unpacked into 3 dimensions the full information that is contained is shown:

$$\boxed{\begin{aligned}
x:\ \ \rho\left(\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u\right) &= \rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\\
y:\ \ \rho\left(\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + w\frac{\partial}{\partial z}v\right) &= \rho g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)\\
z:\ \ \rho\left(\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w\right) &= \rho g_z - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)
\end{aligned}}$$