A derivation of the surface area of the circle and the sphere

Ben Thomas

July 2023

1 The Cicle's Area

1.1 Cartesian Coordinates

A circle is created when a constant length is revolved around a centre of rotation. The largest distance from this centre of rotation to the end of the length is called the radius.

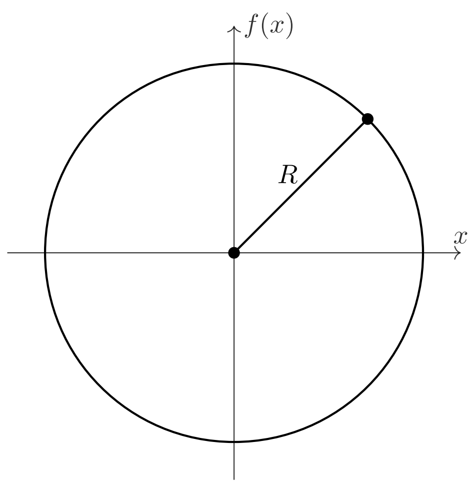

Figure 1: Circle of radius R

Using the knowledge that the radius is always constant, with Pythagoras' Theorem one can determine the equation for a circle. Therefore with that corresponding equation relating x, y and R one obtain the mathematical expression for a circle:

$$x^2 + y^2 = R^2 \tag{1}$$

Consider now how to get an analytic formula for it's area. An option that can be chosen is to cut up the circle squares as a first approximation. To make this the actual area thought a limit can be taken. As the number of sums goes to infinity, the size of each portion of area to be added goes to (dS, or $dxdy$).

$$S = \sum_{k}^{n} S_k \implies \lim_{n \to \infty} \sum_{k}^{n} S_k = \int_S dS$$

The differential of area can be expressed as a product of the respective differentials of the surface to be considered:

$$dS = dxdy$$

To evaluate the integral, it's limits must be defined. Consider defining the following:

$$\begin{cases} -1 \leq x \leq 1 \\ -1 \leq y \leq 1 \end{cases}$$

The latter would cover a square portion of area 4 as follows:

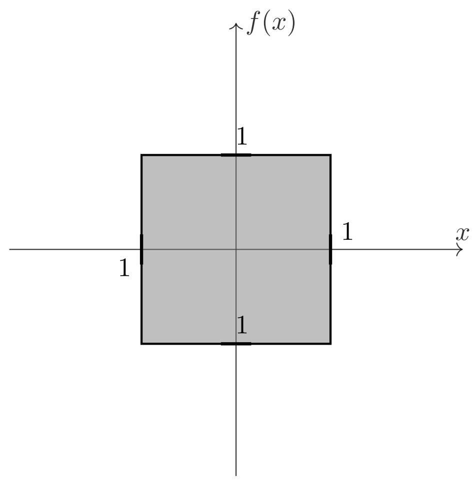

Figure 2: Centred rectangle of side length 2 limits of integration

The question then is finding the corresponding limits for the circle, consider fixing one of the variables, let it be x for the sake of argument. The other variable y is now automatically fixed following a function of x, otherwise the rectangle case would happen. In this case, using the equation for the circle, one obtains y as a function of x in the following manner:

$$y(x) = \pm\sqrt{R^2 - x^2}$$

Notice that this just gives 2 solutions, the upper and lower curve corresponding to the circle. Let it be considered just 1 (the positive for sake of argument) and multiply the area times 2 to obtain the full one. The whole integral will now be:

$$S = 2\int_0^R \int_0^{\sqrt{R^2-x^2}} dydx = 2\int_0^R \sqrt{R^2-x^2}dx$$

Using a change of variable:

$$\begin{cases} x = R\sin\theta \\ dx = R\cos\theta d\theta \end{cases}$$

The new limits of integration are given when $x = R$, therefore when $\sin\theta = 1$, meaning the integration bounds after the trig substitution are $0 \leq \frac{\pi}{2}$ The integral can be expressed as:

$$\begin{aligned} S &= 4\int_0^{\frac{\pi}{2}} \left(\sqrt{R^2 - R^2\sin\theta}\right) R\cos\theta d\theta \\ &= 4R^2\int_0^{\frac{\pi}{2}} \left(\cos^2\theta\right) d\theta = 4R^2\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right) d\theta \\ &= 2R^2\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta = 2R^2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} \\ &= \boxed{\pi R^2} \end{aligned}$$

1.2 Polar Coordinates

When using polar coordinates, an angle and a distance is defined always. This makes the limits of integration much easier and therefore the integral too:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} 0 \leq r \leq R \\ 0 \leq r \leq 2\pi \end{cases}$$

Therefore the area of the circle would be:

$$\int_S dS = \int_0^{2\pi}\int_0^R \det(\mathbf{J}) dr d\theta$$

Where $\mathbf{J}$ is the Jacobian which is in charge of taking into account a scaling factor when applying the coordinate system change and is defined as:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial \mathbf{f}(\mathbf{x})}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{f}(\mathbf{x})}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla^T f_1(\mathbf{x}) \\ \vdots \\ \nabla^T f_m(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1(\mathbf{x})}{\partial x_1} & \cdots & \dfrac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m(\mathbf{x})}{\partial x_1} & \cdots & \dfrac{\partial f_m(\mathbf{x})}{\partial x_n} \end{bmatrix}$$

In the case of the change in coordinate system that has been applied, which is a vector function ($\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$, with $(x, y) = \mathbf{T}(r, \theta) = (r\cos\theta, r\sin\theta)$):

$$\mathbf{J} = \begin{pmatrix} \frac{\partial}{\partial r}(r\cos\theta) & \frac{\partial}{\partial \theta}(r\cos\theta) \\ \frac{\partial}{\partial r}(r\sin\theta) & \frac{\partial}{\partial \theta}(r\sin\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

And therefore the determinant will give:

$$\det(\mathbf{J}) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Finally, the integral would give:

$$S = \int_0^{2\pi}\int_0^R r dr d\theta = 2\pi\left(\frac{1}{2}R^2\right) = \boxed{\pi R^2}$$

Using the latter one could also compute the area of any circular section as follows:

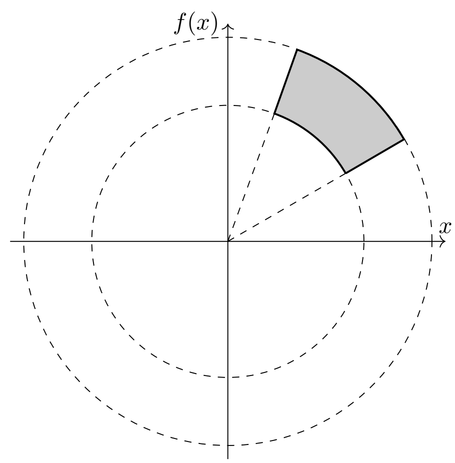

Figure 3: Circular Section

All that needs to be changed are the limits of integration:

$$\begin{cases} \theta_1 \le \theta \le \theta_2 \\ R_1 \le r \le R_2 \end{cases}$$

And therefore:

$$S = \int_{\theta_1}^{\theta_2}\int_{R_1}^{R_2} r dr d\theta = \boxed{\frac{1}{2}(\theta_2 - \theta_1)\left(R_2^2 - R_1^2\right)}$$

2 The Sphere's Area

Using the latter, the surface of a sphere will be calculated. For this a revolution will be used, the formula for the surface of revolution given any function of 1 variable is derived as follows.

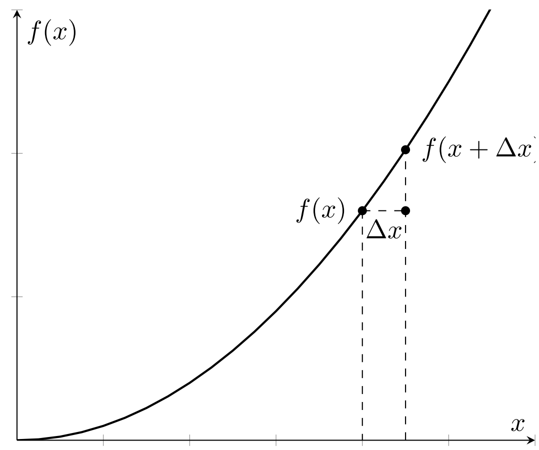

Figure 4: Line of revolution

The surface area that the line will create can be decomposed into many slices and added up. Consider a slice as shown in Figure 4 (comprised between x and $x + dx$), The surface area it would create would be (considering dx is small):

$$2\pi f(x)w \qquad \Longrightarrow \qquad S = 2\pi \lim_{n\to\infty}\sum_{i}^{n} f(x_i)w = 2\pi \int_a^b f(x)w$$

Where w represents the width. Using Pythagoras' Theorem, one can find the expression for the width in the following manner:

$$w(x) = \sqrt{\Delta x^2 + \left(f(x+\Delta x) - f(x)\right)^2}$$

But, if one recalls the formal definition of the derivative:

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$$

The expression can be rewritten as:

$$w(x) = \sqrt{\Delta x^2 + \Delta x^2 f'(x)^2} = \sqrt{1 + f'(x)^2}\Delta x$$

And adding up all the slices:

$$S = \lim_{n\to\infty}\sum_{i}^{n} f(x_i)\sqrt{1 + f'(x_i)}\Delta x = \boxed{2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}dx}$$

The function that will be revolved in this particular case needs to generate a sphere once rotated. To simplify calculations, a quarter of a sphere has been taken, and the final area will be multiplied times 2. Note that if more than half the sketch of a circle is taken, superposition will come into play and give an erroneous result.

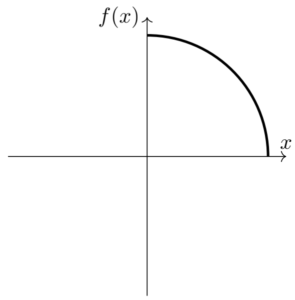

Figure 5: Line for revolution

The function defining this line is $f(x) = \sqrt{R^2 - x^2}$ with x defined between 0 and R, therefore the surface area is:

$$
\begin{aligned}
A &= 2 \cdot 2\pi \int_0^R \sqrt{R^2 - x^2} \sqrt{1 + \left[\frac{d}{dx}\left(\sqrt{R^2 - x^2}\right)\right]^2} dx \\
&= 4\pi \int_0^R \sqrt{R^2 - x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{R^2 - x^2}}\right)^2} dx \\
&= 4\pi \int_0^R \sqrt{R^2 - x^2 + x^2} dx = 4\pi R \int_0^R dx \\
&= \boxed{4\pi R^2}
\end{aligned}
$$